Tutorial of Hopfield Network

A binary Hopfield network is used to store 4 binary 4-dimensional patterns, $x_1 = [1,1, -1, -1]$, $x_2 = [-1, -1,1,1]$, $x_3 = [-1,1,1, -1]$ and $x_4 = [1,1,1, -1]$.

1. Compute the weight matrix W for the Hopfield network.

2. Recover the value of the 2nd dimension when the input state is $x_5 = [1,?, -1, -1]$

3. Given a noisy input $[1, -1,1, -1]$, using asynchronous mode update rule to determine the final stable state of the network (node selection sequence is 1>2>3>4).

4. Assuming we use a Hopfield network with no external inputs, please answer Question 3 again.

Answer:

Q1. Let I denotes identity matrix, the the weight matrix W is computed as the followed equation:

$$W = \frac{1}{N}\Sigma_{n=1,\ldots,N}\, x_n x_n^T - I$$

or the element of the weight matrix W is computed as:

$$w_{ij} = \begin{cases} \frac{1}{N}\Sigma_{n=1,\ldots,N}\, x_n(i)x_n(j) & if\ i \neq j \\ 0, & if\ i = j \end{cases}$$

And both the following to calculating method, (1) and (2), are correct.

a.

$$x_1 x_1^T = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$x_2 x_2^T = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$x_3 x_3^T = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$x_4 x_4^T = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

According the equation, the weight matrix is given:

$$W = \frac{1}{4}\begin{bmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & 0 & -4 \\ -2 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 \end{bmatrix}$$

b.

$w_{11} = w_{22}=w_{33}=w_{44}= 0$

$w_{12} = w_{21}= \frac{1}{4}(1 + 1 - 1 + 1) = 2/4$

$w_{13} = \quad w_{31}= \frac{1}{4}(-1 - 1 - 1 + 1) = \frac{-2}{4}$

$w_{14} = w_{41}= \frac{1}{4}(-1 - 1 + 1 - 1) =- 2/4$

$w_{23} = w_{32}= \frac{1}{4}(-1 - 1 + 1 + 1) = 0$

$w_{24} = w_{42}= \frac{1}{4}(-1 - 1 - 1 - 1) =- 4/4$

$w_{34} = w_{43}= \frac{1}{4}(1 + 1 - 1 - 1) = 0$

Q2. The update rule for each missing dimension is given by:

$$y^j(t + 1) = \sum_{n=1,n\neq j}^{4} w_{jn}x^n(t)$$

$$x^j(t + 1) = sgn(y^j(t + 1)) = \begin{cases} 1, & \text{if } y^j(t + 1) \geq 0 \\ -1, & \text{if } y^j(t + 1) < 0 \end{cases}$$

Therefore: $y^2(0) = \frac{1}{4}$ (2*1+0*(-1)+(-4)*(-1)+(0)) = 6/4

$x^5(0) = sgn(y^2(0)) = 1$

Q3. The update rule for each dimension is given by:

$$y^j(t + 1) = \sum_{n=1,n\neq j}^{4} w_{jn}x^n(t) +I_j$$

$$x^j(t+1) = \text{sgn}(y^j(t+1)) = \begin{cases} 1, & \text{if } y^j(t+1) \geq 0 \\ -1, & \text{if } y^j(t+1) < 0 \end{cases}$$

1 node: $\quad y^1(0) = \frac{1}{4} \ (2*(-1)+(-2)*(1)+(-2)*(-1))+1 = 2/4$

$\quad\quad\quad\quad x^1(0) = \text{sgn}(y^1(0)) = 1$

2 node: $\quad y^2(0) = \frac{1}{4} \ (2*1+0*(1)+(-4)*(-1))+(-1) = 2/4$

$\quad\quad\quad\quad x^2(0) = \text{sgn}(y^2(0)) = 1$

3 node: $\quad y^3(0) = \frac{1}{4} \ ((-2*1)+0*(1)+(0)*(-1))+(1) = 2/4$

$\quad\quad\quad\quad x^3(0) = \text{sgn}(y^3(0)) = 1$

4 node: $\quad y^4(0) = \frac{1}{4} \ ((-2)*1+(-4)*(1)+(0)*(1))+(-1) =- 10/4$

$\quad\quad\quad\quad x^4(0) = \text{sgn}(y^4(0)) = -1$

Q4.   As there is no any external input,   the update rule for each dimension is given by:

$$y^j(t+1) = \sum_{n=1,n\neq j}^{4} w_{jn}x^n(t)$$

$$x^j(t+1) = \text{sgn}(y^j(t+1)) = \begin{cases} 1, & \text{if } y^j(t+1) \geq 0 \\ -1, & \text{if } y^j(t+1) < 0 \end{cases}$$

1 node: $\quad y^1(0) = \frac{1}{4} \ (2*(-1)+(-2)*(1)+(-2)*(-1)) = -2/4$

$\quad\quad\quad\quad x^1(0) = \text{sgn}(y^1(0)) = -1$

2 node: $y^2(0) = \frac{1}{4}$ (2*(-1)+0*(1)+(-4)*(-1)) = 2/4

$x^2(0) = \text{sgn}(y^2(0)) = 1$

3 node: $y^3(0) = \frac{1}{4}$ ((-2)*(-1)+0*(1)+(0)*(-1))= 2/4

$x^3(0) = \text{sgn}(y^3(0)) = 1$

4 node: $y^4(0) = \frac{1}{4}$ ((-2)*(-1)+(-4)*(1)+(0)*(1))=- 2/4

$x^4(0) = \text{sgn}(y^4(0)) = $ -1