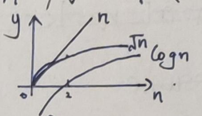
Question 1.

(a) $O(n^2)$

(b) To prove $f(n) = 9n + 3n^2 + 2n\log n + 3\sqrt{n}$ is $O(n^2)$, we show that there exists a contant c that $\forall n \geq n_0$, $n_0 \in Z^+$, $9n + 3n^2 + 2n\log n + 3\sqrt{n} \leq c \cdot n^2$

As $9n \leq 9n^2$, $3n^2 \leq 3n^2$, $2n\log n \leq 2n^2$, $3\sqrt{n} \leq 3n^2$; $\forall n \geq 1$



Then $9n + 3n^2 + 2n\log n + 3\sqrt{n} \leq 17n^2$, $\forall n \geq 1$

Since 17 is a constant, Therefore $f(n)$ is of the order of magnitude.

(The inequality reason: $n \geq 1 \Rightarrow n^2 \geq n \Rightarrow 9n^2 \geq 9n$.

$\Rightarrow 3n^2 \geq 3n^2$

$\Rightarrow D_n(n - \log n) = 1 - \frac{1}{n} \geq 0$ ascending ↑ $\min(n - \log n) = 1$.

$\Rightarrow n - \log n \geq 1 \Rightarrow n^2 - n\log n \geq n \geq 1 \Rightarrow 2n^2 - 2n\log n \geq 2 \geq 0$

$\Rightarrow n - \sqrt{n} + \frac{1}{4} = (\sqrt{n} - \frac{1}{2})^2 \geq \frac{1}{4} \Rightarrow n - \sqrt{n} \geq 0 \Rightarrow n \geq \sqrt{n} \Rightarrow n^2 \geq \sqrt{n}$.

$\Rightarrow 3n^2 \geq 3\sqrt{n}$)

Question 2.

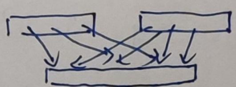
(a) In merge sort algorithm,

I. Basic Condition: We only have 1 components, then we do not to sort,

$T(1) = 1$

II. $n \geq 2$: Firstly we need divde the sequence into half, each time complectity is $O(n/2)$, in sum: $2T(\frac{n}{2})$; this is divide step.

Secondly; After $2T(\frac{n}{2})$ time, we finish each half sequence sorting, we need to combine them. For each $\frac{n}{2}$ sequence, we compare them from the lowest each: like the figure.



As the two sequence is ordered, each operation we can considere it as choose one to put in the n-sequence according to the comparison. We need to do this step n operation, in sum: n, this is conquer step.

In all, $n \geq 2$, $T(n) = 2T(\frac{n}{2}) + n$, we need to combine all the operations

$n = 1$, $T(1) = 1$

(b) We will use mathmatical induction to first prove :

$$\forall n \geqslant 2,\ n \in \mathbb{Z}^+,\ T(n) \leq 2n \log n$$

① $n=2$. $T(n) = 2(T(\frac{n}{2})) + n$.

$T(2) = 2T(1) + 2 = 4 \leq 4\log 2 = 2 \cdot 2 \log 2$ comes true.

② Assume $\exists k,\ n=k,\ k \in \mathbb{Z}^+,\ T(k) \leq 2k \log k$.

we will not prove $n=k+1$ but $n=2k$ (as the time complexity ~~is~~ $T(2k)$ is correspoing to $T(k)$, the other, from $k+1$ to $2k-1$ can be considered the weak ~~same in operations to be handled.~~ demonstration coming from $T(2k)$ demonstration)

then $n=2k$, $T(2k) = 2T(k) + 2k \leq 4k\log k + 2k$.

$2\cdot(2k)\log 2k = 4k\log 2k = 4k\log k + 4k\log 2 = 4k\log k + 4k$.

as $2k \leq 4k,\ \forall k \geqslant 1 \Rightarrow 4k\log k + 2k \leq 4k\log k + 4k$.

$\Rightarrow 2T(k) + 2k \leq 4k\log k + 4k = 2\cdot(2k)\log 2k$.

$\Rightarrow T(2k) \leq 4k\log 2k$ comes true.

We have proved $\forall n \geqslant 2,\ n \in \mathbb{Z}^+,\ T(n) \leq 2n\log n$.

2 is a constant, then $n \geqslant 2$, $T(n) = O(n\log n)$.

$T(1) = 1 \Rightarrow O(1) = T(1)$ And $O(n\log n) = O(1)$ at $n=1$.

$\Rightarrow$ $n=1$, $T(1) = O(n\log n)$

Combine $n=1$ and $n \geqslant 2$ we can get $T(n) = O(n\log n)$, $\forall n$.

Question 3

(a) 2, 4, 6, 2, 4, 6   swap

2, 4, 2, 6, 4, 6   1.

2, 2, 4, 6, 4, 6   2.

2, 2, 4, 4, 6, 6   3.

In all, 3 swap operations.

(b). As in each $i=0$ to $n-2$ ~~and~~ with $j=n-1 \to i+1$, a key comparison handled, we just calculate this numer.

		Operations
$i=0$	$j=n-1\to 1$	$n-1$
$i=1$	$j=n-1\to 2$	$n-2$
$\vdots$	$\vdots$	$\vdots$
$i=n-2$	$j=n-1\to n-1$	1

$$Sum = (n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$$

In this question, $n=6 \Rightarrow Sum = \frac{6\times 5}{2} = 15$.

In all 15 comparisons needed to sort the numbers.

Question 4.

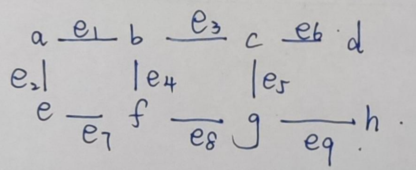
(a) adjacency matrix.

$$
\begin{array}{c|cccccccc}
 & a & b & c & d & e & f & g & h \\
\hline
a & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
b & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
c & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
d & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
e & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
f & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
g & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

adjacency list.

- a → b → e
- b → a → c → f
- c → b → d → g
- d → c
- e → a → f
- f → b → e → g
- g → c → f → h
- h → g

(b) Let $e_1 = (a, b)$, $e_2 = (a, e)$, $e_3 = (b, c)$, $e_4 = (b, f)$, $e_5 = (c, g)$, $e_6 = (c, d)$, $e_7 = (e, f)$, $e_8 = (f, g)$, $e_9 = (g, h)$



incidence matrix.

$$
\begin{array}{c|cccccccc}
 & a & b & c & d & e & f & g & h \\
\hline
e_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
e_3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_4 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
e_5 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
e_6 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
e_7 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
e_8 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
e_9 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

incidence list.

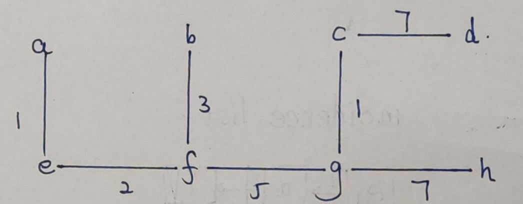
- e_1 → a → b
- e_2 → a → e
- e_3 → b → c
- e_4 → b → f
- e_5 → c → g
- e_6 → c → d
- e_7 → e → f
- e_8 → f → g
- e_9 → g → h

Question 5

(a) By Prim's algorithm, the order in which the vertices are selected is as follows : a, e, f, b, g, c, d, h ~~(or h, g according to your setting in comparisons);~~

Order selected	a(0,-)	b(-,∞)	c(-,∞)	d(-,∞)	e(-,∞)	f(-,∞)	g(-,∞)	h(-,∞)
a(0,-)		b(a,4)	c(-,∞)	d(-,∞)	e(a,1)	f(-,∞)	g(-,∞)	h(-,∞)
e(a,1)		b(a,4)	c(-,∞)	d(-,∞)		f(e,2)	g(-,∞)	h(-,∞)
f(e,2)		b(f,3)	c(f,10)	d(-,∞)			g(f,5)	h(-,∞)
b(f,3)			c(b,6)	d(-,∞)			g(f,5)	h(-,∞)
g(f,5)			c(g,1)	d(g,10)				h(g,7)
c(g,1)				d(c,7)				h(g,7)
d(c,7)								h(g,7)
h(g,7)								

The MST is as follows:



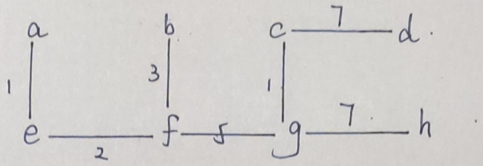
~~The MST drawn is not unique, according to the initial vertice you choose.~~

~~The~~ This is the only MST that can be drawn.

(b)

(a, e)	1
(c, g)	1
(e, f)	2
(b, f)	3
~~(a, b)~~	~~4~~
(f, g)	5
~~(b, e)~~	~~6~~
(c, d)	7
(g, h)	7
~~(d, h)~~	~~8~~
~~(e, f)~~	~~10~~
~~(d, g)~~	~~10~~

The MST is as follows



This is the unique MST that can be drawn.

(c)

order selected	a (0, -)	b (-, ∞)	c (-, ∞)	d (-, ∞)	e (-, ∞)	f (-, ∞)	g (-, ∞)	h (-, ∞)
a (0, -)		b (a, 4)	c (-, ∞)	d (-, ∞)	e (a, 1)	f (-, ∞)	g (-, ∞)	h (-, ∞)
e (a, 1)		b (a, 4)	c (-, ∞)	d (-, ∞)		f (e, 3)	g (-, ∞)	h (-, ∞)
f (e, 3)		b (a, 4)	c (f, 13)	d (-, ∞)			g (f, 8)	h (-, ∞)
b (a, 4)			c (b, 10)	d (-, ∞)			g (f, 8)	h (-, ∞)
g (f, 8)			c (g, 9)	d (g, 18)				h (g, 15)
c (g, 9)				d (c, 16)				h (g, 15)
h (g, 15)				d (c, 16)				
d (c, 16)