

作业 3

2.

(1)根据线性插值条件, 选择 $x_0 = 0.2, x_1 = 0.3$ 作为插值点, 则 $L_1(x)$ 如下

$$L_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

带入 $x_0, x_1, f(x_0), f(x_1)$ 以及 $x = 0.23$, 得出 $f(0.23) = 1.25995$

(2)根据二次插值条件, 选择 $x_0 = 0.1, x_1 = 0.2, x_3 = 0.3$ 作为插值点, 则 $L_2(x)$ 如下

$$L_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

带入 $x_0, x_1, x_2, f(x_0), f(x_1), f(x_2)$ 以及 $x = 0.23$, 得出 $f(0.23) = 1.25866$

7.

(1)根据题意 $f(x) = e^x$ 的等距节点函数表如下

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = e^x$	0.0183	0.0498	0.1353	0.3679	1.0000	2.7183	7.3891	20.0855	54.5982

(2)若用二次插值, 取三个样本点 $x_0 = x_k - h, x_1 = x_k, x_2 = x_k + h$

则由二次插值余项 $R_2(x) = \frac{f^{(3)}(\xi)}{3!} (x - x_0)(x - x_1)(x - x_2)$,

令 $F(x) = (x - x_0)(x - x_1)(x - x_2)$, $F'(x) = 3(x - x_k)^2 - h^2$, 令 $F'(x) = 0$ 可得

$$\max |f(x)| = \left| f \left(x_k + \frac{\sqrt{3}}{3} h \right) \right| = \frac{2\sqrt{3}}{9} h^3$$

\therefore 当 $-4 \leq x \leq 4$ 时, $|R_2(x)| \leq \frac{\sqrt{3}}{27} e^4 h^3 \leq \varepsilon = 10^{-6}$, 故 $h < 0.006585$

10.

根据题意, 由差商和导数之间的关系

$$f[x_0, x_1, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

\therefore 在此题中, $f[2^0 2^1 \dots 2^6] = \frac{1}{6!} f^{(6)}(\xi), f[2^0 2^1 \dots 2^7] = \frac{1}{7!} f^{(7)}(\xi)$

而 $f^{(6)}(x) = 5 \times 6! f^{(7)}(x) = 0$,

$\therefore f[2^0 2^1 \dots 2^6] = \frac{1}{6!} f^{(6)}(\xi) = \frac{5 \times 6!}{6!} = 5$

而 $f[2^0 2^1 \dots 2^7] = \frac{1}{7!} f^{(7)}(\xi) = 0$

24.

根据题意取等距节点 $x_i = a + ih, 0 \leq i \leq n$, 则任给 $x \in [a, b]$, 一定 $\exists i, s. t. x \in [x_i, x_{i+1}]$

以 x_i, x_{i+1} 作为插值基点作 $f(x)$ 的线性插值, 则有

$$L_1(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1})$$

由于

$$\max_{x \in [x_i, x_{i+1}]} |f''(x)| = 2 \quad \max_{x \in [x_i, x_{i+1}]} |(x - x_i)(x - x_{i+1})| \leq \frac{1}{4} h^2$$

\therefore 该插值函数误差有

$$|f(x) - L_1(x)| \leq \frac{1}{2} \max_{x \in [x_i, x_{i+1}]} |f''(x)| \cdot \max_{x \in [x_i, x_{i+1}]} |(x - x_i)(x - x_{i+1})| \leq \frac{1}{4} h^2 = \frac{(b-a)^2}{4n^2}$$

31.

(1) 根据题意令 $f(x) = a_0 \phi_0(x) + a_1 \phi_1(x)$

$$(f, \phi_0) = \int_1^3 \frac{1}{x} dx = \ln 3$$

$$(f, \phi_1) = \int_1^3 \frac{1}{x} \cdot x dx = 2$$

$$(\phi_0, \phi_0) = \int_1^3 1^2 dx = 2$$

$$(\phi_0, \phi_1) = (\phi_1, \phi_0) = \int_1^3 x dx = \frac{1}{2} x^2 \Big|_1^3 = 4$$

$$(\phi_1, \phi_1) = \int_1^3 x^2 dx = \frac{1}{3} x^3 \Big|_1^3 = \frac{26}{3}$$

解如下方程可得

$$\begin{bmatrix} 2 & 4 \\ 4 & \frac{26}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \ln 3 \\ 2 \end{bmatrix}$$

$\therefore a_0 = 1.1410, a_1 = -0.2958$, 故 $f(x) = -0.2958x + 1.1410$

(2) 根据题意令 $f(x) = a_0 \phi_0(x) + a_1 \phi_1(x) + a_2 \phi_2(x)$

$$(f, \phi_0) = \int_0^1 \cos \pi x dx = \frac{\sin \pi x}{\pi} \Big|_0^1 = 0$$

$$(f, \phi_1) = \int_0^1 \cos \pi x \cdot x dx = \frac{1}{\pi} x \cdot \sin \pi x \Big|_0^1 - \frac{1}{\pi} \int_0^1 \sin \pi x dx = \frac{\cos \pi x}{\pi^2} \Big|_0^1 = -\frac{2}{\pi^2}$$

$$(f, \phi_2) = \int_0^1 \cos \pi x \cdot x^2 dx = \frac{1}{\pi} x^2 \sin \pi x \Big|_0^1 - \frac{1}{\pi} \int_0^1 \sin \pi x d(x^2) = \frac{2}{\pi^2} x \cos \pi x \Big|_0^1 - \frac{2}{\pi^2} \int_0^1 \cos \pi x dx = -\frac{2}{\pi^2}$$

且由题意可得该法方程组的系数矩阵为 3 阶 Hilbert 矩阵, 则有如下方程组

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{\pi^2} \\ -\frac{2}{\pi^2} \end{bmatrix}$$

$$a_0 = 1.2159, a_1 = -2.4317, a_2 = 0.0000, \text{ 故 } f(x) = 0.0000x^2 - 2.4317x + 1.2159$$

33.

根据题意有

$$(\phi_0, \phi_0) = \sum_{i=1}^5 1^2 = 5$$

$$(\phi_0, \phi_1) = \sum_{i=0}^4 1 \cdot x_i^2 = 5327$$

$$(\phi_1, \phi_1) = \sum_{i=0}^4 x_i^4 = 7277699$$

$$(f, \phi_0) = \sum_{i=0}^4 y_i = 271.4$$

$$(f, \phi_1) = \sum_{i=0}^4 y_i \cdot x_i^2 = 9776.1$$

$$(f, f) = \sum_{i=0}^4 y_i^2 = 18743.02$$

解如下方程可得

$$\begin{bmatrix} (\phi_0, \phi_0) & (\phi_0, \phi_1) \\ (\phi_1, \phi_0) & (\phi_1, \phi_1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (f, \phi_0) \\ (f, \phi_1) \end{bmatrix}$$

$$a = 0.9726, b = 0.0500, \text{ 故 } y = 0.9726 + 0.500x^2$$

$$\text{根据误差估计式可得 } \|\delta\|_2^2 = \|f\|_2^2 - \sum a_i^* (f, \phi_i) = (f, f) - a(f, \phi_0) - b(f, \phi_1) = 0.015023$$

$$\text{故均方误差 } \delta = \sqrt{\|\delta\|_2^2} = 0.1226$$