Examples

The Product Rule (Multiplication Principle)

Example 1: Telephone Numbering Plan

The *National Numbering Plan* specifies that a telephone number consists of 10 digits:
3-digit area code, 3-digit office code, a 4-digit station code.

There are some restrictions on the digits.

Let X denote a digit from 0 through 9.

Let N denote a digit from 2 through 9.

Let Y denote a digit that is 0 or 1.

Format in old plan (used in 1960s): **NYX-NNX-XXXX**

Format in new plan: **NXX-NXX-XXXX**

How many different telephone numbers are possible under the old plan and the new plan?

$X = \{0, ..., 9\}$, $N = \{2, ..., 9\}$, $Y = \{0, 1\}$
old plan format: **NYX-NNX-XXXX**
new plan format: **NXX-NXX-XXXX**

How many different telephone numbers are possible?

Solution:
Use the Product Rule.

$8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX*.

$8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX.

$8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX*.

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format XXXX

Number of old plan telephone numbers:

$$160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000.$$

Number of new plan telephone numbers:

$$800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000.$$

Example 2

There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

Solution: The procedure of choosing a port consists of two tasks, first picking a computer and then picking a port on this computer. Because there are 32 ways to choose the computer and 24 ways to choose the port no matter which computer has been selected, the product rule shows that there are $32 \cdot 24 = 768$ ports.

The Sum Rule

Sum rule. ***Example***

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list.

How many possible projects are there to choose from?

Solution**:** The student can choose a project by selecting a project from the first list, the second list, or the third list.

Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Sum rule. *Example*

I wish to take two pieces of fruit with me for lunch. I have three bananas, four apples and two pears. How many ways can I select two pieces of fruit of different type?

- If I select one of the three bananas and one of the four apples, then 3×4 selections can be made.
- If I select a banana and a pear, then 3×2 selections can be made.
- If I select an apple and a pear, then 4×2 selections can be made.
- As these sets of possibilities are disjoint

$$12 + 6 + 8 = 26$$

different ways of selecting two pieces of fruit of different types exist.

Combining Sum and Product Rule

Example: Counting Passwords

A password must be 6 - 8 characters long; each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many possible passwords are there?

Example: Counting Passwords

Solution:

Let P be the total number of passwords.

Let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

By the sum rule, $P = P_6 + P_7 + P_8$.

To find each of P_6, P_7, and P_8, find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters.

$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$

$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$

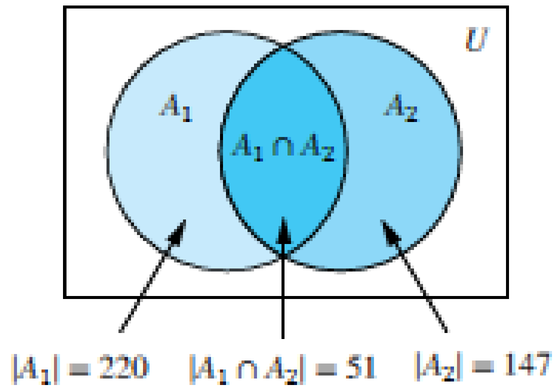
$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360$.

The Subtraction Rule (Subtraction Principle) *Example*

A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business.

How many of these applicants majored neither in computer science nor in business?



Solution: Use the subtraction rule.

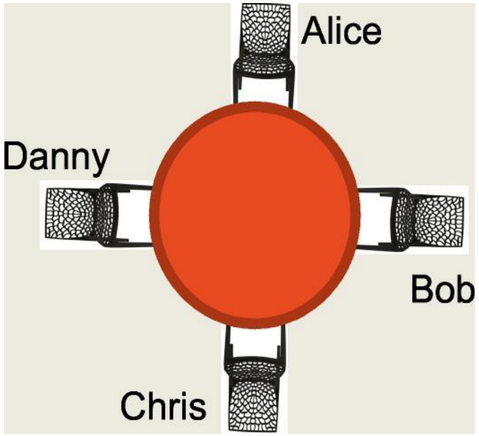
Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business.
$A_1 \cup A_2$ is the set of students who majored in computer science or business (or both),
$A_1 \cap A_2$ is the set of students who majored both in computer science and in business.

The number of students who majored either in computer science or in business (or both)
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$
We conclude that 350 - 316 = 34 of the applicants majored neither in computer science nor in business.

The Division Rule

***Example*:**

How many ways are there to seat four people around a circular table, where two seating are considered the same when each person has the same left and right neighbor?



Solution:

Number the seats around the table from 1 to 4 proceeding clockwise.

There are four ways to select the person for:

- seat 1 - 4 ways
- seat 2 - 3 ways
- seat 3 - 2 ways
- seat 4 – 1 way

Thus, there are 1 • 2 • 3 • 4 = 4! = 24 ways to order the four people.

But two seating are the same when each person has the same left and right neighbor.

Each of the four choices for seat 1 leads to the same seating arrangement.

Therefore, by the division rule, there are **24/4 = 6** different seating arrangements.

Permutations - without repetition

Example:

S={J, K, Q}:

(K, Q, J) is a permutation of S; (J, K) is a *2-permutation* of S

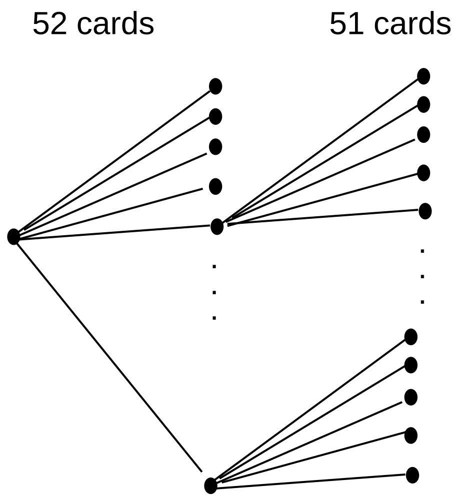
Note: Set is unordered, but permutation is ordered!, e.g. (K, Q) $\neq$ (Q, K)

The number of *2-permutation* of S is:

- **P(3, 2) = 3! / (3 - 2)! = 3!**

Permutation—without repetition

How many *two-card hands* can you draw from a deck when order matters ? (e.g., ace of spades followed by ten of clubs is different than ten of clubs followed by ace of spades)



$$\frac{52!}{(52-2)!} = 52 \cdot 51$$

Example:

Find the number m of permutations of *six* objects, say, A, B, C, D, E, F, taken *three at a time*.

Solution

- Let us represent the general three-letter word by the following three positions:

$$\_\_\_\_, \_\_\_\_, \_\_\_\_$$

The first letter can be chosen in *6 ways*; following this the second letter can be chosen in *5 ways*; and finally, the third letter can be chosen in *4 ways*. Write each number in its appropriate position as follows:

—6 ways—, —5 ways—, —4 ways—

By the Product Rule there are m = 6 · 5 · 4 = 120 possible three-letter words without repetition from the six letters.

This agrees with the formula

$$P(6, 3) = 6! / (6 - 3)! = 6! / 3! = 6 \cdot 5 \cdot 4 = 120$$

Permutations - with repetition

*A zip codes consist of an ordering of **five digits** chosen from 0-9 with replacement (i.e. numbers may be reused).*

How many zipcodes are in the set Z of all possible zip codes?

Solution.

No. zipcodes = permutation of 5 digits from 10 with replacement

$= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5$

$= 100,000$

Permutations - with repetition

We have a set with the numbers from 1 to 6.

How many permutations are there if we take 3 of them?

Solution

$6 \times 6 \times 6 \times 6 = 6^3 = 216$ permutations

Permutations of Identical Objects

Example

How many words can we make by rearranging the letters of the word BEER?

The set $\{B, E, E, R\} = \{B, E, R\}$ but we really have 4 letters with which to work. So let us start with the set $\{B, R, E\ , E\ \}$. We arrange them in $4! = 24$ ways:

BREE	BERE	BEER	RBEE	REBE	REEB	EBRE	EBER	EEBR	ERBE	EREB	EERB
BREE	BERE	BEER	RBEE	REBE	REEB	EBRE	EBER	EEBR	ERBE	EREB	EERB

If we can't tell the difference between E and E (they are both just E), then the words group into *pairs*, e.g., $EEBR$ and $EEBR$ group together — both are the word EEBR.

Thus the number of different words we can form by rearranging the letters must be

$$4!/2 = \frac{4!}{2!}$$

Note that 2! counts the number of ways we can permute the two E's in any given arrangement.