Discrete Mathematics and Statistics (CPT107)

Tutorial 4 - Solutions

1: Show that the relation "is a divisor of" on the set $\mathbb{Z}^+$ of positive integers is a partial order.

Solution:

For $x, y, z \in \mathbb{Z}^+$

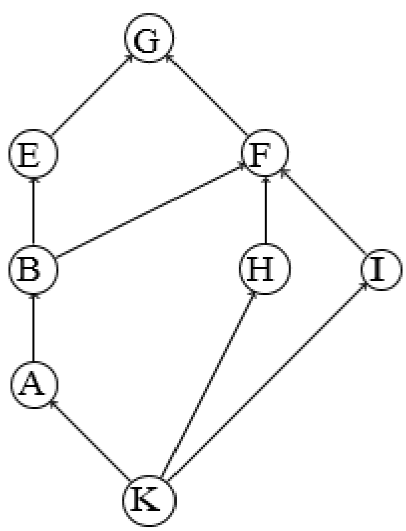
• The relation is reflexive: x is a divisor of x.

• The relation is transitive: If x is a divisor of y and y is a divisor of z then x is a divisor of z.

• The relation is antisymmetric: If x is a divisor of y and y is a divisor of x then x = y.

2: Suppose that A, B, E, F, G, H, I and K represent people on twitter. We assume that everybody follows himself on twitter. Also, everybody follows G. Everybody follows E except H, I, F and G. Everybody follows F except E and G. Also, A follows B and K follows everybody. It turns out that the relation "follows" is a partial order in this case. Draw the Hasse Diagram of this partial order. Construct a pair of people, such that, if the first of them started following the second, then "follows" would no longer be a partial order.

Solution: Here is the Hasse diagram.



If G started following K, this would no longer be a partial order.

3: Let $M = \{a, b, c\}$. What is the transitive closure of the relation $\{(a, a), (a, b), (a, c), (c, a), (b, c)\}$ on M?

Solution: $\{(a, a), (a, b), (a, c), (c, a), (b, c), (c, b), (c, c), (b, a), (b, b)\}$

4: Define the relation on ℝ x ℝ by (a,b) R (x,y) iff a ≤ x and b ≤ y. Prove that R is a partial order on ℝ x ℝ.

Solution: For all (a,b) ∈ ℝ x ℝ, we have (a,b) R (a,b) since a ≤ a and b ≤ b. So, R is reflexive. If (a,b) R (x,y) and (x,y) R (a,b) then a ≤ x and b ≤ y, as well as, x ≤ a and y ≤ b. Since, a ≤ x and x ≤ a we have a = x. Similarly, b = y and we have (a,b) = (x,y). Thus, R is antisymmetric. If (a,b) R (x,y) and (x,y) R (w,z), then a ≤ x and x ≤ w, so a ≤ w. Similarly, b ≤ y and y ≤ z, so b ≤ z. Thus, (a,b) R (w,z) and R is transitive. Therefore, R is a partial order.

5: Given a partially ordered set, is a pair $\mathbf{P} = (X, \leq)$, where X is a nonempty set and $\leq$ is a partial order on X; that is, for x, y, and $z \in X$
(1) $x \leq x$ (reflexivity)
(2) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)
(3) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity)

Prove that: If $(X, \leq)$ is a finite partially ordered set, then X has a maximal and a minimal element.

Solution: For $x, y \in X$, $x \in X$ is maximal if $x \leq y$ implies $x = y$. Pick $x_1 \in X$. If x_1 is not maximal, there is another element $x_2 \in X$ with $x_1 < x_2$. Continuing in this way for r steps, we get $x_1 < x_2 < \cdots < x_r$. Since $<$ is transitive, $x_i < x_j$ whenever $i < j$. In particular they are distinct. Since X is finite this process must stop and the last element is a maximal element. The proof to show that X has a minimal element is similar.

6: Prove that the divides relation on A $= \{1,2,4,8,\ldots,2^n\}$ is a total order; where n is a nonnegative integer.

Solution:

It is not hard to see that “divides relation on A” is a partial order.

Reflexivity: Trivial.

Transitivity: If (a,b) is an element of the divides relation then we know that $\boldsymbol{a}$ divides $\boldsymbol{b}$; similarly if (b,c) is and element of the divides relation, then we know that $\boldsymbol{b}$ divides $\boldsymbol{c}$. Transitivity holds because for elements $(a,b),(b,c)$; (a,c) will be also part of the relation and the divides property will still hold.

Antisymmetry: If (a,b) is a member of the divides relation, then (b,a) would be a member only if $a=b$.

Then we have to show that either $\boldsymbol{aRb}$ or $\boldsymbol{bRa}$:
Let a and b, be particular but arbitrarily chosen elements of A. By definition of A, there are nonnegative integer r and s such that $a=2^r$ and $b=2^s$ (Since r,s are the exponents, nonnegative integers). Now either $r < s$ or $s < r$;
If $r<s$, then $b=2^s=2^r2^{s-r}=a2^{s-r}$ where $s-r>0$. It follows, by definition of divisibility, that a divides b. For $s < r$, the proof is similar.