Tutorial of Hopfield Network

A binary Hopfield network is used to store 4 binary 4-dimensional patterns, $x_1 = [1,1,-1,-1]$, $x_2 = [-1,-1,1,1]$, $x_3 = [-1,1,1,-1]$ and $x_4 = [1,1,1,-1]$.

1. Compute the weight matrix W for the Hopfield network.

2. Recover the value of the 2nd dimension when the input state is $x_5 = [1,?,-1,-1]$

3. Given a noisy input $[1,-1,1,-1]$, using asynchronous mode update rule to determine the final stable state of the network (node selection sequence is 1>2>3>4).

4. Assuming we use a Hopfield network with no external inputs, please answer Question 3 again.

1. $W = \frac{1}{N} \sum_{p=1}^{P} x^p x^{pT} - P I$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} [1 \ 1 \ -1 \ -1] - 4I$$

$$+ x_2 x_2^T + x_3 x_3^T + x_4 x_4^T$$

Answer:

Q1. Let I denotes identity matrix, the the weight matrix W is computed as the followed equation:

$$\mathrm{W} = \frac{1}{\mathrm{N}}\sum_{\mathrm{n}=1,\ldots,\mathrm{N}} \mathrm{x_n x_n^T} - \mathrm{I}$$

or the element of the weight matrix W is computed as:

$$w_{ij} = \begin{cases} \frac{1}{N}\sum_{n=1,\ldots,N} \mathrm{x_n(i)x_n(j)} & if\ \ i \neq j \\ 0, & if\ i = j \end{cases}$$

$$W_{12} = \frac{1}{4}\, X_1 \cdot X_2$$

$$=$$

And both the following to calculating method, (1) and (2), are correct.

a.

$$W_{12} = \frac{1}{4}\sum_{n=1}^{4} x_n(1)x_n(2)$$

$$= \frac{1}{4}(1+1-1+1)$$

$$W_{13} = \frac{1}{4}(-1-1-1+1)$$

$$\mathrm{x_1x_1^T} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\mathrm{x_2x_2^T} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\mathrm{x_3x_3^T} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathrm{x_4x_4^T} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

According the equation, the weight matrix is given:

$$\mathrm{W} = \frac{1}{4}\begin{bmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & 0 & -4 \\ -2 & 0 & 0 & 0 \\ -2 & -4 & 0 & 0 \end{bmatrix}$$

b.

$w_{11} = w_{22} = w_{33} = w_{44} = 0$

$w_{12} = w_{21} = \frac{1}{4}(1+1-1+1) = 2/4$

$w_{13} = \quad w_{31} = \frac{1}{4}(-1-1-1+1) = \frac{-2}{4}$

$w_{14} = w_{41} = \frac{1}{4}(-1-1+1-1) = -2/4$

$w_{23} = w_{32} = \frac{1}{4}(-1-1+1+1) = 0$

$w_{24} = w_{42} = \frac{1}{4}(-1-1-1-1) = -4/4$

$w_{34} = w_{43} = \frac{1}{4}(1+1-1-1) = 0$

$\frac{6}{4}$

$\frac{2}{4} +$ 1

$\frac{2}{4} \times 1$ $\quad 0 \times -1 \quad + \frac{4}{4} \times +1$

$W_{21} \quad W_{23} \quad W_{24}$

$n=1 \quad n=3 \quad n=4$

Q2. The update rule for each missing dimension is given by:

$$y^{j}(t+1) = \sum_{n=1, n\neq j}^{4} w_{jn} x^{n}(t)$$

$$x^{j}(t+1) = \mathrm{sgn}(y^{j}(t+1)) = \begin{cases} 1, & \text{if } y^{j}(t+1) \geq 0 \\ -1, & \text{if } y^{j}(t+1) < 0 \end{cases}$$

Therefore: $y^{2}(0) = \frac{1}{4}$ (2*1+0*(-1)+(-4)*(-1)+(0)) = 6/4

$x^{5}(0) = \mathrm{sgn}(y^{2}(0)) = 1$

Q3. The update rule for each dimension is given by:

$$y^{j}(t+1) = \sum_{n=1, n\neq j}^{4} w_{jn} x^{n}(t) \; + I_j$$

$$x^j(t+1) = \mathrm{sgn}(y^j(t+1)) = \begin{cases} 1, & \text{if } y^j(t+1) \geq 0 \\ -1, & \text{if } y^j(t+1) < 0 \end{cases}$$

1 node: $y^1(0) = \frac{1}{4}$ (2*(-1)+(-2)*(1)+(-2)*(-1))+1 = 2/4

$x^1(0)$ = sgn($y^1(0)$) = 1

2 node: $y^2(0) = \frac{1}{4}$ (2*1+0*(1)+(-4)*(-1))+(-1) = 2/4

$x^2(0)$ = sgn($y^2(0)$) = 1

3 node: $y^3(0) = \frac{1}{4}$ ((-2*1)+0*(1)+(0)*(-1))+(1) = 2/4

$x^3(0)$ = sgn($y^3(0)$) = 1

4 node: $y^4(0) = \frac{1}{4}$ ((-2)*1+(-4)*(1)+(0)*(1))+(-1) =- 10/4

$x^4(0)$ = sgn($y^4(0)$) = -1

Q4. As there is no any external input, the update rule for each dimension is given by:

$$y^j(t+1) = \sum_{n=1,n\neq j}^{4} w_{jn} x^n(t)$$

$$x^j(t+1) = \mathrm{sgn}(y^j(t+1)) = \begin{cases} 1, & \text{if } y^j(t+1) \geq 0 \\ -1, & \text{if } y^j(t+1) < 0 \end{cases}$$

1 node: $y^1(0) = \frac{1}{4}$ (2*(-1)+(-2)*(1)+(-2)*(-1)) = -2/4

$x^1(0)$ = sgn($y^1(0)$) = -1

2 node: $y^2(0) = \frac{1}{4}$ (2*(-1)+0*(1)+(-4)*(-1)) = 2/4

$x^2(0)$ = sgn($y^2(0)$) = 1

3 node: $y^3(0) = \frac{1}{4}$ ((-2)*(-1)+0*(1)+(0)*(-1))= 2/4

$x^3(0)$ = sgn($y^3(0)$) = 1

4 node: $y^4(0) = \frac{1}{4}$ ((-2)*(-1)+(-4)*(1)+(0)*(1))=- 2/4

$x^4(0)$ = sgn($y^4(0)$) = -1