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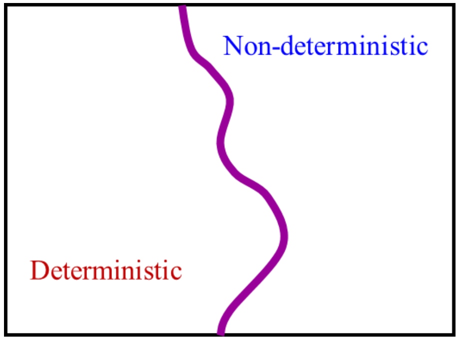
Discrete Mathematics and Statistics – CPT107

Probability Theory

Content

- Basic concepts in probability theory
- Discrete probability: sample spaces and events
- Probability of simple events
- Probability of compound events:
  - Independent events
  - Mutually exclusive events
  - Mutually inclusive events
- Conditional probability
- Probability with and without replacement
- Random value
- Expected value

Experiments



Deterministic Experiment

- There exists a mathematical model that allows "***perfect***" prediction the experiment's outcome.
- Many examples exist in Physics, Chemistry (the exact sciences).
- *Example*: *adding 2 and 2* is deterministic experiment (always produces 4)

Non-deterministic Experiment

- **No** mathematical model exists that allows "***perfect***" prediction the experiment's outcome.
- Example: *tossing a coin* is random experiment (can be either head or tail)

Foundations of Probability

Probability is a measure of **how likely an event is to occur**.

- **Experiment (Random process)**: Any action with unpredictable outcomes (e.g., rolling a die).
- **Sample Space**: is the set, Ω, of all possible outcomes of a random process or experiment.
- **Event**: A specific outcome. A **subset** of the sample space.
- **Probability P(X)**: The probability of outcome X occurring.

Probability in Computer Science

- the core of machine learning and statistics;
- modeling the behavior of systems;
- better understanding the performance of algorithms, etc.

Events

Any *subset* X of the sample space Ω is known as an ***event.***

Special Events

The *Null Event*, The empty event - ϕ ϕ = { } = there are no outcomes

The *Entire Event*, The Sample Space - Ω

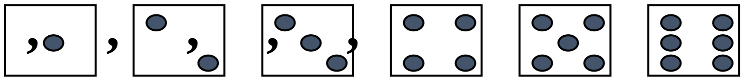
The empty event, ϕ , never occurs.

The entire event, Ω, always occurs.

Examples of Sample Space

1. If the experiment consists of the *flipping of a coin*, then **Ω = {H, T}** where *H* means that the outcome of the toss is a head and *T* that it is a tail.

2. *Rolling a die* – outcomes

Ω ={  }

***Definition*:** Probability of an Event X

Suppose that the sample space $\Omega = \{o_1, o_2, o_3, \ldots o_N\}$ has a finite number, N, of outcomes.

In the case where an experiment has finitely many outcomes and all outcomes are equally likely to occur, = **equiprobable** (each individual outcome has probability 1/N)

If Ω is a finite sample space in which all outcomes are equally likely, and **X** is an **event** in Ω, then the **probability of X**, denoted **P(X)** and describes as:

$$P(X) = \frac{n(X)}{n(\Omega)} = \frac{n(X)}{N} = \frac{\text{number of outcomes in } X}{\text{total number of outcomes}}$$

A probability is a **numerical value assigned to a given event X**.

The probability of an event **ranges from 0 to 1**.

The **sum of probabilities of all possible events equals 1**.

The probability can be expressed as a *fraction*, a *decimal*, or a *percent*

If the probability of an event is **0**, the event is **impossible**.

If the probability of an event is **1**, the event is **certain**.

X' is the **complement** of an event X.

The probability that any event X does not occur is P(X'): $\boldsymbol{P(X') = 1 - P(X)}$

An **event** is a *subset* $X \subseteq \Omega$. The probability of the event is given by

$$P(X) = \sum_{x \in X} P(x).$$

$$P(\emptyset) = 0 \text{ and } P(\Omega) = 1$$

Example

If there is a 40% chance it will rain tomorrow,
What is the probability it will NOT rain?

60% OR 0.4 and 0.6 OR 2/5 and 3/5

Types of Events

An **event** is described as a set of outcomes.

An **event** is any collection (subset) of outcomes contained in the sample space Ω.

A **simple event** is an event that comprises a single result from the sample space.

A **compound event** is an event made up of **two or more simple events (**more than a single result).

- Independent Events
- Dependent Events
- Mutually Inclusive Events
- Mutually Exclusive Events

Simple events

The probability is determined by the outcome of ***one*** trial in the experiment.

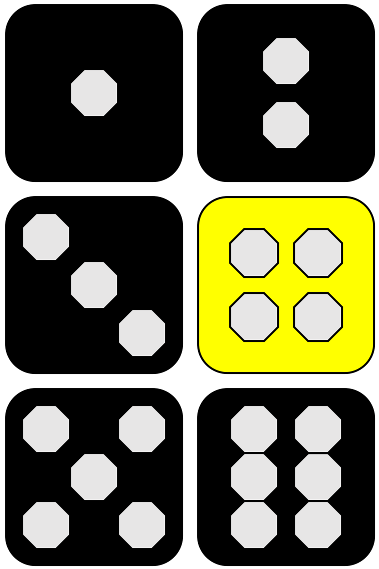
Probability of a Simple Event

Example: Roll a die.

What is the probability of rolling a 4?

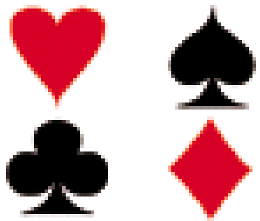
$$\textbf{P(event)} = \frac{\text{no. of outcomes in } X}{\text{total no. of outcomes}}$$

$$\textbf{P(rolling a 4)} = \boxed{\frac{1}{6}}$$



The probability of rolling a 4 is 1 out of 6

Probability of a Simple Event



Example 2: Deck of Cards.

What is the probability of picking a **heart**?

$$P(heart) = \frac{no.of\ outcomes\ in\ X}{total\ no.of\ outcomes} = \frac{13}{52} = \boxed{\frac{1}{4}}$$

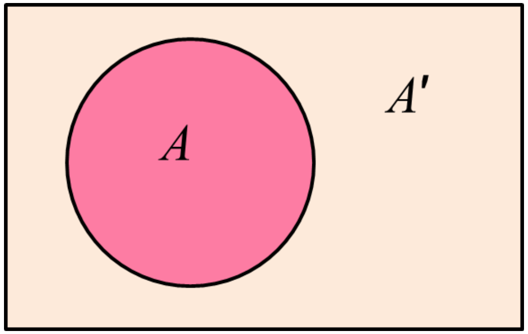
The probability of picking a heart is **1 out of 4 or 0.25 or 25%**

What is the probability of picking a **non heart**?

$$P(non\ heart) = \frac{no.of\ outcomes\ in\ X}{total\ no.of\ outcomes} = \frac{39}{52} = \boxed{\frac{3}{4}}$$

The probability of picking a heart is **3 out of 4 or 0.75 or 75%**

Complement of an Event

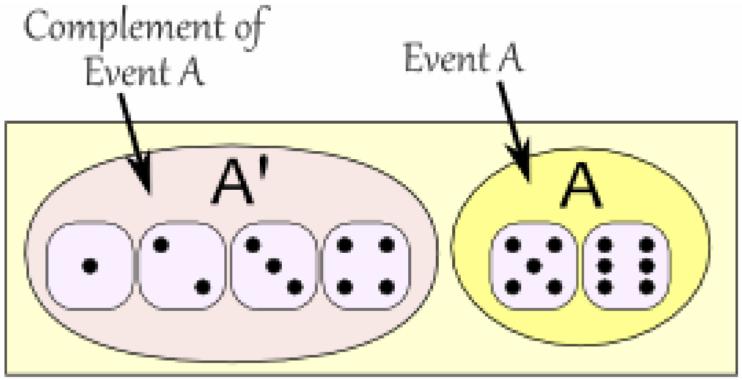


All outcomes that are **NOT** the event.

Example:

- When the event is **Heads**, the complement is **Tails**
- When the event is **{Monday, Wednesday}** the complement is **{Tuesday, Thursday, Friday, Saturday, Sunday}**

$$P(A) + P(A') = 1 \quad \rightarrow \quad P(A') = 1 - P(A)$$



Compound events

A compound event is the combination of ***two or more*** simple events.

- Independent Events
- Dependent Events
- Mutually Inclusive Events
- Mutually Exclusive Events

Compound events

- **Independent events**
  - Events whose occurrence does not dependent on any other event. (sample space is not changed)
    - *Tossing a Coin*, both getting H and T are Independent Events
    - *The rolling a die*, does not affect the next roll
    - *Tossing a Coin and Rolling a Die*
- **Dependent events**
  - Events that are affected by the outcomes of events that had already occurred previously. (sample space is changed)
    - *The chances of pulling a heart from a deck of cards is 13/52. But if you don't put the card back, what is the probability that you pull a heart next time?*

Probability of two or more Independent Events

This is an important idea! A coin does not "know" that it came up heads before ... each toss of a coin is a perfect isolated thing.

P(A *and* B) = P(A) × P(B)

Example:

You toss a **coin three times** and it
comes up "Heads" each time ...
what is the chance that the next toss will also be a "Head"?
The chance is simply 1/2, or 50%,
just like ANY OTHER toss of the coin.

What it did in the past will **not affect** the current toss!

Q: **What is the probability of 7 heads in a row?**

Answer: ½ × ½ × ½ × ½ × ½ × ½ × ½ = 0.0078125 (less than 1%).

Compound events

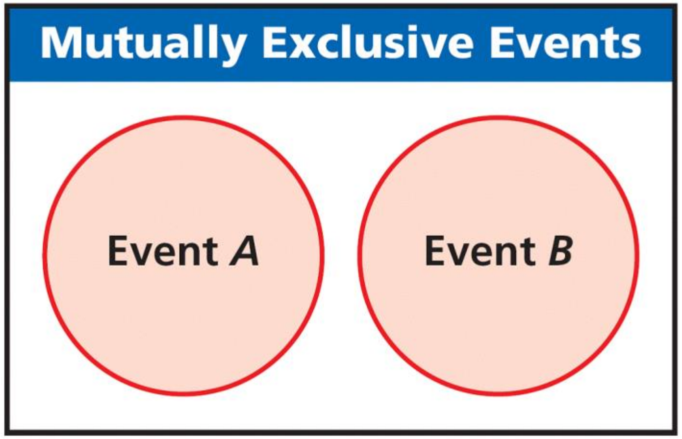
Mutually Exclusive Events

If the two events have nothing in common, then they are called *mutually exclusive events*, mutually exclusive events are similar to mutually exclusive sets.

Formally two events A and B are called mutually exclusive if both of them cannot occur simultaneously.

Examples:
- Turning left or right are Mutually Exclusive (you can't do both at the same time)
- Heads and Tails are Mutually Exclusive
- Kings and Aces are Mutually Exclusive

What **isn't** Mutually Exclusive
Kings and Hearts are not Mutually Exclusive, because you can have a King of Hearts!



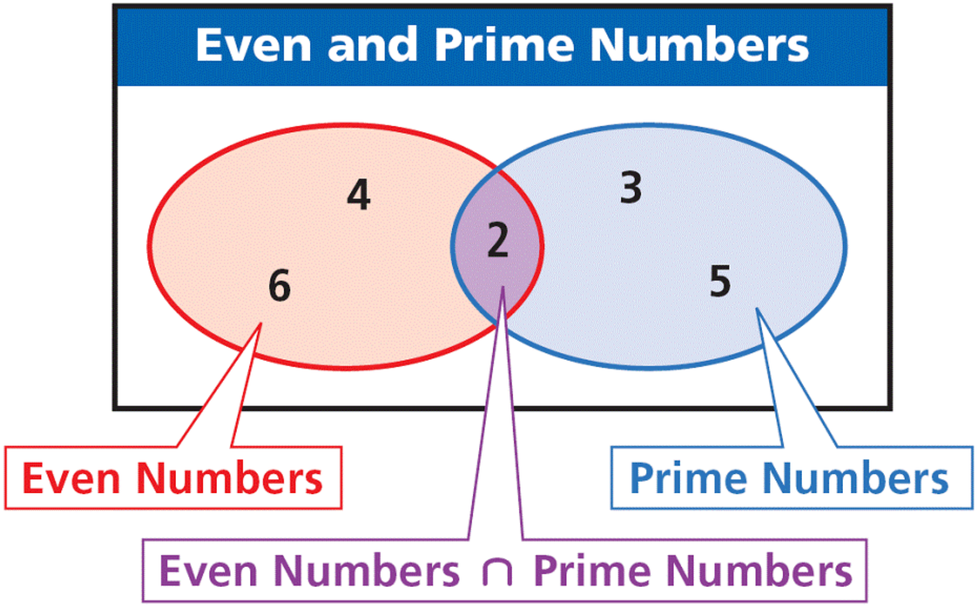
Compound events

Mutually Inclusive Events

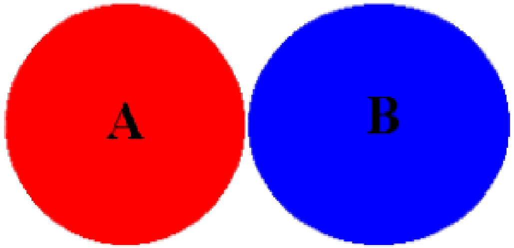
Events have one or more outcomes in common.

- *When you roll a number cube, the outcomes "rolling an even number" and "rolling a prime number" are not mutually exclusive.*

- *The number 2 is both prime and even, so the events are inclusive.*



Probability of Mutually Exclusive Events



Events that **can't happen at the same time**.

If two events A and B are **disjoint** (never occur at the same time), then the probability of either event is *the sum of the probabilities of the two events*:

$$P\,(A \cup B) = P(A \text{ or } B) = P(A) + P(B).$$

Note: a union (∪) of two events occurring means that A or B occurs.

Note: disjoint events does not necessarily mean that they are independent.

Example:

Suppose **five** marbles, each of a different color, are placed in a bowl.

The *sample space* for choosing one marble is:

$$\boldsymbol{\Omega} = \{\text{red, blue, yellow, green, purple}\}.$$

What is the probability of drawing either a purple, red, or green marble from a bowl?

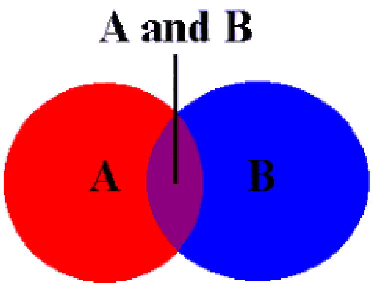
Solution

It is equally likely that any one marble will be selected, then the probability of choosing any one marble is **P(X) = 1/5**.

The possible outcomes of picking a single marble **are disjoint**: only one color is possible on each pick.

The probability of drawing either a purple, red, or green marble from a bowl of five differently colored marbles is the sum of the probabilities of drawing any of these marbles: **1/5 + 1/5 + 1/5 = 3/5**.

Probability of Mutually Inclusive Events



If two events A and B are **not disjoint** (have some overlap with each other / can happen at the same time / occur in a single trial), then the probability of their union (the event that A or B occurs) is equal to *the sum of their probabilities minus the probability of their intersection.*

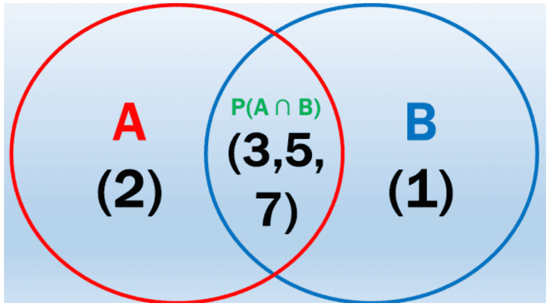
$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

Note: In probability terms, two events are mutually inclusive if their intersection is greater than zero: P(A or B) > 0

Note: Mutually inclusive events cannot happen independently.

Example:

What is the probability of spinning a <u>prime number</u> or an <u>odd number</u> on a spinner numbered 1 to 8?



<u>Solution</u>

For S = {1, 2, 3, 4, 5, 6, 7, 8} - mutually inclusive events.

Let A be the event of spinning a prime number.

A = {2, 3, 5, 7}

P(A) = 4/8

Let B be the event of spinning an odd number.

B = {1, 3, 5, 7}

P(B) = 4/8

For $(A \cap B) = \{3, 5, 7\}$

$P(A \cap B) = 3/8$

$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) = 4/8 + 4/8 - 3/8 = 5/8$

Conditional Probability

The probability of occurrence of any event X when another event Y in relation to X has already occurred is known as conditional probability.

If the probability of "**X**" is conditional (affected by) "**Y**" (what happens before it) then we say:
The probability of "X" given "Y"

$$P(X \mid Y)$$

The probability that this **WILL** happen

Assuming this already **HAS** happened

$$P(X \mid Y) = \frac{\frac{N(X \cap Y)}{N}}{\frac{N(Y)}{N}} = \frac{P(X \cap Y)}{P(Y)}$$

$N(X \cap Y)$ is the number of elements common to both X and Y.
N(Y) is the number of elements in Y, and it cannot be equal to zero.
Let N represent the total number of elements in the sample space.

Example

Two dies are thrown simultaneously, and the sum of the numbers obtained is found to be 7. What is the probability that the number 3 has appeared at least once?

Solution:

The sample space S would consist of all the numbers possible by the combination of two dies.

Therefore, S consists of 6×6, i.e. 36 events.

Event A indicates the combination in which 3 has appeared at least once.

Event B indicates the combination of the numbers which sum up to 7.

A = {(3, 1), (3, 2), (3, 3)(3, 4)(3, 5)(3, 6)(1, 3)(2, 3)(4, 3)(5, 3)(6, 3)}

B = {(1, 6)(2, 5)(3, 4)(4, 3)(5, 2)(6, 1)}

$P(A) = 11/36$

$P(B) = 6/36$

$A \cap B = 2$

$P(A \cap B) = 2/36$

Applying the conditional probability formula we get,

$P(A|B) = P(A \cap B)/P(B) = (2/36)/(6/36) = \frac{1}{3}$

PROBABILITY

- **Without Replacement**
- **With Replacement**

Probability without replacement

Probability without replacement involves dependent events where the preceding event has an effect on the probability of the next event.

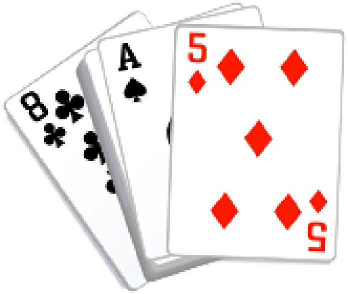
- Once we draw an item, then we do not replace it back to the sample space before drawing a second item.
- The probabilities for the second pick are affected by the result of the first pick.
- In other words, an item cannot be drawn more than once.

The **sample space changes** for different events and the occurrence of the next event depends upon what happens in the preceding event.

Example

Two cards are selected in sequence from a standard deck.

Find the probability that the **second card is a queen**, given that the first card is a king.



Solution:

Because the first card is a king and is not replaced, the remaining deck has 51 cards, 4 of which are queens.

$$P(B \mid A) = P(2^{nd} \text{ card is a Queen} \mid 1^{st} \text{ card is a King}) = \frac{4}{51} \approx 0.078$$

Probability with replacement

In probability theory, two events are said to be independent if one event's outcome does not affect the probability of the other event.

It means once **we draw an item**, then **we put it back to the sample space** before drawing a second item.

The **draws** in probability with replacement are **independent events**.

The **sample space doesn't change** for different events and the occurrence of the next event doesn't depend upon what happens in the preceding event.

Example 1

What is the probability of drawing twice from a deck and getting the **king** of diamonds **then** putting the card back in, then drawing the **queen** of diamonds?

THIS IS CALLED **REPLACEMENT**

The events are conditional

$$P(X) \times P(Y|X)$$

$$\frac{1}{52} \times \frac{1}{52} = \frac{1}{2704}$$

Example 2

Four cards are picked randomly, with replacement, from a regular deck of 52 playing cards.
Find the probability that all four are aces.

Solution:

There are four aces in a deck, and as we are replacing after each sample, so

$$P(first\ ace) = P(second\ ace) = P(third\ ace) = P(fourth\ ace) = \frac{4}{52}$$

All four samples are independent, so

$$P(all\ four\ ace) = \frac{4}{52} \cdot \frac{4}{52} \cdot \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{28561}$$

Random Variable

A random variable is a rule that assigns a numerical value to each outcome in a sample space.

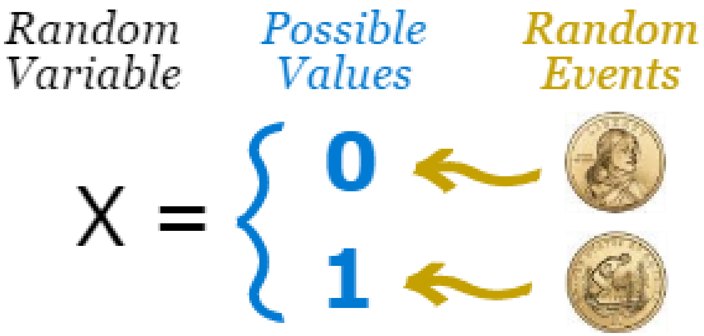
A random variable is needed to be measured, which allows probabilities to be assigned to a set of potential values.

For each element of an experiment's sample space Ω, the random variable can take on exactly one value.

For example, if you roll a die, the outcome is random (not fixed) and there are 6 possible outcomes, each of which occur with probability one-sixth.

Example

Tossing a coin: we could get **Heads** or **Tails**.
Let's give them the values Heads=0 and Tails=1 and we have a *Random Variable* "X":



In short:

X = {0, 1}

Note: We could choose Heads=100 and Tails=150 or other values if we want! It is our choice.

Expected Value

The **expected value** in general, the value that is most likely the result of the next repeated trial of a statistical experiment.

Expected value uses all possible outcomes and their probabilities of occurring to find the weighted average of the data in the data set.

$$E(f) = \sum_{x \in \Omega} f(x) P(x)$$

Example:

At a raffle, 500 tickets are sold for $1 each for two prizes of $100 and $50. What is the expected value of your gain?

Your gain for the $100 prize is $100 – $1 = $99.

Your gain for the $50 prize is $50 – $1 = $49.

Write a probability distribution for the possible gains (or outcomes).

Example continued:

At a raffle, 500 tickets are sold for \$1 each for two prizes of \$100 and \$50. What is the expected value of your gain?

Gain, x	$P(x)$
\$99	$\frac{1}{500}$
\$49	$\frac{1}{500}$
–\$1	$\frac{498}{500}$

Winning **no** prize

$$E(f) = \sum_{x \in \Omega} f(x)P(x)$$

$$= \$99 \cdot \frac{1}{500} + \$49 \cdot \frac{1}{500} + (-\$1) \cdot \frac{498}{500}$$

$$= -\$0.70$$

Because the expected value is negative, you can expect to lose \$0.70 for each ticket you buy.

Linearity of expectation

If f and g are two random variables, then

$$E[f+g] = E[f] + E[g].$$

More linearity of expectation

Suppose f is a random variable. Define the random variable g as follows: For every $x \in \Omega$, $g(x) = cf(x)$. Then $E[g] = cE[f]$.

Not linearity of expectation

If f and g are two random variables, then some students assume that $E[f \times g] = E[f] \times E[g]$. This is not always true.

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End of lecture

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Reading: Discrete Mathematics for Computing R. Haggarty, Chapter 6.

Reading: Discrete Mathematics for Computer Scientists, J.K. Truss, Section 5.1, 5.3