Discrete Mathematics and Statistics

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Part 3. Relations

Reading: Discrete Mathematics for Computing R. Haggarty, Chapter 4.

Contents

Relations

Let S denote the set of undergraduate students at Liverpool University and C denote the set of undergraduate modules on offer.

We can form the set R of ordered pairs $(s, c) \in S \times C$ with the property that student s is registered for the module c. Then

$$R = \{(s, c) \in S \times C \mid \; s \text{ registered for } c\ \}.$$

This subset of $S \times C$ captures the relationship *registered for*.

Definition A binary relation between two sets A and B is a subset R of the Cartesian product $A \times B$. If $A = B$, then R is called a binary relation on A.

Example 1

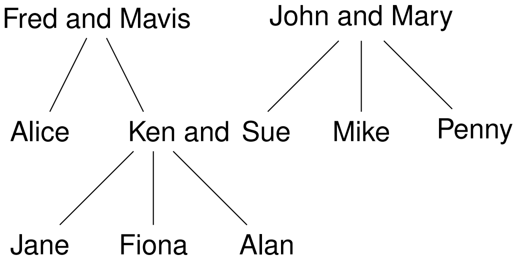


Figure: Family tree

Write down

- $R = \{(x, y) \mid\ x \text{ is a grandfather of } y\ \}$;
- $S = \{(x, y) \mid\ x \text{ is a sister of } y\ \}$.

Example 2

Write down the ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

- $U = \{(x, y) \in A \times B \mid x + y = 9\}$;
- $V = \{(x, y) \in A \times B \mid x < y\}$.

Example 3

Let $A = \{1, 2, 3, 4, 5, 6\}$. Write down the ordered pairs belonging to

$$R = \{(x, y) \in A \times A \mid \ x \text{ is a divisor of } y\ \}.$$

Representation of binary relations: directed graphs

- Let A and B be two finite sets and R a binary relation between these two sets (i.e., $R \subseteq A \times B$).
- We represent the elements of these two sets as vertices of a graph.
- For each $(a, b) \in R$, we draw an arrow linking the related elements.
- This is called the directed graph (or digraph) of R.

Example

Consider the relation V between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ such that $V = \{(x, y) \in A \times B \mid x < y\}$.

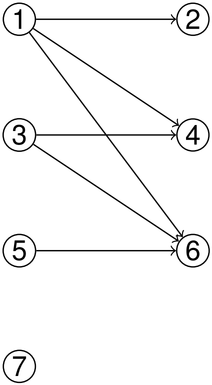


Figure: digraph of V

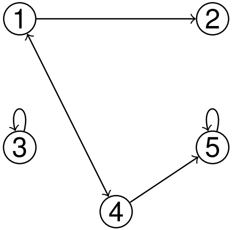
Digraphs of binary relations on a single set

Recall that a binary relation between a set A and itself is called “a binary relation on A”.

To represent such a relation, we use a directed graph in which a single set of vertices represents the elements of A and arrows link the related elements.

Consider the relation $V \subseteq A \times A$ where $A = \{1, 2, 3, 4, 5\}$ and

$$V = \{(1,2), (3,3), (5,5), (1,4), (4,1), (4,5)\}.$$



Representation of binary relations: matrices

- Another way of representing a binary relation between finite sets uses an array.
- Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $R \subseteq A \times B$.
- We represent R by an array M of n rows and m columns. Such an array is called a *n by m matrix*.
- The entry in row i and column j of this matrix is given by $M(i,j)$ where

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

Example 1

Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$, and

$$U = \{(x, y) \in A \times B \mid x + y = 9\}$$

Assume an enumeration $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$ and $b_1 = 2$, $b_2 = 4$, $b_3 = 6$. Then M represents U, where

$$M = \begin{bmatrix} F & F & F \\ F & F & T \\ F & T & F \\ T & F & F \end{bmatrix}$$

Example 2

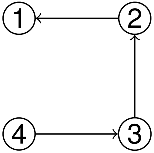
Let $A = \{a, b, c, d\}$ and suppose that $R \subseteq A \times A$ has the following matrix representation:

$$M = \begin{bmatrix} F & T & T & F \\ F & F & T & T \\ F & T & F & F \\ T & T & F & T \end{bmatrix}$$

List the ordered pairs belonging to R.

Example

The binary relation R on $A = \{1, 2, 3, 4\}$ has the following digraph representation.



Unary Relations

The reason that a binary relation is called a “binary” relation is that it is a relation between two sets.

One of the class problems looked at “ternary” relations (relations between three sets).

Unary relations are just subsets of a set.

Example: The unary relation `EvenPositiveIntegers` on the set $\mathbb{Z}^+$ of positive integers is

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even}\}.$$

Infix notation for binary relations

If R is a binary relation then we write xRy whenever $(x, y) \in R$.

The predicate xRy is read as x is R-related to y.

Properties of binary relations

A binary relation R on a set A is

- *reflexive* when xRx for all $x \in A$.
- *symmetric* when xRy implies yRx for all $x, y \in A$;
- *antisymmetric* when xRy and yRx imply $x = y$ for all $x, y \in A$;
- *transitive* when xRy and yRz imply xRz for all $x, y, z \in A$.

In the directed graph representation, R is

- *reflexive* if there is always an arrow from every vertex to itself;
- *symmetric* if whenever there is an arrow from x to y there is also an arrow from y to x;
- *antisymmetric* if whenever there is an arrow from x to y and $x \neq y$, then there is no arrow from y to x;
- *transitive* if whenever there is an arrow from x to y and from y to z there is also an arrow from x to z.

Example

Which of the following define a relation that is reflexive, symmetric, antisymmetric or transtive?

- *x* divides *y* on the set $\mathbb{Z}^+$ of positive integers;
- $x \neq y$ on the set $\mathbb{Z}$ of integers;
- *x has the same age as y* on the set of people.

Transitive Closure

Given a binary relation R on a set A, the *transitive closure* R^* of R is the (uniquely determined) relation on A with the following properties:

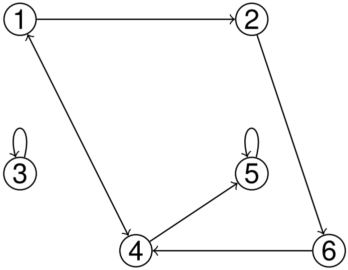
- R^* is transitive;
- $R \subseteq R^*$;
- If S is a transitive relation on A and $R \subseteq S$, then $R^* \subseteq S$.

Example

Let $A = \{1, 2, 3\}$. Find the transitive closure of

$$R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1)\}.$$

Finding the transitive closure is easier with the digraph representation



Equivalence Relations

Definition A binary relation R on a set A is called an *equivalence relation* if it is reflexive, transitive, and symmetric.

Examples:

- the relation R on the non-zero integers given by xRy if $xy > 0$;
- the relation *has the same age* on the set of people.

Definition The *equivalence class* E_x of any $x \in A$ is defined by

$$E_x = \{y \mid yRx\}.$$

Example

Define a relation R on the set $\mathbb{R}$ of real numbers by setting xRy if and only if $x - y$ is an integer. Prove that R is an equivalence relation. Moreover,

- $E_0 = \mathbb{Z}$ is the equivalence class of 0;
- $E_{\frac{1}{2}} = \{\ldots, -2\frac{1}{2} - 1\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots\}$ is the equivalence class of $\frac{1}{2}$.