Discrete Mathematics and Statistics - CPT107

Tutorial week 10

1. Do you think the following are acceptable formulas of propositional logic? If not, why not?

   a. P
   b. $P \,\&\, Q$
   c. $P \oplus Q$
   d. $\forall z.S(z)$
   e. $\neg S(z)$
   f. $\neg(P(a) \vee Q(y)$

SOLUTION:

a. Yes, it is.
b. No – the symbol "&" is not part of propositional logic.
c. No – the symbol "$\oplus$" is not part of propositional logic.
d. No – neither the symbol "$\forall$", nor lower-case letters, nor a full stop, are part of propositional logic.
e. No – lower-case letters are not part of propositional logic. And even if they were, sub-formulas in propositional logic can't simply be put next to each other – they can only be joined by using one of the logical connectives ($\wedge$, or $\vee$, or $\rightarrow$, or $\leftrightarrow$).
f. No, for the same reason as for part (e) – lower-case letters aren't part of propositional logic. Also: the left-most parenthesis is never closed, so this is unlikely to be an acceptable formula of *any* logic.

Parts (d), (e) and (f), although they are not acceptable formulas of propositional logic, are (as we shall see) acceptable formulas of *predicate* logic.

(There are other sorts of logic which *do* use symbols like & and $\oplus$ – one of these is linear logic, a logic which allows us to analyse arguments about the *use of finite resources*. Type systems based on linear logic (and some related logics) are used in some programming languages to help manage resources such as memory.)

2. Identify the predicates needed for expressing the following sentences. Just identify the predicates – don't worry about the logical operators yet.

   a. Iceland is cold but not all countries are cold
   b. No country is a holiday resort unless it is by the sea.
   c. Beautiful holiday resorts are all in European countries.

SOLUTION:

a. $C(x)$ = "x is a cold country".
b. $H(x)$ = "x is a holiday resort", and $S(x)$ = "x is by the sea".
c. $B(x)$ = "x is beautiful", and $E(x)$ = "x is a European country".

These are all *one*-place predicates; they take one argument.

3. Identify the predicates needed for expressing the following sentences.

   a. Germany is adjacent to Italy.
   b. Two is less than three.
   c. Achilles is faster than the Tortoise, but the Tortoise is smarter than Achilles.

SOLUTION:

a. $A(x, y)$ = "x is adjacent to y".
b. $L(x, y)$ = "x is less than y".
c. $F(x, y)$ = "x is faster than y", and $S(x, y)$ = "x is smarter than y".

These are all *two*-place predicates; they need two arguments to make them true or false.

4. Express each of the following statements in predicate logic using the predicates $L(x)$ = "x likes discrete maths" and the constants c = "Charles" and d = "Danica".

   a. Charles likes discrete maths but Danica does not.
   b. Danica likes maths if Charles does too.
   c. Neither Charles nor Danica dislike maths

SOLUTION:

a. $L(c) \wedge \neg L(d)$
b. $L(c) \rightarrow L(d)$
c. $\neg\neg L(c) \wedge \neg\neg L(d)$, or $\neg(\neg L(c) \vee \neg L(d))$

5. Express each of the following statements in predicate logic using the predicates $P(x, y)$ = "x is the parent of y" and the constants a = "Andrew", b = "Bethany", c = "Carol" and d = "Dakota".

   a. Carol is the child of Andrew and Bethany.
   b. If Carol is the child of Bethany, then Andrew is the parent of Dakota.
   c. Andrew is not the parent of Bethany.
   d. Andrew is not the parent of Andrew.

SOLUTION:

a. $P(a, c) \wedge P(b, c)$
b. $P(b, c) \rightarrow P(a, d)$
c. $\neg P(a, b)$
d. $\neg P(a, a)$

6. Express the following in colloquial English as precisely as possible.

   a. $\neg\exists x.(\ short(x) \land clever(x) \land child(x))$
   b. $(\neg\exists x.\ short(x)) \rightarrow \forall x.(\ clever(x) \lor child(x))$
   c. $\forall x.(\ adult(x) \land child(x)) \rightarrow (short(j) \land tall(j))$, where $j =$ "John"

SOLUTION:

a. There are no short clever children.
b. If no-one is short, then everyone is either clever or a child.
c. If everyone is both an adult and a child then John is both short and tall.

7. Express the following colloquial English statements using predicate logic. First identify the predicates then the connectives and quantifiers.

   a. All students are clever
   b. Those who do not work hard are lazy
   c. The lazy students are exactly those who do not work hard
   d. Not being lazy is equivalent to being hardworking
   e. Although all students are hardworking there are some who are lazy if they do not work hard

SOLUTION:

We take our domain as being "all people".

In that case, our predicates will be:

- $S(x) =$ "x is a student".
- $C(x) =$ "x is clever".
- $H(x) =$ "x is hard-working".
- $L(x) =$ "x is lazy".

And the answers, (a)–(e):

a. $\forall x.(S(x) \rightarrow C(x))$
b. $\forall x.(\neg H(x) \rightarrow L(x))$
c. $\forall x.(S(x) \land L(x) \leftrightarrow \neg H(x))$
d. $\forall x.(\neg L(x) \leftrightarrow H(x))$
e. $(\forall x.(S(x) \rightarrow H(x))) \land (\exists x.(S(x) \land (\neg H(x) \rightarrow L(x)))$

SOLUTION:
Alternative approaches: One could consider taking the domain of discourse as being "all students".

In that case, the solution to question (a) would simply be (assume $C(x)$ again is "x is clever"):

a. $\forall x.C(x))$ – "all things (in the domain of discourse) are clever".

But question (b) doesn't refer to students at all – so artificially restricting it to that domain would not be a good translation of the sentence. Instead, we will have to choose a different domain, say "all people", or "all things".

And we will again end up with:

b. $\forall x.(\neg H(x) \to L(x))$

Choosing the same domain (and predicates) for the whole set of problems makes things less complex.

8. With the natural numbers, $\mathbb{N}_{\geq 0}$, as the domain of discourse, and using only the predicates "even" and "odd" and the standard relational and arithmetic operators, express each of the following statements as a predicate logic formula. Which of them are true?

   a. Some odd numbers are greater than 100.
   b. The sum of two odd numbers is an even number.
   c. There is no smallest number.

SOLUTION:
Let $E(x)$ = "x is even" and $O(x)$ = "x is odd".

a. $\exists x.(O(x) \wedge (x > 100))$. This is true; 101 is an odd number, and is greater than 100.
b. $\forall x.\forall y.((O(x) \wedge O(y)) \to E(x + y))$; this is also true. (Although we will not prove it now.)
c. $\neg\exists x.\forall y.(x \leq y)$; this is false. In the domain of natural numbers, there *is* a smallest (least) number, 0.

9. With the integers, $\mathbb{Z}$, as the domain of discourse, write each of the following as a predicate logic formula, using only the standard relational and arithmetic operators; which of them are true?

   a. Some numbers are greater than 100.
   b. The product of two negative numbers is a positive number.
   c. There is no smallest number.

SOLUTION:

a. $\exists x.(x > 100)$; yes, this true. For instance, 101 is an integer, and is greater than 100.
b. $\forall x.\forall y.(x < 0 \land y < 0 \rightarrow (x \times y) > 0)$; this is true.
c. $\neg\exists x.\forall y.(x \leq y)$. This is true; there is no smallest (i.e. least) integer. For any integer you can suggest, we can substract one from it to construct a smaller.

10. Assume that $O(x)$ is the predicate "x is odd", and $L(x)$ is the predicate "x is less than 10". Write an English sentence equivalent to each of the following logic statements. Over the domain of the integers, what is the truth value (i.e. true or false) of each statement?

    a. $\exists x.\ O(x)$
    b. $\forall x.(\ L(x) \rightarrow\ O(x))$
    c. $(\forall x.L(x)) \rightarrow (\forall x.O(x))$

SOLUTION:

a. There exists an odd integer; this is true. For instance, 3 is an integer, and 3 is odd.
b. All integers less than ten are odd; this is false. For instance, 8 is less than ten, but 8 isn't odd.
c. If all integers are less than ten, then they are all odd. This is true. "All integers are less than ten" is false; "all integers are odd" is also false. But $F \rightarrow F$ evaluates to true. (Remember, the only time an implication $P \rightarrow Q$ evaluates to false is when P is true but Q is false; in any other case, it evaluates to true.)