Extension questions

These questions are optional. They are provided to extend those who have completed the first part of the tutorial and would like a challenge. Don't be worried if you can't solve them, based on what we have already covered! Many of the solutions require some independent research or reading, in order to solve problems beyond the scope of what we have seen in lectures.

1. Give an interpretation to prove that each of the following wffs is *not* valid:

   a. $(\exists x.\ A(x) \land \exists x.\ B(x)) \to (\exists x.\ A(x) \land\ B(x))$
   b. $\forall x.\ \neg A(x)\ \leftrightarrow \neg\forall x.A(x)$

SOLUTION:

a. Let $A(x)$ = "x played for the Dockers in August 2015", and $B(x)$ = "x played for the Eagles in August 2015", and the domain of discourse be all people. There are certainly people who satisfy A, and there are certainly people who satisfy B; but there are no people who satisfy both (so far as I know).

   Or, let $A(x)$ = "x is less than 100" and $B(x)$ = "x is greater than 100", and the domain of discouse be the integers. There are certainly integers less than 100, and there are certainly integers greater than 100; but there is no integer which is both less than 100 and greater than 100 at the same time.

b. Let $A(x)$ = "x is 3 or older", and the domain of discourse be all people.
   The right-hand part of the biconditional says "Not all people are 3 or older". The left-hand part says "All people are younger than 3". These are certainly not equivalent, since they have different truth-values; the right-hand part is true, and the left-hand part false.

   Or, let the domain of discourse be the set of all West Australians, and $A(x)$ = "x plays football".
   The right-hand part of the biconditional says that "It's not true that all West Australians play football" (which is true), and the left-hand parts says "No West Australian plays football" (which is false).

2. Prove or disprove the validity of the sentences:

   a. $\exists x.\forall y.\ x < y$
   b. $\forall x.\exists y.\ x < y$

Solution:

a. The sentence is not valid, since we can give an interpretation which makes it false. If we take "$<$" as meaning "less than", then the sentence says "There is an x which is smaller than any y". If we take the domain of discourse to be the integers, then this is not true, since there is no least integer.

   But, we don't even have to interpret "$<$" as meaning "less than". We could take it to mean "is standing on top of", and the domain as being all people. In which case, the sentence would translate as "There exists a person who is standing on top of all people (including themselves)", which isn't true.

   Or, we could interpret "$<$" as meaning "is adjacent to", and the domain as being all countries. In which case, the sentence would translate as "There exists a country which is adjacent to all countries (including itself)", which is also not true.

b. If we take "$<$" as meaning "less than", then the sentence says, "For all x, there exists a y which is larger than x." Now, this happens to be true for the natural numbers ($\mathbb{N}_{\geq 0}$) and the integers ($\mathbb{Z}$); but the sentence is not valid, since we can give interpretations in which it is false.

   We could interpret "$<$" as meaning "greater than", and take the domain of discourse to be the natural numbers. In that case, the sentence says that "For all x, there exists a y which is *smaller* than x", and that certainly isn't true for the natural numbers; there is no natural number lower than 0.

3. Prove or disprove the sentence:

$$\exists x.(P(x) \to Q(x)) \land \forall y.(Q(y) \to R(y)) \land \forall x.P(x) \to \exists x.R(x)$$

Solution:
A formal proof of this would make use of concepts we haven't yet covered; but we can reason informally about it by considering the semantics of the sentence and what we can deduce about it.

First, let's rewrite the sentence to make it a bit easier to see its structure. Quantifiers bind tighter than connectives, and "and" binds more tightly than the conditional, so we can rewrite the sentence (partly in English and partly as a formula) as:

If:
$\exists x.(P(x) \to Q(x))$, and
$\forall y.(Q(y) \to R(y))$, and
$\forall x.P(x)$
then:
$\exists x.R(x)$.

Let's reorder the first part of the conditional (the "antecedent") a bit (which we are allowed to do, since $\wedge$ is commutative):

If:
$\exists x.(P(x) \to Q(x))$, and
$\forall x.P(x)$, and
$\forall y.(Q(y) \to R(y))$,
then:
$\exists x.R(x)$.

Consider the parts (clauses) of the antecedent in turn. If there exists an x such that something is true of it, then we know there's at least one object in the domain; let's give it a name for the moment, and call it a. We know that if a is P (whatever P might be), then it's Q (whatever Q might be), and we also know that all things in the domain are P. Therefore, a must be Q. We also know that if something is Q, then it's also R, so a must be R, as well.

So it follows from all this that something exists (namely, the thing we've called a), and it is R. In other words, $\exists x.R(x)$.

Now, that is exactly what the right-hand part of the conditional (the "consequent") is saying. So if we let A be the antecedent, and B the consequent, then we've shown that A does indeed imply B; and this will be true regardless of what domain of discourse we consider, and regardless of what interpretation we happen to give to P, Q and R. So, the sentence is valid.

4. a. Identify the propositions in each of the following statements and express each statement in propositional logic (from Lewis Carroll).

   i. All humming birds are richly coloured
   ii. No large birds live on honey
   iii. Birds that do not live on honey are dull in colour

   b. Construct a logical argument from the statements above to show that all humming birds are small.

SOLUTION:

a. To express these as propositions, we need to make them implicitly about some thing, x, which we might encounter. i.e., We say something like "Imagine I am wandering about, and I encounter a – *thing*. And let us suppose that all of facts (i)-(iii) are true. What could I conclude?"

   If we think of it this way, we should be able to get by, just by thinking about what we know about propositional logic and when connectives are true and false.

   We get:

   i. H = "x is a humming bird", and R = "x is richly coloured".
   ii. L = "x is a large bird", and O, "x lives on honey".
   iii. O = "x lives on honey", and R = "x is richly coloured". (If we assume that being dull in colour is just the negation of being richly coloured.)

b. We have from (a) that $H \to R$, $O \to \neg L$, and $\neg O \to \neg R$.

   Since $\neg O \to \neg R$, by the contrapositive rule, $R \to O$.

   Since $H \to R$ and $R \to O$, then H should also imply O – i.e. $H \to O$. (This is a technique called proof by conditional, which we haven't seen yet. But you could also use truth tables to show that $(H \to R) \wedge (R \to O)$ is equivalent to $H \to O$.)

   And since we have $H \to O$ and (from (ii)) $O \to \neg L$, it follows by the same reasoning that $H \to \neg L$.

   In other words, all humming birds are small, which is exactly what we had to prove.

5. We have said that predicates can take one or more arguments. What if we allow a predicate to take *no* arguments? Then it would already be a complete sentence, and would be true or false.

   Try expressing each of the following statements in predicate logic using the 0-place predicates C = "Charles likes discrete maths" and D = "Danica likes discrete maths".

   a. Charles likes discrete maths but Danica does not.
   b. Danica likes maths if Charles does too.
   c. Neither Charles nor Danica dislike maths

   What do you get? Do the results remind you of anything we have seen in lectures previously?

SOLUTION:

a. $C \land \neg D$
b. $C \to D$
c. $\neg(\neg C \lor \neg D)$

Note that these look exactly like the answers we had for a similar propositional logic question in week 1, and this is not a coincidence. *Propositions* from propositional logic can be regarded as 0-place predicates; they don't need any arguments to make them false or true.

In fact, propositional logic can be seen as a subset of predicate logic, in which all predicates are 0-place, and no functions, variables, quantifiers or constants are used.

(This makes propositional logic in some ways less powerful than predicate logic, since it can express fewer things; but it also makes it *more* useful in some ways, since we have techniques for solving even very large propositional logic problems – using the "SAT solvers" mentioned in lectures.)

6. We have said that statements like $\forall x.M(x)$ (where our domain of discourse is the set of all people, and M means "is mortal") cannot be represented in propositional logic.

   However, suppose that we had a whole *bunch* of propositions, M_1, M_2, M_3, and so on, one for every person on Earth. (Where we assume we have somehow numbered everyone, and M_1 means "Person 1 is mortal", M_2 means "Person 2 is mortal", and so on.)

   We could then "and" all those propositions together, to get

$$M_1 \land M_2 \land M_3 \land \ldots \land M_{7.4 \text{ billion or so}}$$

   a. Can you think of any way of simply translating this into English?

   b. Now suppose we applied de Morgan's law to that: each "and" becomes an "or", and each individual proposition M_1, M_2 etc. gets negated, and we put a negative around the whole thing. How would you translate the resulting formula into English? Does it remind you of anything else we have seen in lectures?

Solution:

a. We might translate this as "all people are mortal".

b. The "de Morganed" version would be:

$$\neg(\neg M_1 \vee \neg M_2 \vee \neg M_3 \vee \ldots \vee \neg M_{7.4 \text{ billion or so}})$$

Which we could translate as "It is not the case that person 1 is immortal, or person 2 is immortal, or person 3 is immortal . . . " and so on. In other words, none of those people are immortal. Which is very similar to saying "There does not exist a person who is immortal". So the rules for negating quantified predicate logic formulas can be seen as a generalized version of de Morgan's laws – we will see another way of showing this, later.