Discrete image transform

$$\mathbf{y} = \mathbf{T}\mathbf{x}$$

$$y_i = \sum_{j=0}^{N-1} t_{ij} x_j \qquad \text{For } i = 0,1,\ldots,N-1$$

$$\mathbf{x} = \mathbf{T}^{-1}\mathbf{y} \qquad \mathbf{T} \text{ is not singular matrix}$$

Unitary transform:
Unitary matrix:
$\mathbf{T}^{-1} = (\mathbf{T}^*)^t$ and $\mathbf{T}(\mathbf{T}^*)^t = (\mathbf{T}^*)^t\mathbf{T} = \mathbf{I}$

Discrete image transform

Example

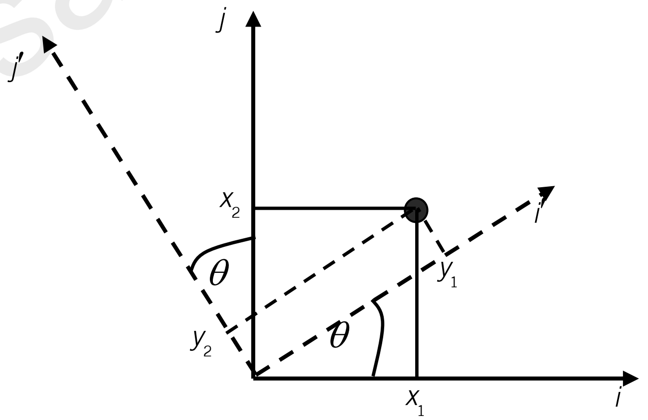
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\mathrm{x} = x_1\mathrm{i} + x_2\mathrm{j}$

New axis: $\mathrm{i}' = \cos(\theta)\,\mathrm{i} + \sin(\theta)\,\mathrm{j}$

New axis: $\mathrm{j}' = -\sin(\theta)\,\mathrm{i} + \cos(\theta)\,\mathrm{j}$

Hence: $\mathrm{y} = y_1\mathrm{i}' + y_2\mathrm{j}'$



Discrete image transform

Fourier Transform

$$F(k) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} f(x) e^{-j\frac{2\pi k}{N} i}$$

Let matrix **W** is a matrix with $w_{i,k}$ is $\frac{1}{\sqrt{N}} e^{-j\frac{2\pi k}{N} i}$

then $\mathbf{F} = \mathbf{W}\mathbf{f}$

If **T** is real and unitary → orthonormal transform

$$\mathbf{T}^{-1} = \mathbf{T}^{t} \text{ and } \mathbf{T}\,\mathbf{T}^{t} = \mathbf{T}^{t}\mathbf{T} = \mathbf{I}$$

Inner product of row i and j of matrix **T** $=$ 0 except $i = j$

$$\mathbf{t}_i^{\,t}\mathbf{t}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

2-D discrete transfrom

$$\mathtt{G}(m,n) = \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} \mathtt{F}(i,k)\, \underline{\mathfrak{I}(i,k,m,n)}$$

i,*k* element in *m*,*n* block

$$\mathfrak{I} = \begin{array}{cc} \begin{bmatrix} [\ ]_{N\times N} & [\ ]_{N\times N} & \cdots & [\ ]_{N\times N} \\ [\ ]_{N\times N} & [\ ]_{N\times N} & \cdots & [\ ]_{N\times N} \\ \vdots & \vdots & \ddots & \vdots \\ [\ ]_{N\times N} & [\ ]_{N\times N} & \cdots & [\ ]_{N\times N} \end{bmatrix} & \begin{array}{l} m=1 \\ m=2 \\ \\ m=N \end{array} \\ \begin{array}{cccc} n=1 & n=2 & \quad & n=N \end{array} & \end{array}$$

$$\mathtt{F}(i,k) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathtt{G}(m,n)\, \mathfrak{R}(i,k,m,n)$$

$\mathfrak{R}$ ➔ inverse transform kernel

2-D discrete transfrom

If $\Im(i,k,m,n)$ is: $\Im(i,k,m,n) = T_r(i,m)\ T_c(k,n) \rightarrow$ seperable transform

$$G(m,n) = \sum_{i=0}^{N-1}\left[\sum_{k=0}^{N-1} F(i,k) T_c(k,n)\right] T_r(i,m)$$

If $T_r = T_c = T \rightarrow$ symmetry transform. But T does not have to be symmetry matrix.

$$G(m,n) = \sum_{i=0}^{N-1} T(i,m)\left[\sum_{k=0}^{N-1} F(i,k) T(k,n)\right]$$

$$G = T^t F T \text{ and } F = (T^t)^{-1} G T^{-1}$$

If T : symmetry matrix. $G = TFT$ and $F = T^{-1} G T^{-1}$

if T is unitary matrix $F = (T^{*t}) G (T^{*t})$

if T is unitary matrix and real $F = (T^t) G (T^t)$

if T is symmetry $F = (T) G (T)$

2-D discrete transfrom

Example Discrete Fourier transform

$$\mathrm{G}(m,n)=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}\left[\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\mathrm{F}(i,k)e^{-j2\pi\left(n\frac{k}{N}\right)}\right]e^{-j2\pi\left(m\frac{i}{N}\right)}$$

$$\Im(i,k,m,n)=\left(\frac{1}{\sqrt{N}}e^{-j2\pi\left(n\frac{k}{N}\right)}\right)\left(\frac{1}{\sqrt{N}}e^{-j2\pi\left(m\frac{i}{N}\right)}\right)=\mathrm{W}(k,n)\mathrm{W}(i,m)$$

$$\mathrm{G}=\mathrm{W}^t\mathrm{F}\mathrm{W}$$

$$\mathrm{F}=(\mathrm{W}^*)\mathrm{G}(\mathrm{W}^{*t})$$

-rows of kernel matrix forms a set of basis vector for an N-dimensional vector space

-vector element are usually formed by the same functional form → basis function

2-D discrete transfrom

if T is unitary matrix $\mathrm{F} = (\mathrm{T}^*)\mathrm{G}(\mathrm{T}^{*t})$

let $\mathrm{A} = (\mathrm{T}^{*t}) \rightarrow \mathrm{F} = (\mathrm{A}^t)\mathrm{G}(\mathrm{A})$

$$\mathrm{A}^t = \begin{bmatrix} \xleftrightarrow{\quad \mathrm{a}_0{}^t \quad} \\ \xleftrightarrow{\quad \mathrm{a}_1{}^t \quad} \\ \vdots \\ \xleftrightarrow{\quad \mathrm{a}_{N-1}{}^t \quad} \end{bmatrix}$$

Let $\mathbf{a}_m{}^t$ and $\mathbf{a}_n{}^t \rightarrow$ m[th] row vector and n[th] row vector of $\mathrm{A}^t \rightarrow \mathrm{F}(m,n) = (\mathrm{a}_m{}^t)\mathrm{G}(\mathrm{a}_n)$

let

$$\begin{array}{cc} \mathrm{C}_{pq} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & & 0 \\ 0 & \vdots & & & 0 \\ 0 & 0 & \cdots & & 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ p \\ \\ \\ \\ \end{array} \\ \qquad\qquad q & \end{array}$$

Let $\mathrm{B}_{pq}(m,n) = (\mathrm{a}_m{}^t)\mathrm{C}_{pq}(\mathrm{a}_n)$ or

$$\mathrm{B}_{pq} = \begin{bmatrix} \mathrm{A}(m,0) & \mathrm{A}(m,1) & \cdots & \mathrm{A}(m,N-1) \end{bmatrix} \mathrm{C}_{pq} \begin{bmatrix} \mathrm{A}(n,0) \\ \mathrm{A}(n,0) \\ \vdots \\ \mathrm{A}(n,0) \end{bmatrix}$$

$$\mathrm{B}_{pq}(m,n) = \mathrm{A}(m,p)\ \mathrm{A}(n,q) = \mathrm{T}^*(p,m)\mathrm{T}^*(q,n)$$

$$\mathrm{B}_{pq} = (\mathrm{t}_p{}^{*t})\mathrm{t}_q{}^*$$

Hence $$\mathrm{F} = \mathrm{G}(0,0)\mathrm{B}_{00} + \mathrm{G}(0,1)\mathrm{B}_{01} + \ldots + \mathrm{G}(N-1,N-1)\mathrm{B}_{N-1N-1}$$

$\mathrm{B}_{pq} \rightarrow$ Basis image

Example $F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$G = T^t F T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}$$

$$F = \left(T^t\right)^{-1} G T^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Use basis image

$$B_{00} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B_{01} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = B_{10}^t$$

$$B_{11} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F = G(0,0)B_{00} + G(0,1)B_{01} + G(1,0)B_{10} + G(1,1)B_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

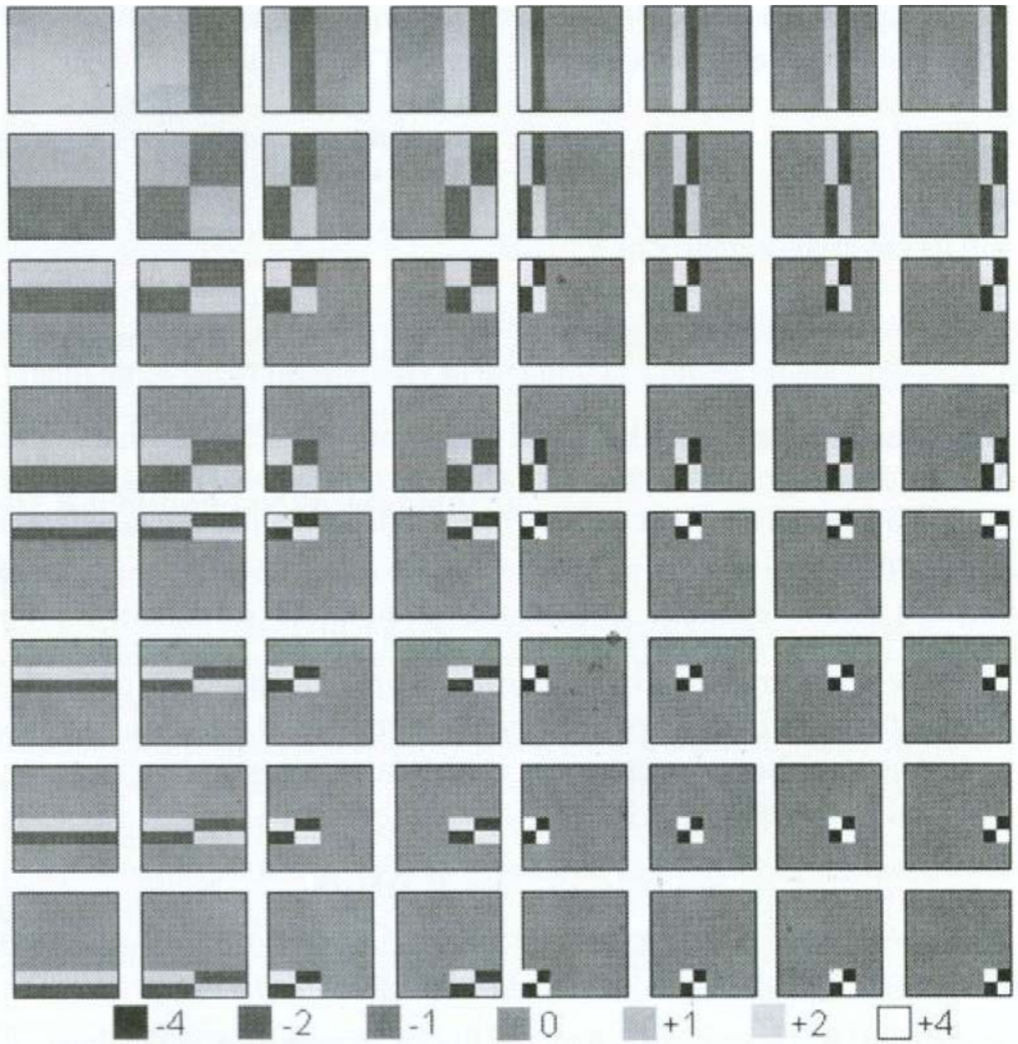


Figure 13–6 The Haar transform basis images for $N = 8$

Discrete Cosine Transform

Forward $$G(m,n)=\alpha(m)\alpha(n)\sum_{i=0}^{N-1}\sum_{k=0}^{N-1}F(i,k)\cos\left(\frac{\pi(2i+1)m}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{2N}\right)$$

Inverse $$F(i,k)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\alpha(m)\alpha(n)G(m,n)\cos\left(\frac{\pi(2i+1)m}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{2N}\right)$$

When $\alpha(0)=\sqrt{\frac{1}{N}}$ and $\alpha(m)=\sqrt{\frac{2}{N}}$ for $1\le m\le N$

$$C(i,m)=\alpha(m)\cos\left(\frac{\pi(2i+1)m}{2N}\right) \qquad C(k,n)=\alpha(n)\cos\left(\frac{\pi(2k+1)n}{2N}\right)$$

$$G = C^t F C$$

-Xform → real

-Cosine Xform is not the real part of DFT

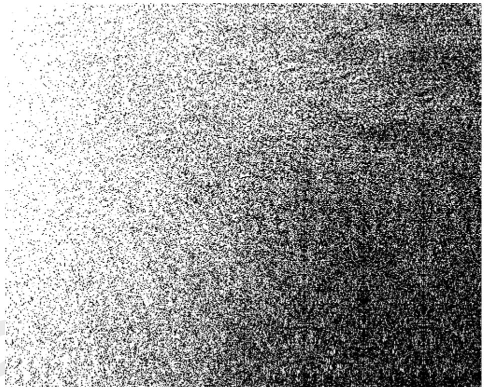
-There is a fast version

-has excellent energy compression properties

-use in image compression

Input

DCT



IDCT and GHPF at 100 cutoff frequency

IDCT and GLPF at 100 cutoff frequency

Hartley Transform

Forward $$\mathtt{G}(m,n)=\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1}\mathtt{F}(i,k)\,\mathrm{cas}\left(\frac{2\pi}{N}(im+kn)\right)$$

Inverse $$\mathtt{F}(i,k)=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\mathtt{G}(m,n)\,\mathrm{cas}\left(\frac{2\pi}{N}(im+kn)\right)$$

When $$\mathrm{cas}\left(\frac{2\pi}{N}(im+kn)\right)=\cos\left(\frac{2\pi}{N}(im+kn)\right)+\sin\left(\frac{2\pi}{N}(im+kn)\right)=\sqrt{2}\cos\left(\left(\frac{2\pi}{N}(im+kn)\right)-\frac{\pi}{4}\right)$$

Basis function $$\mathtt{T}(i,m)=\mathrm{cas}\left(\frac{2\pi}{N}(im)\right)$$

Walsh Hadamard Transform

Forward Walsh transform

$$G(m,n)=\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1}F(i,k)\prod_{j=0}^{n-1}(-1)^{\left[b_j(i)b_{n-1-j}(m)+b_j(k)b_{n-1-j}(n)\right]}$$

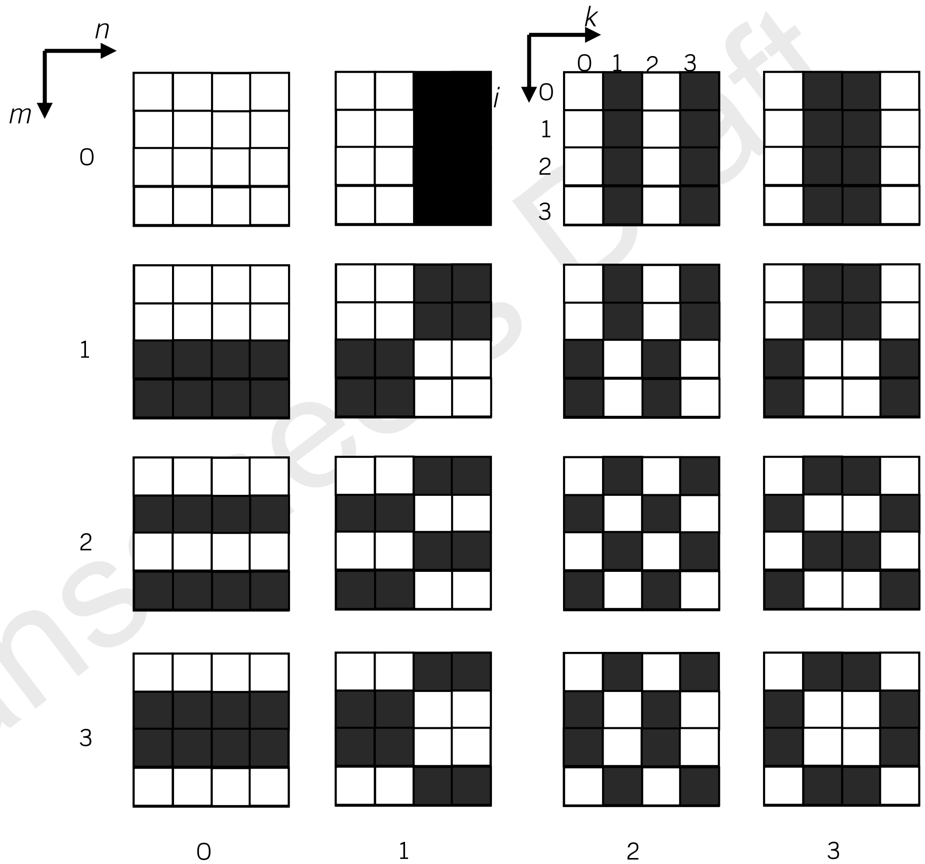
Inverse Walsh transform

$$F(i,k)=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}G(m,n)\prod_{j=0}^{n-1}(-1)^{\left[b_j(i)b_{n-1-j}(m)+b_j(k)b_{n-1-j}(n)\right]}$$

When $b_k(z)$ is k-th bit in the binary representation of z . For example $n = 3$ $(N = 8)$ and $z = 4$ (binary 100), hence $b_0(z) = 0$, $b_1(z) = 0$ and $b_2(z) = 1$

Walsh Hadamard Transform

Basis function of Walsh transform for 4×4 image



Walsh Hadamard Transform

Forward hadamard transform

$$G(m,n)=\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1}F(i,k)(-1)^{\sum_{j=0}^{n-1}\left[b_j(i)b_j(m)+b_j(k)b_j(n)\right]}$$

Inverse Hadamard transform

$$F(i,k)=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}G(m,n)(-1)^{\sum_{j=0}^{n-1}\left[b_j(i)b_j(m)+b_j(k)b_j(n)\right]}$$

When $b_k(z)$ is k-th bit in the binary representation of z . For example $n = 3$ $(N = 8)$ and $z = 4$ (binary 100), hence $b_0(z) = 0$, $b_1(z) = 0$ and $b_2(z) = 1$

Walsh Hadamard Transform

$$T = \frac{1}{\sqrt{2}} H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$T = \frac{1}{\sqrt{N}} H_N = \frac{1}{\sqrt{N}} \begin{bmatrix} H_{N/2} & H_{N/2} \\ H_{N/2} & -H_{N/2} \end{bmatrix}$$

Example create T for 8×8 image transform

$$T = \frac{1}{\sqrt{8}} H_8 = \frac{1}{2\sqrt{2}} \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} \qquad H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Walsh Hadamard Transform

$$T = \frac{1}{\sqrt{8}} H_8 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Reorder rows so that sign change count is in increasing order

$$T = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Harr Transform

Harr function → change in scale and location

→ [0,1]

Let $0 \le k \le N-1$ and $k = 2^p + q - 1$

p → largest integer such that $2^p \le k$,e.g.,

for $k = 0$ → $p = 0$ and $q = 0$

for $k = 1$ → $p = 0$ and $q = 1$

for $k = 2$ → $p = 1$ and $q = 1$

for $k = 3$ → $p = 1$ and $q = 2$

$$T = \begin{bmatrix} \xleftrightarrow{\quad h_0^t \quad} \\ \xleftrightarrow{\quad h_1^t \quad} \\ \vdots \\ \xleftrightarrow{\quad h_{N-1}^t \quad} \end{bmatrix}$$

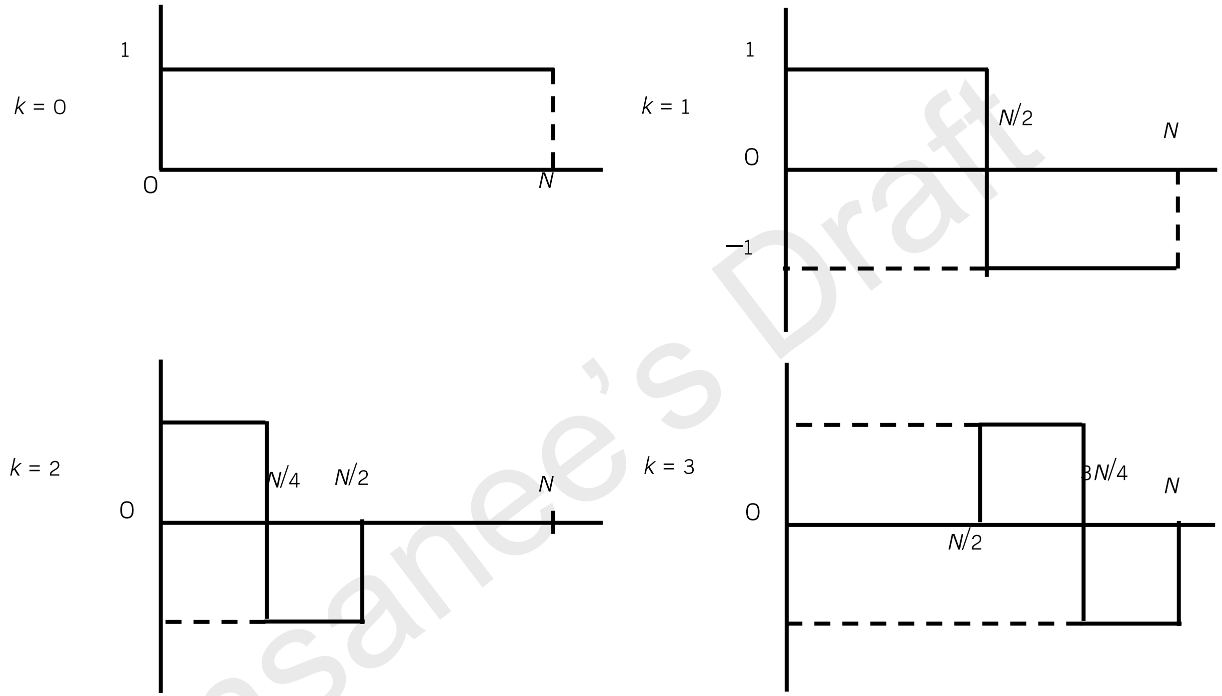
Basis function

$$h_0 = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}_{1\times N}$$

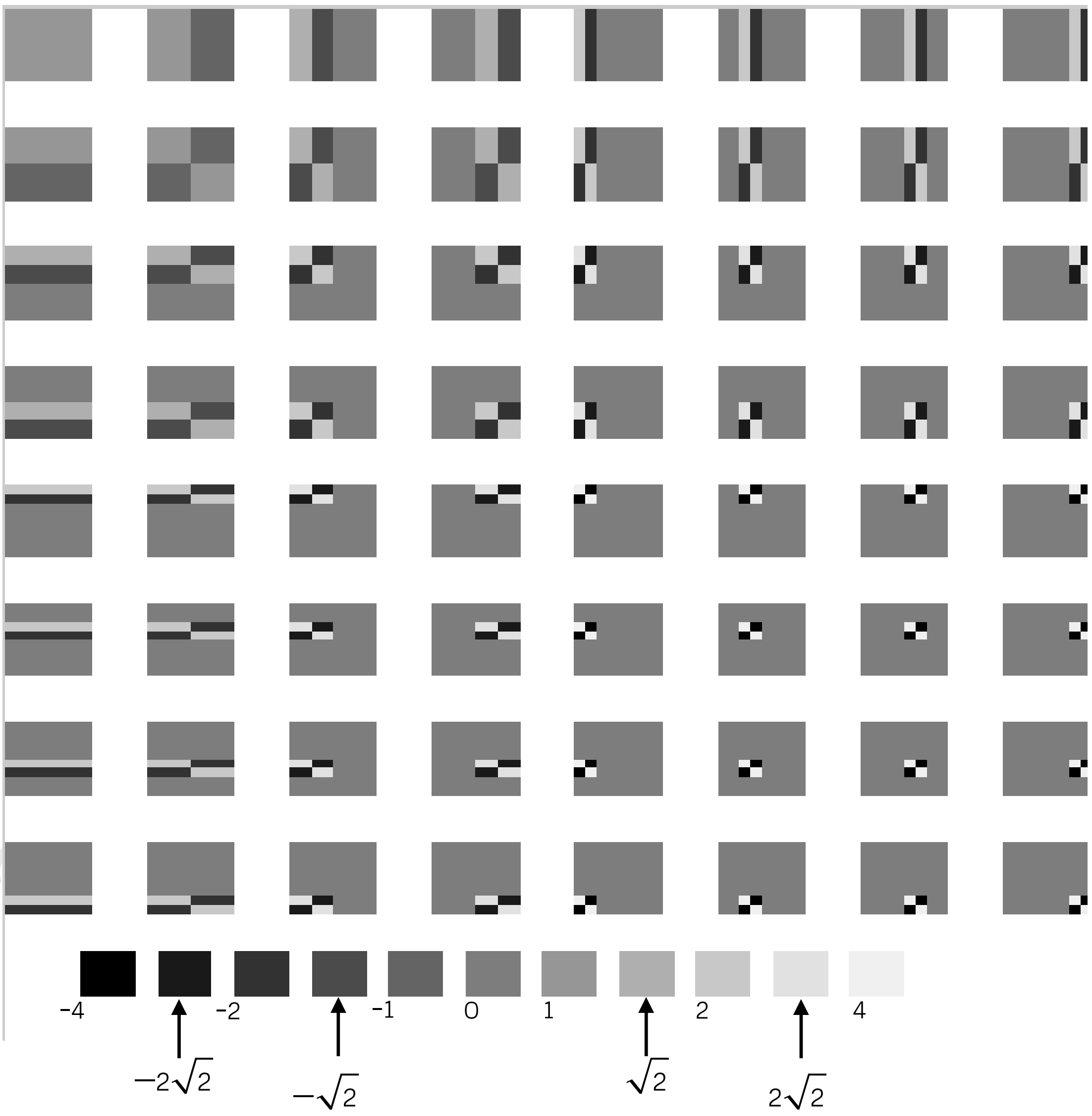
$$h_k = \frac{1}{\sqrt{N}} \begin{bmatrix} h_k(x) \end{bmatrix}_{1\times N}$$

when

$$h_k(x) = \begin{cases} 2^{p/2} & \text{if } \dfrac{q-1}{2^p} \le x < \dfrac{q - \frac{1}{2}}{2^p} \\ -2^{p/2} & \text{if } \dfrac{q - \frac{1}{2}}{2^p} \le x < \dfrac{q}{2^p} \\ 0 & \text{otherwise} \end{cases}$$



k = 0
1
0
N
k = 1
1
0
N/2
N
−1
k = 2
0
N/4
N/2
N
k = 3
0
N/2
3N/4
N



-4
-2
-1
0
1
2
4
$-2\sqrt{2}$
$-\sqrt{2}$
$\sqrt{2}$
$2\sqrt{2}$

$$T = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$