Question 1 * (m_1, m_2) means the greatest common divisor between m_1 and m_2

(a) Proof: Assume $a-b\sqrt{3}$ is rational $(a, b \in \mathbb{Q}, b \neq 0)$. ~~$b_1 \neq 0$~~

Let $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, $a_1, b_1 \in \mathbb{Z}$, $a_2, b_2 \in \mathbb{Z}$ and $a_2, b_2 \neq 0$; $(a_1, a_2) = 1$, $(b_1, b_2) = 1$

Let $a - b\sqrt{3} = \frac{a_1}{a_2} - \frac{b_1}{b_2}\sqrt{3} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$; $(p, q) = 1$

$$\Rightarrow \frac{a_1 b_2 q - a_2 b_1 q\sqrt{3}}{a_2 \cancel{b_2} q} = \frac{a_2 b_2 p}{\cancel{a_2} b_2 q} \Rightarrow \frac{a_1 b_2 q - a_2 b_2 p}{a_2 b_1 q} = \sqrt{3} \quad (\text{as } b \neq 0 \Rightarrow b_1 \neq 0 \Rightarrow a_2 b_1 q \neq 0)$$

As $a_1 b_2 q$, $a_2 b_2 q$, $a_2 b_1 q \in \mathbb{Z}$ Let $M_1 = m_1 \cdot C = a_1 b_2 q - a_2 b_2 p$, ~~$\in \mathbb{Z}$~~

$M_2 = m_2 \cdot C = a_2 b_1 q$, $(M_1, M_2) = C$, $m_1, C, m_2 \in \mathbb{Z}$, $m_2 \neq 0$, $(m_1, m_2) = 1$.

then $\sqrt{3} = \frac{M_1}{M_2} = \frac{m_1}{m_2}$, $(m_1, m_2) = 1$

$m_1^2 = 3m_2^2 \Rightarrow 3 \mid m_1$ Let $m_1 = 3k_1 \Rightarrow 9k^2 = 3m_2^2 \Rightarrow m_2^2 = 3k_1^2$

then $3 \mid m_2$ Let $m_2 = 3k_2$, at that time $(m_1, m_2) = (3k_1, 3k_2) = 3 \neq 1$

Therefore $\sqrt{3}$ can not be written as $\frac{a_1 b_2 q - a_2 b_2 p}{a_2 b_1 q}$, which can conclude that if a and b are rational numbers, $b \neq 0$, then $a - b\sqrt{3}$ is irrational.

(b). I think $\sqrt{75}$ is irrational, let me explain why.

Proof: Assume $\sqrt{75}$ is rational, then $\sqrt{75} = \frac{p}{q}$, $q \neq 0$, $p, q \in \mathbb{Z}$, $(p, q) = 1$.

$\Rightarrow 5\sqrt{3} = \frac{p}{q} \Rightarrow 3 = \frac{p^2}{25q^2} \Rightarrow p^2 = 3 \cdot 25q^2 \Rightarrow 3 \mid p^2 \Rightarrow 3 \mid p$.

Let $p = 3k_1 \Rightarrow 9k_1^2 = 3 \cdot 25q^2 \Rightarrow 3k_1^2 = 25q^2 \Rightarrow 3 \mid 25q^2$ as $(3, 25) = 1$

$\Rightarrow 3 \mid q^2 \Rightarrow 3 \mid q \Rightarrow$ Let $q = 3k_2 \Rightarrow (p, q) = (3k_1, 3k_2) = 3 \neq 1$.

Therefore $\sqrt{75}$ is irrational.

(c)

Proof: $n=0$, $n^3-n=0$, can be ~~divided~~ divisible by 3.

$n=1$, $n^3-n=0$, can be divisible by 3.

Assume $n=k$, ~~$(n\geq k, k\in Z)$~~ $(k\geq 1, k\in Z)$, k^3-k is divisible by 3

Let $k^3-k=3p$, $p\in Z$

Considering $n=k+1$,

$$n^3-n=(k+1)^3-(k+1)=k^3+3k^2+3k+\cancel{1}-k-\cancel{1}$$
$$=k^3-k+3k^2+3k$$
$$=3p+3(k^2+k)$$
$$=3(p+k^2+k)$$

As, $p+k^2+k\in Z$ and $p+k^2+k=\frac{k^3}{3}-\frac{k}{3}+k^2+k=\frac{k^3}{3}+k^2+\frac{2}{3}k$.

$$=\frac{k}{3}(k^2+3k+2)=\frac{k}{3}(k+1)(k+2)\geq\frac{1}{\cancel{3}}\cdot 2\cdot\cancel{3}=2\neq 0.$$

therefore $p+k^2+k\in Z^+ \Rightarrow n^3-n$ is divisible by 3.

So In all, for every $n\in N$, n^3-n is divisible by 3

(d) Proof: $x=0$, then $2(1)=2$ $\quad 2^{x+2}-2=2^2-2=2$, so true $2=2$.

$x=1$, $2(1+\cdots+2^x)=2(1+2)=6$ $\quad 2^{x+2}-2=2^3-2=8-2=6$, so true.

Assume $x=k$, $(k\geq 1, k\in Z^+)$ $2(1+\cdots+2^k)=2^{k+2}-2$.

Considering $x=k+1$, then $2(1+\cdots+2^k+2^{k+1})=2(1+\cdots+2^k)+2\cdot 2^{k+1}$

$$=2^{k+2}-2+2^{k+2}$$
$$=2\cdot 2^{k+2}-2=2^{k+3}-2$$
$$=2^{(k+1)+2}-2$$, so true.

Therefore, $x\in Z$, $x\geq 0$, $2(1+2+4+\cdots+2^x)=2^{x+2}-2$

(2)

I think it's false, let me give a counter example:

Let $A=\{1,2\}$, $C=\{3\}$, $B=\{4,5\}$

$A-C=\{x\in A \mid x\notin C\}=\{1,2\}$

$B-C=\{x\in B \mid x\notin C\}=\{4,5\}$

$(A-C)\cup(B-C)=\{1,2,4,5\}\neq\{1,2\}=A-C$

therefore, it's false

(b) I think it's true.

Proof: Firstly, I will proof: "if $X\subseteq Y$, then $X\cap Y=X$"

i $X=\emptyset$, $\emptyset\cap Y=\emptyset$ $X\cap Y=X$, goes true

ii $X\neq\emptyset$, for every a in X, $a\in X$, as $X\subseteq Y \Rightarrow a\in Y$.

Let $Y-X=M$, $M=\{m\in Y \mid m\notin X\}$

$X\cap Y=\underline{X\cap(X\cup M)=(X\cap X)\cup(X\cap M)}$ "The algebra of set 2"

As $X\cap X=X$, $X\cap M=\{a\in X\}\cap\{m\in Y \mid m\notin X\}$

for every $a\in X$, $a\notin M \Rightarrow X\cap M=\emptyset$

then $X\cap Y=X\cup\emptyset=X$, goes true

Secondly, I will proof "if $X\cap Y=X$, then $X\subseteq Y$"

i. $X=\emptyset$. $\emptyset\cap Y=\emptyset$, $\emptyset\subseteq Y$ for every Y, goes true

ii $X\neq\emptyset$, $X\cap Y=X \Rightarrow Y\neq\emptyset$

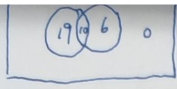
for every a in X, $a\in X$, $X\cap Y=X$ then all a in Y which can write

as for every $a\in X$, $X\cap Y=X$, then $a\in Y$

As $\forall a\in X$, we can get $a\in Y \Rightarrow X\subseteq Y$.

then "if $X\cap Y=X$, then $X\subseteq Y$" goes true

In all, $X\subseteq Y$ if only if $X\cap Y=X$ (X and Y are sets)



attack $\downarrow$ midfield $\downarrow$

Let $|U| = 35$, $|A| = 29$, $|B| = 16$, $|A \cap B| = 10$

then $|A \cup B| = |A| + |B| - |A \cap B| = 45 - 10 = 35$

$|U| - |A \cup B| = 0$, which means neither attack nor midfield is 0

Attack only : $|A - B| = |A| - |A \cap B| = 29 - 10 = 19$

Midfield only : $|B - A| = |B| - |A \cap B| = 16 - 10 = 6$

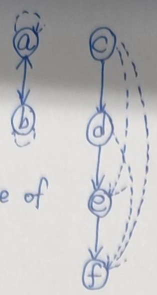
In all : 1. in attack only 19.

2 in midfield only 6.

3. in neither attack nor midfield 0

n 3.

he directed graph is:



Let S^t is the transitive closure of S on X

1 jumps: As $(a,b) \in S$ and $(b,a) \in S$, then $(a,a) \in S^t$.

As $(b,a) \in S$ and $(a,b) \in S$, then $(b,b) \in S^t$.

As $(c,d) \in S$ and $(d,e) \in S$, then $(c,e) \in S^t$.

As $(d,e) \in S$ and $(e,f) \in S$, then $(d,f) \in S^t$

2 jumps: As $(c,d),(d,e),(e,f) \in S$, then $(c,f) \in S^t$

Therefore, $S_t = S \cup \{(a,a),(b,b),(c,e),(d,f),(c,f)\}$

$= \{(a,a),(a,b),(b,a),(b,b),(c,d),(c,e),(c,f),(d,e),(d,f),(e,f)\}$

(b) I think S is a partial order.

To prove it, just to prove that the relation S is reflective, transitive and antisymmetric

① Reflective means $\forall x,\ x \in X \Rightarrow (x,x) \in S$

As $x = a,b,c,d,e$, and $(a,a),(b,b),(c,c),(d,d),(e,e) \in S$ therefore S is reflective.

② Transitive means $\forall x \forall y \forall z,\ x \in X \cap y \in X \cap z \cap X \cap (x,y) \in S \cap (y,z) \in S \Rightarrow (x,z) \in S$

Based on Transitivity and Composition, a relation S is transitive if and only if $S \circ S \subseteq S$

Use Logical matrix:

$$S = \begin{pmatrix} T & F & T & T & T \\ F & T & T & T & T \\ F & F & T & T & T \\ F & F & F & T & F \\ F & F & F & F & T \end{pmatrix} \quad S \circ S = \begin{pmatrix} T & F & T & T & T \\ F & T & T & T & T \\ F & F & T & T & T \\ F & F & F & T & F \\ F & F & F & F & T \end{pmatrix} = S \Rightarrow S \circ S = S \subseteq S$$

Therefore S is transitive.

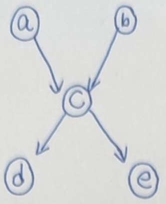
③ Antisymmetric means $\forall x \forall y\ x \in X \cap y \in X \cap (x,y) \in S \cap (y,x) \in S \Rightarrow x = y$

Let $1 \le i \le 5$, $1 \le j \le 5$, from S we can see there doesn't exist $S(i,j)$ and $S(j,i)$ both are T $(i \ne j)$. This is because $S(i,j) = F$ $(i > j)$. So no matter what $S(j,i)$ is, $S(i,j)$ and $S(j,i)$ aren't both T. As in ① we prove S is reflective, then $S(i,i) = T$ $(1 \le i \le 5)$ which is the only condition that promises

Therefore S is Antisymmetric

From ①, ②, ③, we can conclude S is a partial order.

The Hasse diagram:



(C)/(1) I think it's true.

Proof: Firstly, we should know that $A \cap B \neq \emptyset$ otherwise it's meaningless

the transitive relations that we can get $\begin{cases} \forall x,y,z \in S,\ xAy,\ yAz \Rightarrow xAz & ① \\ \forall x,y,z \in S,\ xBy,\ yBz \Rightarrow xBz & ② \end{cases}$

Let $R = A \cap B$, Now we need to prove "$\forall x,y,z \in S,\ xRy,\ yRz \Rightarrow xRz$"

As $R = A \cap B \Rightarrow xRy$ fulfill xAy and xBy

yRz fulfill yAz and yBz

from ①, ② then xRy, yRz fufill xAz and xBz

As $R = A \cap B$, $xAz, xBz \Rightarrow xRz$

Therefore We prove "$\forall x,y,z \in S.\ xRy,\ yRz \Rightarrow xRz$"

In all, $A \cap B$ is transitive

C/(2) I ~~don~~ think it's false

Let $S = \{1, 2, 3\}$.

$B = \{(1,2), (1,3), (2,3)\}$ $A = \{(2,3), (3,1), (2,1)\}$

~~$A = \{(2,3), (1,2), (1,3)\}$~~

then $A \circ B = \{(a,c) \mid \text{exists } b \text{ that } aBb \text{ and } bAc\} = \{(1,3), (1,1), (2,1)\}$

~~$= \{(1,3), (1,4), (2,4)\}$~~

In $A \circ B$, if $A \circ B$ is transitive, then $(2,1) \in A \circ B, (1,3) \in A \circ B \Rightarrow (2,3) \in A \circ B$

but In fact, $(2,3) \notin A \circ B$

In all $A \circ B$ isn't transitive

I think it's true.

s "A ∘ B is also transitive" is false. and "A ∩ B is also transitive" is true.

from (C)/(1), (C)/(2) from truth table that

P	Q	$P \Rightarrow Q$
F	T	T
F	F	T
T	T	T
T	F	F

We can conclude that "A ∘ B is also transitive" implies "A ∩ B is also transitive" is True.

(d) As A is equivalence relation, then it fufills the following condition:

$\forall x \Rightarrow xAx \ (x \in S)$ [reflective]

$\forall x, y \in S, \ xAy \Rightarrow yAx$ [symmetric]

$\forall x, y, z \in S, \ xAy, yAz \Rightarrow xAz$ [transitive]

$A^{-1} = \{(b,a) \mid (a,b) \in R\}$ therefore $xA^{-1}x \Longleftrightarrow xAx, \forall x \in S$ ①

To prove A^{-1} is an equivalence relation of S, is to prove reflective, symmetric, transitive relation.

from ①, reflective has been proved.

To prove symmetric, we know $\forall x, y \in S, \ xAy \Rightarrow yAx$

As $yA^{-1}x = xAy$, $yAx = xA^{-1}y$, then $yA^{-1}x \Rightarrow xA^{-1}y$, which means A^{-1} is symmetric.

To prove transitive, we know $\forall x, y, z \in S, \ xAy, yAz \Rightarrow xAz$ ②

$xAy = yA^{-1}x$, $yAz = zA^{-1}y$, $xAz = zA^{-1}x$.

then we can turn ② into $\forall z, y, x \in S, \ zA^{-1}y, yA^{-1}x \Rightarrow zA^{-1}x$

which means A^{-1} is transitive.

In all, A^{-1} is an equivalence relation of S