Discrete Mathematics and Statistics

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Reading: Discrete Mathematics for Computing R. Haggarty, Chapter 5.

Contents

Inverse Relation

Definition Given a relation $R \subseteq A \times B$, we define the *inverse relation* $R^{-1} \subseteq B \times A$ by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Example: The inverse of the relation *is a parent of* on the set of people is the relation *is a child of*.

Composition of Relations

Definition Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The composition of R and S, denoted by $S \circ R$, is the binary relation between A and C given by

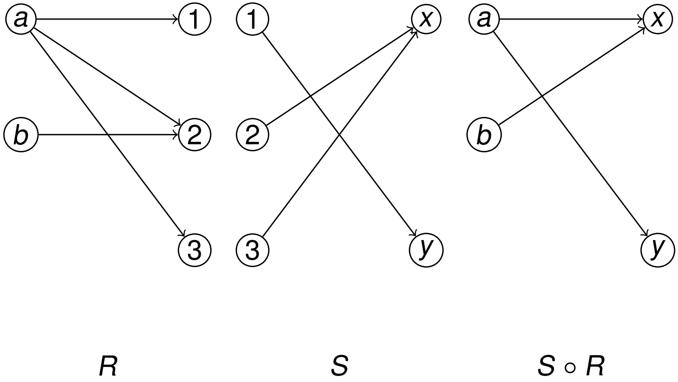
$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Example: If R is the relation *is a sister of* and S is the relation *is a parent of*, then

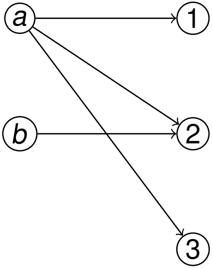
$S \circ R$ is the relation *is an aunt of*;

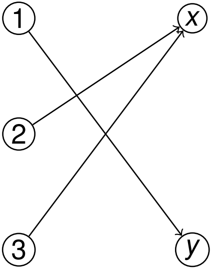
$S \circ S$ is the relation *is a grandparent of*.

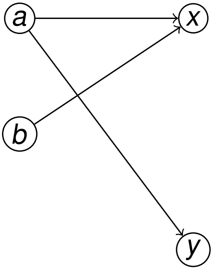
Digraph representation of compositions



Now let's go back and see how this works for matrices representing relations







$$R : \begin{bmatrix} T & T & T \\ F & T & F \end{bmatrix} \qquad S : \begin{bmatrix} F & T \\ T & F \\ T & F \end{bmatrix} \qquad S \circ R : \begin{bmatrix} T & T \\ T & F \end{bmatrix}$$

The formal description

Given two matrices with entries "T" and "F" representing the relations we can form the matrix representing the composition. This is called the *logical* (*Boolean*) *matrix product*.

Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_p\}$.

The logical matrix M representing R is given by:

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

The logical matrix N representing S is given by

$$N(i,j) = \begin{cases} T & \text{if} \quad (b_i, c_j) \in S \\ F & \text{if} \quad (b_i, c_j) \notin S \end{cases}$$

Matrix representation of compositions

Then the entries $P(i,j)$ of the logical matrix P representing $S \circ R$ are given by

> $P(i,j) = T$ if there exists l with $1 \leq l \leq m$ such that $M(i,l) = T$ and $N(l,j) = T$.
>
> $P(i,j) = F$, otherwise.

We write $P = MN$.

The example from before

Let R be the relation between $A = \{a, b\}$ and $B = \{1, 2, 3\}$ represented by the matrix

$$M = \left[\begin{array}{ccc} T & T & T \\ F & T & F \end{array} \right]$$

Similarly, let S be the relation between B and $C = \{x, y\}$ represented by the matrix

$$N = \left[\begin{array}{cc} F & T \\ T & F \\ T & F \end{array} \right]$$

Example

Then the matrix $P = MN$ representing $S \circ R$ is

$$P = \begin{bmatrix} T & T \\ T & F \end{bmatrix}$$

Another example

The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is represented by the matrix

$$\begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix}$$

Determine the matrix $R \circ R$ and hence explain why R is not transitive.

$$\begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix} \begin{bmatrix} T & F & F & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ T & F & T & F & F \\ F & T & F & T & F \end{bmatrix} = \begin{bmatrix} T & F & \color{red}{T} & T & F \\ F & T & F & T & T \\ F & F & T & F & F \\ T & F & T & T & F \\ T & T & T & F & T \end{bmatrix}$$

$$R \circ R = \{(a, c) \mid \text{ exists } b \in A \text{ such that } aRb \text{ and } bRc\}.$$

Note (in red) that there are pairs (a, c) that are in $R \circ R$ but not in R. Hence, R is not transitive.

Transitivity and Composition

We saw earlier that a relation S is transitive if and only if $S \circ S \subseteq S$.

This is because

$$S \circ S = \{(a, c) \mid \text{ exists } b \text{ such that } aSb \text{ and } bSc\}.$$

Let S be a relation. Set $S^1 = S$, $S^2 = S \circ S$, $S^3 = S \circ S \circ S$, and so on.