Xi'an Jiaotong-Liverpool University
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CPT205 Computer Graphics

Mathematics for Computer Graphics

Lecture 02
2024-25

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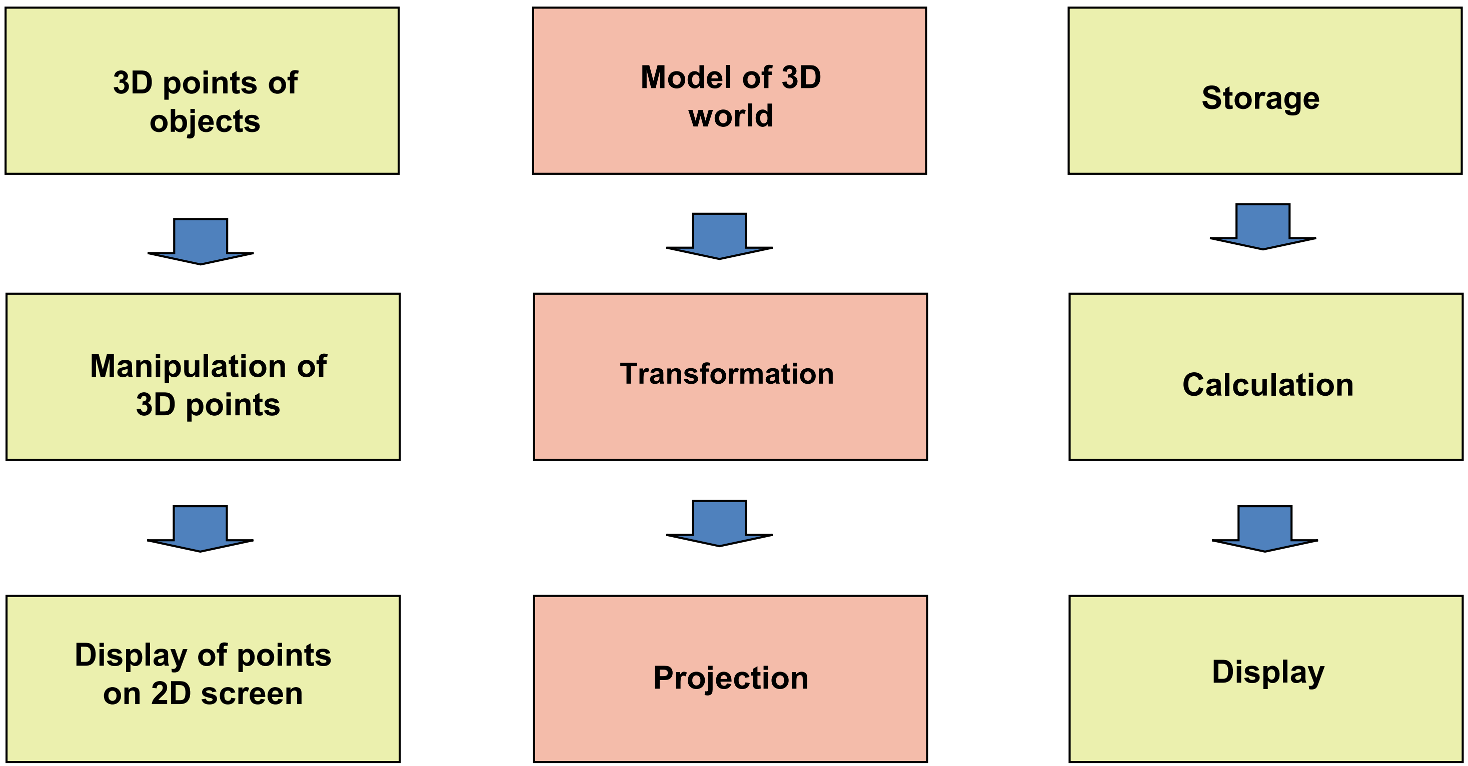
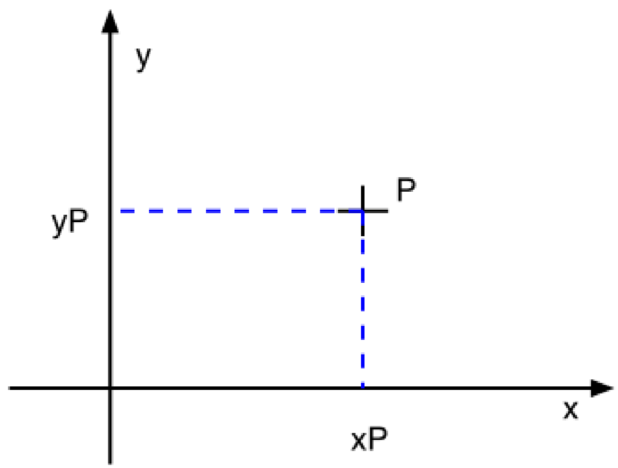
Topics for today

- Computer representation of objects
- Cartesian co-ordinate system
- Points, lines and angles
- Trigonometry
- Vectors (unit vector) and vector calculations (addition, subtraction, scaling, dot product and cross product)
- Matrices (dimension, transpose, square/symmetric/identity and inverse) and matrix calculations (addition, subtraction and multiplication)

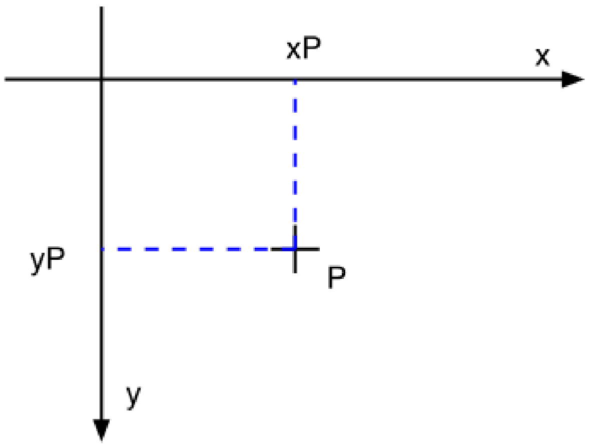
Computer representation of objects



Cartesian co-ordinate system

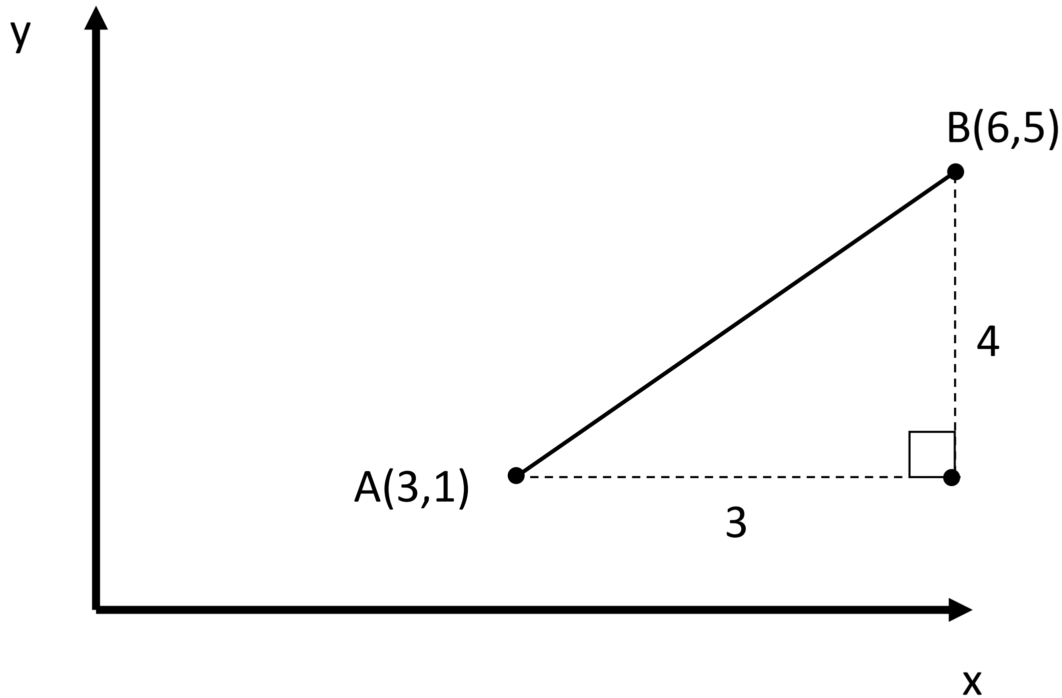


Representation of point
$P(x_p, y_p)$ in Cartesian co-ordinates



Representation of point
$P(x_p, y_p)$ on computer screen

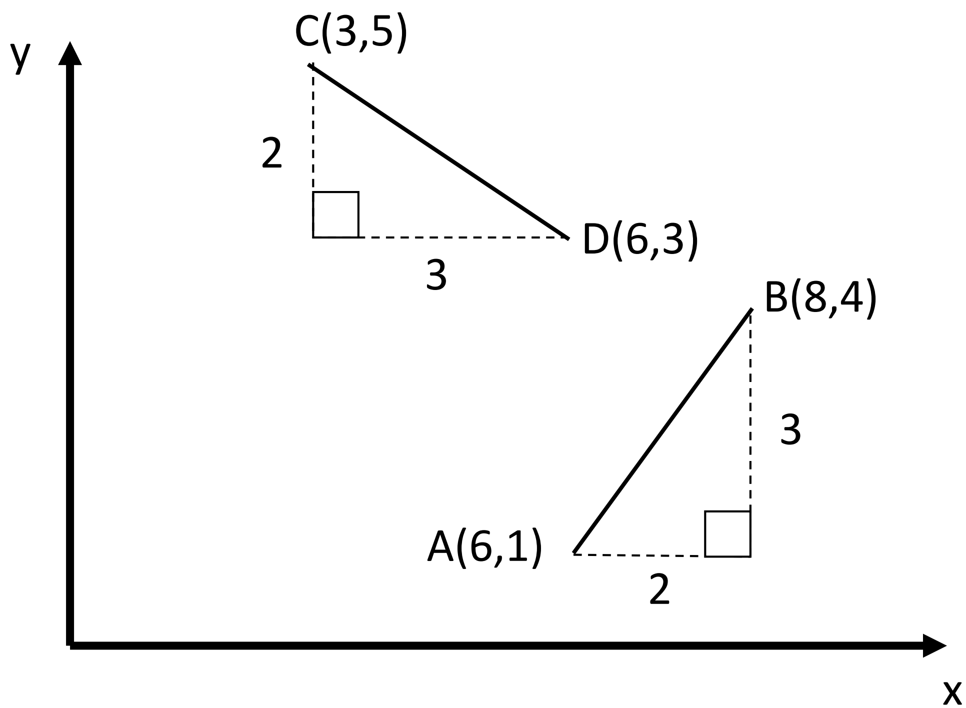
Straight line



Using Pythagoras theorem:

$AB = \text{sqrt}\ (4^2 + 3^2) = 5$

Gradient of a line



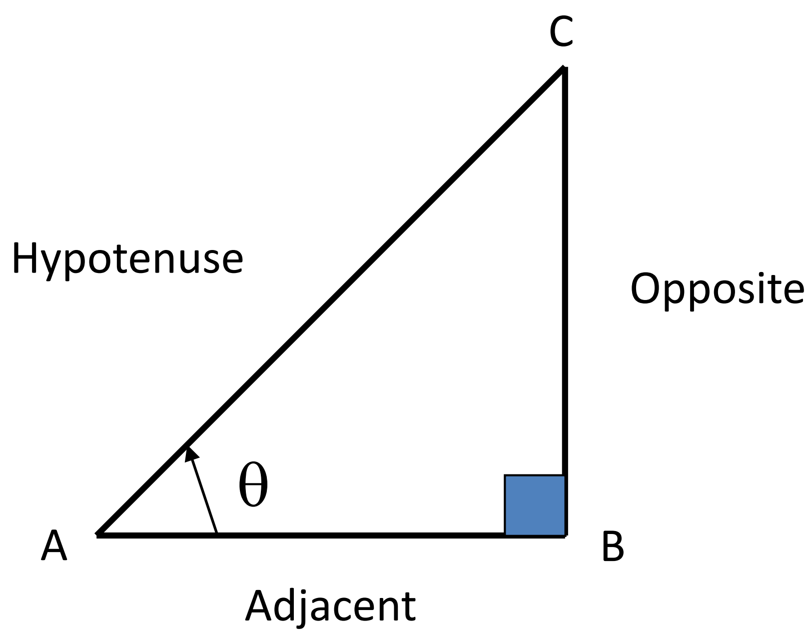
Gradient AB = $\Delta y/\Delta x$ = (4-1) / (8-6) = 3/2
Gradient CD = $\Delta y/\Delta x$ = (3-5) / (6-3) = -(2/3)

An uphill line (direction is ‘bottom left to top right’) has a **positive** gradient.
A downhill line (direction is ‘top left to bottom right’) has a **negative** gradient.

Perpendicular lines

- Given that the gradient of AB=3/2 and gradient of CD=-2/3, when the two gradients are multiplied together we have: (3/2) * (-2/3) = -1.
- Thus we, conclude that lines AB and CD are perpendicular.
- Prove this using graph paper.
- What can you say about lines with same gradient?

Angles and trigonometry



sine (θ) = Opposite / Hypotenuse = BC / AC

cosine (θ) = Adjacent / Hypotenuse = AB / AC

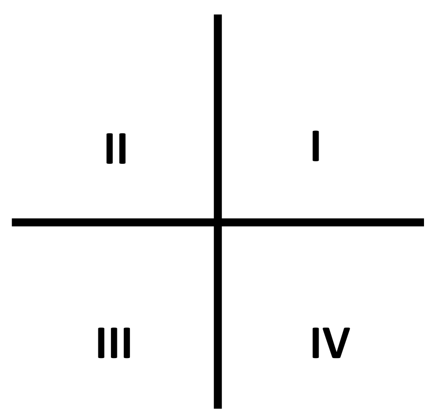
tangent (θ) = Opposite / Adjacent = BC / AB

Angles and trigonometry

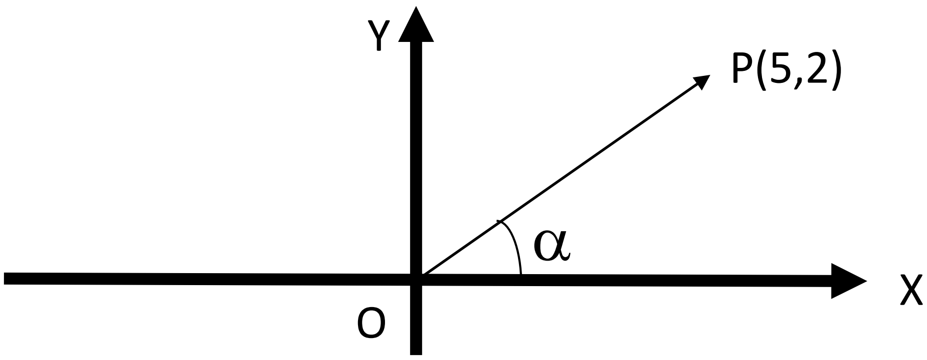
A complete revolution gives 360° or 2π (rad).

The following diagram is used to find the values of the trigonometric ratios:

- All trigonometric ratios of angles in quadrant 1 have positive ratios.
- Only sine of angles in quadrant 2 have positive ratios.
- Only tangent of angles in quadrant 3 have positive ratios.
- Only cosine of angles in quadrant 4 have positive ratios.



Vectors



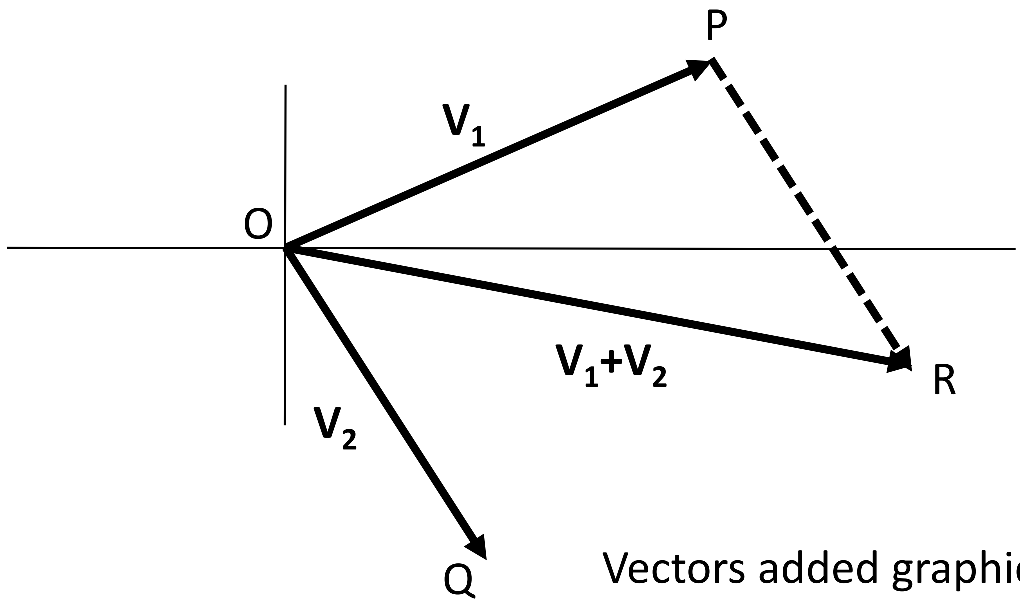
- **OP** = x**i** + y**j**
  where **i** and **j** are **unit vectors** along the x- and y-axes, respectively.

- The magnitude or modulus of **OP** = 5**i** + 2**j** is
  |**OP**| = sqrt(5^2 + 2^2) = 5.39

- Unit vector of OP is
  (OP) = **OP** / |**OP**| = (5**i** + 2**j**) / 5.39 = 0.93**i** + 0.37**j**

- sin(α) = 2 / |**OP**| = 2/5.39 = 0.37
  cos(α) = 5 / |**OP**| = 5/5.39 = 0.93

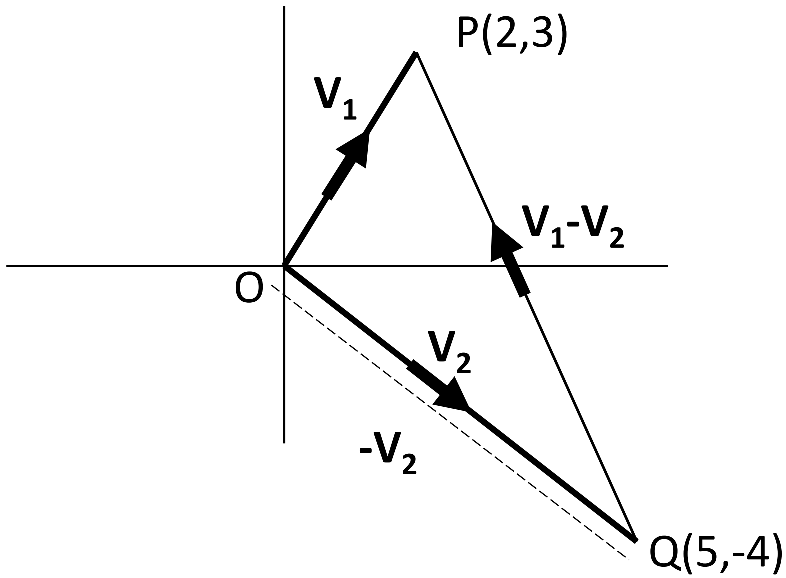
Vector addition



Vectors added graphically

- For two vectors **OP** and **OQ** such as
  OP = $\mathbf{V_1}$ = 5**i** + 2**j** and **OQ** = $\mathbf{V_2}$ = 2**i** - 4**j**
- The vector addition is the sum of vectors **OP** and **OQ**
  $\mathbf{V_1} + \mathbf{V_2} = (5\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} - 4\mathbf{j}) = 7\mathbf{i} - 2\mathbf{j}$
- The direction of $\mathbf{V_1} + \mathbf{V_2}$ with respect to the x-axis is
  $\cos(\alpha) = 7 / |\mathbf{V_1} + \mathbf{V_2}| = 7 / \mathrm{sqrt}(7^2 + (-2)^2) = 0.962$

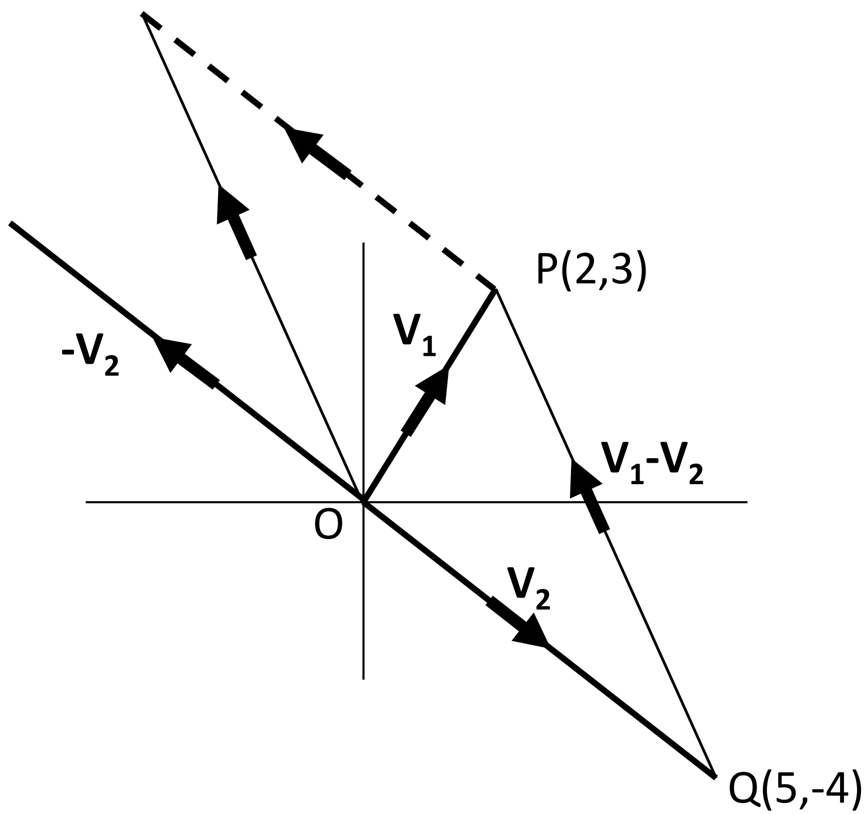
Vector subtraction



Vectors subtracted graphically

- For two vectors **OP** and **OQ** such as **OP** = $\mathbf{V_1}$ = 2**i** + 3**j** and **OQ** = $\mathbf{V_2}$ = 5**i** - 4**j**
- $\mathbf{V_1} - \mathbf{V_2}$ = (2**i** + 3**j**) - (5**i** - 4**j**) = -3**i** + 7**j**
- The direction of $\mathbf{V_1}$ - $\mathbf{V_2}$ with respect to the x-axis is $\cos(\alpha) = -3 / |\mathbf{V_1} + \mathbf{V_2}| = -3 / \mathrm{sqrt}((-3)^2 + 7^2) = -0.394$

Vector subtraction



Vectors subtracted graphically

Vector scaling

A vector may be scaled up or down by multiplying it with a scalar number. Assume the following vector

$$\mathbf{V} = 4\mathbf{i} + 3\mathbf{j}$$

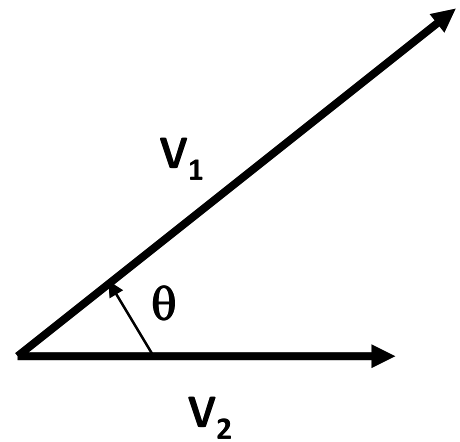
multiplying by 3, we have

$$3*\mathbf{V} = 3*(4\mathbf{i} + 3\mathbf{j}) = 12\mathbf{i}+9\mathbf{j}$$

multiplying by 1/2, we have

$$(1/2)*\mathbf{V} = (1/2)*(4\mathbf{i} + 3\mathbf{j}) = 2\mathbf{i} + 1.5\mathbf{j}$$

Dot product of two vectors



Given vectors $\mathbf{V_1}$ and $\mathbf{V_2}$, their dot product is a scalar.

$\mathbf{V_1} \bullet \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\cos(\alpha)$ where $0 \leq \alpha \leq 180^o$

$\cos(\alpha) = \mathbf{V_1} \bullet \mathbf{V_2} / (|\mathbf{V_1}||\mathbf{V_2}|)$

Dot product of two vectors

- The product $\mathbf{V_1} \bullet \mathbf{V_2}$ for $\mathbf{V_1} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{V_2} = x_2\mathbf{i} + y_2\mathbf{j}$ is

$$\begin{aligned}\mathbf{V_1} \bullet \mathbf{V_2} &= (x_1\mathbf{i})*(x_2\mathbf{i} + y_2\mathbf{j}) + (y_1\mathbf{j})*(x_2\mathbf{i} + y_2\mathbf{j}) \\ &= (x_1*x_2)*\mathbf{i}*\mathbf{i} + (y_1*y_2)*\mathbf{j}*\mathbf{j} + (x_1*y_2)*\mathbf{i}*\mathbf{j} + (y_1*x_2)*\mathbf{j}*\mathbf{i}\end{aligned}$$

- Because $\mathbf{i}*\mathbf{i} = \mathbf{j}*\mathbf{j} = 1$ and $\mathbf{i}*\mathbf{j} = \mathbf{j}*\mathbf{i} = 0$, therefore

$$\mathbf{V_1} \bullet \mathbf{V_2} = x_1*x_2 + y_1*y_2$$

- The dot product is also expressed as

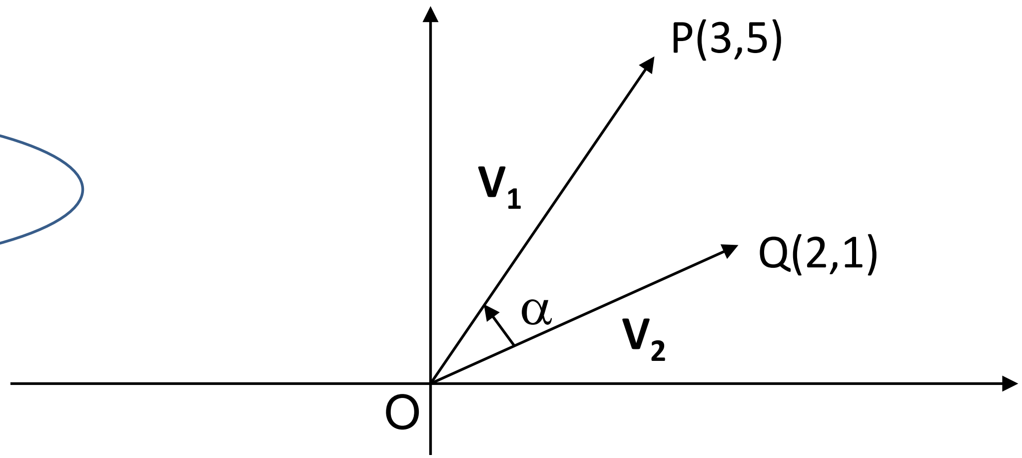
$$\mathbf{V_1} \bullet \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\cos(\alpha)$$

therefore $\cos(\alpha) = \mathbf{V_1} \bullet \mathbf{V_2} / (|\mathbf{V_1}||\mathbf{V_2}|)$

$= (x_1*x_2 + y_1*y_2) / (|\mathbf{V_1}||\mathbf{V_2}|)$

Example use of dot product

Find angle α?



$\mathbf{V_1} = 3\mathbf{i} + 5\mathbf{j}$

$\mathbf{V_2} = 2\mathbf{i} + \mathbf{j}$

$\mathbf{V_1} \bullet \mathbf{V_2} = 3*2 + 5*1 = 11$

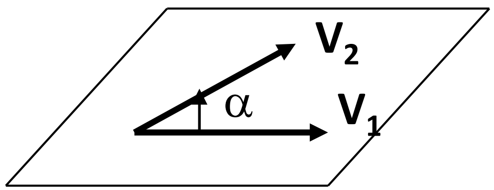
$|\mathbf{V_1}| = \mathrm{sqrt}(3^2 + 5^2) = \mathrm{sqrt}(34) = 5.831$

$|\mathbf{V_2}| = \mathrm{sqrt}(2^2 + 1) = \mathrm{sqrt}(5) = 2.236$

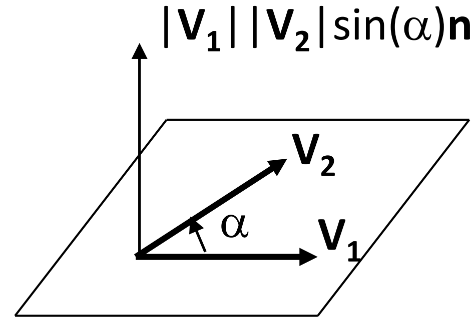
$\cos(\alpha) = \mathbf{V_1} \bullet \mathbf{V_2} / (|\mathbf{V_1}||\mathbf{V_2}|) = 11 / (5.831*2.236) = 0.8437$

$\alpha = 32.47^o$

Cross product of two vectors



(i)



(ii)

For two vectors $\mathbf{V_1}$ and $\mathbf{V_2}$ lying on a plane (Figure i), their cross product is another vector, which is perpendicular to the plane (Figure ii).

The cross product is defined as

$$\mathbf{V_1} \times \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\sin(\alpha)\mathbf{n}$$

where $0 <= \alpha <= 180$ and **n** is a unit vector along the direction of the plane normal obeying the right-hand rule.

Cross product of two vectors

- $\mathbf{V_1} \times \mathbf{V_2} = -\mathbf{V_2} \times \mathbf{V_1}$
- $\mathbf{V_1} \times \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\sin(\alpha)\mathbf{n}$, thus $|\mathbf{V_1} \times \mathbf{V_2}| = |\mathbf{V_1}||\mathbf{V_2}|\sin(\alpha)$
- When $\alpha = 0$, $\sin(\alpha) = 0$. Hence $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$
  where **i**, **j** and **k** are unit vectors along the x, y and z axes, respectively.
- When $\alpha = 90$, $\sin(\alpha) = 1$. Hence $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$ and $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
- From the identity above, the reverse is true,
  i.e. $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$, $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$ and $\mathbf{i}\times\mathbf{k} = -\mathbf{j}$

Matrices

- Matrices are techniques for applying transformations.
- A matrix is simply a set of numbers arranged in a rectangular format.
- Each number is known as an element.
- Capital letters are used to represent matrices, bold letters when printed **M**, or underlined when written $\underline{M}$.
- A matrix has dimensions that refer to the number of rows and the number of columns it has.

Dimensions of matrices

$$\mathbf{A} = \begin{array}{cc} \begin{array}{ccc}\text{Col 1} & \text{Col 2} & \text{Col 3}\end{array} & \\ \begin{bmatrix} 1 & 6 & 3 \\ -1 & 2 & 4 \end{bmatrix} & \begin{array}{l}\text{Row 1} \\ \text{Row 2}\end{array} \end{array}$$

The dimensions of A are (2×3)

$$\mathbf{B} = \begin{array}{cc} \begin{array}{ccc}\text{Col 1} & \text{Col 2} & \text{Col 3}\end{array} & \\ \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & -4 & 6 \\ 0 & 2 & 1 \end{bmatrix} & \begin{array}{l}\text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4}\end{array} \end{array}$$

The dimensions of B are (4×3)

$$\mathbf{C} = \begin{array}{cc} \begin{array}{cc}\text{Col 1} & \text{Col 2}\end{array} & \\ \begin{bmatrix} 1 & 6 \\ 2 & 9 \\ -3 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{array}{l}\text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \\ \text{Row 5}\end{array} \end{array}$$

The dimensions of C are (5×2)

$$\mathbf{D} = \begin{array}{cc} \begin{array}{ccc}\text{Col 1} & \text{Col 2} & \text{Col 3}\end{array} & \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0.5 \\ 0.3 & 0 & 0.2 \end{bmatrix} & \begin{array}{l}\text{Row 1} \\ \text{Row 2} \\ \text{Row 3}\end{array} \end{array}$$

The dimensions of D are (3×3)

Transpose matrix

- When a matrix is rewritten so that its rows and columns are interchanged, then the resulting matrix is called the transpose of the original.

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 3 \\ -1 & 2 & 4 \end{bmatrix}$$

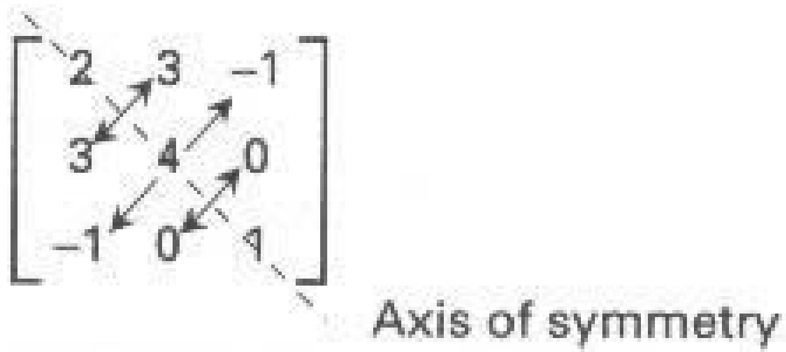
The dimensions of A are (2×3)

$$\mathbf{A}' = \begin{bmatrix} 1 & -1 \\ 6 & 2 \\ 3 & 4 \end{bmatrix}$$

The dimensions of A' are (3×2)

Square and symmetric matrices

- A **square matrix** is a matrix where the number of rows equals the number of columns (e.g. Matrix D in slide 21).
- A **symmetric matrix** is a **square matrix** where the rows and columns are such that its transpose is the same as the original matrix, i.e. elements $a_{ij} = a_{ji}$ where $i \neq j$.



Identity matrices

- An **identity matrix, I** is a square and symmetric matrix with zeros everywhere except its diagonal elements which have a value of 1.
- Examples of 2x2, 3x3, and 4x4 matrices are

$$\mathbf{I}_{(2\times 2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{(3\times 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{(4\times 4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding matrices

- Matrices **A** and **B** may be added if they have the same dimensions.
- That is, the corresponding elements may be added to yield a resulting matrix.
- The sum is **commutative**, i.e. **A** + **B** = **B** + **A**

$$\begin{bmatrix} 2 & \boxed{1} & 3 \\ 4 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & \boxed{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & \boxed{3} & 3 \\ 4 & 2 & -1 \end{bmatrix}$$

Subtracting matrices

- Matrix **B** may be subtracted from matrix **A** if they have the same dimensions, i.e. the corresponding elements of **B** may be subtracted from those of **A** to yield a resulting matrix.

$$\begin{bmatrix} 2 & 1 \\ 6 & \boxed{5} \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & \boxed{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & \boxed{3} \\ -1 & 1 \end{bmatrix}$$

- The result is **not commutative**. Reversing the order of the matrices yields different results, i.e. **A** - **B** $\neq$ **B** -**A**

$$\begin{bmatrix} 6 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & -1 \end{bmatrix}$$

Reversing the operation

$$\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -2 & 1 \end{bmatrix}$$

Different result

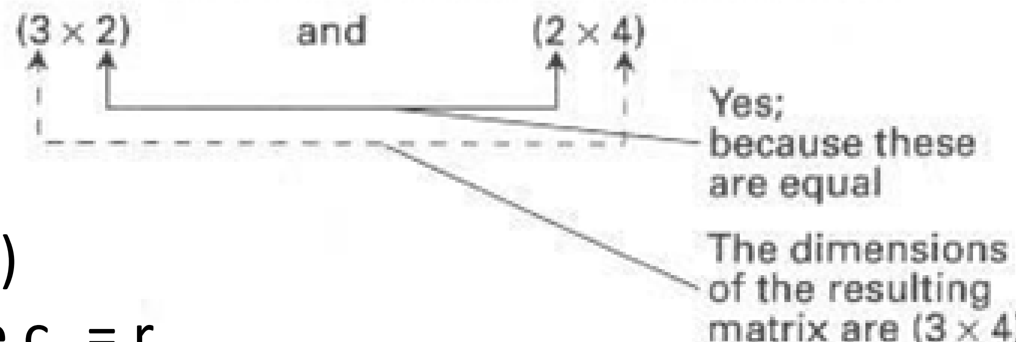
Multiplying matrices

- By a constant

$$3\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}3 & 6\\6 & 3\end{bmatrix}$$

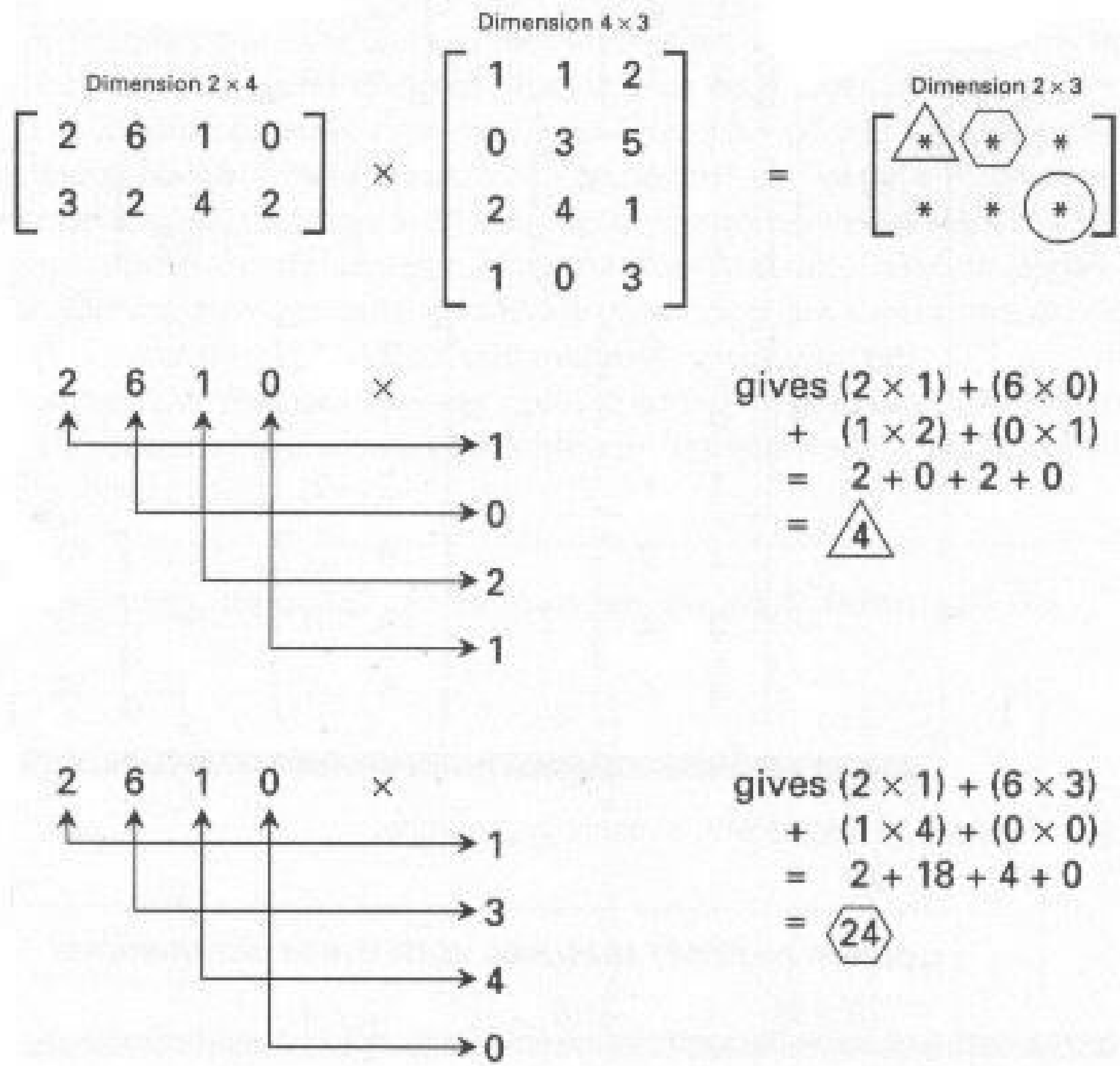
$$-1\begin{bmatrix}1 & 0 & 1\\2 & 1 & 2\end{bmatrix} = \begin{bmatrix}-1 & 0 & -1\\-2 & -1 & -2\end{bmatrix}$$

- By a matrix - The rule for multiplying one matrix to another is simple: if the **number of columns** in the first matrix is the **same as the number of rows** in the second matrix, the multiplication can be done.

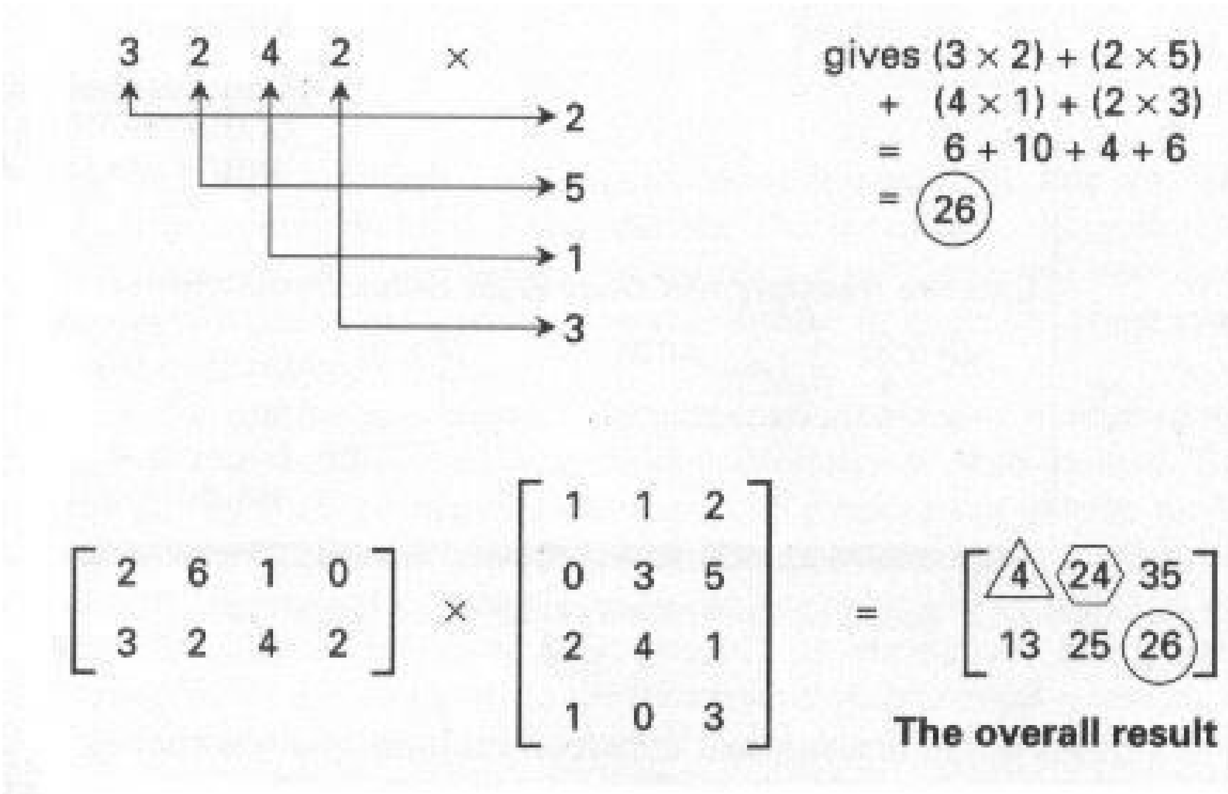


So $\mathbf{M_1}(r_1*c_1) * \mathbf{M_2}(r_2*c_2)$
$= \mathbf{M_3}(r_1*c_2)$ where $c_1 = r_2$

Multiplying matrices



Multiplying matrices – example



Non-commutative property of matrix multiplication

Matrix multiplication is not **commutative.**

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 14 \\ 7 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 9 \\ 22 & 14 \end{bmatrix}$$

Reversing the order of the matrices yields different results.

Non-commutative property of matrix multiplication

Reversing the order of the matrices yields different results (e.g. Slide 30) or the condition for matrix multiplication will not be satisfied (e.g. Slide 28).

Further example:

$$\mathbf{A} = [1 \;\; 2 \;\; 3], \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Will the following multiplications be possible?

A*B
B*A

Inverse matrices

If two matrices **A** and **B**, when multiplied together, results in an identity matrix **I**, then matrix **A** is the inverse of matrix **B** and vice versa, i.e.

$$\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A} = \mathbf{I}$$

$$\mathbf{A} = \mathbf{B}^{-1} \text{ and } \mathbf{B} = \mathbf{A}^{-1}$$

e.g.

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

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- Matrices (dimension, transpose, square/symmetric/identity and inverse) and matrix calculations (addition, subtraction and multiplication)