Introduction to Graph Theory

Lecture 24

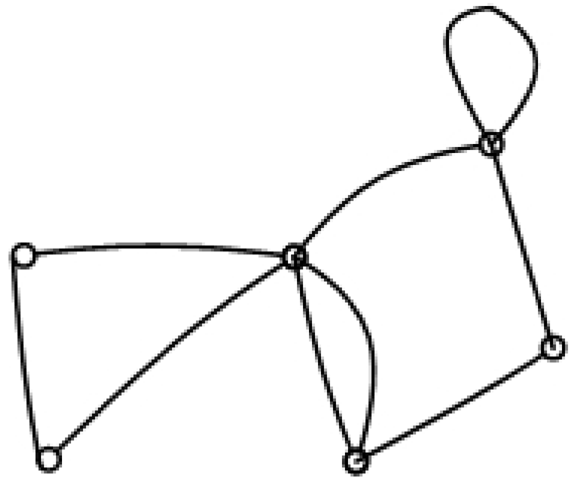
Menu

- Basic definitions of graph theory
- Properties of graphs
- Paths
- Trees
- Digraphs and their applications, network flows

Definitions

- A **graph** G consist of :
  - a *finite* set of **vertices** $V(G)$, which cannot be empty,
  - and a *finite* set of **edges** $E(G)$, which connect pairs of vertices.
- The number of vertices in G is called the **order** of G, denoted by $|V|$.

Give the number of vertices and the number of edges of the following graph:



- $|V| = 6$;
- $|E| = 9$.

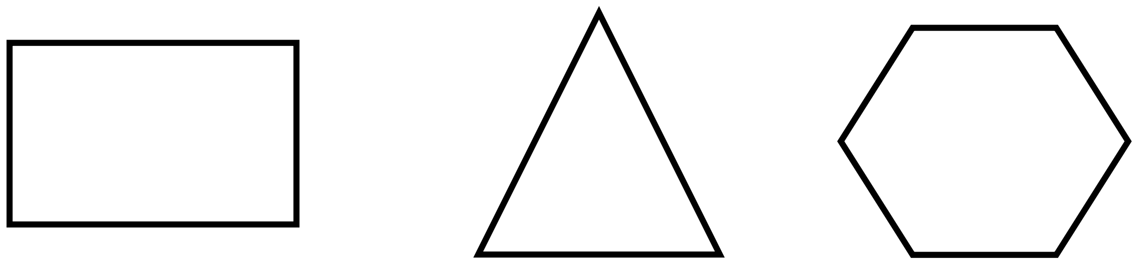
Incidence, adjacency and neighbors

- Two vertices are **adjacent** if they are joined by an edge.
- Adjacent vertices are said to be **neighbors**.
- The edge which joins vertices is said to be **incident** to them.

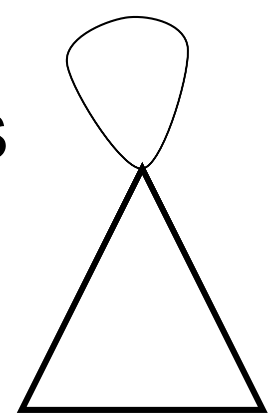
Multiple edges, loops and simple graphs

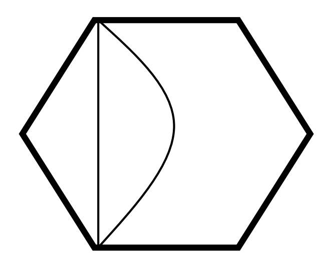
- Two or more edges joining the same pair of vertices are **multiple edges**.
- An edge joining a vertex to itself is called a **loop**.
- A graph containing no multiple edges or loops is called a **simple graph**

Simple graph: examples

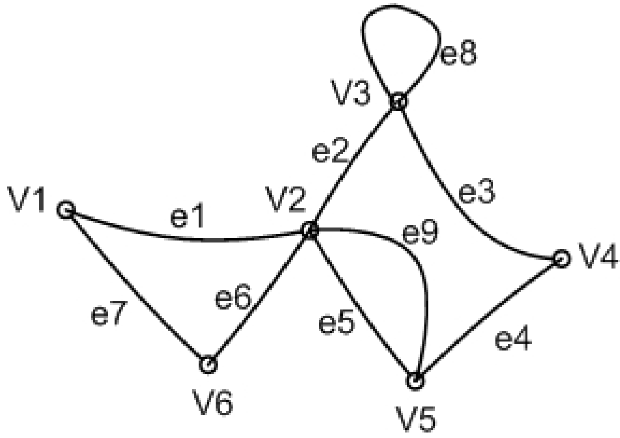


- Non-simple graphs





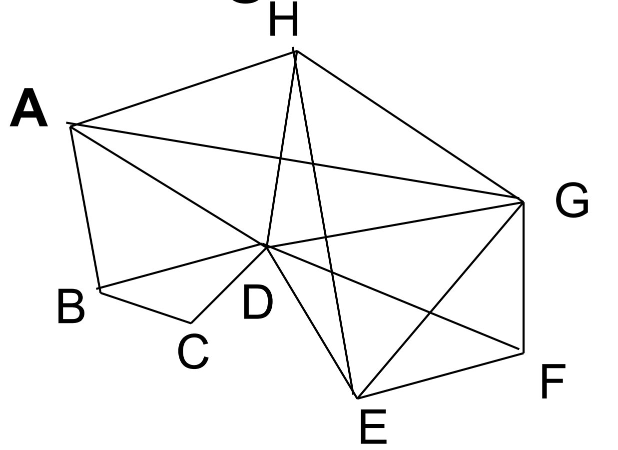
In the following graph:
Identify the neighbours of *V*4
Identify the edge incident to *V*3 and *V*4
Identify multiple edges
Identify the loop



- The neighbours of *V*4 are: *V*3 and *V*5
- The edge incident to *V*3 and *V*4 is: *e*3
- *e*5 and *e*9 are multiple edges
- *e*8 is a loop

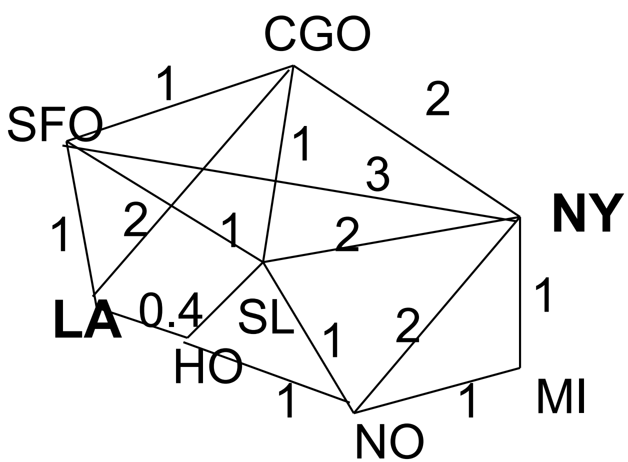
Why study graph theory?

- There are many engineering or computer science related problems that can be modelled using ‘graphs’
- For example, **travelling salesman problem**: find the minimal cost path to cover all the cities A-H, starting from A, ending at A



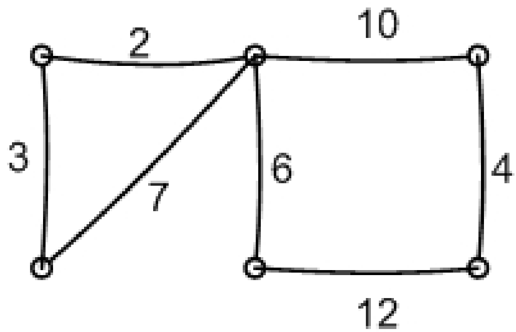
Why study graph theory?

- Routing problem: find the minimal delay path from LA to NY.



Weighted graphs

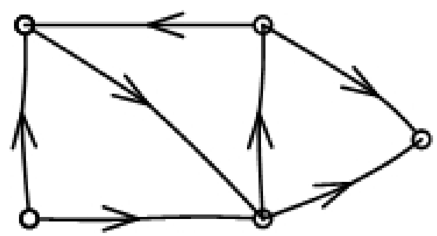
- A **weighted graph** has a number assigned to each of its edges, called its **weight**.
- The weight can be used to represent distances, capacities or costs.
- Is the following weighted graph a simple graph? Justify your answer

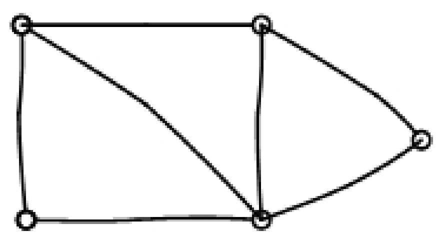


- The weighted graph is a simple graph because it has no multiple edges or loops

Digraphs

- A **digraph** is a *directed* graph, a graph where instead of edges we have directed edges with arrows (**arcs**) indicating the direction of flow.
- Sketch the underlying graph of the digraph:





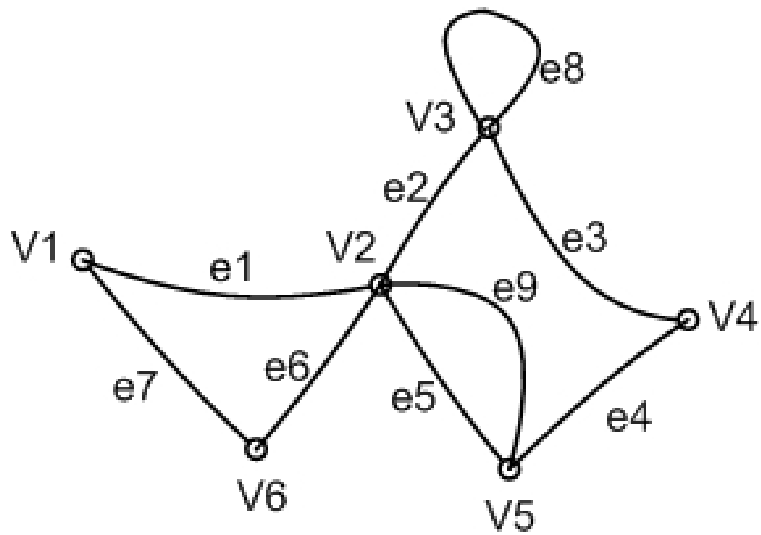
Degree

- The *number of times* edges are incident to a vertex V is called its **degree**, denoted by $d(V)$.
- The **degree sequence** of a graph consists of the degrees of the vertices written in non-*increasing order*, with repeats where necessary.
- The sum of the values of the degrees, $d(V)$, over all the vertices of a simple graph is twice the number of edges:

$$\sum_i d(V_i) = 2|E|$$

- Why?

Give the degrees of the vertices *V*1 and *V*3 of the graph of



- $d(V1) = 2$ and $d(V3) = 4$

Degree

- A vertex of a digraph has an in-degree of $d-(V)$ and an out-degree $d+(V)$.

Degree

- For a digraph we get

$$\sum_i d_-(V_i) = |A|$$

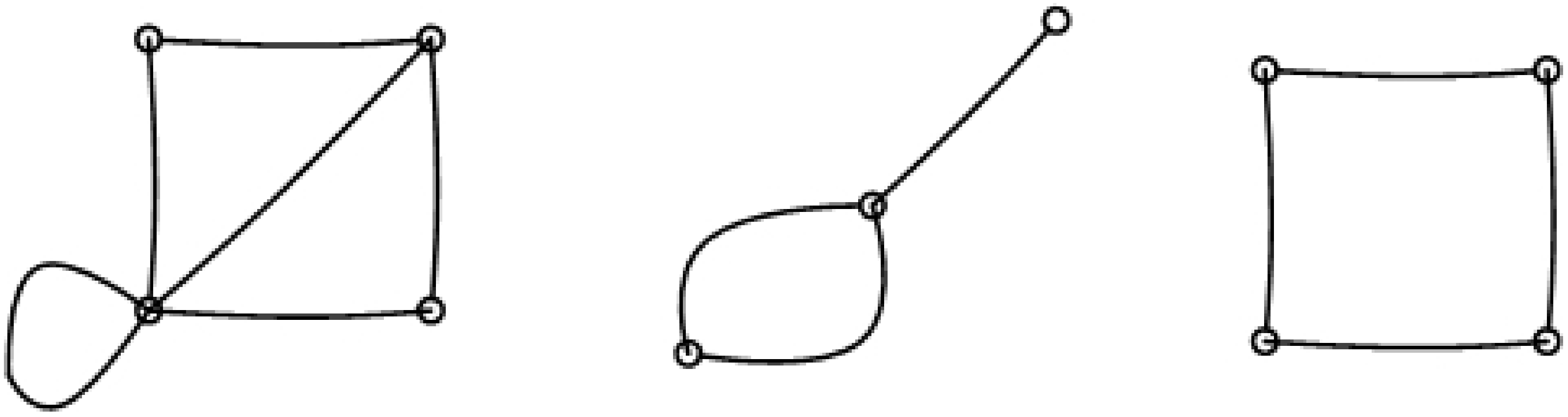
$$\sum_i d_+(V_i) = |A|$$

- where |A| is the number of arcs.

Subgraphs

- A **subgraph** of G is a graph, H, whose vertex set is a subset of G's vertex set, and whose edge set is a subset of the edge set of G.
- If a subgraph H of G spans all of the vertices of G, i.e. $V(H) = V(G)$, then H is called a **spanning subgraph** of G.

For the graph (a) which of the subgraphs (b) and (c) is a spanning subgraph?



(a) (b) (c)

- Subgraph (c) is a spanning subgraph of graph (a).

Summary

- Definitions of **graphs: vertices, edges, order**
- Definitions of: **multiple edges, loops**
- Definitions of: **simple graphs**
- **Digraph**: directed graph
- **Weighted graphs**
- The number of times edges are incident to a vertex V is called its **degree**

Summary

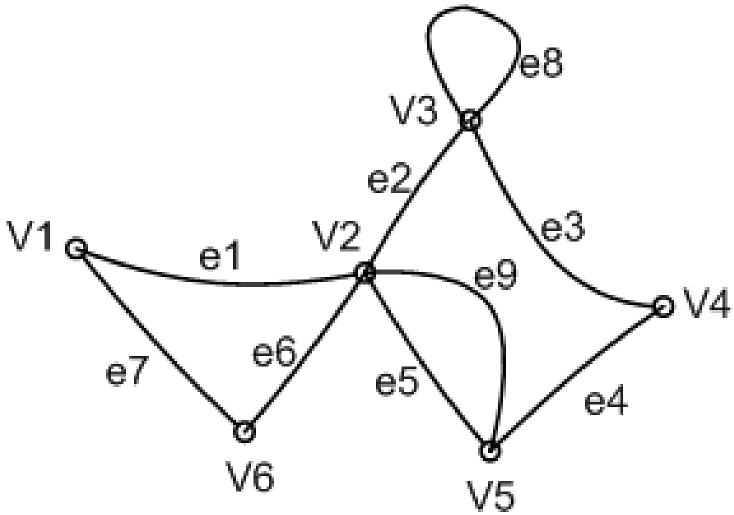
- The sum of the values of the degrees, *d*(*V*), over all the vertices of a simple graph is twice the number of edges: $$\sum_i d(V_i) = 2|E|$$
- Definitions of: **subgraphs, spanning subgraphs**

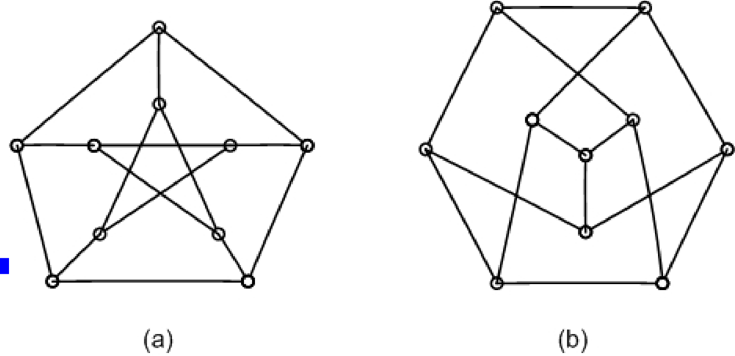
Readings

- [Mar07] Read 9.1
- [Mar13] Read 9.1

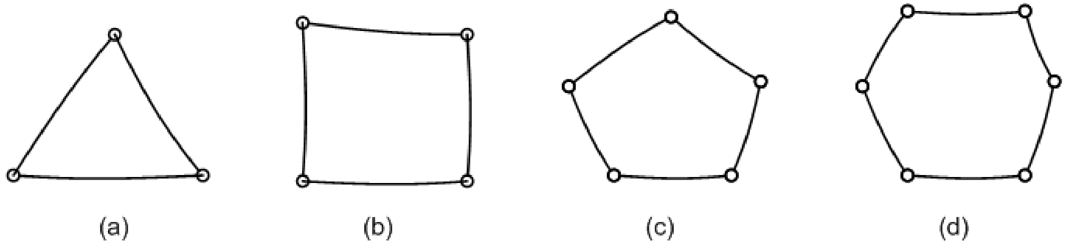
Self test

- 1. Write down the vertex set and edge set of the graph in:

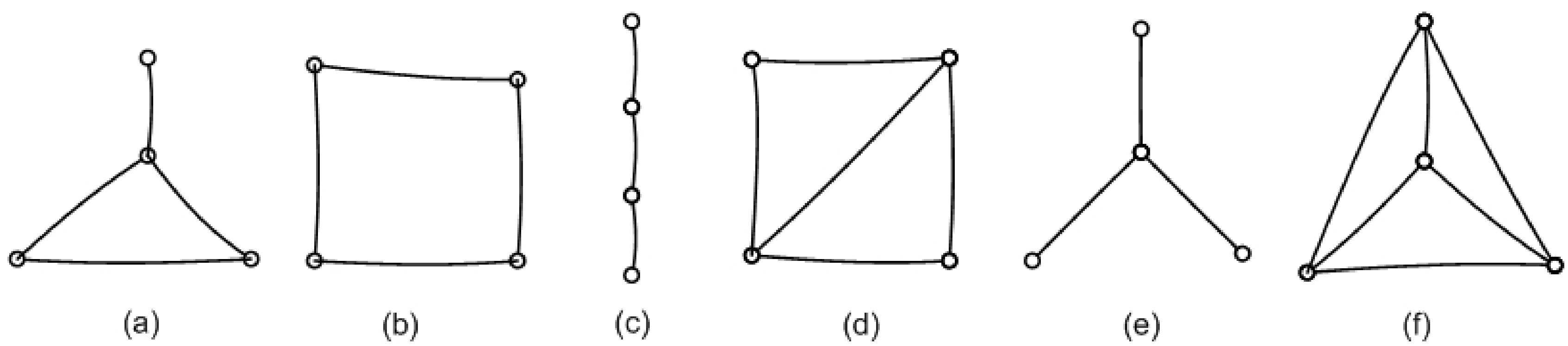




- 2. Which graphs below are subgraphs of those shown above.



- 3. Write down the degree sequence in the graphs below. Verify that the sum of the values of the degrees are equal to twice the number of edges in the graph.



1. Answers:

- The vertex set is {V1,V2,V3,V4,V5,V6},

- The edge set is {e1,e2,e3,e4,e5,e6,e7,e8, e9}.

Answers

- 2. (c), (d)

- **<u>Answers</u>**

- 3. (a) (3, 2, 2, 1);

  (b) (2, 2, 2, 2);

  (c) (2, 2, 1, 1);

  (d) (3, 3, 2, 2);

  (e) (3, 1, 1, 1);

  (f) (3, 3, 3, 3);