Week 8 – Tutorial
Solutions

○ Functions: definitions and examples
○ Domain, codomain, and range / image
○ Injective, surjective, and bijective functions
○ Invertible functions
○ Compositions of functions

Q.1.

Let A = {0, 2, 4,6} and B = {1, 3, 5, 7}. Determine which of the following relations between A and B forms a function with domain A and codomain B:

(a) {(6, 3), (2, 1), (0, 3), (4, 5)},

(b) {(2, 3), (4, 7), (0, 1), (6, 5)},

(c) {(2, 1), (4, 5), (6, 3)},

(d) {(6, 1), (0, 3), (4, 1), (0, 7), (2, 5)}.

For those which are functions, which are injective and which are surjective?

A.1.

(a) a function, but neither surjective nor injective

(b) a function, surjective and injective

(c) not a function since no element of B is associated with 0 in A,

(d) not a function since 3 and 7 are associated with 0.

Q.2.

Which of the following functions are injective? Which are surjective?

a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2 + 1$.

b) $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = 2^x$.

c) $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = 5x - 1$

A.2.

(a) $f(-1) = f(1)$, therefore f is not injective. $f(x) = 3$ has no solution in $\mathbf{Z}$. Therefore f is not surjective.

(b) If $g(a) = g(b)$, then $2^a = 2^b$ and $a = b$. Hence, g is injective. $g(x) = 3$ has no solution in $\mathbf{N}$. Hence, g is not surjective.

(c) h is surjective. $h(b) = h(a)$ implies $a = b$. Hence, h is injective.

Q.3.

The function $f : A \rightarrow B$ is given by $f(x) = 1 + \dfrac{2}{x}$ where A denotes the set of real numbers excluding 0 and B denotes the set of real numbers excluding 1. Show that f is bijective and determine the inverse function.

A.3.

If $f(a) = f(b)$ then $1+\frac{2}{a} = 1+\frac{2}{b}$ and so $a = b$. Hence, f is injective. If $f(x) = y$ then $1 + \frac{2}{x} = y$ and so $x = \frac{2}{(y-1)}$. Hence, f is surjective. Therefore, f is bijective. The inverse function is given by $f^{-1} : B \rightarrow A$ where $f^{-1}(x) = \frac{2}{(x-1)}$.

Q.4.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x + 9$.

Calculate $g \circ f, f \circ g, f \circ f$ and $g \circ g$.

A.4.

1. $g \circ f(x) = g(f(x)) = g(3x) = 3x + 9$
2. $f \circ g(x) = f(g(x)) = f(x + 9) = 3x + 27$
3. $f \circ f(x) = f(f(x)) = f(3x) = 9x$
4. $g \circ g(x) = g(g(x)) = g(x + 9) = x + 18$

Q.5.

Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be functions. Prove that:

a) if f and g are injective, then $g \circ f$ is also injective.
b) if f and g are surjective, then $g \circ f$ is also surjective.
c) if $g \circ f$ is injective, then f is injective.
d) if $g \circ f$ is surjective, then g is surjective.

A.5. (I)

a) if f and g are injective, then $g \circ f$ is also injective.

We must show that for all $a_1, a_2 \in A$, if $g(f(a_1)) = g(f(a_2))$, then $a_1 = a_2$.

Assume we have $g(f(a_1)) = g(f(a_2))$, Let $f(a_1) = b_1$ and $f(a_2) = b_2$. So we have $g(b_1) = g(b_2)$.

Because g is injective, this implies $f(a_2) = b_1 = b_2 = f(a_2)$.

Because f is also injective, this implies $a_1 = a_2$.

b) if f and g are surjective, then $g \circ f$ is also surjective.

We must show that for all $c \in C$, these exists at least one a in A such that $g(f(a)) = c$.

Since $g : B \rightarrow C$ is surjective, there exists $b \in B$ such that $g(b) = c$.

Since $f : A \rightarrow B$ is surjective, there exists $a \in A$ such that $f(a) = b$.

Then $g(f(a)) = g(b) = c$.

A.5.(II)

c) if $g \circ f$ is injective, then f is injective.

We have to prove when $f(a_1) = f(a_2)$, $a_1 = a_2$.

Let $a_1, a_2 \in A$. Then

$$f(a_1) = f(a_2) \implies g(f(a_1)) = g(f(a_2))$$
$$\iff g \circ f(a_1) = g \circ f(a_2) \quad \iff a1 = a2.$$

d) if $g \circ f$ is surjective, then g is surjective.

Let $c \in C$. As $g \circ f$ is surjective there is an $a \in A$ with $c = g \circ f(a)$.

Then b $\overset{\text{def}}{=}$ f(a) $\in$ B and $g(b) = c$.

Q.6.

For some domains and codomains, show the composition:

a) is commutative (e.g. $(g \circ f) = (f \circ g)$);

b) is associative (e.g. $(h \circ g) \circ f = h \circ (g \circ f)$).

A.6.

a)   Counter example : $f(x) = x^2$ and $g(x) = x + 1$.

$$g(f(x)) = x^2 + 1$$

$$f(g(x)) = (x + 1)^2 = x^2 + 2x + 1$$

b)   Let a from A.

$$[(h \circ g) \circ f](a) = (h \circ g)(f(a))$$

$$= h\Big(g(f(a))\Big)$$

$$= h(g \circ f(a))$$

$$= [h \circ (g \circ f)](a)$$

Q.7.

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Show that:
   a) if f and g are surjective, then fg are surjective.
   b) if f and g are injective, then $f+g$ are injective.

A.7.

Counter example:

$f(x) = x$ which is surjective and injective

$g(x) = -x$ which is surjective and injective

$fg(x) = -x^2$ which is not surjective.

$(f + g)(x) = 0$ which is not injective.

Q.8.

Let $m \neq 0$ and b be real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$.

(a) Prove that f is a bijection.

(b) Since f is a bijection, it is invertible. Find its inverse f^{-1}, and show it is an inverse by demonstrating that

$$f^{-1}(f(x)) = x.$$

A.8.(a)

Discussion . To show that $f(x) = mx + b$ is a bijection, we must show that it is both an injection and a surjection.

- To show that f is a surjection, we must find, for every $a \in \mathbb{R}$, an x such that $f(x) = mx + b = a$. Solving for x, we see that $x = \dfrac{a - b}{m}$ will work.

- To show that f is an injection, we will assume that $f(x) = f(y)$ and check algebraically that $x = y$.

Solution To show that $f(x) = mx + b$, we will show that it is both a surjection and an injection.

To show that $f(x)$ is a surjection, let $a \in \mathbb{R}$. We will find a pre-image for a. Notice that $x = \frac{a-b}{m} \in \mathbb{R}$ is indeed a real number since $m \neq 0$. Furthermore, x is a pre-image because

$$f\left(\frac{b - a}{m}\right) = m\left(\frac{b - a}{m}\right) + b = b - a + b = a.$$

Thus, every $a \in \mathbb{R}$ has a pre-image and f is surjective.

To show that $f(x)$ is an injection, assume that $f(x) = f(y)$. Then, $mx + b = my + b$. Subtracting b, we get $mx = my$ and dividing by $m \neq 0$, we get $x = y$, as desired. Thus, f is an injection.

Since f is both a surjection and an injection, it is a bijection.

A.8.(b)

Discussion — To find the inverse, we can simply set $y = mx + b$ and solve for x. Doing so, we get that $x = \frac{y-b}{m}$ and thus it may be wise to defined $f^{-1}(x) = \frac{x-b}{m}$.

Proof For $f(x) = mx + b$, consider the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = \frac{x-b}{m}$. Notice that

$$f^{-1}(f(x)) = f^{-1}(mx + b) = \frac{(mx + b) - b}{m} = \frac{mx}{m} = x.$$

Thus, f^{-1} is indeed the inverse of f.