The Multiplication Principle

Two step multiplication principle: Assume that a task can be broken up into two consecutive steps. If step 1 can be performed in m ways and for each of these, step 2 can be performed in n ways, then the task itself can be performed in $m \times n$ ways.

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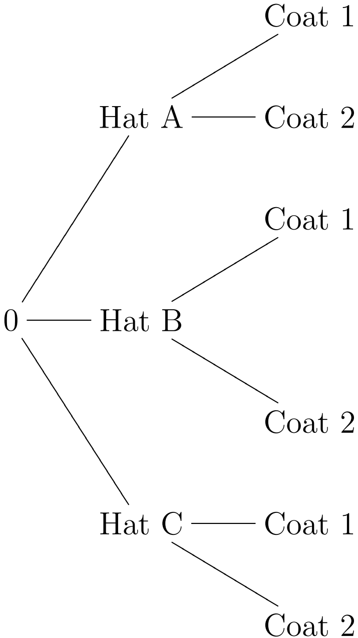
Example 1 Suppose you have 3 hats, hats A, B and C, and 2 coats, Coats 1 and 2, in your closet. Assuming that you feel comfortable with wearing any hat with any coat. How many different choices of hat/coat combinations do you have? List all combinations.

The Multiplication Principle

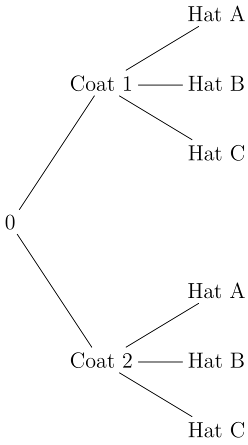
We can get some insight into why the formula holds by representing all options on a tree diagram. We can break the decision making process into two steps here:
Step 1: Choose a hat,
Step 2: choose a coat.
From the starting point 0, we can represent the three choices for step 1 by three branches whose endpoints are labelled by the choice names. From each of these endpoints we draw branches representing the options for step two with endpoints labelled appropriately. The result for the above example is shown below:

The Multiplication Principle



The Multiplication Principle

Here is the problem done with Coats first and then Hats.



The Multiplication Principle

Each path on the tree diagram corresponds to a choice of hat and coat. Each of the three branches in step 1 is followed by two branches in step 2, giving us 3×2 distinct paths.

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If we had m hats and n coats, we would get $m \times n$ paths on our diagram. Of course if the numbers m and n are large, it may be difficult to draw.

The Multiplication Principle

Example 2 The South Shore line runs from South Bend Airport to Randolph St. Station in Chicago. There are 20 stations at which it stops along the line. How many one way tickets could be printed, showing a point of departure and a destination? (Assuming you can not depart and arrive at the same station.)

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You can start at any of twenty stations. Once this is picked, you can pick any of nineteen destinations. The answer is $20 \cdot 19 = 380$.

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The Multiplication Principle

Example 3 You want to design a 30 minute workout. For the first 15 minutes, you will choose an aerobic exercise from running, kickboxing, skipping or circuit training. For the second 15 minutes, you will work on strength and/or balance choosing from weight training, TRX, Bosu, resistance bands or your core routine. How many such workouts are possible.

The Multiplication Principle

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There are 4 things you can do for your first 15 minutes.
There are 5 things you can do for the second 15 minutes.
The answer is $4 \cdot 5 = 20$.

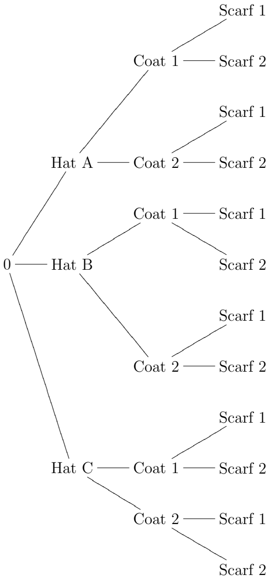
The Multiplication Principle

Example 4 If your closet contains 3 hats, 2 coats and 2 scarves. Assuming you are comfortable with wearing any combination of hat, coat and scarf, (and you need a hat, coat and scarf today), how many different outfits could you select from your closet? (Break the decision making process into steps and draw a tree diagram representing the possible choices.)

The Multiplication Principle

Example 4 If your closet contains 3 hats, 2 coats and 2 scarves. Assuming you are comfortable with wearing any combination of hat, coat and scarf, (and you need a hat, coat and scarf today), how many different outfits could you select from your closet? (Break the decision making process into steps and draw a tree diagram representing the possible choices.) Before you do this, try to predict the answer.

The Multiplication Principle



The General Multiplication Principle

If a task can be broken down into R consecutive steps, Step 1, Step 2, ..., Step R, and if
I can perform step 1 in m_1 ways,
and for each of these I can perform step 2 in m_2 ways,
and for each of these I can perform step 3 in m_3 ways,
and so forth
Then the task can be completed in

$$m_1 \cdot m_2 \cdot \cdots \cdot m_R$$

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Note in example 4, $R = 3$, $m_1 = 3$, $m_2 = 2$ and $m_3 = 2$.

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$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$$

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The current population of Indiana seems to be just short of $6,500,000$. Since there are families with more cars than people, this is probably not enough. In fact Indiana now often uses 3 letters which yields

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$$

The General Multiplication Principle

Example 6 A group of 5 boys and 3 girls is to be photographed.

(a) How many ways can they be arranged in one row?

There are 8 people so there are

$$8 \cdot 7 \cdot \cdots \cdot 2 \cdot 1 = 8! = 40,320$$

possible ways to do this. The fact that some of them are boys and others girls is irrelevant.

Example 6 continued

(b) How many ways can the 5 boys and 3 girls be arranged with the girls in front and the boys in the back row?

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There are 3 girls so there are $3 \cdot 2 \cdot 1 = 3!$ ways to arrange the first row. There are 5 boys so there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways to arrange the second row. The two rows can be arranged independently so the answer is $3! \cdot 5! = 6 \cdot 120 = 720$ possibilities.

The General Multiplication Principle

Example 7 How many different 4 letter words (including nonsense words) can you make from the letters of the word

MATHEMATICS

if (a) letters cannot be repeated (MMMM is not considered a word but MTCS is).

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'MATHEMATICS' has 8 distinct letters $\{M, A, T, H, E, I, C, S\}$. Hence the answer is $8 \cdot 7 \cdot 6 \cdot 5 = 1,680$

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(c) Letters cannot be repeated and the word must start with a vowel.

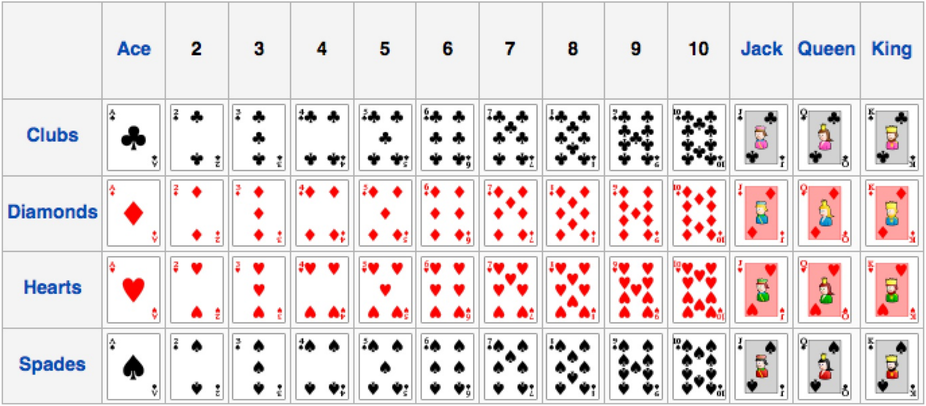
The 8 distinct letters {M, A, T, H, E, I, C, S} have 3 vowels {A, E, I}. You can select a vowel in any of 3 ways. Once you have done this you have 7 choices for the second letter; 6 choices for the third letter; and 5 choices for the fourth letter. Hence the answer is $3 \cdot 7 \cdot 6 \cdot 5 = 630$.

The General Multiplication Principle

A standard deck of 52 cards can be classified according to suits or denominations as shown in the picture from Wikipedia below. We have 4 suits, Hearts Diamonds, Clubs and Spades and 13 denominations, Aces, Kings, Queens, $\ldots$, twos.

The General Multiplication Principle

Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades



The General Multiplication Principle

Example 8 Katy and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them.

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(a) How many different outcomes can result? $52 \cdot 51$

(b) In how many of the possible outcomes do both players have Hearts? $13 \cdot 12$

Combining Counting Principles

Recall that the inclusion-exclusion principle says that if A and B are sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)\ .$$

If the sets A and B are **disjoint** then this principle reduces to $n(A \cup B) = n(A) + n(B)$. Thus in counting disjoint sets, we can just count the number of elements in each and add. This principle extends easily to $R > 2$ disjoint sets:
If A_1, A_2, $\ldots A_R$ are disjoint sets, then

$$n(A_1 \cup A_2 \cup \cdots \cup A_R) = n(A_1) + n(A_2) + \cdots + n(A_R)$$

Combining Counting Principles

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There are four distinct possibilities. The possibilities are 2 clubs, 2 diamonds, 2 hearts or 2 spades and these are distinct. In each of these the first card has 13 possibilities while the second has 12. Hence the answer is
$(13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12)$.

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A second approach is that there are 52 ways to pick the first card and then there are 12 ways to pick the second. Hence the answer is $52 \cdot 12$.

Combining Counting Principles

Example 10 Suppose you are going to buy a single carton of milk today. You can either buy it on campus when you are at school, or at the mall when you go to get a gift for a friend or in the neighborhood near your apartment on your way home. There are 5 different shops on campus to buy from, 2 at the mall and 3 in your neighborhood. In how many different shops can you buy the milk?

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There are three distinct outcomes. You buy the milk on campus with 5 choices, or you buy the milk at the mall with 2 choices or you buy the milk in your neighborhood with 3 choices, so the answer is $5 + 2 + 3$.

Combining Counting Principles

Example 10 Suppose you are going to buy a single carton of milk today. You can either buy it on campus when you are at school, or at the mall when you go to get a gift for a friend or in the neighborhood near your apartment on your way home. There are 5 different shops on campus to buy from, 2 at the mall and 3 in your neighborhood. In how many different shops can you buy the milk?

There are three distinct outcomes. You buy the milk on campus with 5 choices, or you buy the milk at the mall with 2 choices or you buy the milk in your neighborhood with 3 choices, so the answer is $5 + 2 + 3$.

If you answered $5 \cdot 2 \cdot 3$ you answered the question of how many ways could you buy one carton of milk on campus, one carton at the mall and one carton near home. In particular you end up with three cartons.

Combining Counting Principles

Example 11 Suppose you wish to photograph 5 schoolchildren on a soccer team. You want to line the children up in a row and Sid insists on standing at the end of the row(either end will do). If this is the only restriction, in how many ways can you line the children up for the photograph? (You can think through this as the number of ways to carry out the task or the number of photographs in a set).

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There are two distinct possibilities, Sid is on the left or Sid is on the right. There are $4!$ ways to arrange the other children. Hence the answer is $4! + 4!$.

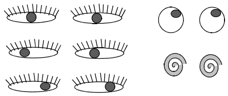
Extras, Multiplication Principle

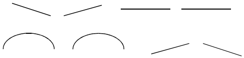
Example 12 How many faces can you make?

Below you are given 5 pairs of eyes, 4 sets of eyebrows, 2 noses, 5 mouths and 7 hairstyles to choose from. How many possible faces can you make using combinations of the features given if each face you make has a pair of eyes, a pair of eyebrows, a nose, a mouth, and one of the given hairstyles?

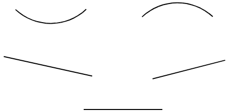
Example 12 continued - your choices

Noses





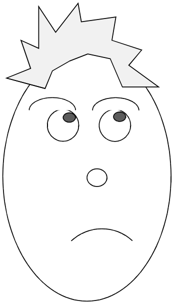
Eyebrows



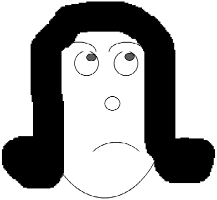
Mouths

Example 12 continued

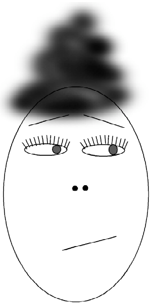
Here is an example of 3 faces, draw three different faces with the features given!



I don’t want a Lisa Simpson Hairdo!



If you say “Multiplication Principle” one more time.....



How many roads must a face walk down.....

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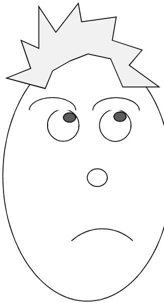

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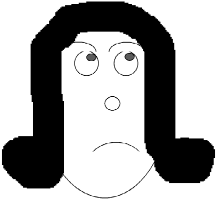

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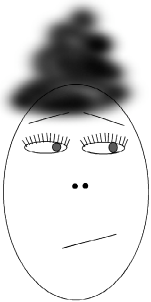

How many roads must a face walk down.....

$5 \cdot 4 \cdot 2 \cdot 5 \cdot 7 = 1,400.$