Information Technology Engineering

FH2021

Engineering Mathematics II

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Syllabus Content

Differential Equations of First Order and First Degree
1.1 Exact differential Equations, Equations reducible to exact form by using integrating factors.
1.2 Linear differential equations (Review), equation reducible to linear form, Bernoulli's equation.

 # Self learning topics: Simple application of differential equation of first order and first degree to electrical and Mechanical Engineering problem

Course Outcome

CO1: Identify and solve the differential equation of first order and first degree, exact and linear differential equation

1. Higher Engineering Mathematics, Dr.B.S.Grewal, Khanna Publication
2. Advanced Engineering Mathematics, Erwin Kreyszig, Wiley EasternLimited, 9th Ed.
3. Engineering Mathematics by Srimanta Pal and SubodhBhunia, Oxford University Press
4. Applied Numerical Methods with MATLABfor Engineers and Scientists by Steven Chapra, McGraw Hill
5. Elementary Linear Algebra with Application by Howard Anton and Christ Rorres. 6th edition.
6. John Wiley & Sons,INC.

(1) A differential equation *is an equation which involves differential coefficients or differentials.*

Thus (*i*) $e^x\,dx + e^y\,dy = 0$ (*ii*) $\left(\frac{d^2x}{dt^2}\right)' + n^2x = 0$

(*iii*) $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$ (*iv*) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2} = c$

(*v*) $\frac{dx}{dt} - wy = a\cos pt,\ \frac{dy}{dt} + wx = a\sin pt$ (*vi*) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

(*vii*) $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ **are all examples of differential equations.**

(2) *An* **ordinary differential equation** *is that in which all the differential coefficients have reference to a single independent variable.* Thus the equations (i) to (v) are all ordinary differential equations.

A **partial differential equation** *is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.* Thus the equations (vi) and (vii) are partial differential equations.

(3) *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

Thus, from the examples above,

(i) is of the first order and first degree ; (ii) is of the second order and first degree ;

(iii) written as $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (iv) written as $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

(1) Def. *A differential equation of the form* **M**(x, y) $\boldsymbol{dx}$ + **N** (x, y) $\boldsymbol{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y \text{ cons.})} M\,dx + \int (\text{terms of N not containing x})\,dy = c$$

provided $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

$\int M dx$

$N = \cancel{x^2} + \cancel{y\cos x} + y^2$

$\int (N)_{no\,x}\, dy = \int y^2 dy = \frac{y^3}{3} + c.$

Example 01: Solve $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

$Mdx + Ndy = 0$

$\rightarrow M = y^2e^{xy^2}+4x^3$ $\qquad N = 2xy\,e^{xy^2}-3y^2$

$\frac{\partial M}{\partial y} = y^2\cdot e^{xy^2}\cdot(2xy) + e^{xy^2}\cdot(2y) + 0 = 2xy^3e^{xy^2}+2ye^{xy^2}$

$\frac{\partial N}{\partial x} = 2y\left[x\cdot e^{xy^2}(y^2) + e^{xy^2}\cdot(1)\right] + 0 = 2xy^3e^{xy^2} + 2ye^{xy^2}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ d'eqn is exact

$\int M dx = \int\left(y^2e^{xy^2}+4x^3\right)dx$

$= y^2\left[\frac{e^{xy^2}}{y^2}\right] + 4\cdot\frac{x^4}{4}$

$= e^{xy^2}+x^4 + c_1$

$\int$[Terms in N free from x] dy

$= \int -3y^2\,dy$

$= -3\cdot\frac{y^3}{3}$

$= -y^3 + c_2$

$\therefore$ Soln is

$\int M dx + \int N dy = \boxed{c}$

$\boxed{e^{xy^2}+x^4-y^3 = c}$

$+c_1+c_2$

$= c - c_1 - c_2$

Example 02: Solve $\left\{y\left(1+\frac{1}{x}\right)+cosy\right\}dx+(x+logx-xsiny)dy=0$ $\equiv Mdx + Ndy = 0$

$\rightarrow$ $M = y + \frac{y}{x} + \cos y$ $\qquad$ $N = x + \log x - x\sin y$

$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y \quad \longleftrightarrow \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$ ✓

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ d' eqn is exact.

$\int M \cdot dx = \int \left(y + \frac{y}{x} + \cos y\right) dx$

$= xy + y \cdot \log x + x\cos y$

$\int$(Terms in N free from x) dy

$= c_1$

$\therefore$ soln is

$$\boxed{xy + y\log x + x\cos y = c}$$

Example 03: Solve $(1 + 2xycosx^2 - 2xy)dx + (sinx^2 - x^2)dy = 0$

$\Rightarrow \quad M = 1 + 2xy\cos x^2 - 2xy \qquad N = \sin x^2 - x^2$

$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x \qquad \frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact

$\int M \cdot dx + \int [\text{Terms of } y \text{ in } N]\, dy = c$

$\int (1 + 2xy\cos x^2 - 2xy)\, dx + \int 0\, dy = c$

$x + y \cdot \sin x^2 - x^2 y + c_1 = c'$

$\therefore \boxed{x + y\sin x^2 - x^2 y = c}$

$\int 2x \cdot \cos x^2\, dx \quad , \quad t = x^2$

$\int \cos t\, dt$

$= \sin t$

$= \sin x^2$

Example 04: Solve $\frac{dy}{dx} + \frac{ycosx+siny+y}{sinx+xcosy+x} = 0$

$Mdx+Ndy=0$

$\rightarrow \quad (\sin x + x\cos y + x)dy + (y\cos x + \sin y + y)dx = 0$

$M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$

$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ or $M_y = N_x \Rightarrow$ exact

$\int M dx + \int [\text{Terms of } y \text{ in } N] dy = c$

$\int (y\cos x + \sin y + y) dx + \int 0 \, dy = c'$

$y \cdot \sin x + x\sin y + xy + c_1 = c'$

$\therefore \quad \boxed{y\sin x + x\sin y + xy = c}$

Non Exact Differential Equation : The concept of Integrating Factor

Sometimes the DE is not an exact DE, but it can be reduced to exact by multiplying a term known as Integrating Factor

(a) if $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ be a function of x only $= f(x)$ say, then $e^{\int f(x)dx}$ is an integrating factor.

(b) if $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ be a function of y only $= F(y)$ say, then $e^{\int F(y)dy}$ is an integrating factor.

Non Exact Differential Equation : The concept of Integrating Factor

c) *I.F. of homogeneous: If* $Mdx + Ndy = 0$ *is homogeneous equation in x and y then* $\frac{1}{Mx+Ny}$ *is an IF, provided that* $Mx + Ny \neq 0$.

d) *I.F. for an equation of the type* $f_1(x,y)ydx + f_2(x,y)xdy = 0$:

If $Mdx + Ndy = 0$ *is of the above form, then* $\frac{1}{Mx-Ny}$ *is an IF, provided that* $Mx - Ny \neq 0$.

Example 05: Solve $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$ ① $= Mdx + Ndy$

Type I: If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = f(x)$

$M = xy^2 - e^{1/x^3}$ $\qquad N = -x^2y$

$\frac{\partial M}{\partial y} = 2xy$ $\qquad \frac{\partial N}{\partial x} = -2xy$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ d' eqⁿ is non exact,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2y} = \frac{4xy}{-x^2y} = \boxed{-\frac{4}{x}} = f(x)$$

$\therefore$ I.F = $e^{\int f(x)\cdot dx} = e^{\int -\frac{4}{x}\cdot dx} = e^{-4\cdot\log x}$

integrating factor

$= e^{\log x^{-4}}$

$= x^{-4}$

$= \frac{1}{x^4}$

Multiply ① by I.F

$\left[\frac{xy^2 - e^{1/x^3}}{x^4}\right]dx - \frac{x^2y}{x^4}dy = 0$ is exact

$M = \frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}$; $N = -\frac{y}{x^2}$

$\int x^{-3} = \frac{x^{-3+1}}{-2}$

$\int Mdx = \int \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx$

$= y^2\left[\frac{x^{-2}}{-2}\right] - \left[-\frac{1}{3}e^{1/x^3}\right]$

$= -\frac{y^2}{2x^2} + \frac{e^{1/x^3}}{3}$ —②

$\frac{1}{x^3} = t = x^{-3}$

$-3x^{-4} = dt$

$-\frac{3}{x^4} = dt$

$\frac{1}{x^4} = -\frac{dt}{3}$

$\int -e^t \cdot \frac{dt}{3}$

$= -\frac{1}{3}e^t = -\frac{1}{3}e^{1/x^3}$

$\int Ndy = c'$ —③

soln is ② + ③ = c

$\therefore \boxed{\frac{-y^2}{2x^2} + \frac{e^{1/x^3}}{3} = c}$

Example 06: Solve $y(2x^2 - xy + 1)dx + (x - y)dy = 0$ —(1)

Type I: If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = f(x)$

$M = 2x^2y - xy^2 + y$ $\qquad N = x - y$

$\frac{\partial M}{\partial y} = 2x^2 - 2xy + 1$ $\qquad \frac{\partial N}{\partial x} = 1$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x^2 - 2xy + 1 - 1}{x - y} = \frac{2x(x-y)}{x-y} = 2x = f(x)$$

$\therefore$ I.F $= e^{\int f(x)dx} = e^{\int 2x\,dx} = e^{x^2}$

multiply (1) by I.F.

$(2yx^2 - xy^2 + y)e^{x^2}dx + (x-y)e^{x^2}dy = 0$ is exact.

$\int M\,dx = 2y\int x^2 e^{x^2}dx - y^2\int x\cdot e^{x^2}dx + y\int e^{x^2}dx$

$= 2y\cdot[$

$x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$

$I_1 = \int t\cdot e^t\cdot\frac{dt}{2\sqrt{t}}$

$= \frac{1}{2}\int t^{1/2}\cdot e^t\cdot dt\}$

Example 07: $(x^2 + y^2 + 1)dx - 2xydy = 0$ —(1)

Type I: If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = f(x)$

$\rightarrow$ $M = x^2+y^2+1$ $\qquad N = -2xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$

$\therefore \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2y-(-2y)}{-2xy} = \dfrac{4y}{-2xy} = \dfrac{-2}{x} = f(x)$

$\therefore$ I.F. $= e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$

$\therefore$ multiply (1) by I.F.

$\left[\frac{x^2+y^2+1}{x^2}\right]dx - \frac{2xy}{x^2}dy = 0$ is exact.

$\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \boxed{\frac{2y}{x}}dy = 0$

$\therefore \int M dx = \int\left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right]dx$

$= x - \frac{y^2}{x} - \frac{1}{x}$

$\int$[Term of y in N] $dx = \int 0\, dx = c_1$

$\therefore$ soln is

$$\boxed{x - \frac{y^2}{x^2} - \frac{1}{x} = C}$$

Example 08: Solve $(2xlogx - xy)dy + 2ydx = 0$

Type I: If $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/N = f(x)$

$M = 2y \qquad N = 2x\log x - xy = x(2\log x - y)$

$\frac{\partial M}{\partial y} = 2 \quad ; \quad \frac{\partial N}{\partial x} = 2\left[x \cdot \frac{1}{x} + \log x\right] - y$

$= 2 + 2\log x - y$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x) \text{ only}$$

$\therefore$ I.F $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ multiply ① by I.F.

$\frac{2y}{x}dx + \left(\frac{2x\log x - xy}{x}\right)dy = 0$

$dx\left(\frac{2y}{x}\right) + (2\log x - y)dy = 0$ is exact

$\int M\,dx = \int \frac{2y}{x}dx = 2y \cdot \log x$

$\int$(Term in N free from x) dy

$= \int -y\,dy$

$= -\frac{y^2}{2}$

$\therefore$ soln is

$$\boxed{2y\log x - \frac{y^2}{2} = c}$$

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