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Discrete Mathematics and Statistics – CPT107

COMBINATORICS

Part 6. Combinatorics

Reading: Discrete Mathematics for Computing R. Haggarty, Chapter 6.

Reading: Discrete Mathematics for Computer Scientists, J.K. Truss, Section 5.1, 5.3

Contents

- **Combinatorics** is the study of collections of objects. Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules.
- **Counting** objects is important in order to analyze algorithms and compute discrete probabilities
- *Counting* is used to determine the complexity of algorithms (the run time of an algorithm, need to count the number of times certain steps or loops were executed). *Counting* also plays a crucial role in probability theory.
- Solving combinatorial problems always requires knowledge of basic combinatorial configurations such as *arrangements*, *permutations*, *and combinations*.

Problems we want to solve

How many cards must Bob select from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

How many ways are there to seat four people around a circular table, where two seating are considered the same when each person has the same left and right neighbor?

Alice likes 4 types of bagels. How many ways are there to select a dozen bagels when she chooses only among the 4 types she likes?

Basic Counting Principles and Counting Rules

- Product Rule (Multiplication Principle)
- Sum Rule (Addition Principle)
- Subtraction Rule (Subtraction Principle)
- Division Rule (Division Principle)

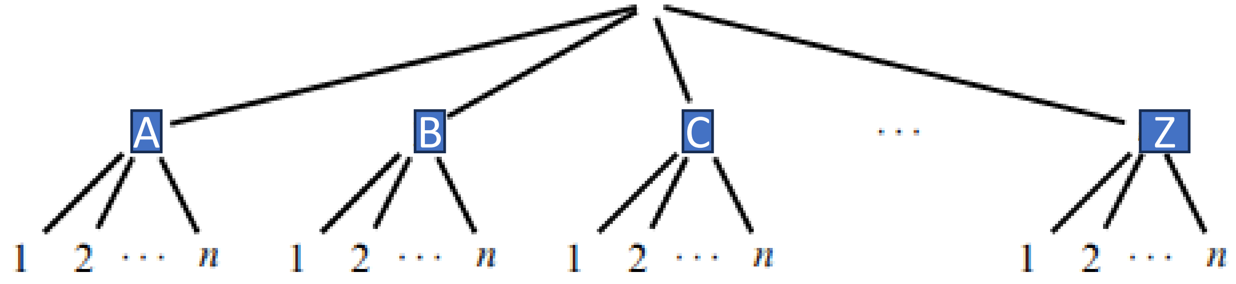
The Product Rule (Multiplication Principle)

If there are **n(A) ways to do A** and **n(B) ways to do B**, then

the number of ways to do A and B is **n(A) × n(B)**.

This is true if <u>the number of ways of doing A and B are independent</u>; the number of choices for doing B is the same regardless of which choice you made for A.

Again, this generalizes. There are n(A) × n(B) × n(C) ways to do A and B and C



Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \ldots \times A_m$ is done by choosing an element in A_1, an element in A_2 , ..., and an element in A_m.
- By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|.$$

The Product Rule (Multiplication Principle)

Example 1: How many 7-bit strings can be made ?

Solution:

Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

Example 2**:** How many different license plates can be made if each plate is a sequence of three uppercase English letters followed by three digits?

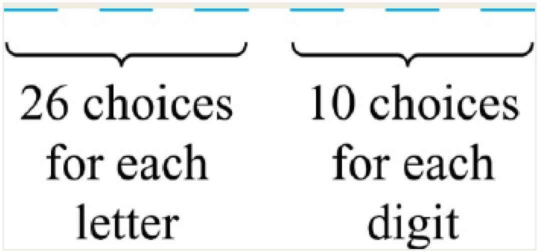
e.g., AGF349

Solution:

By the product rule, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17.576.000$$

different possible license plates.



The Sum Rule (the Addition Principle)

If there are n_1 ways for one task and n_2 ways for another task and the two tasks cannot be done at the same time, then there are $\mathbf{n_1 + n_2}$ ways to select one of these tasks.

Sum Rule in terms of sets

The sum rule can be phrased in terms of sets.

$$|A \cup B| = |A| + |B| \text{ as long as } A \text{ and } B \text{ are disjoint sets.}$$

Or more generally,

$$|A_1 \cup A_2 \cup \dots \cup A_m| = |A_1| + |A_2| + \dots + |A_m|$$

$$\text{when } A_i \cap A_j = \varnothing \text{ for all } i, j.$$

The Sum Rule (the Addition Principle)

Example 1: Friday night you can see one of five movies, go to one of two concerts, or stay home.

How many choices do you have for spending Friday night?

Solution

There are $5 + 2 + 1 = 8$ choices

Example 2: The CS department chooses either a student or a faculty as a representative for a committee.

How many choices are there for this representative if there are 47 CS faculty and 558 CS students and no one is both a faculty member and a student.

Solution

There are $47 + 558 = 605$ possible ways

Notation for Sums and Products

Let $f : D \to \mathbb{R}$ be a function with some domain D. For $S \subseteq D$, $\sum_{i \in S} f(i)$ denotes the sum of $f(i)$ over all $i \in S$ and $\prod_{i \in S} f(i)$ denotes the product of $f(i)$ over all $i \in S$.

Combining Sum and Product Rule

Example:

Suppose an ID can be either a two letters or a letter followed by a digit. Find the number of possible IDs.

Solution:

$(26 \cdot 26) + (26 \cdot 10) = 936$

The Subtraction Rule (Subtraction Principle)

If a task can be done either in one of $\mathbf{n_1}$ ways or in one of $\mathbf{n_2}$ ways, then *the total number of ways to do the task* is $n_1 + n_2$ *minus the number of ways to do the task that are common to the two different ways*.

Also known as, the ***principle of inclusion-exclusion***:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Counting Bit Strings

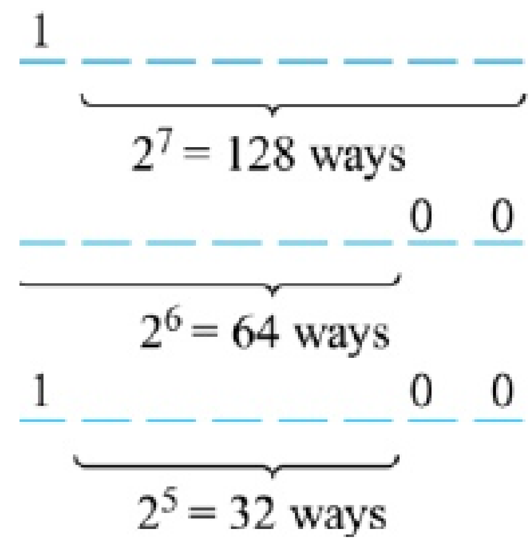
How many bit strings of length eight either start with a 1 or end with 00?

Solution: Use the subtraction rule.

Number of bit strings of length eight
that start with a 1: $2^7 = 128$

Number of bit strings of length eight
that end with 00: $2^6 = 64$

Number of bit strings of length eight
that start with a 1 and end with 00: $2^5 = 32$



Hence, the number is 128 + 64 - 32 = 160.

The Division Rule (Division Principle)

If a task can be carried out in n ways, and for every way **w**, exactly d of the n ways correspond to way **w**, then there

$$\frac{n}{d}$$

ways to do the task.

Example: Bob counts the number of people in the room by courting the number of ears (assume every person has two ears).

To get the number of people, he needs to divide the number of ears by 2.

Factorial Function

The product of the positive integers from 1 to n inclusive is denoted by $n!$, read "n factorial." Namely:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2)(n-1)n = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Accordingly, $1! = 1$ and $n! = n(n-l)!$. It is also convenient to define $0! = 1$.

How many ways are there of placing a, b and c in a row? There are six ways, namely

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba.$$

Note that there are three choices for the first place, then two for the second, and then just one for the third; so by the multiplication principle there are $3 \times 2 \times 1 = 6$ possible orderings. In general, if we define $n!$ ("n factorial") by

$$n! = n(n-1)(n-2)\ldots 2.1$$

Combinatorics studies the way in which discrete structures/objects can be combined or arranged.

- **Permutations**
  - Permutation without repetition
  - Permutation with repetition
  - Permutation of identical objects
- **Combinations**
  - Combinations without repetition
  - Combinations with repetition
  - Binomial coefficients

COUNTING METHODS

- **PERMUTATIONS - order matters!**
  - **Without repetition**
  - **With repetition**
  - **of Identical Objects**
- **COMBINATIONS - order doesn’t matter!**
  - **Without repetition**
  - **With repetition**

Permutations

PERMUTATIONS - Order matters!

A permutation is an ordered arrangement of ***r*** objects chosen from ***n*** available objects.

The word "permutation" also refers to the act or process of changing the linear order of an ordered set.

If $0 \leq r \leq n$ (and r is a natural number) then an ***r-permutation* of *n* objects** is an arrangement of r of the n objects into an ordered line.

Without repetition - **when select objects one time** (after you draw an ace of spades out of a deck, there is 0 probability of getting it again).

With repetition – **when select the same objects multiple times** (after you roll a 6, you can roll a 6 again on the same die).

Of identical objects - **when some (or all) of the objects to be arranged are identical**

Permutations - without repetition

This means that *once one of the elements of the set has left, it cannot reappear.*

Formula for **permutations without repetition**

$$P(n, r) = \frac{n!}{(n-r)!}$$

- *n* the number of elements of the set
- *r* the elements we choose, $0 \leq r \leq n$
- $P(n, n) = n!$
- $P(n, 0) = 1$

Reminder:
$n! = n(n-1)(n-2)\ldots 1.$ - n factorial.
$(n+1)! = (n+1)n!$
$0! = 1$

Permutation—without repetition

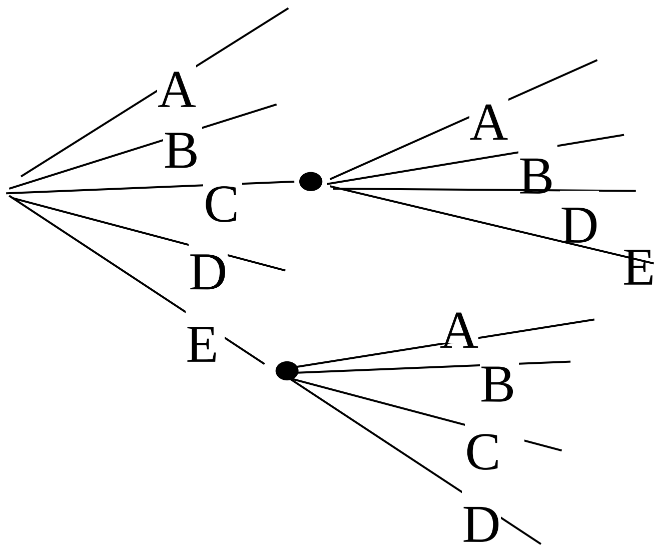
Example

How many ways there are to order **5** people in **5** chairs

Seat One:
5 possible

Seat Two:
only 4 possible

Etc….



…….

# of permutations = 5 x 4 x 3 x 2 x 1 = 5!

$$\frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!$$

There are 5! ways to order 5 people in 5 chairs (since a person cannot repeat)

Permutation—without repetition

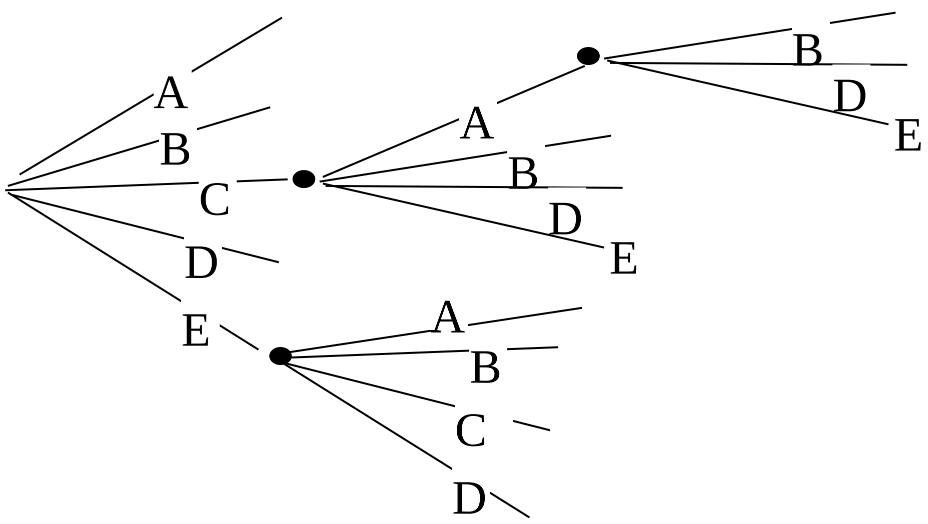
Example

What if you had to arrange **5** people in only **3** chairs (meaning 2 are out)?

<u>Seat One</u>: 5 possible

<u>Seat Two</u>: only 4 possible

<u>Seat Three</u>: only 3 possible



$$5 \cdot 4 \cdot 3 =$$

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} =$$

$$\frac{5!}{(5-3)!}$$

Permutation with repetition

A permutation with repetition is the one in which we have *n* elements that we take a number of *r* times.

Permutation with repetition, *if we pick up an element, it can come back out again in the second election and so on until reaching r times.*

Formula for **permutations with repetition**

$$\boxed{P(n, r) = n^r}$$

- *n* the number of elements of the set
- *r* the elements we choose
- $0 \leq r \leq n$

Permutations - with repetition

Example

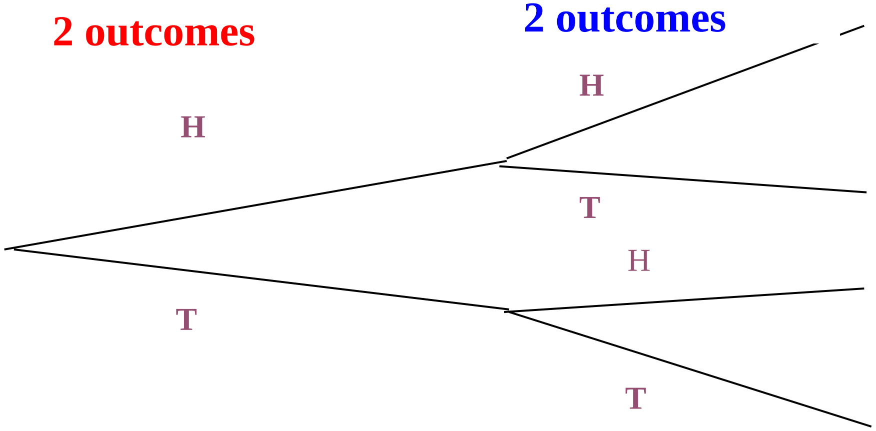
“memoryless” – After you get heads, you have an equally likely chance of getting a heads on the next toss (unlike in cards example, where you can’t draw the same card twice from a single deck).

How many possible outcomes of getting two heads in a row when tossing a coin?

2^2 total possible outcomes: {HH, HT, TH, TT}

Toss 1:
2 outcomes

Toss 2:
2 outcomes



$$\boldsymbol{Outcomes = 2^2}$$

Permutations of Identical Objects

The number of permutations of n objects with n_1 identical objects of type 1, n_2 identical objects of type 2, …, and n_k identical objects of type k is:

$$\frac{n!}{n_1!\,n_2!\ldots n_k!}$$

In general, the number of permutations of n objects with r of the objects identical objects is:

$$\frac{n!}{r!}$$

Permutations of Identical Objects

Example How many words (including nonsense words) can be made from rearrangements of the word ALPACA?

$\frac{6!}{3!} = \frac{720}{6} = 120.$ There are 6 letters in ALPACA and one of them, 'A' is repeated 3 times.

Example How many words can be made from rearrangements of the word BANANA?

$\{B, A, N, A, N, A\} = \{A, B, N\}.$
The 'A' is repeated 3 times.
The 'N' is repeated 2 times.
The 'B' is repeated once.
Hence the answer is $\frac{6!}{1! \cdot 2! \cdot 3!} = 60$.

Combinations

Combinations – without repetition

- An unordered selection of r elements from a set containing n distinct elements.

Example:

S = {A, J, Q, K}. {A, J, K}={K, A, J}={J, K, A} are all *3-combinations of set S*.

The number of r-combinations of a set of size n is called **binomial coefficient**, denoted $C(n,r)$, and given by:

$$\boldsymbol{C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}}$$

$0 \leq r \leq n$

We pronounce it "***n choose r***" because it is the number of ways to choose a size-k subset of n elements.

Example

A menu in a Chinese restaurant allows you to order exactly three of seven main dishes. How many different combinations of main dishes can you order?

Solution:

$$\binom{7}{3} = \frac{7!}{(7-3)!3!} = 35.$$

Combinations – with repetition

- An **unordered** selection of *r* elements from a set containing *n* distinct elements, with **repetition permitted**.

$$\boldsymbol{C(n+r-1,r)=\binom{n+r-1}{r}=\frac{(n+r-1)!}{r!\,(n-1)!}}$$

- *n* the number of elements of the set
- *r* the elements we choose, $0 \leq r \leq n$

Example

You are to choose three pieces of candy from a jar containing 12 (or more) pieces of candy.

There are four flavors of candy: *watermelon*, *butterscotch*, *licorice* and *cinnamon*.

There are at least *three* pieces of each flavor in the jar.

In how many ways can you choose the candy?

Solution

There are

$$\boldsymbol{C(4+3-1,3) = \binom{4+3-1}{3} = \frac{(4+3-1)!}{3!\,3!} = \frac{6!}{3!\,3!} = 20}$$

Example

Four family members have just completed lunch and are ready to choose their after-lunch fruit.
There are *bananas*, *apples*, *pears*, *kiwi*, *apricots*, and *oranges* in the house.
In how many ways can a selection of four pieces of fruit be chosen?

Solution

Each fruit variety for every family member to have his or her first choice, and only the selection of varieties (not which person eats what fruit) is of interest, this is a combination with repetition.
The solution is thus

$$\boldsymbol{C(6+4-1,4)=\binom{6+4-1}{4}=\frac{(6+4-1)!}{4!\,5!}=\frac{9!}{4!\,5!}=126}$$

COMBINATIONS

If you have **n** objects to choose from

And you have **k** slots to put them in

The order doesn't matter

(each slot is the same)

All the prizes or slots or positions are the same.

i.e., every winner gets the same award

Without repetition

$$\boxed{\frac{n!}{k!\,(n-k)!}}$$

With repetition

$$\boxed{\frac{(n+k-1)!}{k!\,(n-1)!}}$$

PERMUTATIONS

If you have **n** objects to choose from

And you have **k** slots to put them in

The order does matter

(each slot is different)

All the prizes or slots or positions are different.

i.e., first place, second place etc.

Without repetition

$$\boxed{\frac{n!}{(n-k)!}}$$

With repetition

$$\boxed{n^k}$$

End of lecture

- **Summary**
- Basic Counting Principles and Counting Rules
- Permutations
  - Permutation without repetition
  - Permutation with repetition
  - Permutation of identical objects
- Combinations
  - Combinations without repetition
  - Combinations with repetition