

Discrete Mathematics and Statistics (CPT 107)

Tutorial 4 bis

1. Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n .

Solution: Let R be a symmetric relation on set A . Proof by induction:

Basis Step: $R^1 = R$ is symmetric is True.

Inductive Step: Assume that R^n is symmetric.

To prove that R^{n+1} is symmetric.

R^{n+1} is symmetric if for all (x,y) in R^{n+1} , we have (y,x) is in R^{n+1} as well.

Assume that (x,y) is in R^{n+1} .

Now, $R^{n+1} = R^n \circ R = R \circ R^n$

We know that if $(x,y) \in R \circ R^n$, then by the definition of composition there exists a z in A such that xRz and $z(R^n)y$, i.e (x,z) is in R and (z,y) is in R^n

Also we know that R and R^n are symmetric, which implies that (z,x) is in R and also (y,z) is in R^n .

Therefore, by definition of composition,

$(y,x) \in R \circ R^n$; i.e.; $(y,x) \in R^{n+1}$.

Hence proved.

2. The relation R on a set A is transitive $\Leftrightarrow R^n \subseteq R$ for $n = 1, 2, 3, \dots$

Solution:

Proof. (\Leftarrow):

Suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \dots$ We want to show that R is transitive.

In particular, suppose that $R^2 \subseteq R$. Now if $(a, b) \in R$ and $(b, c) \in R$, then by definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

(\Rightarrow):

Suppose that R is transitive. We want to show that $R^n \subseteq R$ for $n = 1, 2, 3, \dots$

We use mathematical induction to prove this part.

Base Step: Obviously, $R^1 = R \subseteq R$.

Inductive Step: Assume that $R^k \subseteq R$, where $k \in \mathbb{Z}^+$. We want to show that $R^{k+1} \subseteq R$.

Assume $(a, b) \in R^{k+1}$. We want to show that $(a, b) \in R$.

Because $R^{k+1} = R^k \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^k$.

By inductive hypothesis, $R^k \subseteq R$. Therefore $(x, b) \in R$. So we have $(a, x) \in R$ and $(x, b) \in R$. Because R is transitive, it follows that $(a, b) \in R$. This is what we wanted to show: assuming $(a, b) \in R^{k+1}$, we showed that $(a, b) \in R$. Thus $R^{k+1} \subseteq R$, completing the proof.

3. Suppose A, B, and C are non empty sets with R a relation from A to B and S a relation from B to C. Prove that:
- $(R^{-1})^{-1} = R$
 - $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Solution:

Definition I . We say two sets A and B are equal, and write $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$. We apply Definition I to the sets involved.

To prove a) suppose $(a, b) \in R$. Then by definition $(b, a) \in R^{-1}$. The definition of $(R^{-1})^{-1}$ is $(R^{-1})^{-1} = \{(a, b)|(b, a) \in R^{-1}\}$. Thus we see $(a, b) \in (R^{-1})^{-1}$ and so R. Similar arguments show any $(a, b) \in (R^{-1})^{-1}$ is also in R. That is, $(R^{-1})^{-1} \subseteq R$, and part a) follows.

To prove b), suppose $(c, a) \in (S \circ R)^{-1}$. Then by definition of the inverse relation, $(a, c) \in (S \circ R)$. By the definition of the composite relation, there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. This of course means $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$. Applying again the definition of the composite relation, we find $(c, a) \in R^{-1} \circ S^{-1}$. Thus $(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$. Similar arguments show $R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}$ and the result follows.

4. Give examples in accordance with:

- $(R^{-1})^{-1} = R$
- $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Solution:

Suppose $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $C = \{7, 8, 9\}$, and

$$R = \{(1, 4), (1, 5), (2, 5)\}, S = \{(4, 7), (5, 7), (6, 9)\}$$

$$\text{Then } S \circ R = \{(1, 7), (2, 7)\},$$

$$(S \circ R)^{-1} = \{(7, 1), (7, 2)\}.$$

$$\text{Moreover, } S^{-1} = \{(7, 4), (7, 5), (9, 6)\} \text{ and } R^{-1} = \{(4, 1), (5, 1), (5, 2)\}.$$

$$\text{Thus } R^{-1} \circ S^{-1} = \{(7, 1), (7, 2)\} \text{ (in accordance with a) and b))}$$