Discrete Mathematics and Statistics

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Relations (bis)

Many credits go to Prof. Aaron Bloomfield

Properties of relations

Let R be a binary relation on a set S. Then:

- R is **reflexive** means $(\forall x)(x \in S \rightarrow (x,x) \in R)$

- R is **symmetric** means

  $$(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x,y) \in R \rightarrow (y,x) \in R)$$

- R is **transitive** means

  $$(\forall x)(\forall y)(\forall z)(x \in S \wedge y \in S \wedge z \in S \wedge (x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R)$$

- R is **antisymmetric** means

  $$(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x,y) \in R \wedge (y,x) \in R \rightarrow x=y)$$

Examples

Are the following relations on {1, 2, 3, 4} symmetric and/or antisymmetric?

$R = \{(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)\}$???

$R = \{(1, 1)\}$???

$R = \{(1, 3), (3, 2), (2, 1)\}$???

$R = \{(4, 4), (3, 3), (1, 4)\}$???

Examples

Are the following relations on {1, 2, 3, 4} symmetric and/or antisymmetric?

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)} - ***symmetric***

R = {(1, 1)} - both ***symmetric*** and ***antisymmetric***

R = {(1, 3), (3, 2), (2, 1)} - ***antisymmetric***

R = {(4, 4), (3, 3), (1, 4)} - ***antisymmetric***

Exercises

Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\begin{array}{llll} \text{For all } a, b \in Z^+, & a \; R_1 \; b & \Leftrightarrow & a \mid b. \\ \text{For all } a, b \in Z, & a \; R_2 \; b & \Leftrightarrow & a \mid b. \end{array}$$

a. Is R_1 antisymmetric? Prove it or give a counterexample.

b. Is R_2 antisymmetric? Prove it or give a counterexample.

Exercises - solutions

a. R_1 is antisymmetric. (see *Tutorial 4*)

b. R_2 is not antisymmetric.

Counterexample:
Let a = 2 and b = –2. Then a | b [since –2 = (–1) • 2] and b | a [since 2 = (–1)(–2)].

Hence a R_2 b and b R_2 a but a $\neq$ b.

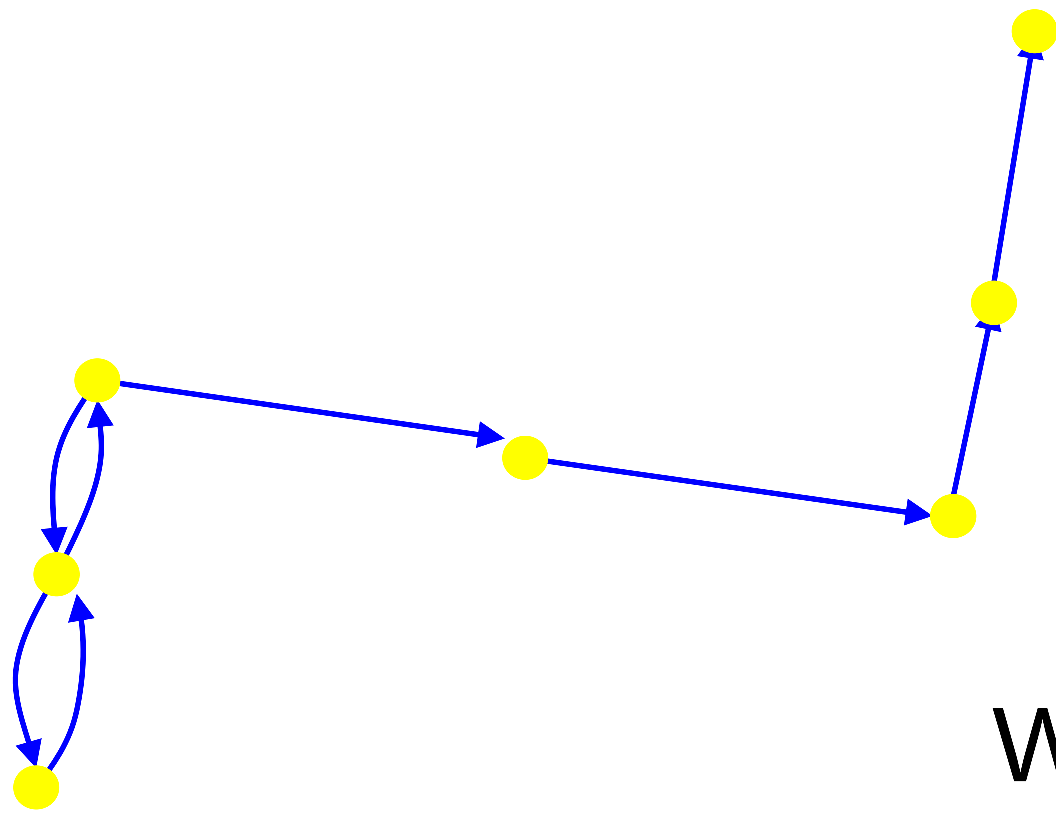
Relational closures

- A binary relation R on a set S may not have a particular property such as reflexivity, symmetry and/or transitivity. However, it may be possible to extend the relation so that it does have some properties.
- Three types we will study
  – Reflexive
    - Easy

  – Symmetric
    - Easy

  – Transitive
    - Hard

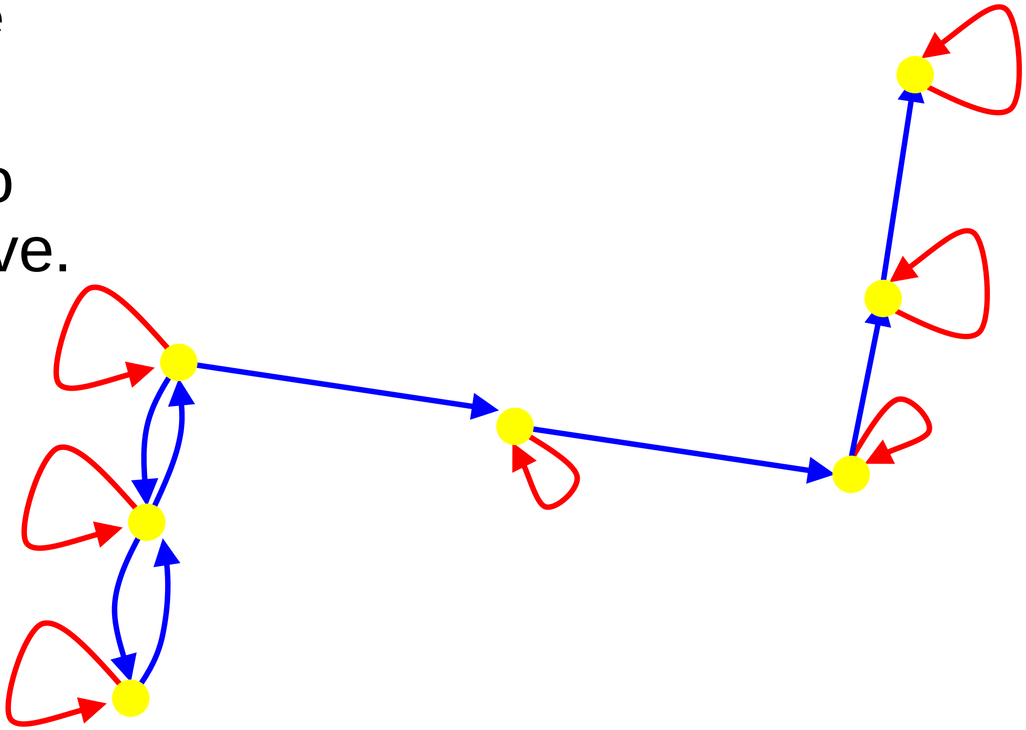
Consider the following relation R :



What properties???

Reflexive closure

- Consider the relation *R* (in the last slide set):
  - Note that it is not reflexive
- We want to add edges to make the relation reflexive.
- By adding those edges, we have made a non-reflexive relation *R* into a reflexive relation.



- This new relation is called the **reflexive closure** of *R.*

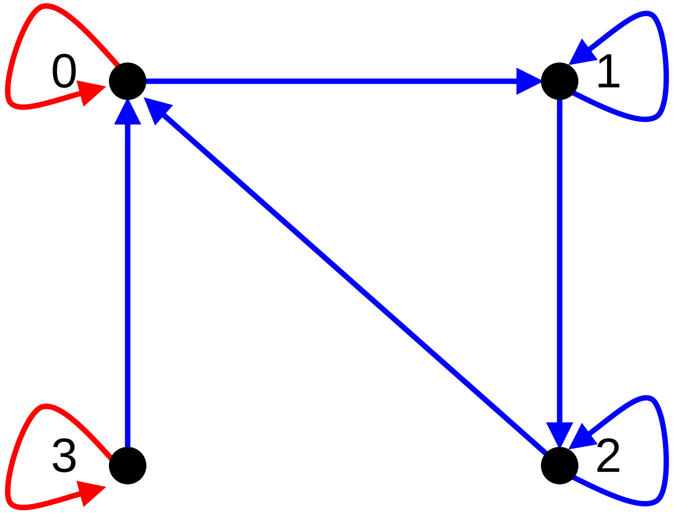
Reflexive closure

- In order to find the reflexive closure of a relation *R*, we add a loop at each node that does not have one.

- The reflexive closure of *R* is $R \cup \Delta$
  - Where $\Delta = \{ (a,a) \mid a \in R \}$
    - Called the "diagonal relation"

Example

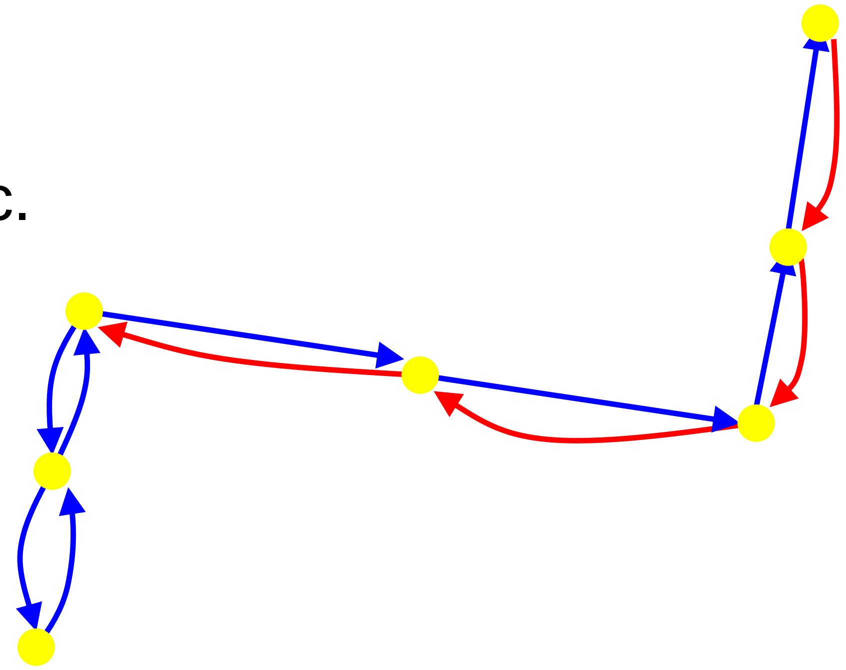
- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0).
- What is the reflexive closure of *R*?
- We add all pairs of edges (*a*,*a*) that do not already exist.



We add edges:
(0,0), (3,3)

Symmetric closure

- Consider the relation *R* :
  - Note that it is not *symmetric*
- We want to add edges to make the relation symmetric.
- By adding those edges, we have made a non-symmetric relation *R* into a symmetric relation.



- This new relation is called the **symmetric closure** of *R.*

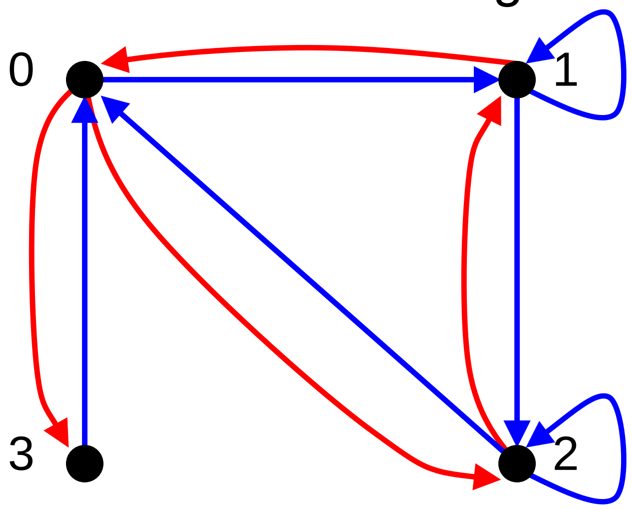
Symmetric closure

- In order to find the symmetric closure of a relation R, we add an edge from a to b, where there is already an edge from b to a.

- The symmetric closure of R is $R \cup R^{-1}$
  - If $R = \{ (a,b) \mid \ldots \}$
  - Then $R^{-1} = \{ (b,a) \mid \ldots \}$

Example

- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the symmetric closure of *R*?
- We add all pairs of edges (*a,b*) where (*b,a*) exists.
  - We make all “single” edges into anti-parallel pairs

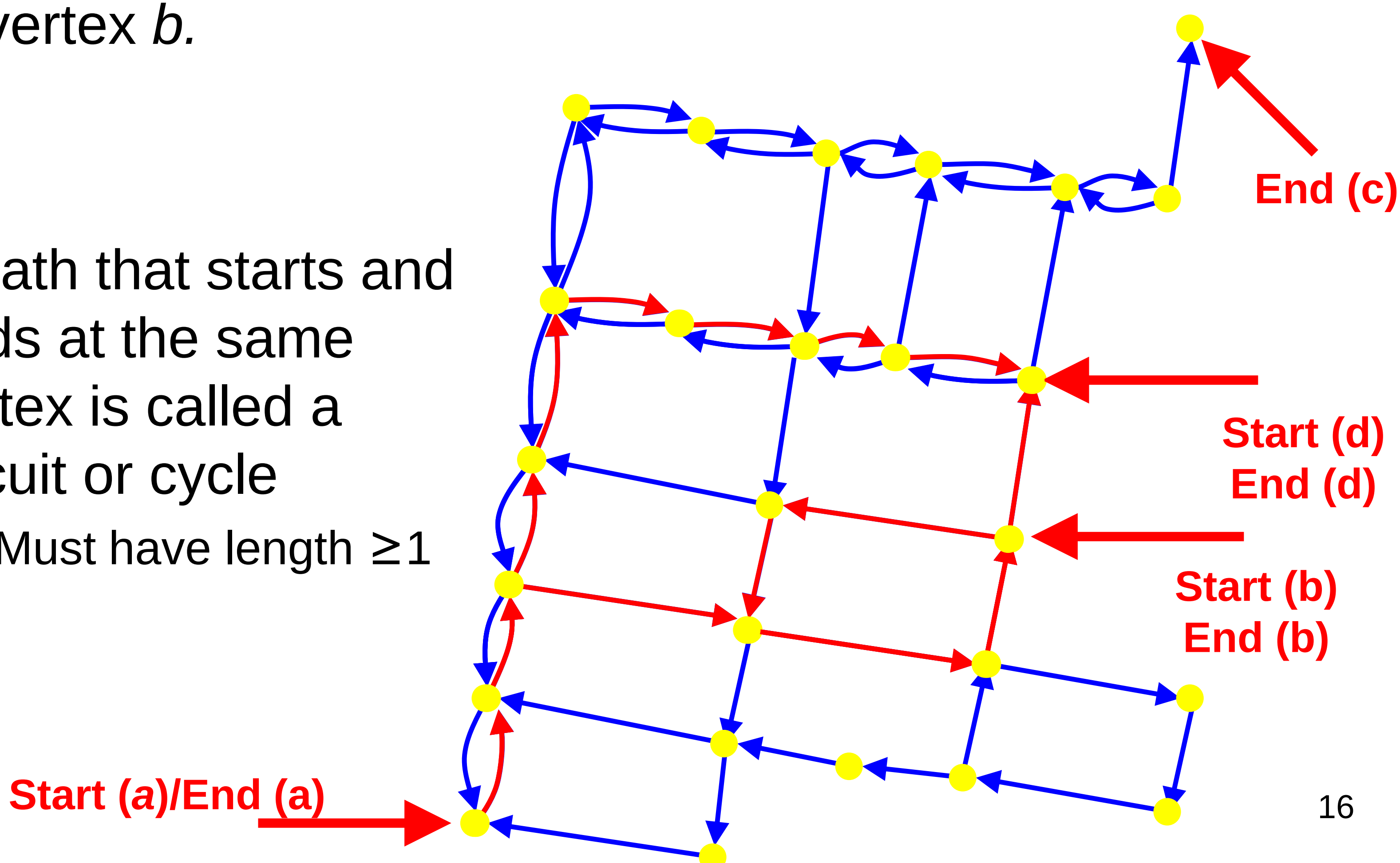


We add edges:
(0,2), (0,3)
(1,0), (2,1)

Paths in directed graphs

- A *path* is a sequences of connected edges from vertex *a* to vertex *b.*

- A path that starts and ends at the same vertex is called a circuit or cycle
  - Must have length ≥1

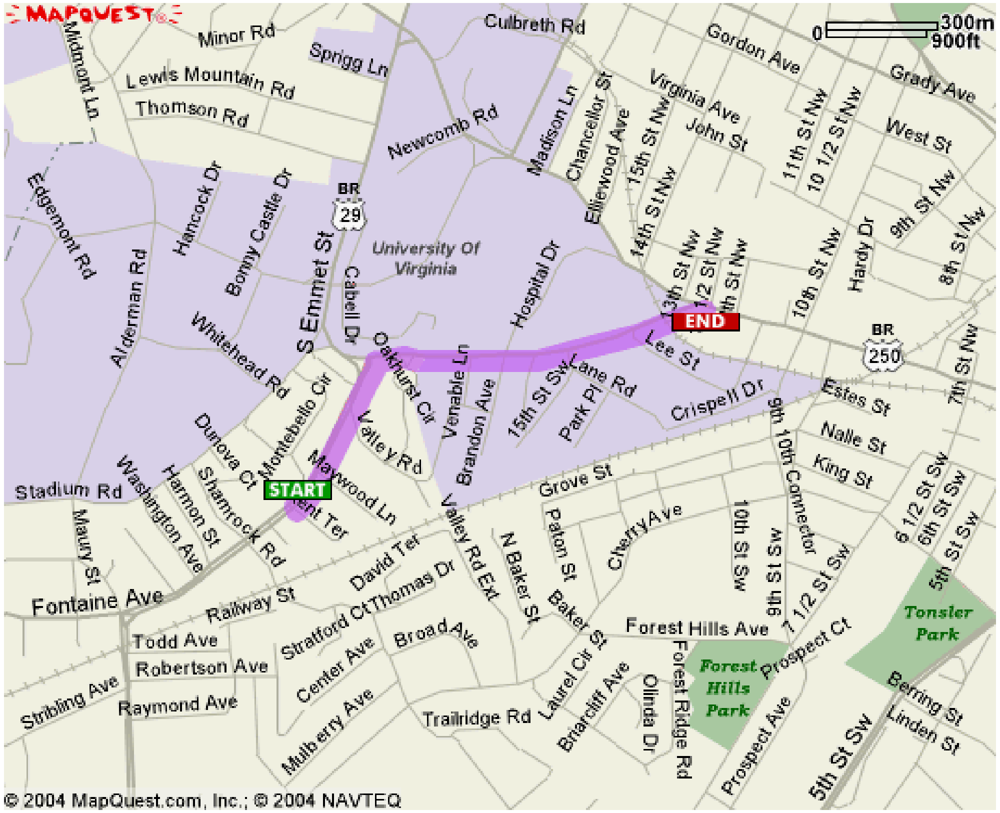


More on paths…

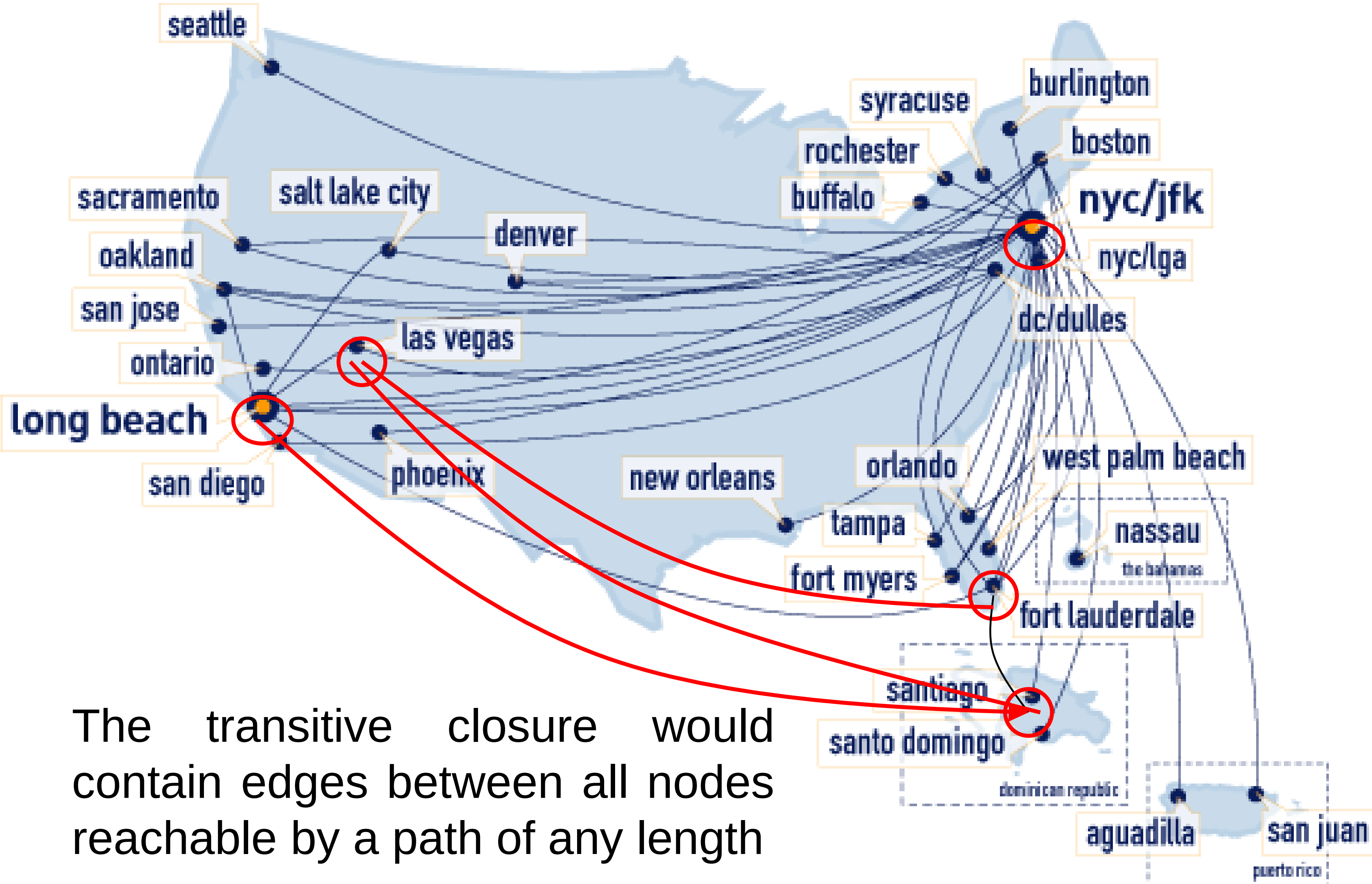
- The length of a path is the number of **edges** in the path, not the number of nodes.

Shortest paths

- What is really needed in most applications is finding the shortest path between two vertices



Transitive closure

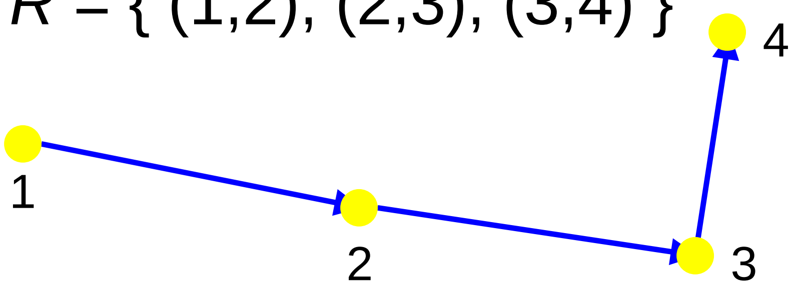


The transitive closure would contain edges between all nodes reachable by a path of any length

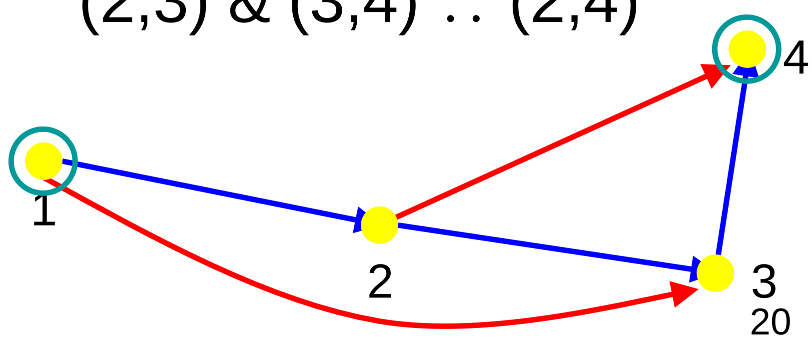
Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure.
- First take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c.*
- But there is a path from 1 to 4 with no edge!

$R = \{ (1,2), (2,3), (3,4) \}$

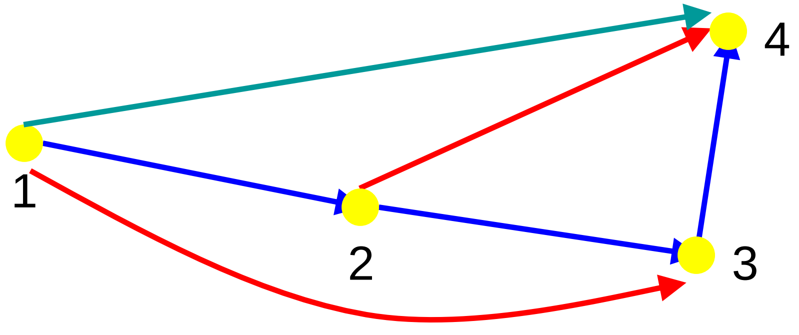


(1,2) & (2,3) $\therefore$ (1,3)
(2,3) & (3,4) $\therefore$ (2,4)



Transitive closure

- We will study different algorithms for determining the transitive closure (in *S2*)
- red means added on the first repeat
- teal means added on the second repeat



Transitive Closure

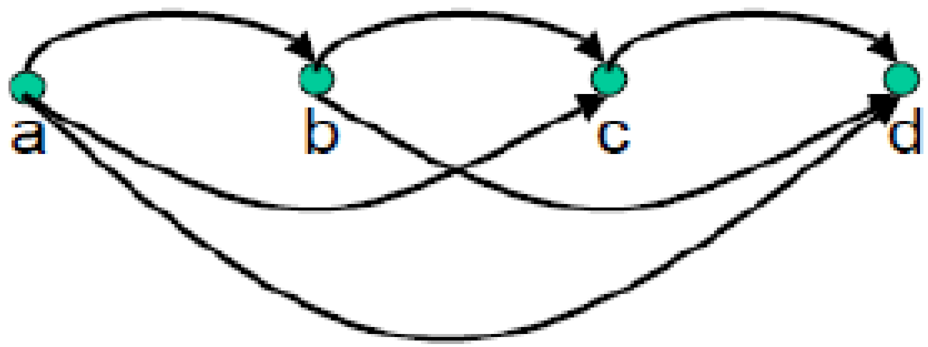
- Formal definition: the transitive closure of a binary relation R on the set S is the smallest transitive relation R^t on S that contains R:
  - R^t is transitive
  - $R \subseteq R^t$
  - R^t is a subset of any other relation on S that includes R, and is transitive.
- Transitive closures are particularly interesting in that they provide “connection” information.
  - Is it possible to travel from city X to city Y?
- How to find transitive closure – we need to add the minimum number of tuples to R, giving us R^t, such that if (a,b) is in R^t and (b,c) is in R^t, then (a,c) is in R^t
- $R^t = R \cup \Delta$

  $(a,b) \in R^t \wedge (b,c) \in R^t \rightarrow (a,c) \in R^t$

Graphical Construction of Transitive Closure

Original Relation

Original Relation plus 2-jumps



Original Relation plus 2-jumps, 3-jump

Example of Transitive Closure:

Let $S = \{1, 2, 3\}$.

$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}$.

$(2,3) \in R \wedge (3,1) \in R \rightarrow (2,1) \in R^t$

$(3,1) \in R \wedge (1,2) \in R \rightarrow (3,2) \in R^t$

$(3,1) \in R \wedge (1,3) \in R \rightarrow (3,3) \in R^t$

$(2,1) \in R^t \wedge (1,2) \in R \rightarrow (2,2) \in R^t$ (*Must be done iteratively)

So, $R^t = R \cup \{(2,1), (3,2), (3,3), (2,2)\}$

Partial orderings

A partial ordering (or partial order) is a relation that is *reflexive*, *antisymmetric*, and *transitive*:

- Recall that antisymmetric means that if $(a,b) \in R$, then $(b,a) \notin R$ unless $b = a$.
- Thus, (a,a) is allowed to be in R.
- But since it is reflexive, all possible (a,a) must be in R.

A set S with a partial ordering R is called a *partially ordered set*, or *poset:*

- Denoted by (S,R)

Partial ordering examples

- Show that ≥ is a partial order on the set of integers:
  - It is reflexive: $a \geq a$ for all $a \in \mathbf{Z}$.
  - It is antisymmetric: if $a \geq b$ then the only way that $b \geq a$ is when $b = a$.
  - It is transitive: if $a \geq b$ and $b \geq c$, then $a \geq c$.
- Note that ≥ is the partial ordering on the set of integers.
- $(\mathbf{Z}, \geq)$ is the partially ordered set, or poset.

More examples

The symbol $\succcurlyeq$ is used to represent *any* relation on the set *S* when discussing partial orders:

- Not just the less than or equals to relation.
- Can represent $\leq$, $\geq$, $\subseteq$, etc.
- Thus, $a \succcurlyeq b$ denotes that $(a,b) \in \succcurlyeq$.
- The poset is $(S,\succcurlyeq)$.

Comparable / Incomparable

- The elements a and b of a poset (*S*, ➢) are called ***comparable*** if either a ➢ b or b ➢ a.

- When a and b are elements of *S* such that neither a ➢ b nor b ➢ a, a and b are called ***incomparable***.

Predecessor, immediate predecessor and successor

- Let (**Z**,**≤**) be a partially ordered set. If x ≤ y but x≠y, we write x $\lessdot$ y and say that x is a **predecessor** of y or y is a **successor** of x.

- A given y may have many predecessors, but if x $\lessdot$ y and there is no z with x $\lessdot$ z $\lessdot$ y, then x is an **immediate predecessor** of y.

Example

Consider the partial order "≤" on **Z**:

- 5 is an immediate predecessors of 6 since $5 \leq 6$ and there is no integer c not equal to 5 or 6 which satisfies $5 \leq c \leq 6$.
- 3 is NOT an immediate predecessors of 6 since $3 \leq c \leq 6$ is satisfied by 4 or 5.

Consider the partial order given by integer division on **Z**:

- 3 is an immediate predecessors of 6 since there is no integer c which 3 divides and which divides 6 other than 3 or 6.
- 3 is also an immediate predecessors of 9, but not of 12 (why???).

Example

Consider the relation "x divides y" on {1,2,3,6,12,18}.

- Write the ordered pairs (x,y) of this relation.

  (1,1),(1,2),(1,3),(1,6),(1,12),(1,18),(2,2),(2,6),
  (2,12),(2,18),(3,3),(3,6),(3,12),(3,18),(6,6),(6,12),
  (6,18),(12,12),(18,18).

- Write all the predecessors of 6.

  1,2,3

- Write all the immediate predecessors of 6.

  2,3

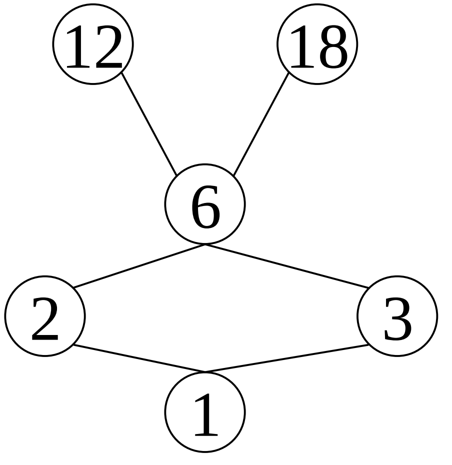
Hasse Diagram

- Visual depiction of a partially ordered set.
- Each of the elements of S is represented by a dot, called a node, or vertex, of the diagram.
- If x is an *immediate predecessor* of y, then the node for y is placed above the node for x and the two nodes are connected by a straight-line segment.
- Two nodes in a Hasse diagram should never be joined by a horizontal line.

Example

Draw the Hasse diagram for the relation “x divides y” on {1,2,3,6,12,18}.

1. Write the ordered pairs (x,y) of this relation.
2. Find the predecessors.
3. Find the immediate predecessors.
4. Draw the diagram.

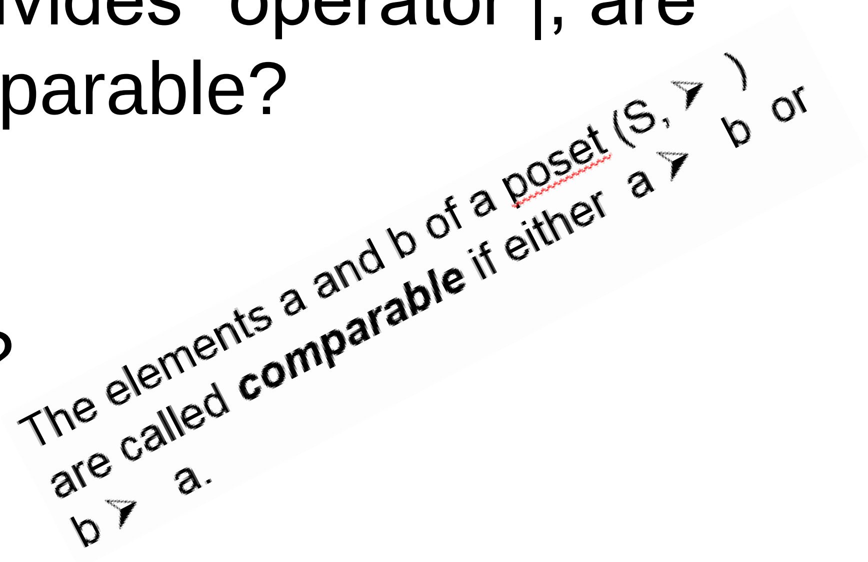


Totally ordered

- If (*S*,➢) is a poset and every two elements of *S* are *comparable*, *S* is called totally *ordered* or *linearly ordered* set, and ➢ is called a ***total order*** or a ***linear order***.
- A totally ordered set is also called a chain.

Example

- In the poset (Z^+,|) with “divides” operator |, are the integers 3 and 9 comparable?
  - Yes, as 3 | 9
- Are 7 and 5 comparable?
  - No, as 7 | 5 and 5 | 7



- Thus, as there are pairs of elements in Z^+ that are not comparable, the poset (Z^+,|) is a partial order. However, it is not a chain or total/linear.

Example

- In the poset $(Z^+, \leq)$, are the integers 3 and 9 comparable?
    - Yes, as $3 \leq 9$

- Are 7 and 5 comparable?
    - Yes, as $5 \leq 7$

- As all pairs of elements in Z^+ are comparable, the poset $(Z^+, \leq)$ is a total order (also called totally ordered poset, linear order, or chain).

Hasse Diagram for (A = {1, 2, 3, 4}, ≤)

- The poset (***A***, ≤) is totally ordered, because a ≤ b or b ≤ a whenever a and b are integers.

- When we KNOW it's a poset, we can simplify the graph