

# Koopa: Learning Non-stationary Time Series Dynamics with Koopman Predictors

Yong Liu\*, Chenyu Li\*, Jianmin Wang, Mingsheng Long✉

School of Software, BNRist, Tsinghua University, China

{liuyong21, lichenyu20}@mails.tsinghua.edu.cn, {jimwang, mingsheng}@tsinghua.edu.cn

## Abstract

Real-world time series is characterized by intrinsic non-stationarity that poses a principal challenge for deep forecasting models. While previous models suffer from complicated series variations induced by changing temporal distribution, we tackle non-stationary time series with modern Koopman theory that fundamentally considers the underlying time-variant dynamics. Inspired by Koopman theory of portraying complex dynamical systems, we disentangle time-variant and time-invariant components from intricate non-stationary series by *Fourier Filter* and design *Koopman Predictor* to advance respective dynamics forward. Technically, we propose **Koopa** as a novel **Koopman** forecaster composed of stackable blocks that learn hierarchical dynamics. Koopa seeks measurement functions for Koopman embedding and utilizes Koopman operators as linear portraits of implicit transition. To cope with time-variant dynamics that exhibits strong locality, Koopa calculates context-aware operators in the temporal neighborhood and is able to utilize incoming ground truth to scale up forecast horizon. Besides, by integrating Koopman Predictors into deep residual structure, we ravel out the binding reconstruction loss in previous Koopman forecasters and achieve end-to-end forecasting objective optimization. Compared with the state-of-the-art model, Koopa achieves competitive performance while saving 77.3% training time and 76.0% memory.

## 1 Introduction

Time series forecasting has become an essential part of real-world applications, such as weather forecasting, energy consumption, and financial assessment. With numerous available observations, deep learning approaches exhibit superior performance and bring the boom of deep forecasting models. TCNs [4, 43, 47] utilize convolutional kernels and RNNs [12, 22, 37] leverage the recurrent structure to capture underlying temporal patterns. Afterward, attention mechanism [42] becomes the mainstream of sequence modeling and Transformers [31, 48, 53] show great predictive power with the capability of learning point-wise temporal dependencies. And the recent revival of MLPs [32, 51, 52] presents a simple but effective approach to exhibit temporal dependencies by dense weighting.

In spite of elaboratively designed models, it is a fundamental problem for deep models to generalize on varying distribution [1, 25, 33], which is widely reflected in real-world time series because of inherent non-stationarity. Non-stationary time series is characterized by time-variant statistics and temporal dependencies in different periods [2, 14], inducing a huge distribution gap between training and inference and even among each lookback window. While previous methods [16, 28] tailor existing architectural design to attenuate the adverse effect of non-stationarity, few works research on the theoretical basis that can be applied to deal with time-variant temporal patterns naturally.

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\*Equal Contribution

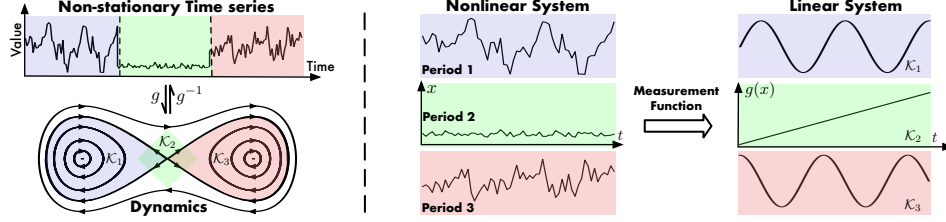


Figure 1: The measurement function  $g$  maps between non-stationary time series and the nonlinear dynamical system so that the timeline will correspond to a system trajectory. Therefore, time series variations in different periods are reflected as sub-regions of nonlinear dynamics, which can be portrayed and advanced forward in time by linear Koopman operators  $\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$  respectively.

From another perspective, real-world time series acts like time-variant dynamics [6]. As one of the principal approaches to analyze complex dynamics, Koopman theory [20] provides an enlightenment to transform nonlinear system into measurement function space, which can be described by a linear Koopman operator. Several pilot works accomplish the integration with deep learning approaches by employing autoencoder networks [40] and operator-learning [26, 49]. More importantly, it is supported by Koopman theory that for time-variant dynamics, there exists a coordinate transformation of the system, where localized Koopman operators are valid to describe the whole measurement function space into several subspaces with linearization [23, 36]. Therefore, Koopman-based methods are appropriate to learn non-stationary time series dynamics (Figure 1). Besides, the linearity of measurement function space enables us to utilize spectral analysis to interpret nonlinear systems.

In this paper, we disentangle non-stationary series into time-invariant and time-variant dynamics and propose **Koopa** as a novel **Koopman** forecaster, which is composed of modular **Koopman Predictors (KP)** to hierarchically describe and advance forward series dynamics. Concretely, we utilize Fourier analysis for dynamics disentangling. And for time-invariant dynamics, the model learns Koopman embedding and linear operators to reveal the implicit transition underlying long-term series. As for the remaining time-variant components that exhibit strong locality, Koopa performs context-aware operator calculation and adaptation within different lookback windows. Besides, Koopman Predictor goes beyond the canonical design of Koopman Autoencoder without the binding reconstruction loss, and we incorporate modular blocks into deep residual architecture [32] to realize end-to-end time series forecasting. Our contributions are summarized as follows:

- From the perspective of modern dynamics Koopman theory, we propose *Koopa* composed of modular *Fourier Filter* and *Koopman Predictor*, which can hierarchically disentangle and exploit time-invariant and time-variant dynamics for time series forecasting.
- Based on the linearity of Koopman operators, the proposed model is able to utilize incoming series and adapt to varying dynamics for scaling up forecast horizon.
- Compared with state-of-the-art methods, our model achieves competitive performance while saving 77.3% training time and 76.0% memory averaged from six real-world benchmarks.

## 2 Related Work

### 2.1 Time Series Forecasting with DNNs

Deep neural networks (DNNs) have made great breakthroughs in time series forecasting. TCN-based models [4, 43, 47] explore hierarchical temporal patterns and adopt shared convolutional kernels with diverse receptive fields. RNN-based models [12, 22, 37] utilize the recurrent structure with memory to reveal the implicit transition over time points. MLP-based models [32, 51, 52] learn point-wise weighting and the impressive performance and efficiency highlight that MLP performs well for modeling simple temporal dependencies. However, their practical applicability may still be constrained on non-stationary time series, which is endowed with time-variant properties and poses challenges for model capacity and efficiency. Unlike previous methods, Koopa fundamentally considers the complicated dynamics underlying time series and implements efficient and interpretable transition learners in both time-variant and time-invariant manners inspired by Koopman theory.

Recently, Transformer-based models have also achieved great success in time series forecasting. Initial attempts [19, 27, 48, 53] renovate the canonical structure and reduce the quadratic complexity

for long-term forecasting. However, recent studies [16, 21] find it a central problem for Transformer and other DNNs to generalize on varying temporal distribution and several works [15, 16, 21, 28] tailor to empower the robustness against shifted distribution. Especially, PatchTST [31] boosts Transformer to the state-of-the-art performance by channel-independence and instance normalization [41] but may lead to unaffordable computational cost when the number of series variate is large. In this paper, our proposed model supported by Koopman theory works naturally for non-stationary time series and achieves the state-of-the-art forecasting performance with remarkable model efficiency.

## 2.2 Learning Dynamics with Koopman Operator

Koopman theory [20] has emerged over decades as the dominant perspective to analyze modern dynamical systems [6]. Together with Dynamic Mode Decomposition (DMD) [38] as the leading numerical method to approximate the Koopman operator, significant advances have been accomplished in aerodynamics and fluid physics [3, 9, 30]. Recent progress made in Koopman theory is inherently incorporated with deep learning approaches in the data science era. Pilot works [29, 40, 50] leverage data-driven approaches such as Koopman Autoencoder to learn the measurement function and operator simultaneously. PCL [3] further introduces a backward procedure to improve the consistency and stability of the operator. Based on the capability of Koopman operator to advance nonlinear dynamics forward, it is also widely applied to sequence prediction. By means of Koopman spectral analysis, MDKAE [5] disentangles dominant factors underlying sequential data and is competent to forecast with specific factors. K-Forecast [24] utilizes Koopman theory to handle nonlinearity in temporal signals and propose to optimize data-dependent basis for long-term time series forecasting. By leveraging predefined measurement functions, KNF [44] learns Koopman operator and attention map to cope with time series forecasting with changing temporal distribution.

Different from previous Koopman forecasters, we design modular Koopman Predictors to tackle time-variant and time-invariant components with hierarchically learned operators, and renovate Koopman Autoencoder by removing the reconstruction loss to achieve fully predictive training.

## 3 Background

### 3.1 Koopman Theory

A discrete-time dynamical system can be formulated as  $x_{t+1} = \mathbf{F}(x_t)$ , where  $x_t$  denotes the system state and  $\mathbf{F}$  is a vector field describing the dynamics. However, it is challenging to identify the system transition directly on the state because of nonlinearity or noisy data. Instead, Koopman theory [20] hypothesizes the state can be projected into the space of measurement function  $g$ , which can be governed and advanced forward in time by an infinite-dimensional linear operator  $\mathcal{K}$ , such that:

$$\mathcal{K} \circ g(x_t) = g(\mathbf{F}(x_t)) = g(x_{t+1}). \quad (1)$$

Koopman theory provides a bridge between finite-dimensional nonlinear dynamics and infinite-dimensional linear dynamics, where spectral analysis tools can be applied to obtain in-depth analysis.

### 3.2 Dynamic Mode Decomposition

Dynamic Mode Decomposition (DMD) [38] seeks the best fitted finite-dimensional matrix  $K$  to approximate infinite-dimensional operator  $\mathcal{K}$  by collecting the observed system state (a.k.a. *snapshot*). Although DMD is the standard numerical method to analyze dynamics, it only works on linear space assumptions, which can be hardly identified without prior knowledge. Therefore, eDMD [45] is proposed to avoid handcrafting measurement functions and harmonious incorporations are made with the learning approach by employing autoencoders, which yields Koopman Autoencoder (KAE). By the universal approximation theorem [13] of deep networks, KAE finds desired *Koopman embedding*  $g(x_t)$  with learned measurement function in a data-driven approach.

### 3.3 Time Series as Dynamics

It is challenging to predict real-world time series because of inherent non-stationarity. But if we zoom in the timeline, we will find the localized time series exhibited weak stationarity. It coincides with Koopman theory to analyze large nonlinear dynamics. That is, the measurement

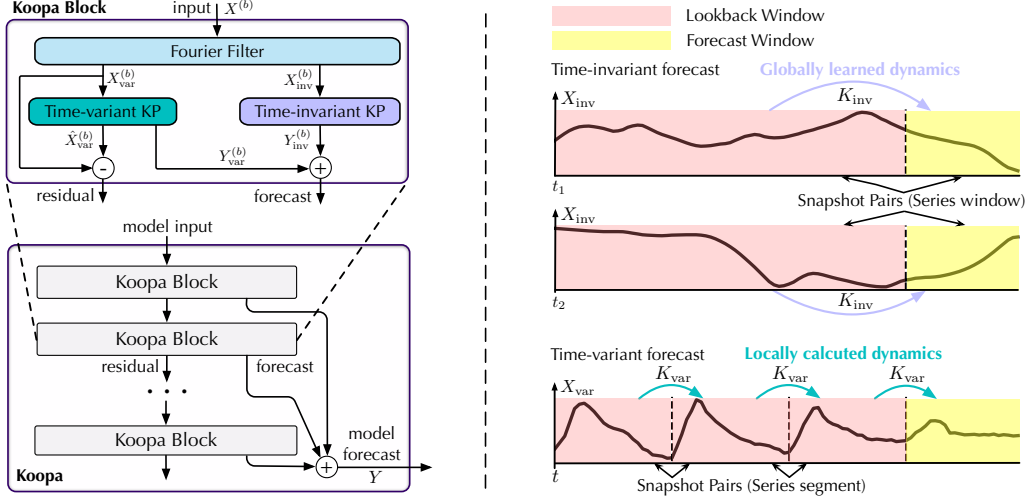


Figure 2: Left: Koopa structure. By taking the residual of previous block fitted dynamical system, each block learns hierarchical dynamics and Koopa aggregates the forecast of all blocks. Right: For time-invariant forecast, Koopa learns globally shared dynamics from each lookback-forecast window pair. For time-variant forecast, the model calculates localized and segment-wise dynamics.

function space can be divided into several neighborhoods, which are discriminately portrayed by localized linear operators [23]. Therefore, we leverage Koopman-based approaches that tackle large nonlinear dynamics by disentangling time-variant and time-invariant dynamics. Inspired by Wold’s Theorem [46] that every covariance-stationary time series  $X_t$  can be formally decomposed as:

$$X_t = \eta_t + \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}, \quad (2)$$

where  $\eta_t$  denotes the deterministic component and  $\varepsilon_t$  is the stochastic component as the stationary process input of linear filter  $\{b_j\}$ , we introduce globally learned and localized linear Koopman operators to exploit respective dynamics underlying different components.

## 4 Koopa

We propose *Koopa* composed of stackable *Koopa Blocks* (Figure 2). Each block is obliged to learn the input dynamics and advance it forward for prediction. Instead of struggling to seek one unified operator that governs the whole measurement function space, each Koopa Block is encouraged to learn operators hierarchically by taking the residual of previous block fitted dynamics as its input.

**Koopa Block** As aforementioned, it is essential to disentangle different dynamics and adopt proper operators for non-stationary series forecasting. The proposed block shown in Figure 2 contains *Fourier Filter* that utilizes frequency domain statistics to disentangle time-variant and time-invariant components and implements two types of *Koopman Predictor (KP)* to obtain Koopman embedding respectively. In Time-invariant KP, we set the operator as a model parameter to be globally learned from lookback-forecast windows. In Time-variant KP, analytical operator solutions are calculated locally within the lookback window, with series segments arranged as snapshots. In detail, we formulate the  $b$ -th block input  $X^{(b)}$  as  $[x_1, x_2, \dots, x_T]^T \in \mathbb{R}^{T \times C}$ , where  $T$  and  $C$  denote the lookback window length and the variate number. The target is to output a forecast window of length  $H$ . Our proposed Fourier Filter conducts disentanglement at the beginning of each block:

$$X_{var}^{(b)}, X_{inv}^{(b)} = \text{FourierFilter}(X^{(b)}). \quad (3)$$

Respective KPs will predict with time-invariant input  $X_{inv}^{(b)}$  and time-variant input  $X_{var}^{(b)}$ , and Time-variant KP simultaneously outputs the fitted input  $\hat{X}_{var}^{(b)}$ :

$$\begin{aligned} Y_{inv}^{(b)} &= \text{TimeInvKP}(X_{inv}^{(b)}), \\ \hat{X}_{var}^{(b)}, Y_{var}^{(b)} &= \text{TimeVarKP}(X_{var}^{(b)}). \end{aligned} \quad (4)$$

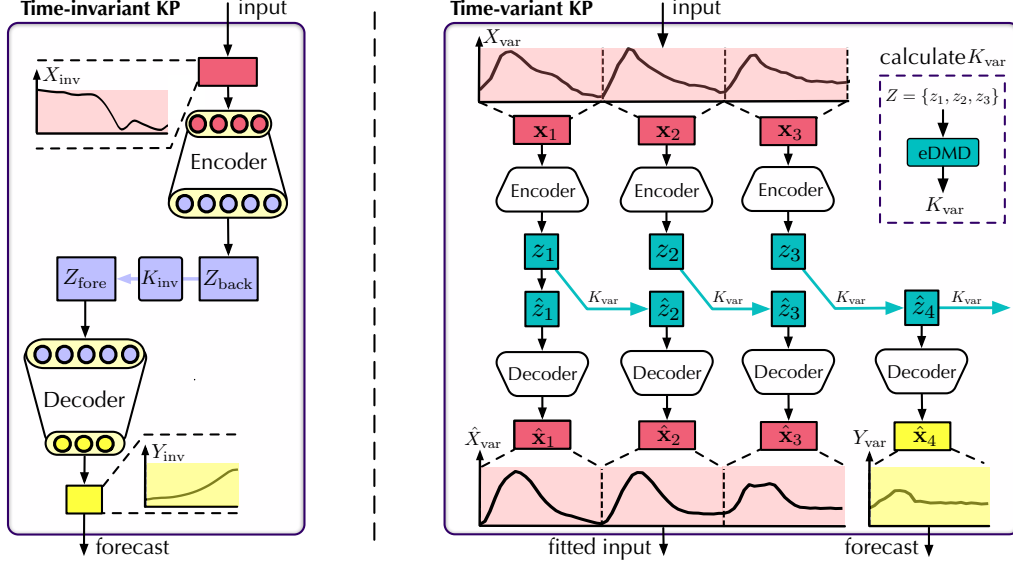


Figure 3: Left: Time-invariant KP learns Koopman embedding and operator with time-invariant components globally from all windows. Right: Time-variant KP conducts localized operator calculation within lookback window and advances dynamics forward with the obtained operator for predictions.

Unlike KAEs [45, 44] that introduce a loss term for rigorous reconstruction of the lookback-window series, we feed the residual  $X^{(b+1)}$  as the input of next block for learning a corrective operator. And the model forecast  $Y$  is the sum of predicted components  $Y_{\text{var}}^{(b)}, Y_{\text{inv}}^{(b)}$  gathered from all Koopa Blocks:

$$X^{(b+1)} = X_{\text{var}}^{(b)} - \hat{X}_{\text{var}}^{(b)}, \quad Y = \sum (Y_{\text{var}}^{(b)} + Y_{\text{inv}}^{(b)}). \quad (5)$$

**Fourier Filter** To disentangle the series components, we leverage Fourier analysis to find the globally shared and localized frequency spectrums reflected on different periods. Concretely, we precompute the Fast Fourier Transform (FFT) of each lookback window of the training set, calculate the averaged amplitude of each spectrum  $\mathcal{S} = \{0, 1, \dots, [T/2]\}$ , and sort them by corresponding amplitude. We take the top percent of  $\alpha$  as the subset  $\mathcal{G}_\alpha \subset \mathcal{S}$ , which contains dominant spectrums shared among all lookback windows and exhibits time-invariant dynamics underlying the dataset. And the remaining spectrums are the specific ingredient for varying windows in different periods. Therefore, we divide the spectrums  $\mathcal{S}$  into  $\mathcal{G}_\alpha$  and its complementary set  $\bar{\mathcal{G}}_\alpha$ . During training and inference, FourierFilter( $\cdot$ ) conducts the disentanglement of input  $X$  (block superscript omitted) as

$$\begin{aligned} X_{\text{inv}} &= \mathcal{F}^{-1}(\text{Filter}(\mathcal{G}_\alpha, \mathcal{F}(X))), \\ X_{\text{var}} &= \mathcal{F}^{-1}(\text{Filter}(\bar{\mathcal{G}}_\alpha, \mathcal{F}(X))) = X - X_{\text{inv}}, \end{aligned} \quad (6)$$

where  $\mathcal{F}$  means FFT,  $\mathcal{F}^{-1}$  is its inverse and Filter( $\cdot$ ) only passes corresponding frequency spectrums with the given set. We validate the disentangling effect of our proposed Fourier Filter in Section 5.2 by calculating the variation degree of temporal dependencies in the disentangled series.

**Time-invariant KP** Time-invariant KP is designed to portray the globally shared dynamics, which discovers the direct transition from lookback window to forecast window as  $\mathbf{F} : X_{\text{inv}} \mapsto Y_{\text{inv}}$ . Concretely, we introduce a pair of Encoder :  $\mathbb{R}^{T \times C} \mapsto \mathbb{R}^D$  and Decoder :  $\mathbb{R}^D \mapsto \mathbb{R}^{H \times C}$  to learn the common Koopman embedding for the time-invariant components of running window pairs, where  $D$  denotes the embedding dimension. Working on the data-driven measurement function, we introduce the operator  $K_{\text{inv}} \in \mathbb{R}^{D \times D}$  as a learnable parameter in each Time-invariant KP, which regards the embedding of lookback and forecast window  $Z_{\text{back}}, Z_{\text{fore}} \in \mathbb{R}^D$  as running snapshot pairs. The procedure is shown in Figure 3 and TimeInvKP( $\cdot$ ) is formulated as follows:

$$Z_{\text{back}} = \text{Encoder}(X_{\text{inv}}), \quad Z_{\text{fore}} = K_{\text{inv}} Z_{\text{back}}, \quad Y_{\text{inv}} = \text{Decoder}(Z_{\text{fore}}). \quad (7)$$

**Time-variant KP** As time-variant dynamics changes continuously, we utilize localized snapshots in a window, which constitute a temporal neighborhood more likely to be linearized. To obtain semantic snapshots and reduce iterations, the input  $X_{\text{var}}$  is divided into  $\frac{T}{S}$  segments  $\mathbf{x}_j$  of length  $S$ :

$$\mathbf{x}_j = [x_{(j-1)S+1}, \dots, x_{jS}]^\top \in \mathbb{R}^{S \times C}, j = 1, 2, \dots, T/S. \quad (8)$$

We assume  $S$  is divisible by  $T$  and  $H$ ; otherwise, we pad the input or truncate the output to make it compatible. Time-variant KP aims to portray localized dynamics, which is manifested analytically as the segment-wise transition  $\mathbf{F} : \mathbf{x}_t \mapsto \mathbf{x}_{t+1}$  with observed snapshots. We utilize another pair of Encoder :  $\mathbb{R}^{S \times C} \mapsto \mathbb{R}^D$  to transform each segment into Koopman embedding  $z_j$  and Decoder :  $\mathbb{R}^D \mapsto \mathbb{R}^{S \times C}$  to transform the fitted or predicted embedding  $\hat{z}_j$  back to time segments  $\hat{\mathbf{x}}_j$ :

$$z_j = \text{Encoder}(\mathbf{x}_j), \hat{\mathbf{x}}_j = \text{Decoder}(\hat{z}_j). \quad (9)$$

Given snapshots collection  $Z = [z_1, \dots, z_{\frac{T}{S}}] \in \mathbb{R}^{D \times \frac{T}{S}}$ , we leverage eDMD [45] to find the best fitted matrix that advances forward the system. We apply one-step operator approximation as follows:

$$Z_{\text{back}} = [z_1, z_2, \dots, z_{\frac{T}{S}-1}], Z_{\text{fore}} = [z_2, z_3, \dots, z_{\frac{T}{S}}], K_{\text{var}} = Z_{\text{fore}} Z_{\text{back}}^\dagger, \quad (10)$$

where  $Z_{\text{back}}^\dagger \in \mathbb{R}^{(\frac{T}{S}-1) \times D}$  is the Moore–Penrose inverse of lookback window embedding collection. The calculated  $K_{\text{var}} \in \mathbb{R}^{D \times D}$  varies with windows and helps to analyze local temporal variations as a linear system. With the calculated operator, the fitted embedding is formulated as follows:

$$[\hat{z}_1, \hat{z}_2, \dots, \hat{z}_{\frac{T}{S}}] = [z_1, K_{\text{var}} z_1, \dots, K_{\text{var}} z_{\frac{T}{S}-1}] = [z_1, K_{\text{var}} Z_{\text{back}}]. \quad (11)$$

To obtain a prediction of length  $H$ , we iterate operator forwarding to get  $\frac{H}{S}$  predicted embedding:

$$\hat{z}_{\frac{T}{S}+t} = (K_{\text{var}})^t z_{\frac{T}{S}}, t = 1, 2, \dots, H/S. \quad (12)$$

Finally, we arrange the segments transformed by Decoder( $\cdot$ ) as the module outputs  $\hat{X}_{\text{var}}, Y_{\text{var}}$ . The whole procedure is shown in Figure 3 and TimeVarKP( $\cdot$ ) can be formulated as Equation 8–13.

$$\hat{X}_{\text{var}} = [\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_{\frac{T}{S}}]^\top, Y_{\text{var}} = [\hat{\mathbf{x}}_{\frac{T}{S}+1}, \dots, \hat{\mathbf{x}}_{\frac{T}{S}+\frac{H}{S}}]^\top. \quad (13)$$

**Forecasting Objective** In Koopa, Encoder, Decoder and  $K_{\text{inv}}$  are learnable parameters, while  $K_{\text{var}}$  is calculated on-the-fly. To maintain the Koopman embedding consistency in different blocks, we share Encoder, Decoder in Time-variant and Time-invariant KPs, which are formulated as  $\phi_{\text{var}}$  and  $\phi_{\text{inv}}$  respectively, and use the MSE loss with the ground truth  $Y_{\text{gt}}$  for parameter optimization:

$$\text{argmin}_{K_{\text{inv}}, \phi_{\text{var}}, \phi_{\text{inv}}} \mathcal{L}_{\text{MSE}}(Y, Y_{\text{gt}}). \quad (14)$$

Optimizing by a single forecasting objective based on the assumption that if reconstruction failed, the prediction must also fail. Thus eliminating forecast discrepancy helps for fitting observed dynamics.

## 5 Experiments

**Datasets** We conduct extensive experiments to evaluate the performance and efficiency of Koopa. For multivariate forecasting, we include six real-world benchmarks used in Autoformer [48]: ECL (UCI), ETT [53], Exchange [22], ILI (CDC), Traffic (PeMS), and Weather (Wetterstation). For univariate forecasting, we evaluate the performance on the well-acknowledged M4 dataset [39], which contains four subsets of periodically collected univariate marketing data. And we follow the data processing and split ratio used in TimesNet [47].

Notably, instead of setting a fixed lookback window length, for every forecast window length  $H$ , we set the length of lookback window  $T = 2H$  as the same with N-BEATS [32], because historical observations are always available in real-world scenarios and it can be beneficial for deep models to leverage more observed data with the increasing forecast horizon.

**Baselines** We extensively compare Koopa with the state-of-the-art deep forecasting models, including Transformer-based model: Autoformer [48], PatchTST [31]; TCN-based model: TimesNet [47], MICN [43]; MLP-based model: DLinear [51]; Fourier forecaster: FiLM [54], and Koopman forecaster: KNF [44]. We also introduce additional specialized models N-HiTS [7] and N-BEATS [32] for univariate forecasting as competitive baselines. All the baselines we reproduced are implemented based on the original paper or official code. We repeat each experiment three times with different random seeds and report the test MSE/MAE. And we provide detailed code implementation and hyperparameters sensitivity in Appendix C.

## 5.1 Time Series Forecasting

**Forecasting results** We list the results in Table 1–2 with the best in **bold** and the second underlined. Koopa shows competitive forecasting performance in both multivariate and univariate forecasting. Concretely, Koopa achieves state-of-the-art performance in more than **70%** multivariate settings and consistently outperforms other deep models in the univariate settings.

Notably, Koopa surpasses the state-of-the-art Koopman-based forecaster KNF by a large margin in real-world time series, which can be attributed to our hierarchical dynamics learning and disentangling mechanism. Also, as the representative of efficient linear models, the performance of DLinear is still subpar in ILI, Traffic and Weather, indicating that nonlinear dynamics underlying the time series poses challenges for model capacity and point-wise weighting may not be appropriate to portray time-variant dynamics. Besides, compared with painstakingly trained PatchTST with channel-independence mechanism, our model can achieve a close and even better performance with naturally addressed non-stationary properties of real-world time series.

Table 1: Multivariate forecasting results with different forecast lengths  $H \in \{24, 36, 48, 60\}$  for ILI and  $H \in \{48, 96, 144, 192\}$  for others. We set the lookback length  $T = 2H$ . Additional results (ETTm1, ETTm2, ETTth1) are provided in Appendix D.1.

Models	<b>Koopa</b>		PatchTST		TimesNet		DLinear		MICN		KNF		FiLM		Autoformer	
Metric	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
ECL	48	<b>0.130</b> <b>0.234</b>	<u>0.147</u>	0.246	0.149	0.254	0.158	<u>0.241</u>	0.156	0.271	0.175	0.265	0.197	0.270	0.164	0.272
	96	<b>0.136</b> <b>0.236</b>	<u>0.143</u>	<u>0.241</u>	0.170	0.275	0.153	0.245	0.165	0.277	0.198	0.284	0.238	0.341	0.182	0.289
	144	<u>0.149</u>	0.247	<b>0.145</b> <b>0.241</b>	0.183	0.287	0.152	<u>0.245</u>	0.163	0.274	0.204	0.297	0.234	0.338	0.210	0.315
	192	<u>0.156</u>	0.254	<b>0.147</b> <b>0.240</b>	0.189	0.291	<u>0.153</u>	<u>0.246</u>	0.171	0.284	0.245	0.321	0.240	0.339	0.221	0.324
ETTh2	48	<u>0.226</u>	<u>0.300</u>	<b>0.223</b> <b>0.297</b>	0.241	0.319	<u>0.226</u>	0.305	0.260	0.336	0.385	0.376	0.261	0.324	0.355	0.380
	96	<u>0.297</u>	<b>0.349</b>	0.300	0.353	0.325	0.376	<b>0.294</b>	<u>0.351</u>	0.343	0.393	0.433	0.446	0.322	0.372	0.427
	144	<b>0.333</b> <b>0.381</b>	<u>0.346</u>	<u>0.390</u>	0.374	0.408	0.354	0.397	0.374	0.411	0.441	0.456	0.352	0.397	0.457	0.461
	192	<b>0.356</b> <b>0.393</b>	<u>0.383</u>	<u>0.406</u>	0.394	0.434	0.385	0.418	0.455	0.464	0.528	0.503	0.361	0.410	0.503	0.491
Exchange	48	<b>0.042</b> <b>0.143</b>	0.044	<u>0.144</u>	0.059	0.172	<u>0.043</u>	0.145	0.117	0.248	0.128	0.271	0.071	0.192	0.125	0.252
	96	<b>0.083</b> <u>0.207</u>	0.085	<b>0.204</b>	0.120	0.255	<u>0.084</u>	0.220	0.108	0.251	0.294	0.394	0.112	0.245	0.280	0.386
	144	<b>0.130</b>	0.261	<u>0.132</u>	<u>0.260</u>	0.206	0.334	<u>0.132</u>	<b>0.253</b>	0.152	0.301	0.597	0.578	0.174	0.306	0.520
	192	0.184	0.309	<b>0.174</b>	<u>0.300</u>	0.377	0.463	<u>0.178</u>	<b>0.299</b>	0.187	0.331	0.654	0.595	0.241	0.364	0.653
ILI	24	<b>1.621</b> <b>0.800</b>	<u>2.063</u>	<u>0.881</u>	2.464	1.039	2.624	1.118	4.380	1.558	3.722	1.432	3.590	1.424	2.831	1.085
	36	<b>1.803</b> <b>0.855</b>	<u>2.178</u>	<u>0.943</u>	2.388	1.007	2.693	1.156	3.314	1.313	3.941	1.448	4.200	1.383	2.801	1.088
	48	<b>1.768</b> <u>0.903</u>	<u>1.916</u>	<b>0.896</b>	2.370	1.040	2.852	1.229	2.457	1.085	3.287	1.377	3.317	1.417	2.322	1.006
	60	<b>1.743</b> <b>0.891</b>	<u>1.981</u>	<u>0.917</u>	2.193	1.003	2.554	1.144	2.379	1.040	2.974	1.301	4.077	1.444	2.470	1.061
Traffic	48	<b>0.415</b> <b>0.274</b>	<u>0.426</u>	<u>0.286</u>	0.567	0.306	0.488	0.352	0.496	0.301	0.621	0.382	0.498	0.312	0.640	0.361
	96	<b>0.401</b> <b>0.275</b>	<u>0.413</u>	<u>0.283</u>	0.611	0.337	0.485	0.336	0.511	0.312	0.645	0.376	0.451	0.297	0.668	0.367
	144	<b>0.397</b> <b>0.276</b>	<u>0.405</u>	<u>0.278</u>	0.603	0.322	0.452	0.317	0.498	0.309	0.683	0.402	0.430	0.288	0.681	0.379
	192	<b>0.403</b>	<u>0.284</u>	<u>0.404</u>	<b>0.277</b>	0.604	0.321	0.438	0.309	0.494	0.312	0.699	0.405	0.425	0.288	0.692
Weather	48	<b>0.126</b> <b>0.168</b>	0.140	<u>0.179</u>	<u>0.138</u>	0.191	0.156	0.198	0.157	0.217	0.201	0.288	0.160	0.206	0.185	0.240
	96	<b>0.154</b> <b>0.205</b>	<u>0.160</u>	<u>0.206</u>	0.180	0.231	0.186	0.229	0.187	0.250	0.295	0.308	0.189	0.233	0.230	0.279
	144	<b>0.172</b> <u>0.225</u>	<u>0.174</u>	<b>0.221</b>	0.190	0.244	0.199	0.244	0.197	0.257	0.394	0.401	0.200	0.245	0.268	0.308
	192	<b>0.193</b> <b>0.241</b>	<u>0.195</u>	<u>0.243</u>	0.212	0.265	0.217	0.261	0.214	0.270	0.462	0.437	0.219	0.263	0.325	0.347
1 <sup>st</sup> Count	<b>34</b>		<u>11</u>		0		3		0		0		0		0	

Table 2: Univariate forecasting results for the M4 dataset. We report the weighted average forecasting error from all four subsets and full results are provided in Appendix D.1.

Models		Koopa	N-HITS	N-BEATS	PatchTST	TimesNet	DLinear	MICN	KNF	FiLM	Autoformer
Weighted Average	sMAPE	<b>11.863</b>	11.960	<u>11.910</u>	13.022	11.930	12.418	13.023	12.126	12.489	14.057
	MASE	<b>1.595</b>	1.606	1.613	1.814	<u>1.597</u>	1.656	1.836	1.641	1.690	1.954
	OWA	<b>0.858</b>	<u>0.861</u>	0.862	0.954	<u>0.867</u>	0.891	0.960	0.874	0.902	1.029

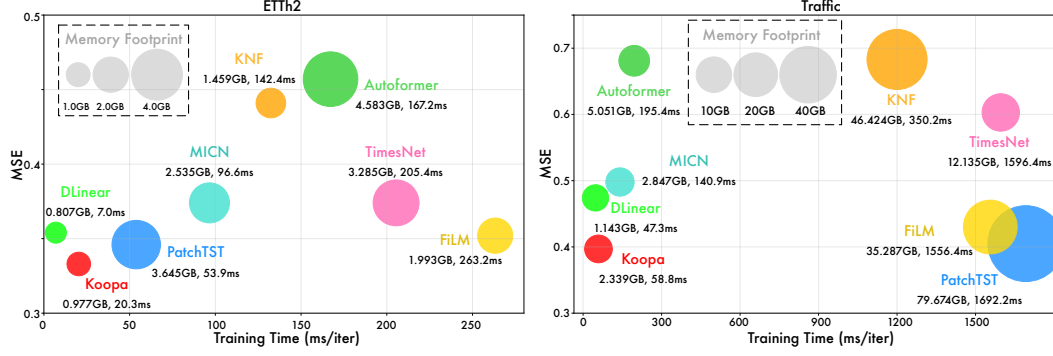


Figure 4: Model efficiency comparison. The performance comes from Table 1 with forecast window length  $H = 144$ . Training time and memory footprint are recorded with the same batch size and official code configuration. Full results of all six datasets are provided in Appendix D.3.

**Model efficiency** We comprehensively evaluate the model efficiency from three aspects: forecasting performance, training speed, and memory footprint. In Figure 4, we compare the efficiency under two representative datasets with different variate numbers (7 in ETTh2 and 862 in Traffic).

Compared with the state-of-the-art forecasting model PatchTST, Koopa saves **62.3%** and **96.5%** training time respectively in the ETTh2 and Traffic datasets with only **26.8%** and **2.9%** memory footprint. Concretely, the averaged training time and memory ratio of Koopa compared to PatchTST are **22.7%** and **24.0%** in all six datasets (see Appendix D.3 for the detail). Besides, as an efficient MLP-based forecaster, Koopa is also capable of learning nonlinear dynamics from time-variant and time-invariant components, and thus achieves a better performance.

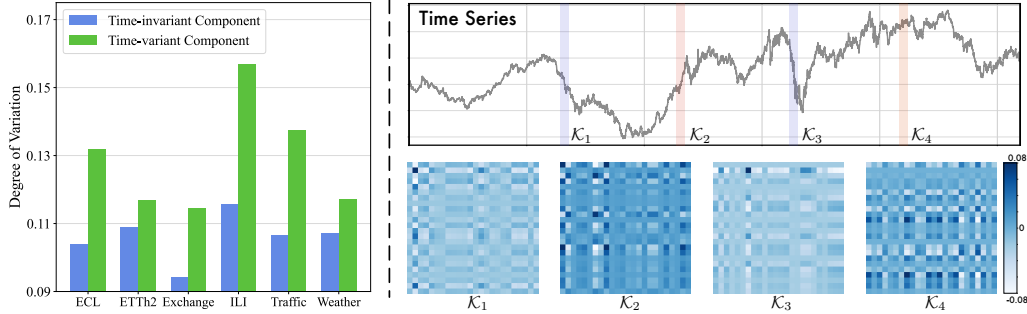


Figure 5: Left: Comparison of Degree of Variation (the standard deviation of linear weighting fitted on different periods), we plot respective values of disengaged components on all six datasets. Right: A case of localized Koopman operators calculated on the Exchange dataset at the interval of one year.

## 5.2 Model Analysis

**Dynamics disentanglement** To validate the disentangling effect of our proposed Fourier Filter, we divide the whole time series into 20 subsets of different periods and conduct respective linear regression on the components disentangled by Fourier Filter. The standard deviation of the linear weighting reflects the variation of point-to-point temporal dependencies, which works as the manifestation of time-variant property. We plot the value as *Degree of Variation* (Figure 5 Left). It can be observed that larger deviations occur in the time-variant component, which indicates the proposed module successfully disentangles two types of dynamics from the perspective of frequency domain.

**Case study** We present a case study on real-world time series (exchange rate) on the right of Figure 5. We sample the lookback window at the interval of one year and visualize the Koopman operators calculated in Time-variant KP. It can be clearly observed that localized operators can exhibit changing temporal patterns in different periods, indicating the necessity of utilizing varying operators to describe time-variant dynamics. And interpretable insights are also presented as series uptrends correspond to heatmaps with large value and downtrends are reflected with small value.



**Ablation study** We conduct ablations on Koopa structure: *Only  $K_{inv}$*  uses one-block Time-invariant KP; *Only  $K_{var}$*  stacks Time-variant KPs only; *Truncated Filter* replaces Fourier Filter with High-Low Pass Filter; *Branch Switch* changes the order of KPs to deal with disentangled components. As shown in Table 3, Koopa conducts effective disentanglement and tackles the right dynamics with complementary KPs, and thus achieves the best forecasting performance.

Table 3: Model ablation. We list the averaged MSE/MAE of different forecast lengths.

Dataset	ECL		ETTh2		Exchange		ILI		Traffic		Weather	
Metric	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
Only $K_{inv}$	0.148	0.250	0.312	0.358	0.120	0.241	2.146	0.963	0.740	0.446	0.170	0.213
Only $K_{var}$	1.547	0.782	0.371	0.405	0.205	0.316	2.370	1.006	0.947	0.544	0.180	0.232
Truncated Filter	0.155	0.255	0.311	0.362	0.129	0.246	1.988	0.907	0.536	0.334	0.172	0.220
Branch Switch	0.696	0.393	0.344	0.385	0.231	0.325	2.130	0.964	0.451	0.304	0.173	0.221
<b>Koopa</b>	<b>0.146</b>	<b>0.246</b>	<b>0.303</b>	<b>0.356</b>	<b>0.111</b>	<b>0.230</b>	<b>1.734</b>	<b>0.862</b>	<b>0.419</b>	<b>0.293</b>	<b>0.162</b>	<b>0.211</b>

### 5.3 Scaling Up Forecast Horizon

Most deep forecasting models work as a settled function once trained (e.g. input- $T$ -output- $H$ ). For scenarios where the prediction horizon is mismatched or long-term, it poses two challenges for the trained model: (1) reuse parameters learned from observed series; (2) utilize incoming ground truth for model adaptation. The practical scenarios, which we name as *scaling up forecast horizon*, may lead to failure on most deep models but can be naturally tackled by Koopa. In detail, we first train Koopa with forecast length  $H_{tr}$  and attempt to apply it on a larger forecast length  $H_{te}$ .

**Method** Koopa scales up forecast horizon as follows: Since Time-invariant KP has learned the globally shared dynamics and Time-variant KP can calculate localized operator  $K_{var}$  within the lookback window, we freeze the parameters of trained Koopa but only use the incoming ground truth to adapt  $K_{var}$ . The naïve implementation uses incremental Koopman embedding with dimension  $D$  and conducts Equation 10 to obtain an updated operator, which has a complexity of  $\mathcal{O}(H_{te}D^3)$ . We further propose an iterative algorithm with improved  $\mathcal{O}((H_{te} + D)D^2)$  complexity. The detailed method implementations and complexity analysis can be found in Appendix A.

**Results** As shown in Table 4, the proposed operator adaption mechanism further boosts the performance on the scaling up scenario, which can be attributed to more accurately fitted time-variant dynamics with incoming ground truth snapshots. Besides, the promotion becomes more significant when applied to non-stationary datasets (manifested as large ADF Test Statistic [10]).

Table 4: Scaling up forecast horizon:  $(H_{tr}, H_{te}) = (24, 48)$  for ILI and  $(H_{tr}, H_{te}) = (48, 144)$  for others. *Koopa* conducts vanilla rolling forecast and *Koopa OA* further introduces operator adaptation.

Dataset	Exchange		ETTh2		ILI		ECL		Traffic		Weather	
ADF Test Statistic	(-1.889)		(-4.135)		(-5.406)		(-8.483)		(-15.046)		(-26.661)	
Metric	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
Koopa	0.214	0.348	0.437	0.429	2.836	1.065	0.199	0.298	0.709	0.437	0.237	0.276
<b>Koopa OA</b>	<b>0.172</b>	<b>0.319</b>	<b>0.372</b>	<b>0.404</b>	<b>2.427</b>	<b>0.907</b>	<b>0.182</b>	<b>0.271</b>	<b>0.699</b>	<b>0.426</b>	<b>0.225</b>	<b>0.264</b>
Promotion (MSE)	19.6%		14.9%		14.1%		8.5%		1.4%		5.1%	

## 6 Conclusion

This paper tackles time series as dynamical systems. With disentangled time-variant and time-invariant components from non-stationary series, the Koopa model reveals the complicated dynamics hierarchically and leverages MLP modules to learn Koopman embedding and operator. Experimentally, our model shows competitive performance with remarkable efficiency and the potential to scale up the forecast length by operator adaptation. In the future, we will explore Koopa with the dynamic modes underlying non-stationary data using the toolbox of Koopman spectral analysis.

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## A Scaling Up Forecast Horizon

In this section, we introduce the capability of Koopa to scale up forecast horizon. In detail, we train a Koopa model with forecast length  $H_{tr}$  and attempt to apply it on a larger length  $H_{te}$ . The basic approach conducts rolling forecast by taking the model prediction as the input of the next iteration until the desired forecast horizon is all filled. Instead, we further assume that after the model gives a prediction, the model can utilize the incoming ground truth for *model adaptation* and continue rolling forecast for the next iteration. It is notable that we do not retrain parameters during model adaptation, since it will lead to overfitting on the incoming ground truth and Catastrophic Forgetting [11, 18, 35].

Koopa can naturally cope with the scenario by learning Koopman embedding and operator  $K_{inv}$  in Time-invariant KPs while calculating localized operator  $K_{var}$  to describe the dynamics in the temporal neighborhood. Therefore, we freeze the parameters of Koopa but only use the incoming ground truth for operator adaptation of  $K_{var}$  in Time-variant KPs.

### A.1 Implementation of Operator Adaptation

At the beginning of rolling forecast, the Encoder in Time-variant KP outputs  $D$ -dimensional Koopman embedding for each observed series segment as  $[z_1, z_2, \dots, z_F]$ , where  $F = \frac{H_{tr}}{S}$  is the segment number with  $S$  as the segment length. The operator  $K_{var}$  in Time-variant KP is calculated as follows:

$$Z_{back} = [z_1, z_2, \dots, z_{F-1}], Z_{fore} = [z_2, z_3, \dots, z_F], K_{var} = Z_{fore} Z_{back}^\dagger, \quad (15)$$

where  $Z_{back}, Z_{fore} \in \mathbb{R}^{D \times (F-1)}$ ,  $K_{var} \in \mathbb{R}^{D \times D}$ . With the calculated operator, we obtain the next predicted Koopman embedding by one-step forwarding:

$$\hat{z}_{F+1} = K_{var} z_F. \quad (16)$$

After decoding the embedding  $\hat{z}_{F+1}$  to the series prediction, we can utilize the true value of incoming Koopman embedding  $z_{F+1}$  obtained by Koopa with frozen parameters. Instead of using  $K_{var}$  to obtain the next embedding

$\hat{z}_{F+2}$ , we use incremental embedding collections  $Z_{\text{back}+}, Z_{\text{fore}+} \in \mathbb{R}^{D \times F}$  to obtain a more accurate operator  $K_{\text{var}+} \in \mathbb{R}^{D \times D}$  to describe the local dynamics:

$$Z_{\text{back}+} = [Z_{\text{back}}, z_F], Z_{\text{fore}+} = [Z_{\text{back}}, z_{F+1}], K_{\text{var}+} = Z_{\text{fore}+} Z_{\text{back}+}^\dagger. \quad (17)$$

The procedure repeats for  $L$  times ( $L \propto H_{\text{te}}$ ) until the forecast horizon is all filled, we formulate it as Algorithm 1. And experimental results (KoopA OA) in Section 5.3 have demonstrated the promotion of forecasting performance due to more precisely fitted dynamics.

## A.2 Computational Acceleration

The naïve implementation shown in Algorithm 1 repeatedly conducts Equation 17 on the incremental embedding collection to obtain new operators, which has a complexity of  $\mathcal{O}(LD^3)$ . We propose an equivalent algorithm with improved complexity of  $\mathcal{O}((L+D)D^2)$  as shown in Algorithm 2.

**Theorem.** *Algorithm 2 gives the same  $K_{\text{var}}$  as Algorithm 1 in each iteration with  $\mathcal{O}(D^2)$  complexity.*

**Proof.** We start with the first iteration analysis. By the definition of Moore–Penrose inverse, we have  $Z_{\text{back}}^\dagger Z_{\text{back}} = I_{F-1}$ , where  $I_{F-1}$  is an identity matrix with the dimension of  $F-1$ . When the model receives the incoming embedding  $z_{F+1}$ , incremental embedding  $m = z_F, n = z_{F+1}$  will be appended to  $Z_{\text{back}}$  and  $Z_{\text{fore}}$  respectively. Instead of calculating new  $K_{\text{var}+}$  from incremental collections, we utilize calculated  $K_{\text{var}}$  to find the iteration rule on  $K_{\text{var}+}$ . Concretely, we suppose

$$Z_{\text{back}+}^\dagger = \begin{bmatrix} Z_{\text{back}}^\dagger & \Delta \\ b^\top & \end{bmatrix} \in \mathbb{R}^{F \times D}, \quad (18)$$

where  $\Delta \in \mathbb{R}^{(F-1) \times D}, b \in \mathbb{R}^D$  are variables to be identified. By the definition of Moore–Penrose inverse, we have  $Z_{\text{back}+}^\dagger Z_{\text{back}+} = I_F$ . By unfolding it, we have the following equations:

$$\Delta Z_{\text{back}} = \mathbf{0}, b^\top Z_{\text{back}} = \vec{0}, b^\top m = 1, Z_{\text{back}}^\dagger m - \Delta m = \vec{0}. \quad (19)$$

We suppose  $\Delta = \delta b^\top$ , where  $\delta \in \mathbb{R}^{F-1}$ , such that when  $b^\top Z_{\text{back}} = \vec{0}$ , then  $\Delta Z_{\text{back}} = \mathbf{0}$ . Then we have  $Z_{\text{back}}^\dagger m - \delta b^\top m = Z_{\text{back}}^\dagger m - \delta = \vec{0}$ , thus  $\Delta = Z_{\text{back}}^\dagger m b^\top$ . Given equations that  $b^\top Z_{\text{back}} = \vec{0}$  and  $b^\top m = 1$ , we have the analytical solution of  $b$ :

$$b = r / \|r\|^2, \text{ where } r = m - Z_{\text{back}} Z_{\text{back}}^\dagger m. \quad (20)$$

Therefore, we find the equation between the incremental version  $K_{\text{var}+}$  and calculated  $K_{\text{var}}$ :

$$Z_{\text{back}+}^\dagger = \begin{bmatrix} Z_{\text{back}}^\dagger (I_D - m b^\top) \\ b^\top \end{bmatrix}, K_{\text{var}+} = Z_{\text{fore}+} Z_{\text{back}+}^\dagger = K_{\text{var}} + (n - K_{\text{var}} m) b^\top, \quad (21)$$

where  $m, n$  are the incremental embedding of  $Z_{\text{back}}, Z_{\text{fore}}$  and  $b$  can be calculated by Equation 20. We also derive the iteration rule on  $X = Z_{\text{back}} Z_{\text{back}}^\dagger$  to obtain  $b$ , which is formulated as follows:

$$X_+ = Z_{\text{back}+} Z_{\text{back}+}^\dagger = X + (m - X m) b^\top = X + r b^\top. \quad (22)$$

By adopting Equation 21–22 and permuting the matrix multiplication order, we reduce the complexity of each iteration to  $\mathcal{O}(D^2)$ . Therefore, Algorithm 2 has a overall complexity of  $\mathcal{O}((L+D)D^2)$ . Since  $L \propto H_{\text{te}}$ , Algorithm 1–2 have  $\mathcal{O}(H_{\text{te}}D^3)$  and  $\mathcal{O}((H_{\text{te}}+D)D^2)$  complexity respectively.

## B Implementation Details

KoopA is trained with L2 loss and optimized by ADAM [17] with an initial learning rate of 0.001 and batch size set to 32. The training process is early stopped within 10 epochs. We repeat each experiment three times with different random seeds to obtain average test MSE/MAE and detailed results with standard deviations are listed in Table 5. All experiments are implemented in PyTorch [34] and conducted on NVIDIA TITAN RTX 24GB GPUs.

All the baselines that we reproduced are implemented based on the benchmark of TimesNet [47] Repository, which is fairly built on the configurations provided by each model’s original paper or official code. Since several baselines adopt Series Stationarization from Non-stationary Transformers [28] while others do not, we equip all models with the method for a fair comparison.

## C Hyperparameter Sensitivity

We verify the robustness of Koopa with respect to hyperparameters as follows: the dimension of Koopman embedding  $D$ , the hidden layer number  $l$  and the hidden dimension  $d$  used in Encoder and Decoder. Considering

**Algorithm 1** Koopa Operator Adaptation.**Require:** Observed embedding  $Z = [z_1, \dots, z_F]$  and successively incoming ground truth embedding

$[z_{F+1}, \dots, z_{F+L}]$  with each embedding  $z_i \in \mathbb{R}^D$ .

- 1:  $Z_{\text{back}} = [z_1, \dots, z_{F-1}], Z_{\text{fore}} = [z_2, \dots, z_F]$   $\triangleright Z_{\text{back}}, Z_{\text{fore}} \in \mathbb{R}^{D \times (F-1)}$
- 2:  $K_{\text{var}} = Z_{\text{fore}} Z_{\text{back}}^\dagger$   $\triangleright K_{\text{var}} \in \mathbb{R}^{D \times D}$
- 3:  $\hat{z}_{F+1} = K_{\text{var}} n$   $\triangleright \hat{z}_{F+1} \in \mathbb{R}^D$
- 4: **for**  $l$  **in**  $\{1, \dots, L\}$ :  $\triangleright z_{F+l}$  comes successively
- 5:      $m = z_{F+l-1}, n = z_{F+l}$   $\triangleright m, n \in \mathbb{R}^D$
- 6:      $Z_{\text{back}} \leftarrow [Z_{\text{back}}, m], Z_{\text{fore}} \leftarrow [Z_{\text{fore}}, n]$   $\triangleright Z_{\text{back}}, Z_{\text{fore}} \in \mathbb{R}^{D \times (F+l-1)}$
- 7:      $K_{\text{var}} = Z_{\text{fore}} Z_{\text{back}}^\dagger$   $\triangleright K_{\text{var}} \in \mathbb{R}^{D \times D}$
- 8:      $\hat{z}_{F+l+1} = K_{\text{var}} n$   $\triangleright \hat{z}_{F+l+1} \in \mathbb{R}^D$
- 9: **End for**
- 10: **Return**  $[\hat{z}_{F+1}, \dots, \hat{z}_{F+L+1}]$   $\triangleright$  Return predicted embedding

**Algorithm 2** Accelerated Koopa Operator Adaptation.**Require:** Observed embedding  $Z = [z_1, \dots, z_F]$  and successively incoming ground truth embedding

$[z_{F+1}, \dots, z_{F+L}]$  with each embedding  $z_i \in \mathbb{R}^D$ .

- 1:  $Z_{\text{back}} = [z_1, \dots, z_{F-1}], Z_{\text{fore}} = [z_2, \dots, z_F]$   $\triangleright Z_{\text{back}}, Z_{\text{fore}} \in \mathbb{R}^{D \times (F-1)}$
- 2:  $K_{\text{var}} = Z_{\text{fore}} Z_{\text{back}}^\dagger, X = Z_{\text{back}} Z_{\text{back}}^\dagger$   $\triangleright K_{\text{var}}, X \in \mathbb{R}^{D \times D}$
- 3:  $\hat{z}_{F+1} = K_{\text{var}} n$   $\triangleright \hat{z}_{F+1} \in \mathbb{R}^D$
- 4: **for**  $l$  **in**  $\{1, \dots, L\}$ :  $\triangleright z_{F+l}$  comes successively
- 5:      $m = z_{F+l-1}, n = z_{F+l}$   $\triangleright m, n \in \mathbb{R}^D$
- 6:      $r = m - X m$   $\triangleright r \in \mathbb{R}^D$
- 7:      $b = r / \|r\|^2$   $\triangleright b \in \mathbb{R}^D$
- 8:      $K_{\text{var}} \leftarrow K_{\text{var}} + (n - K_{\text{var}} m) b^\top$   $\triangleright K_{\text{var}} \in \mathbb{R}^{D \times D}$
- 9:      $X \leftarrow X + r b^\top$   $\triangleright X \in \mathbb{R}^{D \times D}$
- 10:      $\hat{z}_{F+l+1} = K_{\text{var}} n$   $\triangleright \hat{z}_{F+l+1} \in \mathbb{R}^D$
- 11: **End for**
- 12: **Return**  $[\hat{z}_{F+1}, \dots, \hat{z}_{F+L+1}]^\top$   $\triangleright$  Return predicted embedding

Table 5: Detailed performance of Koopa. We report the MSE/MAE and standard deviation of different forecast horizons  $\{H_1, H_2, H_3, H_4\} = \{24, 36, 48, 60\}$  for ILI and  $\{48, 96, 144, 192\}$  for others.

Dataset	ECL		ETTh2		Exchange	
Horizon	MSE	MAE	MSE	MAE	MSE	MAE
$H_1$	0.130±0.003	0.234±0.003	0.226±0.003	0.300±0.003	0.042±0.002	0.143±0.003
$H_2$	0.136±0.004	0.236±0.005	0.297±0.004	0.349±0.004	0.083±0.004	0.207±0.004
$H_3$	0.149±0.003	0.247±0.003	0.333±0.004	0.381±0.003	0.130±0.005	0.261±0.003
$H_4$	0.156±0.004	0.254±0.003	0.356±0.005	0.393±0.004	0.184±0.009	0.309±0.005

Dataset	ILI		Traffic		Weather	
Horizon	MSE	MAE	MSE	MAE	MSE	MAE
$H_1$	1.621±0.008	0.800±0.006	0.415±0.003	0.274±0.005	0.126±0.005	0.168±0.004
$H_2$	1.803±0.040	0.855±0.020	0.401±0.005	0.275±0.004	0.154±0.006	0.205±0.003
$H_3$	1.768±0.015	0.903±0.008	0.397±0.004	0.276±0.003	0.172±0.005	0.225±0.005
$H_4$	1.743±0.040	0.891±0.009	0.403±0.007	0.284±0.009	0.193±0.003	0.241±0.004

the efficiency of hyperparameters search, we fix the segment length  $S = T/2$  and the number of Koopa blocks  $B = 3$ . As the detailed hyperparameter sensitivity analysis is shown in Figure 6, we find the proposed model is insensitive to the choices of above hyperparameters, which can be beneficial for practitioners to reduce hyperparameters search burden in real-world applications.

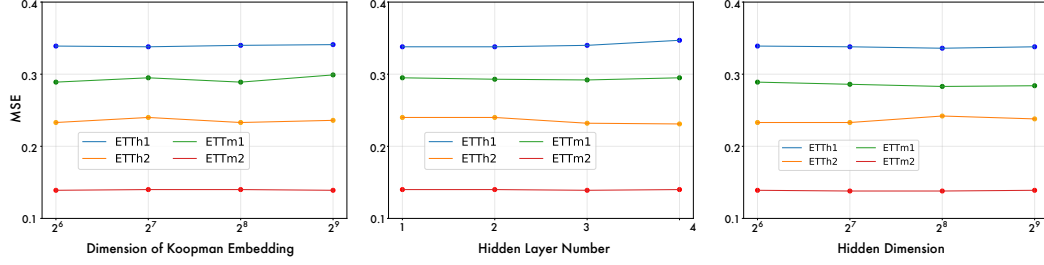


Figure 6: Hyperparameter sensitivity with respect to the dimension of Koopman embedding, hidden layer number, and hidden dimension of Encoder and Decoder in Koopa. The results are recorded with forecast window length  $H = 48$  in ETT datasets.

## D Supplementary Experimental Results

### D.1 Full Forecasting Results

Due to the limited pages, we list additional multivariate benchmarks on ETT datasets [53] in Table 6, which includes the hourly recorded ETTh2 and 15-minutely recorded ETTm1/ETTh2, and the full univariate results of M4 [39] in Table 7, which contains the yearly, quarterly and monthly collected univariate marketing data. Notably, Koopa still achieves competitive performance compared with state-of-the-art deep forecasting models and specialized univariate models.

Table 6: Forecasting results with different forecast window lengths  $H \in \{48, 96, 144, 192\}$  on ETT dataset. We set the lookback window length  $T = 2H$ .

Models	KooPA	PatchTST [31]	TimesNet [47]	DLinear [51]	MICN [43]	KNF [44]	FiLM [54]	Autoformer [48]
Metric	MSE MAE	MSE MAE	MSE MAE	MSE MAE	MSE MAE	MSE MAE	MSE MAE	MSE MAE
ETTh1	48  <b>0.283</b> <b>0.333</b>	0.286 0.336	0.308 0.354	0.322 0.355	0.294 0.353	1.026 0.792	0.324 0.353	0.592 0.419
	96  <b>0.294</b> <b>0.345</b>	0.299 0.346	0.329 0.370	0.309 0.346	0.306 0.364	0.957 0.782	0.311 0.346	0.493 0.469
	144  <b>0.322</b> 0.366	0.325 0.363	0.358 0.387	0.327 <b>0.359</b>	0.342 0.390	0.921 0.760	0.328 0.358	0.735 0.569
	192  <b>0.337</b> 0.378	0.343 0.375	0.462 0.441	<b>0.337</b> <b>0.365</b>	0.386 0.415	0.896 0.731	0.339 0.366	0.592 0.506
ETTh2	48 0.134 <b>0.226</b>	0.135 0.231	0.142 0.234	0.144 0.240	<b>0.131</b> 0.238	0.621 0.623	0.146 0.243	0.191 0.280
	96  <b>0.171</b> <b>0.254</b> <b>0.171</b>	0.255 0.187	0.269 0.172	0.256 0.197	0.295 1.535	1.012 0.174	0.257 0.241	0.311 0.311
	144 0.206 0.280	0.205 0.282	0.216 0.291	<b>0.200</b> <b>0.276</b>	0.210 0.297	1.337 0.876	0.204 0.279	0.300 0.352
	192 0.226 0.298	0.221 0.294	0.243 0.313	<b>0.219</b> <b>0.290</b>	0.248 0.328	1.355 0.908	0.224 0.293	0.324 0.370
ETTh1	48  <b>0.336</b> 0.377	0.337 0.375	0.365 0.399	0.343 <b>0.371</b>	0.375 0.406	0.876 0.709	0.407 0.427	0.442 0.438
	96  <b>0.371</b> 0.405	0.372 <b>0.393</b>	0.411 0.430	0.379 <b>0.393</b>	0.406 0.429	0.975 0.744	0.429 0.431	0.634 0.523
	144 0.405 0.418	0.394 0.412	0.442 0.447	<b>0.393</b> <b>0.403</b>	0.437 0.448	0.801 0.662	0.451 0.448	0.522 0.491
	192 0.416 0.429	0.416 0.439	0.469 0.470	<b>0.407</b> <b>0.416</b>	0.518 0.496	0.941 0.744	0.460 0.459	0.525 0.501

### D.2 Full Ablation Results

We elaborately conduct model ablations in Table 8 to verify the effect of our proposed modules: Time-invariant KP, Time-variant KP, Fourier Filter and the choices to tackle dynamics.

**Dynamics underlying time series** As shown in Table 8, Time-variant and Time-invariant KPs perform as complementary modules for exploring the dynamics underlying time series, discarding any one of them (Only  $K_{inv}$  and Only  $K_{var}$ ) will lead to the inferior forecasting performance. It is also a surprising finding that only utilizing Time-invariant KP surpasses only utilizing Time-variant KP in more cases (ECL, ETTh2, Exchange, Traffic), indicating the time-variant dynamics plays a dominant role in these time series datasets and it also emphasizes the significance to first establish the time-invariant dynamics and then assisted it with adapted local time-variant dynamics.

Table 7: Full univariate forecasting results for M4 dataset. We follow the same data processing and forecasting length settings used in TimesNet [47] benchmark. Additional forecasting models N-HiTS [7] and N-BEATS [32] are also included.

Models		KooPA	N-HiTS	N-BEATS	PatchTST	TimesNet	DLinear	MICN	KNF	FiLM	Autoformer
Year	sMAPE	<b>13.352</b>	13.371	13.466	13.517	13.394	13.866	14.532	13.986	14.012	14.786
	MASE	<b>2.997</b>	3.025	3.059	3.031	3.004	3.006	3.359	3.029	3.071	3.349
	OWA	<b>0.786</b>	0.790	0.797	0.795	0.787	0.802	0.867	0.804	0.815	0.874
Quarter	sMAPE	10.159	10.454	<b>10.074</b>	10.847	10.101	10.689	11.395	10.343	10.758	12.125
	MASE	1.189	1.219	<b>1.163</b>	1.315	1.183	1.294	1.379	1.202	1.306	1.483
	OWA	0.895	0.919	<b>0.881</b>	0.972	0.890	0.957	1.020	0.965	0.905	1.091
Month	sMAPE	<b>12.730</b>	12.794	12.801	14.584	12.866	13.372	13.829	12.894	13.377	15.530
	MASE	<b>0.953</b>	0.960	0.955	1.169	0.964	1.014	1.082	1.023	1.021	1.277
	OWA	0.901	0.895	<b>0.893</b>	1.055	0.894	0.940	0.988	0.985	0.944	1.139
Others	sMAPE	4.861	<b>4.696</b>	5.008	6.184	4.982	4.894	6.151	4.753	5.259	5.841
	MASE	<b>3.124</b>	3.130	3.443	4.818	3.323	3.358	4.263	3.138	3.608	4.308
	OWA	1.004	<b>0.988</b>	1.070	1.140	1.048	1.044	1.319	1.019	1.122	1.294
Weighted Average	sMAPE	<b>11.863</b>	11.960	11.910	13.022	11.930	12.418	13.023	12.126	12.489	14.057
	MASE	<b>1.595</b>	1.606	1.613	1.814	1.597	1.656	1.836	1.641	1.690	1.954
	OWA	<b>0.858</b>	0.861	0.862	0.954	0.867	0.891	0.960	0.874	0.902	1.029

Table 8: Model ablation with detailed forecasting performance. We report forecasting results with different prediction lengths  $\{24, 36, 48, 60\}$  for ILI and  $H \in \{48, 96, 144, 192\}$  for others. For columns: Only  $K_{inv}$  uses one-block Time-invariant KP; Only  $K_{var}$  stacks Time-variant KPs only; *Truncated Filter* replaces Fourier Filter with High-Low Frequency Pass Filter; *Branch Switch* changes the order of KPs to deal with disentangled components.

Models		KooPA		Only $K_{inv}$		Only $K_{var}$		Truncated Filter		Branch Switch	
Metric		MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
ECL	48	<b>0.130</b>	<b>0.234</b>	0.150	0.243	1.041	0.777	0.149	0.245	0.137	0.234
	96	<b>0.136</b>	<b>0.236</b>	0.137	0.242	4.643	1.669	0.172	0.280	2.240	0.724
	144	<b>0.149</b>	0.247	0.150	0.252	0.238	0.327	<b>0.149</b>	<b>0.246</b>	0.226	0.331
	192	0.156	0.254	0.158	0.260	0.267	0.355	<b>0.152</b>	<b>0.248</b>	0.181	0.284
ETTh2	48	<b>0.226</b>	<b>0.300</b>	0.235	0.304	0.271	0.334	0.340	0.310	0.245	0.317
	96	<b>0.297</b>	<b>0.349</b>	0.311	0.353	0.382	0.405	0.301	0.352	0.343	0.384
	144	<b>0.333</b>	0.381	0.337	<b>0.379</b>	0.427	0.444	0.338	0.386	0.403	0.418
	192	<b>0.356</b>	<b>0.393</b>	0.363	0.397	0.402	0.437	0.363	0.400	0.384	0.420
Exchange	48	<b>0.042</b>	<b>0.143</b>	0.046	0.150	0.065	0.184	0.048	0.150	0.055	0.165
	96	<b>0.083</b>	<b>0.207</b>	<b>0.083</b>	0.210	0.147	0.274	0.087	0.210	0.151	0.277
	144	<b>0.130</b>	<b>0.261</b>	0.149	0.281	0.222	0.351	0.150	0.278	0.254	0.369
	192	<b>0.184</b>	<b>0.309</b>	0.200	0.322	0.385	0.456	0.229	0.345	0.463	0.490
ILI	24	<b>1.621</b>	<b>0.800</b>	2.165	0.882	1.972	0.919	2.140	0.874	2.092	0.894
	36	<b>1.803</b>	0.855	1.815	0.882	2.675	1.091	1.692	<b>0.844</b>	2.116	0.950
	48	<b>1.768</b>	0.903	2.107	0.981	2.446	1.045	1.762	<b>0.895</b>	2.394	1.084
	60	<b>1.743</b>	<b>0.891</b>	2.496	1.108	2.387	0.970	2.357	1.018	1.917	0.926
Traffic	48	<b>0.415</b>	<b>0.274</b>	0.445	0.295	0.915	0.536	0.668	0.363	0.468	0.300
	96	<b>0.401</b>	<b>0.275</b>	0.403	0.277	0.833	0.465	0.441	0.323	0.429	0.298
	144	<b>0.397</b>	<b>0.276</b>	0.400	0.278	0.816	0.452	0.436	0.321	0.438	0.307
	192	<b>0.403</b>	<b>0.284</b>	1.371	0.788	1.224	0.723	0.597	0.331	0.469	0.312
Weather	48	0.126	0.168	0.142	0.181	0.140	0.190	<b>0.125</b>	<b>0.166</b>	0.130	0.173
	96	0.154	0.205	0.164	0.209	0.169	0.224	<b>0.154</b>	<b>0.202</b>	0.163	0.210
	144	<b>0.172</b>	<b>0.225</b>	0.178	0.226	0.194	0.247	0.176	0.226	0.187	0.238
	192	<b>0.193</b>	<b>0.241</b>	0.195	0.245	0.217	0.268	0.195	0.244	0.212	0.261



**Appropriate disentanglement** By replacing the proposed Fourier Filter with another disentangling method (Truncated Filter), we validate the effectiveness of our proposed modules to disentangle time-variant and time-invariant dynamics, since the amplitude statistics of frequency spectrums work as a global view to exhibit the time-agnostic information, and thus reveals perfectly the specific and common dynamics underlying each series window. Besides, Koopa fundamentally considers the properties of disentangled time series with the right Koopman Predictors, and switching the order of KPs (Branch Switch) will lead to degrading performance.

### D.3 Model Efficiency

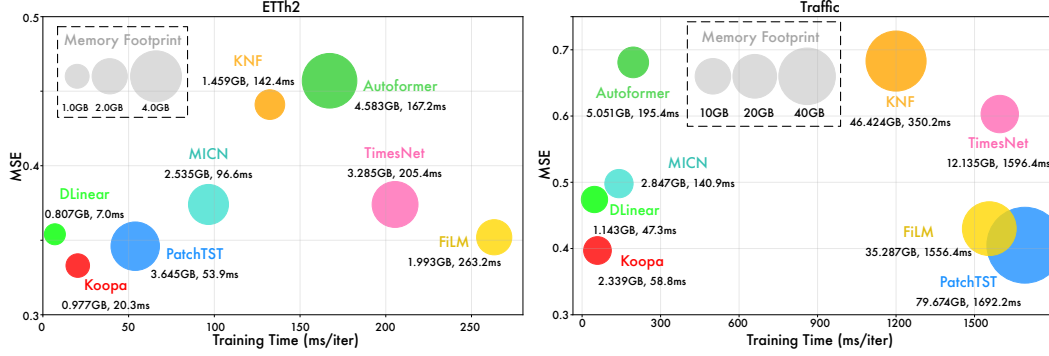


Figure 7: Model efficiency comparison with forecast length  $H = 144$  for ETTh2 and Traffic.

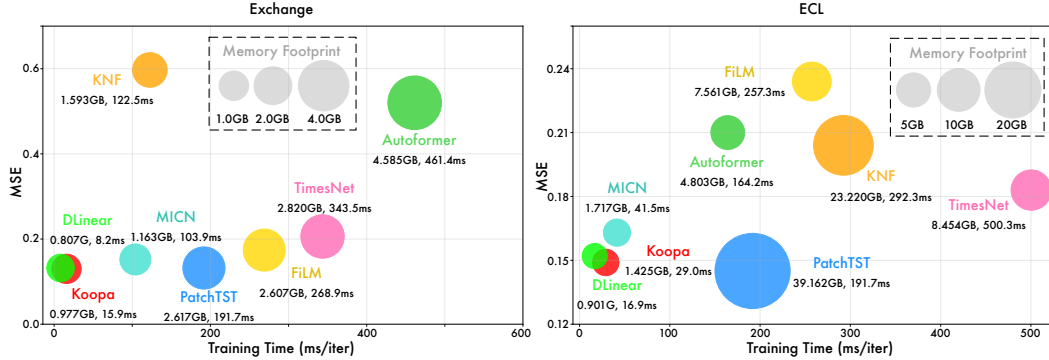


Figure 8: Model efficiency comparison with forecast length  $H = 144$  for Exchange and ECL.

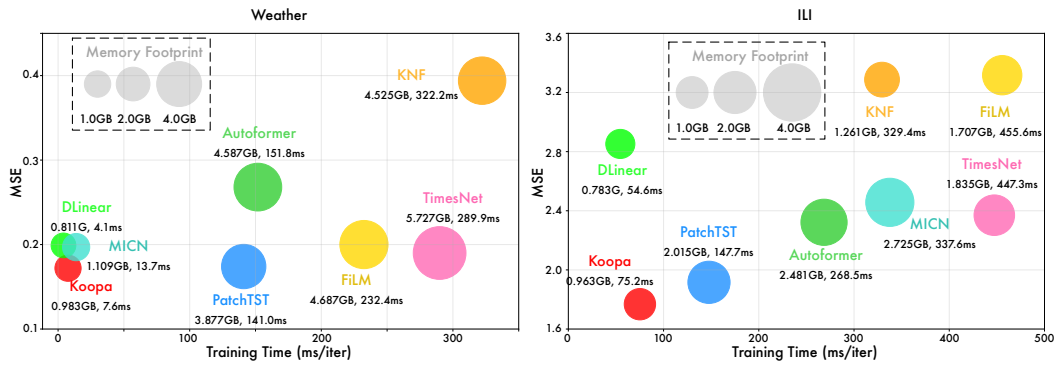


Figure 9: Model efficiency comparison with forecast length  $H = 144$  for Weather and 48 for ILI.

We comprehensively compare the forecasting performance, training speed, and memory footprint of our model with well-acknowledged deep forecasting models. The results are recorded with the official model configuration and the same batch size. We visualize the model efficiency under all six multivariate datasets in Figure 7–9. In detail, compared with the previous state-of-the-art model PatchTST [8], Koopa consumes only 15.2% training time and 3.6% memory footprint respectively in ECL, 37.8% training time and 26.8% memory in ETTh2, 23.5% training time and 37.3% memory in Exchange, 50.9% training time and 47.8% memory in ILI, 3.5% training time and 2.9% memory in Traffic, and 5.4% training time and 25.4% memory in Weather, leading to the averaged **77.3%** and **76.0%** saving of training time and memory footprint in all six datasets. The remarkable efficiency can be attributed to Koopa with MLPs as the building blocks, and we find the budget saving becoming more significant on datasets with more series variables (ECL, Traffic).

Besides, as an efficient linear model, the performance of Koopa still surpasses other MLP-based models. Especially, Compared with DLinear [51], our model reduces 38.0% MSE (2.852→1.768) in ILI and 13.6% MSE (0.452→0.397) in Weather. And the average MSE reduction of Koopa compared with the previous state-of-the-art MLP-based model reaches **12.2%**.

Therefore, our proposed Koopa is efficiently built with MLP networks and shows great model capacity to exploit nonlinear dynamics and complicated temporal dependencies in real-world time series.

## E Broader Impact

### E.1 Impact on Real-world Applications

This paper copes with real-world time series forecasting, which is characterized by intrinsic non-stationarity that poses fundamental challenges for deep forecasting models. Since previous studies hardly research the theoretical basis that can naturally address the time-variant property in non-stationary data, we propose a novel Koopman forecaster that fundamentally considers the implicit time-variant and time-invariant dynamics based on Koopman theory. Our model achieves the state-of-the-art performance on six real-world forecasting tasks, covering energy, economics, disease, traffic, and weather, and demonstrates remarkable model efficiency in training time and memory footprint. Therefore, the proposed model makes it promising to tackle real-world forecasting applications, which helps our society prevent risks in advance and make better decisions with limited computational budgets. Our paper mainly focuses on scientific research and has no obvious negative social impact.

### E.2 Impact on Future Research

In this paper, we find modern Koopman theory natural to learn the dynamics underlying non-stationary time series. The proposed model explores complex non-stationary patterns with temporal localization inspired by Koopman approaches and implements respective deep network modules to disentangle and portray time-variant and time-invariant dynamics with the enlightenment of Wold’s Theorem. The remarkable efficiency and insights from the theory can be instructive for future research.

## F Limitation

Our proposed model does not respectively considers dynamics in different variates, which leaves improvement for better multivariate forecasting with the consideration of various evolution patterns and series relationships. And Koopman spectral theory is still under leveraging in our work, which can discover Koopman modes to interpret the linear behavior underlying non-stationary data in a high-dimensional representation. Besides, Koopman theory for control considering factors outside the system can be promising for series forecasting with covariates, which leaves our future work.