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Discrete Mathematics and Statistics - CPT107

FUNCTIONS

Contents

- Functions: definitions and examples
- Domain, codomain, and range / image
- Injective, surjective, and bijective functions
- Invertible functions
- Compositions of functions
- Pigeonhole principle (PHP)

Cartesian product of Sets

Let's assume A and B to be two non-empty sets, the sets of all ordered pairs (x, y) where $x \in A$ and $y \in B$ is called a **Cartesian product** of the sets.

$$A \times B = \{x, y \mid x \in A \text{ and } y \in B\}$$

Example: Find the cartesian products of set A ={1,2,3} and B={3,4,5}.

Solution:

Following the above definition, let Cartesian product be X,

$X = A \text{ x } B$

$= \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5)\}$

Relation

A relation from set A to set B is a <u>subset of the cartesian product</u> set A x B.

The subset is made up by describing a relationship between the first element and the second element of elements in A x B.

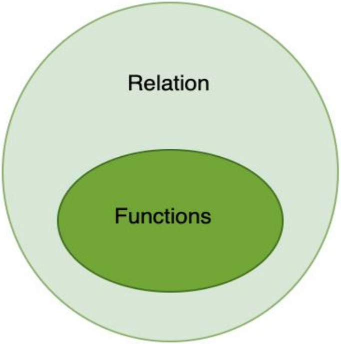
Example:

R = {(1,2), (2, -3), (3,5)}

- set of *all first elements* i.e {1, 2, 3} is called **Domain** while the set of *all second elements* i.e {2, -3, 5} is called the **Range of the relation**.

What is a function?

A function is a special kind of relation.



Why do we need functions in CS?

A function transforms an input value into an output value; that is, a function f takes an argument or parameter x and returns a value $f(x)$.

Functions

Definition(1)

A function from a set ***A*** to a set ***B*** is a binary relation ***R*** in which *every element* of ***A*** is associated with a *uniquely specified* element of ***B.***

In other words: for each a ∊ ***A*** there is precisely one pair of the form *(a, b)* in ***R.***

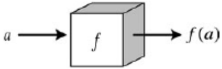
Definition (2)

Let **A** and **B** be sets.

A function ***f*** from **A** to **B**, written ***f : A → B***, is a rule that assigns to each input value a∈A *exactly one* output value b∈B; the unique value **b** assigned to **a** is denoted by ***f (a)***.

A function is also called a **mapping** or a **transformation**.

Functions



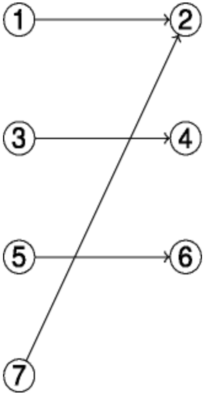


Figure: A function

Example

f (1) = 2
f (3) = 4
f (5) = 6
f (7) = 2

Figure: No functions

Domain, codomain, and range

Suppose $f: A \rightarrow B$.

- A is called the ***domain*** of f.
- B is called the ***codomain*** of f.
- The ***image* of *x* under *f*** is the value $f(x)$;
- if <u>R is any subset of A</u>, then $f(R) = \{f(r) | r \in R\}$ is **the set of images of elements of *R*** and is called **the *image of R***.
- The ***image*** *or* ***range*** ***f(A)* of *f*** is $f(A) = \{f(x) | x \in A\}$

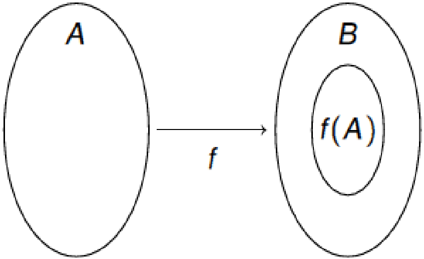
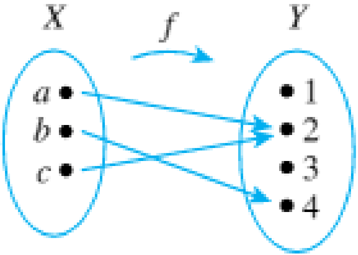


Figure: The image values of f is called **image** or **range** of f

Image of *f* = { b ∈ B: there exists a ∈ A such that f(a) = b}
f(A) ⊆ B – Image of f is a subset of B

image of *A* under *f* = range of *f*

Example



domain of f = {a, b, c},

co-domain of f = {1, 2, 3, 4}

range of f = {2, 4}

What Makes a Relation a Function?

All functions are relations, but not all relations are functions.

A relation in which an element is mapped to only range value is called a function.

To determine if a relation is a function, we just need to make sure that no element has two corresponding range values.

A relation that is not a function

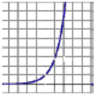
x	y
-3	7
-1	5
0	-2
5	9
5	3

x	y
-2	0
-1	-2
0	3
4	-1
5	-3

A relation that is a function

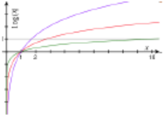
Example of functions

$$f(x) = e^x$$



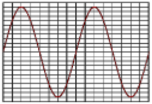
domain = R
codomain = $R^{>0}$

$$f(x) = \log(x)$$



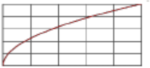
domain = $R^{>0}$
codomain = R

$$f(x) = \sin(x)$$



domain = R
codomain = [-1,1]

$$f(x) = \sqrt{x}$$



domain = $R^{>=0}$
codomain = $R^{>=0}$

Injective (one-to-one) functions

Definition. Let $f : A \to B$ be a function. We call f an ***injective*** (or **one-to-one**) function if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$

This is logically equivalent to $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ and so injective functions never repeat values.

In other words, different inputs give different outputs.

Examples

$f : Z \to Z$ given by $f(x) = x^2$ is not injective.

$h : Z \to Z$ given by $h(x) = 2x$ is injective.

Example

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for each real number x, then f is one-to-one.

Proof: Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]* By definition of f,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

[as was to be shown].

Surjective (or onto) functions

Definition. $f : A \to B$ is **surjective** (or onto) if the range of f coincides with the codomain of f.

This means that for every $b \in B$ there exists $a \in A$ with $b = f(a)$.

We call f ***bijective*** if f is **both injective and surjective**.

Examples

$f : Z \to Z$ given by $f(x) = x^2$ is not surjective.
$h : Z \to Z$ given by $h(x) = 2x$ is not surjective.
$h' : Q \to Q$ given by $h'(x) = 2x$ is surjective.

Example

If $f: \mathbf{R} \rightarrow \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for each real number x, then f is onto.

Proof: Let $y \in \mathbf{R}$. *[We must show that* $\exists x$ *in* $\mathbf{R}$ *such that* $f(x) = y$.*]* Let $x = (y + 1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$
\begin{aligned}
f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\
&= 4\cdot\left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\
&= (y+1) - 1 = y && \text{by basic algebra,}
\end{aligned}
$$

[as was to be shown].

Inverse functions

Since any function $f : A \rightarrow B$ is a relation, we can form the inverse relation f^{-1}.

If this inverse relation is itself a function, we say that f is an *invertible* function and write

$f^{-1} : B \rightarrow A$

for the *inverse* function.

If $f(a) = b$ then (a, b) is a pair in the function, so (b, a) is a pair in the inverse, so $f^{-1}(b) = a$.

Example

f : R —, R given by $f(x) = 4x + 3$ is invertible and

$$f^{-1}(y) = \frac{1}{4}(y - 3)$$

Invertibility and bijections

Theorem. A function f is invertible if and only if it is a **bijection**.

Hint: When you want to prove that something is unique, the standard technique is to assume that there are two different such things and then obtain a contradiction.

Suppose f has two inverse functions, f_1^{-1} and f_2^{-1} (existence of either means that f is a bijection).

Example

Let A = $\{x \mid x \in \mathbb{R}, x \neq 1\}$ and $f: A \rightarrow A$ be given by

$$f(x) = \frac{x}{x-1}$$

Show that f is bijective and determine the inverse function.

Composition of functions

If $f : A \rightarrow$ B and g : $B \rightarrow$ C are functions, then the composition relation $g\ o\ f$ between A and C consists of all pairs (a, c) where,

for some $b \in B$, (a, $b) \in f$ and $(b, c) \in$ g, g o f is a function as well.

It is given by

$$(g\ o\ f)(x) = g(f(x)).$$

Example

Consider the function $f: \mathrm{R} \rightarrow \mathrm{R}$ given by $f(x) = x^2$, and

the function $g : \mathrm{R} \rightarrow \mathrm{R}$ given by $g(x) = 4x + 3$.

Calculate $g \circ f$, $f \circ g$, $f \circ f$ and $g \circ g$.

Definition

Let $f : A \to B$.

A function $g : B \to A$ is the inverse of f if

$$f \circ g = 1_B \text{ and } g \circ f = 1_A.$$

identity on B

identity on A

The pigeonhole principle PHP

Let $f : A \to \mathrm{B}$ be a function where A and B are finite sets.

Suppose that A contains n elements labelled a_1 ,..., a_n.

The *pigeonhole principle* states that if $|\mathrm{A}|>|B|$ then at least one value of f occurs more than once.

In other words, we have $\mathrm{f(a_i) = f(a_j)}$ for some i and j where $i \neq j$.

So, there can be **no one-to-one function** $f : A \to \mathrm{B}$.

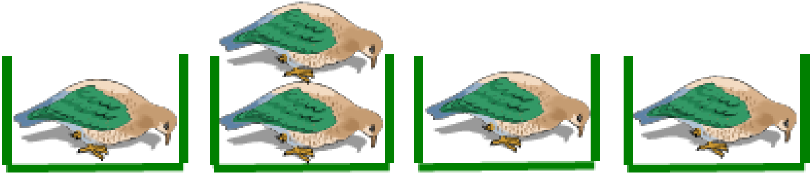
The pigeonhole principle

If more pigeons

than pigeonholes,

The pigeonhole principle

then some hole must have at least two pigeons!



Pigeonhole principle

A function from a larger set to a smaller set cannot be injective.

(There must be at least two elements in the domain that are mapped to the same element in the codomain.)

Extended pigeonhole principle

Consider a function $f: A \to B$ where A and B are finite sets and $|A| > k|B|$ for some natural number k.

Then, there is a value of f which occurs at least $k + 1$ times.

Example

Question: Let $A = \{1,2,3,4,5,6,7,8\}$

If five distinct integers are selected from A, must a pair of integers have a sum of 9?

Consider the pairs $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$. The sum of each pair is equal to 9.

If we choose 5 numbers from the set A, then by the pigeonhole principle, both elements of some pair will be chosen, and their sum is equal to 9.

Example

Question: There are 10 kids in the family. Prove that at least two of the kids were born on the same day of the week

Solution: There are 10 kids (pigeons) which we are placing into 7 days of the week (pigeonholes), so by the PHP, some day of the week has two kids.

Example

Question: XJTLU CSSE has 400 students. Show that at least two of them were born on the same day of the year.

Solution: There are 400 students, and only 366 days they could have been born on, so by the PHP some two students were born on the same day.

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End of lecture

- **Summary**
  - Functions: definitions and examples
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  - Injective, surjective, and bijective functions
  - Invertible functions
  - Compositions of functions
  - Pigeonhole principle (PHP)

- **Reading**
  - Discrete Mathematics for Computing R. Haggarty, Chapter 5