Discrete Mathematics and Statistics - CPT107

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Part 2. Set Theory

Reading: Discrete Mathematics for Computing R. Haggarty, Chapter 3.

Contents

Notation

A *set* is a collection of objects, called the *elements* of the set. For example:

- $\{7, 5, 3\}$;
- $\{\text{Liverpool}, \text{Manchester}, \text{Leeds}\}$.

We have written down the elements of each set and contained them between the *braces* $\{\ \}$.

We write $a \in S$ to denote that the object a is an element of the set S:

$$7 \in \{7, 5, 3\}, \quad 4 \notin \{7, 5, 3\}.$$

Notation

For a large set, especially an infinite set, we cannot write down all the elements. We use a predicate P instead.

$$S = \{x \mid P(x)\}$$

denotes the set of objects x for which the predicate $P(x)$ is true.

Examples: Let $S = \{1, 3, 5, 7, \ldots\}$. Then

$$S = \{x \mid \ x \text{ is an odd positive integer}\}$$

and

$$S = \{2n - 1 \mid \ n \text{ is a positive integer }\}.$$

More examples

Find simpler descriptions of the following sets by listing their elements:

- $A = \{x \mid\ x \text{ is an integer and } x^2 + 4x = 12\}$;
- $B = \{x \mid\ x \text{ a day of the week not containing ``}u\text{''}\}$;
- $C = \{n^2 \mid\ n \text{ is an integer}\}$.

Important sets

The empty set has no elements. It is written as $\emptyset$ or as $\{\}$.

We have seen some other examples of sets in Part 1.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (the natural numbers)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers)
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (the positive integers)
- $\mathbb{Q} = \{x/y \mid x \in \mathbb{Z}, y \in \mathbb{Z}, y \neq 0\}$ (the rationals)
- $\mathbb{R}$: (real numbers)

Subsets

Definition A set B is called a *subset* of a set A if every element of B is an element of A. This is denoted by $B \subseteq A$.

Examples:

$$\{3,4,5\} \subseteq \{1,5,4,2,1,3\},\ \{3,3,5\} \subseteq \{3,5\},\ \{5,3\} \subseteq \{3,5\}.$$

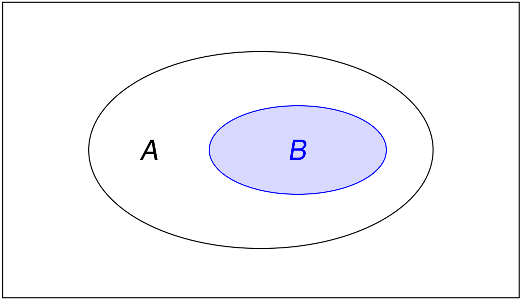

Figure: Venn diagram of $B \subseteq A$.

Equality

Definition A set A is called *equal* to a set B if $A \subseteq B$ and $B \subseteq A$. This is denoted by $A = B$.

Examples:

$$\{1\} = \{1, 1, 1\},$$

$$\{1, 2\} = \{2, 1\},$$

$$\{5, 4, 4, 3, 5\} = \{3, 4, 5\}.$$

Data types

Modern computing languages require that variables are declared as belonging to a particular *data type*; a data type consists of

- a set of objects;
- a list of standard operations on those objects.

For example, the set of integers with the operations $+$ and – form a data type.

Specifying the type of a variable is equivalent to specifying the set from which the variable is drawn.

The union of two sets

Definition The union of two sets A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

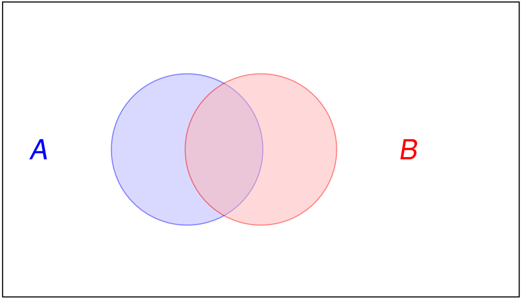

Figure: Venn diagram of $A \cup B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A \cup B = \{4, 7, 8, 9, 10\}.$$

The intersection of two sets

Definition The intersection of two sets A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

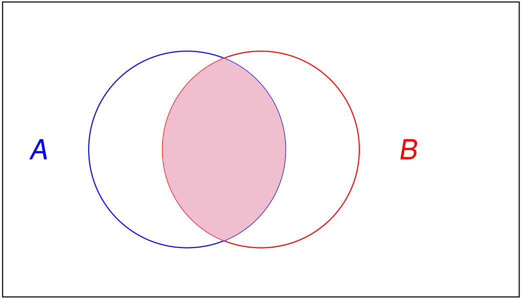

Figure: Venn diagram of $A \cap B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A \cap B = \{4\}$$

The relative complement

Definition The relative complement of a set B relative to a set A is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

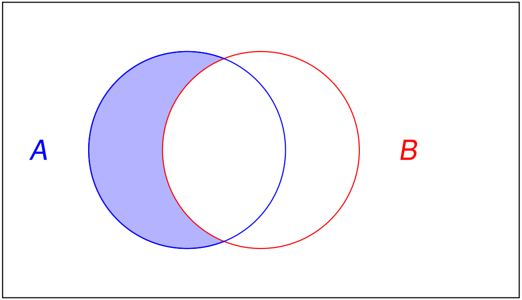

Figure: Venn diagram of $A - B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A - B = \{7, 8\}$$

The complement

When we are dealing with subsets of some large set U, then we call U the *universal set* for the problem in question.

Definition The complement of a set A is the set

$$\sim A = \{x \mid x \notin A\} = U - A.$$

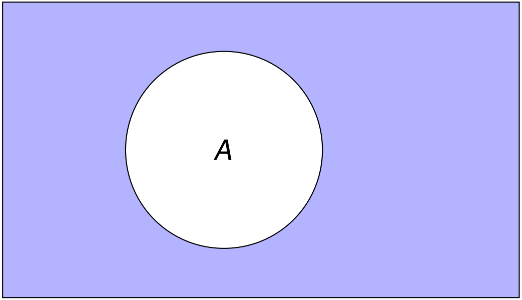

Figure: Venn diagram of $\sim A$.

The symmetric difference

Definition The symmetric difference of two sets A and B is the set

$$A \Delta B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}.$$

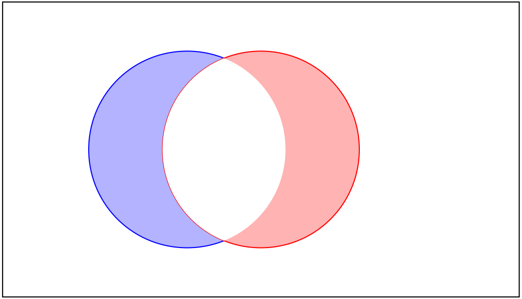

Figure: Venn diagram of $A \Delta B$.

Example

Suppose

$$A = \{4, 7, 8\}$$

and

$$B = \{4, 9, 10\}.$$

Then

$$A \Delta B = \{7, 8, 9, 10\}$$

The algebra of sets

Suppose that A, B and U are sets with $A \subseteq U$ and $B \subseteq U$.

Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A;$$

The algebra of sets

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$.

Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C;$$

The algebra of sets

Suppose that A and U are sets with $A \subseteq U$.

Identity laws:

$$A \cup \emptyset = A, \; A \cup U = U, \; A \cap U = A, \; A \cap \emptyset = \emptyset;$$

The algebra of sets

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, and $C \subseteq U$.

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \; A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

The algebra of sets

Suppose that A and U are sets with $A \subseteq U$. Let $\sim A = U - A$. Then

Complement laws:

$$A \cup \sim A = U, \ \sim U = \emptyset, \ \sim(\sim A) = A, A \cap \sim A = \emptyset, \ \sim \emptyset = U;$$

The algebra of sets

Suppose that A, B and U are sets with $A \subseteq U$, and $B \subseteq U$. Recall that $\sim X = U - X$. Then

De Morgan's laws:

$$\sim (A \cup B) = \sim A \cap \sim B, \ \sim (A \cap B) = \sim A \cup \sim B.$$

Reflection

The following statements hold:

- $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \emptyset$;
- $\emptyset \subseteq \{5\}$;
- $\{2\} \nsubseteq \{\{2\}\}$ but $\{2\} \in \{\{2\}\}$;
- $\{3, \{3\}\} \neq \{3\}$.

The Power Set

Definition The power set $Pow(A)$ of a set A is the set of all subsets of A. In other words,

$$Pow(A) = \{C \mid C \subseteq A\}.$$

Example:

Let $A = \{1, 2, 3\}$. Then

$$Pow(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}.$$

The Power Set

Observation For all sets A and B:
$Pow(A \cap B) = Pow(A) \cap Pow(B)$.

Proof We have to show for any set C that $C \in Pow(A \cap B)$ if and only if $C \in Pow(A)$ and $C \in Pow(B)$. But

$$\begin{aligned} C \in Pow(A \cap B) &\Leftrightarrow C \subseteq A \cap B \\ &\Leftrightarrow C \subseteq A \text{ and } C \subseteq B \\ &\Leftrightarrow C \in Pow(A) \text{ and } C \in Pow(B). \end{aligned}$$

The Power Set

Observation There exist sets A and B such that $Pow(A \cup B) \neq Pow(A) \cup Pow(B)$.

Using the algebra of sets

Prove that $A \Delta B = (A \cup B) \cap \sim (A \cap B)$. (See the next slide.)

$$
\begin{aligned}
(A \cup B) \cap \sim (A \cap B) &= (A \cup B) \cap (\sim A \cup \sim B) \text{ De Morgan} \\
&= ((A \cup B) \cap \sim A) \cup ((A \cup B) \cap \sim B) \text{ distributive} \\
&= (\sim A \cap (A \cup B)) \cup (\sim B \cap (A \cup B)) \text{ commutative} \\
&= ((\sim A \cap A) \cup (\sim A \cap B)) \cup ((\sim B \cap A) \cup (\sim B \cap B)) \text{ distributive} \\
&= ((A \cap \sim A) \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup (B \cap \sim B)) \text{ commutative} \\
&= (\emptyset \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup \emptyset) \text{ complement} \\
&= (A \cap \sim B) \cup (B \cap \sim A) \text{ commutative and identity} \\
&= A \Delta B \text{ definition}
\end{aligned}
$$

Cardinality of sets

Definition The cardinality of a *finite* set S is the number of elements in S, and is denoted by $|S|$.

Computing the cardinality of a union of two sets

If A and B are sets then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example

Suppose there are 100 third-year students. 40 of them take the module “Sequential Algorithms” and 80 of them take the module “Multi-Agent Systems”. 25 of them took both modules. How many students took neither modules?

Computing the cardinality of a union of three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

These are special cases of the principle of inclusion and exclusion which we will study later.

Ordered pairs

In discussing sets, the order in which elements are listed is unimportant. In order to handle ordered lists of objects we first introduce:

Definition The cartesian product $A \times B$ of sets A and B is the *set* consisting of all pairs (a, b) with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Example

- $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

Example

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

Cartesian plane

The set $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$ as it is often written, consists of all pairs of real numbers (x, y). $\mathbb{R}^2$ is called the *Cartesian plane*.

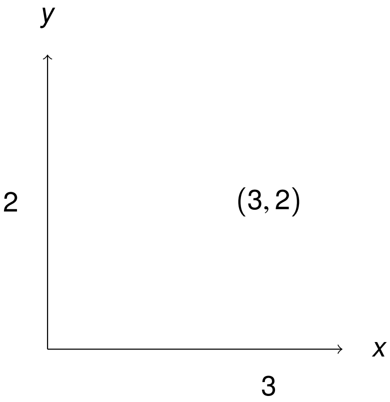

Figure: The Cartesian plane

Definition

The cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of sets $A_1, A_2, \ldots, A_n$ is defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Here $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

If $A_1, \ldots, A_n$ are all the same set A, then we write A^n to refer to the set

$$A_1 \times \cdots \times A_n.$$

Bit strings of length n

Let $B = \{0, 1\}$. B^n consists of all lists of zeros and ones of length n. These are called *bit strings* of length n.

Bit strings can be used to represent the subsets of a set.

Suppose we have a set $S = \{s_1, \ldots, s_n\}$.

(I have given the n elements of the set names. s_1 is the name of the first element. s_2 is the name of the second element (which is different from the first element) and so on.)

If we have a subset $A \subseteq S$, the characteristic vector of A is the n-bit string $(b_1, \ldots, b_n) \in B^n$ where

$$b_i = \begin{cases} 1 & \text{if} \quad s_i \in A \\ 0 & \text{if} \quad s_i \notin A \end{cases}$$

Example

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- The characteristic vector of A is $(1, 0, 1, 0, 1)$.
- The characteristic vector of B is $(0, 0, 1, 1, 0)$.
- The characteristic vector of $A \cap B$ is $(0, 0, 1, 0, 0)$.
- The characteristic vector of $A \cup B$ is $(1, 0, 1, 1, 1)$.