Discrete Mathematics and Statistics **(CPT107)**

Tutorial 3 bis - Solutions

1: Let $A = \{a \in \mathbb{Z} \mid a \neq 0 \}$ and $R = \{(a, b) \mid$ a and b have the same parity$\}$. Show that R over A is an equivalence notion.

Solution: The parity relation is an equivalence relation because it preserves the properties of *Reflexivity*, *Symmetry* and *Transitivity*.

1. *Reflexivity*: For any $x \in A$, x has the same parity as itself, so $(x, x) \in R$.
2. *Symmetry*: If $(x, y) \in R$, x and y have the same parity, so $(y, x) \in R$.
3. *Transitivity*: If $(x, y) \in R$, and $(y, z) \in R$, then x and z have the same parity as y, so they have the same parity as each other (if y is odd, both x and z are odd; if y is even, both x and z are even), thus $(x, z) \in R$.

2: A relation R is defined on the integers by $aRb\ if\ a^2 - b^2 \leq 3$. Show that R is reflexive, but not symmetric and not transitive.

Solution: It is reflexive since $a^2 - a^2 = 0 \leq 3$. Through counter-examples, it is not hard to see that: it is not symmetric since $0^2 - 10^2 \leq 3$ but $10^2 - 0^2 > 3$. Also, it is not transitive since $2^2 - 1^2 \leq 3$ and $1^2 - 0^2 \leq 3$ but $2^2 - 0^2 > 3$.

3: Is the intersection of two equivalence relations on the same set (let us say A) an equivalence relation?

Solution: This is true. Note that if R_1 and R_2 are equivalence relations on a set A, and if we let $R = R_1 \cap R_2$, then we have that xRy iff xR_1y and xR_2y. We now need to check the three properties:

Reflexivity: Since R_1 and R_2 are equivalence relations, we have xR_1x and xR_2x, for each $x \in A$. Therefore, xRx for each $x \in A$.

Symmetry: Let $x, y \in A$ be given so that xRy. Then xR_1y and xR_2y. Therefore, since R_1 and R_2 are equivalence relations, it follows that yR_1x and yR_2x. Therefore, yRx, and we have proved symmetry.

Transitivity: Let $x, y, z \in A$ be given so that xRy and yRz. It then follows that xR_1y, xR_2y, yR_1z and yR_2z. Since R_1 and R_2 are equivalence relations, it follows that xR_1z and xR_2z. Therefore, xRz, and transitivity has been proved.

Challenging/optional exercises to do at home/after class:

1. Let R be an equivalence relation on a non-empty set A. Then the equivalence classes $\{E_x \mid x \in A\}$ form a partition of A.

Proof

The proof is in four parts:

(1) We show that the equivalence classes $E_x = \{y \mid yRx\}$, $x \in A$, are non-empty subsets of A: by definition, each E_x is a subset of A. Since R is reflexive, xRx. Therefore $x \in E_x$ and so E_x is non-empty.

(2) We show that A is the union of the equivalence classes E_x, $x \in A$: We know that $E_x \subseteq A$, for all E_x, $x \in A$. Therefore the union of the equivalence classes is a subset of A. Conversely, suppose $x \in A$. Then $x \in E_x$. So, A is a subset of the union of the equivalence classes.

The purpose of the last two parts is to show that distinct equivalence classes are disjoint, satisfying the definition of partition.

(3) We show that if xRy then $E_x = E_y$: Suppose that xRy and let $z \in E_x$. Then, zRx and xRy. Since R is a transitive relation, zRy. Therefore, $z \in E_y$. We have shown that $E_x \subseteq E_y$. An analogous argument shows that $E_y \subseteq E_x$. So, $E_x = E_y$.

(4) We show that any two distinct equivalence classes are disjoint: To this end we show that if two equivalence classes are not disjoint then they are identical. Suppose $E_x \cap E_y \neq 0$. Take a $z \in E_x \cap E_y$. Then, zRx and zRy. Since R is symmetric, xRz and zRy. But then, by transitivity of R, xRy. Therefore, by (3), $E_x = E_y$.

2. Suppose that $A_1, \ldots, A_n$ is a partition of A. Define a relation R on A by setting: xRy if and only if there exists i such that $1 \leq i \leq n$ and $x, y \in A_i$. Then R is an equivalence relation.

Proof

Reflexivity: if $x \in A$, then $x \in A_i$ for some i. Therefore xRx.

Transitivity: if xRy and yRz, then there exists A_i and A_j such that $x, y \in A_i$ and $y, z \in A_j$. $y \in A_i \cap A_j$ implies $i = j$. Therefore $x, z \in A_i$ which implies xRz.

Symmetry: if xRy, then there exists A_i such that $x, y \in A_i$. Therefore yRx.