Tutorial – Week 11 - SOLUTIONS

The phrase **at least n** *means* "n or more."

The phrase **at most n** *means* "n or fewer."

1. How many permutations of the letters **ABCDEFGH** contain the string **ABC**?

Solution:

Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC = X and the individual letters D, E, F, G, and H.

Because these six objects XDEFGH can occur in any order, there are **6! = 720** permutations of the letters ABCDEFGH in which ABC occurs as a block.

|**ABC**DEFGH|**ABC**DEFHG|**ABC**GDEFH|**ABC**GHDEF|**ABC**HDEFG|**ABC**HGDEF|DEF**ABC**GH|DEF**ABC**HG|DEFGABCH|DEFGH**ABC**|DEFHABCG|DEFHGABC|GABCDEFH|GABCHDEF|GDEFABCH|GDEFHABC|GHABCDEF|GHDEFABC|HABCDEFG|HABCGDEF|HDEFABCG|HDEFGABC|HGABCDEF|HGDEFABC|.....

2. Suppose that a saleswoman has to visit **eight different cities**.

She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes.

How many possible orders can the saleswoman use when visiting these cities?

Solution:

The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily.

Consequently, there are **7! = 7 x 6 x 5 x 4 x 3 x 2 x 1 = 5040** ways for the saleswoman to choose her tour.

If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths!

3. In how many ways can we select ***three students*** from a group of five students to stand in line for a picture?

In how many ways can we arrange ***all five of these students*** in a line for a picture?

Solution

a) First, note that the order in which we select the students matters.

There are five ways to select the first student to stand at the start of the line.

Once this student has been selected, there are four ways to select the second student in the line.

After the first and second students have been selected, there are three ways to select the third student in the line.

There are **5 x 4 x 3 = 60** ways to select three students from a group of five students to stand in line for a picture.

b) To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way.

Consequently, there are **5 x 4 x 3 x 2 x 1 = 120** ways to arrange all five students in a line for a picture.

4. Find the number **m** of *seven-letter words* that can be formed using the letters of the word "**BENZENE.**"

Solution

We seek the number of permutations of 7 objects of which 3 are alike (the three E's), and 2 are alike (the two N's).

By **P(7;3, 2)=7! / 3!2! = 7·6·5·4·3·2·1 / 3·2·1·2·1 = 420 = m**

5. Three cards are chosen one after the other from a 52-card deck.

Find the number *m* of ways this can be done:

(a) with replacement.

(b) without replacement.

Solution

(a) Each card can be chosen in 52 ways. Thus m = **52 x 52 x 52 =140 608**.

(b) Here there is no replacement.

Thus, the first card can be chosen in 52 ways,

the second in 51 ways, and

the third in 50 ways.

Therefore: **m = 52 x 51 x 50 = 132 600**

6. A group of 30 people have been trained as astronauts to go on the first mission to Mars.

How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution:

The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter.

The number of such combinations is

C(30, 6) = 30! / 6! 24! = 30 · 29 · 28 · 27 · 26 · 25/6 · 5 · 4 · 3 · 2 · 1 = 593 775.

7. What is the total number of ways in which 5 balls of different colors can be distributed among 3 people so that each person gets **at least** one ball?

Solution

Each person gets at least one ball.

The selection of balls can be done by:

Case1:

Person	I	II	III
No. of balls	1	1	3

OR

Case 2:

Person	I	II	III
No. of balls	1	2	2

The number of ways to distribute the balls in first case:

$= C(5,1) \times C(4,1) \times C(3,3) = 20$ ways

The number of ways to distribute the balls in second case:

$= C(5,1) \times C(4,2) \times C(2,2) = 30$ ways

The case 1 can occur in 3 ways: (1,1,3) (1,3,1) (3,1,1)

The case 2 can occur in 3 ways: (1,2,2) (2,2,1) (2,1,2)

The required numbers of ways $3 \times 20 + 3 \times 30 = 60 + 90 = 150$.

8. A coin is flipped 10 times where each flip comes up either heads or tails.

How many possible outcomes

a) are there in total?

b) contain **exactly** two heads?

c) contain **at most** three tails?

d) contain **the same number** of heads and tails?

Solution:

a) Each flip can be either heads or tails, so there are $\mathbf{2^{10} = 1024}$ possible outcomes.

b) To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads.

There are $\mathbf{C(10, 2) = 45}$ such outcomes.

c) To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part (b), we see that there are

$\mathbf{C(10, 3) + C(10, 2) + C(10, 1) + C(10, 0) = 120 + 45 + 10 + 1 = 176}$ such outcomes.

d) To have an equal number of heads and tails in this case means to have five heads.

Therefore, the answer is

$\mathbf{C(10, 5) = 252}$.

9.

a). How many distinguishable ways can the letters of the word **HULLABALOO** be arranged in order?

b). How many distinguishable orderings of the letters of **HULLABALOO** begin with **U** and end with L?

c). How many distinguishable orderings of the letters of **HULLABALOO** contain the two letters **HU** next to each other in order?

Solution

a. since there are 2 A's, 1 B, 1 H, 3 L's, 2 O's, and 1 U

$$10! / 2!1!1!3!2!1! = 151\,200$$

b. $$8! / 2!1!1!2!2! = 5\,040$$

c. $$9! / 1!2!1!3!2! = 15\,120$$

10. Suppose the group of **twelve** consists of **five** men and **seven** women.

> **a.** How many five-person teams can be chosen that consist of three men and two women?
>
> **b.** How many five-person teams contain at least one man?
>
> **c.** How many five-person teams contain at most one man?

Solution

a). Forming a team is a two-step process:

Step 1: Choose the men.

Step 2: Choose the women.

There are

> C(5, 3) ways to choose the **three men out of the five** and
>
> C(7, 2) ways to choose the **two women out of the seven**.

By the product rule, the number of teams of five that contain three men, and two women is

$$C(5, 3) \cdot C(7, 2) = (5! / 3!\, 2!) \cdot (7! / 2!\, 5!) = 210$$

b). We can use either the addition rule or by the subtraction rule.

Observe that the set of five-person teams containing at least one man equals the set difference between the set of all five-person teams and the set of five-person teams that do not contain any men.

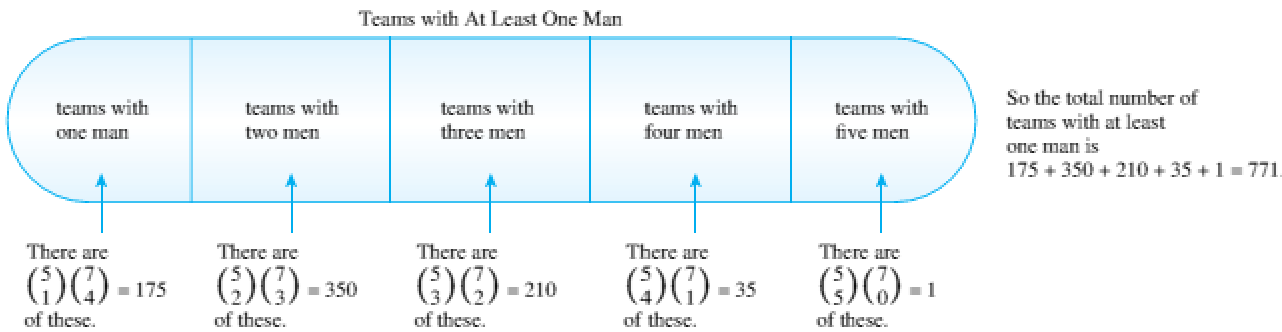
Now a team with no men consists entirely of *five women chosen from the seven women in the group*, so there are **C(7, 5)** such teams.

Also, the total number of five-person teams is **C(12, 5) = 792.**

Hence, by the difference rule,

[the number of teams with at least one man] = [the total number of teams of five] – [the number of teams of five that do not contain any men] =

$$= C(12, 5) - C(7, 5) = 792 - 21 = 771$$



$\binom{5}{1}\binom{7}{4}$ teams with one man and four women

$\binom{5}{2}\binom{7}{3}$ teams with two men and three women

$\binom{5}{3}\binom{7}{2}$ teams with three men and two women

$\binom{5}{4}\binom{7}{1}$ teams with four men and one woman

$\binom{5}{5}\binom{7}{0}$ teams with five men and no women.

c). The set of teams containing **at most one man** can be partitioned into the set without any men and the set with exactly one man.

By the addition rule,

[the number of teams with at most one man] = [the number of teams without any men] + [the number of teams with one man] =

$$= \mathbf{C(5, 0) \cdot C(7, 5) + C(5, 1) \cdot C(7, 4) = 21 + 175 = 196}$$