作业 3

2.

(1)根据线性插值条件，选择$x_0=0.2, x_1=0.3$作为插值点，则$L_1(x)$如下

$$L_1(x)=\frac{x-x_1}{x_0-x_1}f(x_0)+\frac{x-x_0}{x_1-x_0}f(x_1)$$

带入$x_0, x_1, f(x_0), f(x_1)$以及$x=0.23$，得出$f(0.23)=\mathbf{1.25995}$

(2)根据二次插值条件，选择$x_0=0.1, x_1=0.2, x_3=0.3$作为插值点，则$L_2(x)$如下

$$L_2(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

带入$x_0, x_1, x_2, f(x_0), f(x_1), f(x_2)$以及$x=0.23$，得出$f(0.23)=\mathbf{1.25866}$

7.

(1)根据题意$f(x)=e^x$的等距节点函数表如下

$\boldsymbol{x}$	-4	-3	-2	-1	0	1	2	3	4
$\boldsymbol{f(x)=e^x}$	0.0183	0.0498	0.1353	0.3679	1.0000	2.7183	7.3891	20.0855	54.5982

(2)若用二次插值，取三个样本点$x_0=x_k-h, x_1=x_k, x_2=x_k+h$

则由二次插值余项$R_2(x)=\frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$，

令$F(x)=(x-x_0)(x-x_1)(x-x_2)$，$F'(x)=3(x-x_k)^2-h^2$，令$F'(x)=0$可得

$$max|f(x)|=\left|f\left(x_k+\frac{\sqrt{3}}{3}h\right)\right|=\frac{2\sqrt{3}}{9}h^3$$

$\therefore$ 当$-4\le x\le 4$时，$|R_2(x)|\le\frac{\sqrt{3}}{27}e^4h^3\le\varepsilon=10^{-6}$，故$h<0.006585$

10.

根据题意，由差商和导数之间的关系

$$f[x_0,x_1,\dots,x_n]=\frac{1}{n!}f^{(n)}(\xi)$$

$\therefore$ 在此题中，$f[2^0 2^1\cdots 2^6]=\frac{1}{6!}f^{(6)}(\xi), f[2^0 2^1\cdots 2^7]=\frac{1}{7!}f^{(7)}(\xi)$

而$f^{(6)}(x)=5\times 6!\ f^{(7)}(x)=0$，

$\therefore f[2^0 2^1\cdots 2^6]=\frac{1}{6!}f^{(6)}(\xi)=\frac{5\times 6!}{6!}=5$

而$f[2^0 2^1\cdots 2^7]=\frac{1}{7!}f^{(7)}(\xi)=0$

24.

根据题意取等距节点$x_i=a+ih, 0\le i\le n$，则任给$x\in[a,b]$，一定$\exists i, s.t.\, x\in[x_i,x_{i+1}]$

以x_i,x_{i+1}作为插值基点作$f(x)$的线性插值，则有

$$L_1(x)=\frac{x-x_{i+1}}{x_i-x_{i+1}}f(x_i)+\frac{x-x_i}{x_{i+1}-x_i}f(x_{i+1})$$

由于

$$\max_{x\in[x_i,x_{i+1}]}|f''(x)|=2 \qquad \max_{x\in[x_i,x_{i+1}]}|(x-x_i)(x-x_{i+1})|\le\frac{1}{4}h^2$$

∴该插值函数误差有

$$|f(x)-L_1(x)|\le\frac{1}{2}\max_{x\in[x_i,x_{i+1}]}\left|f'^{\prime}(x)\right|\cdot\max_{x\in[x_i,x_{i+1}]}|(x-x_i)(x-x_{i+1})|\le\frac{1}{4}h^2=\frac{(b-a)^2}{4n^2}$$

31.

(1)根据题意令$f(x)=a_0\phi_0(x)+a_1\phi_1(x)$

$$(f,\phi_0)=\int_1^3\frac{1}{x}dx=\ln 3$$

$$(f,\phi_1)=\int_1^3\frac{1}{x}\cdot xdx=2$$

$$(\phi_0,\phi_0)=\int_1^3 1^2\,dx=2$$

$$(\phi_0,\phi_1)=(\phi_1,\phi_0)=\int_1^3 xdx=\frac{1}{2}x^2\bigg|_1^3=4$$

$$(\phi_1,\phi_1)=\int_1^3 x^2\,dx=\frac{1}{3}x^3\bigg|_1^3=\frac{26}{3}$$

解如下方程可得

$$\begin{bmatrix}2 & 4\\ 4 & \frac{26}{3}\end{bmatrix}\begin{bmatrix}a_0\\ a_1\end{bmatrix}=\begin{bmatrix}\ln 3\\ 2\end{bmatrix}$$

$\therefore a_0=1.1410,a_1=-0.2958$，故$f(x)=-0.2958x+1.1410$

(2)根据题意令$f(x)=a_0\phi_0(x)+a_1\phi_1(x)+a_2\phi_2(x)$

$$(f,\phi_0)=\int_0^1\cos\pi xdx=\frac{\sin\pi x}{\pi}\bigg|_0^1=0$$

$$(f,\phi_1)=\int_0^1\cos\pi x\cdot xdx=\frac{1}{\pi}x\cdot\ sin\pi x\bigg|_0^1-\frac{1}{\pi}\int_0^1\sin\pi xdx=\frac{\cos\pi x}{\pi^2}\bigg|_0^1=-\frac{2}{\pi^2}$$

$$(f,\phi_2)=\int_0^1\cos\pi x\cdot x^2dx=\frac{1}{\pi}x^2\sin\pi x\bigg|_0^1-\frac{1}{\pi}\int_0^1\sin\pi xd(x^2)=\frac{2}{\pi^2}x\cos\pi x\bigg|_0^1-\frac{2}{\pi^2}\int_0^1\cos\pi xdx=-\frac{2}{\pi^2}$$

且由题意可得该法方程组的系数矩阵为 3 阶$Hilbert$矩阵，则有如下方程组

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} 0 \\ -\frac{2}{\pi^2} \\ -\frac{2}{\pi^2} \end{bmatrix}$$

$a_0 = 1.2159, a_1 = -2.4317, a_2 = 0.0000$，故$f(x) = 0.0000x^2 - 2.4317x + 1.2159$

33.

根据题意有

$$(\phi_0, \phi_0) = \sum_{i=1}^{5} 1^2 = 5$$

$$(\phi_0, \phi_1) = \sum_{i=0}^{4} 1 \cdot x_i^2 = 5327$$

$$(\phi_1, \phi_1) = \sum_{i=0}^{4} x_i^4 = 7277699$$

$$(f, \phi_0) = \sum_{i=0}^{4} y_i = 271.4$$

$$(f, \phi_1) = \sum_{i=0}^{4} y_i \cdot x_i^2 = 369321.5$$

$$(f, f) = \sum_{i=0}^{4} y_i^2 = 18743.02$$

解如下方程可得

$$\begin{bmatrix} (\phi_0, \phi_0) & (\phi_0, \phi_1) \\ (\phi_1, \phi_0) & (\phi_1, \phi_1) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} (f, \phi_0) \\ (f, \phi_1) \end{bmatrix}$$

$a = 0.9726, b = 0.0500$，故$y = 0.9726 + 0.0500x^2$

根据误差估计式可得$\left\|\delta\right\|_2^2 = \left\|f\right\|_2^2 - \sum a_i^*(f, \phi_i) = (f, f) - a(f, \phi_0) - b(f, \phi_1) = 0.015023$

故均方误差$\delta = \sqrt{\left\|\delta\right\|_2^2} = 0.1226$