作业二

5.

Doolittle 分解：

根据题意对 A 进行 Doolittle 分解得到

$$A=\begin{bmatrix}5&7&9&10\\6&8&10&9\\7&10&8&7\\5&7&6&5\end{bmatrix}=\begin{bmatrix}1&&&\\\frac{6}{5}&1&&\\\frac{7}{5}&-\frac{1}{2}&1&\\1&0&\frac{3}{5}&1\end{bmatrix}\begin{bmatrix}5&7&9&10\\&-\frac{2}{5}&-\frac{4}{5}&-3\\&&-5&-\frac{17}{2}\\&&&\frac{1}{10}\end{bmatrix}$$

通过$Ly=b$即

$$\begin{bmatrix}1&&&\\\frac{6}{5}&1&&\\\frac{7}{5}&-\frac{1}{2}&1&\\1&0&\frac{3}{5}&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

解得$y=\begin{bmatrix}1&-\frac{3}{5}&-\frac{1}{2}&\frac{3}{10}\end{bmatrix}^T$

通过$Ux=y$即

$$\begin{bmatrix}5&7&9&10\\&-\frac{2}{5}&-\frac{4}{5}&-3\\&&-5&-\frac{17}{2}\\&&&\frac{1}{10}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$$

解得$x=\begin{bmatrix}20&-12&-5&3\end{bmatrix}^T$

Crout 分解：

根据题意对 A 进行 Crout 分解得到

$$A=\begin{bmatrix}5&7&9&10\\6&8&10&9\\7&10&8&7\\5&7&6&5\end{bmatrix}=\begin{bmatrix}5&&&\\6&-\frac{2}{5}&&\\7&\frac{1}{5}&-5&\\5&0&-3&\frac{1}{10}\end{bmatrix}\begin{bmatrix}1&\frac{7}{5}&\frac{9}{5}&2\\&1&2&\frac{15}{2}\\&&1&\frac{17}{10}\\&&&1\end{bmatrix}$$

通过$Ly=b$即

$$\begin{bmatrix}5&&&\\6&-\frac{2}{5}&&\\7&\frac{1}{5}&-5&\\5&0&-3&\frac{1}{10}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

解得$y=\begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{10} & 3 \end{bmatrix}^T$

通过$Ux=y$即

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{9}{5} & 2 \\ & 1 & 2 & \frac{15}{2} \\ & & 1 & \frac{17}{10} \\ & & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

解得$x=\begin{bmatrix} 20 & -12 & -5 & 3 \end{bmatrix}^T$

7

平方根法：

根据题意将A进行平方根法分解

$$A=\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.75 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}=\begin{bmatrix} 2 & & \\ -\frac{1}{2} & \frac{3\sqrt{2}}{2} & \\ \frac{1}{2} & \sqrt{2} & \frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} 2 & -\frac{1}{2} & \frac{1}{2} \\ & \frac{3\sqrt{2}}{2} & \sqrt{2} \\ & & \frac{\sqrt{5}}{2} \end{bmatrix}$$

通过$\widehat{L}y=b$即

$$\begin{bmatrix} 2 & & \\ -\frac{1}{2} & \frac{3\sqrt{2}}{2} & \\ \frac{1}{2} & \sqrt{2} & \frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 6 \\ 7.25 \end{bmatrix}$$

解得$y=\begin{bmatrix} 2 & 3.2998 & 1.4162 \end{bmatrix}^T$

通过$\widehat{L}^Tx=y$即

$$\begin{bmatrix} 2 & -\frac{1}{2} & \frac{1}{2} \\ & \frac{3\sqrt{2}}{2} & \sqrt{2} \\ & & \frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

解得$x=\begin{bmatrix} 0.8611 & 0.7111 & 1.2667 \end{bmatrix}^T$

改进平方根法：

根据题意将A进行改进平方根法分解

$$A=\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.75 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}=\begin{bmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} 4 & & \\ & \frac{9}{2} & \\ & & \frac{5}{4} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ & 1 & \frac{2}{3} \\ & & 1 \end{bmatrix}$$

通过$\widehat{L}y=b$即

$$\begin{bmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 6 \\ 7.25 \end{bmatrix}$$

解得$y=\begin{bmatrix} 4 & 7 & 1.5833 \end{bmatrix}^T$

再通过$L^Tx=D^{-1}y$即

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ & 1 & \frac{2}{3} \\ & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{4} & & \\ & \frac{2}{9} & \\ & & \frac{4}{5} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

解得$x=\begin{bmatrix} 0.8611 & 0.7111 & 1.2667 \end{bmatrix}^T$

8.

(1)根据题意将A进行追逐法分解

$$A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}=\begin{bmatrix} 2 & & & \\ 1 & \frac{7}{2} & & \\ & 1 & \frac{26}{7} & \\ & & 1 & \frac{45}{26} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & & \\ & 1 & \frac{2}{7} & \\ & & 1 & \frac{7}{26} \\ & & & 1 \end{bmatrix}$$

通过$\widehat{L}y=b$即

$$\begin{bmatrix} 2 & & & \\ 1 & \frac{7}{2} & & \\ & 1 & \frac{26}{7} & \\ & & 1 & \frac{45}{26} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}=\begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}$$

解得$y=\begin{bmatrix} \frac{1}{2} & -\frac{5}{7} & 1 & -\frac{26}{45} \end{bmatrix}^T$

通过$\widehat{U}x=y$即

$$\begin{bmatrix} 1 & \frac{1}{2} & & \\ & 1 & \frac{2}{7} & \\ & & 1 & \frac{7}{26} \\ & & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

解得$x=\begin{bmatrix} \frac{46}{45} & -\frac{47}{45} & \frac{52}{45} & -\frac{26}{45} \end{bmatrix}^T$

(2)根据题意将A进行追逐法分解

$$A=\begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}=\begin{bmatrix} 4 & & & & \\ -1 & \frac{15}{4} & & & \\ & -1 & \frac{56}{15} & & \\ & & -1 & \frac{209}{56} & \\ & & & -1 & \frac{780}{209} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{4} & & & \\ & 1 & -\frac{4}{15} & & \\ & & 1 & -\frac{15}{56} & \\ & & & 1 & -\frac{56}{209} \\ & & & & 1 \end{bmatrix}$$

通过$\widehat{L}y=b$即

$$\begin{bmatrix} 4 & & & & \\ -1 & \frac{15}{4} & & & \\ & -1 & \frac{56}{15} & & \\ & & -1 & \frac{209}{56} & \\ & & & -1 & \frac{780}{209} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}=\begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \\ 200 \end{bmatrix}$$

解得$y=\begin{bmatrix} 25 & \frac{20}{3} & \frac{25}{14} & \frac{100}{209} & \frac{2095}{39} \end{bmatrix}^T$

通过$\widehat{U}x=y$即

$$\begin{bmatrix} 1 & -\frac{1}{4} & & & \\ & 1 & -\frac{4}{15} & & \\ & & 1 & -\frac{15}{56} & \\ & & & 1 & -\frac{56}{209} \\ & & & & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

解得$x=\left[\ 27.0513\ \ 8.2051\ \ 5.7692\ \ 14.8718\ \ 53.7179\ \right]^T$

10,

根据题意，利用$Gauss-Jordan$消元法求矩阵的逆的过程如下

$$\left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]\to\left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]\to\left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 1 \end{array}\right]\to\left[\begin{array}{ccc:ccc} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 1 \end{array}\right]\to$$

$$\left[\begin{array}{ccc:ccc} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & -1 & 1 & -\frac{1}{2} & 0 \end{array}\right]\to\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right]\to\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -1 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right]\to\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

故可到矩阵的逆如下

$$A^{-1}=\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ -1 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

15,

(1)

迭代格式：

***Jacobi*迭代：**

$$\begin{cases} x_1^{(k+1)} = 1.3 - 0.4x_2^{(k)} - 0.4x_3^{(k)} \\ x_2^{(k+1)} = 1.1 - 0.4x_1^{(k)} - 0.8x_3^{(k)} \\ x_3^{(k+1)} = 2.5 - 0.4x_1^{(k)} - 0.8x_2^{(k)} \end{cases}$$

***Gauss − Seidel*迭代：**

$$\begin{cases} x_1^{(k+1)} = 1.3 - 0.4x_2^{(k)} - 0.4x_3^{(k)} \\ x_2^{(k+1)} = 1.1 - 0.4x_1^{(k+1)} - 0.8x_3^{(k)} \\ x_3^{(k+1)} = 2.5 - 0.4x_1^{(k+1)} - 0.8x_2^{(k+1)} \end{cases}$$

***SOR*迭代：**

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + 0.135(13 - 10x_1^{(k)} - 4x_2^{(k)} - 4x_3^{(k)}) \\ x_2^{(k+1)} = x_2^{(k)} + 0.135(11 - 4x_1^{(k+1)} - 10x_2^{(k)} - 8x_3^{(k)}) \\ x_3^{(k+1)} = x_3^{(k)} + 0.135(25 - 4x_1^{(k+1)} - 8x_2^{(k+1)} - 10x_3^{(k)}) \end{cases}$$

(2)

证明收敛性：

根据题意A是具有正对角元素的实对称矩阵，且A的各阶顺序主子式如下，且其均大于 0

$$\Delta_1 = 10 > 0,\ \ \Delta_2 = 84 > 0,\ \ \Delta_3 = 296 > 0$$

故A是对称正定矩阵，$\omega = 1.75, 0 < \omega < 2$，所以$Gauss - Seidel$迭代和$SOR$迭代均收敛，

而$2D - A$中各阶顺序主子式如下，

$$\Delta_1 = 10 > 0,\ \ \Delta_2 = 84 > 0,\ \ \Delta_3 = -216 < 0$$

故$2D - A$非正定，$Jacobi$迭代发散