Discrete Mathematics and Statistics (CSE107)
Tutorial I
Part 1. Number Systems and Proof Techniques

1. $x = 4, y = 5.$
2. $x = 2, y = 3.$
3.
   - The natural numbers? No. Take x = 2, y = 5.
   - The positive integers? No. Take x = 2, y = 5.
   - The integers? Yes.
   - The rationals? Yes.
4. Since q and p are rational, then can be written as $q = \frac{x}{y}$ and $p = \frac{w}{z}$ where w, x, y, z are integers and $y \neq 0$ and $z \neq 0$. Then
$$q - p = \frac{x}{y} - \frac{w}{z} = \frac{xz}{yz} - \frac{wy}{yz} = \frac{xz - wy}{yz}.$$
Now note that $xz - wy$ is an integer (since the integers are closed under multiplication and subtraction) and yz is an integer which is not 0 since y and z are not 0. Thus, $q - p$ is rational.
5. The only prime number that is even is "2".
6. Since x and y are even positive integers, we can write $x = 2w$ and $y = 2z$ where w and z are positive integers. Thus $x + y = 2w + 2z = 2(w + x)$. So $x + y$ has 2 as a factor, so it is even.
7. We can write $x = 2w$ where w is a positive integer. Since $x > 2$, we know $w > 1$. Then $x - 2 = 2w - 2 = 2(w - 1)$, which is a positive integer with 2 as a factor, so it is even.
8. Proof:

(*) $$\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Base case: When $n = 1$, the left side of (*) is $(-1)1^2 = -1$, and the right side is $\frac{(-1)1(1+1)}{2} = -1$, so both sides are equal and (*) is true for $n = 1$.

Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose (*) is true for $n = k$. Then
$$\sum_{i=1}^{k+1}(-1)^i i^2 = \sum_{i=1}^{k}(-1)^i i^2 + (-1)^{k+1}(k+1)^2$$
$$= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$$
$$= \frac{(-1)^k (k+1)}{2}\big(k - 2(k+1)\big)$$
$$= \frac{(-1)^k (k+1)}{2}(-k-2)$$
$$= \frac{(-1)^{k+1}(k+2)}{2}$$

Thus, (*) holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: By the principle of induction, (*) is true for all $n \in \mathbb{Z}_+$

9. Let x be a real number in the range given, namely $x > -1$. We will prove by induction that for any positive integer n,
(*) $$(1+x)^n \geq 1 + nx.$$

holds for any $n \in \mathbb{Z}_+$.
Base case: For $n = 1$, the left and right sides of (*) are both $1 + x$, so (*) holds.
Induction step: Let $k \in \mathbb{Z}_+$. be given and suppose (*) is true for $n = k$. We have
$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1 + (k+1)x + kx^2$$
$$\geq 1 + (k+1)x$$

Hence (*) holds for $n = k + 1$, and the proof of the induction step is complete.
Conclusion: By the principle of induction, it follows that (*) holds for all $n \in \mathbb{Z}_+$.

10. Proof:
Suppose √3 is rational, then $\sqrt{3} = a/b$ for some (a,b) suppose we have a/b in simplest form.
$$\sqrt{3} = a/b$$
$$a^2 = 3b^2$$
If b is even, then a is also even in which case a/b is not simplest form.
If b is odd then a is also odd. Therefore:
$$a = 2n + 1$$
$$b = 2m + 1$$
$$(2n+1)^2 = 3(2m+1)^2$$
$$4n^2 + 4n + 1 = 12m^2 + 12m + 3$$
$$2n^2 + 2n = 6m^2 + 6m + 1$$
$$2(n^2 + n) = 2(3m^2 + 3m) + 1$$
Since $(n^2 + n)$ is an integer, the left hand side is even. Since $(3m^2 + 3m)$ is an integer, the right hand side is odd and we have found a contradiction. Therefore our hypothesis is false.