

MVA/IMA – 3D Vision

Graph Cuts and Application to Disparity Map Estimation

Renaud Marlet

LIGM/École des Ponts ParisTech (IMAGINE) & Valeo.ai

<http://imagine.enpc.fr/~marletr>

(with many borrowings from Boykov & Veksler 2006)

Introduction

3D reconstruction

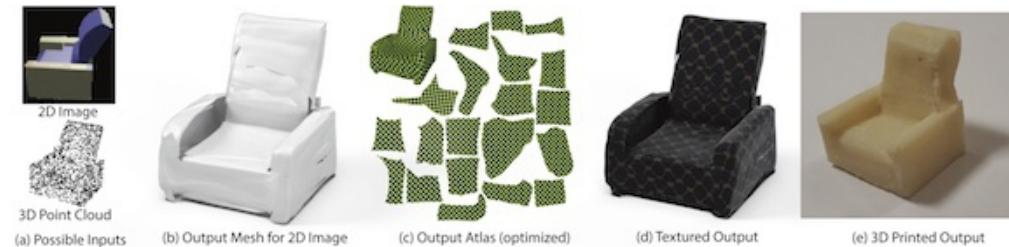
- capturing reality
 - for diagnosis, simulation, movies, video games, interaction in virtual/augmented reality, ...

This course :

- camera calibration
 - relevance of accuracy: 1° error, at 10m → 17cm error
- low-level 3D (disparity/depth map, mesh)
 - as opposed to high-level geometric primitives, semantics...

Mathematical tools for 3D reconstruction

- Deep learning:
 - very good for matching image regions
→ subcomponent of 3D reconstruction algorithm
 - a few methods for direct disparity/depth map estimation
 - fair results on 3D reconstruction from single view



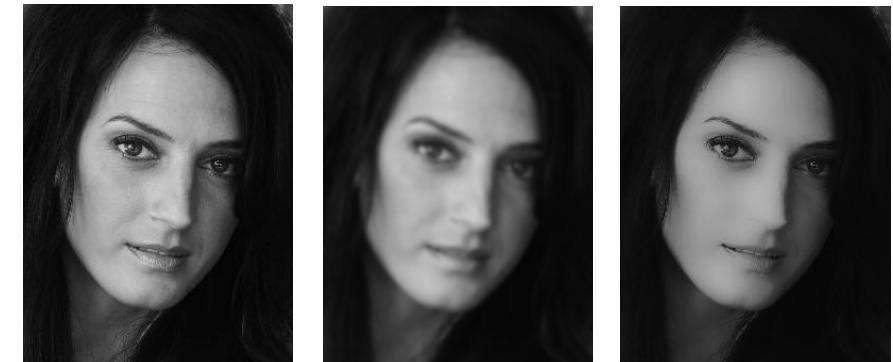
Groueix et al. 2017

- Graph cuts (this lecture):
 - practical, well-founded, general (→ maps, meshes...)

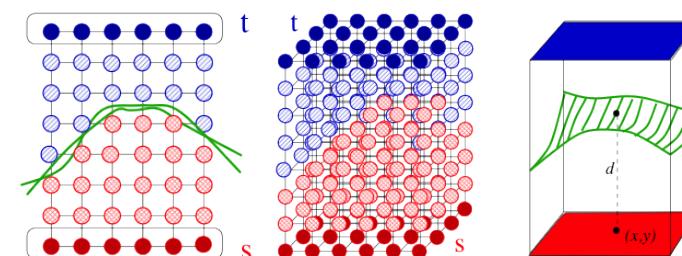
Motivating graph cuts

- Powerful **multidimensional energy minimization** tool
 - wide class of binary and non binary energies
 - in some cases, globally optimal solutions
 - some provably good approximations (and good in practice)
 - allowing regularizers with contrast preservation
 - enforcement of piecewise smoothness while preserving relevant sharp discontinuities

$$E(f) = \sum_{p \in P} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$$



- Geometric interpretation
 - hypersurface in n -D space



Many links to other domains

(cf. Boykov & Veksler 2006)

- Combinatorial algorithms (e.g., dynamic programming)
- Simulated annealing
- Markov random fields (MRFs)
- Random walks and electric circuit theory
- Bayesian networks and belief propagation
- Level sets and other variational methods
- Anisotropic diffusion
- Statistical physics
- Submodular functions
- Integral/differential geometry, etc.

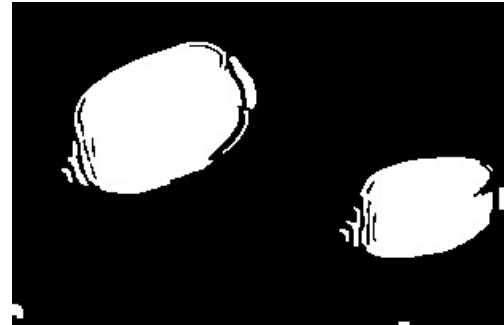
dynamic programming = programmation dynamique
simulated annealing = recuit simulé
Markov random field = champ (aléatoire) de Markov
random walk = marche aléatoire
Bayesian network = réseaux bayésien
level set = ligne de niveau
submodular function = fonction sous-modulaire

Overview of the course

- Notions
 - graph cut, minimum cut
 - flow network, maximum flow
 - optimization: exact (global), approximate (local)
- Illustration with emblematic applications



segmentation



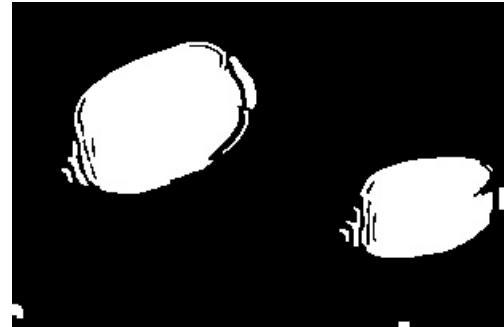
disparity map estimation



Overview of the course

- Notions
 - graph cut, minimum cut
 - flow network, maximum flow
 - optimization: exact (global), approximate (local)
- Illustration with emblematic applications

No time to go deep
into every topic →
general ideas,
read the references



segmentation



disparity map estimation

Part 1

Graph cuts basics

Max-flow min-cut theorem

Application to image restoration
and image segmentation

Graph cut basics

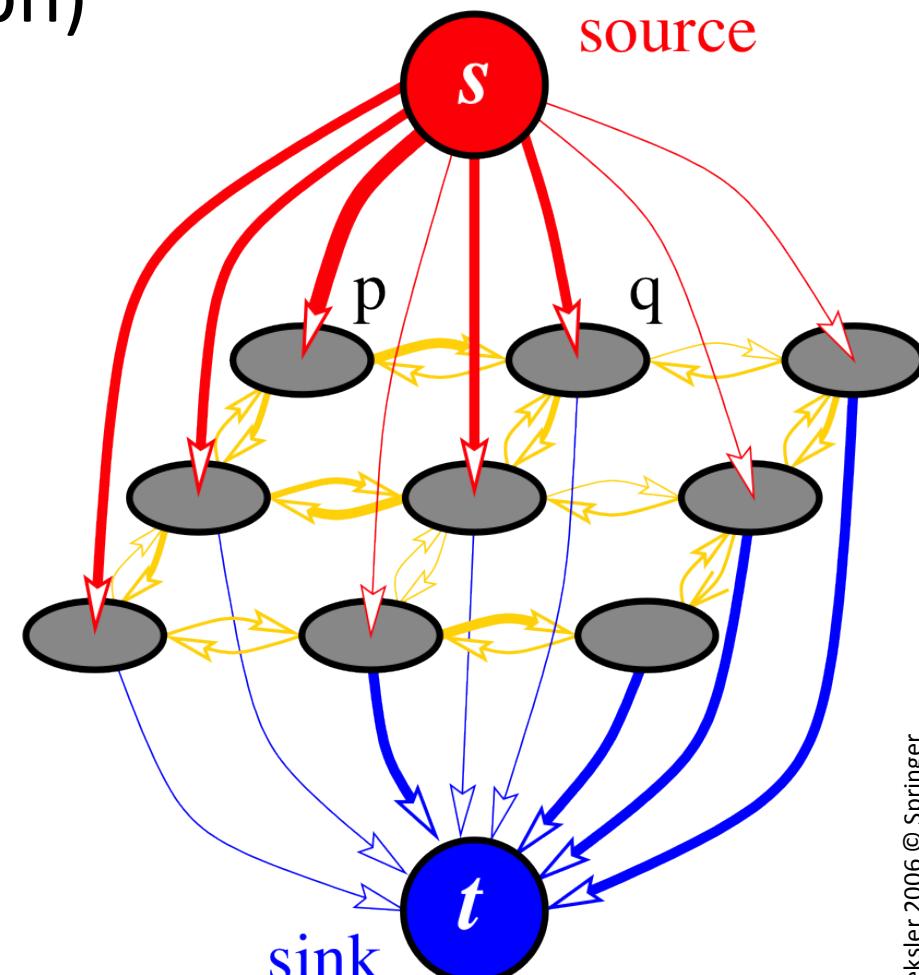
node = nœud
 vertex (vertices) = sommet(s)
 edge = arête
 directed = orienté
 digraph (directed graph) =
 graphe orienté
 sink = puits

- Graph $G = \langle \mathcal{V}, \mathcal{E} \rangle$ (digraph)

- set of nodes (vertices) \mathcal{V}
- set of directed edges \mathcal{E}
 - $p \rightarrow q$

- $\mathcal{V} = \{s, t\} \cup \mathcal{P}$

- terminal nodes: $\{s, t\}$
 - s : source node
 - t : target node (= sink)
- non-terminal nodes: \mathcal{P}
 - ex. \mathcal{P} = set of pixels, voxels, etc.
(can be very different from an image)

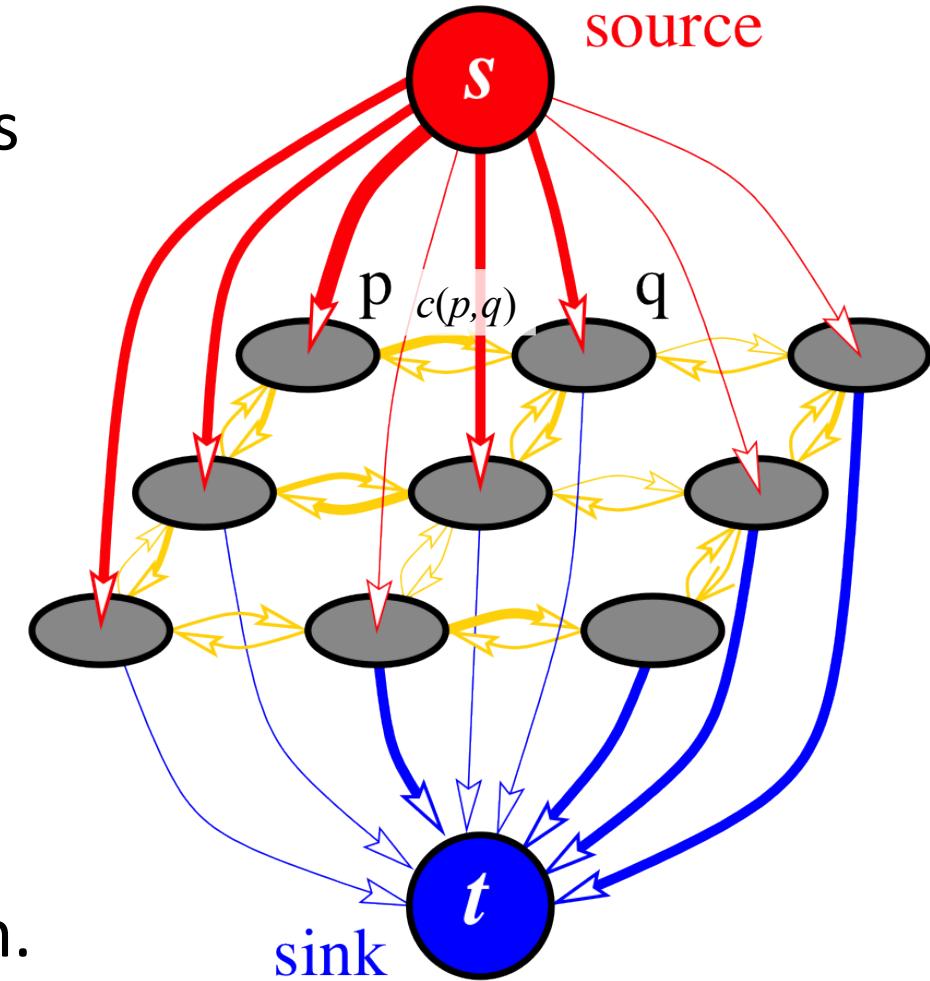


Example of connectivity

Graph cut basics

label = étiquette
 weight = poids
 link = lien

- Edge labels, for $p \rightarrow q \in \mathcal{E}$
 - $c(p,q) \geq 0$: **nonnegative costs**
also called weights $w(p,q)$
 - $c(p,q)$ and $c(q,p)$, if any, may differ
- Links
 - t-link: term. \leftrightarrow non-term.
 - $\{s \rightarrow p \mid p \neq t\}, \{q \rightarrow t \mid q \neq s\}$
 - n-link: non-term. \rightarrow non-term.
 - $\mathcal{N} = \{p \rightarrow q \mid p, q \neq s, t\}$



Cut and minimum cut

cut = coupe
severed = coupé, sectionné

- **s - t cut** (or just “cut”): $C = \{S, T\}$

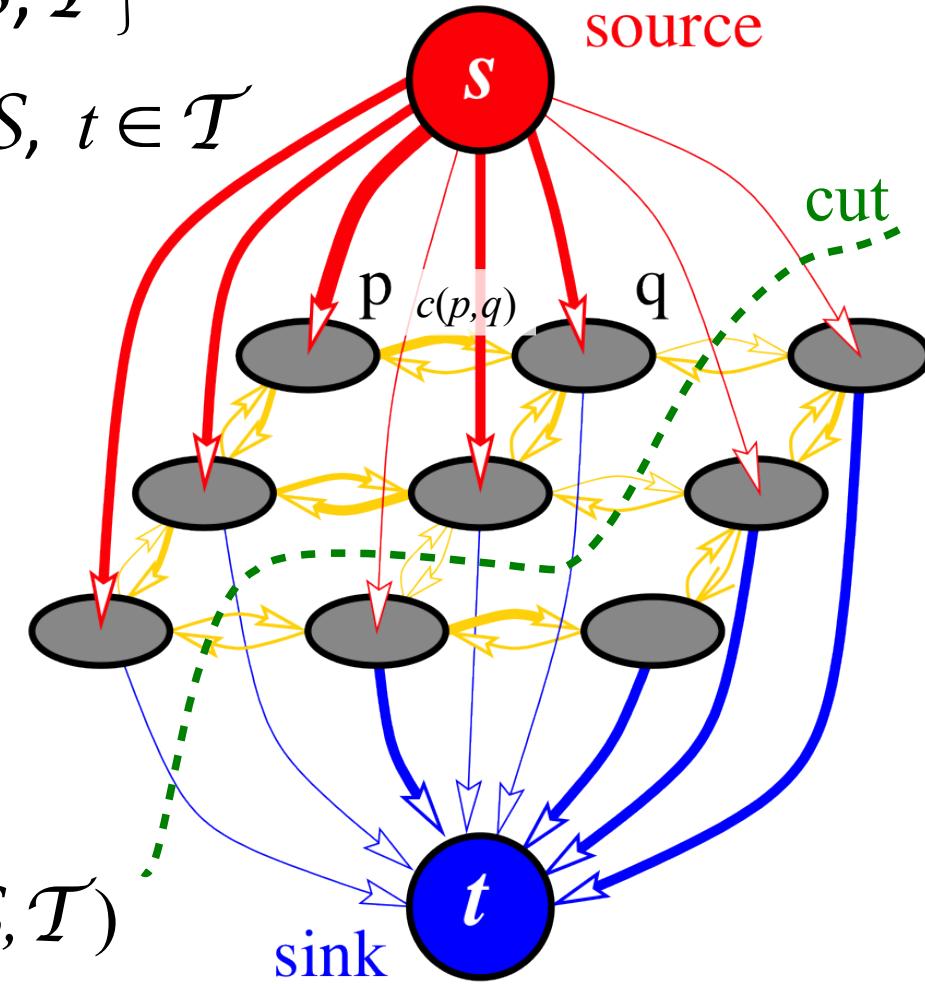
node partition such that $s \in S, t \in T$

- **Cost of a cut** $\{S, T\}$:

- $c(S, T) = \sum_{p \in S, q \in T} c(p, q)$
- N.B. cost of severed edges:
only from S to T

- **Minimum cut:**

- i.e., with min cost: $\min_{S, T} c(S, T)$
- intuition: cuts only “weak” links



Different view: flow network

(or transportation network)

flow = flot
 network = réseau
 transportation = transport
 vertex = sommet
 node = nœud
 edge = arête

- Different vocabulary and features

- graph \leftrightarrow network

vertex = node p, q, \dots

edge = arc $p \rightarrow q$ or (p, q)

cost = capacity $c(p, q)$

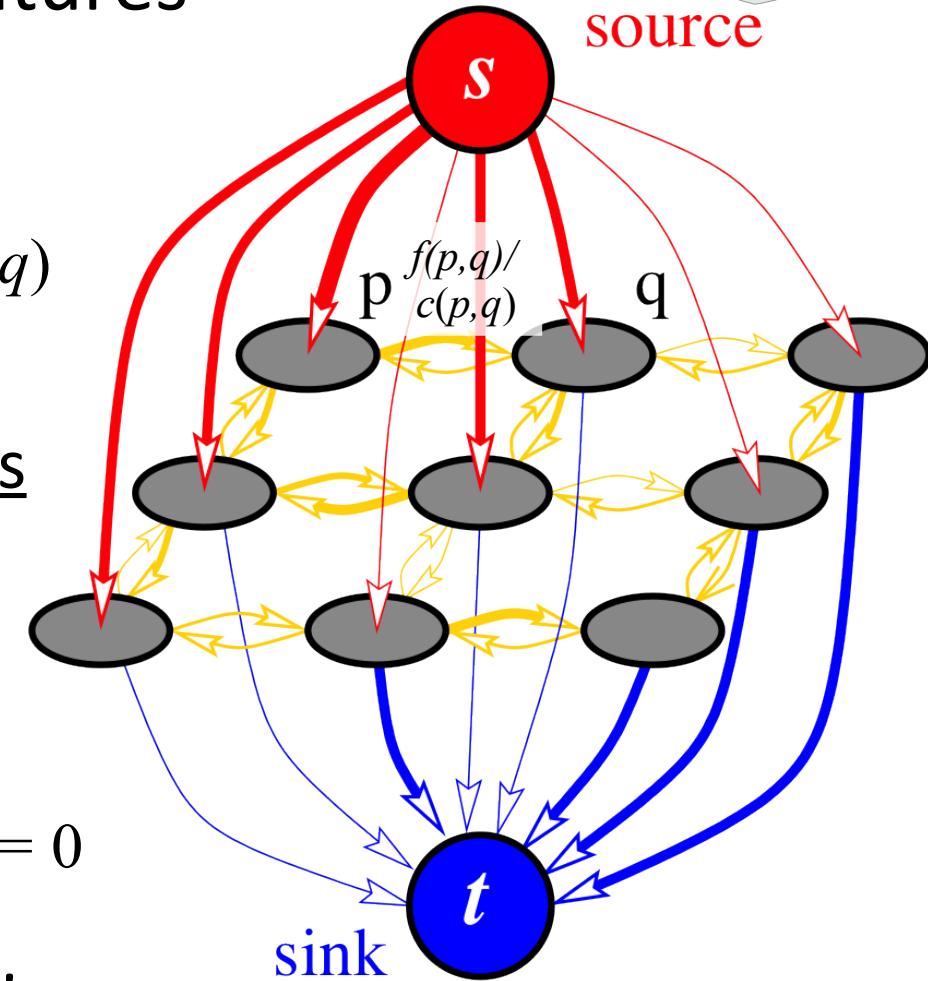
- possibly many sources & sinks

- Flow $f: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$

- $f(p, q)$: amount of flow $p \rightarrow q$

- $(p, q) \notin \mathcal{E} \Leftrightarrow c(p, q) = 0, f(p, q) = 0$

- e.g. road traffic, fluid in pipes, current in electrical circuit, ...



Flow network constraints

skew symmetry = antisymétrie

- Capacity constraint

- $f(p,q) \leq c(p,q)$

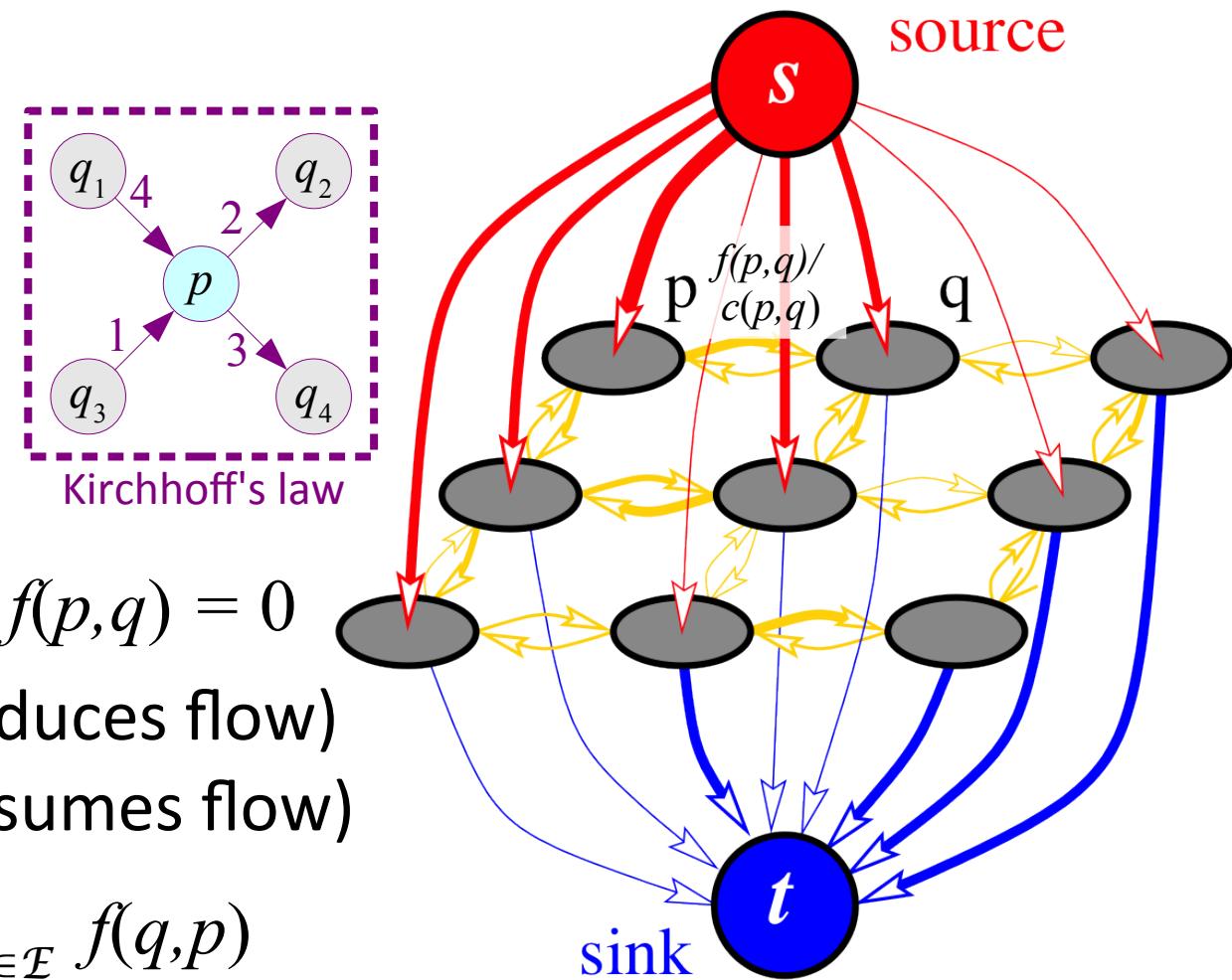
- Skew symmetry

- $f(p,q) = -f(q,p)$

- Flow conservation

- $\forall p, \text{ net flow } \sum_{q \in \mathcal{V}} f(p,q) = 0$
unless $p = s$ (s produces flow)
or $p = t$ (t consumes flow)

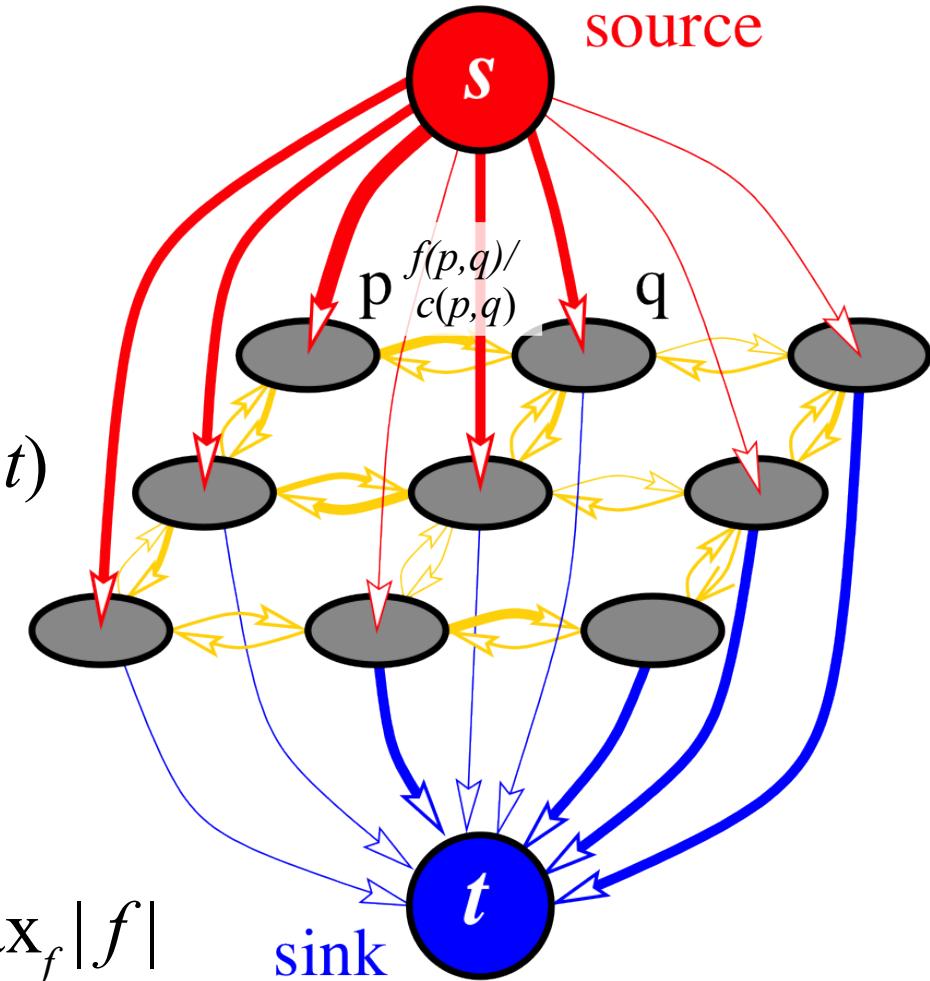
- i.e., incoming $\sum_{(q,p) \in \mathcal{E}} f(q,p)$
= outgoing $\sum_{(p,q) \in \mathcal{E}} f(p,q)$



Flow network constraints

skew symmetry = antisymétrie

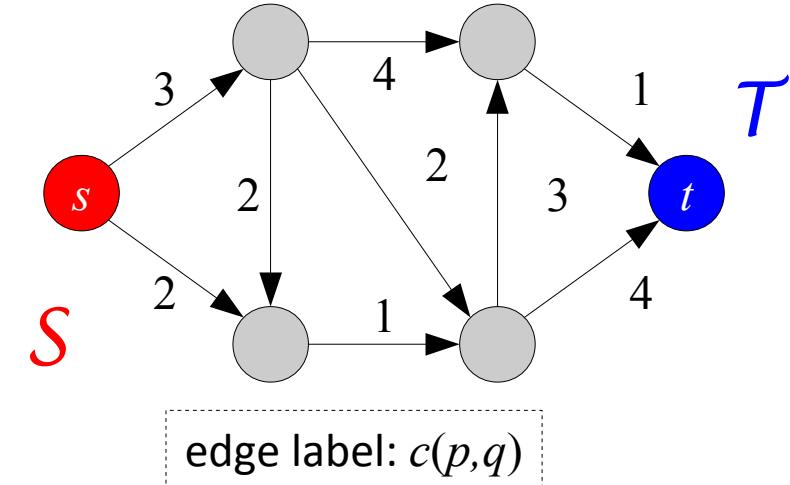
- **s - t flow** (or just “flow) f
 - $f: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ satisfying flow constraints
- **Value of s - t flow**
 - $|f| = \sum_{q \in \mathcal{V}} f(s, q) = \sum_{p \in \mathcal{V}} f(p, t)$
 - amount of flow from source
= amount of flow to sink
- **Maximum flow:**
 - i.e., with maximum value: $\max_f |f|$
 - intuition: arcs saturated as much as possible



Max-flow min-cut theorem

- Theorem
 - The maximum value of an s - t flow is equal to the minimum capacity (i.e., min cost) of an s - t cut.

- Example
 - $- |f| = c(S, T) = ?$

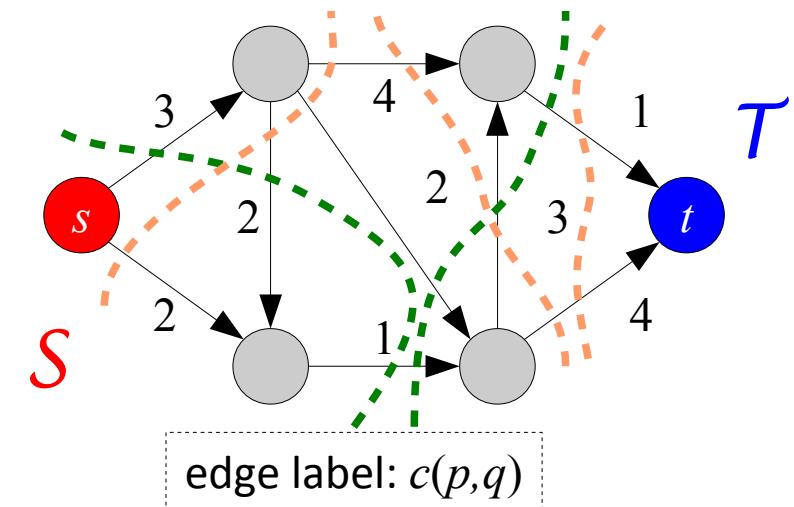


Max-flow min-cut theorem

- Theorem

The maximum value of an s - t flow is equal to the minimum capacity (i.e., min cost) of an s - t cut.

- Example
 - $|f| = c(S, T) = 4$
 - min: enumerate partitions...



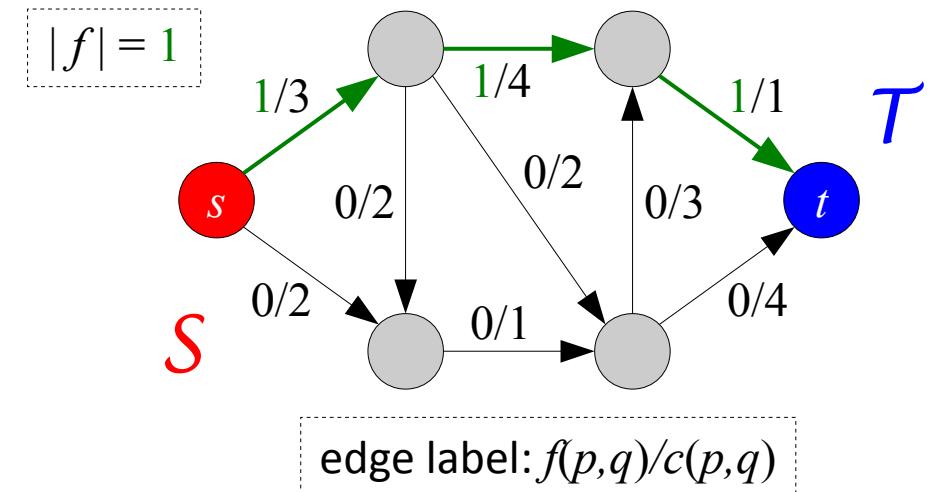
Max-flow min-cut theorem

- **Theorem**

The maximum value of an s - t flow is equal to the minimum capacity (i.e., min cost) of an s - t cut.

- **Example**

- $|f| = c(S, T) = 4$
- min: enumerate partitions...
- max: try increasing $f(p, q)$...



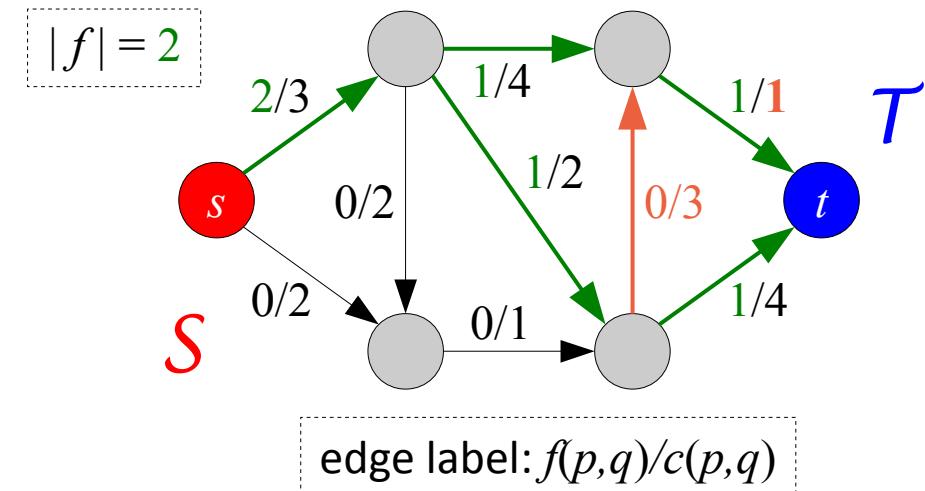
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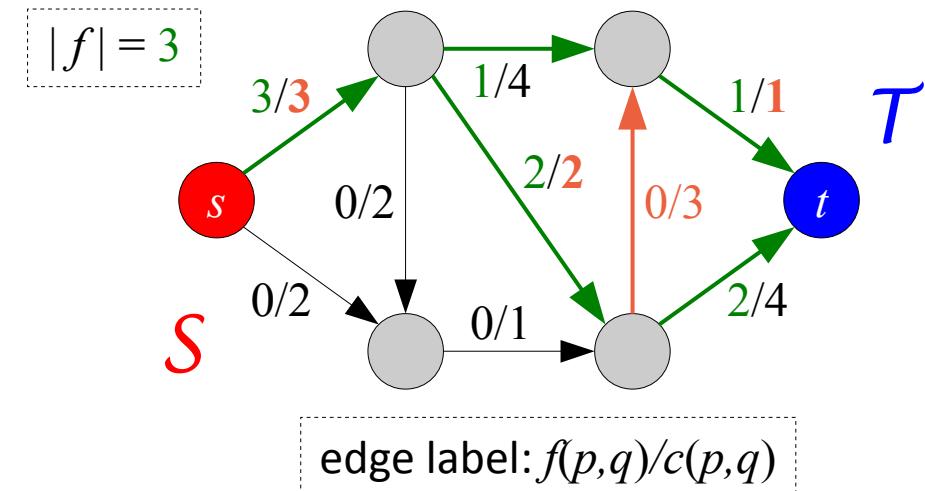
Max-flow min-cut theorem

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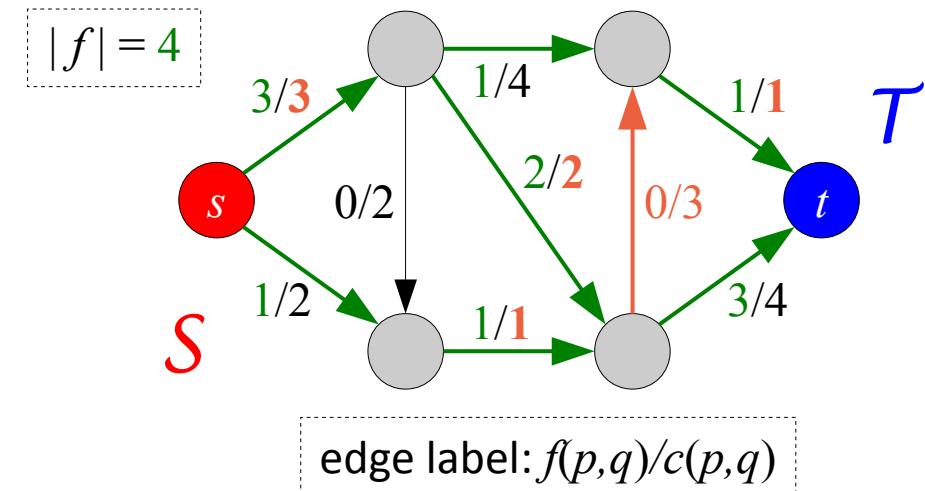
Max-flow min-cut theorem

- Theorem

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- Example

- $|f| = c(S, T) = 4$
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Max-flow min-cut theorem

- **Theorem**

The maximum value of an s - t flow is equal to the minimum capacity (i.e., min cost) of an s - t cut.

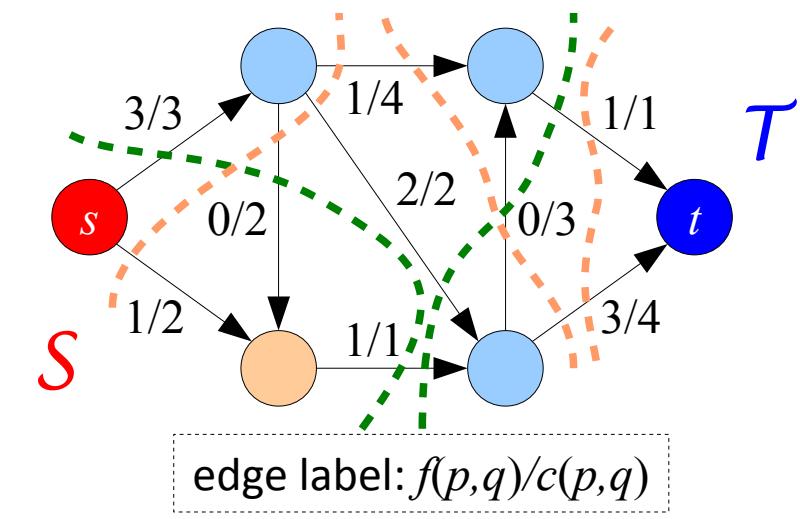
to pull = tirer
to tear = (se) déchirer
weak = faible

- **Example**

- $|f| = c(S, T) = 4$
- min: enumerate partitions...
- max: try increasing $f(p, q)$...

- **Intuition**

- pull s and t apart: the graph tears where it is weak
- min cut: cut corresponding to a small number of weak links
- max flow: flow bounded by low-capacity links in a cut



Max-flow min-cut theorem

linear programming =
programmation linéaire

- **Theorem**

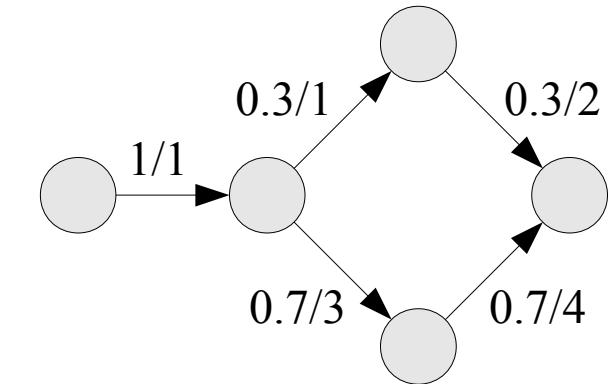
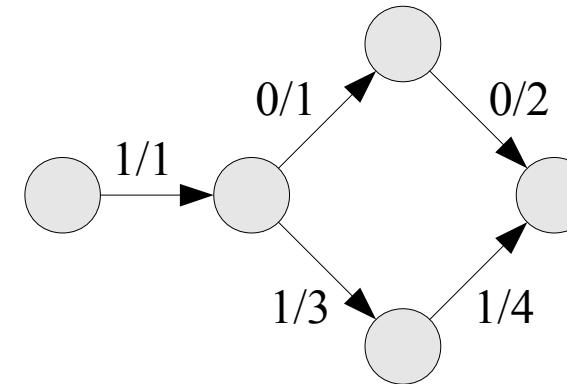
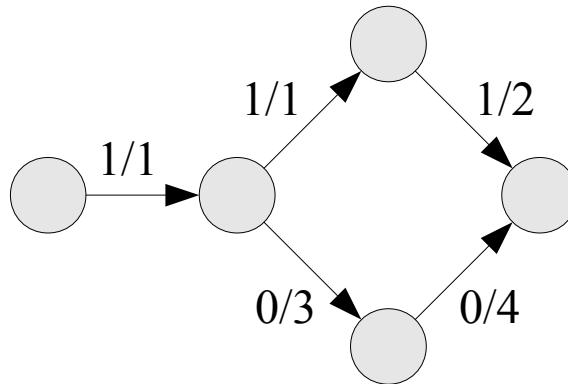
The maximum value of an s - t flow is equal to the minimum capacity (i.e., min cost) of an s - t cut.

- proved independently by Elias, Feinstein & Shannon, and Ford & Fulkerson (1956)
- special case of strong duality theorem in linear programming
- can be used to derive other theorems

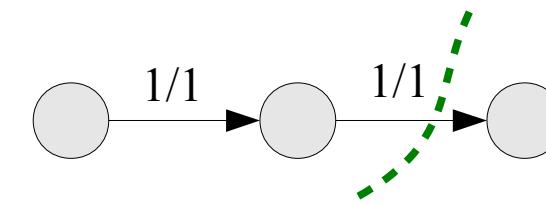
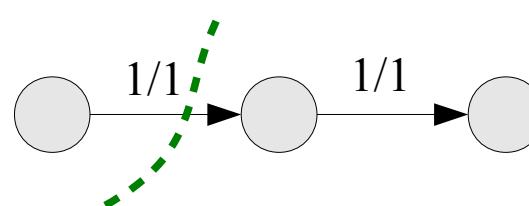
Max flows and min cuts configurations are not unique

- Different configurations with same maximum flow

edge label: $f(p,q)/c(p,q)$



- Different configurations with same min-cut cost



Algorithms for computing max flow

- Polynomial time
- Push-relabel methods
 - better performance for general graphs
 - e.g. Goldberg and Tarjan 1988: $O(VE \log(V^2/E))$
 - where V : number of vertices, E : number of edges
- Augmenting paths methods
 - iteratively push flow from source to sink along some path
 - better performance on specific graphs
 - e.g. Ford-Fulkerson 1956: $O(E \max|f|)$ for integer capacity c

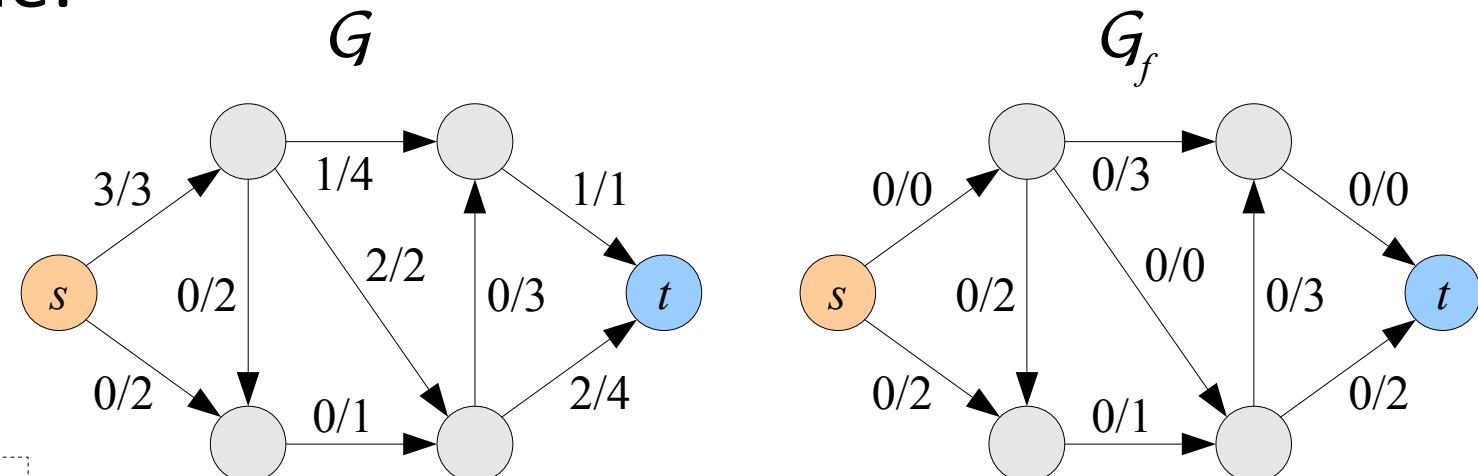
To go further on this subject

Residual network/graph

- Given flow network $G = \langle \mathcal{V}, \mathcal{E}, c, f \rangle$

Define residual network $G_f = \langle \mathcal{V}, \mathcal{E}, c_f, 0 \rangle$ with

- residual capacity $c_f(p,q) = c(p,q) - f(p,q)$
- no flow, i.e., value 0 for all edges
- Example:



To go further on this subject

Ford-Fulkerson algorithm (1956)

termination = terminaison
semi-algorithm: termination
not guaranteed for all inputs

$f(p,q) \leftarrow 0$ for all edges

[P : augmenting path]

while \exists path P from s to t such that $\forall (p,q) \in P \ c_f(p,q) > 0$

$c_f(P) \leftarrow \min\{c_f(p,q) \mid (p,q) \in P\}$

[min residual capacity]

for each edge $(p,q) \in P$

$f(p,q) \leftarrow f(p,q) + c_f(P)$

[push flow along path]

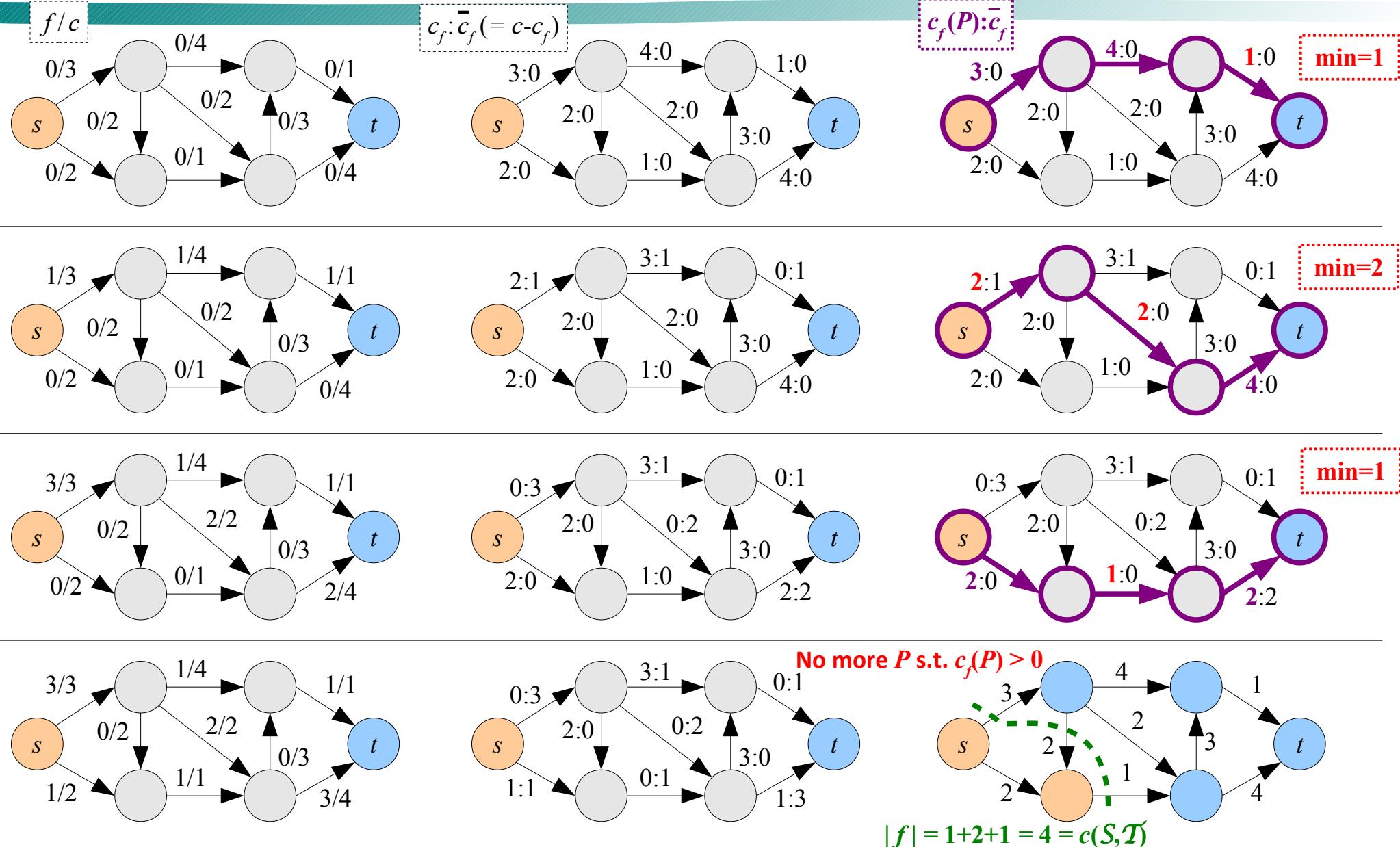
$f(q,p) \leftarrow f(q,p) - c_f(P)$

[keep skew symmetry]

- N.B. termination not guaranteed
 - maximum flow reached if (semi-)algorithm terminates
(but may “converge” to less than maximum flow if it does not terminate)
 - always terminates for integer values (or rational values)

To go further on this subject

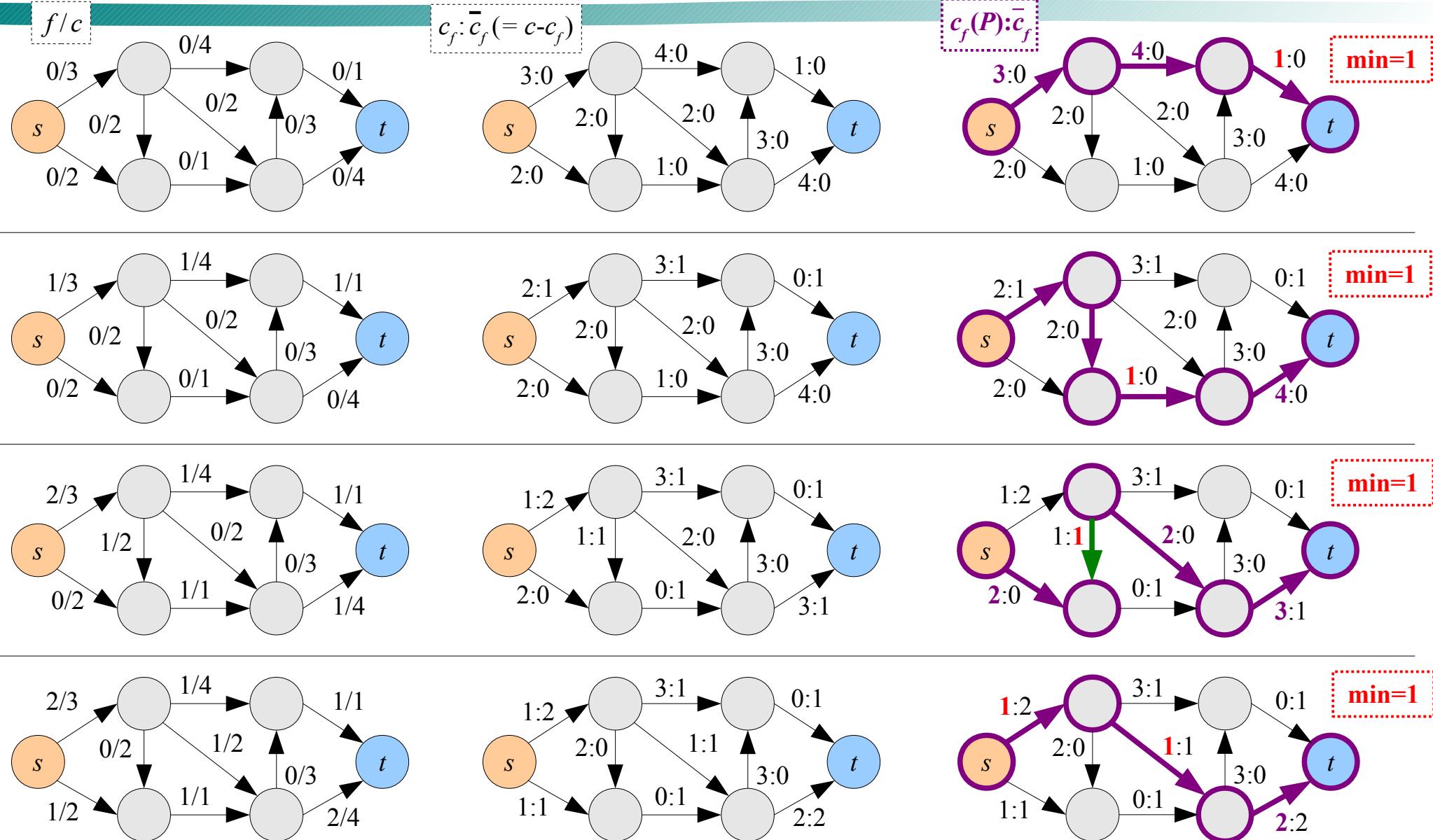
Ford-Fulkerson algorithm: an example



To go further on this subject

Ford-Fulkerson algorithm: an example

Taking edges backwards = OK (and sometimes needed)



To go further on this subject

Edmonds-Karp algorithm (1972)

breadth-first = en largeur d'abord
sparse = épars, peu dense

- As Ford-Fulkerson but **shortest path** with >0 capacity
 - breadth-first search for augmenting path (cf. example above)
- Termination: now guaranteed
- Complexity: $O(VE^2)$
 - slower than push-relabel methods for general graphs
 - faster in practice for sparse graphs
- Other variant (Dinic 1970), complexity: $O(V^2 E)$
 - other flow selection (blocking flows)
 - $O(VE \log V)$ with dynamic trees (Sleator & Tarjan 1981)

Maximum flow for grid graphs

- Fast augmenting path algorithm
(Boykov & Kolmogorov 2004)
 - often significantly outperforms push-relabel methods
 - observed running time is linear
 - many variants since then
- But push-relabel algorithm can be run in parallel
 - good setting for GPU acceleration

The “best” algorithm depends on the context

To go further on this subject

Variant: Multiway cut problem

- More than two terminals: $\{s_1, \dots, s_k\}$
- Multiway cut:
 - set of edges leaving each terminal in a separate component
- Multiway cut problem
 - find cut with minimum weight
 - same as min cut when $k = 2$
 - NP-hard if $k \geq 3$ (in fact APX-hard, i.e., NP-hard to approx.)
 - but can be solved exactly for planar graphs

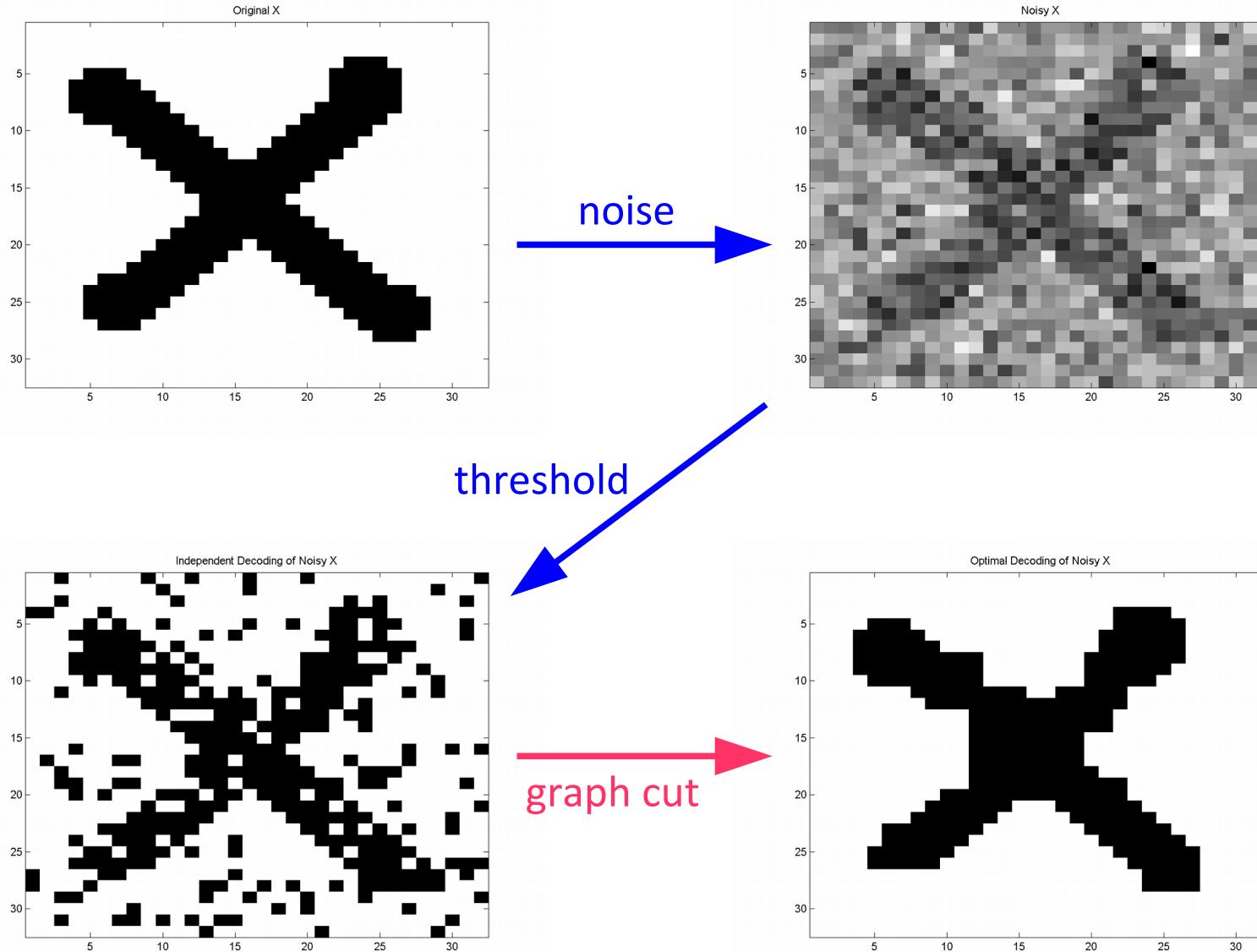
planar = planaire

Graph cuts for binary optimization

- Inherently a **binary** technique
 - splitting in two
- 1st use in image processing:
binary image restoration (Greig et al. 1989)
 - black&white image with noise → image with no noise
- Can be generalized to large classes of binary energy
 - regular functions

Binary image restoration

noise = bruit
threshold = seuil



Binary image restoration: The graph cut view

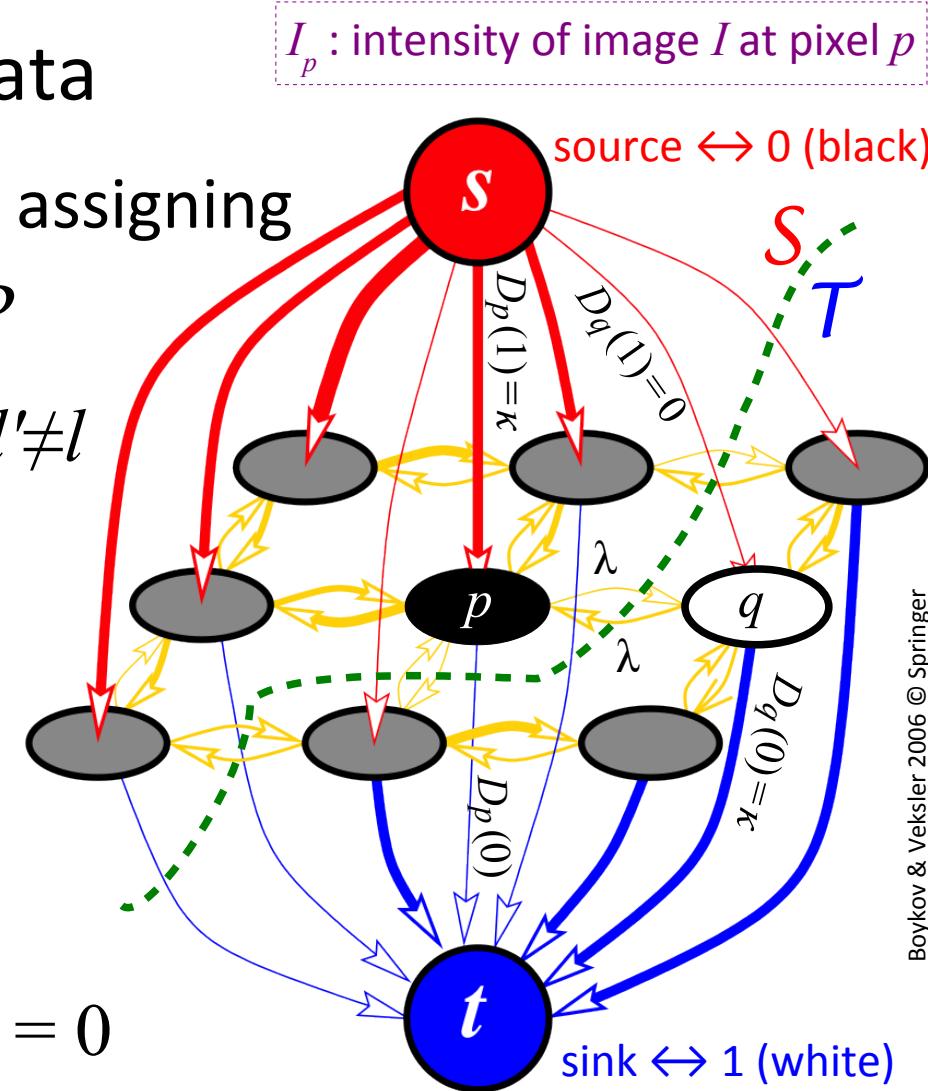
penalty = pénalité, coût
reward = récompense

- Agreement with observed data

- $D_p(l)$: penalty (= –reward) for assigning label $l \in \{0,1\}$ to pixel $p \in \mathcal{P}$
- if $I_p = l$ then $D_p(l) < D_p(l')$ for $l' \neq l$
- $w(s,p)=D_p(1)$, $w(p,t)=D_p(0)$

- Example:

- if $I_p = 0$, $D_p(0) = 0$, $D_p(1) = \kappa$
if $I_p = 1$, $D_p(0) = \kappa$, $D_p(1) = 0$
- if $I_p = 0$ and $p \in S$, cost = $D_p(0) = 0$
if $I_p = 0$ and $p \in T$, cost = $D_p(1) = \kappa$



Binary image restoration: The graph cut view

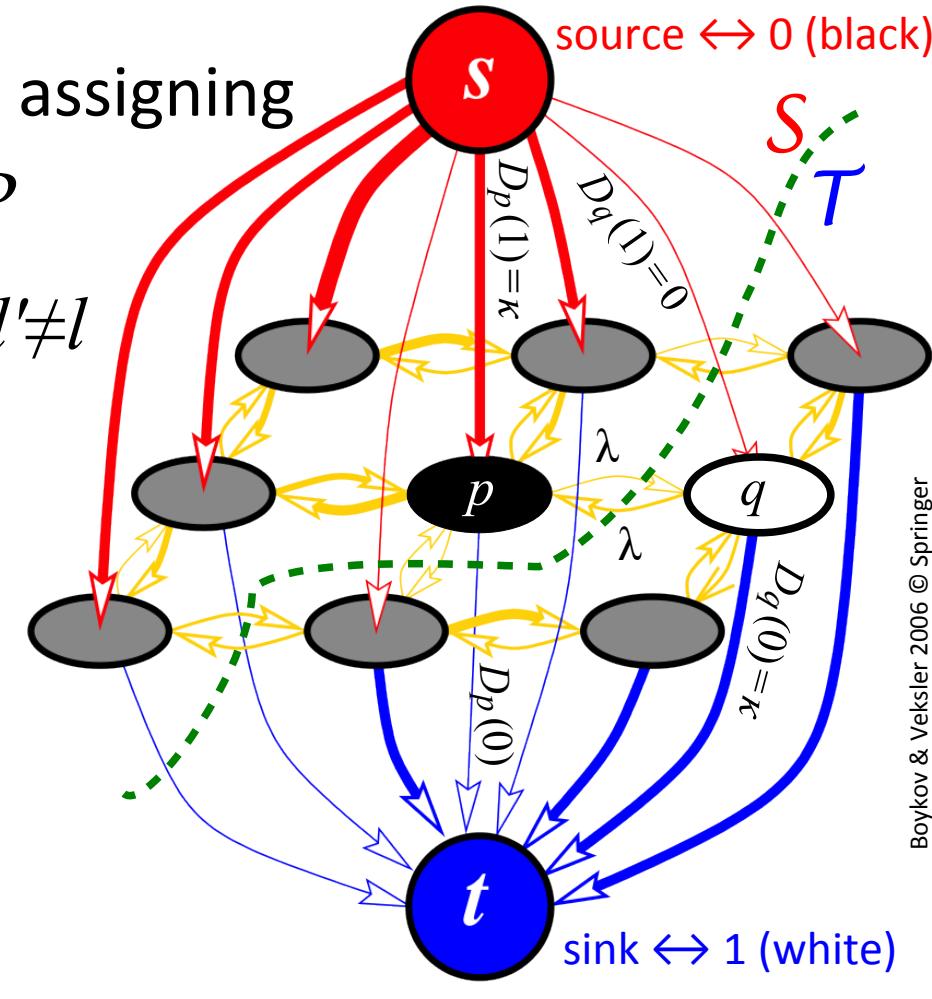
penalty = pénalité, coût
 reward = récompense
 regularizing constraint =
 contrainte de régularisation
 smoothing = lissage

- Agreement with observed data

- $D_p(l)$: penalty (= –reward) for assigning label $l \in \{0,1\}$ to pixel $p \in \mathcal{P}$
- if $I_p = l$ then $D_p(l) < D_p(l')$ for $l' \neq l$
- $w(s,p)=D_p(1)$, $w(p,t)=D_p(0)$

- Minimize discontinuities

- penalty for (long) contours
 - $w(p,q) = w(q,p) = \lambda > 0$
- spatial coherence,
regularizing constraint, smoothing factor... (see below)



Binary image restoration: The graph cut view

labeling = étiquetage

- Binary labeling f [N.B. different from “flow f ”]

- assigns label $f_p \in \{0,1\}$ to pixel $p \in \mathcal{P}$
 - $f: \mathcal{P} \rightarrow \{0,1\}$ $f(p) = f_p$

- Cut $C = \{S, T\} \leftrightarrow$ labeling f

- 1-to-1 correspondence: $f = \mathbf{1}_{|T|}$

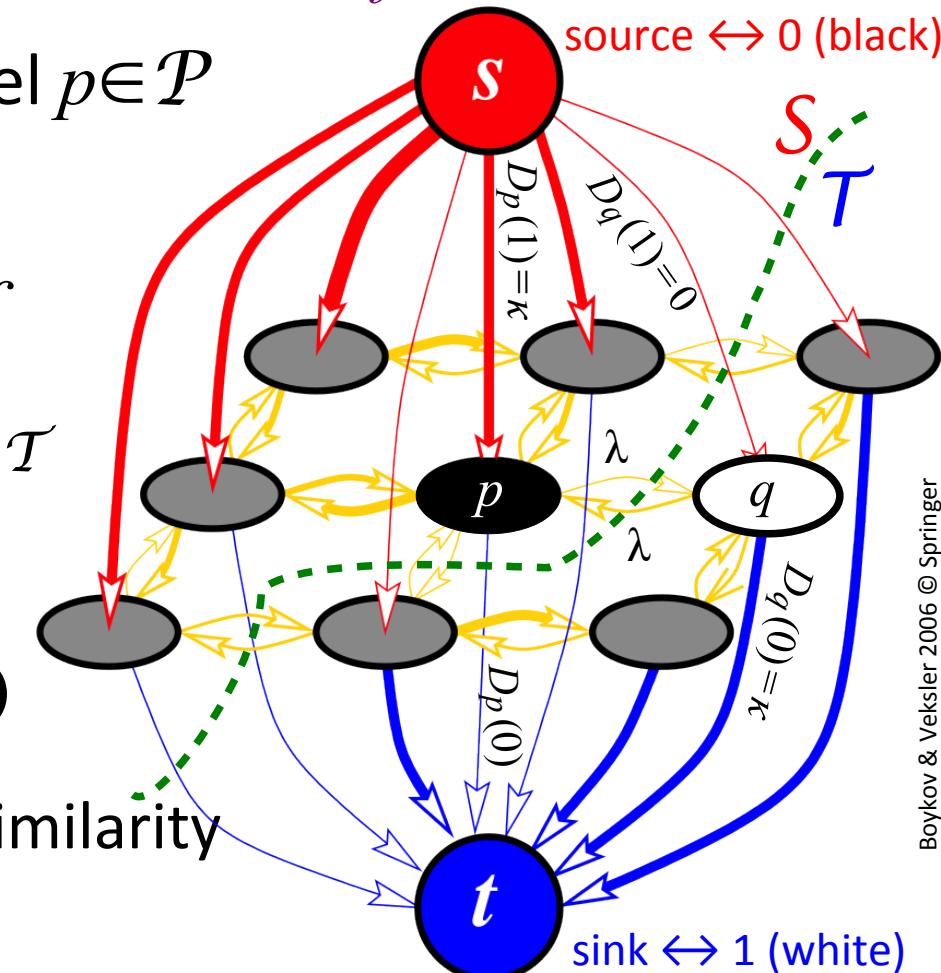
- Cost of a cut: $|C| =$

$$\sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in S \times T} w(p,q)$$

= cost of flip + cost of local dissimilarity

- Restored image:

= labeling corresponding to a minimum cut



Binary image restoration: The energy view

- Energy of labeling f

- $E(f) \stackrel{\text{def}}{=} |C| =$
 $\sum_{p \in \mathcal{P}} D_p(f_p) +$
 $\lambda \sum_{(p,q) \in \mathcal{N}} \mathbf{1}(f_p = 0 \wedge f_q = 1)$

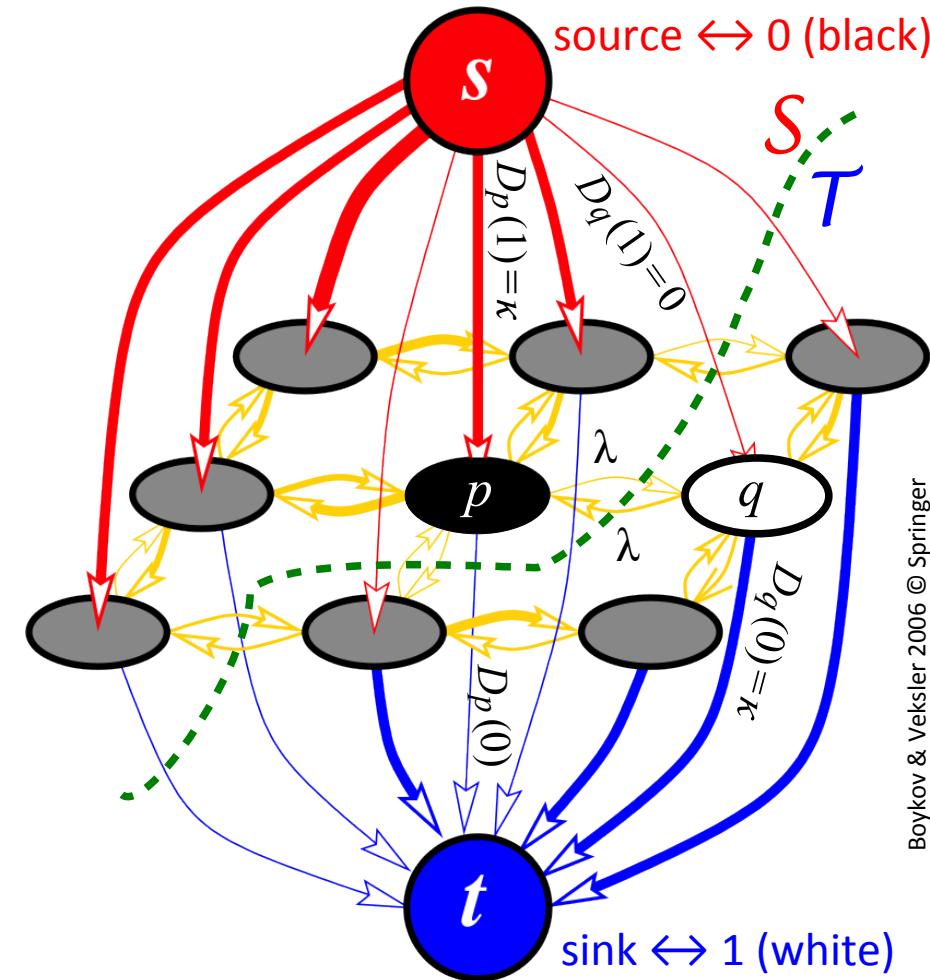
where

$$\mathbf{1}(\text{false}) = 0 \quad | \quad \mathbf{1}(\text{true}) = 1$$

$$[\text{ou: } \frac{1}{2} \lambda \sum_{(p,q) \in \mathcal{N}} \mathbf{1}(f_p \neq f_q)]$$

- Restored image:

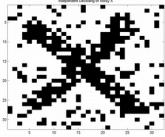
- labeling corresponding to minimum energy (= minimum cut)



Binary image restoration: The smoothing factor

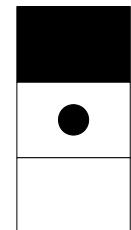
cluster = amas
outlier = point aberrant

- Small λ (actually λ/κ):



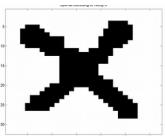
- pixels chose their label independently of their neighbors

- Large λ :

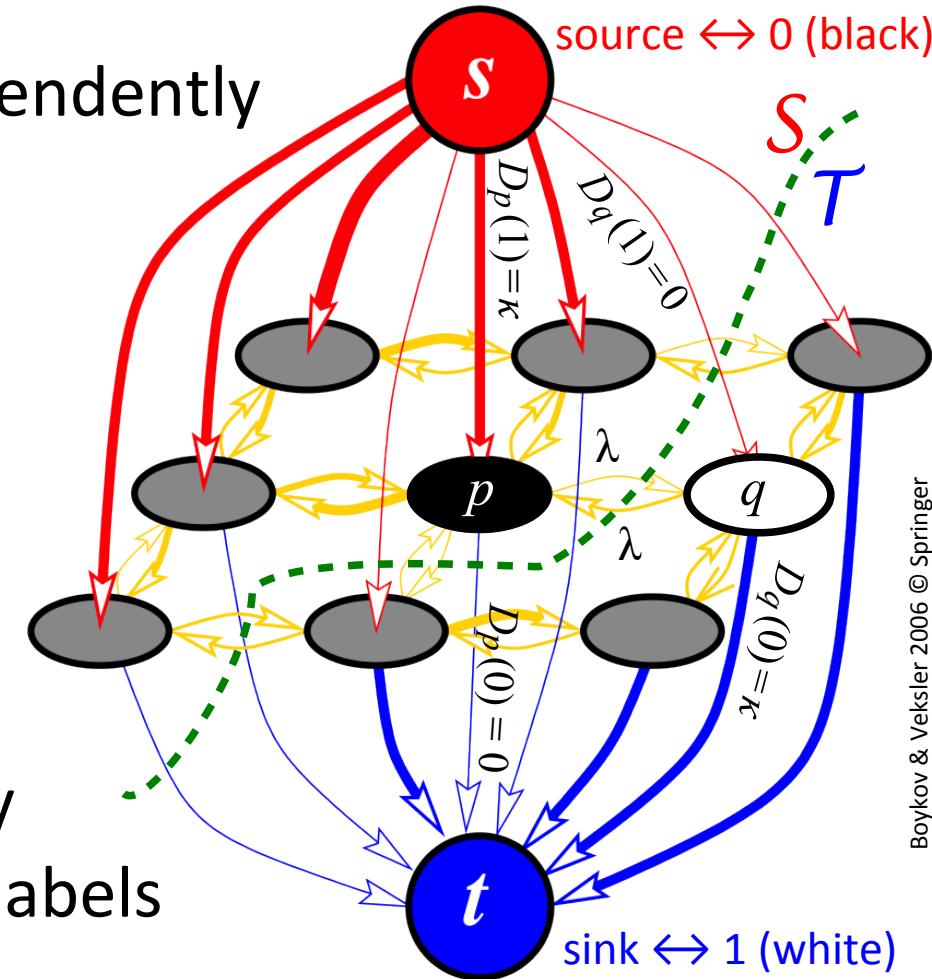


- pixels chose the label with smaller average cost

- Balanced λ value:



- pixels form compact, spatially coherent clusters with same labels
- noise/outliers conform to neighbors

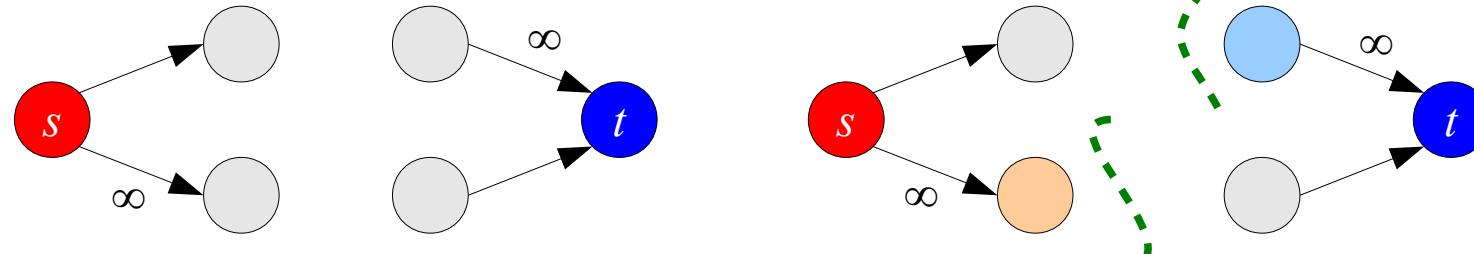


Graph cuts for energy minimization

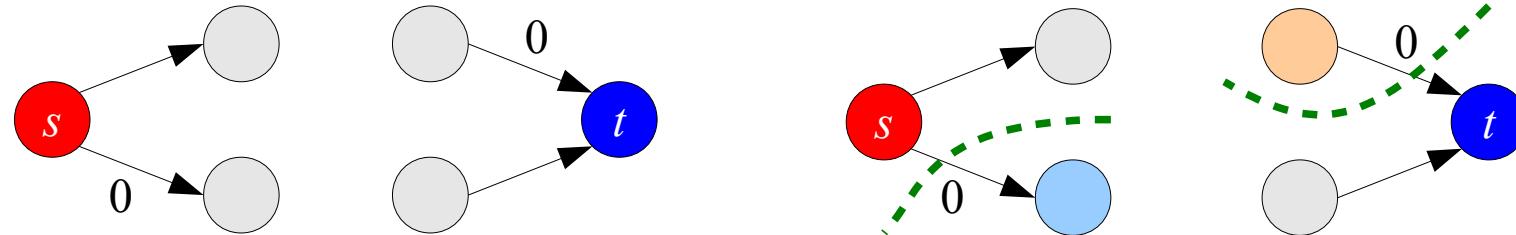
- Given some energy $E(f)$ such that
 - $f: \mathcal{P} \rightarrow \mathcal{L} = \{0,1\}$ binary labeling
 - $E(f) = \underbrace{\sum_{p \in \mathcal{P}} D_p(f_p)}_{E_{\text{data}}(f)} + \underbrace{\sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)}_{E_{\text{regul}}(f)}$
 - regularity condition (see below)
 - $V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0)$
- Theorem: then there is a graph whose minimum cut defines a labeling f that reaches the minimum energy
(Kolmogorov & Zabih 2004)
 - [N.B. Vladimir Kolmogorov, not Andrey Kolmogorov]
 - [structure of graph somehow similar to above form]

Graph construction

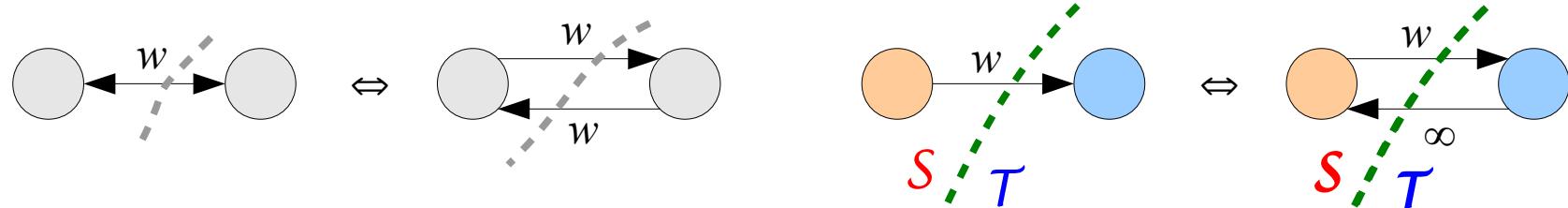
- Preventing a t-link cut: “infinite” weight



- Favoring a t-link cut: null weight (\approx no edge)



- Bidirectional edge vs monodirectional & back edges

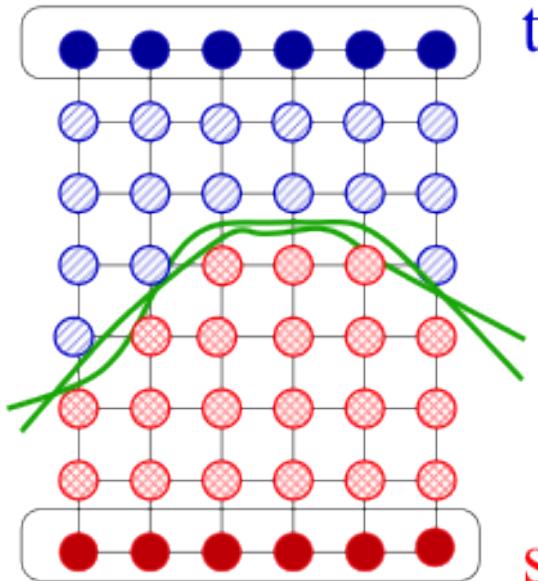


To go further on this subject

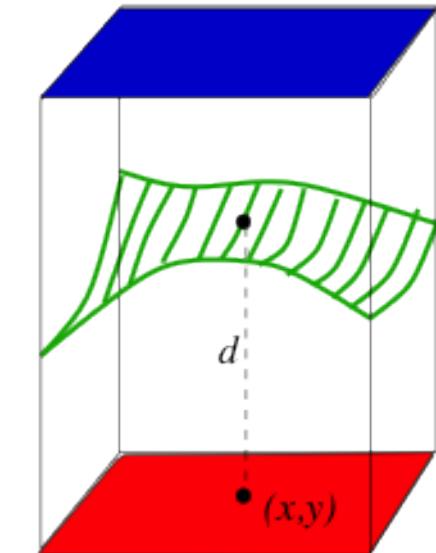
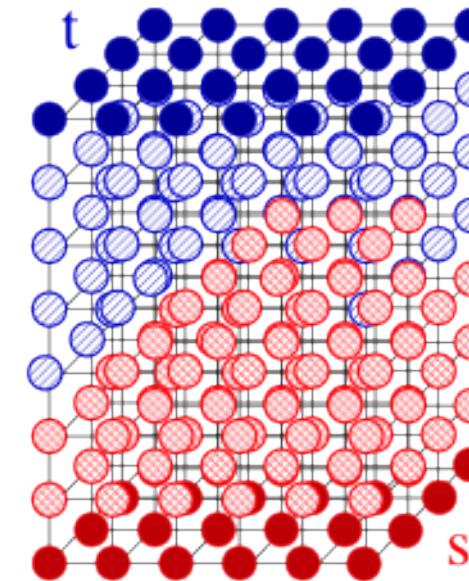
Graph cuts as hypersurfaces

(cf. Boykov & Veksler 2006)

- Cut on a 2D grid



- Cut on a 3D grid



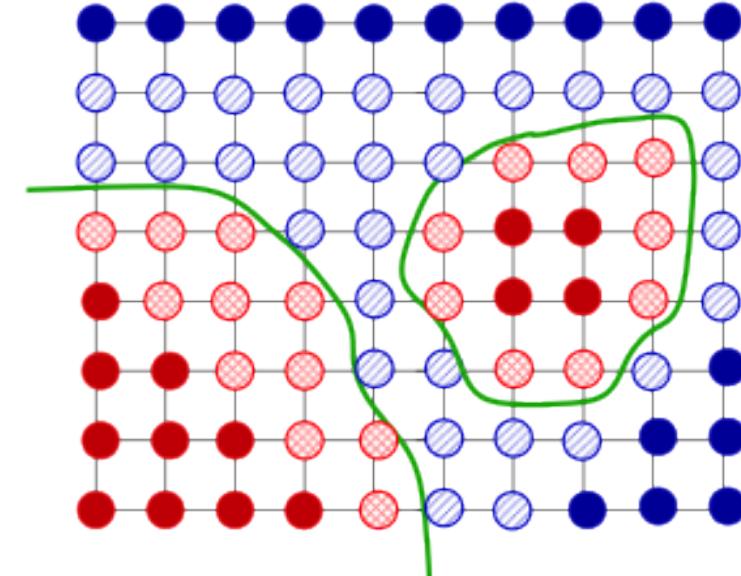
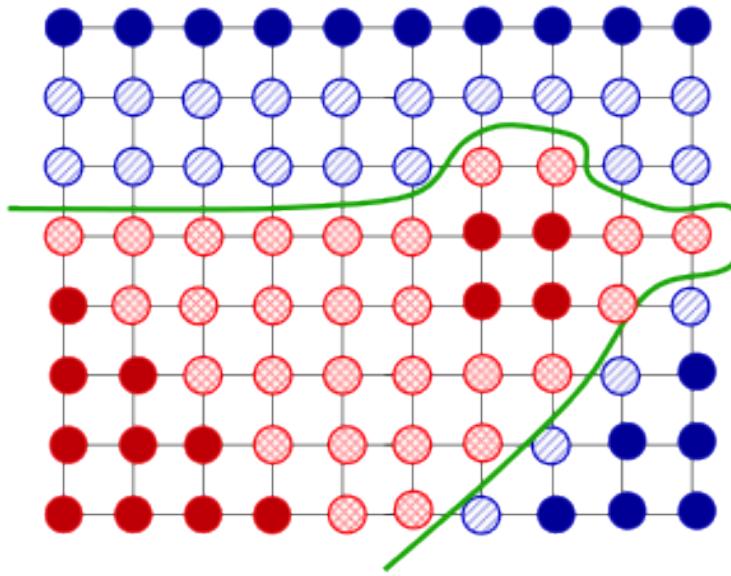
N.B. Several “seeds” (sources and sinks)

To go further on this subject

Example of topological issue

seed = graine

- Connected seeds
- Disconnected seeds

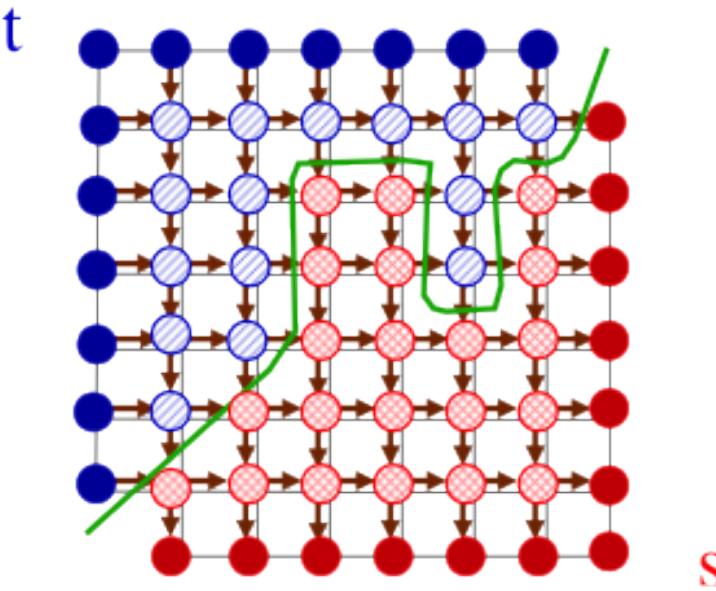
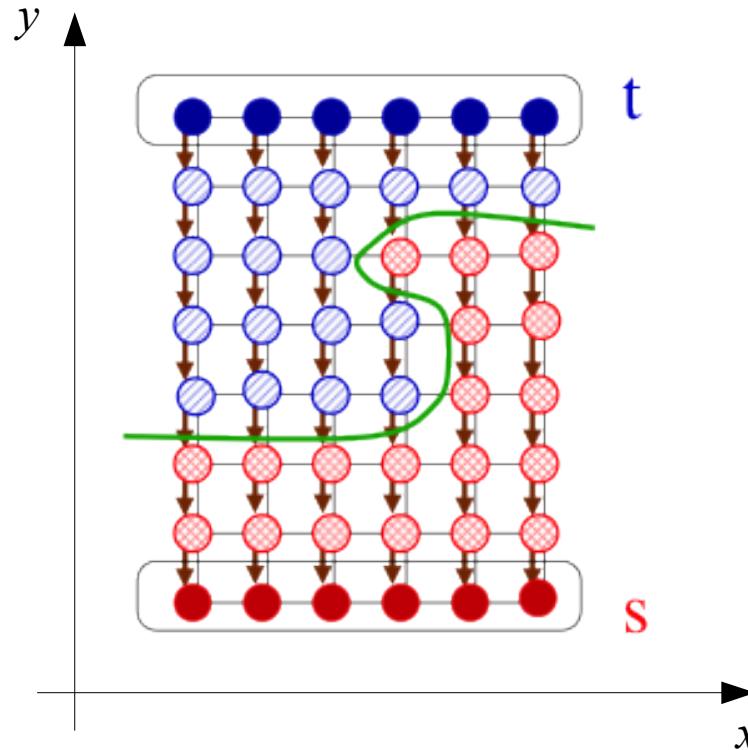


To go further on this subject

Example of topological constraint: fold prevention

disparity map = carte de disparité

- Ex. in disparity map estimation: $d = f(x,y)$
- In 2D: $y = f(x)$, only one value for y given one x



A “revolution” in optimization

simulated annealing = recuit simulé

- Previously (before Greig et al. 1989)
 - exact optimization like this was not possible
 - used approaches:
 - iterative algorithms such as simulated annealing
 - very far from global optimum, even in binary case like this
 - work of Greig et al. was (primarily) meant to show this fact
- Remained unnoticed for almost 10 years in the computer vision community...
 - maybe binary image restoration was viewed as too restrictive ?
(Boykov & Veksler 2006)

Graph cut techniques: now very popular in computer vision

- Extensive work since 1998
 - Boykov, Geiger, Ishikawa, Kolmogorov, Veksler, Zabih and others...
- Almost linear in practice (in nb nodes/edges)
 - but beware of the graph size :
it can be exponential in the size of the problem
- Many applications
 - regularization, smoothing, restoration
 - segmentation
 - stereovision: disparity map estimation, ...

Warning: global optimum \neq best real-life solution

- Graph cuts provide exact, global optimum
 - to binary labeling problems (under regularity condition)
 - But the problem remains a model
 - approximation of reality
 - limited number of factors
 - parameters (e.g., λ)
- 👉 Global optimum of **abstracted problem**,
not necessarily best solution **in real life**

Not for free

- Many papers construct
 - their own graph
 - for their own specific energy function
 - The construction can be fairly complex
- 👉 Powerful tool but does not exempt from thinking
(contrary to some aspects
of deep learning ☺)



Graph cut vs deep learning

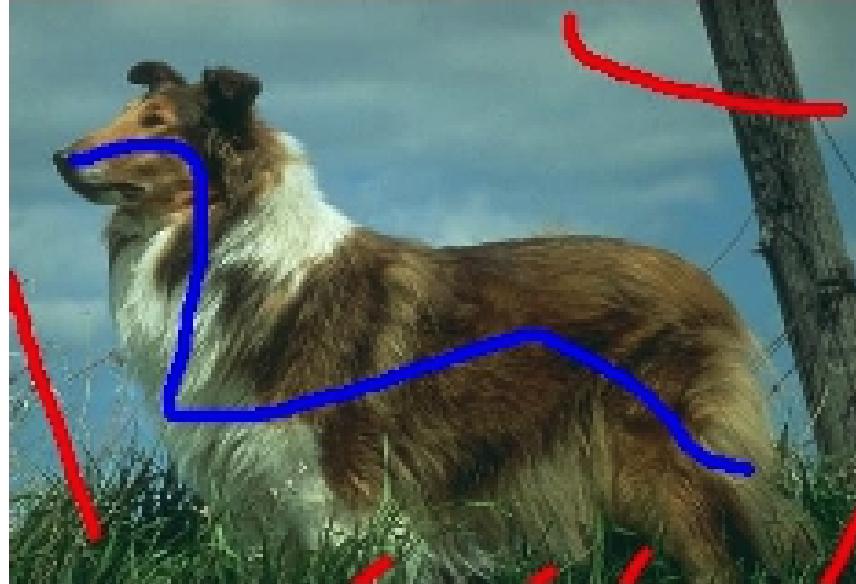
- Graph cut
 - works well, with proven optimality bounds
- Deep learning
 - works extremely well, but mainly empirical
- Somewhat complementary
 - graph cut sometimes used to regularize network output

Application to image segmentation

- Problem:

- given an image with foreground objects and background
- given sample areas of both kinds
- separate objects from background

background = arrière-plan
sample = échantillon
area = zone

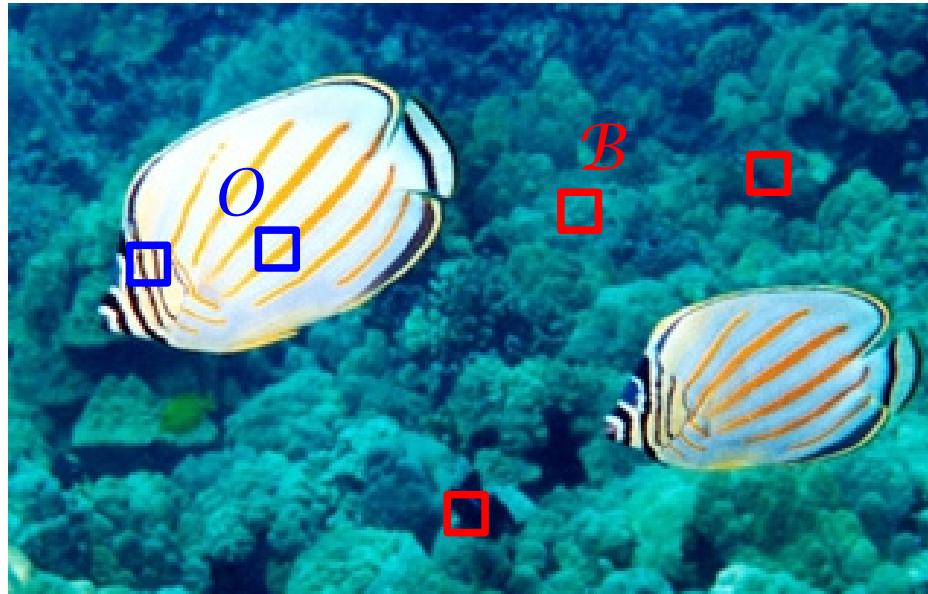


Application to image segmentation

- Problem:

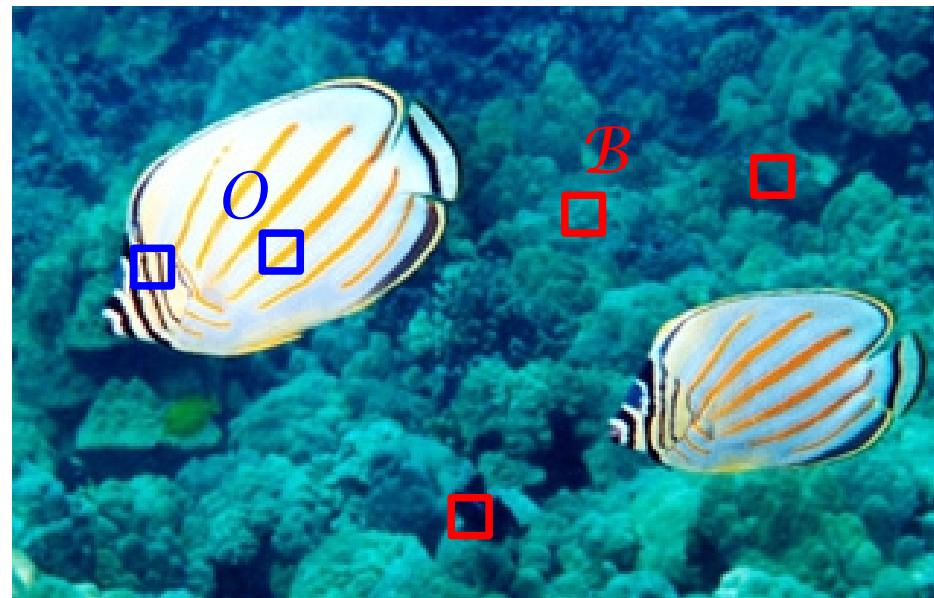
- given an image with foreground objects and background
- given sample areas of both kinds (O, \mathcal{B})
- separate objects from background

background = arrière-plan
sample = échantillon
area = zone



Intuition

What characterizes an object/background segmentation ?

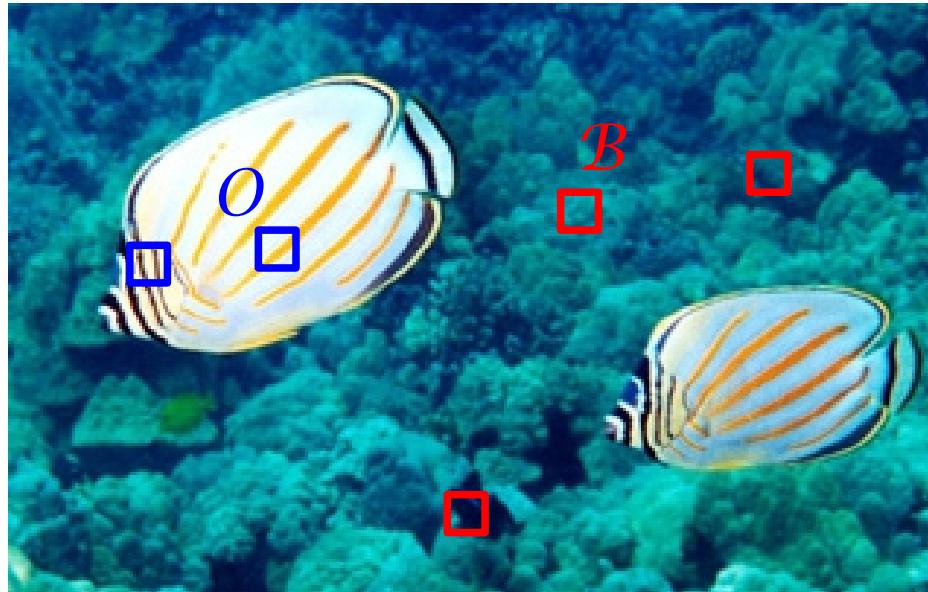


Intuition

background = arrière-plan
sample = échantillon
area = zone

What characterizes an object/background segmentation ?

- pixels of segmented object and background look like corresponding sample pixels O and \mathcal{B}
- segment contours have high gradient, and are not too long



General formulation

[Boykov & Jolly 2001]

- Pixel labeling with binary decision $f_p \in \mathcal{L} = \{0,1\}$
 - 1 = object, 0 = background
- Energy formulation
 - minimize $E(f) = D(f) + \lambda R(f)$
 - $D(f)$: **data term** (a.k.a. data fidelity term) = regional term
 - penalty for assigning labels f in image I given pixel sample assignments in \mathcal{L} : O (object pixels), B (background pixels)
 - $R(f)$: **regularization term** = boundary term
 - penalty for label discontinuity of neighboring pixels
 - λ : relative importance of regularization term vs data term

data term = terme d'attache aux données
 regularization term = terme de régularisation
 a.k.a. = also known as
 penalty = pénalité, coût
 to assign = affecter (une valeur à qq chose)
 sample = échantillon
 background =
 boundary = frontière
 neighboring pixel = pixel voisin

To go further on this subject

Probabilistic justification/framework

posterior probability =
 probabilité a posteriori
 likelihood =
 vraisemblance
 (log-)likelihood =
 (log-)vraisemblance

- Minimize $E(f) \leftrightarrow$ maximize posterior proba. $\Pr(f|I)$
- Bayes theorem:

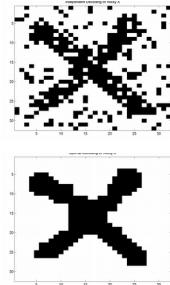
$$\Pr(f|I) \Pr(I) = \Pr(I|f) \Pr(f)$$

The term want to maximize w.r.t. f

A constant (independent of f)

\leftrightarrow data term, probability to observe image I knowing labeling f

\leftrightarrow regularization term, depending on type of labeling and with various hypotheses (e.g., locality, cf. MRF below)



- Consider likelihoods $L(f|I) = \Pr(I|f)$
- Actually consider log-likelihoods (\rightarrow sums)

$$E(f) = D(f) + \lambda R(f) \Leftrightarrow -\log \Pr(f|I) + c = -\log \Pr(I|f) - \log \Pr(f)$$

To go further on this subject

Data term:

linking estimated labels to observed pixels

- $D(f)$ and likelihood

- penalty for assigning labels f in I given sample assignments
 \Leftrightarrow (log-)likelihood that f is consistent with image samples
- $D(f) = -\log L(f|I) = -\log \Pr(I|f)$

- Pixel independence hypothesis (common approximation)

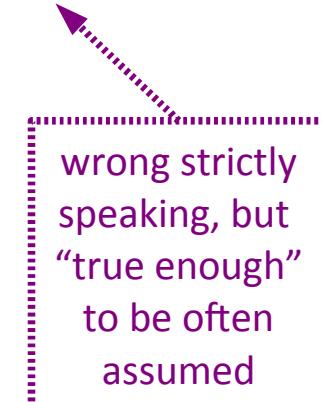
- $\Pr(I|f) = \prod_{p \in \mathcal{P}} \Pr(I_p|f_p)$ if pixels iid (independent and identically distributed random variables)

- $D(f) = \sum_{p \in \mathcal{P}} D_p(f_p)$ where $D_p(f_p) = -\log \Pr(I_p|f_p)$

- $D_p(f_p)$: penalty for observing I_p for a pixel of type f_p

- 👉 Find an estimate of $\Pr(I_p|f_p)$

penalty = pénalité, coût
to assign = affecter
sample = échantillon
likelihood = vraisemblance
random variable = variable aléatoire



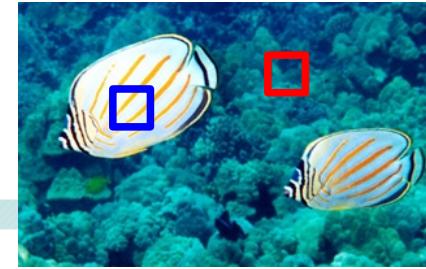
wrong strictly speaking, but
“true enough”
to be often assumed

To go further on this subject

Data term:

likelihood/color model

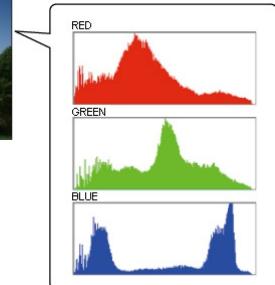
empirical probability =
probabilité empirique
(fréquence relative)
Gaussian mixture =
mélange de gaussiennes



- Approaches to find an estimate of $\Pr(I_p | f_p)$

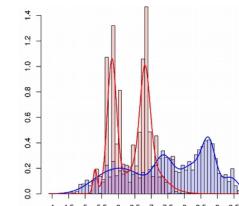
- histograms

- build an empirical distribution of the color of object/background pixels, based on pixels marked as object/background
- estimate $\Pr(I_p | f_p)$ based on histograms: $\Pr_{\text{emp}}(rgb|O), \Pr_{\text{emp}}(rgb|B)$



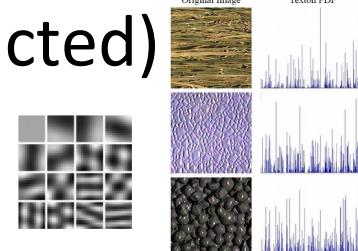
- Gaussian Mixture Model (GMM)

- model the color of object (resp. background) pixels with a distribution defined as a mixture of Gaussians



- texon (or texton): texture patch (possibly abstracted)

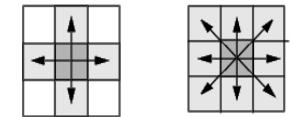
- compare with expected texture property: response to filters (spectral analysis), moments...



To go further on this subject

Regularization term: locality hypotheses

- Markov random field (MRF), or Markov network
 - neighborhood system: $\mathcal{N} = \{\mathcal{N}_p \mid p \in \mathcal{P}\}$
 - \mathcal{N}_p : set neighbors of p such that $p \notin \mathcal{N}_p$ and $p \in \mathcal{N}_q \Leftrightarrow q \in \mathcal{N}_p$
 - $X = (X_p)_{p \in \mathcal{P}}$: field (set) of random variables such that each random variable X_p depends on other random variables only through its neighbors \mathcal{N}_p
 - ☛ **locality hypothesis:** $\Pr(X_p = x \mid X_{\mathcal{P} \setminus \{p\}}) = \Pr(X_p = x \mid X_{\mathcal{N}_p})$
 - $\mathcal{N} \approx$ undirected graph: (p,q) edge iff $p \in \mathcal{N}_q$ ($\Leftrightarrow q \in \mathcal{N}_p$)
(MRF also called undirected graphical model)



Markov random field =
 champ de Markov
 random variable =
 variable aléatoire
 neighborhood = voisinage
 undirected graph =
 graph non orienté
 graphical model =
 modèle graphique

To go further on this subject

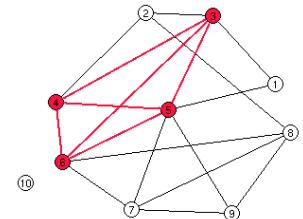
Regularization term: locality hypotheses

Gibbs random field = champ de Gibbs
 undirected graph = graph non orienté
 clique = clique (!)
 clique potential = potentiel de clique
 prior probability = probabilité a posteriori

- Gibbs random field (GRF)
 - \mathcal{G} undirected graph, $X = (X_p)_{p \in \mathcal{P}}$ random variables such that

$$\Pr(X=x) \propto \exp\left(-\sum_{C \text{ clique of } \mathcal{G}} V_C(x)\right)$$
 - clique = complete subgraph: $\forall p \neq q \in C \quad (p,q) \in \mathcal{G}$
 - V_C : clique potential = prior probability of the given realization of the elements of the clique C (fully connected subgraph)
- Hammersley-Clifford theorem (1971)
 - If probability distribution has positive mass/density, i.e., if $\Pr(X=x) > 0$ for all x , then:

$$X \text{ MRF w.r.t. graph } \mathcal{N} \text{ iff } X \text{ GRF w.r.t. graph } \mathcal{N}$$
 - ➔ provides a characterization of MRFs as GRFs



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To go further on this subject

Regularization term: locality hypotheses

[Boykov, Veksler & Zabih 1998]

- Hypothesis 1: only 2nd-order cliques (i.e., edges)

$$\begin{aligned} R(f) &= -\log \Pr(f) = -\log \exp(-\sum_{(p,q) \text{ edge of } G} V_{(p,q)}(f)) \quad [\text{GRF}] \\ &= \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q) \quad [\text{MRF pairwise potentials}] \end{aligned}$$

- Hypothesis 2: (generalized) Potts model

$$V_{p,q}(f_p, f_q) = B_{p,q} \mathbf{1}(f_p \neq f_q)$$

$$\begin{aligned} \text{i.e., } V_{p,q}(f_p, f_q) &= 0 && \text{si } f_p = f_q \\ V_{p,q}(f_p, f_q) &= B_{p,q} && \text{si } f_p \neq f_q \end{aligned}$$

pairwise = par paire
 pairwise potential = potentiel d'ordre 2
 Potts model = modèle de Potts
 statistical mechanics = physique statistique

(Origin: statistical mechanics

- spin interaction in crystalline lattice
- link with “energy” terminology)

To go further on this subject

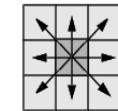
Examples of boundary penalties (ad hoc)

- Penalize label discontinuity at intensity continuity

[Boykov & Jolly 2001]

- $B_{p,q} = \exp(-(I_p - I_q)^2/2\sigma^2) / \text{dist}(p,q)$

- large between pixels of similar intensities, i.e., when $|I_p - I_q| < \sigma$
- small between pixels of dissimilar intensities, i.e., when $|I_p - I_q| > \sigma$
- decrease with pixel distance $\text{dist}(p,q)$ [here: 1 or $\sqrt{2}$]
- \approx distribution of noise among neighboring pixels



- Penalize label discontinuity at low gradient

- $B_{p,q} = g(\|\nabla I_p\|)$ with g positive decreasing

- e.g., $g(x) = 1/(1 + c x^2)$
- penalization for label discontinuity at low gradient

To go further on this subject

Wrapping up

- Pixel labeling with binary decision $f_p \in \{0,1\}$
 - 0 = background, 1 = object
- Energy formulation
 - minimize $E(f) = D(f) + \lambda R(f)$
 - data term: $D(f) = \sum_{p \in \mathcal{P}} D_p(f_p)$
 - $D_p(f_p)$: penalty for assigning label f_p to pixel p given its color/textture
 - regularization term: $R(f) = \sum_{(p,q) \in \mathcal{N}} B_{p,q} \mathbf{1}(f_p \neq f_q)$
 - $B_{p,q}$: penalty for label discontinuity between neighbor pixels p, q
 - λ : relative importance of regularization term vs data term

Graph-cut formulation (version 1)

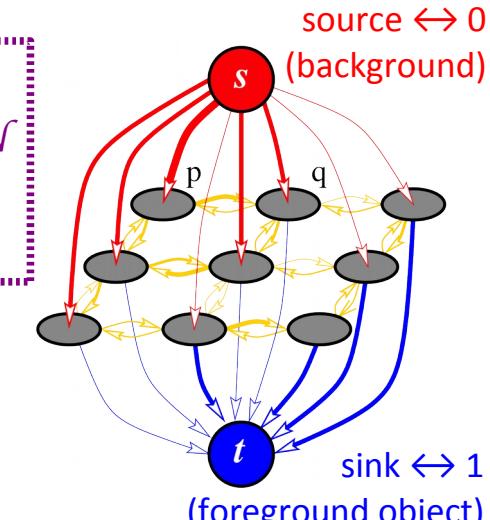
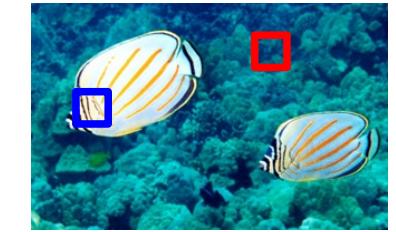
- Direct expression as graph-cut problem:

$$- \quad \mathcal{V} = \{s, t\} \cup \mathcal{P}$$

$$- \quad \mathcal{E} = \{(s, p) \mid p \in \mathcal{P}\} \cup \{(p, q) \mid p, q \in \mathcal{N}\} \cup \{(p, t) \mid p \in \mathcal{P}\}$$

Edge	Weight	Sites
(p, q)	$\lambda B_{p,q}$	$(p, q) \in \mathcal{N}$
(s, p)	$D_p(1)$	$p \in \mathcal{P}$
(p, t)	$D_p(0)$	$p \in \mathcal{P}$

Warning:
the connexity \mathcal{N}
is specific to
the problem !



$$- \quad E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \lambda \sum_{(p, q) \in \mathcal{N}} B_{p,q} \mathbf{1}(f_p \neq f_q)$$

- ex. $D_p(l) = -\log \Pr_{\text{emp}}(I_p \mid f_p = l)$ [empirical probability for O et \mathcal{B}]
- ex. $B_{p,q} = \exp(-(I_p - I_q)^2 / 2\sigma^2) / \text{dist}(p, q)$

Graph-cut formulation (version 1)

- Direct expression as graph-cut problem:

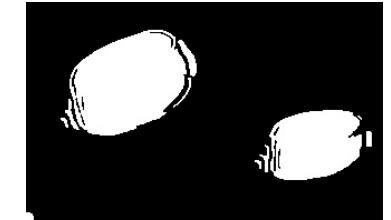
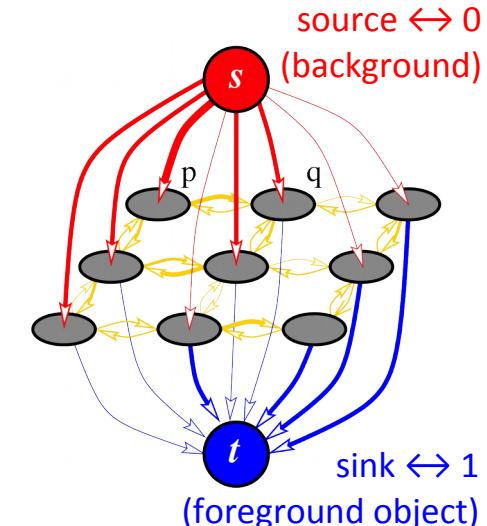
$$- \quad \mathcal{V} = \{s, t\} \cup \mathcal{P}$$

$$- \quad \mathcal{E} = \{(s, p) \mid p \in \mathcal{P}\} \cup \{(p, q) \mid p, q \in \mathcal{N}\} \cup \{(p, t) \mid p \in \mathcal{P}\}$$

Edge	Weight	Sites
(p, q)	$\lambda B_{p,q}$	$(p, q) \in \mathcal{N}$
(s, p)	$D_p(1)$	$p \in \mathcal{P}$
(p, t)	$D_p(0)$	$p \in \mathcal{P}$

$$- \quad E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \lambda \sum_{(p, q) \in \mathcal{N}} B_{p, q} \mathbf{1}(f_p \neq f_q)$$

- Any problem/risk with this formulation ?



Graph-cut formulation (version 1)

- Direct expression as graph-cut problem:

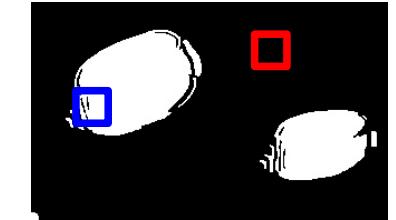
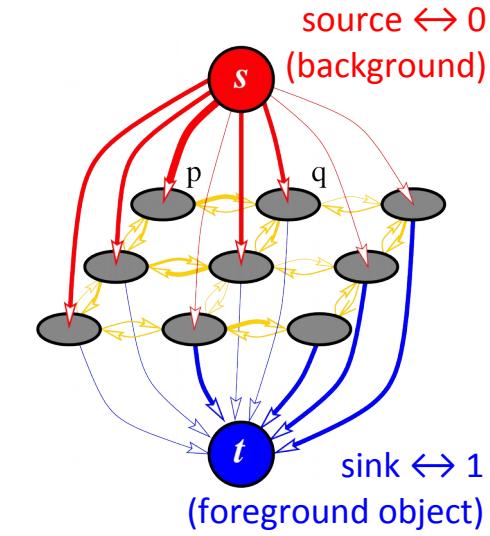
$$- \quad \mathcal{V} = \{s, t\} \cup \mathcal{P}$$

$$- \quad \mathcal{E} = \{(s, p) \mid p \in \mathcal{P}\} \cup \{(p, q) \mid p, q \in \mathcal{N}\} \cup \{(p, t) \mid p \in \mathcal{P}\}$$

Edge	Weight	Sites
(p, q)	$\lambda B_{p,q}$	$(p, q) \in \mathcal{N}$
(s, p)	$D_p(1)$	$p \in \mathcal{P}$
(p, t)	$D_p(0)$	$p \in \mathcal{P}$

$$- \quad E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \lambda \sum_{(p, q) \in \mathcal{N}} B_{p, q} \mathbf{1}(f_p \neq f_q)$$

- Pb: pixels of object/background samples not necessarily assigned with good label !

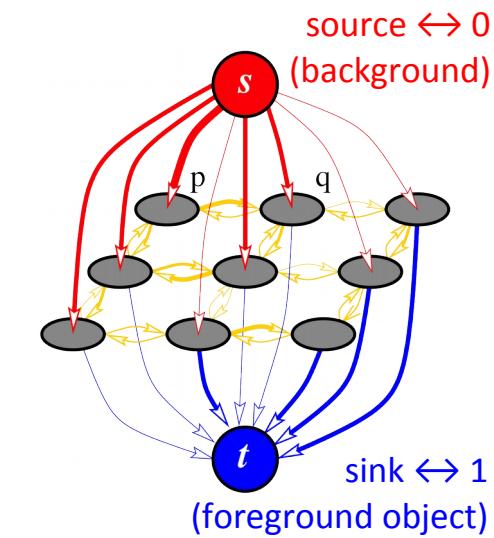


Graph-cut formulation (version 2)

[Boykov & Jolly 2001]

- Obj/Bg samples now always labeled OK in minimal f^*

Edge	Weight	Sites
(p,q)	$\lambda B_{p,q}$	$(p,q) \in \mathcal{N}$
(s,p)	$D_p(1)$	$p \in \mathcal{P}, p \notin (\mathcal{O} \cup \mathcal{B})$
	K	$p \in \mathcal{B}$
	0	$p \in \mathcal{O}$
(p,t)	$D_p(0)$	$p \in \mathcal{P}, p \notin (\mathcal{O} \cup \mathcal{B})$
	0	$p \in \mathcal{B}$
	K	$p \in \mathcal{O}$



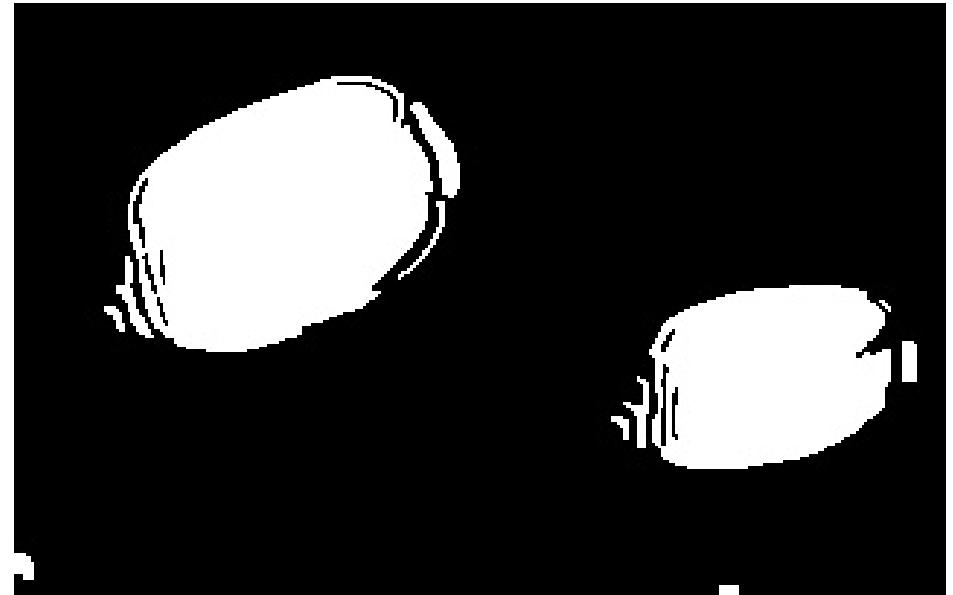
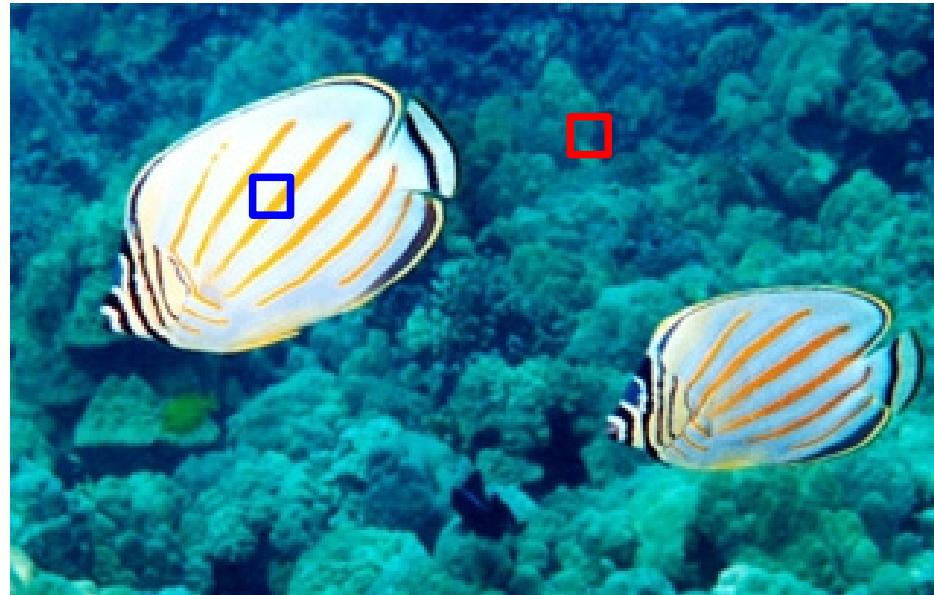
- where $K = 1 + \max_{p \in \mathcal{P}} \lambda \sum_{(p,q) \in \mathcal{N}} B_{p,q}$
 $K \approx +\infty$, i.e., too expensive to pay \Rightarrow label never assigned

To go further on this subject

Some limitations

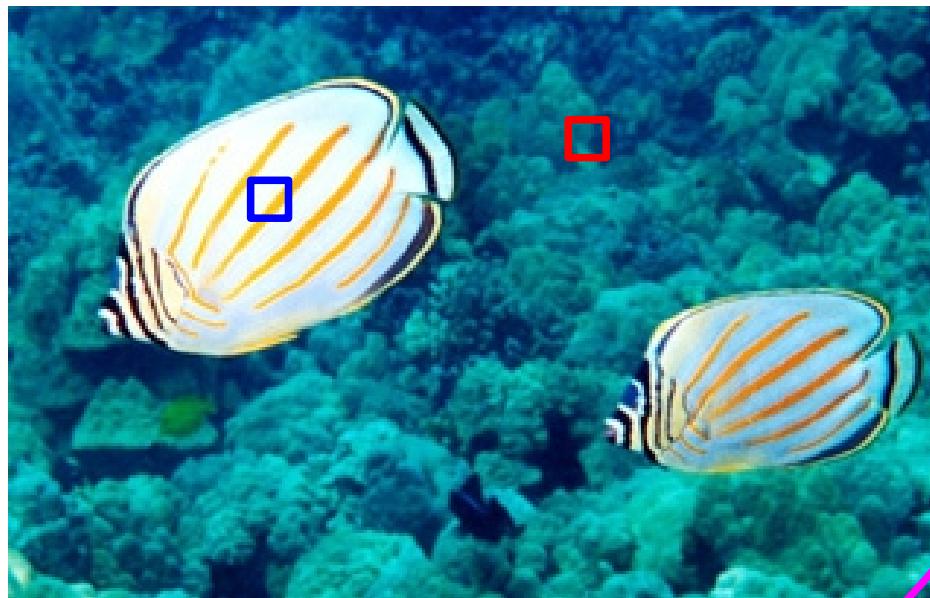
(here with simple color model)

- Is the segmentation OK ?



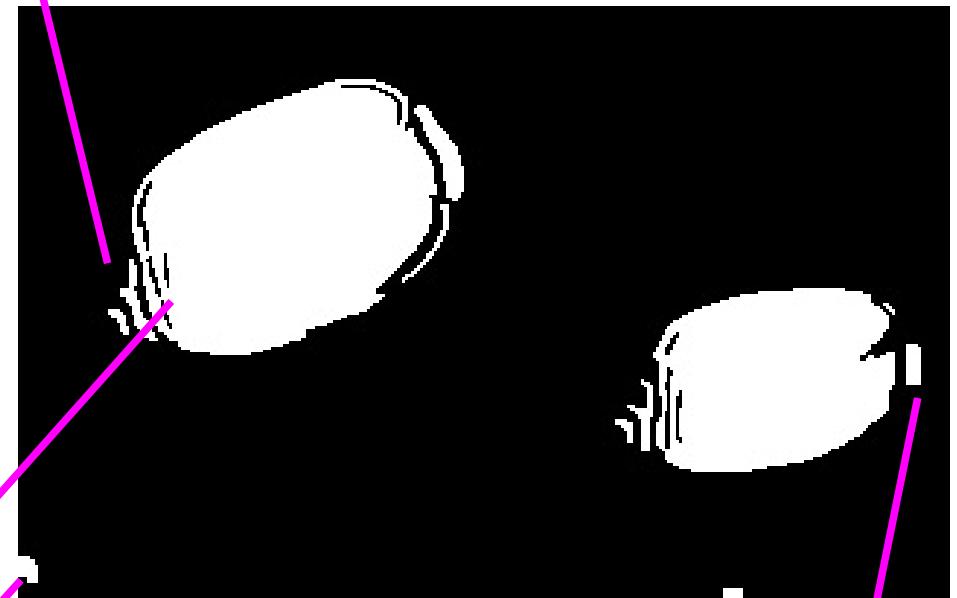
To go further on this subject

Some limitations (here with simple color model)



color model
not complex
enough

sensitivity to
regularization
parameter



neighboring
model
not complex
enough

To go further on this subject

Exercise 1: implement simple image segmentation using graph cuts

- Load image and display it
- Get object/background samples with mouse clicks (e.g., small square area around mouse position) and draw them
- Define graph as in course (version 2)
 - choose neighbor connectivity [4:  or 8: ]
 - choose one of $B_{p,q}$ expressions defined in course
 - model color of object/background with 1 single gaussian centered around mean color
 - write $\Pr(I_p | f_p)$ w.r.t. mean color of obj/bg: what is $D_p(f_p)$ then ?
- Compute max flow, extract cut, display image segments
- Experiment with various parameters ($\lambda, \sigma, c, \dots$)
- And **comment what you observe**

Part 2

Multi-label problems

Exact vs approximate solutions

Application to stereovision
(disparity/depth map estimation):
disparity/depth \leftrightarrow label

Two-label (binary) problem

- \mathcal{P} : set of sites (pixels, voxels...)
- \mathcal{N} : set of neighboring site pairs
- $\mathcal{L} = \{0,1\}$: binary labels
- $f: \mathcal{P} \rightarrow \mathcal{L}$ binary labeling [notation: $f_p = f(p) = l$]
- $E: (\mathcal{P} \rightarrow \mathcal{L}) \rightarrow \mathbb{R}$: energy
 - $$E(f) = \underbrace{\sum_{p \in \mathcal{P}} D_p(f_p)}_{E_{\text{data}}(f)} + \underbrace{\sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)}_{E_{\text{regul}}(f)}$$
 - $D_p(l)$: label penalty for site p
 - $V_{p,q}(l, l')$: prior knowledge about optimal pairwise labeling
- Pb: find f^* that reaches the minimum energy $E(f^*)$

Two-label problem assumptions

- $E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$
- $D_p(l)$: label penalty for site p
 - small/null for preferred label, large for undesired label
 - assumption $D_p(l) \geq 0$ (else add constant \rightarrow same optimum)
- $V_{p,q}(l, l')$: prior knowledge on optimal pairwise labeling
 - in general, smoothness: non-decreasing function of $\mathbf{1}(l \neq l')$
 - e.g., $V_{p,q}(l, l') = u_{p,q} \mathbf{1}(l \neq l')$ [Potts model]
- Regularity condition, required for min-cut ($\Rightarrow c(p, q) \geq 0$)
 - $V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0)$ [see below]

Multi-label problem

disparity = disparité

- \mathcal{P} : set of sites (pixels, voxels...)
- \mathcal{N} : set of neighboring site pairs
- \mathcal{L} : finite set of labels (\rightarrow can model scalar or even vector)
 - e.g., **discretization** of intensity, **stereo disparity**, motion vector...
- $f: \mathcal{P} \rightarrow \mathcal{L}$ labeling
- $E: (\mathcal{P} \rightarrow \mathcal{L}) \rightarrow \mathbb{R}$: energy
 - $E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q) = E_{\text{data}}(f) + E_{\text{regul}}(f)$
 - $D_p(l)$: label penalty for site p
 - $V_{p,q}(l_p, l_q)$: prior knowledge about optimal pairwise labeling
- Pb: find f^* that reaches the minimum energy $E(f^*)$

Multi-label problem assumptions

- $E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$

smoothness = lissage
- $D_p(l)$: label penalty for site p
 - small for preferred label, large for undesired label
 - assumption $D_p(l) \geq 0$ (else add constant \rightarrow same optimum)
- $V_{p,q}(l_p, l_q)$: prior knowledge on optimal pairwise labeling
 - in general, smoothness prior:
non-decreasing function of $\|l_p - l_q\|$ [norm used if vector]
 - e.g., $V_{p,q}(l_p, l_q) = \lambda_{p,q} \|l_p - l_q\|$
 - smaller penalty for closer labels

Graph cuts for “general” energy minimization

- Problem: find labeling $f^* : \mathcal{P} \rightarrow \mathcal{L}$ minimizing energy

$$E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$$

- Question: can a **globally optimal** labeling f^* be found using some graph-cut construction?
- Answer:

- binary labeling: yes iff $V_{p,q}$ is regular (Kolmogorov & Zabih 2004)

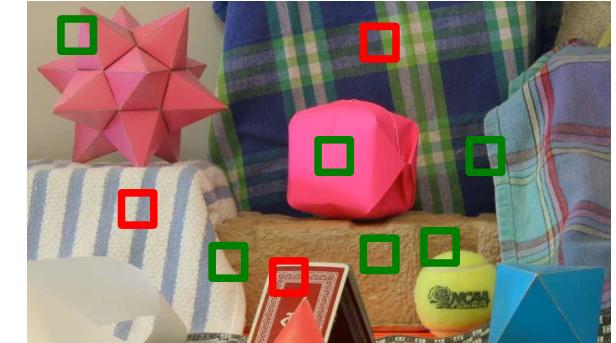
$$V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0) \quad \text{[otherwise it's NP-hard]}$$

- multi-labeling: yes if $V_{p,q}$ convex (Ishikawa 2003)
and if \mathcal{L} linearly ordered (\Rightarrow 1D only \Rightarrow not 2D motion vector)
 - otherwise: approximate solutions (but some very good)

Piecewise-smooth vs everywhere-smooth

piecewise = par morceaux

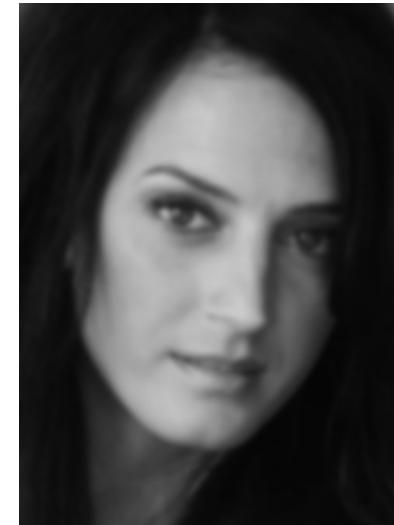
- Observation: object properties often smooth everywhere except on boundaries



- ➔ Consequence: piecewise-smooth models more appropriate than everywhere-smooth models



original



uniform smoothing

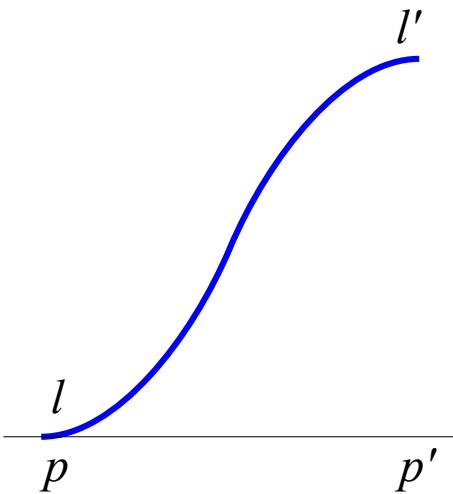


piecewise smoothing

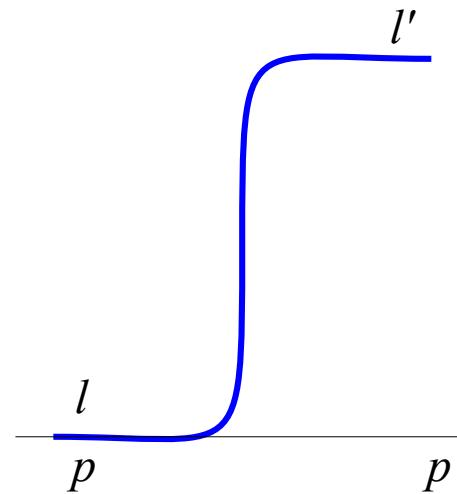
Piecewise-smooth models vs everywhere-smooth models

steep = raide, très pentu

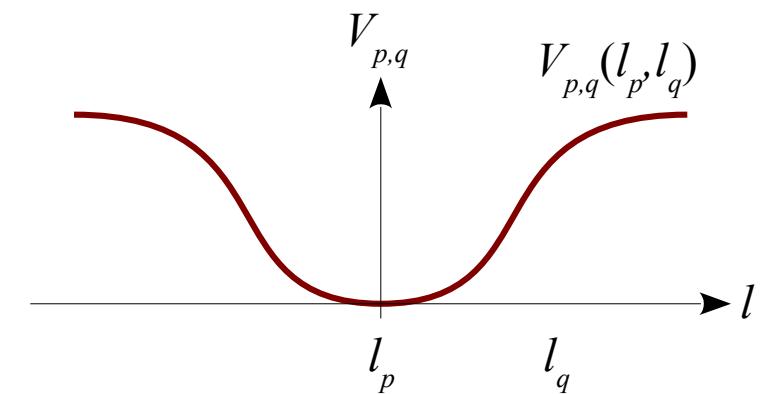
- Local variation of potentials $V_{p,q}$ depending on label variation



locally smooth from l to l'
when going from p to p'



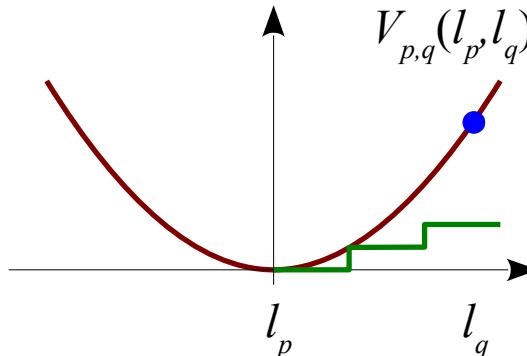
locally steep from l to l'
when going from p to p'



piecewise-smooth potential

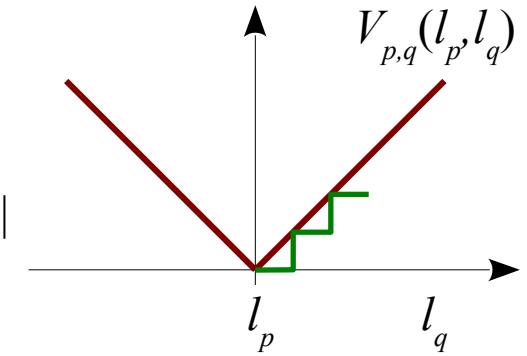
Piecewise-smooth potentials vs everywhere-smooth potentials

- General graph construction for any convex $V_{p,q}$ (Ishikawa 2003)
 - convex \Rightarrow large penalty for sharp jump
 - a few small jumps cheaper than one large jump
 - 👉 discontinuities smoothed with “ramp” \Rightarrow oversmoothing



In practice,
best results with
“least convex”
function, e.g.,

$$V_{p,q}(l_p, l_q) = \lambda_{p,q} \|l_p - l_q\|$$



Discontinuity-preserving energy

- At boundaries, very different labels for adjacent pixels are OK
- To not overpenalize in E adjacent but very different labels:

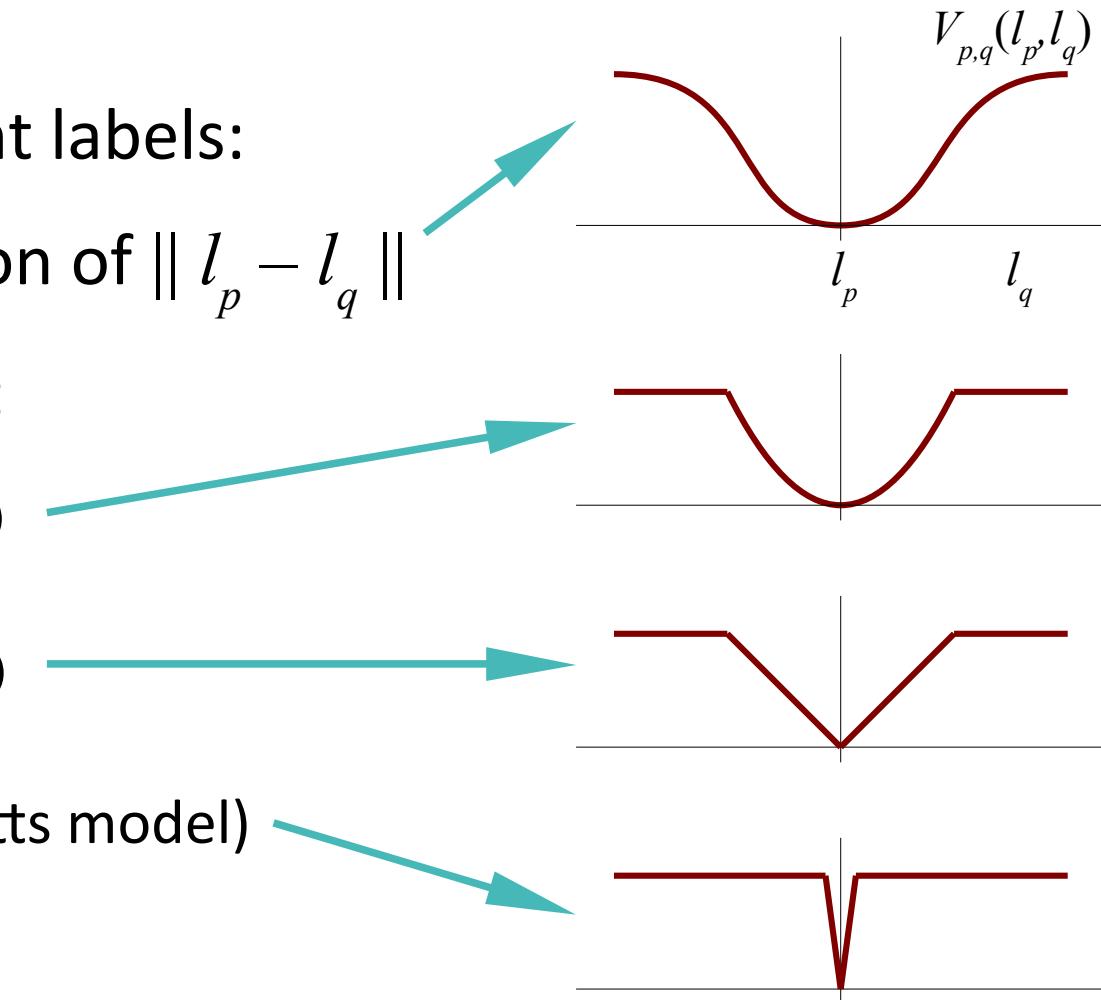
– $V_{p,q}$ non-convex function of $\| l_p - l_q \|$

– for instance (cap max):

$$\bullet \quad V_{p,q} = \min(K, \| l_p - l_q \|^2)$$

$$\bullet \quad V_{p,q} = \min(K, \| l_p - l_q \|)$$

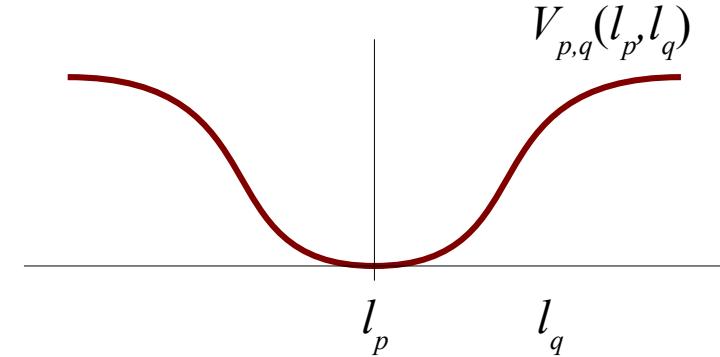
$$\bullet \quad V_{p,q} = u_{p,q} \mathbf{1}(l_p \neq l_q) \text{ (Potts model)}$$



Difficulty of minimization

simulated annealing = recuit simulé

- $E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$ with
 - $f: \mathcal{P} \rightarrow \mathcal{L}$
 - $V_{p,q}(f_p, f_q)$ non convex
- $\min_f E(f)$: minimization of non-convex function in large-dimension space (dimension = $|\mathcal{P}|$)
 - NP-hard even in simple cases
 - e.g. $V_{pq}(f_p, f_q) = \mathbf{1}(f_p \neq f_q)$ (Potts model) with $|\mathcal{L}| > 2$
 - general case: simulated annealing...



Exact binary optimization (reminder)

- Pb: find labeling $f^* : \mathcal{P} \rightarrow \mathcal{L} = \{0,1\}$ minimizing energy
$$E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$$
- Question:
 - can a **globally optimal** labeling f^* be found using some graph-cut construction?
- Answer (Kolmogorov & Zabih 2004):
 - yes iff V_{pq} is regular
 - $V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0)$
 - otherwise it's NP-hard
- But what about **general energies** on **binary variables** ?

Exact binary optimization

[Kolmogorov & Zabih 2004]

- Question:
 - what functions can be minimized using graph cuts?
- Classes of functions on binary variables:
 - \mathcal{F}^2 : $E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \sum_{i < j} E^{i,j}(x_i, x_j)$
 - \mathcal{F}^3 : $E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \sum_{i < j} E^{i,j}(x_i, x_j) + \sum_{i < j < k} E^{i,j,k}(x_i, x_j, x_k)$
 - \mathcal{F}^m : $E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \dots + \sum_{u_1 < \dots < u_m} E^{u_1, \dots, u_m}(x_{u_1}, \dots, x_{u_m})$
- “Using graph cuts”: E graph-representable iff
 \exists graph $G = \langle \mathcal{V}, \mathcal{E} \rangle$ with $\mathcal{V} = \{v_1, \dots, v_n, s, t\}$ such that
 \forall configuration $\mathbf{x} = x_1, \dots, x_n$, $E(x_1, \dots, x_n) = \text{cost}(\min s-t\text{-cut in which } v_i \in S \text{ if } x_i = 0 \text{ and } v_i \in T \text{ if } x_i = 1) + k \text{ constant } \in \mathbb{R}$

***m*-th order potentials**

To go further on this subject

Exact binary optimization

[Kolmogorov & Zabih 2004]

- E regular iff
 - \mathcal{F}^2 : $\forall i,j \ E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(0,1) + E^{i,j}(1,0)$
 - \mathcal{F}^m : for all terms E^{u_1, \dots, u_m} in E , all projections (specializations) of E^{u_1, \dots, u_m} to a two-variable function (i.e., all variables fixed but two) are regular
- Question:
 - what functions can be minimized using graph cuts?
- Answer (Kolmogorov & Zabih 2004):
 - $\mathcal{F}^2, \mathcal{F}^3$: E graph-representable $\Leftrightarrow E$ regular
 - any binary E : E not regular $\Rightarrow E$ not graph-representable

To go further on this subject

Link with submodularity

submodular = sous-modulaire

- $g : 2^{\mathcal{P}} \rightarrow \mathbb{R}$ submodular
 - iff $g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y)$
for any $X, Y \subset \mathcal{P}$
 - iff $g(X \cup \{j\}) - g(X) \geq g(X \cup \{i,j\}) - g(X \cup \{j\})$
for any $X \subset \mathcal{P}$ and $i, j \in \mathcal{P} \setminus X$
- g submodular $\Leftrightarrow E$ regular, with $E(\mathbf{x}) = g(\{p \in \mathcal{P} \mid x_p = 1\})$
 - $E^{i,j}(0,1) + E^{i,j}(1,0) \geq E^{i,j}(0,0) + E^{i,j}(1,1)$
- \exists independent results on submodular functions
 - minimization in polynomial time but slow, best known $O(n^6)$

Exact multi-label optimization (for 2nd-order potentials)

- Problem: find labeling $f^* : \mathcal{P} \rightarrow \mathcal{L}$ minimizing energy

$$E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} V_{p,q}(f_p, f_q)$$

- Assumption: \mathcal{L} linearly ordered — w.l.o.g. $\mathcal{L} = \{1, \dots, k\}$

(1D only \Rightarrow not suited, e.g., for 2D motion vector estimation)

- Solution: reduction/encoding to binary label case

- for $V_{p,q}(l_p, l_q) = \lambda_{p,q} |l_p - l_q|$ (Boykov et al. 1998, Ishikawa & Geiger 1998)
- for any convex $V_{p,q}$ (Ishikawa 2003)

- See also

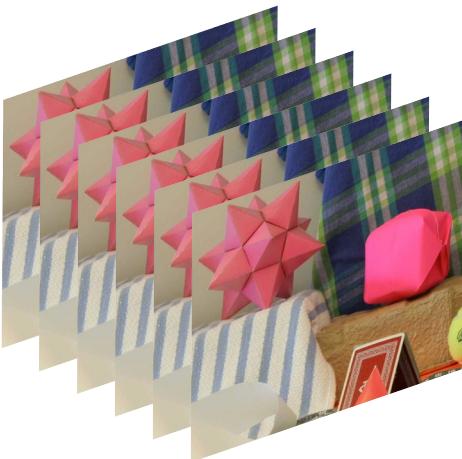
- MinSum pbs (Schlesinger & Flach 2006)
- submodular $V_{p,q}$ (Darbon 2009)

linearly ordered = linéairement ordonné
w.l.o.g. = without loss of generality

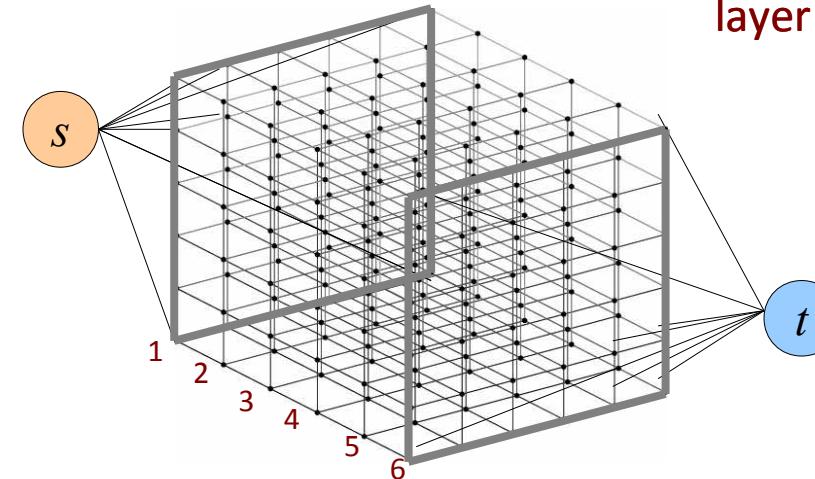
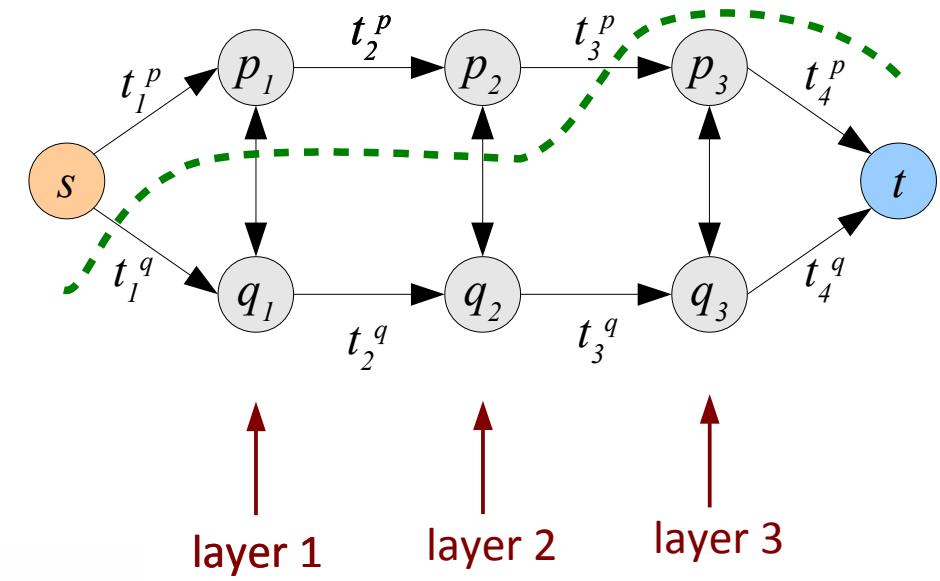
Linear multi-label graph construction

(cf. Boykov et al. 1998)

- Given $\mathcal{L} = \{1, \dots, k\}$
- General idea:
 - construct one layer per label value
 - read label value from cut location



e.g., $k = 4$ cut: $f_p = 3, f_q = 1$



layer = couche

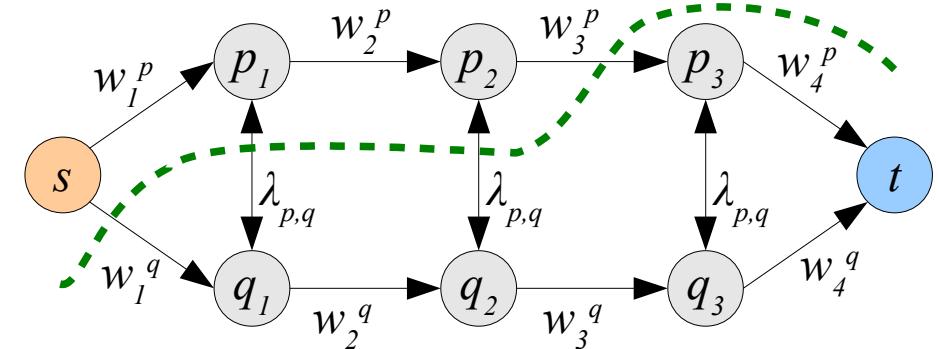
Linear multi-label graph construction

(cf. Boykov et al. 1998)

- $E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} \lambda_{p,q} |f_p - f_q|$ cut: $f_p = 3, f_q = 1$
with $f_p \in \mathcal{L} = \{1, \dots, k\}$

Attempt 1:

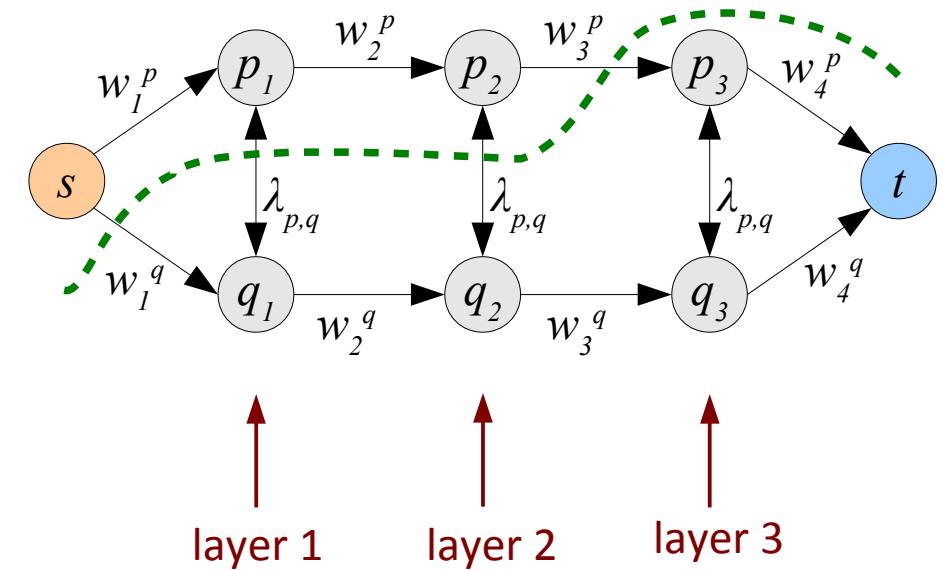
- For each site p
 - create nodes p_1, \dots, p_{k-1}
 - create edges $t_1^p = (s, p_1), t_j^p = (p_{j-1}, p_j), t_k^p = (p_{k-1}, t)$
 - assign weights $w_j^p = w(t_j^p) = D_p(j)$
- For each pair of neighboring sites p and q
 - create edges $(p_j, q_j)_{j \in \{1, \dots, k-1\}}$ with weight $\lambda_{p,q}$
- Read label value from cut location, e.g., $p_2 \in S, p_3 \in T \Rightarrow f_p = 3$



Linear multi-label graph construction

(cf. Boykov et al. 1998)

- Given $\mathcal{L} = \{1, \dots, k\}$ e.g., $k = 4$ cut: $f_p = 3, f_q = 1$
- General idea:
 - construct one layer per label value
 - read label value from cut location
- Any problem ?



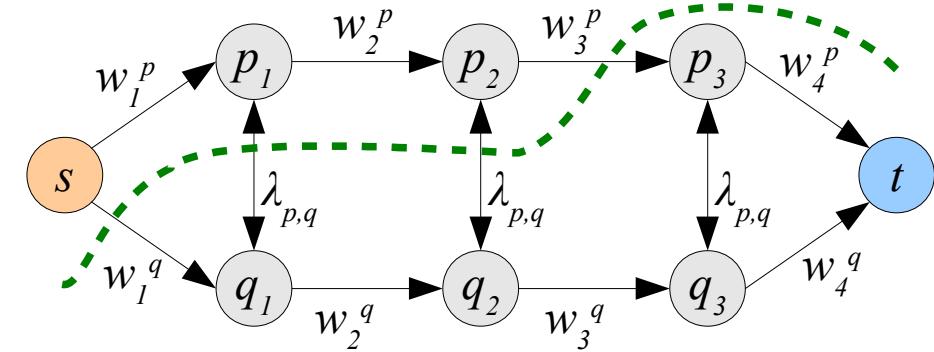
Linear multi-label graph construction

(cf. Boykov et al. 1998)

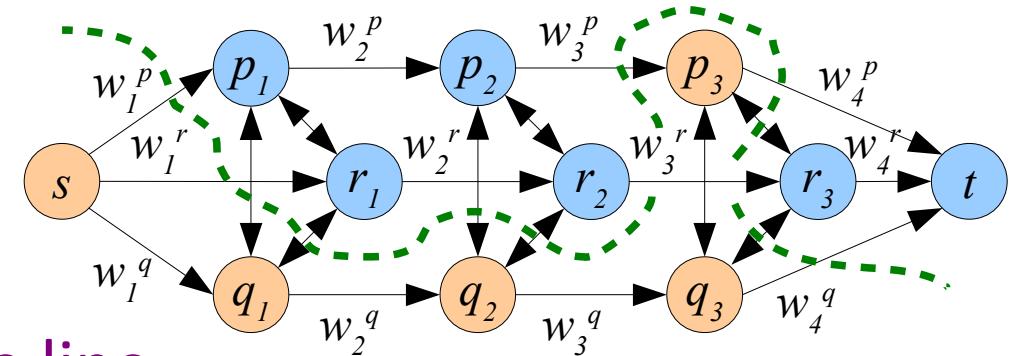
- Given $\mathcal{L} = \{1, \dots, k\}$
- General idea:
 - construct one layer per label value
 - read label value from cut location

e.g., $k = 4$

cut: $f_p = 3, f_q = 1$



- Any problem ?
 - there could be several cut locations on the same line



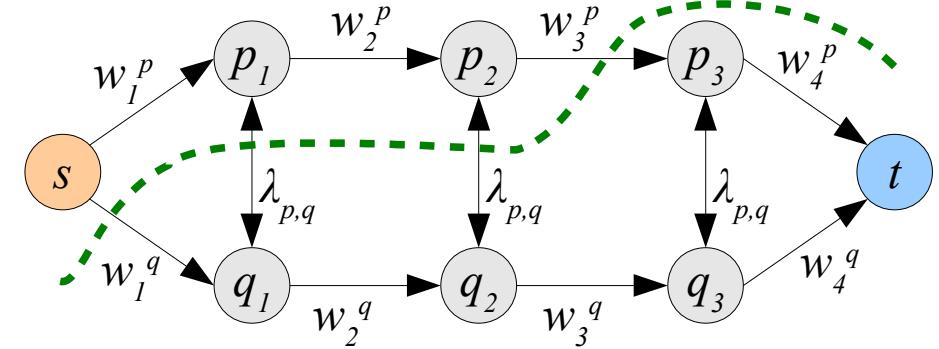
Linear multi-label graph construction

(cf. Boykov et al. 1998)

- $E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{(p,q) \in \mathcal{N}} \lambda_{p,q} |f_p - f_q|$ cut: $f_p = 3, f_q = 1$
with $f_p \in \mathcal{L} = \{1, \dots, k\}$

Attempt 2:

- For each site p
 - create nodes p_1, \dots, p_{k-1}
 - create edges $t_1^p = (s, p_1), t_j^p = (p_{j-1}, p_j), t_k^p = (p_{k-1}, t)$
 - assign weights $w_j^p = w(t_j^p) = D_p(j) + K_p$ [penalize more cutting t_j^p]
with $K_p = 1 + (k-1) \sum_{q \in \mathcal{N}_p} \lambda_{p,q}$ (where \mathcal{N}_p set of neighbors of p)
- For each pair of neighboring sites p and q
 - create edges $(p_j, q_j)_{j \in \{1, \dots, k-1\}}$ with weight $\lambda_{p,q}$

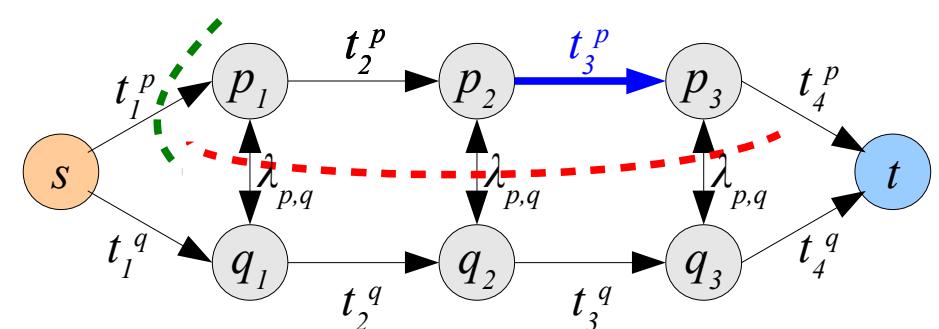
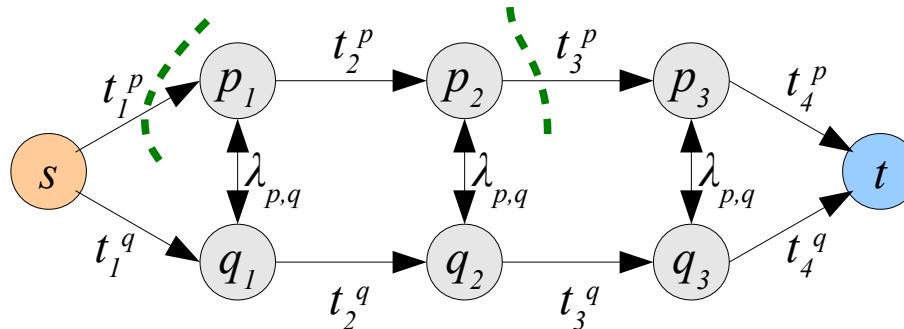


Linear multi-label graph properties

(cf. Boykov et al. 1998)

- Lemma: for each site p , a minimum cut severs exactly one t_j^p
 - [≥ 1] Any cut severs at least one t_j^p
 - [≤ 1] Suppose t_a^p, t_b^p are cut (same line p), then build new cut with t_b^p restored and links $(p_j, q_j)_{j \in \{1, \dots, k-1\}}$ broken for $q \in \mathcal{N}_p$

severed = coupé, sectionné



Impact on (minimum) cost: $-w(t_b^p) + (k-1) \sum_{q \in \mathcal{N}_p} \lambda_{p,q}$
 $= -D_p(j) - 1 < 0 \rightarrow$ strictly smaller cost \rightarrow contradiction

- Theorem (Boykov et al. 1998): a minimum cut minimizes $E(f)$

Application to stereovision: disparity map estimation

- Problem

- given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



- Graph-cut setting

- discrete disparities: $d_p \in \mathcal{L} = \{d_{\min}, \dots, d_{\max}\}$
- data term: $D_p(d_p)$
 - small when pixel p in I similar to pixel $p' = p + (d_p, 0)$ in I'
- smoothness term: $V_{p,q}(d_p, d_q)$
 - small when disparities d_p and d_q are similar

rectified images \leftrightarrow aligned cameras

Application to stereovision: disparity map estimation

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 $d(p) = d_p$ for each pixel $p = (u, v)$



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e.g., what definition?

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e.g., what definition?

- smoothness term: $V_{p,q}(d_p, d_q)$
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Application to stereovision: disparity map estimation

- Problem

- given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



$$p' = p + (d_p, 0)$$

- Graph-cut setting

- discrete disparities: $d_p \in \mathcal{L} = \{d_{\min}, \dots, d_{\max}\}$
- data term: $D_p(d_p)$

e.g., what definition?

- e.g., $D_p(d_p) = E_{ZNSSD}(P_p; (d_p, 0))$ where P_p patch around pixel p

e.g., what definition?

- smoothness term: $V_{p,q}(d_p, d_q)$
- e.g., $V_{p,q}(d_p, d_q) = \lambda |d_p - d_q|$

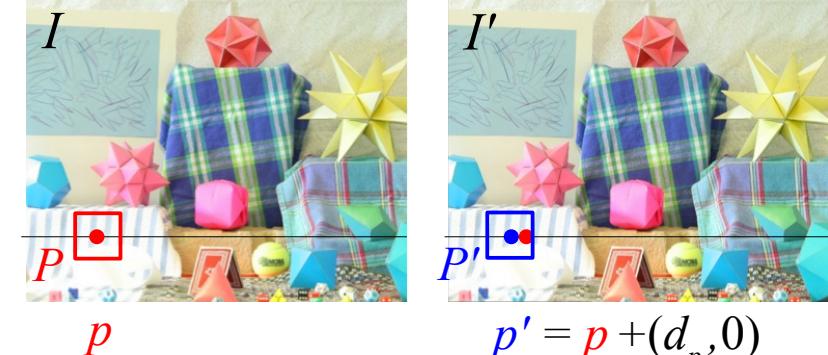
[Boykov et al. → optimal disparities]

SSD = sum of square differences
NSSD = normalized ...
ZNSSD = zero-normalized ...

Application to stereovision: disparity map estimation

- Problem

- given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



- Graph-cut setting

- discrete disparities: $d_p \in \mathcal{L} = \{d_{\min}, \dots, d_{\max}\}$
- data term: $D_p(d_p)$
 - e.g., $D_p(d_p) = E_{ZNSSD}(P_p; (d_p, 0))$
- smoothness term: $V_{p,q}(d_p, d_q)$
 - e.g., $V_{p,q}(d_p, d_q) = \lambda |d_p - d_q|$

[Boykov et al. → optimal disparities]

Is it the “optimal” solution
to the disparity map
estimation problem ?

Application to stereovision: disparity map estimation

- Problem

- given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



- Graph-cut setting

- discrete disparities: $d_p \in \mathcal{L} = \{d_{\min}, \dots, d_{\max}\}$
- data term: $D_p(d_p)$
 - e.g., $D_p(d_p) = E_{ZNSSD}(P_p; (d_p, 0))$
- smoothness term: $V_{p,q}(d_p, d_q)$
 - e.g., $V_{p,q}(d_p, d_q) = \lambda |d_p - d_q|$

- Meaningful but arbitrary choices:
 patch size, similarity, smoothness...

- Optimal solution for energy $\not\Rightarrow$
 optimal solution for problem

[Boykov et al. \rightarrow optimal disparities]

Application to stereovision: disparity map estimation

CC = cross-correlation
 NCC = normalized ...
 ZNCC = zero-normalized ...

- Problem

$$E_{ZNCC}(P; \mathbf{u}) = 1/|P| \sum_{qaP} [(I'_{q+\mathbf{u}} - \bar{I}_P)/\sigma' \cdot (I_q - \bar{I}_P)/\sigma]$$

$$\bar{I}_P = 1/|P| \sum_{qaP} I_q \quad \sigma = [1/|P| \sum_{qaP} (I_q - \bar{I}_P)^2]^{1/2}$$

$$E_{ZNSSD}(P; \mathbf{u}) = 2 - 2 E_{ZNCC}(P; \mathbf{u})$$

- given 2 rectified images I, I' ,
 estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



p

$p' = p + (d_p, 0)$

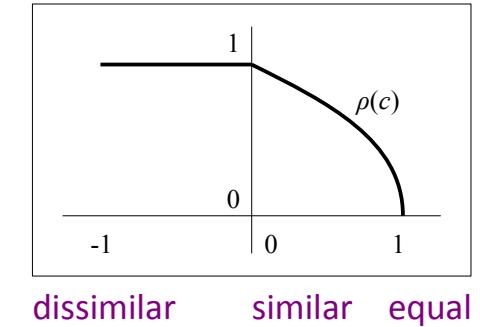
- Graph-cut setting (alternative)

- discrete disparities: $d_p \in \mathcal{L} = \{d_{\min}, \dots, d_{\max}\}$
- $D_p(d_p) = w_{cc} \rho(E_{ZNCC}(P; (d_p, 0)))$ with $\rho(c) \in [0, 1] \setminus$

- e.g. $\rho(c) = \begin{cases} 1 & \text{if } c < 0 \\ \sqrt{1-c} & \text{if } c \geq 0 \end{cases}$

- $V_{p,q}(d_p, d_q) = \lambda |d_p - d_q|$

N.B. only w_{cc}/λ matters



Approximate optimization

- Exact multi-label optimization:
 - only limited cases
 - in practice, may require large number of nodes
- How to go beyond exact optimization constraints?

bound = borne

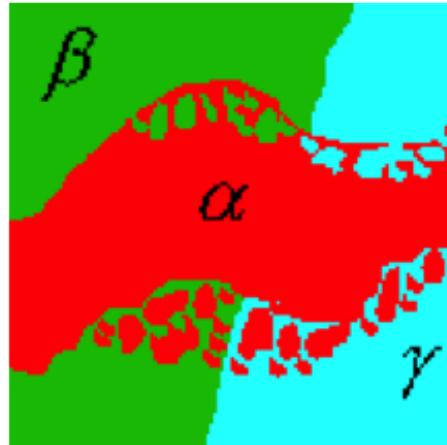
👉 Iterate exact optimizations on subproblems (Boykov et al. 2001)

- → local minimum ☹
- but within known bounds of global minimum ☺

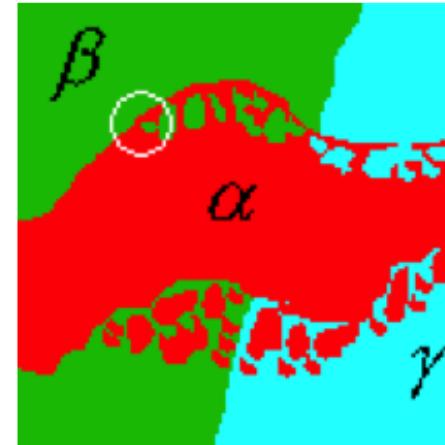
Notion of move – Examples

at once = à la fois
move = déplacement
(\approx modification)
de la solution

Move: maps a labeling $f: \mathcal{P} \rightarrow \mathcal{L}$ to a labeling $f': \mathcal{P} \rightarrow \mathcal{L}$
Idea: iteratively apply moves to get closer to optimum f^*



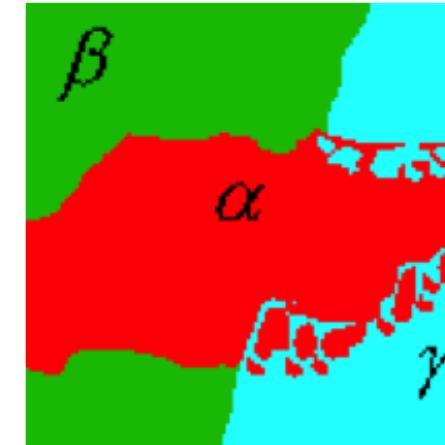
(a) initial labeling



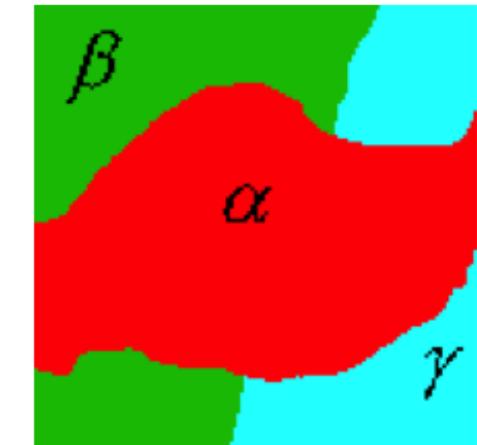
(b) standard move

$$\alpha \rightarrow \beta$$

at one site
only

(c) α - β -swap
$$\alpha \leftrightarrow \beta$$

at many sites
at once

(d) α -expansion
$$\text{any } l \rightarrow \alpha$$

at many sites
at once

Moves

Given a labeling $f: \mathcal{P} \rightarrow \mathcal{L}$ and labels α, β

move = déplacement
 (~ modification)
 de la solution
 α - β -swap =
 permutation α - β

- f' is a **standard move** from f iff
 f and f' differ at most on one site p
- f' is an **expansion move** (or α -expansion) from f iff
 $\forall p \in \mathcal{P}, f'_p = f_p$ or α
 \rightarrow in f' , compared to f , extra sites p can now be labeled α
- f' is a **swap move** (or α - β -swap) from f iff
 $\forall p \in \mathcal{P}, f_p \neq \alpha, \beta \Rightarrow f_p = f'_p$
 \rightarrow some sites that were labeled α are now β and vice versa

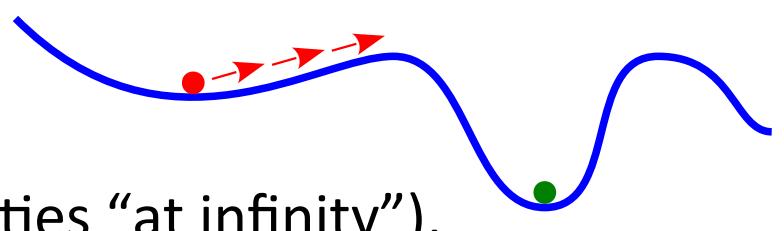
N.B. Other kinds of moves can be defined...

Optimization w.r.t. moves

simulated annealing =
recuit simulé
sampling =
échantillonage

(cf. Boykov et. al 2001)

- Iterative optimization over moves
 - random cycle over all labels until convergence → local min
- **Iterating standard moves**
 - = usual discrete optimization method
 - iterated conditional modes (ICM) = iterative maximization of the probability of each variable conditioned on the rest
 - local minimum w.r.t. standard move,
i.e., energy cannot decrease with a single pixel label difference
⇒ weak condition, low quality
 - simulated annealing, ...
 - slow convergence (optimal properties “at infinity”),
modest quality, some sampling strategies but mostly random



Optimization w.r.t. moves

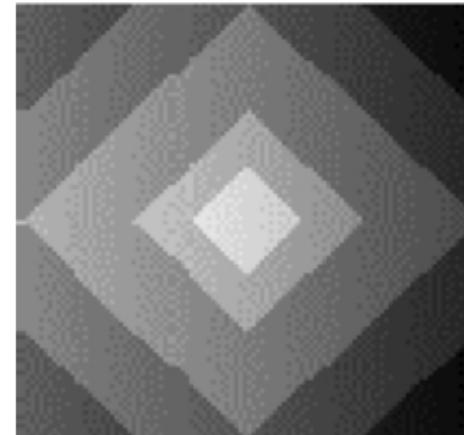
(cf. Boykov et. al 2001)

- Iterative optimization over moves
 - random cycle over all labels until convergence → local min
- **Iterating expansion/swap moves (strong moves)**
 - number of possible moves exponential in number of sites
 - compute optimal move using graph cut = **binary problem!**
 - see Boykov et. al 2001 for graph construction and details
 - significantly fewer local minima than with standard moves
 - sometimes within constant factor of global minimum
 - e.g., expansion moves & Potts model → optimum within factor 2

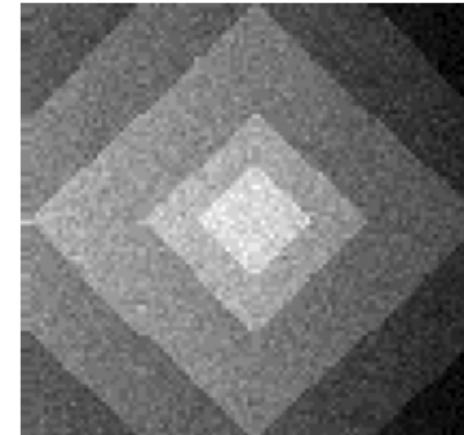
Image restoration with moves

- Restoration with standard moves vs α -expansions

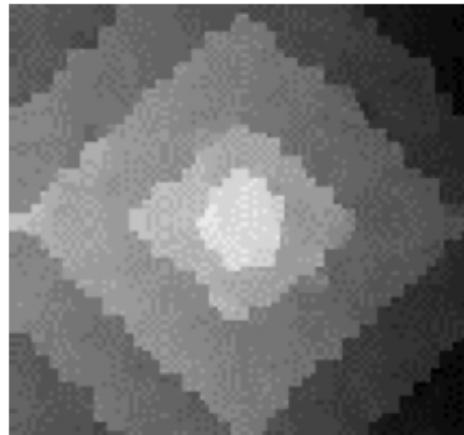
original image



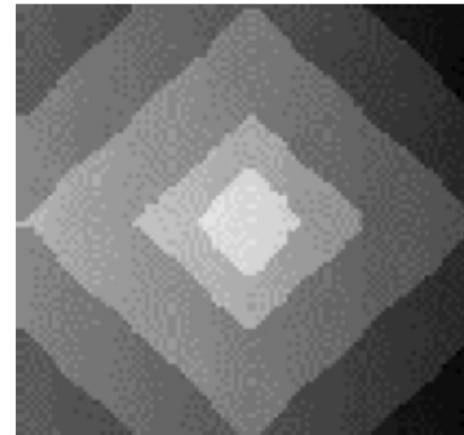
noisy image



restoration with
standard moves



restoration with
 α -expansions



Constraints on interaction potential

(see details in Boykov et. al 2001)

- Expansion move: V metric, \rightarrow expansion inequality:

metric = métrique
 $(=$ fonct distance)
 $d(x,y) = 0 \Leftrightarrow x = y$
 $d(x,y) = d(y,x) \geq 0$
 $d(x,z) \leq d(x,y) + d(y,z)$

$$V_{p,q}(\alpha, \alpha) + V_{p,q}(\beta, \gamma) \leq V_{p,q}(\alpha, \gamma) + V_{p,q}(\beta, \alpha) \quad \text{for all } \alpha, \beta, \gamma \in \mathcal{L}$$

- Swap move: V semi-metric, \rightarrow swap inequality:

semi-metric =
semimétrique
 $d(x,y) = 0 \Leftrightarrow x = y$
 $d(x,y) = d(y,x) \geq 0$

$$V_{p,q}(\alpha, \alpha) + V_{p,q}(\beta, \beta) \leq V_{p,q}(\alpha, \beta) + V_{p,q}(\beta, \alpha) \quad \text{for all } \alpha, \beta \in \mathcal{L}$$

[= as metric but triangle inequality not required: $V_{p,q}(\alpha, \gamma) \not\leq V_{p,q}(\alpha, \beta) + V_{p,q}(\beta, \gamma)$]
[weaker condition than for expansion move]

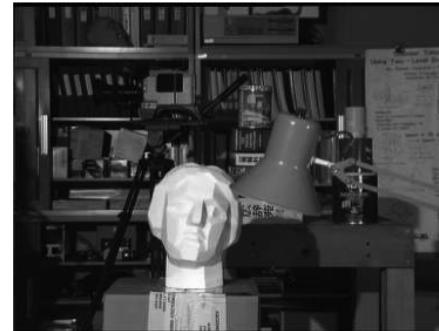
- Examples

– Potts model: $V_{p,q}(\alpha, \beta) = \lambda_{p,q} \mathbf{1}(\alpha \neq \beta)$

discontinuity-preserving!

– truncated L₂ distance: $V_{p,q}(\alpha, \beta) = \min(K, \| \alpha - \beta \|)$

Disparity map estimation with moves



(a) Left image: 384x288, 15 labels



(b) Ground truth



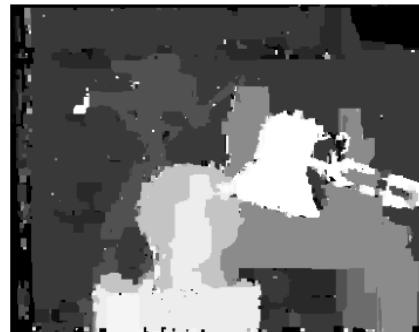
(c) Swap algorithm



(d) Expansion algorithm



(e) Normalized correlation



(f) Simulated annealing

Tsukuba images
from famous
Middlebury
benchmark
(also contains
Moebius images)



To go further on this subject

Disparity map estimation: alternative data term

(cf. Boykov et al. 1999,
Boykov et al. 2001)

- Idea: direct intensity comparison, but sensitive to sampling

$$- D_p(d_p) = \min(K, |I_p - I'_{p+d_p}|^2)$$

- With image sampling insensitivity:

No patch similarity here:
the local consistency is given
by the smoothness term

- disparity range discretized to 1 pixel accuracy
→ sensitivity to high gradients
- (sub)pixel dissimilarity measure for greater accuracy,
e.g., by linear interpolation (Birchfield & Tomasi 1998)

$$- C_{\text{fwd}}(p,d) = \min_{d-1/2 \leq u \leq d+1/2} |I_p - I'_{p+u}|$$

$$- C_{\text{rev}}(p,d) = \min_{p-1/2 \leq x \leq p+1/2} |I_x - I'_{p+d}| \quad [\text{for symmetry}]$$

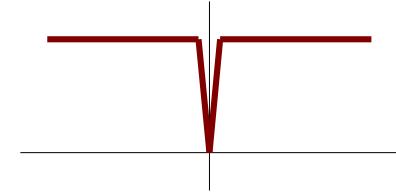
$$- D_p(d_p) = C(p,d_p) = \min(K, C_{\text{fwd}}(p,d_p), C_{\text{rev}}(p,d_p))^2$$

Disparity map estimation: smoothness term

- Scene with fronto-parallel objects

- **piecewise-constant** model = OK
 - e.g., Potts model:

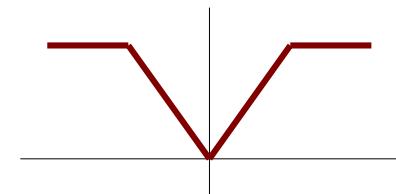
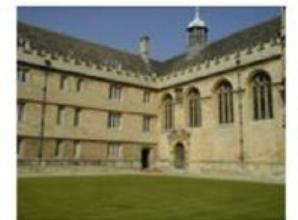
$$V_{p,q}(d_p, d_q) = u_{p,q} \mathbf{1}(d_p \neq d_q)$$



- Scene with slanted surfaces (e.g., ground)

- **piecewise-smooth** model = better
 - e.g., smooth cap max value:

$$V_{p,q} = \lambda \min(K, |d_p - d_q|)$$



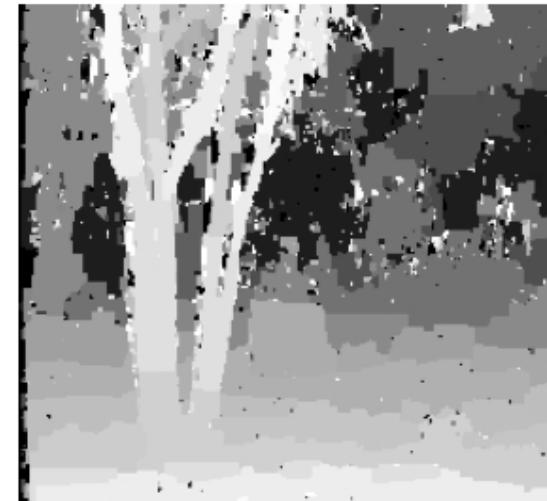
- Metric \Rightarrow both swap and expansion algorithms usable

Potts model vs smooth cap max value

- Potts model : **piecewise-constant**
 - suited for uniform areas (\Rightarrow fewer disparities on large areas)
- Smooth cap max value: **piecewise-smooth** model
 - suited for slowly-varying areas (e.g., slope)



(a) Left image: 256x233, 29 labels



(b) Piecewise constant model



(c) Piecewise smooth model

To go further on this subject

Disparity map estimation: smoothness term

(cf. Boykov et al. 1998,
Boykov et al. 2001)

- Contextual information
 - neighbors p, q more likely to have same disparity if $I_p \approx I_q$
 → make $V_{p,q}(d_p, d_q)$ also depend on $|I_p - I_q|$
 - meaningful in low texture areas (where $|I_p - I_q|$ meaningful)
- E.g., with Potts model: $V_{p,q}(d_p, d_q) = u_{p,q} \mathbf{1}(d_p \neq d_q)$
 - $u_{p,q}$: penalty for assigning different disparities to p and q
 - textured regions: $u_{p,q} = K$
 - textureless regions: $u_{p,q} = U(|I_p - I_q|)$
 - $u_{p,q}$ smaller for pixels p, q with large intensity difference $|I_p - I_q|$
 - e.g., $U(|I_p - I_q|) = \begin{cases} 2K & \text{if } |I_p - I_q| \leq 5 \\ K & \text{if } |I_p - I_q| > 5 \end{cases}$

Many extensions to more complex energies

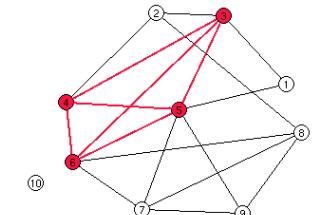
(cf. Pansari & Kumar 2017)

- Truncated Convex Models (TCM)
 - several other approximate algorithms to minimize

$$E(\mathbf{x}) = \sum_{a \in \mathcal{V}} \theta_a(x_a) + \sum_{(a,b) \in \mathcal{E}} \omega_{ab} \min\{d(x_a - x_b), M\}$$

- Truncated Max of Convex Models (TMCM)
 - no clique size restriction (high-order > pairwise)

$$\theta_c(\mathbf{x}_c) = \omega_c \sum_{i=1}^m \min\{d(p_i(\mathbf{x}_c) - p_{c-i+1}(\mathbf{x}_c)), M\}$$

(a) Ground truth
(Energy, Time (s))(b) Cooccurrence
(2098800, 101)(c) Parsimonious
(1364200, 225)(d) $m = 1, h' = 4$
(1257249, 256)(e) $m = 3, h' = 4$
(1267449*, 335)

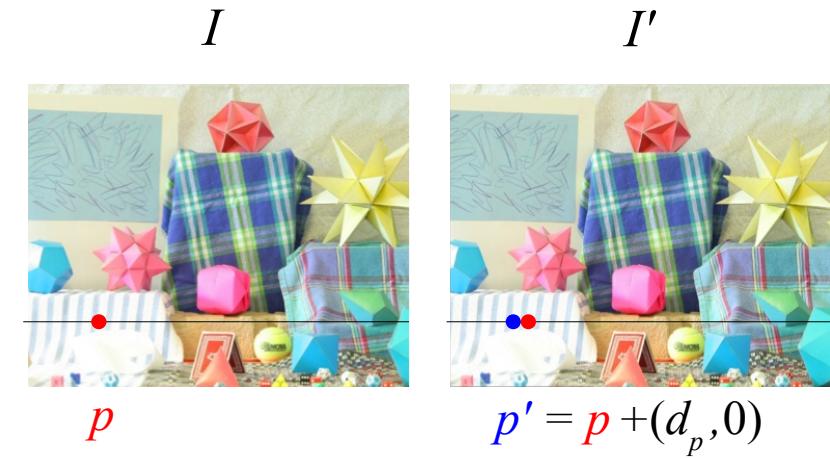
© Battiti & Brunato 2009

c : clique
 \mathbf{x}_c : labeling of a clique
 ω_c : clique weight
 d : convex function
 M : truncation factor
 $p_i(\mathbf{x}_c)$: i -th largest label in \mathbf{x}_c
 $c = |\mathbf{c}|$

Disparity map estimation

- Problem

- given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



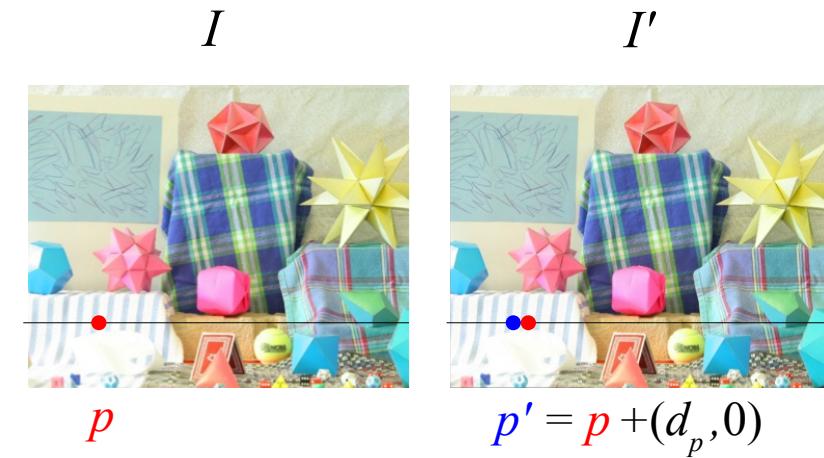
- Are the preceding formulations OK?
 - anything not modeled?
 - any bias?

Disparity map estimation

(Boykov et. al 2001)

- Problem

- given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for each pixel $p = (u, v)$



- Are the preceding formulations OK?

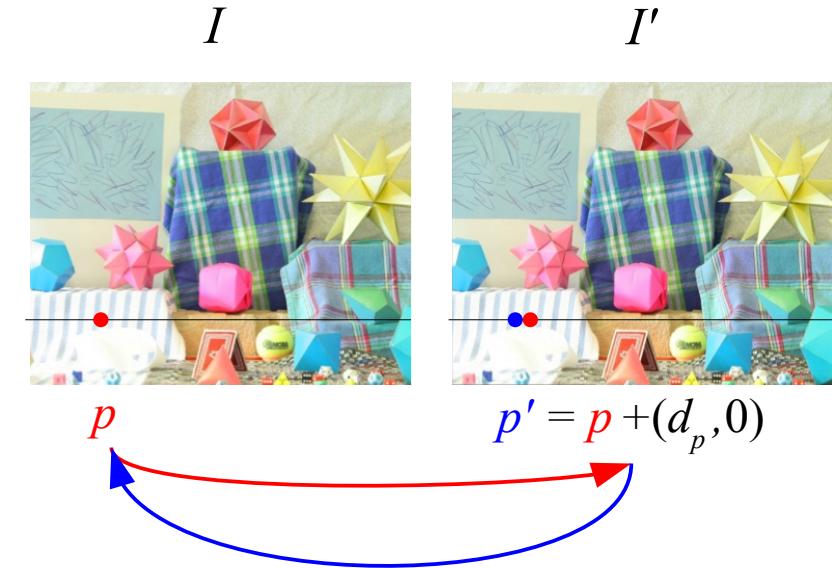
- no treatment of occlusion
- no symmetry: one center image, one auxiliary image
 - treatment of second image relative to the first (main) one
 - difficulty to incorporate occlusion naturally

Cross-checking

(Bolles & Woodfill, 1993)

occlusion =
occultation

- Problem
 - given 2 rectified images I, I' , estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u, v)$
- Cross-checking method:
 - compute left-to-right disparity
 - compute right-to-left disparity
 - mark as occlusion pixels in one image mapping to pixels in the other image but which do not map back to them
- Common and easy to implement



Stereovision with occlusion handling

(cf. Kolmogorov & Zabih 2001)

occluded =
occulté

- Occlusion
 - pixel visible in one image only
 - occurs usually at discontinuities
- Uniqueness model hypothesis
 - pixel in one image → at most one pixel in other image
[sometimes too restrictive]
 - pixel with no correspondence: labeled as occluded
- Main idea:
 - use labels representing corresponding pixels (= pixel pairs),
not pixel disparity

To go further on this subject

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- \mathcal{A} : correspondence candidates (pixel pairs in $I \times I'$) = pixel assignments
 - $\mathcal{A} = \{(p, p') \mid p_y = p'_y \text{ and } 0 \leq p'_x - p_x < k\}$ (same line, different position)
 - disparity: for $a = (p, p') \in \mathcal{A}$, $d(a) = p'_x - p_x$
 - hypothesis: disparities lie in limited range $[0, k]$
 - goal: find subset of \mathcal{A} containing only corresponding pixels
 - use: subsets defined as labelings $f: \mathcal{A} \rightarrow \mathcal{L} = \{0, 1\}$ such that $\forall a = (p, p') \in \mathcal{A}, f_a = 1$ if p and p' correspond, otherwise $f_a = 0$
 - symmetric treatment of images (& applicable to non-aligned cameras)
- $A(f)$: active assignments, i.e., pixel pairs considered as corresponding
 - $A(f) = \{a \in \mathcal{A} \mid f_a = 1\}$

To go further on this subject

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- $N_p(f)$: set of correspondences for pixel p
 - $N_p(f) = \{a \in A(f) \mid \exists p' \in \mathcal{P}, a = (p, p')\}$
 - configuration f unique iff $\forall p \in \mathcal{P} \mid N_p(f) \leq 1$
 - occluded pixels defined as pixels such that $|N_p(f)| = 0$
- \mathcal{N} : a neighborhood system on assignments (used for smoothness term)
 - $\mathcal{N} \subset \{\{a_1, a_2\} \subset \mathcal{A}\}$
 - for efficient energy minimization via graph cuts:
 - neighbors having the same disparity
 - $\mathcal{N} = \{(p, p'), (q, q')\} \subset \mathcal{A} \mid p, p' \text{ are neighbors and } d(p, p') = d(q, q')\}$
 $(\rightarrow \text{then } q, q' \text{ are also neighbors})$

To go further on this subject

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- $E(f) = E_{\text{data}}(f) + E_{\text{smooth}}(f) + E_{\text{occ}}(f)$
 - $E_{\text{data}}(f) = \sum_{a=(p,p') \in A(f)} (I_p - I'_{p'})^2$
 - single pixel similarity
 - $E_{\text{smooth}}(f) = \sum_{\{a_1, a_2\} \in \mathcal{N}} V_{a_1, a_2} \mathbf{1}(f_{a_1} \neq f_{a_2})$
 - $\mathcal{N} = \{(p, p'), (q, q')\} \subset \mathcal{A} \mid p, p' \text{ are neighbors and } d(p, p') = d(q, q')\}$
 - penalty if: $f_{a_1} = 1, a_2$ close to $a_1, d(a_2) = d(a_1)$, but $f_{a_2} = 0$
 - Potts model **on assignments** (pixel pairs), **not on pixel disparity**
 - $E_{\text{occ}}(f) = \sum_{p \in \mathcal{P}} C_p \cdot \mathbf{1}(|N_p(f)| = 0)$ [occlusion penalty]
 - penalty C_p if p occluded

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- $E(f) = E_{\text{data}}(f) + E_{\text{smooth}}(f) + E_{\text{occ}}(f)$
- Optimizable by graph cuts as multi-label problem (cf. paper)
 - graph construction on assignments (pixel pairs), not pixels
 - \mathcal{A}^α : set of all assignments with disparity α
 - $\mathcal{A}^{\alpha,\beta} = \mathcal{A}^\alpha \cup \mathcal{A}^\beta$
 - expansion move:
 - f' within single α -expansion move of f iff $A(f') \subset A(f) \cup \mathcal{A}^\alpha$
 - currently active assignments can be deleted
 - new assignments with disparity α can be added
 - swap move:
 - f' within single swap move of f iff $A(f') \cup \mathcal{A}^{\alpha,\beta} = A(f) \cup \mathcal{A}^{\alpha,\beta}$
 - only changes: adding or deleting assignments having disparities α or β

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- Expansion-move algorithm:

$$\begin{aligned} f \text{ unique} \Leftrightarrow \\ \forall p \in \mathcal{P} \quad |N_p(f)| \leq 1 \end{aligned}$$

1. start with arbitrary, unique configuration f_0
 2. set success \leftarrow false
 3. for each disparity α
 - 3.1. find $f^\alpha = \operatorname{argmin}_f E(f)$
subject to f unique and within single α -move of f_0
 - 3.2. if $E(f^\alpha) < E(f_0)$, then set $f_0 \leftarrow f^\alpha$, success \leftarrow true
 4. if success go to 2
 5. return f_0
- Critical step: efficient computation of α -move with smallest energy

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- Swap-move algorithm:

$$\begin{aligned} f \text{ unique} \Leftrightarrow \\ \forall p \in \mathcal{P} \quad |N_p(f)| \leq 1 \end{aligned}$$

1. start with arbitrary, unique configuration f_0
 2. set success \leftarrow false
 3. for each pair of disparities α, β ($\alpha \neq \beta$)
 - 3.1. find $f^{\alpha\beta} = \operatorname{argmin}_f E(f)$
subject to f unique and within single $\alpha\beta$ -swap of f_0
 - 3.2. if $E(f^{\alpha\beta}) < E(f_0)$, then set $f_0 \leftarrow f^{\alpha\beta}$, success \leftarrow true
 4. if success go to 2
 5. return f_0
- Critical step: efficient computation of $\alpha\beta$ -swap with smallest energy

Stereovision with occlusion

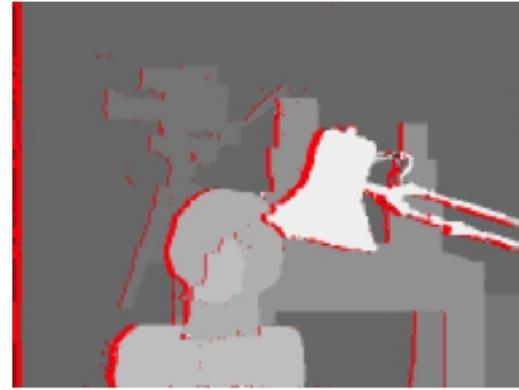
(cf. Kolmogorov & Zabih 2001)



(a) Left image of *Head* pair



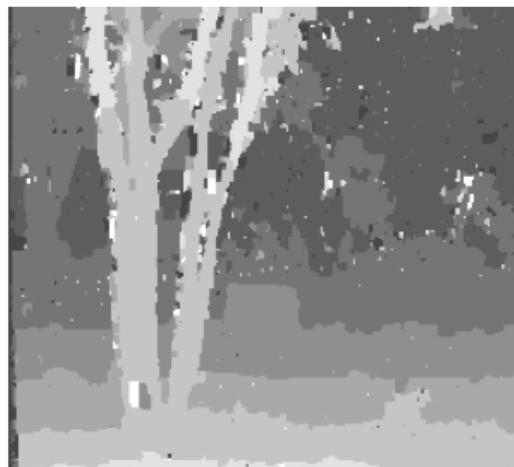
(b) Potts model stereo
Disparity maps obtained for the *Head* pair



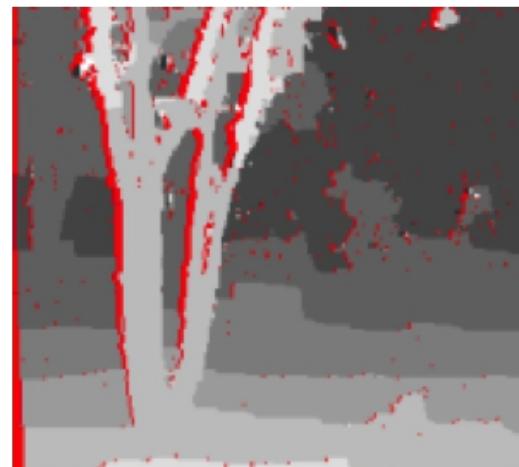
(c) Stereo with occlusions
Disparity maps obtained for the *Head* pair



(d) Left image of *Tree* pair



(e) Potts model stereo
Disparity maps obtained for the *Tree* pair



(f) Stereo with occlusions
Disparity maps obtained for the *Tree* pair

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- Expansion moves vs swap moves



with α -expansions



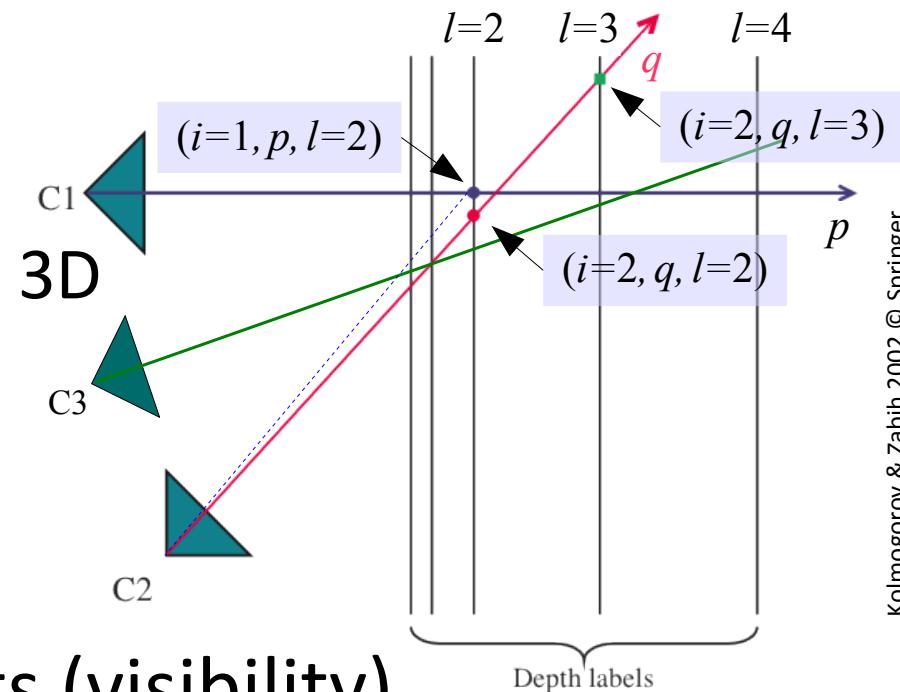
with $\alpha\beta$ -swaps

- Swap moves not powerful enough to escape local minima for this class of energy function

Multi-view reconstruction

(cf. Kolmogorov & Zabih 2002)

- Given n calibrated images on the “same side” of scene
- Global model
 - \mathcal{L} = discretized set of **depths (not disparities)**
 - image i , pixel p , depth l
- Difficulty = point interaction
 - pb: def $(i,p,l), (j,q,l)$ “close” in 3D
→ too many interactions → ☹
 - sol.: def q closest pixel of projection of (i,p,l) on j → ☺
- Photo-consistency constraints (visibility)
 - red point, at depth $l=2$, blocks C2's view of green point, at depth $l=3$



Multi-view reconstruction

(cf. Kolmogorov & Zabih 2002)

- Terms in the energy: data, smoothness, visibility
- Optimization by α -expansion

See paper
for details



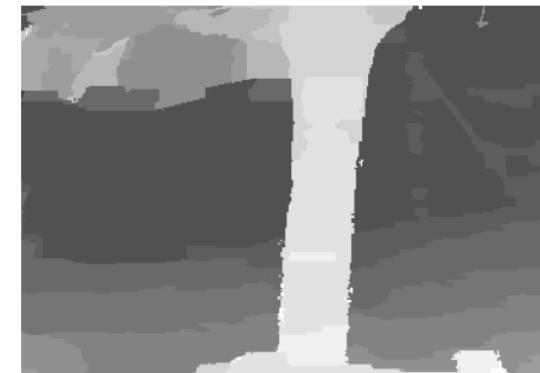
(a) Middle image of *Head* dataset



(b) Scene reconstruction for *Head* dataset



(c) Middle image of *Garden* sequence



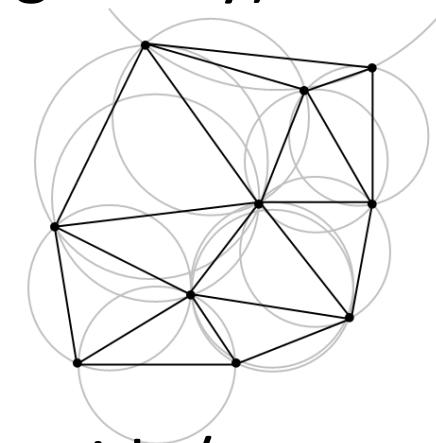
(d) Scene reconstruction for *Garden* sequence

point cloud =
nuage de points
sweep = balayage
outliers = donnée (ici
points) aberrantes
tetrahedralization =
tétraédrisation

Beyond disparity maps: 3D mesh reconstruction

(cf. Vu et al. 2012)

- Merging of depth maps into single point cloud
 - possibly sparse depth maps, e.g., obtained by plane sweep
- Problems:
 - multi-view visibility (to be taken into account globally)
 - outliers
- Solution:
 - Delaunay tetrahedralization of point cloud
 - binary labelling of tetrahedra: inside/full or outside/empty
 - 3D surface = interface inside/outside

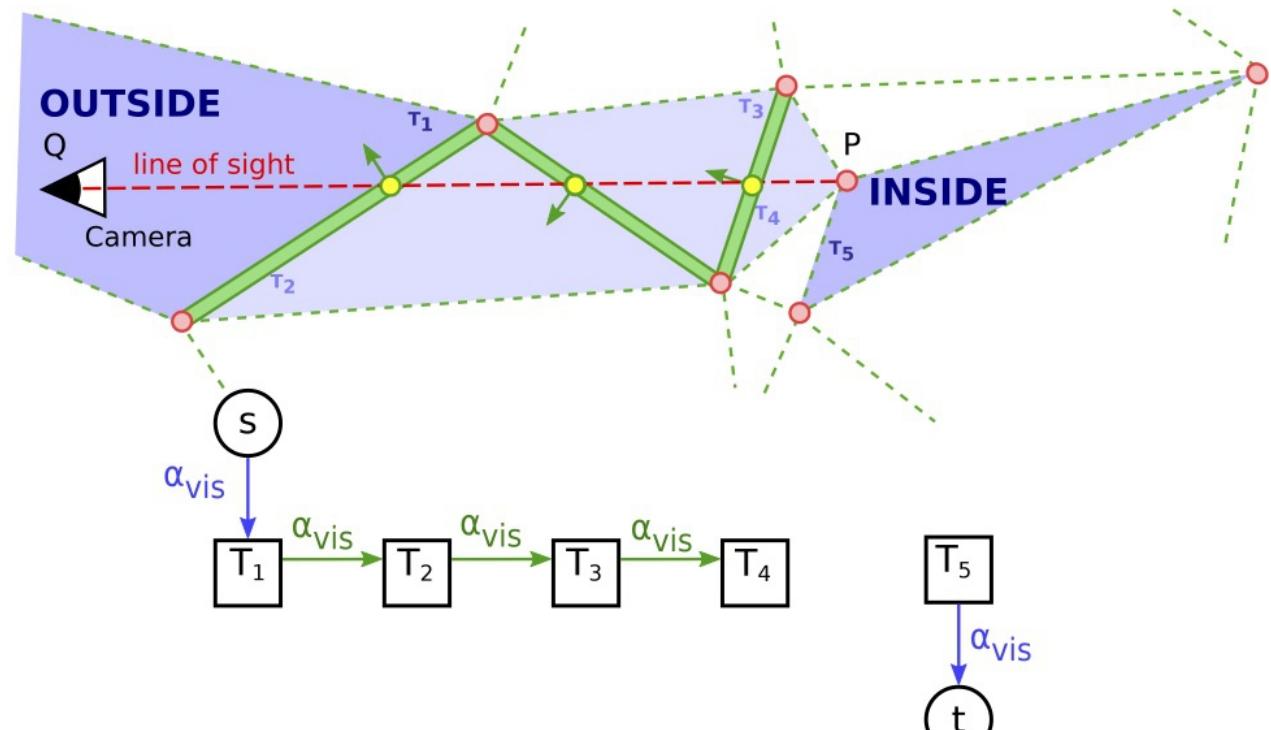


Visibility consistency via graph cut

(cf. Vu et al. 2012)

- Lines of sight from cameras to visible points \Rightarrow outside

Q, P : points
 T : tetrahedron
 S : surface
 P : point cloud
 v : line of sight
 $l_T = 0$: T outside
 (empty space)
 $l_T = 1$: T inside
 (occupied space)



$$D_{\text{out}}(l_T) = \alpha_{\text{vis}} \mathbf{1}[l_T = 0]$$

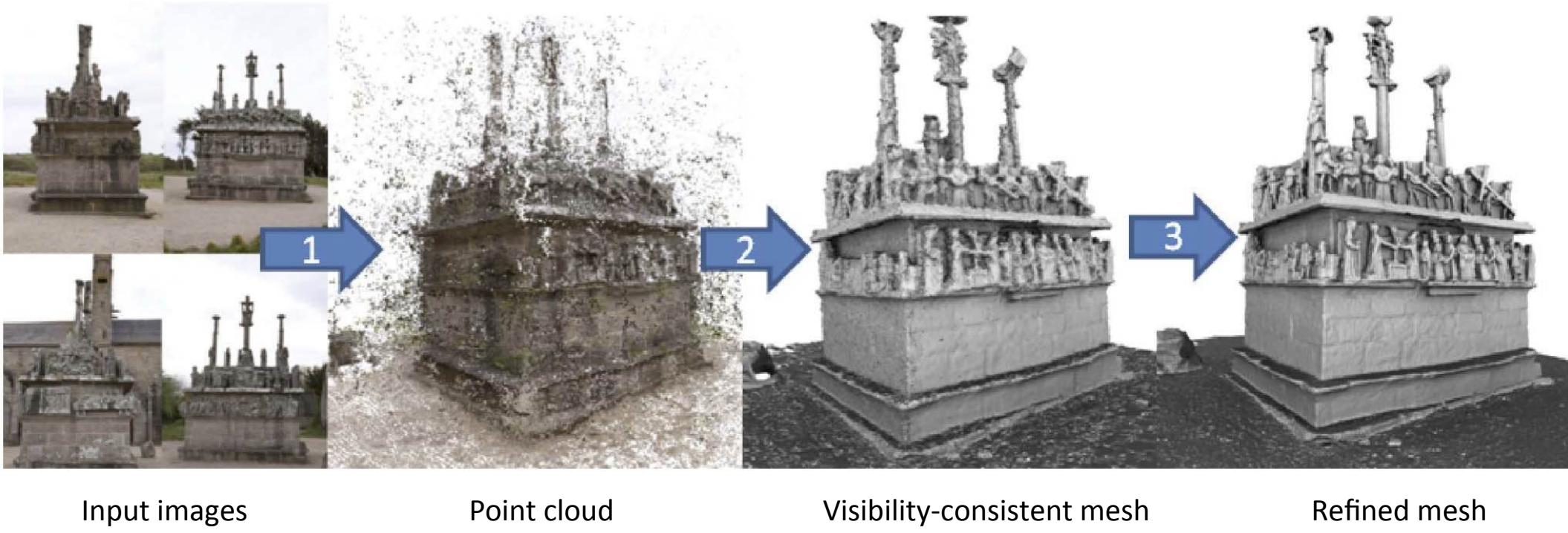
$$D_{\text{in}}(l_T) = \alpha_{\text{vis}} \mathbf{1}[l_T = 1]$$

$$V_{\text{align}}(l_{T_i}, l_{T_j}) = \alpha_{\text{vis}} \mathbf{1}[l_{T_i} = 0 \wedge l_{T_j} = 1]$$

$$E_{\text{vis}}(S, P, v) = \sum_{P \in P} \left(\sum_{Q \in v_P} D_{\text{out}}(l_{T_1^{Q \rightarrow P}}) + \sum_{i=1}^{N_{[PQ]}} V_{\text{align}}(l_{T_i^{Q \rightarrow P}}, l_{T_{i+1}^{Q \rightarrow P}}) + D_{\text{in}}(l_{T_{N_{[PQ]}+1}^{Q \rightarrow P}}) \right)$$

Beyond disparity maps: 3D mesh reconstruction

(cf. Vu et al. 2012)



Input images

Point cloud

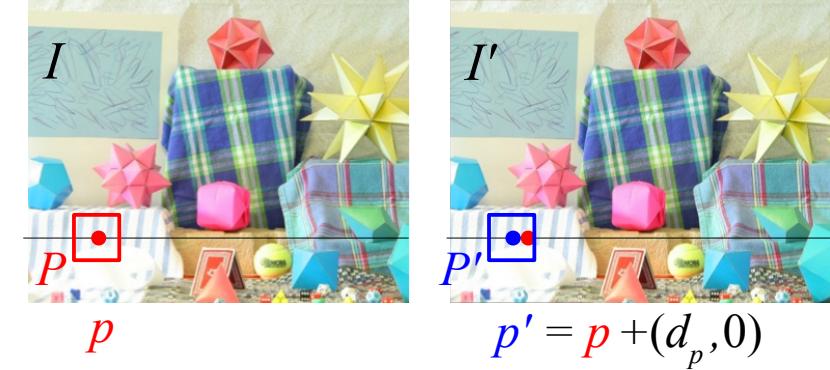
Visibility-consistent mesh

Refined mesh

- Best reconstruction results on international benchmarks
- Startup company with IMAGINE members (2011)
 - 15 employees, 90% revenue = international
 - bought by Bentley Systems (2015), still success

Exercise 2: simple disparity map estimation (without moves nor occlusion)

- Given 2 rectified images I, I' ,
estimate optimal disparity
 $d(p) = d_p$ for pixels $p = (u, v)$



- Setting: linear multi-label graph construction (cf. pp. 85-96)
 - discrete disparities: $d_p \in \mathcal{L} = \{d_{\min}, \dots, d_{\max}\}$
 - \mathcal{N}_p : 4 neighbors of pixel p
 - $D_p(d_p) = w_{cc} \rho(E_{ZNCC}(P; (d_p, 0)))$ with $\rho(c) = \begin{cases} 1 & \text{if } c < 0 \\ \sqrt{1-c} & \text{if } c \geq 0 \end{cases}$
 - $V_{p,q}(d_p, d_q) = \lambda |d_p - d_q|$

- See material provided for the exercise on web site
(template code and detailed exercise description)

N.B. only w_{cc} / λ matters

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