Model Answers for InClass Exercises

(a) Let $x \in \mathbb{Z}$. If x^2 is even, then x is even. If you think that it is true, prove it. Otherwise, give a counterexample.

Answer:

True.

Proof by contradiction -

Step 1: Suppose x^2 is even, then x is NOT even (the assumption).

Step 2: There exists an integer y such that $x = 2y + 1$

Step 3: Then $x^2 = (2y + 1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$, so x^2 is odd.

Step 4: This is a contradiction, so the assumption must be wrong.

(b) There exist two distinct positive integers whose sum and difference are perfect squares. Prove it or give an example.

Answer:

$5+4 = 9$ and $5-4 = 1$ (examples)

(c) For every integer $n \geq 1$, use proof by induction to show that $5^n - 1$ is divisible by 4.

Answer:

Proof by induction -

Base Case: *n=1,* i.e., 5^1 -1 = 4=4$\times$ 1

Inductive Step: Assume it is true for *n=k* for $5^k - 1 = 4r$ where *r* is an integer

Proof continues for *n=k+1*:

$5^{k+1} - 1 = 5 \times 5^k - 1$=5$\times$ $(4r + 1) - 1$=

4x5r-5-1=4x(5r+1), which is divisible by 4

Additional Exercises:

(a) Use proof by induction to show that $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$ for $x \neq 1$.

(b) Let x and y be two real numbers such that $x, y \geq 0$. Then $\frac{x+y}{2} \geq \sqrt{xy}$. If you think that it is true, prove it. Otherwise, give a counterexample.

Model Answers:

(a) Answer:

Proof by induction -

Base Case: *n=0,* i.e., $\sum_{i=0}^{0} x^i = x^0 = 1 = \frac{1-x^{0+1}}{1-x}$

Inductive Step: Assume it is true for *n=k*.

Proof continued for *n=k+1*:

$$\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1} = \frac{1-x^{k+1}}{1-x} + x^{k+1} = \frac{1-x^{k+1}+(1-x)(x^{k+1})}{1-x} = \frac{1-x^{k+2}}{1-x}$$

(b) Answer:

True.

Proof by contradiction -

Step 1: Suppose $\frac{x+y}{2} < \sqrt{xy}$ (the assumption).

Step 2: $\frac{(x+y)^2}{4} < xy$ (take the square).

Step 3: $(x+y)^2 - 4xy < 0$

Step 4: Then $(x-y)^2 < 0$. This is a contradiction, so the assumption must be wrong.