r-Combinations with Repetition Allowed

Write a complete list to find the number of 3-combinations with repetition allowed, or multisets of size 3, that can be selected from {1, 2, 3, 4}.

Observe that because the order in which the elements are chosen does not matter, the elements of each selection may be written in increasing order, and writing the elements in increasing order will ensure that no combinations are overlooked.

Solution

[1, 1, 1] ;[1, 1, 2]; [1, 1, 3]; [1, 1, 4]	all combinations with 1, 1
[1, 2, 2] ;[1, 2, 3]; [1, 2, 4];	all additional combinations with 1, 2
[1, 3, 3] ;[1, 3, 4]; [1, 4, 4];	all additional combinations with 1, 3 or 1, 4
[2, 2, 2] ;[2, 2, 3]; [2, 2, 4];	all additional combinations with 2, 2
[2, 3, 3] ;[2, 3, 4]; [2, 4, 4];	all additional combinations with 2, 3 or 2, 4
[3, 3, 3] ;[3, 3, 4]; [3, 4, 4];	all additional combinations with 3, 3 or 3, 4
[4, 4, 4]	the only additional combination with 4, 4

Thus there are twenty 3-combinations with repetition allowed.

How could the number twenty have been predicted other than by making a complete list?

Consider the numbers 1, 2, 3, and 4 as categories and imagine choosing a total of three numbers from the categories with multiple selections from any category allowed.

The results of several such selections are represented by the table below.

Category 1		Category 2		Category 3		Category 4	Result of the Selection
	|	×	|		|	× ×	1 from category 2 2 from category 4
×	|		|	×	|	×	1 each from categories 1, 3, and 4
× × ×	|		|		|		3 from category 1

As you can see, each selection of three numbers from the four categories can be represented by a string of vertical bars and crosses.

Three vertical bars are used to separate the four categories, and three crosses are used to indicate how many items from each category are chosen.

Each distinct string of three vertical bars and three crosses represents a distinct selection.

For instance, the string

$$\times \times \mid \mid \times \mid$$

represents the selection: two from category 1, none from category 2, one from category 3, and none from category 4.

Thus the number of distinct selections of three elements that can be formed from the set

{1, 2, 3, 4} with repetition allowed equals the number of distinct strings of six symbols consisting of three **|**'s and three **×**'s.

But this equals the number of ways to select three positions out of six because once three positions have been chosen for the **×**'s, the **|**'s are placed in the remaining three positions. Thus the answer is

$$C(6, 3) = 6!/3!(6-3)! = 20$$

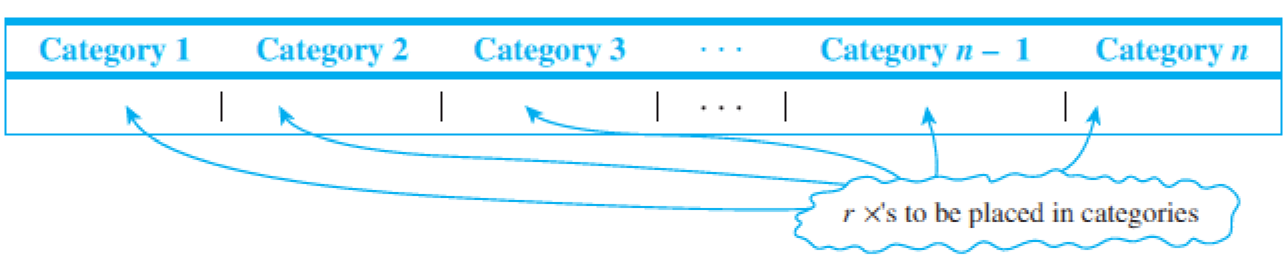
as was obtained earlier by listing.

The analysis of this example extends to the general case. To count the number of

r-combinations with repetition allowed, or *multisets of size r*, that can be selected from a set of *n* elements, think of the elements of the set as categories.

Then each *r-combination with repetition* allowed can be represented as a string of $n - 1$ vertical bars (to separate the *n* categories) and *r* crosses (to represent the *r elements* to be chosen).

The number of **×**'s in each category represents the number of times the element represented by that category is repeated.



The number of strings of *n – 1 vertical bars* and *r crosses* is the number of ways to choose r positions, into which to place the *r* crosses, out of a total of r + (n−1) positions, leaving the remaining positions for the vertical bars.

Theorem

The number of *r-combinations with repetition allowed* (multisets of size r) that can be selected from a set of n elements is

$$\binom{n+r-1}{r}$$

This equals the number of ways *r* objects can be selected from n categories of objects with repetition allowed.

Example

A person giving a party wants to set out *15 cans of soft drinks* for his guests.

He shops at a store that sells *five different types of soft drinks*.

How many different selections of cans of 15 soft drinks can he make?

Solution

Think of the five different types of soft drinks as the *n* categories and the 15 cans of soft drinks to be chosen as the *r* objects (so n = 5 and r = 15).

Each selection of cans of soft drinks is represented by a string of 5 – 1 = 4 vertical bars (to separate the categories of soft drinks) and 15 crosses (to represent the cans selected).

For instance, the string

× × × | × × × × × × × | | × × × | × ×

represents a selection of three cans of soft drinks of type 1, seven of type 2, none of type 3, three of type 4, and two of type 5.

The total number of selections of 15 cans of soft drinks of the five types is the number of strings of 19 symbols, 5 – 1 = 4 of them **|** and 15 of them **×**:

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{15!\cdot 16\ \cdot 17\cdot\ 18\cdot\ 19}{15!\cdot 4\ \cdot 3\cdot 2\cdot 1} = 3876$$