

# Discrete Mathematics and Statistics (CPT 107)

## Tutorial 4 bis

1. Let  $R$  be a symmetric relation. Show that  $R^n$  is symmetric for all positive integers  $n$ .

**Solution:** Let  $R$  be a symmetric relation on set  $A$ . Proof by induction:

**Basis Step:**  $R^1 = R$  is symmetric is True.

**Inductive Step:** Assume that  $R^n$  is symmetric.

**To prove that  $R^{n+1}$  is symmetric.**

$R^{n+1}$  is symmetric if for all  $(x,y)$  in  $R^{n+1}$ , we have  $(y,x)$  is in  $R^{n+1}$  as well.

Assume that  $(x,y)$  is in  $R^{n+1}$ .

Now,  $R^{n+1} = R^n \circ R = R \circ R^n$

We know that if  $(x,y) \in R \circ R^n$ , then by the definition of composition there exists a  $z$  in  $A$  such that  $xRz$  and  $z(R^n)y$ , i.e.  $(x,z)$  is in  $R$  and  $(z,y)$  is in  $R^n$

Also we know that  $R$  and  $R^n$  are symmetric, which implies that  $(z,x)$  is in  $R$  and also  $(y,z)$  is in  $R^n$ .

Therefore, by definition of composition,

$(y,x) \in R \circ R^n$ ; i.e.;  $(y,x) \in R^{n+1}$ .

Hence proved.

2. The relation  $R$  on a set  $A$  is transitive  $\Leftrightarrow R^n \subseteq R$  for  $n = 1, 2, 3, \dots$

**Solution:**

Proof. ( $\Leftarrow$ ):

Suppose that  $R^n \subseteq R$  for  $n = 1, 2, 3, \dots$ . We want to show that  $R$  is transitive.

In particular, suppose that  $R^2 \subseteq R$ . Now if  $(a, b) \in R$  and  $(b, c) \in R$ , then by definition of composition,  $(a, c) \in R^2$ . Because  $R^2 \subseteq R$ , this means that  $(a, c) \in R$ . Hence,  $R$  is transitive.

( $\Rightarrow$ ):

Suppose that  $R$  is transitive. We want to show that  $R^n \subseteq R$  for  $n = 1, 2, 3, \dots$

We use mathematical induction to prove this part.

Base Step: Obviously,  $R^1 = R \subseteq R$ .

Inductive Step: Assume that  $R^k \subseteq R$ , where  $k \in \mathbb{Z}^+$ . We want to show that  $R^{k+1} \subseteq R$ .

Assume  $(a, b) \in R^{k+1}$ . We want to show that  $(a, b) \in R$ .

Because  $R^{k+1} = R^k \circ R$ , there is an element  $x$  with  $x \in A$  such that  $(a, x) \in R$  and  $(x, b) \in R^k$ .

By inductive hypothesis,  $R^k \subseteq R$ . Therefore  $(x, b) \in R$ . So we have  $(a, x) \in R$  and  $(x, b) \in R$ . Because  $R$  is transitive, it follows that  $(a, b) \in R$ . This is what we wanted to show: assuming  $(a, b) \in R^{k+1}$ , we showed that  $(a, b) \in R$ . Thus  $R^{k+1} \subseteq R$ , completing the proof.

3. Suppose  $A$ ,  $B$ , and  $C$  are non empty sets with  $R$  a relation from  $A$  to  $B$  and  $S$  a relation from  $B$  to  $C$ . Prove that:
- $(R^{-1})^{-1} = R$
  - $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

**Solution:**

Definition I . We say two sets  $A$  and  $B$  are equal, and write  $A = B$ , if and only if  $A \subseteq B$  and  $B \subseteq A$ .  
We apply Definition I to the sets involved.

To prove a) suppose  $(a, b) \in R$ . Then by definition  $(b, a) \in R^{-1}$ . The definition of  $(R^{-1})^{-1}$  is  $(R^{-1})^{-1} = \{(a, b) | (b, a) \in R^{-1}\}$ . Thus we see  $(a, b) \in (R^{-1})^{-1}$  and so  $R$ . Similar arguments show any  $(a, b) \in (R^{-1})^{-1}$  is also in  $R$ . That is,  $(R^{-1})^{-1} \subseteq R$ , and part a) follows.

To prove b), suppose  $(c, a) \in (S \circ R)^{-1}$ . Then by definition of the inverse relation,  $(a, c) \in (S \circ R)$ . By the definition of the composite relation, there exists  $b \in B$  such that  $(a, b) \in R$  and  $(b, c) \in S$ . This of course means  $(c, b) \in S^{-1}$  and  $(b, a) \in R^{-1}$ . Applying again the definition of the composite relation, we find  $(c, a) \in R^{-1} \circ S^{-1}$ . Thus  $(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$ . Similar arguments show  $R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}$  and the result follows.

4. Give examples in accordance with:
- $(R^{-1})^{-1} = R$
  - $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

**Solution:**

Suppose  $A = \{1, 2, 3\}$ ,  $B = \{4, 5, 6\}$ ,  $C = \{7, 8, 9\}$ , and

$R = \{(1, 4), (1, 5), (2, 5)\}$ ,  $S = \{(4, 7), (5, 7), (6, 9)\}$

Then  $S \circ R = \{(1, 7), (2, 7)\}$ ,

$(S \circ R)^{-1} = \{(7, 1), (7, 2)\}$ .

Moreover,  $S^{-1} = \{(7, 4), (7, 5), (9, 6)\}$  and  $R^{-1} = \{(4, 1), (5, 1), (5, 2)\}$ .

Thus  $R^{-1} \circ S^{-1} = \{(7, 1), (7, 2)\}$  (in accordance with a) and b))