Generalization Bounds in Hybrid Quantum-Classical Machine Learning Models

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Abstract—**Hybrid classical-quantum models aim to harness the strengths of both quantum computing and classical machine learning, but their practical potential remains poorly understood. In this work, we develop a unified mathematical framework for analyzing generalization in hybrid models, offering insight into how these systems learn from data. We establish a novel generalization bound of the form $O\left(\sqrt{\frac{T\log T}{N}} + \frac{\alpha}{\sqrt{N}}\right)$ for N training data points, T trainable quantum gates, and bounded fully-connected layers $\|F\| \leq \alpha$. This bound decomposes cleanly into quantum and classical contributions, extending prior work on both components and clarifying their interaction. We apply our results to the quantum-classical convolutional neural network (QCCNN), an architecture that integrates quantum convolutional layers with classical processing. Alongside the bound, we highlight conceptual limitations of applying classical statistical learning theory in the hybrid setting and suggest promising directions for future theoretical work.**

Index Terms—**Generalization Bounds, Statistical Learning Theory, Hybrid Quantum-Classical Models, Quantum Convolutional Neural Networks (QCCNN), Machine Learning Theory**

I. INTRODUCTION

Quantum-classical hybrid machine learning models have emerged as a promising approach in an era where fully fault-tolerant, large-scale quantum computers remain out of reach. Currently available devices impose significant restriction on width and depth of quantum circuits, making hybridization of quantum and classical computation an ideal candidate for exploiting early capabilities of quantum computing. In the current framework, classical optimization methods act as a complement for shallow-depth circuits. As the quantum technological stack matures into fault-tolerant regime, hybrid algorithms will evolve as well allowing to carefully interleave classical and quantum leveraging the strength of both of these resources. Early proof-of-concept hybrid algorithms show promising results [1–8], however, a solid theoretical understanding of the power of these architectures is lacking. Gaining such insights could shed light onto the differences in computational power between quantum and classical algorithms.

There are several theoretically interesting aspects of a learning model's behavior, e.g. resources demand [1], sampling complexity [8] and learnability [9]. One of the most sought-after metrics is the generalization, which allows to predict its performance beyond the available dataset. Classical approaches in statistical learning theory establish uniform upper bounds on the generalization error based on both the training set size and the inherent complexity of the model. In this context, complexity can be characterized through measures, such as covering numbers and Rademacher complexity [10], which capture the richness of a function class associated with the hypothesis class of the model. Several works have provided rigorous bounds on the generalization error for classical machine learning models [11–13] as well as for fully quantum learning models [1–3]. However, despite the growing popularity of hybrid models, the conditions under which these hybrid architectures can generalize accurately remain largely unexplored.

Inspired by the remarkable success of neural networks (NNs) in classical machine learning, the quantum machine learning community has pursued a similar line of research. Layer-wise organized quantum circuits with parametrizable gates that mimic classical feedforward architecture are known as quantum NNs (QNNs). A diverse zoo of models has emerged inspired by classical ideas, such as hybrid Bolzmann machines [5], hybrid autoencoders [6] and hybrid convolutional NNs [7]. Hybrid quantum-classical convolutional NNs (QCCNNs) have gained particular attention due their favourable qualities, such as data-efficiency [1], absence of barren plateaus [14], and adversarial robustness [15].

In this paper, we derive and prove a generalization bound for a QCCNN that can be extended to a more general QNN case. In the hybrid quantum-classical setting, a model's hypothesis class becomes richer due to integrating quantum with classical machine learning components. In our proof, we first separately derive complexity measures for the hypothesis classes corresponding to the quantum and classical components. Due to the richness of the hybrid model's hypothesis class, we use covering numbers to quantify the complexity of the hypothesis class geometrically, rather than a direct worst-case analysis using Rademacher complexity. After establishing covering number bounds, we combine them to characterize the overall complexity of the hybrid hypothesis class. Using Dudley's entropy integral, we obtain generalization bounds that quantify model's learning ability from finite data. A key implication of our results is that the complexity of a hybrid model that has minimal classical layers will perform closely to a full quantum machine learning model (QMLM),

providing theoretical support for the practical viability and design of quantum-assisted learning algorithms. As such, designing hybrid models with shallow classical components but expressive quantum circuits can achieve strong generalization while reducing implementation overhead, offering a principled framework for balancing quantum and classical resources in real-world QMLM architectures.

This paper is organized as follows: In Section II, we justify the chosen QMLM architecture and provide an overview of previous work on generalization bounds in both quantum and classical contexts. In Section III, we outline the fundamental concepts relevant to this work. In Section IV, we present a derived generalization bound for a hybrid quantum-classical setup and discuss its implications in Section V. Finally, Section VI contains a step-by-step derivation of the aforementioned bound.

II. RELATED WORK

On the quantum side, Caro et al. [1] prove generalization bounds for QMLMs and derive a bound for (full) quantum CNN (QCNN) based on covering numbers with T trainable gates to scale at worst as $\tilde{\mathcal{O}}\left(\sqrt{\frac{T \log T}{N}}\right)$. Furthermore, they show that when re-running the same gates at most M times, the generalization performance scales $\tilde{\mathcal{O}}\left(\sqrt{\frac{T \log MT}{N}}\right)$, only worsening logarithmically. These findings suggest that these models can generalize well even with a small training dataset. In our work, we extend these results to capture the behaviour of a hybrid architecture.

On the hybrid side, while the theoretical foundation is currently lacking (providing motivation for this work) numerous studies have demonstrated promising empirical results. For instance, Matic et al. [7] presented empirical evidence supporting the viability of a hybrid quantum-classical CNNs (QCCNNs) in a medical use case, demonstrating its potential practical utility in real-world scenarios. A subsequent study [16] studied theoretically derived generalization metrics for QMLMs, such as [2, 17], applied to architecture from [7] in an empirical context; however, these efforts revealed a significant lack of correlation between. This implies that the general metrics did not capture the behaviour of a hybrid QCCNN model sufficiently well. In contrast, in this work, we take a fully theoretical approach and develop generalization bounds tailored to a hybrid setup.

On the classical side, extensive research has been conducted on generalization bounds for NNs. Table I illustrates three classical generalization bounds for NNs that illustrate how generalization bounds can be established via norm-based complexity measures, particularly the product of per-layer norms and depth. These results, especially from Bartlett et al. [10] and Neyshabur et al. [18, 19] emphasize that generalization depends not on parameter count alone, but on how weight magnitudes scale with depth. This is reflected in our analysis of the classical component of hybrid QMLMs. The exponential dependence on depth seen in these bounds directly motivates

Reference	Generalization Bound
Neyshabur et al. [18]	$\mathcal{O}\left(\frac{2^L a L}{\sqrt{n}}\right)$
Bartlett et al. [10]	$\tilde{\mathcal{O}}\left(\frac{s^{L-1} L^{\frac{3}{2}} a d^{\frac{1}{2}}}{\sqrt{n}}\right)$
Neyshabur et al. [19]	$\tilde{\mathcal{O}}\left(\frac{s^{L-1} L^{\frac{3}{2}} a d^{\frac{1}{2}}}{\sqrt{n}}\right)$

TABLE I: Simplified generalization bounds for neural networks with L fully connected layers, adapted from [20]. All three bounds exhibit super-linear or exponential dependence on L, despite differing derivation techniques.

the $\alpha^k/\sqrt{N}$ term seen in the hybrid QMLM bound that is found in section V, where α is a norm bound on the classical layers and k is their depth. This connection highlights that hybrid models inherit generalization behavior from classical architectures and that norm control remains a key design consideration even when quantum components are involved. Neyshabur et al. [18] investigated norm-based capacity control in classical neural networks, establishing an exponential dependence on the layer depth.

III. BACKGROUND

A. Generalization

Generalization is the ability of a (trained) model to perform well on unseen data. We measure the generalization error by comparing the model's performance on training data with its expected performance on unseen data. Formally, given a training dataset $S = \{(x_i, y_i)\}, i = 1, \ldots N$ of size N and the model's hypothesis class H, the empirical risk associated with a hypothesis $h \in H$ is given by

$$\hat{R}_S(h) = \frac{1}{N} \sum_{i=1}^{N} \ell(h(x_i), y_i), \tag{1}$$

where ℓ is a predefined loss function that quantifies the discrepancy between the model prediction and the true label. This empirical risk, also known as the training error, serves as an estimate of the expected risk, which is defined as

$$R(h) = \mathbb{E}_{(x,y) \sim \mathcal{D}}[\ell(h(x), y)], \tag{2}$$

where the expectation is taken over the unknown data distribution $\mathcal{D}$. The discrepancy between these two quantities, known as the generalization error,

$$\text{gen}(h) = R(h) - \hat{R}_S(h), \tag{3}$$

measures how well the training performance translates to unseen data. Bounding this generalization error is crucial in statistical learning theory and is the focus of the subsequent theoretical results, where maintaining good generalization to new data while fitting the training data well is a trade-off.

B. Complexity measures

Complexity measures quantify how well functions from a hypothesis class are able to fit various training data, as overfitting can be understood as a class having many functions that are distinct from the target function but can still fit the

training data well. The Rademacher complexity and covering numbers are complexity measures based on probability and geometric characterizations of hypothesis classes and have well-established connections to translate between them. Our approach builds on existing uniform-bound based measures in the quantum setting, where these have proven effective in bounding QMLM's [1].

1) Rademacher complexity:

Definition 1 (Empirical Rademacher Complexity)**.** *Consider arbitrary spaces $\mathcal{X}, \mathcal{Y}$, define $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$. Given a training set $S = \{z_1, \ldots, z_n\}$ and a real-valued hypothesis class H on $\mathcal{Z}$, the empirical Rademacher complexity is defined as the expected supremum of the correlation between the functions in H and the random Rademacher variables σ_i applied to the dataset:*

$$\mathcal{R}_S(H) := \mathbb{E}_\sigma \left[\sup_{h \in H} \frac{1}{n} \sum_{i=1}^{n} \sigma_i h(z_i) \right]. \tag{4}$$

where the Rademacher variables $\sigma_i \sim U_{\{-1,1\}}$ are random variables that take values ± 1 with equal probability.

From the definition, we see that empirical Rademacher complexity varies with the training set, reflecting the actual learning difficulty of each dataset, explaining its name. This data-dependence can yield tighter generalization bounds than distribution-independent worst-case bounds, yet it can still be used to derive bounds that hold for all distributions. As such, Rademacher complexity often serves as a key step in proving general results in learning theory. The following theorem is fundamental in this context.

Theorem 1 (Rademacher Generalization Bounds[21])**.** *Consider arbitrary spaces $\mathcal{X}, \mathcal{Y}$. We define $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$ and a real-valued hypothesis class H on $\mathcal{Z}$. For any $\delta > 0$ and any probability measure $\mathbb{P}$ on $\mathcal{Z}$ we have with probability at least $(1 - \delta)$ for the training data $S \in \mathcal{Z}^n$ obtained by n-times repeated sampling w.r.t. $\mathbb{P}$:*

$$\text{gen}(h) \leq 2\mathcal{R}_S(\ell \circ H) + c\sqrt{\frac{\log \frac{1}{\delta}}{n}}, \quad \forall h \in H, \tag{5}$$

where $c > 0$ is some constant and $\ell \circ H := \{(x, y) \mapsto \ell(h, y, x) \mid h \in H\}$ is the loss transformed hypothesis class.

This theorem allows us to bound the generalization error using Rademacher complexities.

2) Covering Number: We begin by defining the ε-cover and the covering number.

Definition 2. *(ε-cover and covering number [22]) Let $(V, \|\cdot\|)$ be a normed space, and let $U \subseteq V$. Then U is an ε-cover of V if $\forall v \in V$, there exists $u \in U$ such that $\|u - v\| \leq \varepsilon$. The covering number of the normed space $(V, \|\cdot\|)$ with any $\varepsilon > 0$ is the size of the smallest ε-cover.*

$$\mathcal{N}(V, \|\cdot\|, \varepsilon) := \min\{|U| : U \text{ is an } \varepsilon\text{-cover of } V\}. \tag{6}$$

We call the logarithm transformed covering number $\log(\mathcal{N}(V, \|\cdot\|, \varepsilon))$ metric entropy of V.

The larger the covering number, the more complex the hypothesis class, because it indicates that more functions are needed to approximate the entire class within a small error tolerance. Intuitively, if a hypothesis class H has a large covering number for small ε, it means the class is complex, as there are many distinct functions that cannot be closely approximated by one another. On the other hand, if the covering number is small, the class is simpler, with many hypotheses being similar to each other within the margin of error ε. There is a common argument chain to arrive at generalization bounds via Thm. 1 and Dudley's entropy integral once a covering number has been determined for a hypothesis class.

Theorem 2 (Dudley's Entropy Integral [23])**.** *Let H be a hypothesis class on $\mathcal{X}$ equipped with a norm $\|\cdot\|$ and let $\gamma_0 := \sup_{h \in H} \|h\|$. We can then bound the (true) Rademacher complexity by Dudley's entropy integral: The usage of*

$$\hat{\mathcal{R}}_n(H) \leq \inf_{\alpha \in [0, \frac{\gamma_0}{2})} 4\alpha + \frac{12}{\sqrt{n}} \int_\alpha^{\gamma_0} \sqrt{\log \mathcal{N}(H, \|\cdot\|, \varepsilon)} d\varepsilon,$$

where $\hat{\mathcal{R}}_n(H)$ is the true Rademacher complexity.

The integral can be understood as summing the contributions of each infinitesimal resolution to the Rademacher complexity of the hypothesis class. At each level of resolution, the function class will become better at fitting the Rademacher random variables.

IV. RESULTS

With this work, we lay the foundation for more theoretically grounded understanding of the capability of hybrid quantum-classical models. We derive and prove a generalization bound $\tilde{O}\left(\sqrt{\frac{T \log T}{N}} + \frac{\alpha}{\sqrt{N}}\right)$ for a hybrid QMLM with N training data points, T trainable quantum gates, k bounded classical fully-connected layers $\|F\| \leq \alpha$.

Our analysis leverages concepts from statistical learning theory and quantum information, allowing us to theoretically predict the empirical performance of hybrid models. The derived generalization bound indicates how the model's complexity, dictated by the number of quantum gates and the characteristics of the classical layers, influences its ability to generalize from the training data to unseen examples. A key implication of our result is that introducing classical layers into the architecture does not introduce significant generalization disadvantage compared to a fully quantum model with equivalent quantum capacity. In fact, the bound suggests that a well-designed hybrid model with a moderate number of classical layers can match the generalization capabilities of a purely quantum model while offering practical advantages such as reduced quantum circuit depth and better noise tolerance. This provides a theoretical justification for hybridization as a strategy for improving scalability without sacrificing learning performance.

V. DISCUSSION

The bound in the main text is for a single fully connected layer case, where the classical term becomes a constant relative to the quantum term, implying that the generalization performance is predominantly influenced by the quantum component. As the number of classical layers k increases, the classical term scales super-linearly, suggesting that deeper classical networks can dominate the generalization bound: $O\left(\sqrt{\frac{T \log T}{N}} + \frac{\alpha^k}{\sqrt{N}}\right)$. Classical NNs typically exhibit generalization bounds that scale unfavorably with the number of layers, often exponentially. In contrast, the hybrid model's quantum component contributes a polylogarithmic factor, potentially offering more favorable scaling under certain conditions.

While the theoretical bounds investigated in this work provide valuable insights, they have been shown to struggle in the overparameterized regime. Recent works, such as [24] and its quantum variant [25], argue that traditional generalization bounds fall short of explaining the empirical success despite high complexities of modern machine learning models. To bridge the gap between theoretical bounds and empirical observations, future research should explore new paradigms of generalization that synergize current understanding of generalization based on model complexities with emerging evidence from edge cases. This approach will foster a more comprehensive understanding of generalization in diverse scenarios. Recent approach [26] has demonstrated strong empirical support for theoretically established bounds in an overparameterized regime. The authors leveraged the relationship between kernel methods and deep learning via Neural Tangent Kernels, validating the idea presented in [27] that comprehending generalization in deep learning necessitates understanding generalization in kernel methods. This suggest that future research into generalization bounds for NNs can benefit from a synergistic investigation into their more foundational cousin - the kernel methods. Similar connections between models has been identified in quantum case [28, 29], suggesting that this line of research might shed light on generalization bounds of QNNs as well.

Apart from understanding the generalization abilities of quantum models, it is crucial to identify the optimal context (use-case) for utilizing these models. Recent research [30] has shown that QCNNs can be efficiently simulated on classical devices though a process known as dequantization. This is particularly efficient on easy dataset that are usually used to demonstrate the utility of proof-of-concept implementations. These results can be extended to a broader QNN architecture. In pursuit of identifying a "killer application" for QMLMs, it is essential to evaluate both the architecture and its generalization capabilities, alongside the complexity of the specific use case at hand. By integrating the tools developed in the present work with insight from [30], we can create a valuable tool in search for quantum advantage.

Our study has some limitations, which should be addressed in the future work. Firstly, we focused on a purely theoretical derivation of the bound, and empirical validation remains to be conducted. However, the work [1] which is foundational for our study has already performed empirical verification of their bound, suggesting that our results should exhibit similar behavior. Secondly, we consider a hybrid architecture comprising a quantum convolutional layer followed by classical layers. While it is straightforward to extend our results to non-convolutional layers, capturing more complex configurations of quantum and classical layers requires additional effort. While the derived generalization bound provides insight into the learning behavior of the hybrid quantum-classical model, it does not directly answer how to determine the optimal quantum-to-classsical ratio of layers. The bound primarily consists of the insight provided by [1] where the generalization bound on the QMLM scales less than exponentially in the training data and the generalization bounds of NNs. However, since both these bounds are loose [24, 25], they fail to capture the interplay between large circuits and deep classical layers.

VI. METHODS

In this section, we present generalization bounds for a QC-CNN and more generally for QMLMs by combining bounds from quantum (see VI-B) and classical (see VI-A) parts. The generalization bound for the quantum part itself is a significant result, confirming some of the good experimental results of the QCCNN [1, 7, 31]. We demonstrate an application to QCCNNs. For a hybrid QMLM and a sufficiently large training data set, our bounds guarantee accurate generalization that is close to the performance of a fully QMLM with high probability on unseen data.

A. Classical Part: Covering Numbers in Neural Networks

For the classical part, we introduce the metric entropy bound following the proofs in [21], which derived generalization bounds for NNs using covering numbers. The presented Lemma 1 on bounded linear functionals has been adapted from [22]. The covering number considers the matrix product XA, where A will be instantiated as the weight matrix for a layer, and X is the data passed through all layers prior to the present layer. We show how this lemma can be applied to the case of the fully-connected layer.

Lemma 1 (ℓ_2 Metric Entropy for Bounded Linear Functionals). *Consider the set $\mathcal{V} = \left\{v \in \mathbb{R}^n \mid \|v\|_{\ell_2} \leq \alpha\right\}$ of bounded linear functionals (by identifying the vectors v with their duals and interpreting them as functionals on $\mathbb{R}^n$). Then, for any $\varepsilon > 0$, we have*

$$\log \mathcal{N}(\mathcal{V}, \|\cdot\|_{\ell_2}, \varepsilon) \leq n \cdot \log\left(\frac{3\alpha}{\varepsilon}\right). \qquad (7)$$

Proof. Notice that $\mathcal{V} = \{v \in \mathbb{R}^n \mid \|v\|_{\ell_2} \leq \alpha\}$, which is a ℓ_2-ball of radius α. The result follows from plugging into the metric entropy for norm-balls. $\square$

Lemma 2 (Metric Entropy for Fully Connected Layer). *Let $\mathcal{F} = \{F \in \mathbb{R}^{m \times n} \mid \|F\|_F \leq \alpha\}$, where m is the output dimension and n the input dimension. This set represents*

bounded matrices presenting a fully connected layer in a neural network. Its metric entropy bound is given by

$$\log \mathcal{N}(\mathcal{F}, \|\cdot\|_F, \varepsilon) \leq nm \cdot \log\left(\frac{3\alpha}{\varepsilon}\right). \qquad (8)$$

Proof. Recall that $\mathbb{R}^{m \times n}$ is isomorphic to $\mathbb{R}^{mn}$ and notice that $\|F\|_F = \|\phi(F)\|_{\ell_2}$, where ϕ is the isomorphism between the two spaces. Thus, we apply Lemma 1 and derive the result. $\square$

B. Quantum Part: Generalization Bound in QMLMs

We introduce the generalization bound for QMLM derived from the covering number of 2-qubit quantum channels and unitary gates, which is formalized in Lemma 3. This lemma provides an upper bound for the covering number of the set of 2-qubit unitaries by using the fact that the space of 2-qubit unitaries is bounded within a unit ball under the operator norm.

Lemma 3. *(Covering number bounds for 2-qubit unitaries [1]). Let $\|\cdot\|$ be a unitarily invariant norm on complex 4×4-matrices. The covering number of the set of 2-qubit unitaries $\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ w.r.t. the norm $\|\cdot\|$ can be bounded as*

$$\mathcal{N}\left(\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \|\cdot\|, \varepsilon\right) \leqslant \left(\frac{6\|I_{\mathbb{C}^4}\|}{\varepsilon}\right)^{32}, \text{ for } 0 < \varepsilon \leqslant \|I_{\mathbb{C}^4}\| \qquad (9)$$

Theorem 3 (Generalization Bound for QMLM [1]). *For a QMLM $\mathcal{E}_\theta^{\mathrm{QMLM}}$ using T parametrized local quantum channels, we have with high probability over training data of size N that*

$$|R(\mathcal{E}_\theta^{\mathrm{QMLM}}) - \hat{R}(\mathcal{E}_\theta^{\mathrm{QMLM}})| \in \mathcal{O}\left(\sqrt{\frac{T \log(T)}{N}}\right). \qquad (10)$$

Remark: This implies that the required size of the data N scales as

$$\sqrt{\frac{T \log(T)}{N}} < \epsilon \implies N > \frac{T \log(T)}{\epsilon^2}$$

The second related bound addresses situations as in QC-CNNs, where many of the parameterized (local) gates are applied repeatedly. Assume each gate is repeated at most M times.

Theorem 4 (Generalization Bound for Repeated Local Gates [1]). *Let $\mathcal{E}_\theta^{\mathrm{QMLM}}$ be a QMLM with an architecture consisting of T independently parameterized 2-qubit Completely Positive Trace-Preserving (CPTP) maps and at most repeated usage of these channels M times. Then, with probability at least $1 - \delta$ over the choice of i.i.d. training data S of size N according to $\mathbb{P}$,*

$$R\left(\mathcal{E}_\theta^{\mathrm{QMLM}}\right) - \hat{R}_S\left(\mathcal{E}_\theta^{\mathrm{QMLM}}\right)$$
$$\in \mathcal{O}\left(\sqrt{\frac{T \log(TM)}{N}} + \sqrt{\frac{\log(1/\delta)}{N}}\right). \qquad (11)$$

C. Generalization Bound in the Hybrid QMLM

Now that we have derived the covering numbers for each part of the hybrid quantum machine learning model, we investigate how these layers interact to establish the generalization bound.

A hybrid QNN integrates both classical and quantum computational parts. Mathematically, this can be viewed as a hypothesis class that consists of functions mapping classical data to quantum states, then back to a classical result via measurement. Formally, we denote a QMLM as

$$\mathcal{E}_{\theta,x}^{\mathrm{QMLM}}(\cdot),$$

where θ are continuous parameters of the quantum gates, i.e. angles in rotation channels and x is the classical input datum.

We assume in the hybrid setting that the classical datum x is encoded into a quantum state with density matrix $\rho(x)$. As the specific encoding technique has no impact on our proof, we will sometimes omit its dependence on x and just write ρ. The classical learning process receives the expected measurement outcome $\mathrm{tr}\left(M \cdot \mathcal{E}_{\theta,x}^{\mathrm{QMLM}}(\rho(x))\right)$, where M is a measurement operator. We also assume that the classical learning process follows empirical risk minimization (ERM) and the optimization over quantum parameters is performed using a classical optimizer, typically gradient-based methods. The model, therefore, consists of a composite hypothesis class

$$H = \left\{h(x) = F \cdot \mathrm{tr}\left(M \cdot \mathcal{E}_{\theta,x}^{\mathrm{QMLM}}\right) \,\Big|\, F \in \mathcal{F}\right\}, \qquad (12)$$

where the quantum component $\mathcal{E}_{\theta,x}^{\mathrm{QMLM}}$ represents a parametrized quantum channel applied to the encoded classical input x, followed by a measurement operator M. The classical component $F \in \mathcal{F}$ is a classical linear function, typically a fully connected layer with bounded operator norm.

Theorem 5 (Generalization Bound in Quantum-Classical Hybrid Models). *Let H be a hypothesis class for a hybrid machine learning model. Assume that the covering number of H with respect to $\|\cdot\|_{\ell_2}$ satisfies, for any $\varepsilon > 0$,*

$$\mathcal{N}(H, \|\cdot\|_{\ell_2}, \varepsilon) \leq \mathcal{N}\left(\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \|\cdot\|, \frac{\varepsilon}{4T\alpha\beta\sqrt{n}}\right)^T$$
$$\cdot \mathcal{N}\left(\mathcal{F}, \|\cdot\|_F, \frac{\varepsilon}{2\beta\sqrt{n}}\right), \qquad (13)$$

where $\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ denotes the set of 2-qubit unitary operators, $\mathcal{F}$ denotes a class of linear operators (e.g., fully connected layers) with operator norm $\|F\| \leq \alpha$, M is a fixed measurement operator with operator norm $\|M\| \leq \beta$, T is the number of parameterized unitaries, and n is the number of output registers of the quantum circuit.

Furthermore, assume that the loss function $\ell : H \times \mathcal{X} \times \mathcal{Y} \to [0, M]$ is L-Lipschitz continuous in its first argument with respect to the norm $\|\cdot\|$, i.e., for all $h_1, h_2 \in H$ and for all $y \in \mathcal{Y}$,

$$|\ell(h_1(x), y) - \ell(h_2(x), y)| \leq L\|h_1 - h_2\|.$$

Then, with probability at least $1 - \delta$ over the random draw of a sample S with size N, for all hypotheses $h \in H$, the expected loss satisfies:

$$R(h) - \hat{R}(h) \in \tilde{\mathcal{O}}\left(\sqrt{\frac{T \log(T)}{N}} + \frac{\alpha}{\sqrt{N}}\right), \qquad (14)$$

where $\tilde{\mathcal{O}}(n)$ denotes Big-O notation with poly-logarithmic terms in n hidden.

The proof consists of combining the two results Lemma 1 and Lemma 3 to obtain a covering number for the hybrid model, after which we establish the generalization bound. Essentially, we demonstrate that the covering numbers exhibit a submultiplicative property across each layer by addressing how the classical and quantum parts interact as separate layers. This then leads to a standard argument of bounding the Rademacher complexity via Dudley's entropy integral which then delivers the generalization bound.

The generalization bound for the quantum part itself is a significant result, confirming some of the good experimental results of the QCNN [1, 31, 32]. We combine previous work on both classical and quantum machine learning models to claim a similar result for hybrid models with multiple classical layers. For a hybrid QMLM and a sufficiently large training data set, a slightly adjusted proof gives a bound that guarantees accurate generalization close to the performance of a fully QMLM with high probability on unseen data.

Theorem 6 (Generalization Bound in Quantum-Classical Hybrid Models with k-layers). *Let H be a hypothesis class for a hybrid machine learning model with T quantum gates and k classical fully-connected layers and a loss function $\ell : H \times \mathcal{X} \times \mathcal{Y} \to [0, M]$ that is L-Lipschitz continuous in its first argument. Then, with probability at least $1 - \delta$ over the random draw of a sample S with size N, for all hypotheses $h \in H$, the expected loss satisfies:*

$$\mathrm{gen}(h) \in \tilde{\mathcal{O}}\left(\sqrt{\frac{T \log(T)}{N}} + \frac{\alpha^k}{\sqrt{N}}\right), \qquad (15)$$

where $\tilde{\mathcal{O}}(n)$ denotes Big-O notation with poly-logarithmic terms in n hidden.

Remark: For a single fully-connected layer this bound reduces to 4 and implies that the generalization error of a hybrid model is not much worse than a fully quantum model as

$$\tilde{\mathcal{O}}\left(\sqrt{\frac{T \log(T)}{N}} + \frac{\alpha}{\sqrt{N}}\right) = \tilde{\mathcal{O}}\left(\sqrt{\frac{T \log(T)}{N}}\right). \qquad (16)$$

VII. Conclusion

This work contributes a theoretical foundation for understanding generalization in hybrid quantum-classical machine learning models. We provide the first characterization of learning capacity in hybrid models by decomposing into distinct quantum and classical contributions. Our result shows that introducing bounded classical layers on top of a trainable quantum model does not degrade generalization performance. Instead, hybrid architectures can retain the expressivity and learning guarantees of their fully quantum counterparts while offering practical benefits, such as reduced quantum circuit depth and improved robustness to noise. While our bound advances the theoretical understanding of hybrid models, it leaves open the important question of how to determine the optimal balance between quantum and classical components. Addressing this challenge will require new theoretical tools or empirical studies that go beyond the current statistical learning theory tools. Nonetheless, our work provides a stepping stone for principled design and evaluation of hybrid models and opens several promising directions for future research in quantum machine learning theory.

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APPENDIX A

In this section, we present the proof for the claimed generalization bounds for hybrid QMLMs by combining bounds from the quantum VI-B and the classical VI-A components. The proof is based on showing the submultiplicativity property of the covering numbers when combining both parts.

Theorem (Generalization Bound in Quantum-Classical Hybrid Models). *Let H be a hypothesis class for a hybrid machine learning model with T quantum gates and k classical fully-connected layers. Assume that the covering number of H with respect to $\|\cdot\|_{\ell_2}$ satisfies, for any $\varepsilon > 0$,*

$$\mathcal{N}\left(H, \|\cdot\|_{\ell_2}, \varepsilon\right) \leq$$
$$\mathcal{N}\left(\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \|\cdot\|, \frac{\varepsilon}{4T\alpha\beta\sqrt{n}}\right)^T \cdot \mathcal{N}\left(\mathcal{F}, \|\cdot\|_F, \frac{\varepsilon}{2\beta\sqrt{n}}\right) \tag{17}$$

where $\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ denotes the set of 2-qubit unitary operators, $\mathcal{F}$ denotes a class of linear operators (e.g., fully connected layers) with operator norm $\|F\| \leq \alpha$, M is a fixed measurement operator with operator norm $\|M\| \leq \beta$, T is the number of parameterized unitaries, and n is the number of output registers of the quantum circuit.

Furthermore, assume that the loss function $\ell : H \times \mathcal{X} \times \mathcal{Y} \to [0, M]$ is L-Lipschitz continuous in its first argument with respect to the norm $\|\cdot\|$, i.e., for all $h_1, h_2 \in H$ and for all $y \in \mathcal{Y}$,

$$|\ell(h_1(x), y) - \ell(h_2(x), y)| \leq L\|h_1 - h_2\|.$$

Then, with probability at least $1 - \delta$ over the random draw of a sample S with size N, for all hypotheses $h \in H$, the expected loss satisfies:

$$\text{gen}(h) \in \tilde{\mathcal{O}}\left(\sqrt{\frac{T\log(T)}{N}} + \frac{\alpha^k}{\sqrt{N}}\right), \tag{18}$$

where $\tilde{\mathcal{O}}(n)$ denotes Big-O notation with poly-logarithmic terms in n hidden.

Proof. The proof will be split over the next subsections. We first derive the covering number of the hypothesis class of hybrid-quantum classical model H. This is done by realizing that any hypothesis can be covered by coverings of the classical and the quantum component. After deriving the covering number, we will calculate the Rademacher complexity of the hypothesis class via Dudley's entropy integral. Finally, we bound the Rademacher complexity of the loss transformed hypothesis class by scaling the Rademacher complexity of the hypothesis class by a Lipschitz constant L. This delivers the generalization bound.

Submultiplicativity of Covering Numbers for a Hybrid Quantum-Classical Model

The hybrid model under consideration comprises two main components: a quantum component and a classical component. The essential step to finding a covering number for the hybrid model is correctly identifying how a covering can be constructed from coverings of the classical and quantum components. Lem. 3 and Lem. 1 each provide covering numbers and we will show that multiplying both provides an upper bound on the hybrid model's covering number. This submultiplicativity property is the following lemma.

Lemma 4. *Let $\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ denote the set of 2-qubit unitary operators, let n be the number of output registers in the quantum circuit and let $\mathcal{F}$ denote a class of linear operators (e.g. fully connected layers) with operator norm $\|F\| \leq \alpha$ and output size of it. As we are in finite dimension, F is a matrix $\mathbb{R}^{m \times n}$. Let M be a fixed measurement operator with operator norm $\|M\| \leq \beta$. Define the hypothesis class H as introduced in Equation 12 with T the number of parametrized unitaries. Let $\mathcal{N}\left(\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \|\cdot\|, \varepsilon\right)$ and $\mathcal{N}\left(\mathcal{F}, \|\cdot\|_F, \varepsilon\right)$ be the covering numbers for the 2-qubit unitary operators and the fully connected layer derived earlier.*

Then, for any $\epsilon > 0$, the covering number of H with respect to $\|\cdot\|_{\ell_2}$ satisfies

$$\mathcal{N}\left(H, \|\cdot\|_{\ell_2}, \varepsilon\right) \leq \mathcal{N}\left(\mathcal{U}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right), \|\cdot\|, \frac{\varepsilon}{4T\alpha\beta\sqrt{n}}\right)^T \cdot$$
$$\cdot \mathcal{N}\left(\mathcal{F}, \|\cdot\|_F, \frac{\varepsilon}{2\beta\sqrt{n}}\right). \tag{19}$$

Proof. We aim to construct an ε-covering for the hypothesis class H by combining ε_F-coverings for the classical component $\mathcal{F}$ and ε_U-coverings for the quantum component $\mathcal{U}$. The key idea is to ensure that perturbations in F and U individually lead to a controlled perturbation in the composite hypothesis h. Let $\mathcal{C}_{\mathcal{F}}(\varepsilon_F)$ be an ε_F-covering of the fully connected layers $\mathcal{F}$ with respect to $\|\cdot\|_F$, and $\mathcal{C}_{\mathcal{U}}(\varepsilon_U)$ be an ε_U-covering of the unitaries $\mathcal{U}$ with respect to the spectral norm (induced 2-norm) $\|\cdot\|_{\ell_2}$. We determine ε_F and ε_U accordingly, such that a hypothesis in H is covered by its components w.r.t. $\|\cdot\|_{\ell_2}$. Consider two hypotheses $h_1, h_2 \in H$. W.l.o.g., we can define them by:

$$h_1(x) = F \cdot \text{tr}\left[MU_2U_1\rho(x)U_1^\dagger U_2^\dagger\right]$$
$$h_2(x) = K \cdot \text{tr}\left[MV_2V_1\rho(x)V_1^\dagger V_2^\dagger\right], \tag{20}$$

where $\rho(x)$ is the density operator of the encoded data x, $F, K \in \mathcal{F}$ are the fully-connected layers and $U_1, U_2, V_1, V_2 \in \mathcal{U}$ are the 2-qubit unitary operators. As such, in the proof $T = 2$ but the case for more than 2 unitary operators can be extended easily. We generalize at the end of the proof to any T and omit x for notational brevity.[1] We write the output after measurement on n registers as

$$\text{tr}\left[M_i U_2 U_1 \rho(x) U_1^\dagger U_2^\dagger\right]_{i=1}^n. \qquad (21)$$

As a reminder, for the case of $n = 4$ as in the QCCNN case, the output of the quantum circuit is

$$v = \begin{pmatrix} \text{tr}(M_1 \mathcal{E}_{\boldsymbol{\theta},x}^{\text{QMLM}}) \\ \text{tr}(M_2 \mathcal{E}_{\boldsymbol{\theta},x}^{\text{QMLM}}) \\ \text{tr}(M_3 \mathcal{E}_{\boldsymbol{\theta},x}^{\text{QMLM}}) \\ \text{tr}(M_4 \mathcal{E}_{\boldsymbol{\theta},x}^{\text{QMLM}}) \end{pmatrix} \in \mathbb{R}^4.$$

To construct a covering for the hypothesis class H, we want to bound

$$\|h_1(x) - h_2(x)\|_{\ell_2}. \qquad (22)$$

First, define

$$A := U_1^\dagger U_2^\dagger M U_2 U_1, \text{ and } B := V_1^\dagger V_2^\dagger M V_2 V_1, \qquad (23)$$

By cyclicity of the trace, we can see that

$$\text{tr}\left[M U_2 U_1 \rho U_1^\dagger U_2^\dagger\right] = \text{tr}\left[U_1^\dagger U_2^\dagger M U_2 U_1 \rho\right]. \qquad (24)$$

Thus, inserting A, B into Eqn. 22 leads to

$$\|h_1(x) - h_2(x)\|_{\ell_2} = \|F \text{ tr}[A\rho] - K \text{ tr}[B\rho]\|_{\ell_2}. \qquad (25)$$

We can insert an intermediate term to rewrite as

$$\begin{aligned} &\|h_1(x) - h_2(x)\|_{\ell_2} \\ =& \|F \text{ tr}[A\rho] - K \text{ tr}[B\rho]\|_{\ell_2} \\ =& \|F \text{ tr}[A\rho] - F \text{ tr}[B\rho] + F \text{ tr}[B\rho] - K \text{ tr}[B\rho]\|_{\ell_2} \quad (26) \\ =& \|F (\text{tr}[(A-B)\rho]) + (F-K) \text{ tr}[B\rho]\|_{\ell_2} \\ \leq& \|F (\text{tr}[(A-B)\rho])\|_{\ell_2} + \|(F-K) \text{ tr}[B\rho]\|_{\ell_2} \end{aligned}$$

Induced norms such as the spectral norm are consistent with the vector norms that induced it:

$$\|Ax\|_{\ell_2} \leq \|A\|_\sigma \|x\|_{\ell_2}. \qquad (27)$$

Thus, we have

$$\begin{aligned} &\|h_1(x) - h_2(x)\|_{\ell_2} \\ \leq& \|F\|_\sigma \|(\text{tr}[(A-B)\rho])\|_{\ell_2} + \|F-K\|_\sigma \|\text{tr}[B\rho]\|_{\ell_2} \end{aligned} \qquad (28)$$

So far, we have achieved two terms, each being a product of a matrix describing the fully connected layer and an output vector, depending on the quantum circuit. Expanding the ℓ_2-norm, we see that

$$\|\text{tr}[B\rho]\|_{\ell_2} = \sqrt{\sum_{i=1}^n |\text{ tr}[B_i\rho]|^2}. \qquad (29)$$

[1]For complete mathematical accuracy, we should specify that the general form of the local 2-qubit unitary operators consists of tensor products $I \otimes \cdots \otimes U_k \otimes \cdots \otimes I$.

and

$$\|\text{tr}[A - B\rho]\|_{\ell_2} = \sqrt{\sum_{i=1}^n |\text{ tr}[(A_i - B_i)\rho]|^2}. \qquad (30)$$

We apply Hölder's inequality to Eqn. 29 to simplify the summands in terms of bounds on the unitaries in B:

$$\|\text{tr}[B\rho]\|_{\ell_2} \leq \sqrt{\sum_{i=1}^n \|B_i\|_F^2} \leq \sqrt{n} \max_{i=1,\ldots,n} \|B_i\|_F. \qquad (31)$$

Noting that for every i, $B_i = V_1^\dagger V_2^\dagger M_i V_2 V_1$, we use the fact that a product of unitaries with any matrix is norm invariant with Hölder's inequality and arrive at

$$\begin{aligned} \|F-K\|_\sigma \|\text{tr}[B\rho]\|_{\ell_2} &\leq \|F-K\|_\sigma \sqrt{n} \max_{i=1,\ldots,n} \|B_i\|_F \\ &\leq \|F-K\|_\sigma \sqrt{n} \max_{i=1,\ldots,n} \|M_i\|_F \quad (32) \\ &\leq \|F-K\|_\sigma \sqrt{n}\beta. \end{aligned}$$

We apply the same lemmata analogously for the first term and use $\|F\|_\sigma \leq \|F\|_F \leq \alpha$:

$$\begin{aligned} &\|h_1(x) - h_2(x)\|_{\ell_2} \leq \\ &\alpha\sqrt{n} \max_{i=1,\ldots,n} \|A_i - B_i\|_F + \beta\sqrt{n}\|F-K\|_F. \end{aligned} \qquad (33)$$

For the first summand, we first write out $\|A - B\|_F$ to see

$$\|A - B\|_F = \left\|U_1^\dagger U_2^\dagger M U_2 U_1 - V_1^\dagger V_2^\dagger M V_2 V_1\right\|_F. \qquad (34)$$

Inserting an intermediate term again, we arrive at

$$\begin{aligned} &U_1^\dagger U_2^\dagger M U_2 U_1 - V_1^\dagger V_2^\dagger M V_2 V_1 \\ =& U_1^\dagger U_2^\dagger M U_2 U_1 - V_1^\dagger V_2^\dagger M U_2 U_1 \\ &+ V_1^\dagger V_2^\dagger M U_2 U_1 - V_1^\dagger V_2^\dagger M V_2 V_1 \\ =& \left(U_1^\dagger U_2^\dagger - V_1^\dagger V_2^\dagger\right)(M U_2 U_1) \\ &+ \left(V_1^\dagger V_2^\dagger M\right)(U_2 U_1 - V_2 V_1) \end{aligned} \qquad (35)$$

Thus,

$$\begin{aligned} \|A - B\|_F &\leq \left\|U_1^\dagger U_2^\dagger - V_1^\dagger V_2^\dagger\right\|_F \|M\|_F \|U_2\|_F \|U_1\|_F \\ &+ \left\|V_1^\dagger\right\|_F \left\|V_2^\dagger\right\|_F \|M\|_F \|U_2 U_1 - V_2 V_1\|_F \quad (36) \\ &= 2\|M\|_F \|U_2 U_1 - V_2 V_1\|_F \end{aligned}$$

The last step used the fact that the operator norm of a unitary operator is 1 and that the operator norm of a matrix and its transpose is the same. Note that this bound is valid for all i if we bound $\|M\|_F \leq \alpha$. Now we bound $\|U_2 U_1 - V_2 V_1\|$ in terms of differences of each of the unitaries. This can be achieved using

$$\begin{aligned} &\|U_1 U_2 - V_1 V_2\|_F \\ =& \|U_1 U_2 - U_1 V_2 + U_1 V_2 - V_1 V_2\|_F \\ \leq& \|U_1(U_2 - V_2)\|_F + \|(U_1 - V_1)V_2\|_F \quad (37) \\ =& \|U_2 - V_2\|_F + \|U_1 - V_1\|_F \quad \text{(Unitary invariance)} \\ =& \|U_1 - V_1\|_F + \|U_2 - V_2\|_F. \end{aligned}$$

Now we have established an upper bound that depends on the covering for unitaries $\|U_1 - V_1\|_F + \|U_2 - V_2\|_F$ and the fully connected layers $\|F - K\|_F$.

$$\|h - h'\|_{\ell_2} \leq$$
$$2\alpha\beta\sqrt{n}(\|U_1 - V_1\|_F + \|U_2 - V_2\|) + \|F - K\|_F \beta\sqrt{n}. \tag{38}$$

Set

$$\|F - K\|_F \leq \frac{\epsilon}{2\beta\sqrt{n}} = \varepsilon_F, \quad \|U_k - V_k\| \leq \frac{\epsilon}{4 \cdot 2\alpha\beta\sqrt{n}} = \varepsilon_U. \tag{39}$$

Then, for any $h \in H$, there exists a covering $\mathcal{C}_H$ with $h' \in \mathcal{C}_H$ since

$$\|h - h'\|_{\ell_2} \leq 2\alpha\beta\sqrt{n}\left(\frac{2\varepsilon}{8\alpha\beta\sqrt{n}}\right) + \frac{\varepsilon}{2\beta\sqrt{n}}\beta\sqrt{n} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{40}$$

Hence, the constructed cover $\mathcal{C}_H$ satisfies:

$$\sup_{h \in H} \inf_{h' \in \mathcal{C}_H} \|h - h'\|_{\ell_2} \leq \epsilon,$$

and since each hypothesis in H can be approximated within ε by combining an ε_F-cover for $\mathcal{F}$ and T many ε_U-cover for $\mathcal{U}$, the total number of elements in $\mathcal{C}_H(\varepsilon)$ is at most the product of the covering numbers of $\mathcal{F}$ and $\mathcal{U}$ T-times, we proved that:

$$\mathcal{N}(H, \varepsilon, \|\cdot\|_{\ell_2}) \leq$$
$$\mathcal{N}\left(\mathcal{F}, \|\cdot\|_F, \frac{\varepsilon}{2\beta\sqrt{n}}\right) \cdot \mathcal{N}\left(\mathcal{U}, \|\cdot\|, \frac{\varepsilon}{4 \cdot 2\alpha\beta\sqrt{n}}\right)^2. \tag{41}$$

Remark: It is merely a small step to arrive at the claimed $\mathcal{N}\left(\mathcal{U}, \|\cdot\|, \frac{\varepsilon}{4 \cdot T\alpha\beta\sqrt{n}}\right)^T$ factor for the unitary covering, which stems from the Equation 37 being used T times and noticing that each of these unitary differences come from one covering of $\mathcal{U}$, giving the power to T. We have omitted this step in the final equation to keep consistency with the rest of the proof. $\square$

Applying the logarithm and using the product rule gives the metric entropy H by

$$\log\mathcal{N}(H, \|\cdot\|_{\ell_2}, \varepsilon) \leq$$
$$\log\mathcal{N}\left(\mathcal{U}(\mathbb{C}^2 \otimes \mathbb{C}^2), \|\cdot\|, \frac{\varepsilon}{4T\alpha\beta\sqrt{n}}\right) \cdot T +$$
$$\log\mathcal{N}\left(\mathcal{F}, \|\cdot\|_F, \frac{\varepsilon}{2\beta\sqrt{n}}\right). \tag{42}$$

Rademacher Complexity: Having established the covering numbers for the hypothesis class $\mathcal{H}$, our next objective is to bound the Rademacher complexity $\mathcal{R}(\mathcal{H})$. We apply Dudley's entropy integral (Theorem 2) by inserting Eqn. (41) to obtain the Rademacher complexity of H. Note again that we will directly apply Dudley's entropy integral to obtain a bound on the true Rademacher complexity and refer to a measure concentration argument to use these interchangeably later on, sacrificing mathematical rigor for highlighting the primary objective of the generalization bound.

Theorem 7 (Rademacher Complexity Bound via Dudley's Entropy Integral). *Let H be a hypothesis class for a hybrid machine learning model and assume that the covering number of H with respect to $\|\cdot\|_{\ell_2}$ satisfies the bound in Lem. 4 for any $\varepsilon > 0$.*

Then, the Rademacher complexity $\hat{\mathcal{R}}_N(H)$ of the hypothesis class H satisfies:

$$\hat{\mathcal{R}}_N(H) \leq \frac{12}{\sqrt{N}}4T\alpha\beta\sqrt{nT}$$
$$\left(\int_0^\alpha \sqrt{\log\mathcal{N}\left(\mathcal{U}(\mathbb{C}^2 \otimes \mathbb{C}^2), \|\cdot\|, \mu\right)}\,d\mu + \right. \tag{43}$$
$$\left. 2\beta\sqrt{n}\int_0^\alpha \sqrt{\log\mathcal{N}(\mathcal{F}, \|\cdot\|, \mu)}\,d\mu\right),$$

where N is the sample size, n the number of registers in the QMLM and $\|F\|_F \leq \alpha$ the upper bound on the norm of the fully connected layer which is also the coarsest resolution for the coverings.

Proof. We start by inserting the metric entropy derived in Eqn.(42) into Dudley's Thm. 2:

$$\hat{\mathcal{R}}_N(H) \leq \frac{12}{\sqrt{N}}$$
$$\int_0^\alpha \sqrt{T\log\mathcal{N}\left(\mathcal{U}, \frac{\varepsilon}{4T\alpha\beta\sqrt{n}}\right) + \log\mathcal{N}\left(\mathcal{F}, \frac{\varepsilon}{2\beta\sqrt{n}}\right)}\,d\varepsilon \tag{44}$$

Here we used that the upper bound of the integration bounds is bounded by the coarsest resolution α. Then by applying the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$:

$$\hat{\mathcal{R}}_N(H) \leq \frac{12}{\sqrt{N}}\left(\int_0^\alpha \sqrt{T\log\mathcal{N}\left(\mathcal{U}, \frac{\varepsilon}{4T\alpha\beta\sqrt{n}}\right)}\,d\varepsilon + \right.$$
$$\left. \int_0^\alpha \sqrt{\log\mathcal{N}\left(\mathcal{F}, \frac{\varepsilon}{2\beta\sqrt{n}}\right)}\,d\varepsilon\right) \tag{45}$$

We simplify by using a change of variable:

$$\text{Let } \delta_1 = \frac{\varepsilon}{4T\alpha\beta\sqrt{n}} \Rightarrow \varepsilon = 4T\alpha\beta\sqrt{n}\delta_1$$
$$\text{and let } \delta_2 = \frac{\varepsilon}{2\beta\sqrt{n}} \Rightarrow \varepsilon = 2\beta\sqrt{n}\delta_2 \tag{46}$$
$$\text{Then } d\varepsilon = 4T\alpha\beta\sqrt{n}\,d\delta_1 \text{ and } d\varepsilon = 2\beta\sqrt{n}\,d\delta_2.$$

Finally, substituting the variables gives us

$$\hat{\mathcal{R}}_N(H) \leq \frac{12}{\sqrt{N}}\left(4T\alpha\beta\sqrt{nT}\right.$$
$$\left. \int_0^\alpha \sqrt{\log\mathcal{N}(\mathcal{U}, \delta_1)}\,d\delta_1 + 2\beta\sqrt{n}\int_0^\alpha \sqrt{\log\mathcal{N}(\mathcal{F}, \delta_2)}\,d\delta_2\right). \tag{47}$$
$\square$

Generalization Bound: Having determined the true Rademacher complexity $\hat{\mathcal{R}}_N(H)$ of the hypothesis class H, we aim to apply Theorem 1 to establish a corresponding generalization bound. However, it is important to observe that Theorem 1 requires the empirical Rademacher complexity of the loss-transformed hypothesis class $\mathcal{R}_S(\mathcal{L} \circ H)$, rather

$\hat{\mathcal{R}}_N(H)$. To overcome this gap, we will introduce the following lemma. Again, we refer to a measurement concentration argument and write the Rademacher complexity ambiguous in the statement. The lemma provides an upper bound on $\mathcal{R}(\mathcal{L} \circ H)$ in terms of $\mathcal{R}(H)$ for loss functions that are Lipschitz-continuous, such as the multi-class hinge loss, we state it without proof.

Lemma 5 (Talagrand-Ledoux's Contraction Lemma[33])**.** *Let H be a hypothesis class and let $\ell : H \times \mathcal{Y} \to \mathbb{R}$ be an L-Lipschitz loss function with respect to its first argument. For any fixed sample $S = \{x_1, x_2, \ldots, x_n\}$, the empirical Rademacher complexity of the loss-transformed hypothesis class $\ell \circ H$ satisfies:*

$$\hat{\mathcal{R}}_S(\ell \circ H) \leq L \cdot \hat{\mathcal{R}}_S(H) \tag{48}$$

By applying this lemma, we can estimate the Rademacher complexity of the loss-transformed hypothesis class by $L \cdot \hat{\mathcal{R}}_S(H)$, so combining it with the previous step:

$$\hat{\mathcal{R}}_N(\ell \circ H) \leq \frac{12L}{\sqrt{N}}\bigg(4T\alpha\beta\sqrt{nT}$$
$$\int_0^\alpha \sqrt{\log \mathcal{N}\left(\mathcal{U}, \mu\right)}\, d\mu + 2\beta\sqrt{n} \int_0^\alpha \sqrt{\log \mathcal{N}\left(\mathcal{F}, \mu\right)}\, d\mu \bigg). \tag{49}$$

Finally, realizing that the first part is the QMLM bound derived in [1], we focus on integrating the second classical term:

$$\frac{24L\beta\sqrt{n}}{\sqrt{N}} \int_0^\alpha \sqrt{\log \mathcal{N}\left(\mathcal{F}, \|\cdot\|, \mu\right)}$$
$$\leqslant \frac{24L\beta\sqrt{n}}{\sqrt{N}} \cdot \int_0^\alpha \sqrt{nm \log\left(\frac{3\alpha}{\mu}\right)}\, d\mu$$
$$= \frac{24L\beta n\sqrt{m}}{\sqrt{N}} \cdot \left(\alpha\sqrt{\log 3\alpha} + \int_0^\alpha \sqrt{\log\left(\frac{1}{\mu}\right)}\, d\mu \right) \tag{50}$$
$$= \frac{24L\beta n\sqrt{m}}{\sqrt{N}} \cdot \Big(\alpha\sqrt{\log(3\alpha)} + \frac{1}{\alpha}\sqrt{\log\frac{1}{\alpha}} - \frac{\sqrt{\pi}}{2}$$
$$\text{erf}(\sqrt{\log\alpha}) - \frac{\sqrt{\pi}}{2} \Big),$$

where we used the integral $\int \sqrt{\log 1/x}\, dx = x\sqrt{\log 1/x} - (\sqrt{\pi}/2) \cdot \text{erf}(\sqrt{\log 1/x})$, with the error function defined as $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt$.

Using Big O notation, we arrive at the claimed

$$\text{gen}(h) \in \tilde{\mathcal{O}}\left(\sqrt{\frac{T\log(T)}{N}} + \frac{\alpha}{\sqrt{N}} \right) \tag{51}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$