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An inverse time-dependent source problem for a time–space fractional diffusion equation☆

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ABSTRACT

This paper is devoted to identify a time-dependent source term in a time–space fractional diffusion equation by using the usual initial and boundary data and an additional measurement data at an inner point. The existence and uniqueness of a weak solution for the corresponding direct problem with homogeneous Dirichlet boundary condition are proved. We provide the uniqueness and a stability estimate for the inverse time-dependent source problem. Based on the separation of variables, we transform the inverse source problem into a first kind Volterra integral equation with the source term as the unknown function and then show the ill-posedness of the problem. Further, we use a boundary element method combined with a generalized Tikhonov regularization to solve the Volterra integral equation of the fist kind. The generalized cross validation rule for the choice of regularization parameter is applied to obtain a stable numerical approximation to the time-dependent source term. Numerical experiments for six examples in one-dimensional and two-dimensional cases show that our proposed method is effective and stable.

1. Introduction

The time–space fractional diffusion equation $\partial_{0+}^{\alpha} u(x,t) = -(-\Delta)^{\frac{\beta}{2}} u(x,t) + f(x,t)$ with $0<\alpha<1$ and $1<\beta<2$ is used to model anomalous diffusion [16]. The fractional derivative in time can be used to describe particle sticking and trapping phenomena and the fractional space derivative models long particle jumps. The combined effect produces a concentration profile with a sharper peak, and heavier tails. The direc t problems, i.e., initial value problems and initial boundary value problems for time–space fractional diffusion equations have attracted much more attention in recent years, for instances, Jia and Li [11] gave the maximum principles for the classical solution and weak solution. Chen et al. in [1] develops weak solutions of time–space fractional diffusion equations on bounded domains. Ding and Jiang in [5] consider the analytical solutions of multi-term time–space fractional advection-diffusion equations with mixed boundary conditions on a bounded domain.

However, in some practical situations, part of boundary data, or initial data, or diffusion coefficient, or source term may not be given and we want to find them by additional measurement data which will yield some fractional diffusion inverse problems. For the time fractional diffusion equations cases, the inverse source problems have been widely studied, On the

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uniqueness of inverse problems, Cheng et al. in [2] gave the uniqueness results for determining the order of fractional derivative and space-dependent diffusion coefficient in a fractional diffusion equation by means of observation data on the boundary. Sakamoto and Yamamoto in [19] established a few uniqueness results for several inverse problems. Yamamoto and Zhang [28] provided a conditional stability for determining a zeroth-order coefficient in a half-order fractional diffusion equation. On numerical computations of inverse problems, Liu et al. [15] recovered a time-dependent factor in the unknown boundary condition for a time fractional diffusion equation by a nonlocal measurement condition. Yang et al. [29] solved an inverse problem for identifying the unknown source by the Landweber iterative regularization method. Wei and Wang in [26] solved an inverse space-dependent source problem by a modified quasi-boundary value method. Wei and Zhang in [27] proposed a numerical method to solve the inverse time-dependent source problem. Wei et al. in [25] identified a time-dependent source term in a multidimensional time-fractional diffusion equation from the boundary Cauchy data. As we know, the researches on inverse problems for time–space fractional diffusion equations are still lack of wide attention.

In this paper, we consider the following time–space fractional diffusion equation

$$\partial_{0+}^{\alpha} u(x,t) = -(-\Delta)^{\frac{\beta}{2}} u(x,t) + f(x)p(t), \qquad (x,t) \in \Omega_T, \tag{1.1}$$

where $\Omega_T := \Omega \times (0, T]$, $\Omega \subset R^d$ and $\alpha \in (0, 1)$, $\beta \in (1, 2)$ are fractional orders of the time and space derivatives, respectively, $T > 0$ is a fixed final time.

The fractional derivative ∂_{0+}^{α} denotes the Caputo fractional left-sided derivative of order α with respect to t defined by

$$\partial_{0+}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \qquad 0 < \alpha < 1, \quad 0 < t \le T,$$

where Γ is the Gamma function .

The fractional Laplacian operator $(-\Delta)^{\frac{\beta}{2}}$ of order $\beta (1 < \beta \le 2)$ is defined by using the spectral decomposition of the Laplace operator. The definition is summarized in Definition 2.1 in Section 2. One can see Refs. [20–22].

Suppose the unknown function u satisfy the following initial and boundary conditions

$$u(x,0) = \phi(x), \qquad x \in \overline{\Omega}, \tag{1.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in (0,T]. \tag{1.3}$$

If all functions $f(x)$, $p(t)$, $\phi(x)$ are given appropriately, problems (1.1)–(1.3) is a direct problem. The inverse problem here is to determine the source term $p(t)$ in problem (1.1)–(1.3) from the additional data

$$u(x_0,t) = g(t), \qquad 0 < t \le T, \tag{1.4}$$

where $x_0 \in \Omega$ is an interior measurement location.

The inverse source problem mentioned above is an ill-posed problem, refer to Section 5. For the time–space fractional diffusion Eqs. (1.1)–(1.3), Tatar et al. in [20–22] solved an inverse space-dependent source problem and identified the orders of time and space fractional derivatives for a time–space fractional diffusion equation. Tuan and Long in [23] considered an inverse space-dependent problem to determine an unknown source term by Fourier truncation method, and give convergence estimates and regularization parameter choice rules, but no numerical example is provided. Kolokoltsov and Veretennikova in [13] studied the Cauchy problem for non-linear in time and space pseudo-differential equations. Dou and Hon in [6] solving a backward time–space fractional diffusion problem based on a kernel-based approximation technique. In this paper, we focus on the numerical reconstruction for the source term $p(t)$ in (1.1)–(1.4). A generalized Tikhonov regularization method based on a boundary element method is used to determine the source $p(t)$.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries used in Sections 3–5. The existence and uniqueness of a weak solution for the direct problems (1.1)–(1.3) is proved in Section 3. In Section 4, we provide the uniqueness and a stability estimate for the inverse source problem. In Section 5, we propose a regularized method based on the boundary element discretization for recovering a stable approximation to $p(t)$. The numerical results for six examples in one-dimensional and two-dimensional cases are investigated in Section 6. Finally, we give a conclusion in Section 7.

2. Preliminary

Let $AC[0, T]$ be the space of functions f which are absolutely continuous on $[0, T]$. Throughout this paper, we use the following definitions and propositions given in [20–22] and [12].

Definition 2.1. Suppose $\{\bar{\lambda}_k, \varphi_k\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω with Dirichlet boundary condition on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_k = \bar{\lambda}_k \varphi_k, & in \quad \Omega, \\ \varphi_k = 0, & on \quad \partial\Omega. \end{cases}$$

Let

$$\mathcal{H}_0^{\beta}(\Omega) := \left\{ u = \sum_{n=1}^{\infty} a_n \varphi_n : \|u\|^2_{\mathcal{H}_0^{\beta}(\Omega)} = \sum_{n=1}^{\infty} a_n^2 \bar{\lambda}_n^{\beta} < \infty \right\},$$

then if $u \in \mathcal{H}_0^{\beta}(\Omega)$, we define the operator $(-\Delta)^{\frac{\beta}{2}}$ by

$$(-\Delta)^{\frac{\beta}{2}} u = \sum_{n=1}^{\infty} a_n \bar{\lambda}_n^{\beta/2} \varphi_n,$$

which maps $\mathcal{H}_0^{\beta}(\Omega)$ onto $L_2(\Omega)$, with the following equivalence

$$\|u\|_{\mathcal{H}_0^{\beta}(\Omega)} = \|(-\Delta)^{\frac{\beta}{2}} u\|_{L^2(\Omega)}.$$

Note that if α tends to 1 and β tends to 2, the fractional derivative $\partial_{0+}^{\alpha} u$ tends to the first-order derivative u_t and the fractional Laplacian operator $(-\Delta)^{\frac{\beta}{2}}$ tends to the Laplacian operator $-\Delta$, and thus model (1.1) reproduces the standard diffusion equation.

Definition 2.2. The Mittag–Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Proposition 2.3. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a constant $c = c(\alpha, \beta, \mu) > 0$ such that*

$$| E_{\alpha,\beta}(z) | \le \frac{c}{1 + |z|}, \qquad \mu \le | \arg(z) | \le \pi.$$

Proposition 2.4. *Let $\alpha > 0$ and $\lambda > 0$, then we have*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t > 0.$$

Proposition 2.5. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\partial_{0+}^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), \quad t > 0.$$

Proposition 2.6. *(See [18].) For $0 < \alpha < 1$, $t > 0$, we have $0 < E_{\alpha,1}(-t) < 1$. Moreover, $E_{\alpha,1}(-t)$ is completely monotonic that is*

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t) \ge 0, \quad \forall n \in \mathbb{N}.$$

Proposition 2.7. *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \le E_{\alpha,\alpha}(-\eta) \le \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Lemma 2.8. *For $0 < \alpha < 1$ and $\lambda > 0$, if $q(t) \in AC[0, T]$, we have*

$$\begin{aligned}
&\partial_{0+}^{\alpha} \int_0^t q(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) d\tau \\
&= q(t) - \lambda \int_0^t q(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) d\tau, \quad 0 < t \le T.
\end{aligned}$$

Lemma 2.9. *(See [25]) Suppose $p(t) \in L^{\infty}(0, T)$, $0 < \alpha < 1$, $\lambda \ge 0$, denote*

$$g(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) d\tau, \quad t \in (0, T],$$

and define $g(0) = 0$, then $g(t) \in C[0, T]$.

3. Existence and uniqueness of a weak solution for the direct problem

Denote the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition as $\bar{\lambda}_n$ and the corresponding eigenfunctions as $\varphi_n \in H^2(\Omega) \bigcap H_0^1(\Omega)$, that means we have $-\Delta\varphi_n = \bar{\lambda}_n \varphi_n$ and $\varphi_n|_{\partial\Omega} = 0$. Counting according to the multiplicities, we can set : $0 < \bar{\lambda}_1 \le \bar{\lambda}_2 \le \cdots \le \bar{\lambda}_n \le \cdots$ and $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$.

Let us define a weak solution to the direct problems (1.1)–(1.3), then we prove its existence and uniqueness based on the methods in [19].

Definition 3.1. We call $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathcal{H}_0^\beta(\Omega))$ such that $\partial_{0+}^\alpha u \in C((0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$ a weak solution to (1.1)–(1.3) if (1.1) holds in $L^2(\Omega)$ for $0 < t \le T$, (1.2) holds in $L^2(\Omega)$ as $t \to 0^+$.

Theorem 3.2. *If $\phi \in \mathcal{H}_0^\beta(\Omega)$, $f \in L^2(\Omega)$, $p \in AC[0, T]$, then there exists a unique weak solution to (1.1)–(1.3) and the solution is given by*

$$\begin{aligned} u(x, t) = &\sum_{n=1}^{\infty} (\phi, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x) \\ &+ \sum_{n=1}^{\infty} (f, \varphi_n) \int_0^t p(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) d\tau \varphi_n(x), \end{aligned} \tag{3.1}$$

where $\lambda_n = \bar{\lambda}_n^{\beta/2}$. Moreover, we have the following estimates

$$\|u\|_{C([0,T];L^2(\Omega))} \le C_1(\|\phi\|_{L^2(\Omega)} + \|p\|_{L^\infty} \|f\|_{L^2(\Omega)}), \tag{3.2}$$

$$\|u\|_{L^2(0,T;\mathcal{H}_0^\beta(\Omega))} \le C_2(\|\phi\|_{\mathcal{H}_0^\beta(\Omega)} + \|p\|_{L^\infty} \|f\|_{L^2(\Omega)}), \tag{3.3}$$

where C_1, C_2 are positive constants depending on α, T, Ω.

Proof. Based on Proposition 2.5 and Lemma 2.8, by the separation of variables, we can obtain a formal solution for the direct problems (1.1)–(1.3) as (3.1). We will show that (3.1) certainly gives a weak solution to (1.1)–(1.3).

In the following proof, we denote C as a generic positive constant. Denote

$$g_n(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) d\tau, \quad n \in \mathbb{N}, \tag{3.4}$$

then we know

$$|g_n(t)| \le \|p\|_{L^\infty} / \Gamma(\alpha+1) t^\alpha \le \|p\|_{L^\infty} / \Gamma(\alpha+1) T^\alpha, \quad t \in [0, T], \quad n \in \mathbb{N} \tag{3.5}$$

$$|g_n(t)| \le \|p\|_{L^\infty}(1 - E_{\alpha,1}(-\lambda_n T^\alpha))/\lambda_n \le \|p\|_{L^\infty}/\lambda_n, \quad t \in [0, T], \quad n \in \mathbb{N}. \tag{3.6}$$

(1) We first verify $u \in C([0, T]; L^2(\Omega))$ and $\lim_{t\to 0} \|u(\cdot, 0) - \phi(\cdot)\|_{L^2(\Omega)} = 0$. Define

$$\begin{aligned} u_1 &:= \sum_{n=1}^{\infty} (\phi, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x), \\ u_2 &:= \sum_{n=1}^{\infty} (f, \varphi_n) g_n(t) \varphi_n(x). \end{aligned}$$

Then we have $u(\cdot, t) = u_1(\cdot, t) + u_2(\cdot, t)$. We estimate each term separately. For fixed $t \in [0, T]$, by Proposition 2.6 and (3.5), we have

$$\|u_1(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} (\phi, \varphi_n)^2 E_{\alpha,1}^2(-\lambda_n t^\alpha) \le \|\phi\|_{L^2(\Omega)}^2, \tag{3.7}$$

and

$$\|u_2(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} (f, \varphi_n)^2 g_n^2(t) \le \|p\|_{L^\infty}^2 \sum_{n=1}^{\infty} \frac{(f, \varphi_n)^2}{\Gamma^2(\alpha+1)} t^{2\alpha}. \tag{3.8}$$

By (3.8), we know

$$\lim_{t\to 0^+} \|u_2(\cdot, t)\|_{L^2(\Omega)} = 0. \tag{3.9}$$

Thus define $u_2(x, 0) = 0$. From (3.7)–(3.8), we obtain

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le C_1\big(\|\phi\|_{L^2(\Omega)} + \|p\|_{L^\infty} \|f\|_{L^2(\Omega)}\big), \quad t \in [0, T],$$

where $C_1 = \max\{1, T^\alpha / \Gamma(\alpha+1)\}$.

For $t, t+h \in [0, T]$, we have

$$u(x, t+h) - u(x, t) = \sum_{n=1}^{\infty} (\phi, \varphi_n)(E_{\alpha,1}(-\lambda_n (t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)) \varphi_n(x)$$

$$+\sum_{n=1}^{\infty}(f,\varphi_n)(g_n(t+h)-g_n(t))\varphi_n(x)$$
$$=: I_1(x,t;h)+I_2(x,t;h).$$

We estimate each term separately. In fact, by Proposition 2.6, we have

$$\|I_1(\cdot,t;h)\|^2_{L^2(\Omega)}=\sum_{n=1}^{\infty}(\phi,\varphi_n)^2|E_{\alpha,1}(-\lambda_n(t+h)^\alpha)-E_{\alpha,1}(-\lambda_n t^\alpha)|^2\le 4\|\phi\|^2_{L^2(\Omega)},$$

since $\lim_{h\to 0}|E_{\alpha,1}(-\lambda_n(t+h)^\alpha)-E_{\alpha,1}(-\lambda_n t^\alpha)|=0$ for each $n\in\mathbb{N}$, by using the Lebesgue theorem, we have

$$\lim_{h\to 0}\|I_1(\cdot,t;h)\|^2_{L^2(\Omega)}=0.$$

By (3.5), we have

$$\|I_2(\cdot,t;h)\|^2_{L^2(\Omega)}=\sum_{n=1}^{\infty}(f,\varphi_n)^2(g_n(t+h)-g_n(t))^2\le C\|p\|^2_{L^\infty}\|f\|^2_{L^2(\Omega)}.$$

Similarly, by using the Lebesgue theorem and Lemma 2.9, we can prove

$$\lim_{h\to 0}\|I_2(\cdot,t;h)\|^2_{L^2(\Omega)}=0.$$

Therefore, $u\in C([0,T];L^2(\Omega))$.

By Proposition 2.6, we know

$$\|u(\cdot,t)-\phi(\cdot)\|_{L^2(\Omega)}\le\left(\sum_{n=1}^{\infty}(\phi,\varphi_n)^2(E_{\alpha,1}(-\lambda_n t^\alpha)-1)^2\right)^{1/2}+\|u_2(\cdot,t)\|_{L^2(\Omega)}$$
$$\le\|\phi\|_{L^2(\Omega)}+\|u_2(\cdot,t)\|_{L^2(\Omega)}.$$

Since $\lim_{t\to 0}\big(E_{\alpha,1}(-\lambda_n t^\alpha)-1\big)=0$ and (3.9), we have

$$\lim_{t\to 0^+}\|u(\cdot,t)-\phi(\cdot)\|_{L^2(\Omega)}=0.$$

(2) We verify $(-\triangle)^{\frac{\beta}{2}}u\in C((0,T];L^2(\Omega))\cap L^2(0,T;L^2(\Omega))$ and $u\in L^2(0,T;\mathcal{H}_0^\beta(\Omega))$.

By (3.1), we know

$$(-\triangle)^{\frac{\beta}{2}}u(x,t)=\sum_{n=1}^{\infty}\lambda_n(\phi,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x)+\sum_{n=1}^{\infty}\lambda_n(f,\varphi_n)g_n(t)\varphi_n(x)$$
$$=: v_1(x,t)+v_2(x,t),$$

where g_n is defined in (3.4).

For $0<t\le T$, by Proposition 2.3, we obtain

$$\|v_1(\cdot,t)\|^2_{L^2(\Omega)}=\sum_{n=1}^{\infty}(\lambda_n(\phi,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha))^2 \tag{3.10}$$
$$\le\sum_{n=1}^{\infty}\left(\lambda_n(\phi,\varphi_n)^2\left(\frac{c\sqrt{\lambda_n}}{1+\lambda_n t^\alpha}\right)^2\right)\le C\frac{\|\phi\|^2_{\mathcal{H}_0^\beta(\Omega)}}{t^\alpha},$$

where C is a constant depending on α only.

For the second term v_2, by (3.6) we can deduce that

$$\|v_2(\cdot,t)\|^2_{L^2(\Omega)}=\sum_{n=1}^{\infty}\lambda_n^2(f,\varphi_n)^2g_n^2(t)\le\|p\|^2_{L^\infty}\sum_{n=1}^{\infty}(f,\varphi_n)^2\le\|p\|^2_{L^\infty}\|f\|^2_{L^2(\Omega)}. \tag{3.11}$$

Since $v_1(x,t),v_2(x,t)$ are convergent in $L^2(\Omega)$ uniformly on $t\in[t_0,T]$ for any given $t_0>0$, $(-\triangle)^{\frac{\beta}{2}}u\in C\big((0,T];L^2(\Omega)\big)$ can be obtained similarly to the proof in (1).

By estimates (3.10)–(3.11), we know $v_1,v_2\in L^2(0,T;L^2(\Omega))$, hence $(-\triangle)^{\frac{\beta}{2}}u\in L^2(0,T;L^2(\Omega))$. Moreover, we can obtain the following estimate from (3.10)–(3.11)

$$\|u\|_{L^2\left(0,T;\mathcal{H}_0^\beta(\Omega)\right)}=\|(-\triangle)^{\frac{\beta}{2}}u\|_{L^2(0,T;L^2(\Omega))}\le C_2(\|\phi\|_{\mathcal{H}_0^\beta(\Omega)}+\|p\|_{L^\infty}\|f\|_{L^2(\Omega)}),$$

where $C_2=C_2(\alpha,T,\Omega)$ is a positive constant depending on α, T, Ω.

(3) We prove that $\partial_{0+}^{\alpha} u \in C((0,T];L^2(\Omega)) \cap L^2(0,T;L^2(\Omega))$ and (1.1) holds in $L^2(\Omega)$ for $t\in(0,T]$.
By Lemma 2.8, we have

$$\begin{aligned}\partial_{0+}^{\alpha} u(x,t) &= -\sum_{n=1}^{\infty}(\phi,\varphi_n)\lambda_n E_{\alpha,1}(-\lambda_n t^{\alpha})\varphi_n(x)\\ &\quad+\sum_{n=1}^{\infty}(f,\varphi_n)[p(t)-\lambda_n\int_0^t p(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})d\tau]\varphi_n(x)\\ &= p(t)f(x)-(-\triangle)^{\frac{\beta}{2}}u(x,t).\end{aligned}$$

Hence $\partial_{0+}^{\alpha} u \in C((0,T];L^2(\Omega)) \cap L^2(0,T;L^2(\Omega))$ and (1.1) holds in $L^2(\Omega)$ for $t\in(0,T]$.

(4) We prove the uniqueness of the weak solution to (1.1)–(1.3). Under the condition $f(x)p(t)=0$, $\phi=0$, we need to prove that systems (1.1)–(1.3) has only a trivial solution. We take the inner product of (1.1) with $\varphi_n(x)$. Using the Green formula and $\varphi_n|_{\partial\Omega}=0$ and setting $u_n(t):=(u(\cdot,t),\varphi_n)$, we obtain

$$\begin{cases}\partial_{0+}^{\alpha} u_n(t)=-\lambda_n u_n(t), & t\in(0,T],\\ u_n(0)=0.\end{cases}$$

Due to the existence and uniqueness of the ordinary fractional differential equation (e.g., Chapter 3 in [12]), we obtain that $u_n(t)=0, n=1,2,\cdots$. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$, we have $u=0$ in $\Omega\times(0,T]$. Thus the proof is complete. □

4. Uniqueness and a stability estimate for the inverse source problem

Theorem 4.1. *Let u_1, u_2 be the solutions of problems (1.1)–(1.3) for $p=p_1,p_2\in AC[0,T]$ respectively with a fixed $f\in\mathcal{H}_0^{2\gamma}(\Omega)$ in which $\gamma>\frac{d}{2}+\frac{\beta}{2}$ and $\phi\in\mathcal{H}_0^{\beta}(\Omega)$. Assume that there exists a point $x_0\in\Omega$ satisfying $f(x_0)\neq 0$. Then there exists a constant C_3 such that*

$$\|p_1-p_2\|_{C[0,T]}\le C_3\|\partial_{0+}^{\alpha}(u_1(x_0,\cdot)-u_2(x_0,\cdot))\|_{C[0,T]}. \tag{4.1}$$

Proof. In this proof, we denote $u=u_1-u_2$, $p=p_1-p_2$, then by Theorem 3.2, we know

$$u(x,t)=\sum_{n=1}^{\infty}(f,\varphi_n)g_n(t)\varphi_n(x)$$

and

$$\partial_{0+}^{\alpha} u(x,t)=p(t)\sum_{n=1}^{\infty}(f,\varphi_n)\varphi_n(x)-\sum_{n=1}^{\infty}(f,\varphi_n)\lambda_n g_n(t)\varphi_n(x).$$

Since $H^{2m}(\Omega)\hookrightarrow C(\bar{\Omega})$ for $m>\frac{d}{4}$, we have

$$\|\varphi_n\|_{C(\bar{\Omega})}\le C\|\varphi_n\|_{H^{2m}}\le C(\bar{\lambda}_n)^m,\ for\ n=1,2,\dots.$$

Thus, for $m>\frac{d}{4}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\|(f,\varphi_n)\varphi_n\|_{C(\bar{\Omega})} &\le C\sum_{n=1}^{\infty}\|(f,\varphi_n)\varphi_n\|_{H^{2m}}\\ &\le C\sum_{n=1}^{\infty}|(f,\varphi_n)|\bar{\lambda}_n^{\,m}\\ &\le C\left(\sum_{n=1}^{\infty}\frac{1}{\bar{\lambda}_n^{\,2\nu}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\bar{\lambda}_n^{\,2(m+\nu)}(f,\varphi_n)^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\bar{\lambda}_n\ge Cn^{\frac{2}{d}}$, $n\in N$ (see [3]), then we have $\frac{1}{\bar{\lambda}_n^{\,2\nu}}\le C\frac{1}{n^{\frac{4\nu}{d}}}$. If we choose $\nu>\frac{d}{4}$ and $\gamma=m+\nu>\frac{d}{2}$, then by $f\in\mathcal{H}_0^{2\gamma}(\Omega)$, for $(x,t)\in\overline{\Omega}\times[0,T]$, we know the series $\sum_{n=1}^{\infty}(f,\varphi_n)\varphi_n(x)$ is convergent on $\overline{\Omega}\times[0,T]$ uniformly. Further, by (3.6) we know

$$\sum_{n=1}^{\infty}|(f,\varphi_n)\lambda_n g_n(t)\varphi_n(x)|\le\|p\|_{L^\infty}\sum_{n=1}^{\infty}\|(f,\varphi_n)\varphi_n\|_{L^\infty(\Omega)}\le C\|p\|_{L^\infty}\|f\|_{\mathcal{H}_0^{2\gamma}(\Omega)},$$

is convergent on $\overline{\Omega} \times [0, T]$ uniformly. Thus, we have

$$\partial_{0+}^{\alpha} u(x_0, t) = p(t) \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(x_0) - \sum_{n=1}^{\infty} (f, \varphi_n)\lambda_n g_n(t)\varphi_n(x_0) \tag{4.2}$$

for $0 < t \leq T$.

Since $f \in \mathcal{H}_0^{2\gamma}(\Omega)$, $\gamma > \frac{d}{2}$, there is $f \in H^{2\gamma}(\Omega) \hookrightarrow C(\bar{\Omega})$, $H^{2\gamma}(\Omega)$ is the Sobolev space, thus $f(x_0) = \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(x_0)$. Setting

$$Q(t) := \sum_{n=1}^{\infty} -\lambda_n (f, \varphi_n) E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\varphi_n(x_0),$$

similarly, if let $\gamma > \frac{d}{2} + \frac{\beta}{2}$, then $\sum_{n=1}^{\infty} |\lambda_n (f, \varphi_n) E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\varphi_n(x)|$ is also convergent uniformly on $\overline{\Omega} \times [0, T]$, thus we can obtain $Q \in C[0, T]$. Inserting it into (4.2), we can deduce

$$|p(t)| \leq C \left\| \partial_{0+}^{\alpha} u(x_0, \cdot) \right\|_{C[0,T]} + C\|Q\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha-1} |p(\tau)| d\tau, \quad 0 < t \leq T.$$

Applying an inequality of Gronwall type with weakly singular kernel $(t-\tau)^{\alpha-1}$ (e.g. Theorem 1 and Corollary 2 in [30]), we see

$$|p(t)| \leq C_3 \left\| \partial_{0+}^{\alpha} u(x_0, \cdot) \right\|_{C[0,T]}, \quad 0 < t \leq T,$$

where C_3 is a positive constant depending on f, β, T and Ω.

Thus the proof is completed by $p \in C[0, T]$. □

5. Boundary integral equation and the Tikhonov regularization method

From (1.4) and (3.1), we obtain a Volterra integral equation

$$\mathcal{A}p := \int_0^t k(t, \tau) p(\tau) d\tau = y(t), \tag{5.1}$$

where the kernel function is give by

$$k(t, \tau) = (t-\tau)^{\alpha-1} \sum_{n=1}^{\infty} f_n \varphi_n(x_0) E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}), \tag{5.2}$$

for $t > \tau$ and define $k(t, \tau) = 0$ for $t \leq \tau$, and the right-hand side is

$$y(t) = g(t) - \sum_{n=1}^{\infty} \phi_n \varphi_n(x_0) E_{\alpha,1}(-\lambda_n t^{\alpha}), \tag{5.3}$$

where $f_n = (f, \varphi_n)$, $\phi_n = (\phi, \varphi_n)$.

Let $f \in \mathcal{H}_0^{2\gamma}(\Omega)$, $\gamma > \frac{d}{2}$, for any $t \in [0, T]$, by Proposition 2.7, we know

$$\left| \sum_{n=1}^{\infty} f_n \varphi_n(x_0) E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \right| \leq \sum_{n=1}^{\infty} \frac{|f_n \varphi_n(x_0)|}{\Gamma(\alpha)}.$$

By the proof in Theorem 4.1, we know the series $\sum_{n=1}^{\infty} f_n \varphi_n(x_0) E_{\alpha,\alpha}(-\lambda_n t^{\alpha})$ is uniformly convergent on $t \in [0, T]$, then it is continuous on $[0, T]$. By the definition of kernel function $k(t, \tau)$, we know that $k(t, \tau)$ is continuous on $[0, T] \times [0, T] \backslash t \neq \tau$.

Similarly, under the condition $f \in \mathcal{H}_0^{2\gamma}$, $\gamma > \frac{d}{2}$, we know

$$\left| \sum_{n=1}^{\infty} f_n \varphi_n(x_0) E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \right| \leq M, \quad t, \tau \in [0, T].$$

Thus we know that $k(t, \tau)(t-\tau)^{1-\alpha}$ is bounded on $[0, 1] \times [0, 1]$. Refer to Kress [14], the kernel $k(t, \tau)$ is weakly singular in $[0, T] \times [0, T]$ and the operator $\mathcal{A}$ is compact from $C[0, T]$ into $C[0, T]$. Thus, the inverse source problems (1.1)–(1.4) is ill-posed. That means the solut ion $p(t)$ does not depend continuously on the given data and any small perturbation in the given data may cause large change to the solution.

In order to handle with the possible numerical instability of the inverse problem, we employ the first order Tikhonov regularization method based on a boundary element discretization to solve the linear integral Eq. (5.1). Discrete the time interval $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$ by $t_i = \frac{iT}{n}$. Let Eq. (5.1) be satisfied at points t_i, then we have

$$\int_0^{t_i} k(t_i, \tau) p(\tau) d\tau = y(t_i), \quad i = 1, 2, \ldots n. \tag{5.4}$$

Using the constant elements to approximate $p(t)$, i.e. let $p(t) = p_i$ for $t \in (t_{i-1}, t_i]$, denote $\mathbf{p} = (p_1, p_2, \dots, p_n)^T$, then it is easy to obtain the following linear system of equation:

$$A\mathbf{p} = b, \tag{5.5}$$

where $A = (a_{ij})_{n\times n}$ is a matrix and $b = (b_i)_{n\times 1}$ is a vector, it follows from Proposition 2.4 we can be calculated by the following formulate:

$$\begin{aligned} a_{ij} &= \int_{t_{j-1}}^{t_j} k(t_i, \tau)d\tau = \sum_{n=1}^{\infty} f_n\varphi_n(x_0)\int_{t_{j-1}}^{t_j}(t_i-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t_i-\tau)^{\alpha})d\tau \\ &= \sum_{n=1}^{\infty}\frac{f_n\varphi_n(x_0)}{\lambda_n}(E_{\alpha,1}(-\lambda_n(t_i-t_j)^{\alpha}) - E_{\alpha,1}(-\lambda_n(t_i-t_{j-1})^{\alpha})), \quad j \le i, \\ a_{ij} &= 0, \quad j > i, \end{aligned}$$

and

$$b_i = g(t_i) - \sum_{n=1}^{\infty}\phi_n E_{\alpha,1}(-\lambda_n t_i^{\alpha})\varphi_n(x_0).$$

In practical situations, the measured data $g(t)$ are inevitably contaminated by measurement errors. For this case, we replace exact data by noisy data, and the corresponding noisy right hand side is denoted by b^{δ}.

Since the inverse time-dependent source problem of time–space fractional diffusion equation is ill-posed, the ill-conditioning of the matrix $\mathbf{A}$ in Eq. (5.5) still persists. In other words, most standard numerical methods cannot achieve good accuracy in solving the linear system (5.5). Several regularization methods have been developed for solving ill-conditional problems [7–9]. In our computation we adapt the first order Tikhonov regularization technique to solve the matrix Eq. (5.5). The first order Tikhonov regularized solution $\mathbf{p}_{\mu}^{\delta}$ for (5.5) is defined as the solution for the following least-square problem:

$$\mathbf{p}_{\mu}^{\delta} := \min\{\| A\mathbf{p} - b^{\delta} \|^2 + \mu^2 \| L\mathbf{p} \|^2\}, \tag{5.6}$$

where $\|\bullet\|$ denotes the usual Euclidean norm and μ is called the regularization parameter and L is the first order difference matrix:

$$L = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}_{(n-1)\times n}.$$

The determination of a suitable value for the regularization parameter μ is crucial and is still under intensive researches. In our computation we use the generalized cross validation (GCV) method to determine a suitable value of μ. The GCV method was firstly investigated by Wahba in [4,24] and more recently Hansen [8,10] gave the formulation of computation based on the singular value decomposition (SVD).

The GCV is a strategy that give a good regularization parameter by minimizing the following GCV function

$$G(\mu) = \frac{\| A\mathbf{p}_{\mu}^{\delta} - b^{\delta} \|^2}{(trace(I_n - \mathbf{A}\mathbf{A}^I))^2}, \quad \mu > 0 \tag{5.7}$$

where $A^I = (A^TA + \mu^2L^TL)^{-1}A^T$ is a matrix which produces the regularized solution when multiplied with $\mathbf{b}$, i.e., $\mathbf{p}_{\mu}^{\delta} = A^I b$.

In our computation, we used the Matlab code developed by Hansen [9] for solving the discrete ill-conditioned system (5.5) and code give by Podlubny [17] for computing the generalized Mittag–Leffler function.

6. Numerical experiments

For simplicity, we assume that the maximum time is $T = 1$. Let the truncated number for assembling A, b be 50 and the accuracy control in computing the Mittag–Leffler function be 10^{-6}.

To test the computational error of numerical solutions, we calculate the relative root mean square error as

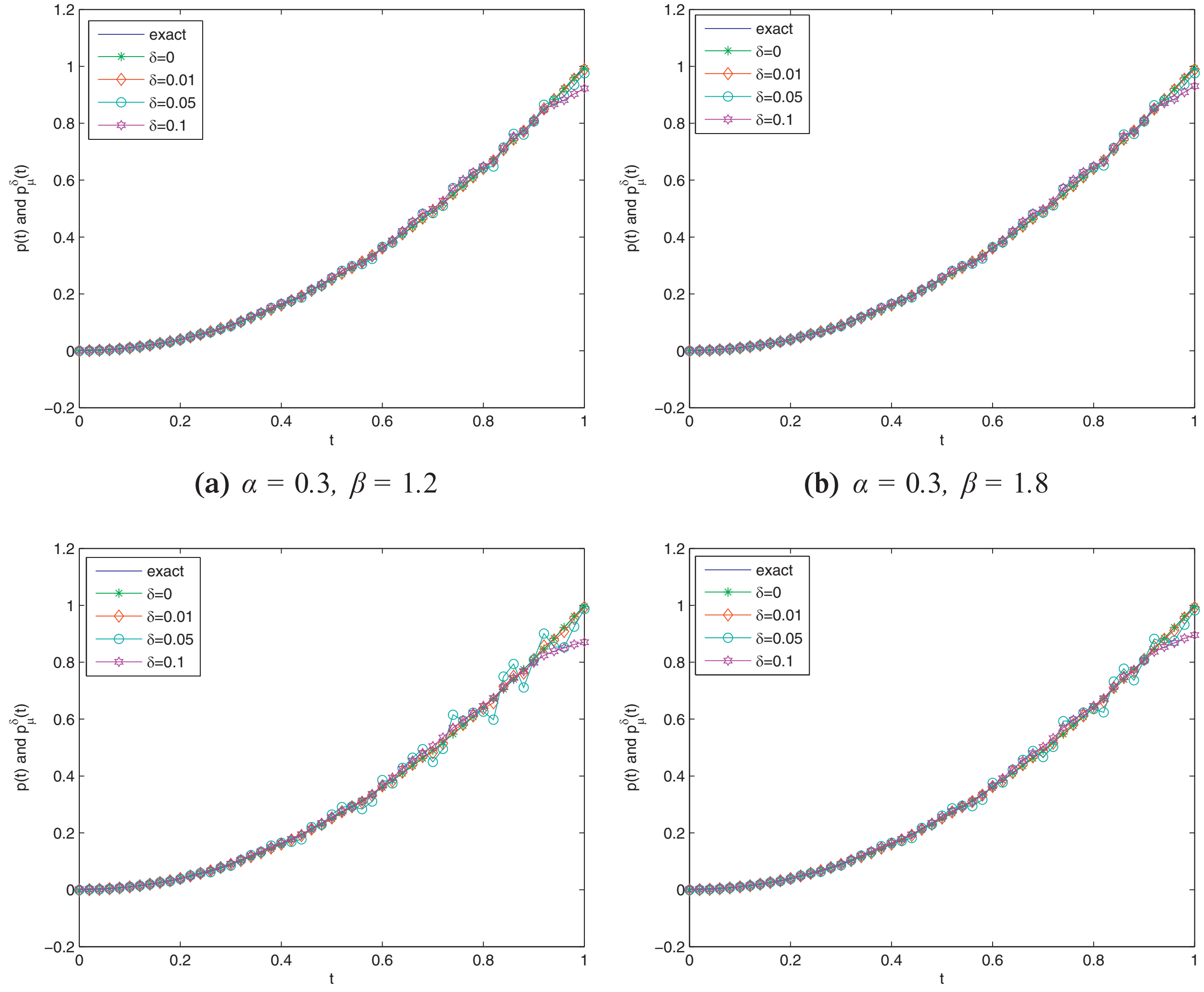
$$\varepsilon(p) = \left(\frac{\sum_{i=1}^{n}\left((p_{\mu}^{\delta})_i - p(t_i)\right)^2}{\sum_{i=1}^{n} p(t_i)^2}\right)^{1/2}, \tag{6.1}$$

where n is the total number of uniformly distributed point on time internal [0,1]. In our computations, we fix $n = 50$ unless otherwise specified.

The additional data $g = u(x_0, t)$ is obtained by formula (3.1) as the exact input data, The noisy data is generated by adding a random perturbation, i.e.,

$$g^{\delta} = g + \delta g \cdot (2\cdot rand(size(g)) - 1),$$

the magnitude δ indicates a relative noise level.



(a) $\alpha = 0.3,\ \beta = 1.2$ (b) $\alpha = 0.3,\ \beta = 1.8$

(c) $\alpha = 0.7,\ \beta = 1.2$ (d) $\alpha = 0.7,\ \beta = 1.8$

Fig. 1. The numerical results for Example 1 for various noise levels.

Table 1
The relative root mean square errors of Example 1 for various values of α with fixed $\beta = 1.2,\ \delta = 10\%$.

α	0.1	0.3	0.5	0.7	0.9
$\varepsilon(p)$	0.0404	0.0373	0.0454	0.0593	0.0809

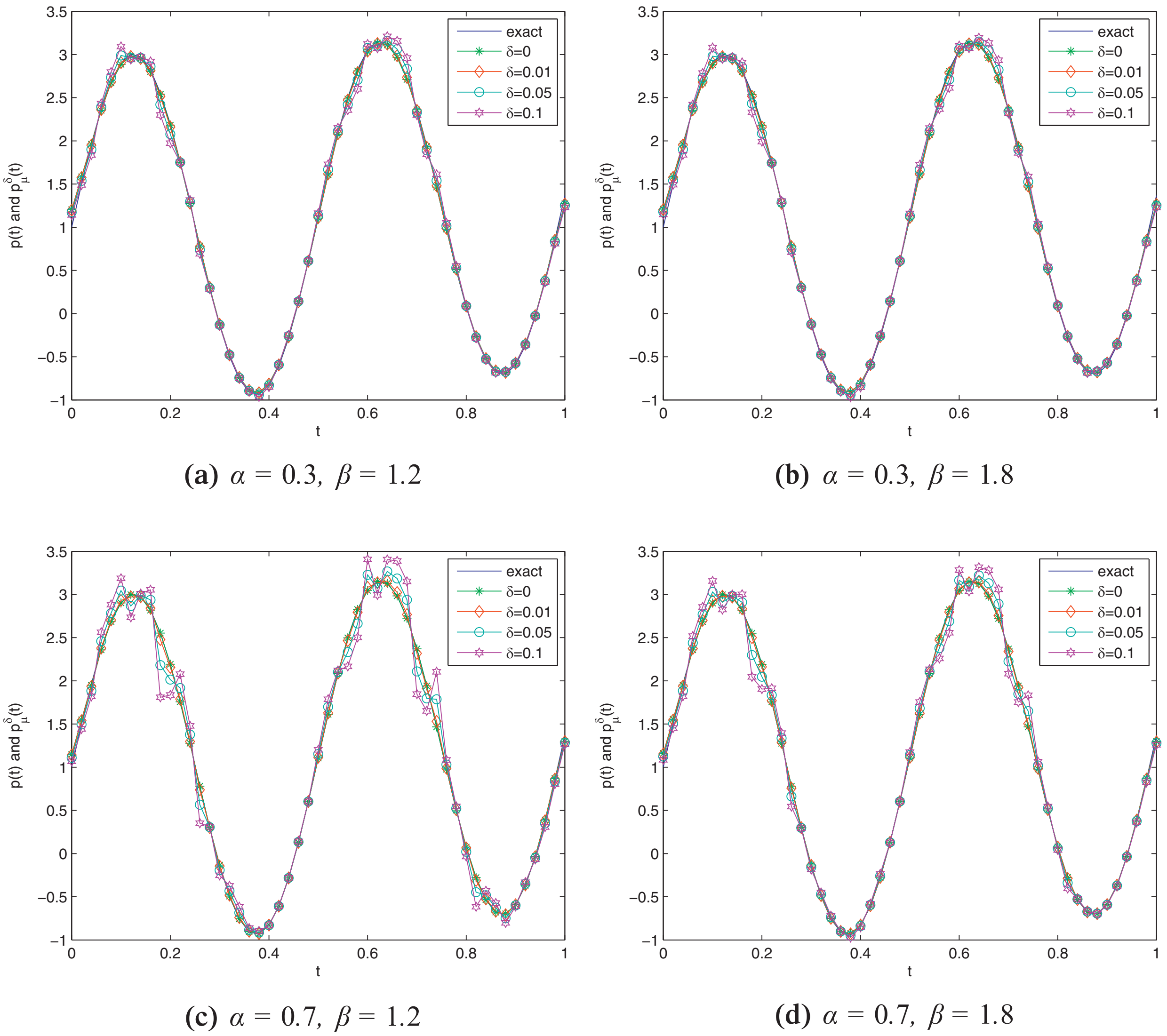
6.1. One-dimensional case

Without loss of generality, the space domain Ω is taken as (0,1) in one-dimensional case. Numerical experiments for four examples are investigated in follows.

Example 1. Let the initial data $\phi(x) = sin(\pi x)$ and take a source function $f(x) = 1,\ p(t) = t^2$, we take $x_0 = 0.5$.

Numerical results for various α and β with various noise levels $\delta = 0, 1\%, 5\%, 10\%$ are presented in Fig 1, It is seen that the numerical approximation match the exact $p(t)$ quit well and nearby the time ends $t = 1$, numerical accuracy become a little bad when relative noise levels $\delta = 10\%$. It is shown that the proposed boundary integral equation method combined with the first order Tikhonov regularization is effective and robust for identifying the source term.

In Table 1, we display the relative root mean square errors $\varepsilon(p)$ for Example 1 with various α in which we fixed a relative noisy level $\delta = 10\%$ and $\beta = 1.2$. It can be seen that the smaller the α is, the better the numerical accuracy is. In Table 2, we



(a) $\alpha = 0.3,\ \beta = 1.2$

(b) $\alpha = 0.3,\ \beta = 1.8$

(c) $\alpha = 0.7,\ \beta = 1.2$

(d) $\alpha = 0.7,\ \beta = 1.8$

Fig. 2. The numerical results for Example 2 for various noise levels.

Table 2
The relative root mean square errors of Example 1 for various values of β with fixed $\alpha = 0.3,\ \delta = 10\%$.

β	1.1	1.3	1.5	1.7	1.9
$\varepsilon(p)$	0.0379	0.0368	0.0359	0.0351	0.0428

display the relative root mean square errors $\varepsilon(p)$ given by (6.1) for Example 1 with various β in which we fixed a relative noisy level $\delta = 10\%$ and $\alpha = 0.3$. It can be seen that the relative root mean square error $\varepsilon(p)$ has a smaller change with respect to the various β .

In Table 3, we display the relative root mean square error $\varepsilon(p)$ for Example 1 with various x_0 in which we fixed a relative noisy level $\delta = 10\%$ and $\alpha = 0.3,\ \beta = 1.2$. We can see that the results depend slightly on x_0. In the following examples we always fixed $x_0 = 0.5$.

Example 2. Let the initial data $\phi(x) = x^2(1-x)^2$ and take a source function $f(x) = e^{-x}$ and $p(t) = 2\sin(4\pi t) + \exp(-t) + t$.

In Fig. 2, we show the numerical results for $\alpha = 0.3, 0.7$ and $\beta = 1.2, 1.8$ with various noise levels $\delta = 0, 1\%, 5\%, 10\%$. It can be see than numerical solutions very good even for a very high noise level 10% when $\alpha = 0.3$, but when $\alpha = 0.7$, the

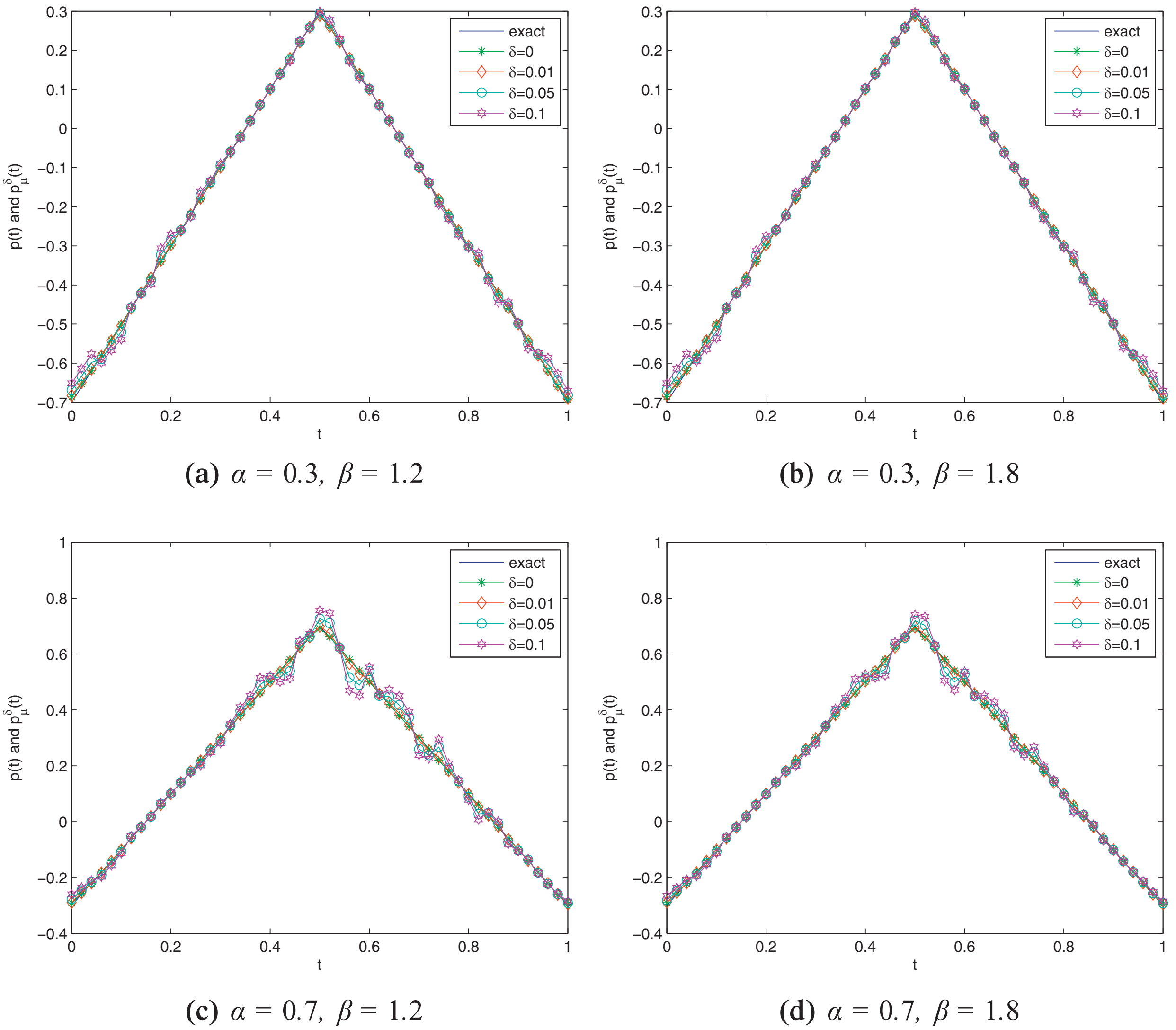


Fig. 3. The numerical results for Example 3 for various noise levels .

Table 3
The relative root mean square errors of Example 1 for various x_0 with fixed $\beta = 1.2,\ \alpha = 0.3,\ \delta = 10\%$.

x_0	0.1	0.3	0.5	0.7	0.9
$\varepsilon(p)$	0.0357	0.0370	0.0373	0.0370	0.0357

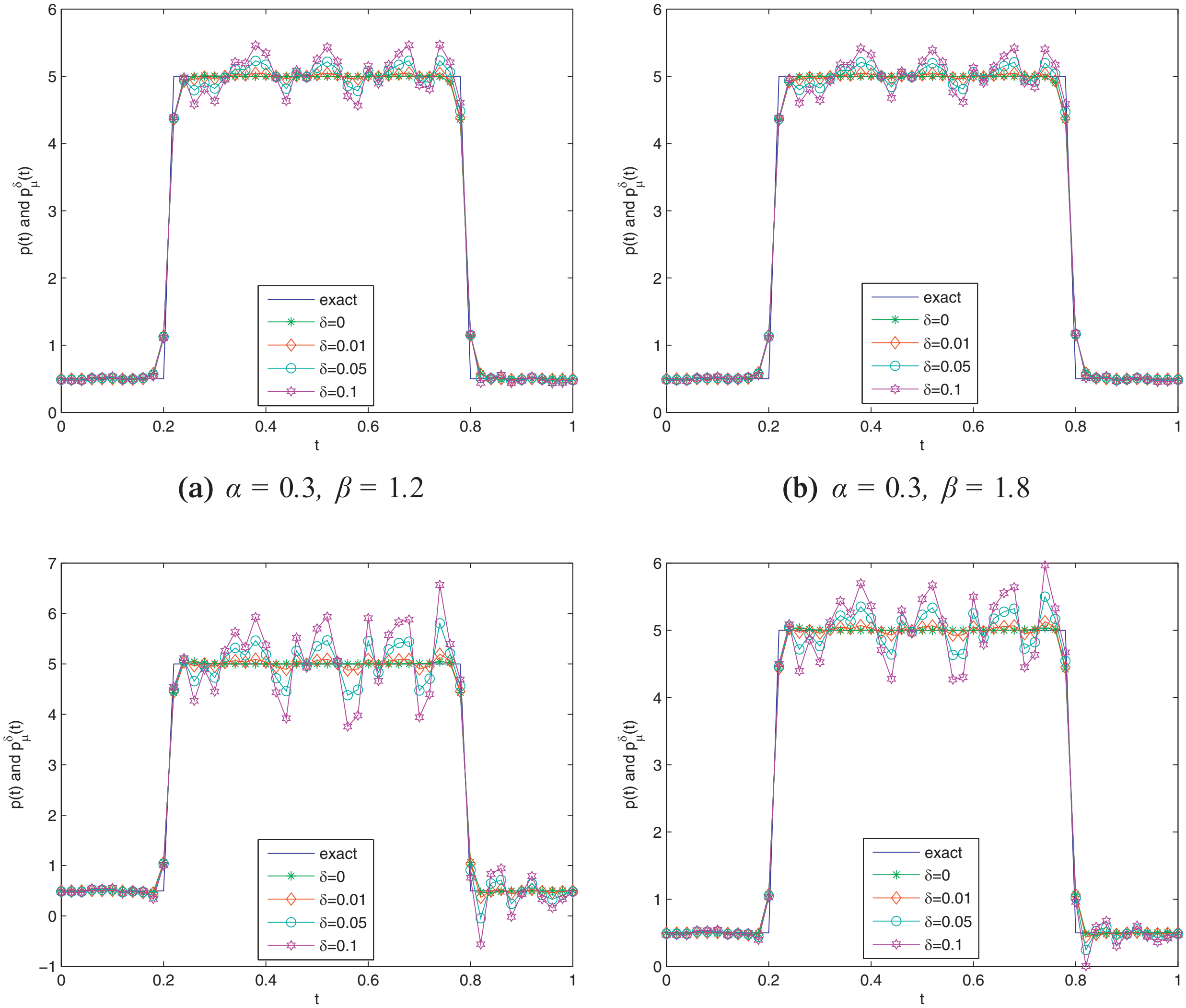
numerical results are less accurate. The numerical results become less accurate if the fractional orders α and the noise level increase.

Example 3. We test a nonsmooth example with a cusp. Let $p(t) = -|2t-1| + \alpha$ and the initial data $\phi(x) = sin(\pi x)$. Take a source function $f(x) = x^2(1-x)^2 + \cos(\pi x) + 1$.

The estimated time-dependent source term $p(t)$ with different values of relative noise levels δ are shown in Fig. 3. It is shown that the calculated $p^{\delta}_{\mu}(t)$ match the exact one quite well everywhere except the time fractional order $\alpha = 0.7$ with the noise level increase.

Example 4. We consider a discontinuous example . The source term is

$$p(t) = \begin{cases} 5, & t \in [0.2, 0.8]; \\ 0.5, & t \in [0, 0.2) \bigcup (0.8, 1]. \end{cases} \tag{6.2}$$



(a) $\alpha = 0.3,\ \beta = 1.2$ (b) $\alpha = 0.3,\ \beta = 1.8$

(c) $\alpha = 0.7,\ \beta = 1.2$ (d) $\alpha = 0.7,\ \beta = 1.8$

Fig. 4. The numerical results for Example 4 for various noise levels .

Let the initial data be $\phi(x) = x^2(1-x)^2$ and take a source function $f(x) = \sin(\pi x) + 1$.

The numerical results for $\alpha = 0.3, 0.7$ and $\beta = 1.2, 1.8$ with various noise levels are shown in Fig. 4. It can be see than our proposed method is also effective for solving the discontinuous example.

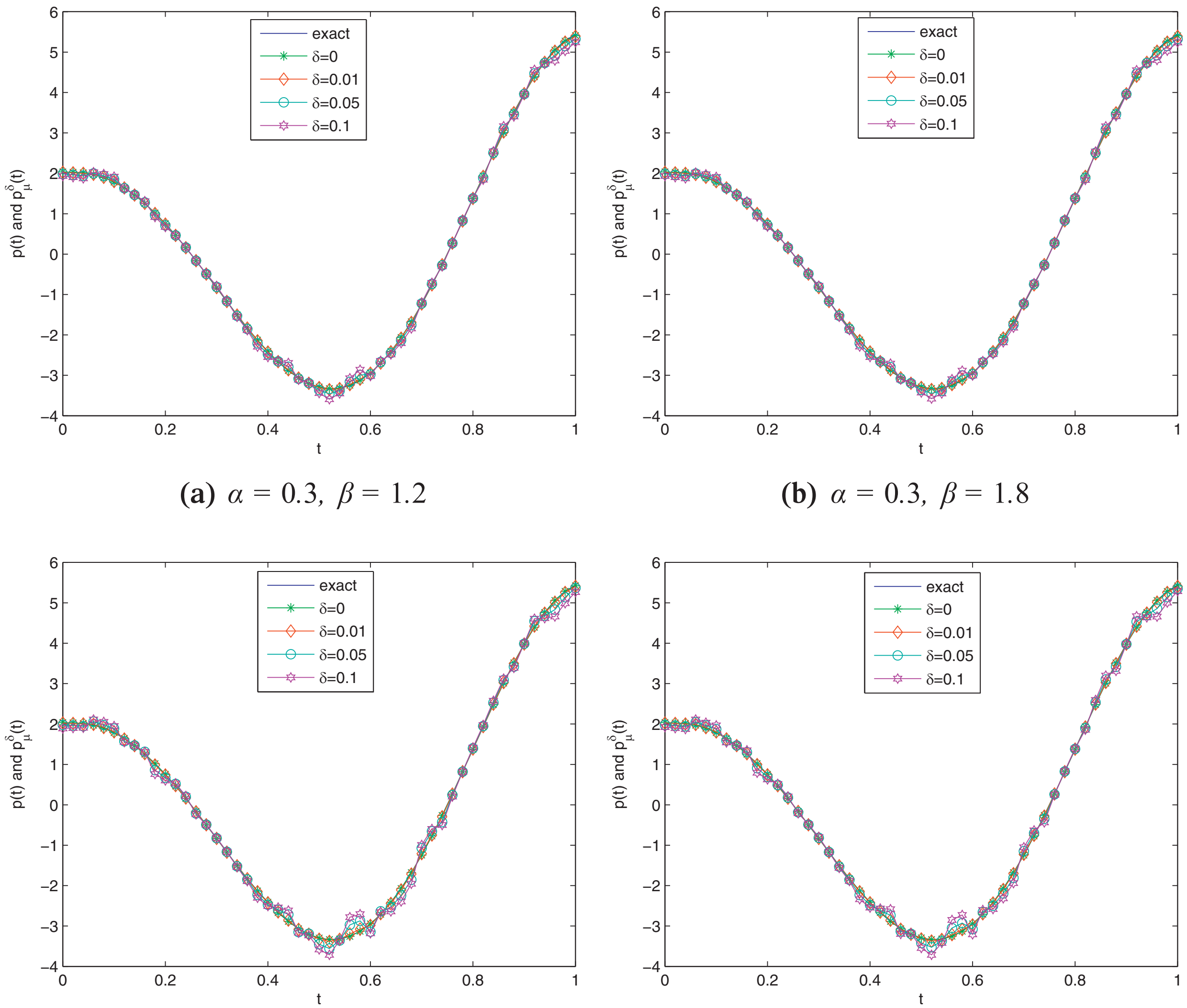
6.2. Two-dimensional case

Denote the space coordinates as (x, y). The space domain Ω is taken as $(0, 1) \times (0, 1)$ in two-dimensional case. We present the numerical results for two examples to show the accuracy and stability of our proposed method.

Example 5. Let the initial data $\phi(x, y) = \sin(\pi x)\sin(\pi y)$ and take a source function $f(x, y) = e^{-(x+y)}$, $p(t) = 2e^{-t}\cos(2\pi t)$, we take $x_0 = (0.5, 0.5)$.

Example 6. We consider a discontinuous time-dependent source term.

$$p(t) = \begin{cases} 1, & t \in [0, 0.25); \\ 0.75, & t \in [0.25, 0.5). \\ 0.5, & t \in [0.5, 0.75); \\ 0.25, & t \in [0.75, 1]). \end{cases}$$



(a) $\alpha = 0.3,\ \beta = 1.2$ **(b)** $\alpha = 0.3,\ \beta = 1.8$

(c) $\alpha = 0.7,\ \beta = 1.2$ **(d)** $\alpha = 0.7,\ \beta = 1.8$

Fig. 5. The numerical results for Example 5 for various noise levels .

Let the initial data $\phi(x, y) = x(1-x)y(1-y)$ and take a source function $f(x, y) = sin(\pi x)sin(\pi y)$, we take $x_0 = (0.5, 0.5)$.

The numerical results for Examples 5 and6 of various noise levels in the case of $\alpha = 0.3, 0.7$ and $\beta = 1.2, 1.8$ are shown in Figs. 5 and 6, respectively. It can be seen that our proposed method is also effective for solving the two-dimensional examples.

From the numerical experiments for Examples 1–6, it can be observed that the results at $\alpha = 0.3$ are better than those at $\alpha = 0.7$. In fact we note that numerical reconstruction becomes more difficult as the value of α increases. One reason may be that if α increases, then the kernel function in the Volterra integral Eq. (5.1) has less singularity which indicates that the ill-posedness of Eq. (5.1) is somewhat higher.

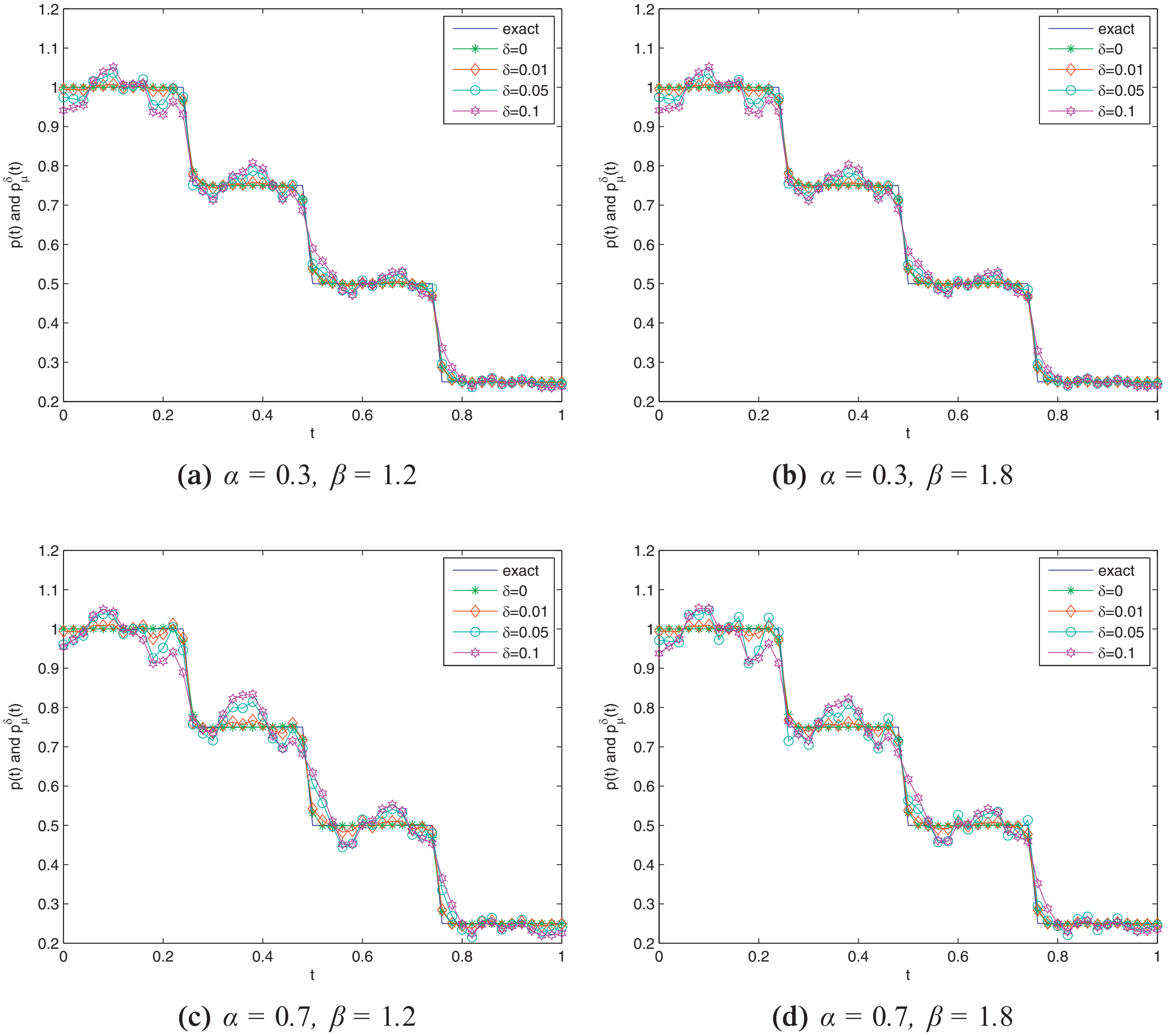


Fig. 6. The numerical results for Example 6 for various noise levels .

7. Conclusions

In this paper, we investigate an inverse time-dependent source problem for a time–space fractional diffusion equation. The existence and uniqueness for the direct problem and the uniqueness for inverse problem are both proved. We combine the boundary element method and the first order Tikhonov regularization to reconstruction time-dependent source term from the additional data. Numerical examples indicate that the proposed method is effective and stable.

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