Wavelet Transform

- Wavelet (Ondelettes) → tell where it occur, how much it occurs
  - Basis function vary in position as well as ferquency
  - Waves of limited duration
  - Haar function →oldest simplest wavelet

Wavelet Transform

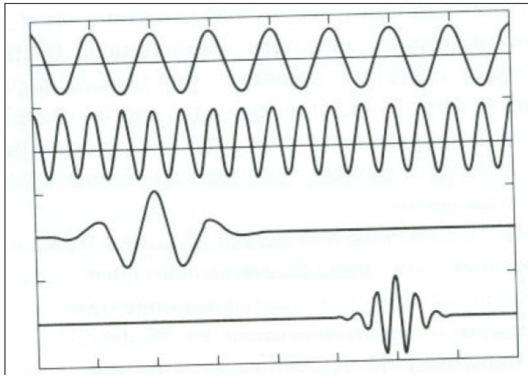


Figure 14–1 Waves and wavelets

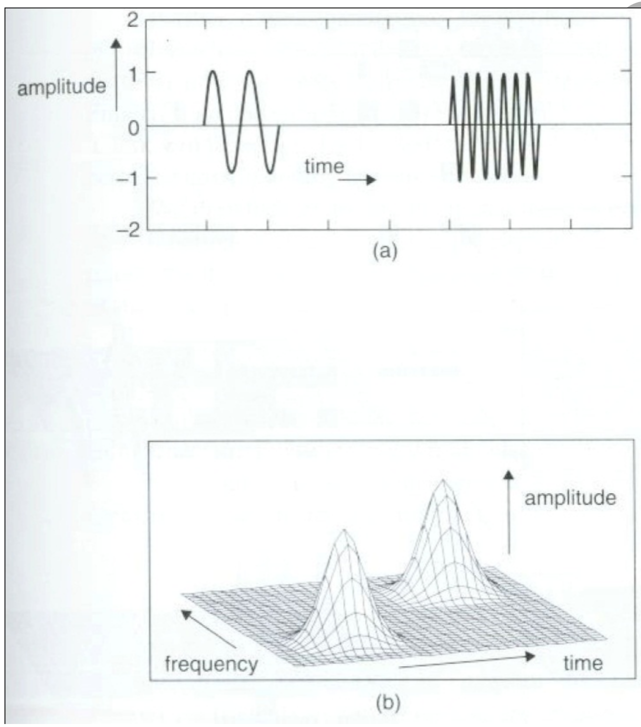


Figure 14–2 Time-frequency space: (a) signal; (b) representation

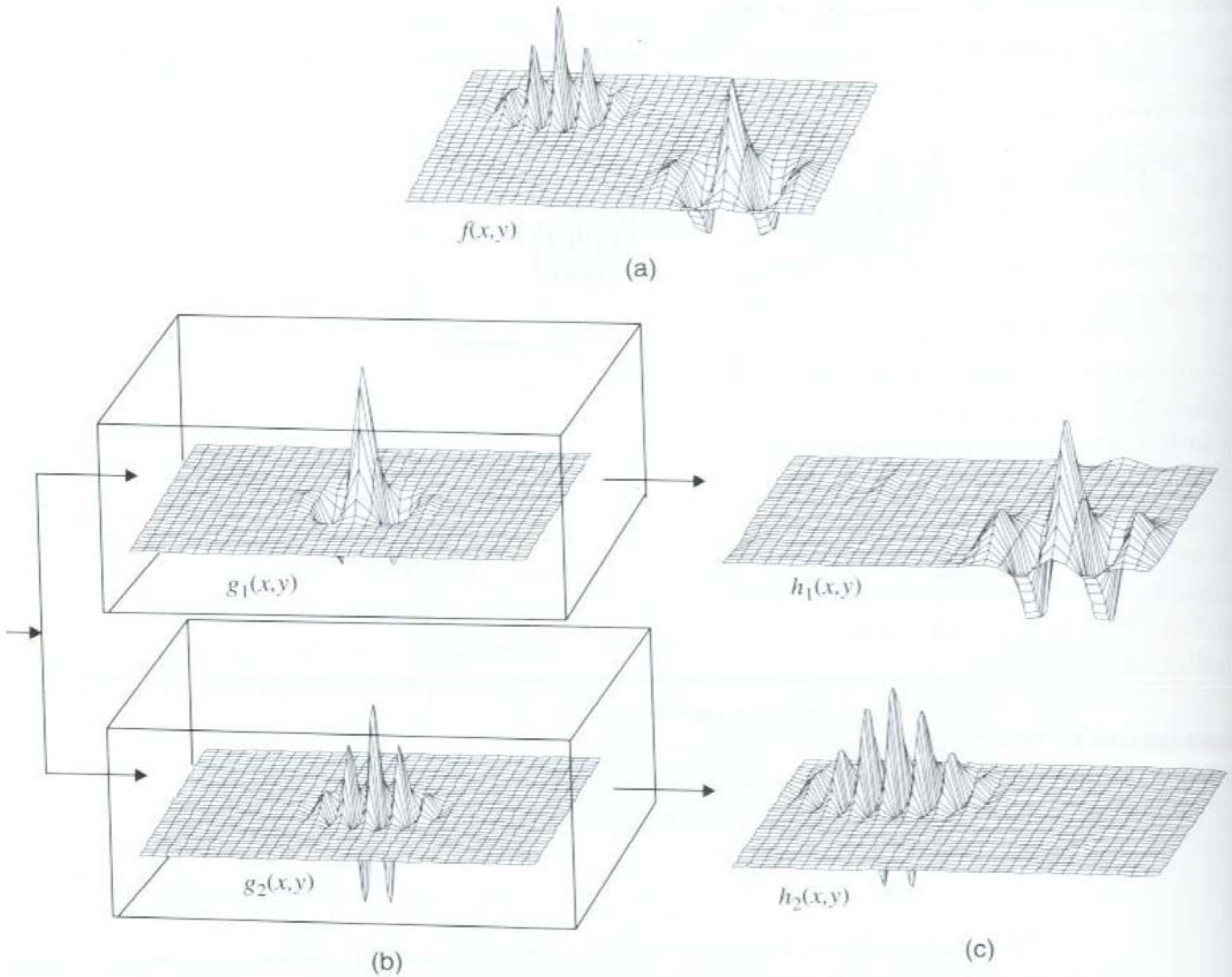
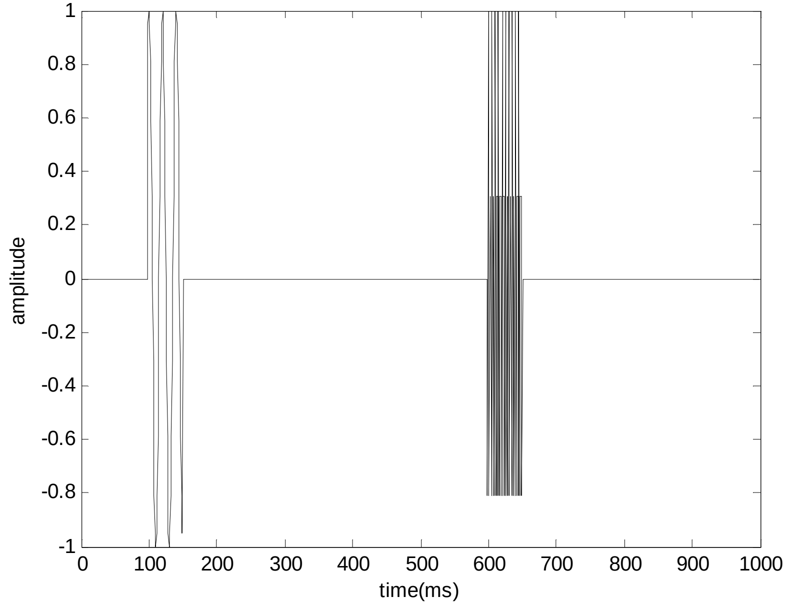
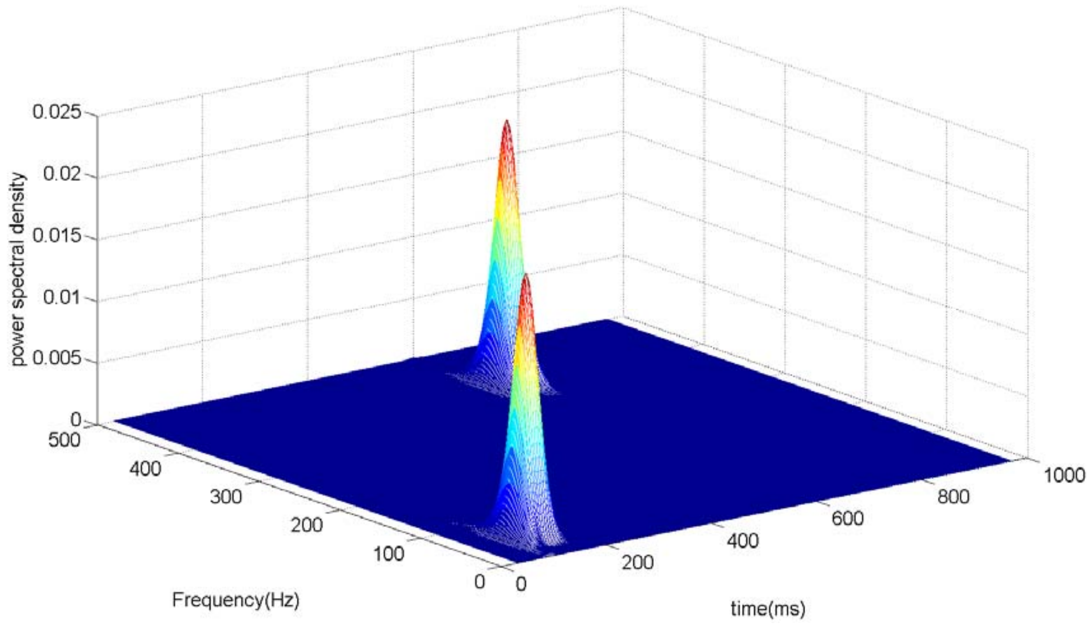


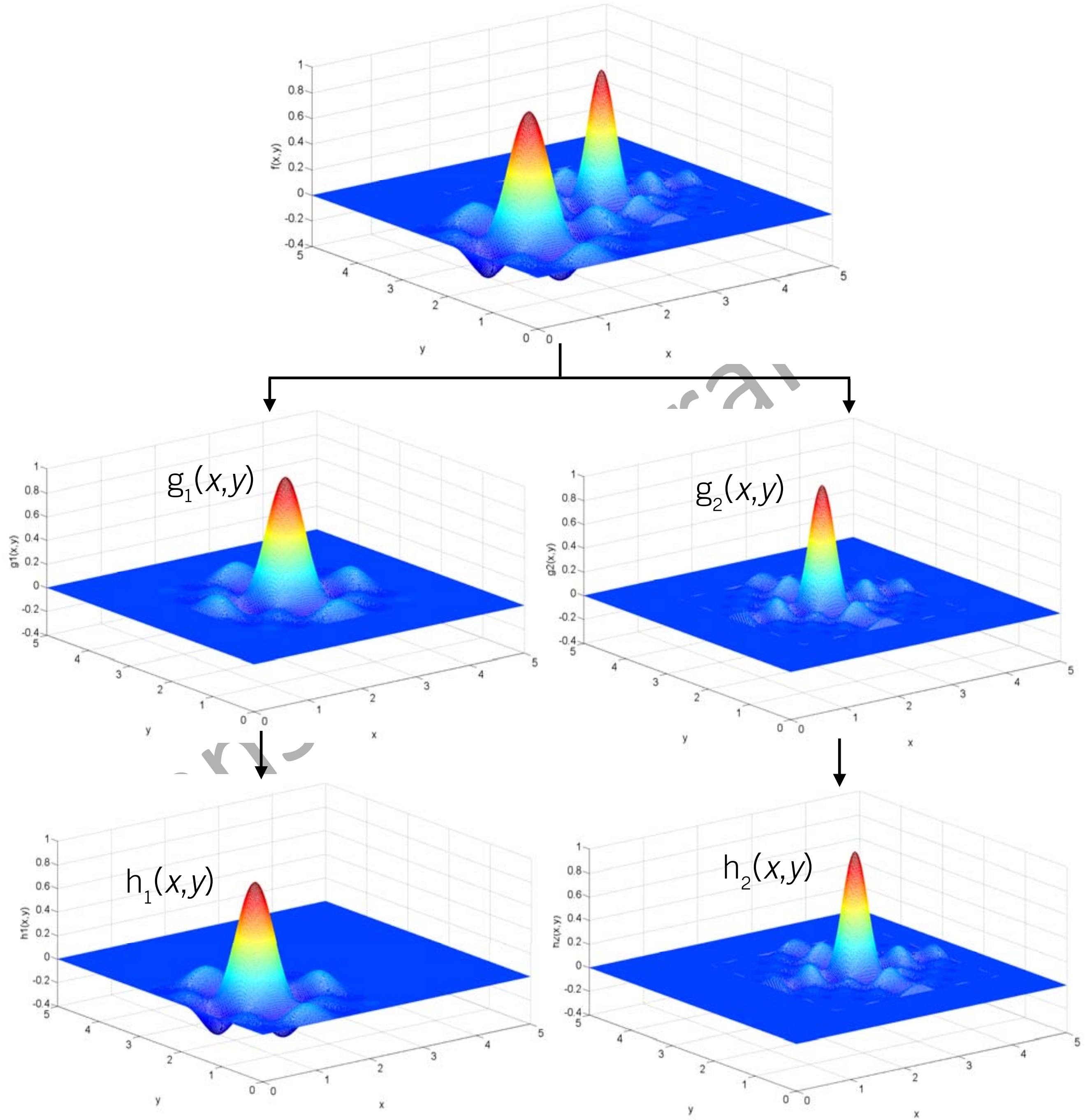
Figure 14–3 Space-frequency analysis of an image

Figure 14–4 Musical notation as a time-frequency plane

Wavelet Transform







f(x,y)
g1(x,y)
g2(x,y)
h1(x,y)
h2(x,y)

Wavelet Transform

- Type
  - Continuous wavelet transform
  - Wavelet series expansion
  - Discrete wavelet transform
- Wavelet basis function
  - Not necessary orthogonal
  - Compact support
- Fourier basis function
  - Not compact support
  - orthogonal

Continuous Wavelet Transform

- $\psi(x)$ →mother function→real-valued function whose Fourier spectrum $\psi(s)$ satisfied the admissibility criterion

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi(s)|^2}{|s|} ds$$

  - Also called basis function
  - If $C_\psi < \infty$ → $\psi(s)\big|_{s=0} = 0$ → $\int_{-\infty}^{\infty} \psi(x)\,dx = 0$ → $\psi(s)\big|_{s\to\infty} = 0$
  - $\psi(x)$→at origin has a value of 0

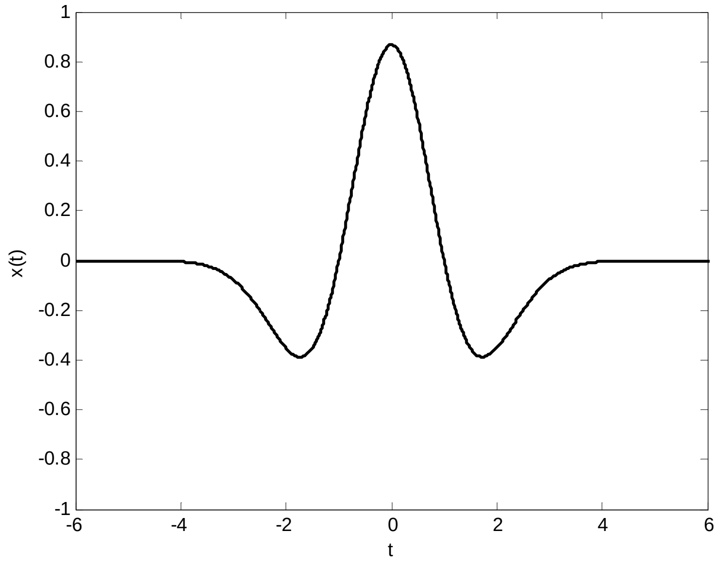
Continuous Wavelet Transform

- Amplitude spectrum of an admissible wavelet is similar to the transfer function of a bandpass filter → any bandpass filter impulse response with 0 mean that decays to zero fast enough with increasing frequency can serve as a basis function
- Each basis function → 1 row of **T**
- a set of wavelet basis function $\{\psi_{a,b}(x)\}$

Continuous Wavelet Transform

- $\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$ with a>0 and b→real number
  - a→scale parameter
  - b→ translated position along x-axis

$$\psi(x) = \frac{2}{\sqrt{3\sqrt{\pi}}}\left(1-x^2\right)e^{-\frac{x^2}{2}}$$



Continuous Wavelet Transform

- Forward

$$w_f(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} f(x)\psi_{a,b}(x)\,dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right)dx$$

- Inverse

$$f(x) = \frac{1}{C_\psi} \int_{0}^{\infty} \int_{-\infty}^{\infty} w_f(a,b)\psi_{a,b}(x)\,db\,\frac{da}{a^2}$$

- From inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$ and

$$\int_{-\infty}^{\infty} \left(f(x) - g(x)\right)^2 dx = \int_{-\infty}^{\infty} f(x)^2\,dx + \int_{-\infty}^{\infty} g(x)^2\,dx - 2\int_{-\infty}^{\infty} f(x)g(x)\,dx$$

Continuous Wavelet Transform

- Hence, if $\int_{-\infty}^{\infty} f(x)g(x)dx$ is big then $\int_{-\infty}^{\infty} (f(x)-g(x))^2 dx$ will be small
- If $\int_{-\infty}^{\infty} (f(x)-g(x))^2 dx$ is closed to 0, *f*(*x*) is similar to *g*(*x*)
- From correlation and $\psi_a(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x}{a}\right)$ then

$$W_f(a,b) = \int_{-\infty}^{\infty} f(x)\psi_a(x+b)dx = f \circledast \psi_a$$

Continuous Wavelet Transform

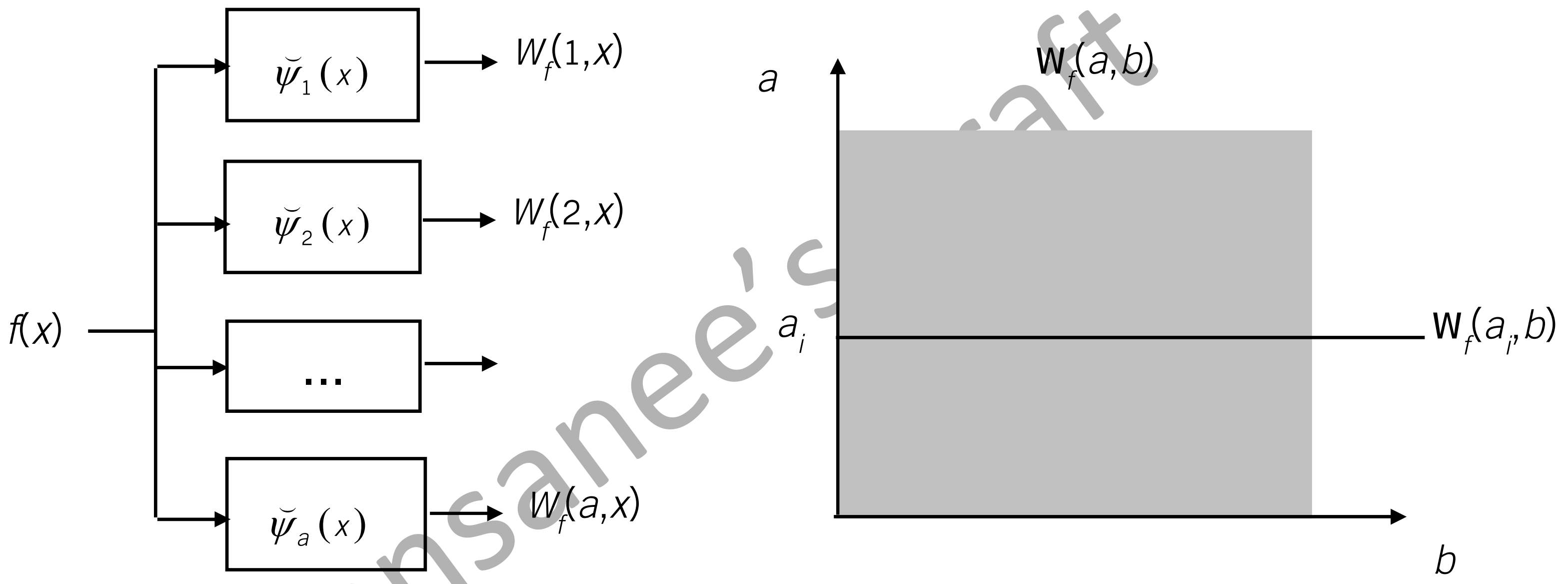
- Or convolution between *f*(*x*) and reflected complex conjugate of the scale wavelet

$$\breve{\psi}_a(x) = \psi_a^*(x) = \frac{1}{\sqrt{a}}\psi^*\left(-\frac{x}{a}\right)$$

$$W_f(a,b) = \int_{-\infty}^{\infty} f(x)\breve{\psi}_a(b-x)\,dx = f*\breve{\psi}_a$$

$$f(x) = \frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^{\infty}\left[f*\breve{\psi}_a\right](b)\psi_a(b-x)\,db\frac{da}{a^2} = \frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^{\infty}\left[f*\breve{\psi}_a*\psi_a\right](x)\frac{da}{a^2}$$

Continuous Wavelet Transform



2-D Continuous Wavelet Transform

- Forward

$$\mathrm{W}_f\left(a,b_x,b_y\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{f}(x,y)\psi_{a,b_x,b_y}(x,y)\,dxdy$$

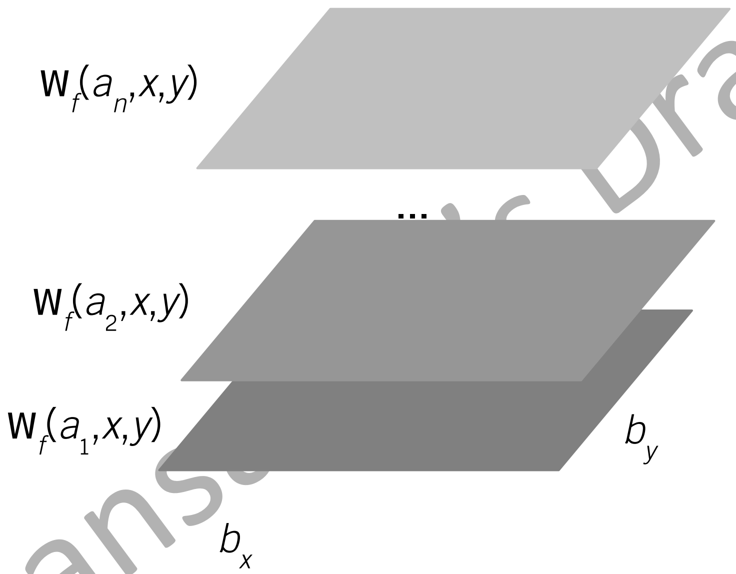
  – When (b_x and b_y → translation in x and y direction

$$\psi_{a,b_x,b_y}(x,y)=\frac{1}{|a|}\psi\left(\frac{x-b_x}{a},\frac{y-b_y}{a}\right)$$

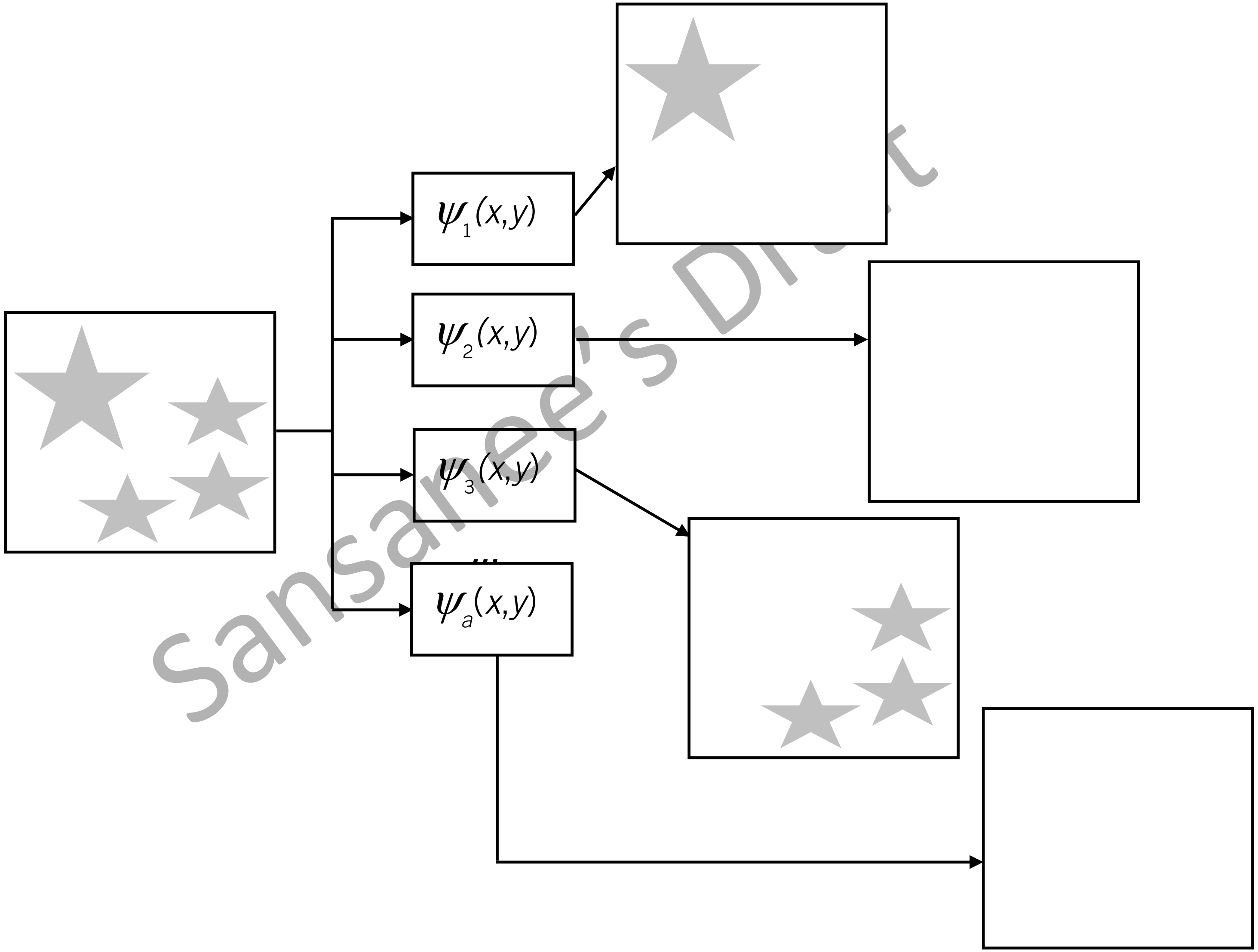
- Inverse

$$\mathrm{f}(x,y)=\frac{1}{C_\psi}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{W}_f\left(a,b_x,b_y\right)\psi_{a,b_x,b_y}(x,y)\,db_x db_y\frac{da}{a^3}$$

2-D Continuous Wavelet Transform



2-D Continuous Wavelet Transform



Wavelet Series Expansion

- Dyadic wavelet→basis wavelet is scaled and translated to form a set of basis function→however the scaling and translation are specified by integer not real number
  - Binary scaling (shrink by factor of 2)
  - Dyadic translation →shift by $k/2^j$

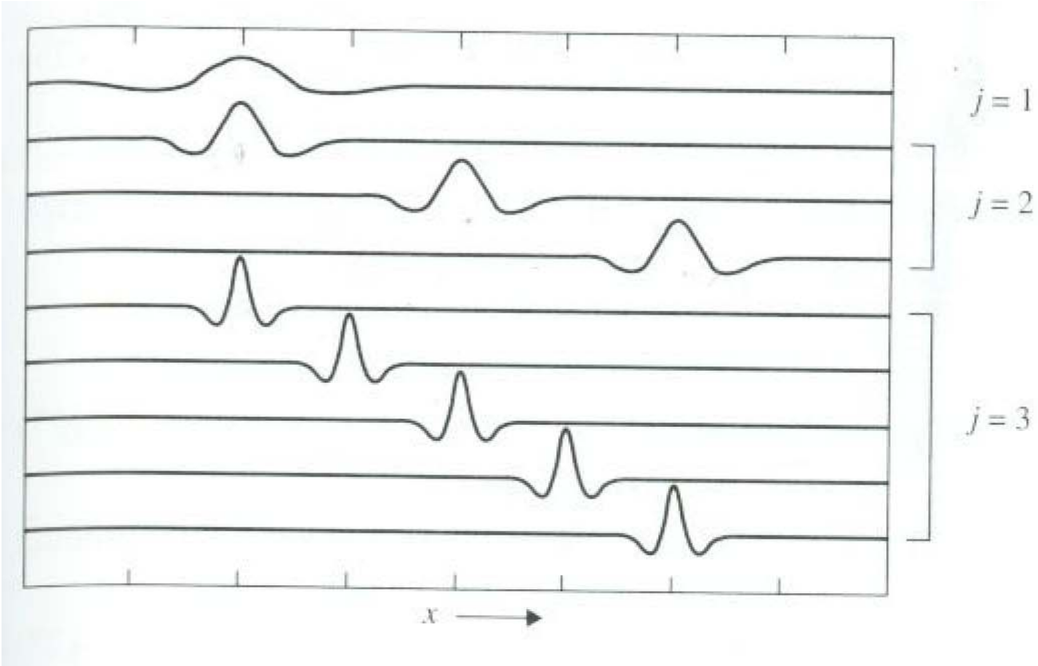


Figure 14–8 Binary scalings and dyadic translations of a wavelet

Wavelet Series Expansion

- A function $\psi(x)$ is an orthogonal wavelet if the set $\{\psi_{j,k}(x)\}$ of function is

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi\left(2^j x - k\right)$$

$-\infty < j,k < \infty$

- Form orthogonal if $\left\langle \psi_{j,k}, \psi_{l,m} \right\rangle = \delta_{j,l}\delta_{k,m}$
- Series expansion

$$f(x) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(x)$$

- when $c_{j,k} = \left\langle f, \psi_{j,k} \right\rangle = 2^{\frac{j}{2}}\int_{-\infty}^{\infty} f(x)\psi\left(2^j x - k\right)dx$

Wavelet Series Expansion

- Compact dyadic wavelets

$$\psi_n(x) = 2^{\frac{j}{2}}\psi\left(2^j x - k\right)$$

  - When $n = 2^j + k$ for j=0,1,2,... and k=0,1,2,...,2^j–1

- Hence $f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x)$

- Assume $\psi_0(x) = 1$
- And

$$c_n = \langle f, \psi_n \rangle = 2^{\frac{j}{2}} \int_{-\infty}^{\infty} f(x)\psi\left(2^j x - k\right)dx$$

Haar transform→dyadic orthonormal wavelet transform

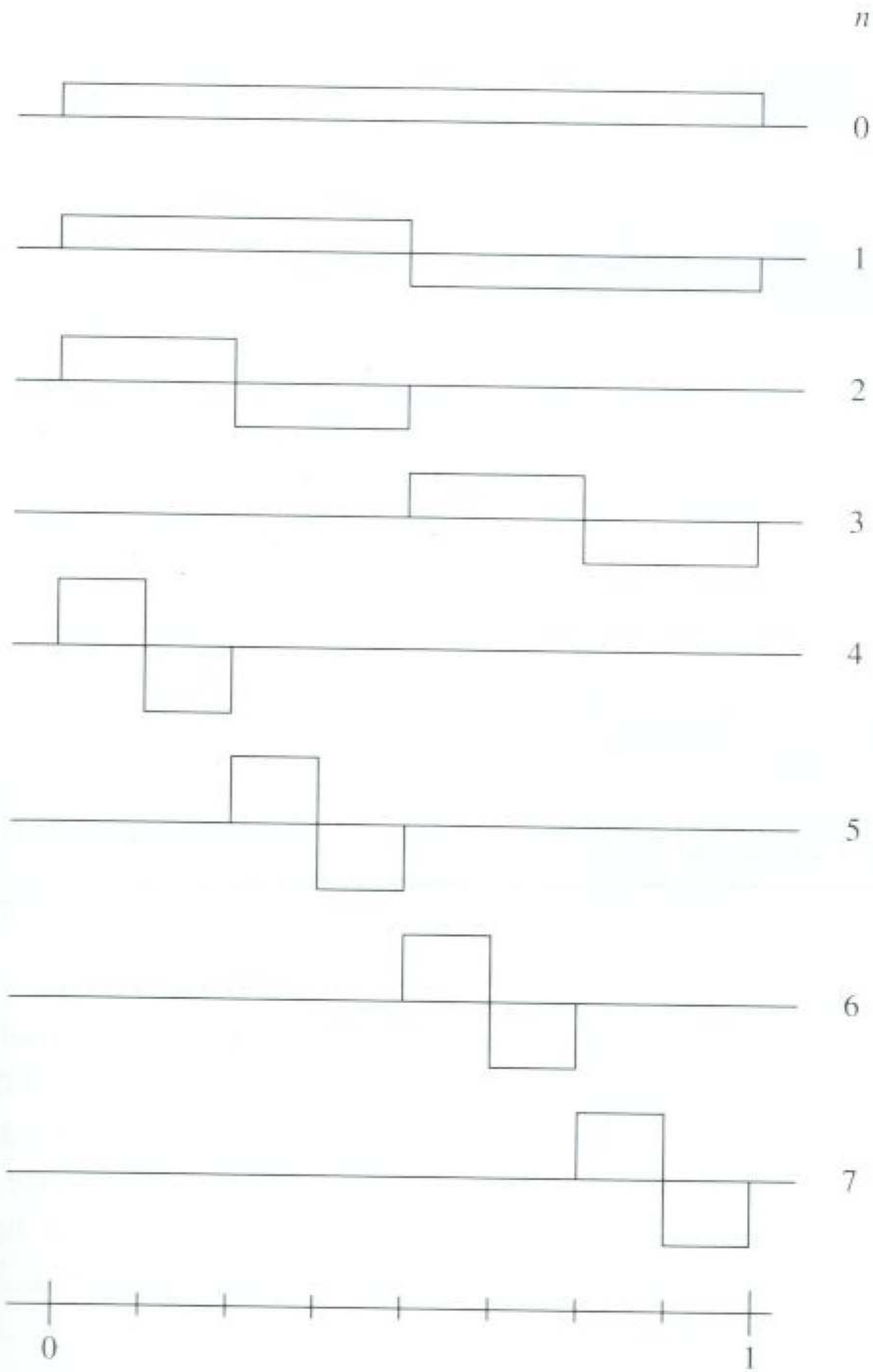


Figure 14–9 The Haar transform basis functions

Example Find Wavelet series expansion of

$$y = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

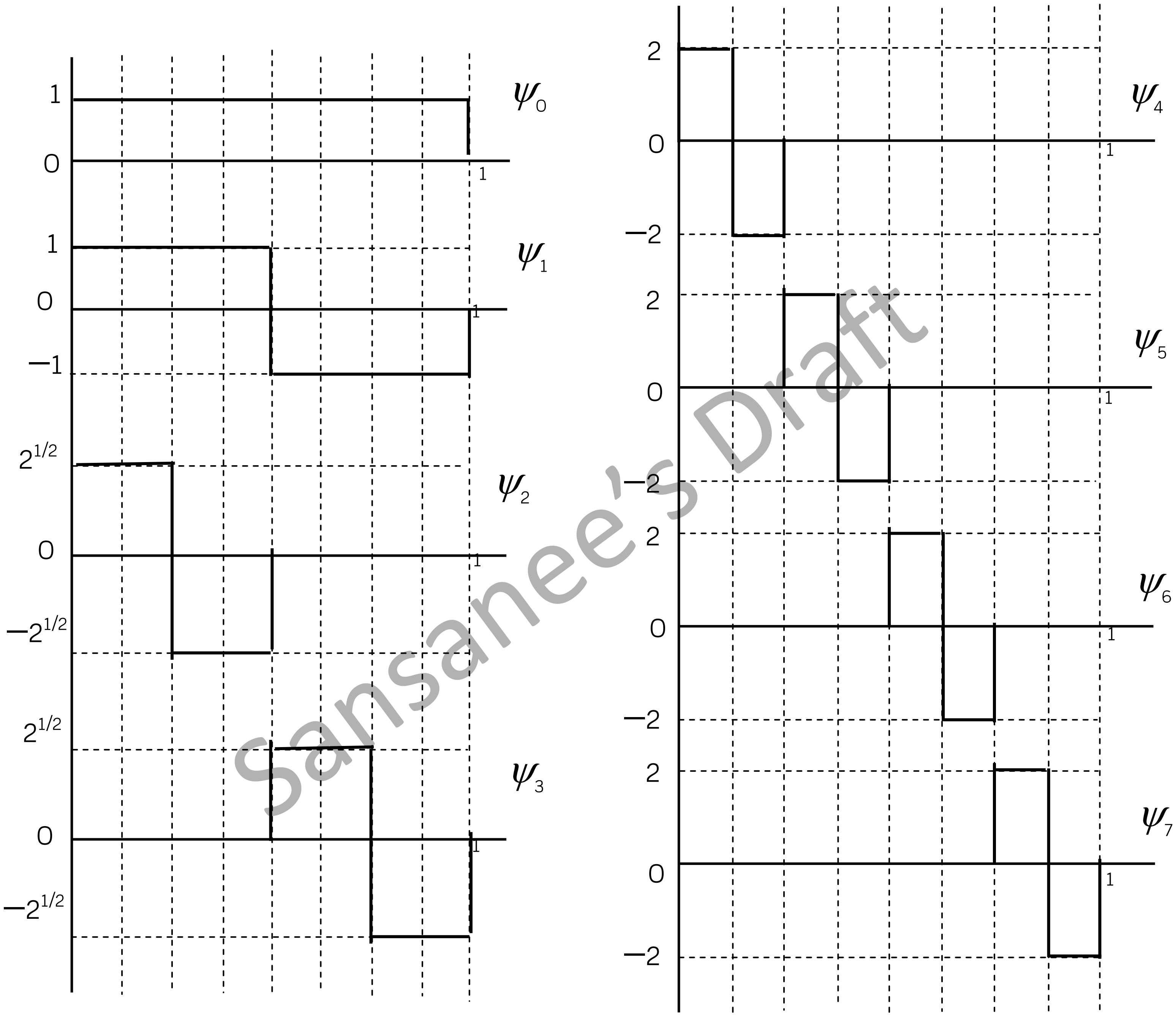
with Haar wavelet function → orthonormal and simplest function → dyadic and compact support wavelet with $\psi_0(x) = 1$

$$\psi(x) = \begin{cases} 1 & \text{for } 0 \le x \le 0.5 \\ -1 & \text{for } 0.5 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$\psi_1(x) = \psi(x) \qquad \psi_2(x) = 2^{\frac{1}{2}}\psi(2x-0) \qquad \psi_3(x) = 2^{\frac{1}{2}}\psi(2x-1)$$

$$\psi_4(x) = 2\psi(4x-0) \qquad \psi_5(x) = 2\psi(4x-1)$$

$$\psi_6(x) = 2\psi(4x-2) \qquad \psi_7(x) = 2\psi(4x-3)$$



ψ_0
ψ_1
ψ_2
ψ_3
ψ_4
ψ_5
ψ_6
ψ_7
1
0
−1
$2^{1/2}$
$-2^{1/2}$
2
−2

$$c_0 = \left\langle y, \psi_0 \right\rangle = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} \qquad c_1 = \left\langle y, \psi_1 \right\rangle = \int_0^{0.5} x^2 dx - \int_{0.5}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_0^{0.5} - \left.\frac{x^3}{3}\right|_{0.5}^{1} = -\frac{1}{4}$$

$$c_2 = \left\langle y, \psi_2 \right\rangle = \int_0^{0.25} \sqrt{2}x^2 dx - \int_{0.25}^{0.5} \sqrt{2}x^2 dx = \left.\frac{\sqrt{2}x^3}{3}\right|_0^{0.25} - \left.\frac{\sqrt{2}x^3}{3}\right|_{0.25}^{0.5} = -\frac{\sqrt{2}}{32}$$

$$c_3 = \left\langle y, \psi_3 \right\rangle = \int_{0.5}^{0.75} \sqrt{2}x^2 dx - \int_{0.75}^{1} \sqrt{2}x^2 dx = \left.\frac{\sqrt{2}x^3}{3}\right|_{0.5}^{0.75} - \left.\frac{\sqrt{2}x^3}{3}\right|_{0.75}^{1} = -\frac{3\sqrt{2}}{32}$$

$$c_4 = \left\langle y, \psi_4 \right\rangle = \int_0^{0.125} 2x^2 dx - \int_{0.125}^{0.25} 2x^2 dx = \left.\frac{2x^3}{3}\right|_0^{0.125} - \left.\frac{2x^3}{3}\right|_{0.125}^{0.25} = -\frac{1}{128}$$

$$c_5 = \left\langle y, \psi_5 \right\rangle = \int_{0.25}^{0.375} 2x^2 dx - \int_{0.375}^{0.5} 2x^2 dx = \left.\frac{2x^3}{3}\right|_{0.25}^{0.375} - \left.\frac{2x^3}{3}\right|_{0.375}^{0.5} = -\frac{3}{128}$$

$$c_6 = \left\langle y, \psi_6 \right\rangle = \int_{0.5}^{0.625} 2x^2 dx - \int_{0.625}^{0.75} 2x^2 dx = \left.\frac{2x^3}{3}\right|_{0.5}^{0.625} - \left.\frac{2x^3}{3}\right|_{0.625}^{0.75} = -\frac{5}{128}$$

$$c_7 = \left\langle y, \psi_7 \right\rangle = \int_{0.75}^{0.875} 2x^2 dx - \int_{0.875}^{1} 2x^2 dx = \left.\frac{2x^3}{3}\right|_{0.75}^{0.875} - \left.\frac{2x^3}{3}\right|_{0.875}^{1} = -\frac{7}{128}$$

$$y=\frac{1}{3}\psi_0(x)-\frac{1}{4}\psi_1(x)-\frac{\sqrt{2}}{32}\psi_2(x)-\frac{3\sqrt{2}}{32}\psi_3(x)-\frac{1}{128}\psi_4(x)$$

$$-\frac{3}{128}\psi_5(x)-\frac{5}{128}\psi_6(x)-\frac{7}{128}\psi_7(x)$$

Discrete Wavelet Transform

- Filter Bank Theory
  - Partition frequency axis into set of disjint (adjacent, non-overlapping) interval and use yjs partitionning to define a set of idea bandpass transfer function)

$$\sum_{i=1}^{\infty} H_i(s) = 1$$

$$\mathfrak{F}\{f(x)\} = F(s) = F(s)\sum_{i=1}^{\infty} H_i(s) = \sum_{i=1}^{\infty} F(s) H_i(s)$$

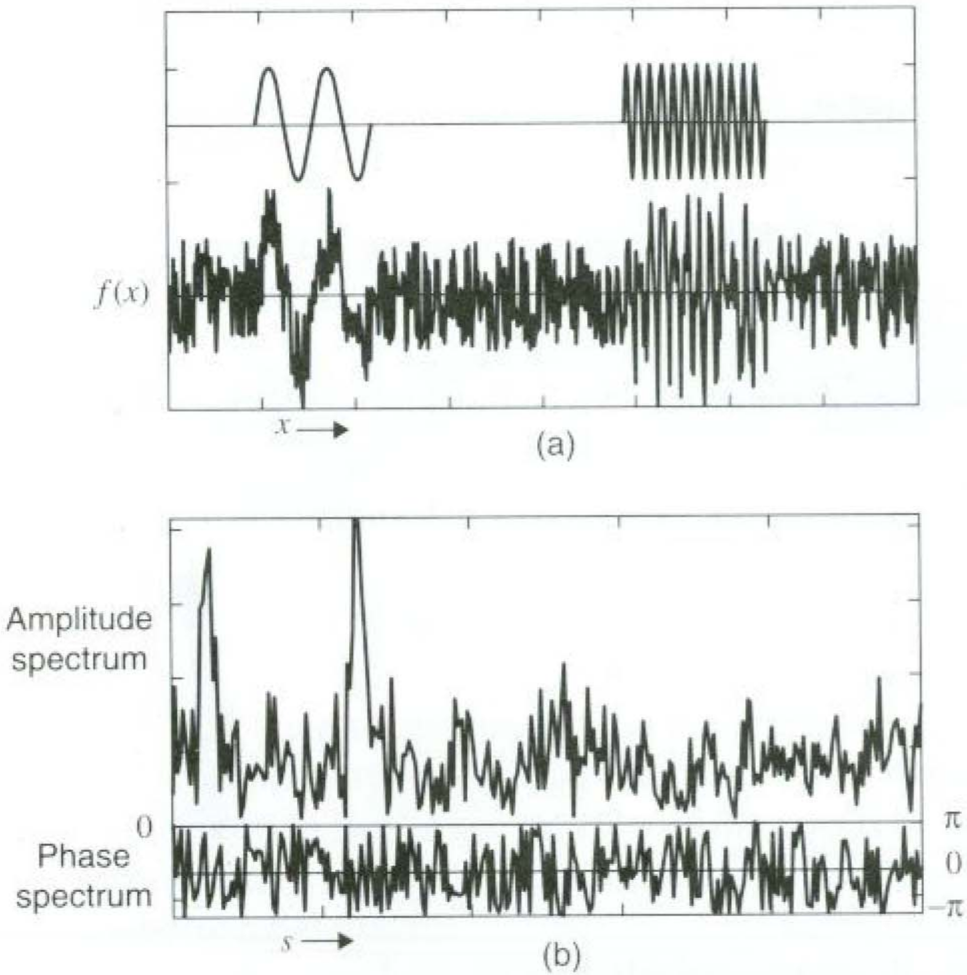


Figure 14–10 Composite signal containing two tone bursts and random noise: (a) the three components; (b) amplitude and phase spectra

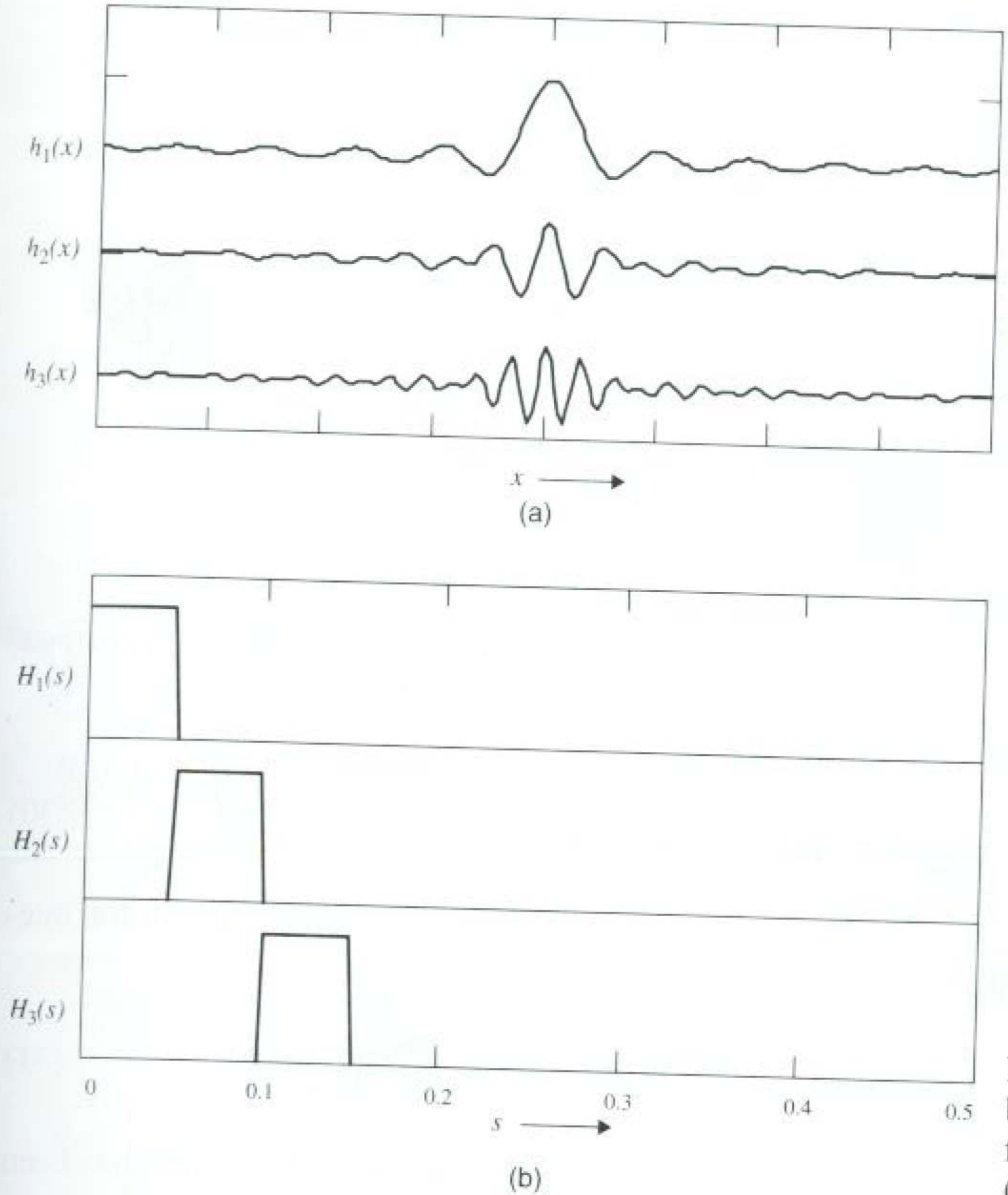
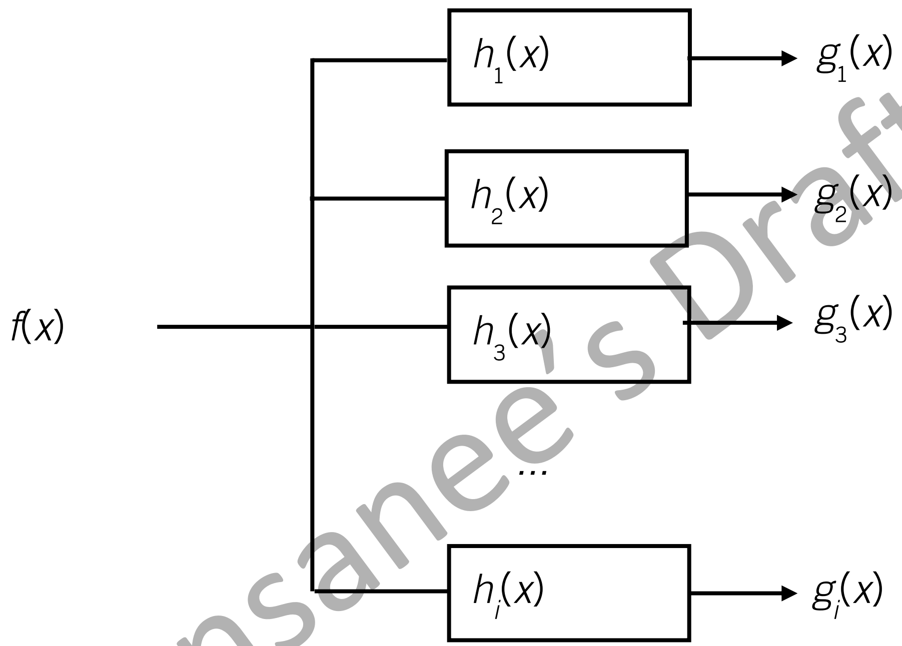


Figure 14–11 Generating a series of bandpass filters by partitioning the frequency axis: (a) impulse responses; (b) transfer functions

Filter Bank Theory



Filter Bank Theory

$$g_i(x) = \int_{-\infty}^{\infty} f(t) h_i(x-t)\,dt$$

$$\mathcal{F}\left(g_i(x)\right) = G_i(s) = \mathcal{F}\left(f(x)\right)\mathcal{F}\left(h_i(x)\right) = F(s)H_i(s)$$

$$\sum_{i=1}^{\infty} G_i(s) = \sum_{i=1}^{\infty} F(s)H_i(s) = F(s)$$

$$\sum_{i=1}^{\infty} g_i(x) = f(x)$$

- $H_i(s)$ is real and even → $h_i(x)$ is also real and even →reflected of $h_i(x)$ does not effect convolution, hence

$$g_i(x) = \int_{-\infty}^{\infty} f(t) h_i(t-x)\,dt = \left\langle f(t), h_i(t-x) \right\rangle$$

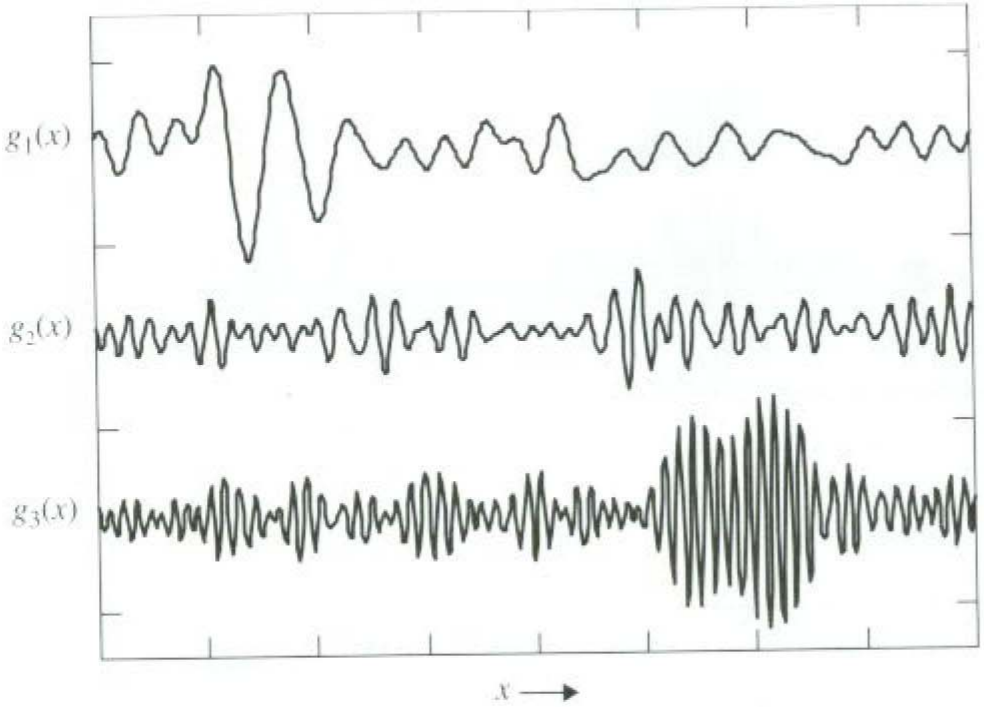


Figure 14–13 Bandpass filter outputs

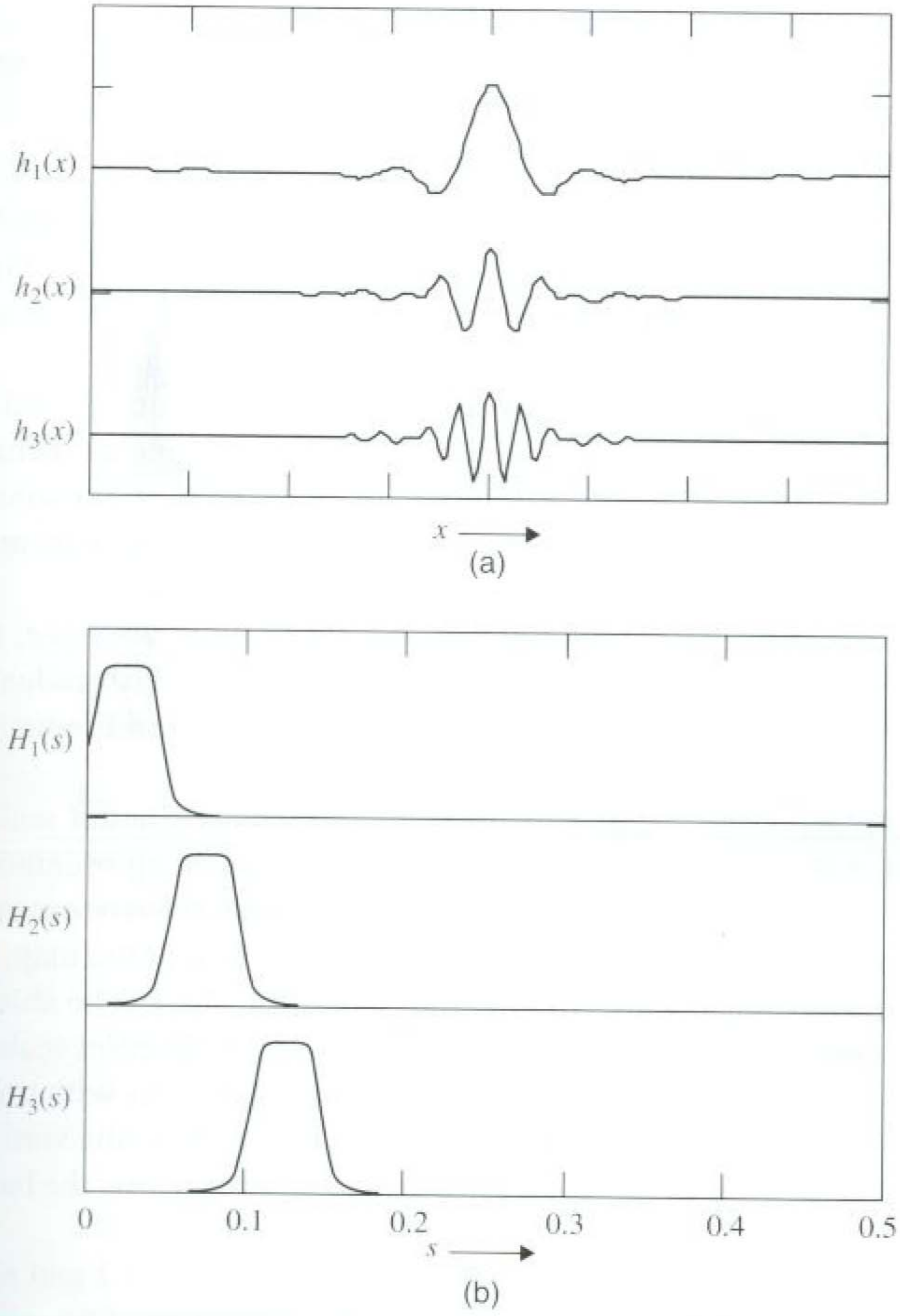


Figure 14–14 Smooth bandpass filters: (a) impulse responses; (b) transfer functions

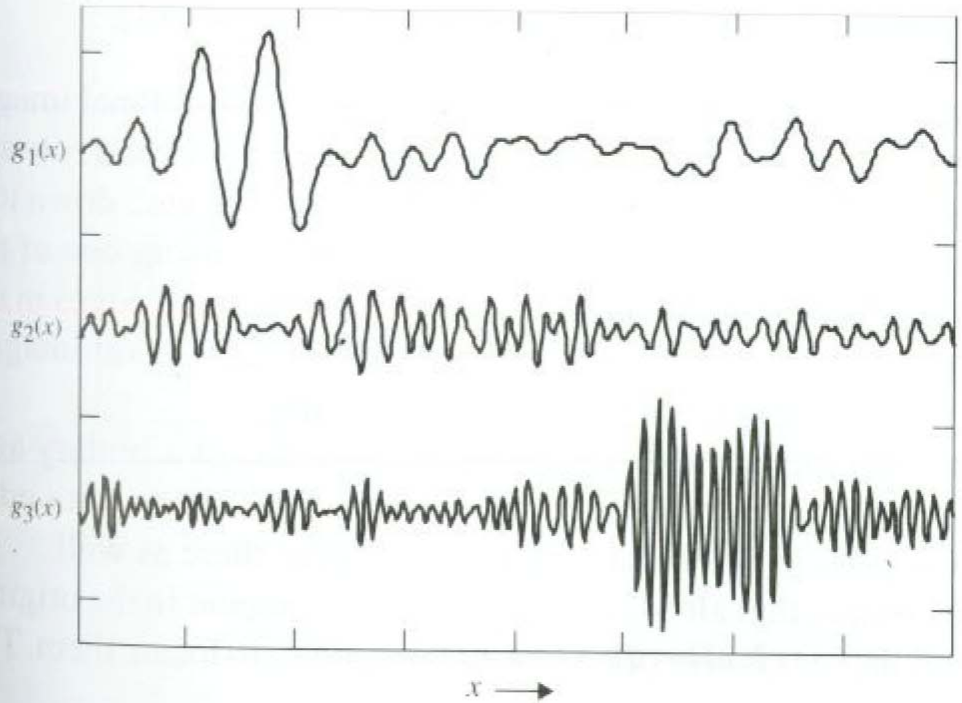
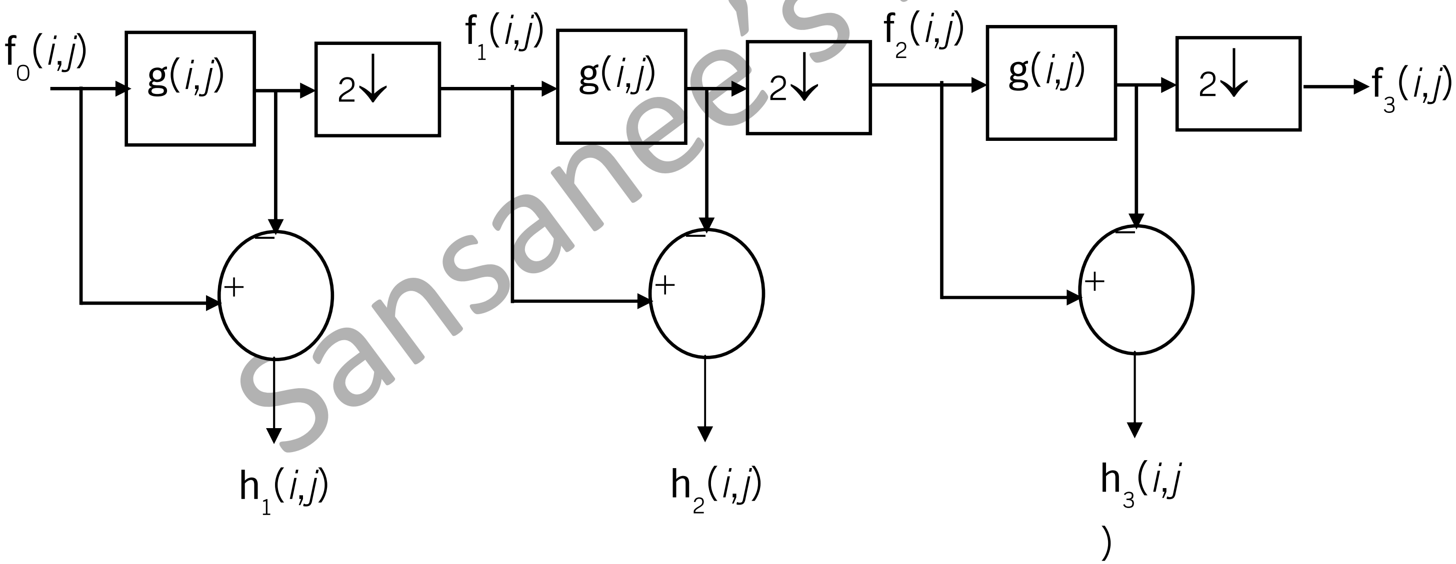


Figure 14–15 Smooth bandpass filter bank output

Multiresolution Analysis

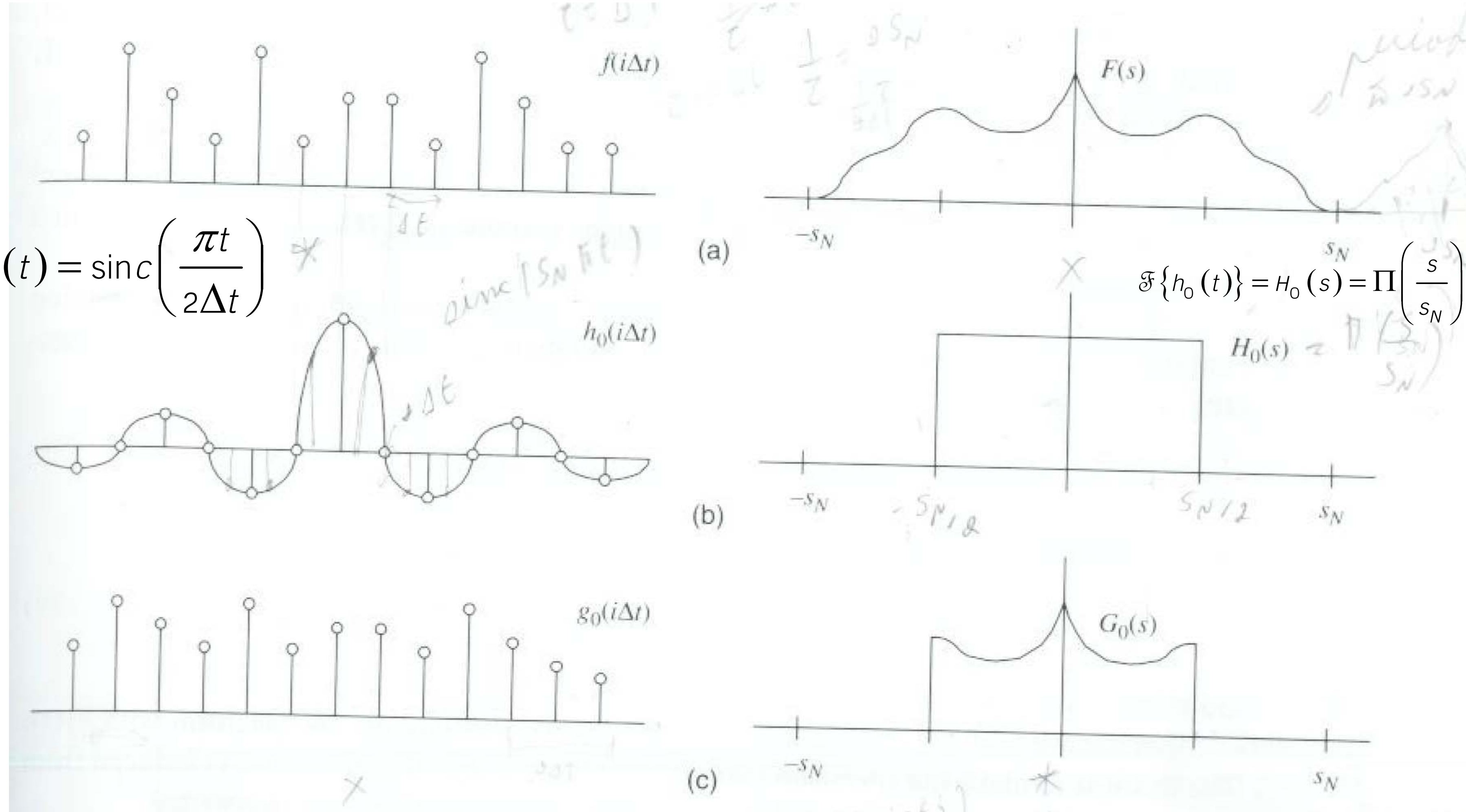
- Pyramid algorithm
- Laplacian pyramid encoding



$$f_1(i,j) = [f_0 * g](2i, 2j) \qquad h_1(i,j) = f_0(i,j) - [f_0 * g](i,j)$$

$$h_0(t) = \operatorname{sinc}\left(\frac{\pi t}{2\Delta t}\right)$$

$$\mathfrak{F}\{h_0(t)\} = H_0(s) = \Pi\left(\frac{s}{s_N}\right)$$



$$f_s(i\Delta t) = \frac{1}{2}\left(1 + \cos\left(2\pi s_N i\Delta t\right)\right)$$

$$\mathfrak{F}\left\{f_s(i\Delta t)\right\} = F_s(s) = \frac{1}{2}\left[\delta(s) + \delta(s - s_N) + \delta(s + s_N)\right]$$

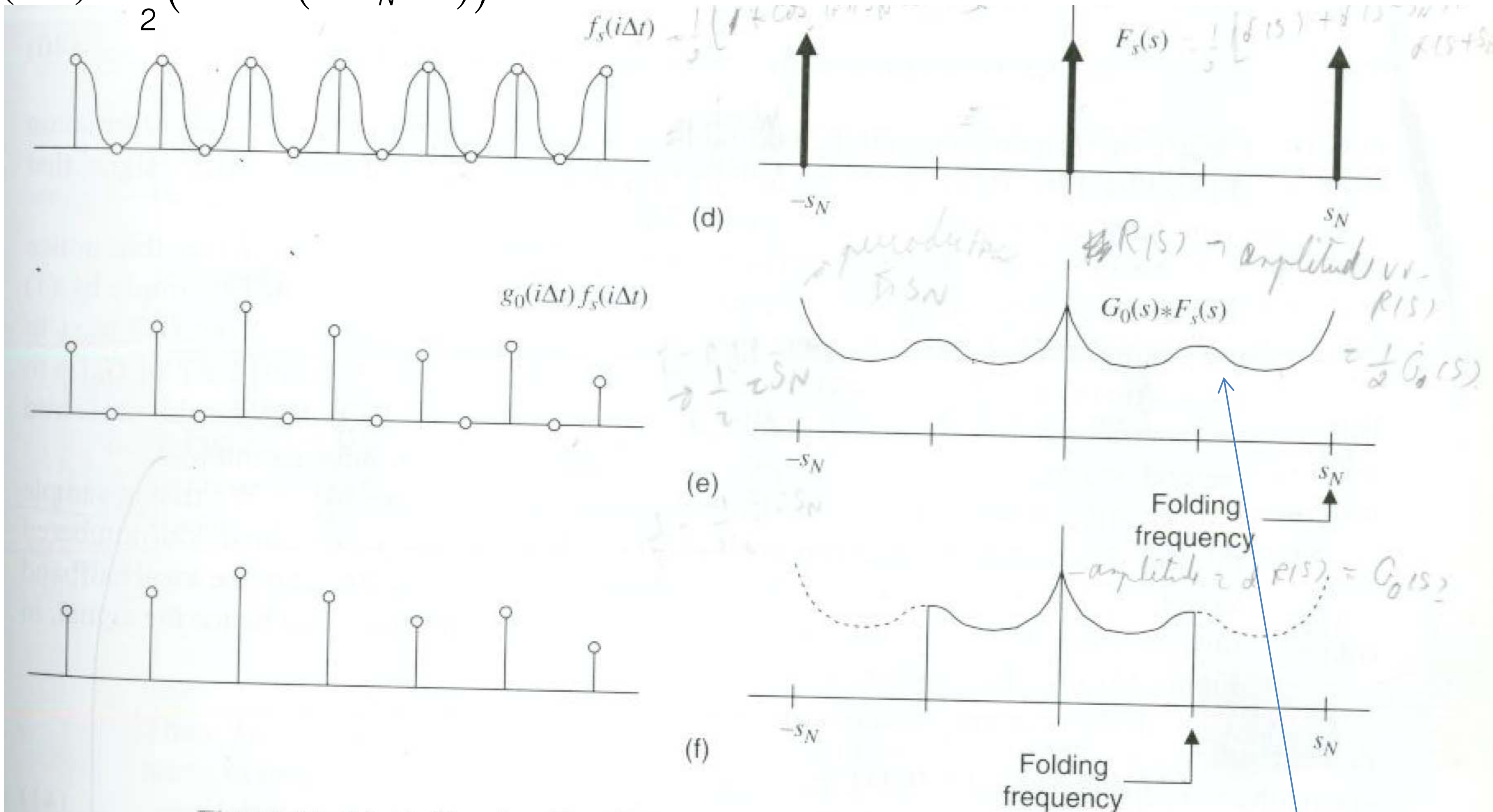


Figure 14–17 Subband coding, the lower halfband: (a) a sampled signal and its bandlimited spectrum; (b) the ideal halfband lowpass filter; (c) the lowpass filtered signal; (d) the subsampling function; (e) odd sample points replaced with zeros; (f) odd sample points discarded

$$\mathfrak{F}\left\{g_0(2i\Delta t)\right\} = F_s(s) * G_0(s) = \frac{1}{2}\left[G_0(s) + G_0(s - s_N) + G_0(s + s_N)\right]$$

Subband coding

- Recover $g_0(i\Delta t)$
  - Compute its $N/2$ point discrete spectrum
  - Padding it with 0 from $s_N/2$ to s_N to reconstruct $G_0(s)$ (14.17c)
  - Inverse (N points) DFT of $G_0(s)$ to get $g_0(i\Delta t)$

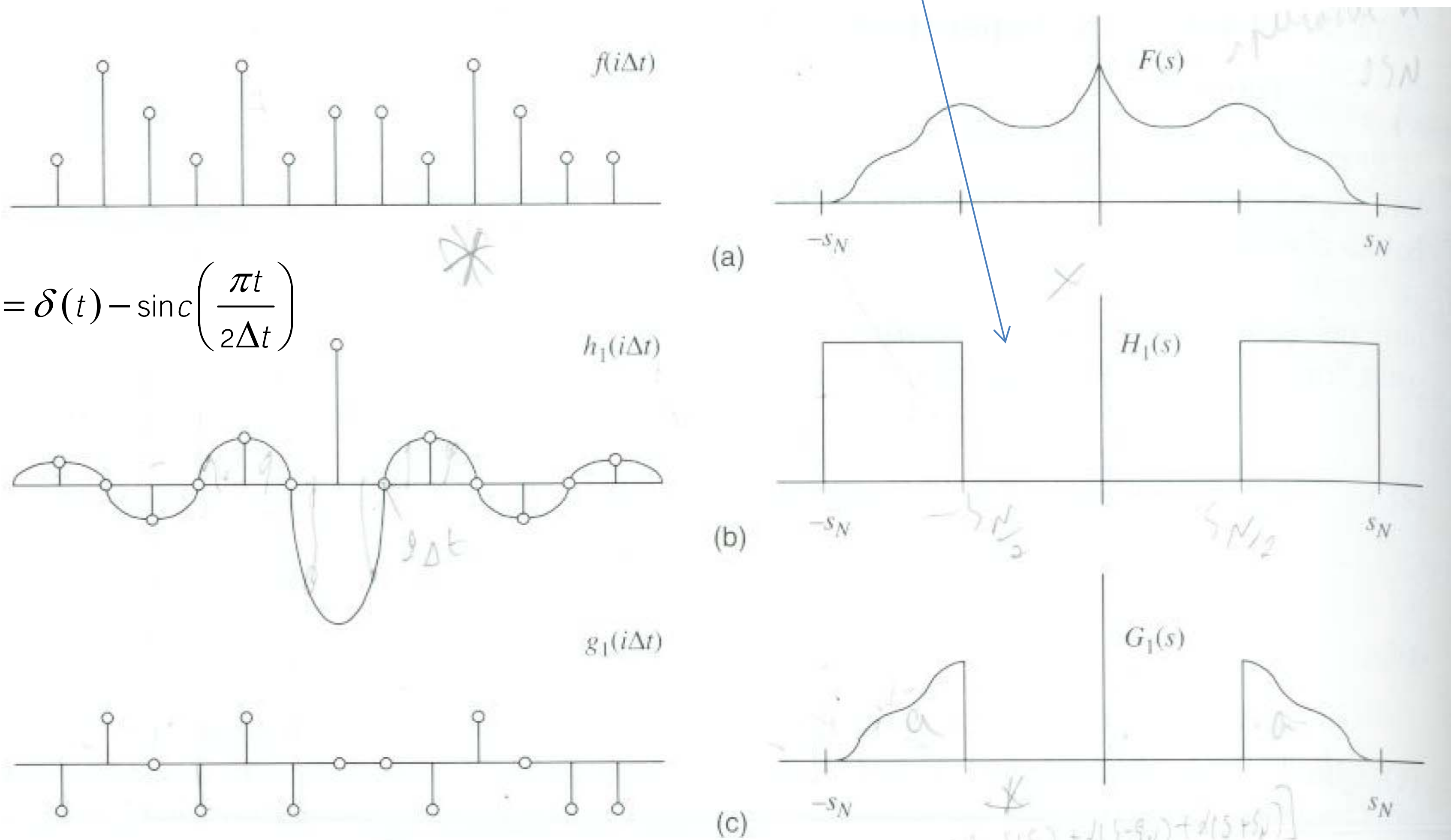
  Or

  - Upsample by inserting 0 at odd numbered sample get 14.17e
  - Filter that signal with $2h_0(i\Delta t)$ get 14.17c and then $g_0(i\Delta t)$

$$\left(F_S(s) * G_0(s)\right) \times H_0(s) = \frac{1}{2}\left[G_0(s) + G_0(s - s_N) + G_0(s + s_N)\right] \times \Pi\left(\frac{s}{s_N}\right) = \frac{1}{2} G_0(s)$$

$$\mathfrak{F}\{h_1(t)\} = H_1(s) = 1 - \Pi\left(\frac{s}{s_N}\right)$$

$$h_1(t) = \delta(t) - \operatorname{sinc}\left(\frac{\pi t}{2\Delta t}\right)$$



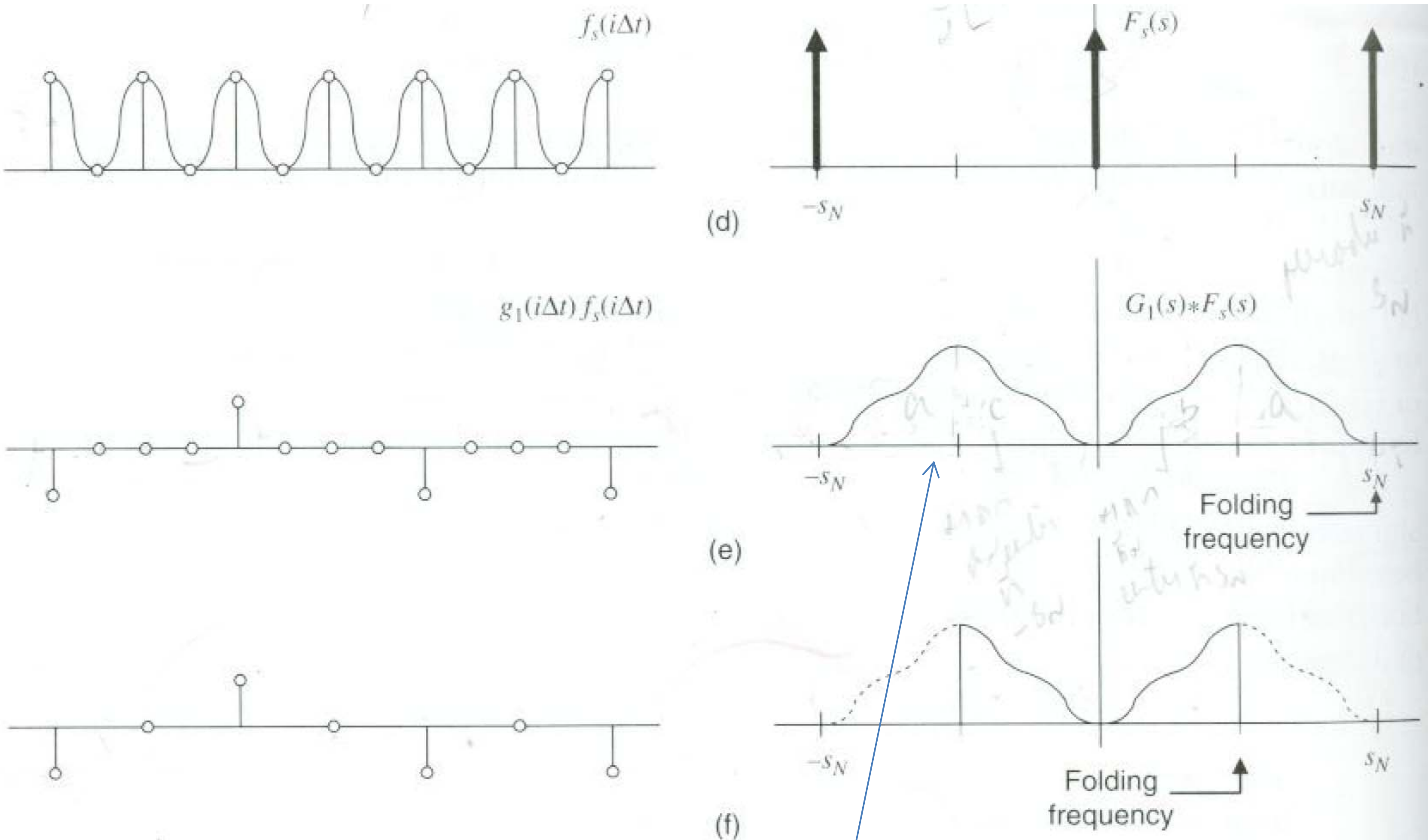


Figure 14–18 Subband coding, the upper halfband: (a) a sampled signal and its bandlimited spectrum; (b) the ideal halfband highpass filter; (c) the highpass filtered signal; (d) the subsampling function; (e) odd sample points replaced with zeros; (f) odd sample points discarded

$$\mathfrak{F}\left\{g_1\left(2i\Delta t\right)\right\} = F_s\left(s\right) * G_1\left(s\right) = \frac{1}{2}\left[G_1\left(s\right) + G_1\left(s - s_N\right) + G_1\left(s + s_N\right)\right]$$

Subband coding

- Recover $g_1(i\Delta t)$
  - Upsample by inserting 0 at odd numbered sample get 14.18e
  - Filter that signal with $2h_1(i\Delta t)$ get 14.18c and then $g_1(i\Delta t)$

$$\left[F_S(s) * G_1(s)\right] \times H_1(s) = \frac{1}{2}\left[G_1(s) + G_1(s - s_N) + G_1(s + s_N)\right] \times H_1(s)$$

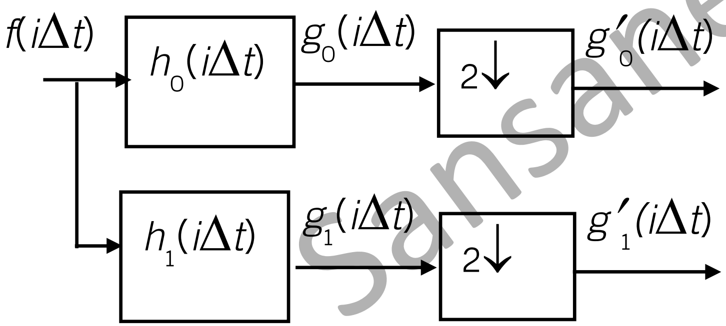
$$= \frac{1}{2} G_1(s)$$

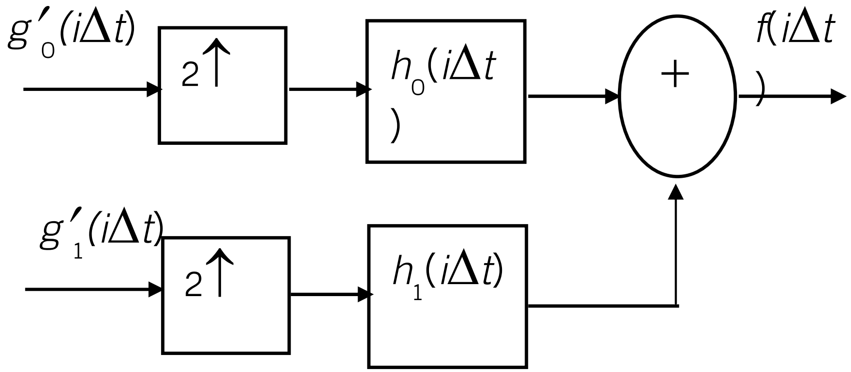
Subband coding

$$g_0(k\Delta t) = \sum_i f(i\Delta t)\, h_0((-i+2k)\Delta t) = [f(i\Delta t) * h_0(k\Delta t)]\downarrow_2$$

$$g_1(k\Delta t) = \sum_i f(i\Delta t)\, h_1((-i+2k)\Delta t) = [f(i\Delta t) * h_1(k\Delta t)]\downarrow_2$$

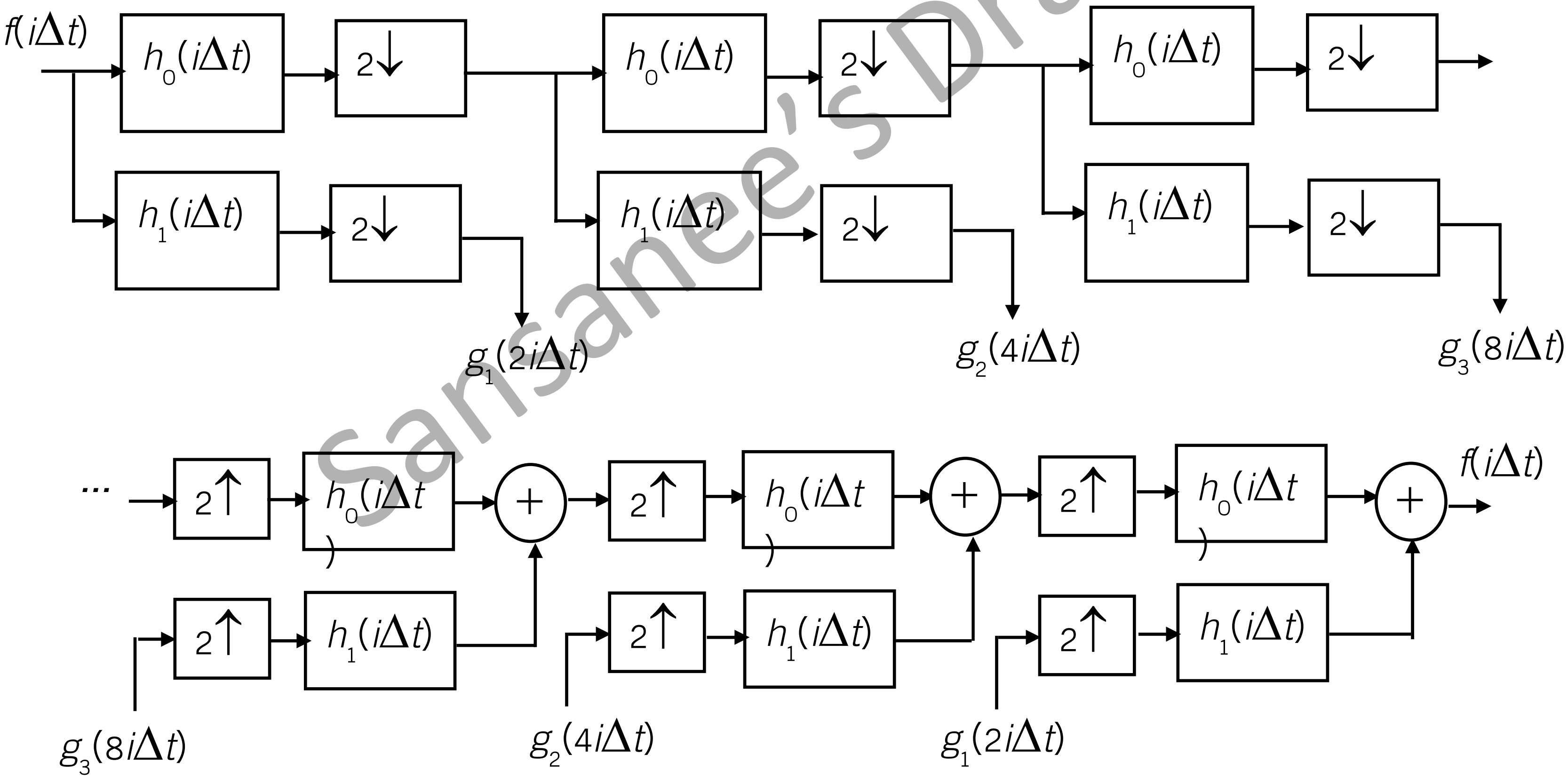
$$f(i\Delta t) = 2\sum_k \left\{ \left[g_0(k\Delta t)\, h_0((-i+2k)\Delta t) \right] + \left[g_1(k\Delta t)\, h_1((-i+2k)\Delta t) \right] \right\}$$





Fast Wavelet Transform

- Mallat's Herribone algorithm



Discrete Wavelet design

- From

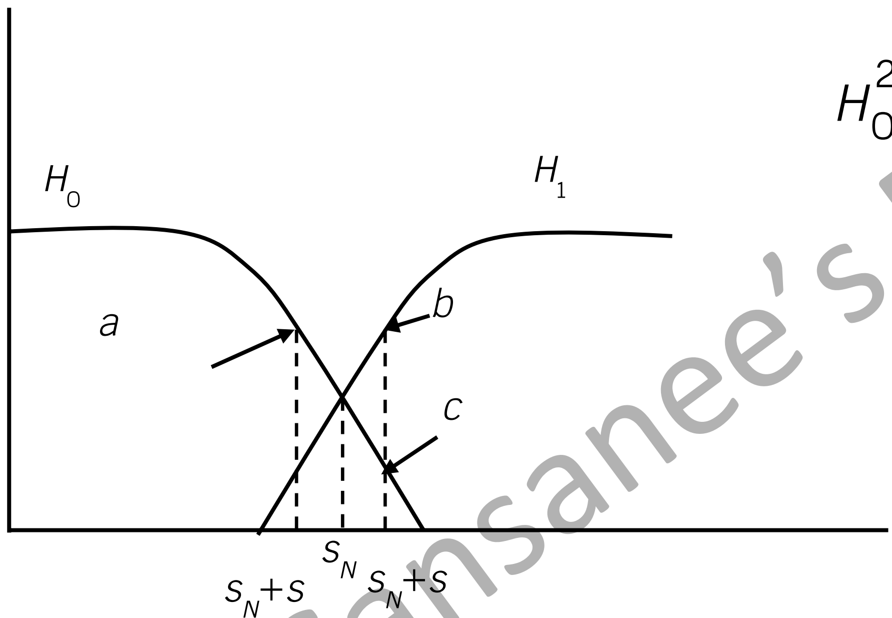
$$F(s) = 2\left[\frac{1}{2}G_0(s)H_0(s) + \frac{1}{2}G_1(s)H_1(s)\right]$$
$$= 2\left[\frac{1}{2}F(s)H_0(s)H_0(s) + \frac{1}{2}F(s)H_1(s)H_1(s)\right]$$

$$F(s) = F(s)\left[H_0^2(s) + H_1^2(s)\right]$$

$$H_0^2(s) + H_1^2(s) = 1 \qquad 0 \le |s| \le s_N$$
$$H_1^2(s) = 1 - H_0^2(s)$$

- $h_1(i\Delta t)$ is translated version of $h_0(i\Delta t)$ with s_N
- $h_1(i\Delta t)$ is mirror filter of $h_0(i\Delta t)$

Discrete Wavelet design



At *a* and *b*

$$H_0^2\left(\frac{s_N}{2} + s\right) + H_1^2\left(\frac{s_N}{2} + s\right) = 1$$

From mirror filter

$$H_0^2\left(\frac{s_N}{2} + s\right) + H_0^2\left(\frac{s_N}{2} - s\right) = 1$$

Discrete Wavelet design

- Scaling vector→sequence such that

$$\sum_k h_0(k) = \sqrt{2} \qquad \sum_k h_0(k)\,h_0(k+2l) = \delta(l)$$

- There exist a scaling function

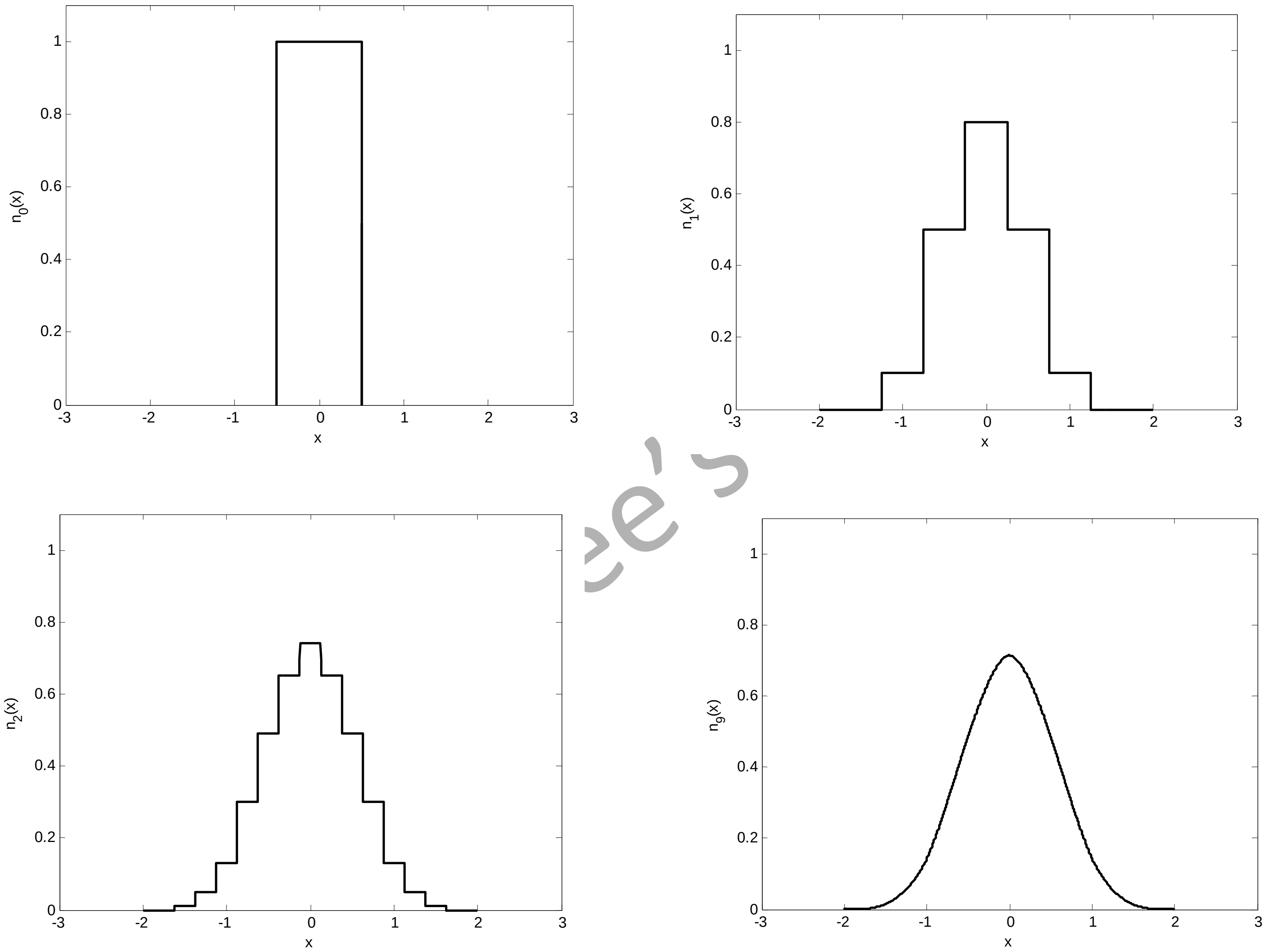
$$\phi(t) = \sum_k h_0(k)\,\phi(2t-k)$$

- example

$$h_0 = \sqrt{2}\,[0.05\ \ 0.25\ \ 0.4\ \ 0.25\ \ 0.05]^t \qquad \phi(x) = \lim_{i\to\infty} \eta_i(x)$$

$$\eta_i(x) = \sqrt{2}\sum_n h_0(n)\,\eta_{i-1}(2x-n)$$

$$\eta_0(x) = \Pi(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$


$n_0(x)$
$n_1(x)$
$n_2(x)$
$n_9(x)$
x
x
x
x

Discrete Wavelet design

- Basic wavelet

$$\psi(t) = \sum_k h_1(k)\phi(2t-k) = \left[h_1(k) * \phi(k)\right]\downarrow_2$$

when $h_1(k) = (-1)^k h_0(-k+1)$

$$\psi_{j,k}(t) = 2^j \psi\left(2^j t - k\right)$$

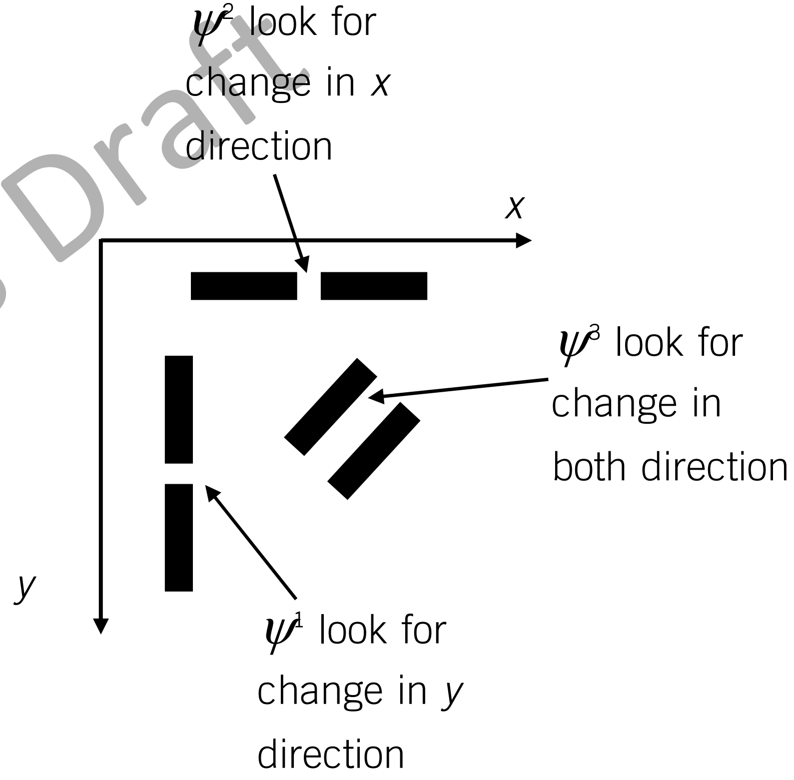
2-D Discrete Wavelet Transform

$$\phi(x,y)=\phi(x)\phi(y)$$

$$\psi^1(x,y)=\phi(x)\psi(y)$$

$$\psi^2(x,y)=\psi(x)\phi(y)$$

$$\psi^3(x,y)=\psi(x)\psi(y)$$



2-D Discrete Wavelet Transform

$$f_2^0(m,n) = \left\langle f_1(x,y), \phi(x-2m, y-2n) \right\rangle = \left[f_1(x,y) * \phi(-x,-y) \right] \downarrow_2$$

$$f_2^1(m,n) = \left\langle f_1(x,y), \psi^1(x-2m, y-2n) \right\rangle = \left[f_1(x,y) * \psi^1(-x,-y) \right] \downarrow_2$$

$$f_2^2(m,n) = \left\langle f_1(x,y), \psi^2(x-2m, y-2n) \right\rangle = \left[f_1(x,y) * \psi^2(-x,-y) \right] \downarrow_2$$

$$f_2^3(m,n) = \left\langle f_1(x,y), \psi^3(x-2m, y-2n) \right\rangle = \left[f_1(x,y) * \psi^3(-x,-y) \right] \downarrow_2$$

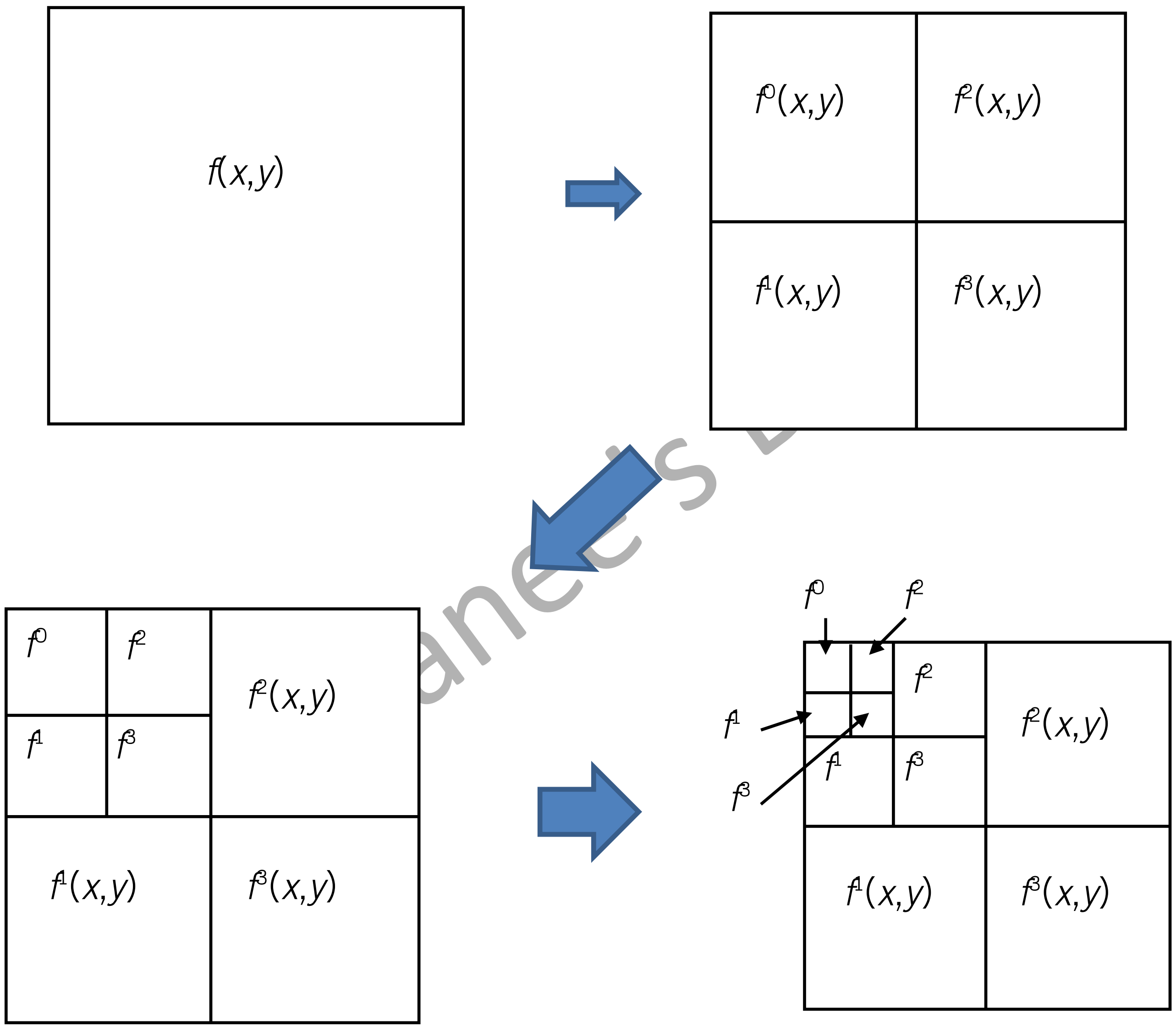
$$f_{2^{j+1}}^0(m,n) = \left[f_{2^j}^0(x,y) * \phi(-x,-y) \right] \downarrow_2$$

$$f_{2^{j+1}}^1(m,n) = \left[f_{2^j}^0(x,y) * \psi^1(-x,-y) \right] \downarrow_2$$

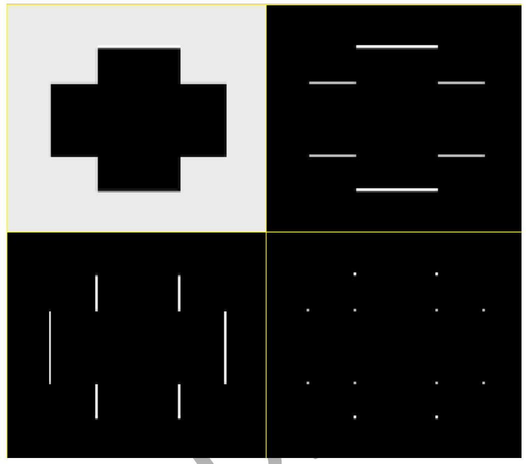
$$f_{2^{j+1}}^2(m,n) = \left[f_{2^j}^0(x,y) * \psi^2(-x,-y) \right] \downarrow_2$$

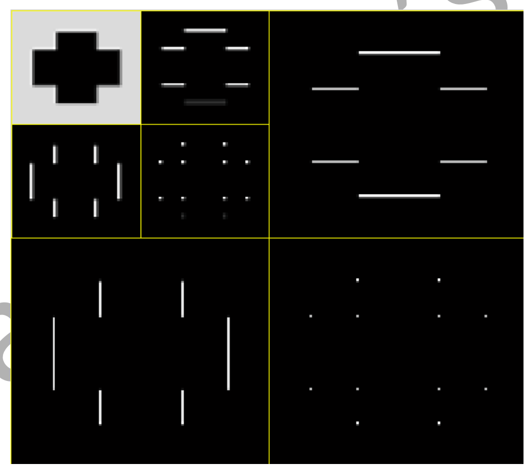
$$f_{2^{j+1}}^3(m,n) = \left[f_{2^j}^0(x,y) * \psi^3(-x,-y) \right] \downarrow_2$$

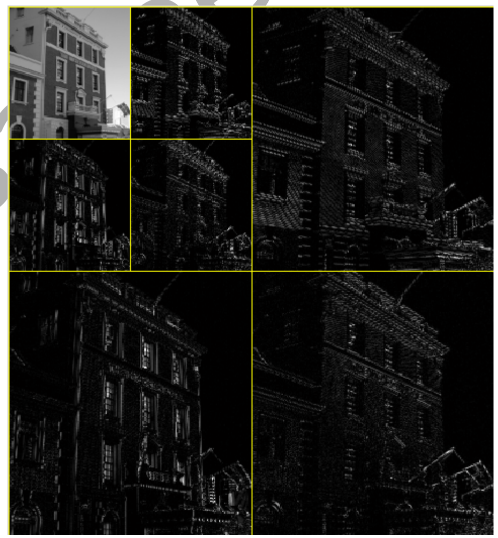
Repeated



f(x,y)
f0(x,y)
f2(x,y)
f1(x,y)
f3(x,y)
f0
f2
f1
f3
f2(x,y)
f1(x,y)
f3(x,y)
f0
f2
f1
f3
f2
f1
f3
f2(x,y)
f1(x,y)
f3(x,y)







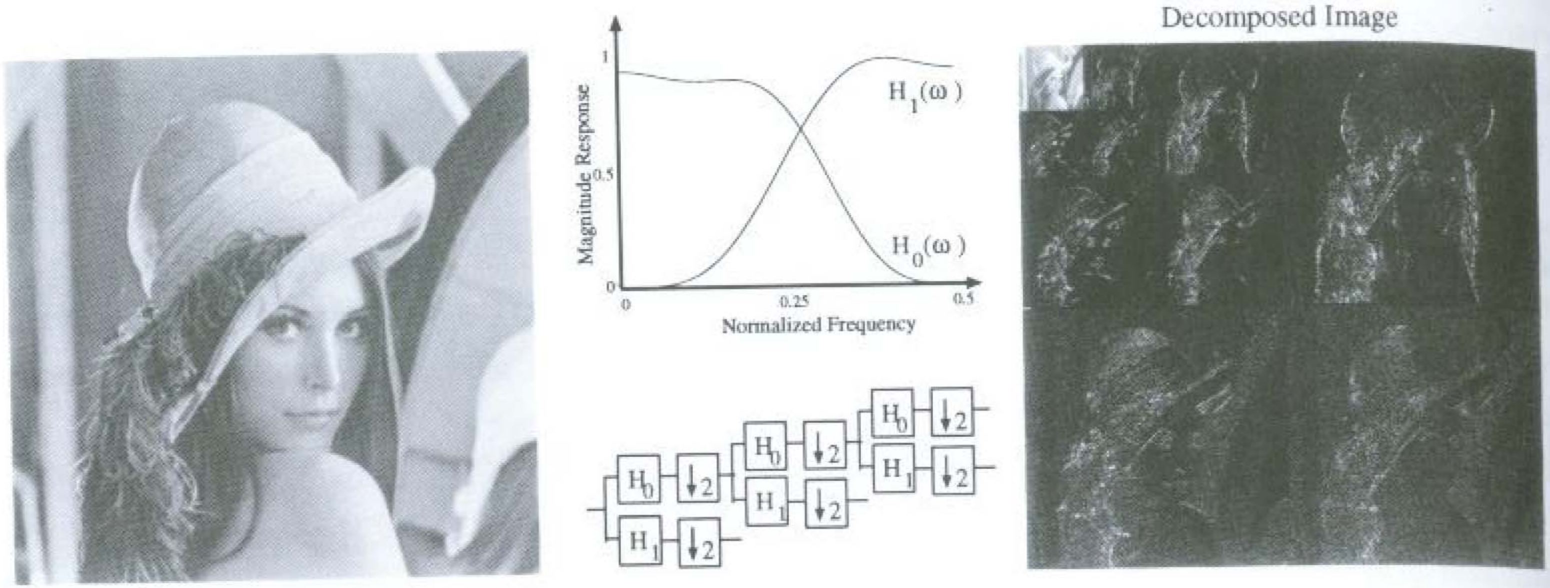


Figure 10.2: Original image "Lenna" and its multiresolution decomposition. The filter bank used is the (9, 7)-tap linear-phase filter bank.

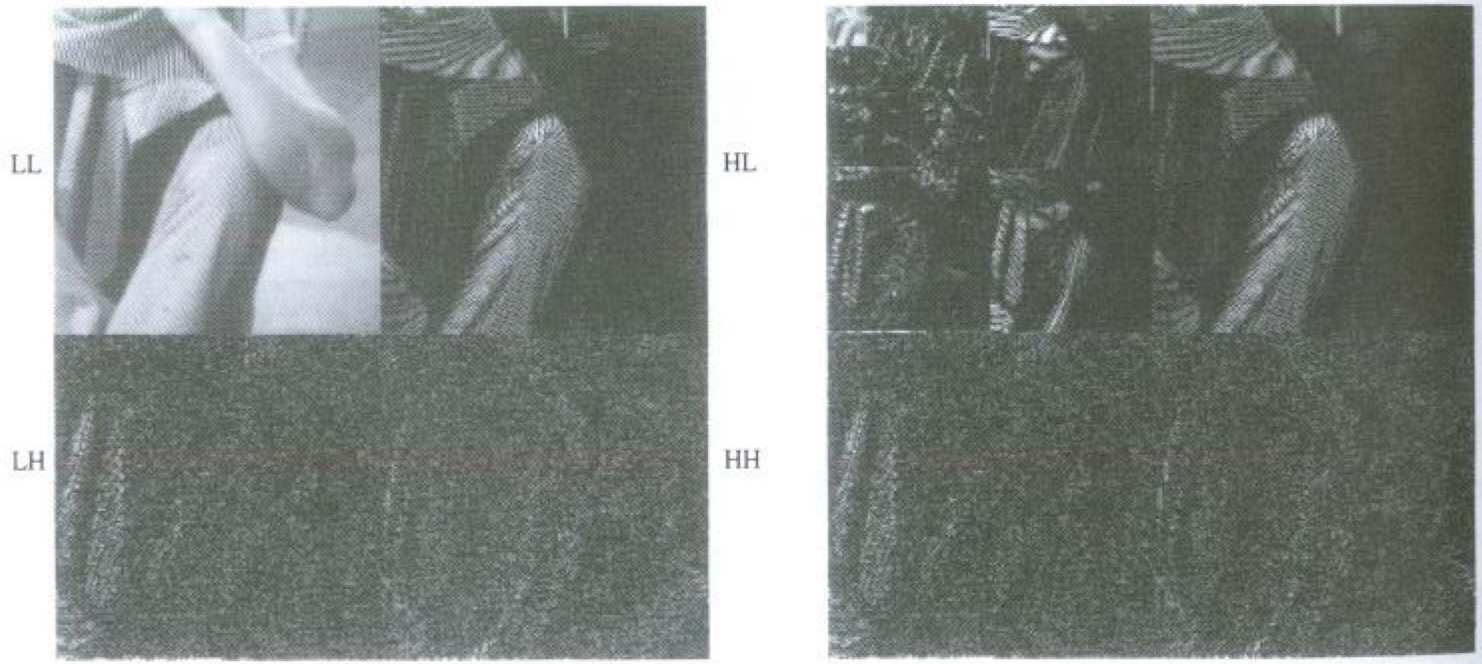


Figure 11.5: The discrete wavelet transform of "Barbara" (one level and four levels).

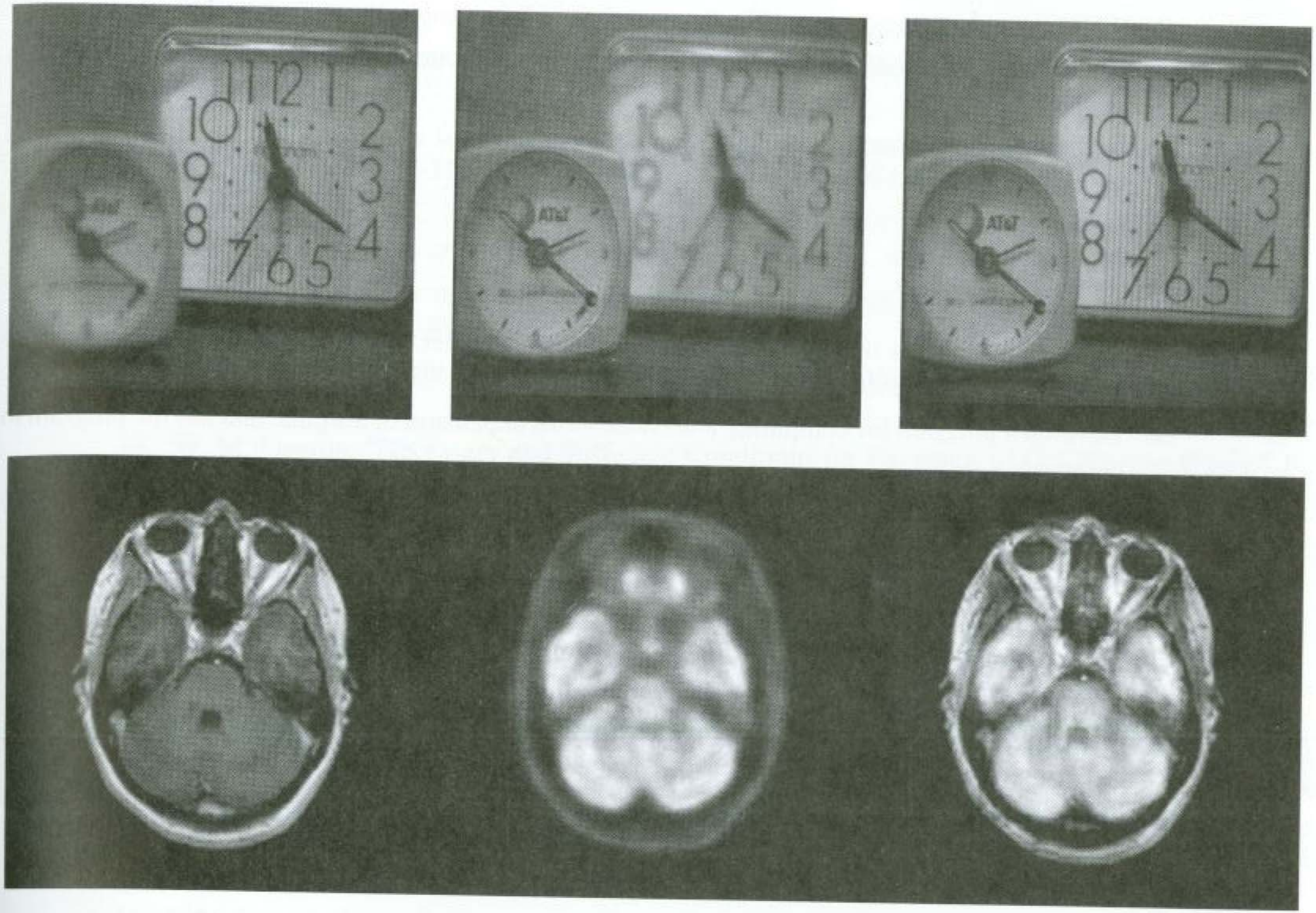


Figure 14–36 Wavelet transform image fusion: (a), (b) images taken at different focus settings; (c) fused image; (d) MRI image; (e) PET image; (f) fused image (Courtesy Henry Hui Li, reprinted by permission from [28])