DUIS

Differentiation Under Integral Sign

$$If\ f(x,\alpha) is\ continuous\ and\ continuously\ differentiable\ throughout\ the\ interval\ [a,b], where\ a\ and\ b\ are\ constants$$

$$Let\ I = \int_a^b f(x,\alpha)dx,$$

$$then\ by\ DUIS$$ rule

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}\,dx$$

Working rule

- Find the parameter and variable x
- Differentiate I *w.r.t.* parameter treating variable x constant
- Complete the partial differentiation under integral *w.r.t.* parameter
- Solve the integral *w.r.t.* variable [now parameter is constant]
- Integrate *w.r.t.* parameter [now x is constant]

parameter

$$I = \int_0^1 \log(1+\alpha)\,dx \quad \text{variable}$$

$$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial}{\partial \alpha}\left[\log(1+\alpha)\right]dx$$

$$= \int_0^1 \frac{1}{1+\alpha}\,dx$$

$$= \frac{1}{1+\alpha}\cdot [x]_0^1$$

$$\frac{dI}{d\alpha} = \frac{1}{1+\alpha}$$

integrate w.r.t. α.

$$I = \log(1+\alpha) + c$$

Example 01 : Prove that $\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1+\alpha), \alpha \geq 0$

→ x: variable ; α: parameter

$$\boxed{I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} \cdot dx} \quad \text{—(1)}$$

diff. w.r.t. α, by DUIS rule

$$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial}{\partial\alpha}\left[\frac{x^\alpha - 1}{\log x}\right] dx \quad \{x \text{ is constant}\}$$

$$= \int_0^1 \frac{1}{\cancel{\log x}} \cdot \{x^\alpha \cdot \cancel{\log x}\} \cdot dx$$

$$= \int_0^1 x^\alpha \cdot dx \quad \{\text{Now } x \text{ is variable}\}$$

$$= \left.\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1} - 0$$

$$\left\{\because \frac{\partial}{\partial x}(2^x) = 2^x \cdot \log 2\right\}$$

$$\frac{dI}{d\alpha} = \frac{1}{\alpha+1}$$

Integrating w.r.t. α $\{\alpha$ is variable$\}$

$$I(\alpha) = \int \frac{1}{\alpha+1} d\alpha$$

$$\boxed{I(\alpha) = \log(\alpha+1) + C} \quad \text{—(2)}$$

put $\alpha = 0$.

$$(2) \Rightarrow \quad I(0) = \log(1) + c$$

$$0 = 0 + c \Rightarrow c = 0$$

$$(2) \Rightarrow \quad \boxed{I = \log(\alpha+1)}$$

Example 02 : Prove that $\int_0^1 \frac{x^\alpha - x^\beta}{\log x} dx = \log\left[\frac{(1+\alpha)}{1+\beta}\right]$

→ Let α be the parameter, x variable, β : constant

$$I(\alpha) = \int_0^1 \frac{x^\alpha - x^\beta}{\log x} \cdot dx \quad -① $$

diff. w.r.t. α.

$$\frac{dI}{d\alpha} = \int_0^1 \frac{1}{\log x} \cdot \frac{\partial}{\partial \alpha}\left[x^\alpha - x^\beta\right] dx$$

$$= \int_0^1 \frac{1}{\log x} \cdot \left[x^\alpha \cdot \log x - 0\right] dx$$

$$= \int_0^1 x^\alpha \cdot dx$$

$$= \left.\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

$$\frac{dI}{d\alpha} = \frac{1}{\alpha+1}$$

Int. w.r.t. α

$$I(\alpha) = \log(\alpha+1) + C \quad -②$$

To find 'c';

$\boxed{\text{put } \alpha = \beta}$

$$② \Rightarrow I(\beta) = \log(\beta+1) + C$$

from eqn ①

$$0 = \log(1+\beta) + c$$

$$\therefore c = -\log(1+\beta)$$

$$② \Rightarrow$$

$$I = \log(\alpha+1) - \log(\beta+1)$$

$$= \log\left(\frac{\alpha+1}{\beta+1}\right)$$

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Example 03 : Evaluate $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-a}\ )dx, \quad a>-1$

$\to$ variable x ; parameter a.

$$I(a) = \int_0^\infty \frac{e^{-x}}{x}\left[1-e^{-ax}\right]dx \quad -(1)$$

diff. w.r.t. a

$$\frac{dI}{da} = \int_0^\infty \frac{e^{-x}}{x}\cdot\frac{\partial}{\partial a}\left[1-e^{-ax}\right]dx$$

$$= \int_0^\infty \frac{e^{-x}}{\cancel{x}}\cdot\left[0-e^{-ax}(-\cancel{x})\right]$$

$$= \int_0^\infty e^{-x-ax}\cdot dx$$

$$= \int_0^\infty e^{-(a+1)x}\,dx$$

$$\frac{dI}{da} = \left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^\infty = 0 + \frac{1}{a+1}$$

Integrate w.r.t. a

$$I(a) = \int \frac{1}{a+1}\cdot da$$

$$I(a) = \log(a+1) + c \quad -(2)$$

To find 'c', put $a=0$

$$I(0) = \log(1) + c \Rightarrow \boxed{c=0}$$

$\because I(0)=0$ by (1)

(2) $\Rightarrow$

$$\boxed{I = \log(a+1)}$$

$$\left\{\ \int_0^\infty e^{(a-1)x}\cdot dx = \left[\frac{e^{(a-1)x}}{a-1}\right]_0^\infty = e^{\infty} - \frac{1}{a-1} = \infty\right.$$

$$\int_0^\infty e^{-(1-a)x}\cdot dx = \left[\frac{e^{-(1-a)x}}{-(1-a)}\right]_0^\infty = e^{-\infty} + \frac{1}{1-a} = \frac{1}{1-a}$$

Example 04 : Prove that $\int_0^\infty e^{-ax}\cdot\frac{\sin mx}{x}dx = \tan^{-1}\frac{m}{a}$, ($a$ is a parameter)

$d(e^{2x}) = e^{2x}\cdot\frac{d}{dx}(2x) = 2\cdot e^{2x}$

$\rightarrow$ a: parameter, x: variable, m = constant

$$\left\{\int \frac{a\,dx}{x^2+a^2} = \tan^{-1}\left(\frac{x}{a}\right)\right\}$$

$$I(a) = \int_0^\infty e^{-ax}\cdot\frac{\sin mx}{x}\,dx \quad \text{--- (1)}$$

$$\frac{dI}{da} = \int_0^\infty \frac{\sin mx}{x}\cdot\frac{\partial}{\partial a}\left[e^{-ax}\right]dx$$

$$= \int_0^\infty \frac{\sin mx}{\cancel{x}}\left[e^{-ax}\cdot(-\cancel{x})\right]dx$$

$$= -\int_0^\infty e^{-ax}\cdot\sin mx\cdot dx$$

$$\left\{\begin{aligned}\int e^{ax}\cdot\sin bx\cdot dx &= \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\\ \int e^{ax}\cdot\cos bx\cdot dx &= \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]\end{aligned}\right.$$

$$= -\left[\frac{e^{-ax}}{a^2+m^2}\left(-a\sin mx - m\cos mx\right)\right]_0^\infty$$

$e^{-\infty}$

$$= -\left[[0] - \left[\frac{1}{a^2+m^2}(0-m)\right]\right]$$

$$\frac{dI}{da} = \frac{-m}{a^2+m^2}$$

Int. w.r.t. a.

$$\boxed{I(a) = -\tan^{-1}\left(\frac{a}{m}\right) + c} \quad \text{--- (2)}$$

To find c, $\boxed{\text{put } a = 0/\infty}$

$$(1) \Rightarrow I(0) = \int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$$

$(2) \Rightarrow$

$$\frac{\pi}{2} = -\tan^{-1}(0) + c$$

$$\Rightarrow c = \pi/2$$

$(2) \Rightarrow$

$$I = \frac{\pi}{2} - \tan^{-1}\left(\frac{a}{m}\right)$$

$$= \cot^{-1}\left(\frac{a}{m}\right)$$

$$I = \tan^{-1}\left(\frac{m}{a}\right) \quad \#$$

Example 05 : Prove that $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cdot sinmx\, dx = \tan^{-1}\frac{m}{a} - \tan^{-1}\frac{m}{b}$

$\left\{ \int \frac{a}{x^2+a^2} dx = \tan^{-1}(x/a) \right.$

→ Let a be a parameter.

$$\boxed{I(a) = \int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) \cdot \sin mx \cdot dx} \quad \text{--- (1)}$$

$$\frac{dI}{da} = \int_0^\infty \frac{\sin mx}{x} \frac{\partial}{\partial a}\left[e^{-ax} - e^{-bx}\right] dx$$

$$= \int_0^\infty \frac{\sin mx}{\cancel{x}} \cdot \left[e^{-ax} \cdot (-\cancel{x})\right]$$

$$= -\int_0^\infty e^{-ax} \cdot \sin mx \cdot dx$$

$\boxed{a = -a}$ $\boxed{b = m}$

$$= -\left[\frac{e^{-ax}}{a^2+m^2}\left[-a \sin mx - m \cos mx\right]\right]_0^\infty$$

$$= -\left[[0] - \left[\frac{1}{a^2+m^2}(0-m)\right]\right]$$

$$\frac{dI}{da} = \frac{-m}{m^2+a^2}$$

Int. w.r.t. a

$$I(a) = -\tan^{-1}\left(\frac{a}{m}\right) + c \quad \text{--- (2)}$$

To find c, put $a = b$

(2) ⇒ $I(b) = -\tan^{-1}\left(\frac{b}{m}\right) + c$

But $I(b) = 0$ from (1)

$\therefore c = \tan^{-1}(b/m)$

(2) ⇒ $\boxed{I = \tan^{-1}\left(\frac{b}{m}\right) - \tan^{-1}\left(\frac{a}{m}\right)}$

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Example 06 : Show that $\int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx = \pi\sqrt{a}$,

$\rightarrow$ Let a be a parameter.

$\therefore \boxed{I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx}$ — (1)

diff. w.r.t. a, & by DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{1}{x^2}\cdot\frac{\partial}{\partial a}\left[\log(1+ax^2)\right]dx$$

$$= \int_0^\infty \frac{1}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{1+ax^2}\cdot dx$$

$$= \int_0^\infty \frac{1}{1+ax^2}\cdot dx$$

$$= \frac{1}{a}\int_0^\infty \frac{1}{\frac{1}{a}+x^2}\cdot dx$$

$$= \frac{1}{a}\int_0^\infty \frac{1}{\left(\sqrt{\frac{1}{a}}\right)^2+x^2}\cdot dx$$

$$= \frac{1}{a}\cdot\frac{1}{1/\sqrt{a}}\left[\tan^{-1}\left(\frac{x}{1/\sqrt{a}}\right)\right]_0^\infty$$

$$= \frac{1}{\sqrt{a}}\left[\frac{\pi}{2}-0\right]$$

$$\therefore \frac{dI}{da} = \frac{\pi}{2\sqrt{a}}$$

Int. w.r.t. a

$$I(a) = \pi\int\frac{1}{2\sqrt{a}}\cdot da$$

$$I(a) = \pi\sqrt{a} + c \quad -(2)$$

$$\int\frac{1}{x^2+a^2}\cdot dx = \frac{1}{a}\cdot\tan^{-1}\frac{x}{a}$$

$\int$

To find c, put $a=0$ in (2)

$I(0) = 0 + c$

$0 = 0 + c \Rightarrow c = 0$

$\therefore$ (2) $\Rightarrow$ $\boxed{I = \pi\sqrt{a}}$

Example 07 : Show that $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a\,,\ (0 \le a \le 1)$

$$\left\{ \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}(x/a) \right.$$

$\rightarrow$ $I(a) = \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\, dx.$ — (1)

diff. I w.r.t. a, by DUIS rule

$$\frac{dI}{da} = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{\partial}{\partial a}\left[\log(1+a\cos x)\right] dx$$

$$= \int_0^{\pi} \frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1+a\cos x}\, dx$$

$$= \int_0^{\pi} \frac{dx}{1+a\cos x}$$

put $\tan\frac{x}{2} = t$, $dx = \frac{2dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

x	0	π
t	0	∞

$$\frac{dI}{da} = \int_0^{\infty} \frac{2dt}{\left[1+a\left(\frac{1-t^2}{1+t^2}\right)\right](1+t^2)}$$

$$= 2\int_0^{\infty} \frac{dt}{(1+t^2)+a(1-t^2)}$$

$$= 2\int_0^{\infty} \frac{dt}{(1+a)+(1-a)t^2}$$

$$= \frac{2}{1-a}\int_0^{\infty} \frac{dt}{\left(\frac{1+a}{1-a}\right)+t^2}$$

$$= \frac{2}{1-a}\int_0^{\infty} \frac{dt}{t^2+\left(\sqrt{\frac{1+a}{1-a}}\right)^2}$$

$$= \frac{2}{1-a} \cdot \frac{1}{\sqrt{\frac{1+a}{1-a}}}\left[\tan^{-1}\left(\frac{t}{\sqrt{\frac{1+a}{1-a}}}\right)\right]_0^{\infty}$$

$$= \frac{\cancel{2}}{\sqrt{1-a}\cdot\sqrt{1+a}}\left[\frac{\pi}{\cancel{2}} - 0\right]$$

$$= \frac{\pi}{\sqrt{1-a^2}} = \frac{dI}{da}$$

Int. w.r.t. a.

$$I(a) = \pi \int \frac{1}{\sqrt{1-a^2}}\, da \qquad \left\{ \int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1}x \right\}$$

$$= \pi \cdot \sin^{-1}a + c$$

To find c, put $a=0$ in (1) $\Rightarrow I(0)=0$

$\therefore c=0$ $\boxed{I = \pi\sin^{-1}a}$

Example 08 : Show that $\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$

$$I(\alpha) = \int_0^{\pi/2} \frac{\log(1+\cos\alpha\cdot\cos x)}{\cos x}\cdot dx \quad \text{--- (1)}$$

Diff. I. w.r.t. α

$$\frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{1}{\cancel{\cos x}}\cdot\frac{-\sin\alpha\cdot\cancel{\cos x}}{(1+\cos\alpha\cdot\cos x)}\cdot dx$$

$$= \int_0^{\pi/2} \frac{-\sin\alpha\, dx}{1+\cos\alpha\cdot\cos x}$$

$\tan\left(\frac{x}{2}\right) = t$, $dx = \frac{2dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

x	0	$\pi/2$
t	0	1

$$\frac{dI}{d\alpha} = \int_0^1 \frac{-2\sin\alpha\cdot dt}{\left[1+\cos\alpha\cdot\left(\frac{1-t^2}{1+t^2}\right)\right](1+t^2)}$$

$$\frac{dI}{d\alpha} = 2\int_0^1 \frac{-\sin\alpha\cdot dt}{(1+t^2)+\cos\alpha(1-t^2)}$$

$$= -2\sin\alpha\cdot\int_0^1 \frac{dt}{[1+\cos\alpha]+[1-\cos\alpha]t^2}$$

$$= \frac{-2\sin\alpha}{1-\cos\alpha}\cdot\int_0^1 \frac{dt}{\left[\frac{1+\cos\alpha}{1-\cos\alpha}\right]+t^2}$$

$$= \frac{-2\sin\alpha}{1-\cos\alpha}\cdot\int_0^1 \frac{dt}{t^2+\left[\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}\right]^2}$$

$$= \frac{-2\sin\alpha}{1-\cos\alpha}\cdot\frac{1}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}\left[\tan^{-1}\left[\frac{t}{\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}}\right]\right]_0^1$$

$$= \frac{-2\sin\alpha}{\sqrt{1-\cos\alpha}\cdot\sqrt{1+\cos\alpha}}\cdot\left[\tan^{-1}\left(\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\right)-0\right]$$

$$= \frac{-2\sin\alpha}{\sqrt{1-\cos^2\alpha}}\cdot\tan^{-1}\sqrt{\frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}}$$

$$= \frac{-2\cancel{\sin\alpha}}{\cancel{\sin\alpha}}\cdot\tan^{-1}[\tan(\alpha/2)]$$

$$= -2\cdot\frac{\alpha}{2} = \boxed{-\alpha} = \frac{dI}{d\alpha}$$

Int. w.r.t. α

$$I(\alpha) = -\frac{\alpha^2}{2}+c \quad \text{--- (2)}$$

to find c put $\alpha = \pi/2$

(2) $\Rightarrow$

$$I\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{8}+c$$

$$0 = -\frac{\pi^2}{8}+c$$

$$c = \frac{\pi^2}{8} \quad \#$$

Example 08 : Show that $\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos\alpha \cos x)}{\cos x}\, dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$

Example 09 : Prove that $\int_0^\infty \frac{1-\cos mx}{x} \cdot e^{-x}\, dx = \frac{1}{2}\log(m^2+1)$

$\rightarrow$ $I(m) = \int_0^\infty \frac{1-\cos mx}{x} \cdot e^{-x} \cdot dx$ — (1)

$$\frac{dI}{dm} = \int_0^\infty \frac{e^{-x}}{x} \cdot \frac{\partial}{\partial m}[1-\cos mx]\,dx$$

$$= \int_0^\infty \frac{e^{-x}}{\cancel{x}} [\sin mx \cdot (\cancel{x})]\,dx$$

$$= \int_0^\infty e^{-x} \cdot \sin mx \cdot dx$$

$\boxed{a=-1}$ $\boxed{b=m}$

$$\left\{ \frac{e^{ax}}{a^2+b^2} \cdot [a\sin bx - b\cos bx] \right\}$$

$$= \left[\frac{e^{-x}}{1+m^2}[-\sin mx - m\cos mx]\right]_0^\infty$$

$$\frac{dI}{dm} = [0] - \left[\frac{1}{1+m^2}(0-m)\right] = \frac{m}{m^2+1}$$

Integrate w.r.t. m

$$\left\{ \int \frac{x}{x^2+1}\cdot dx = \frac{1}{2}\log(x^2+1) \right.$$

$$I(m) = \int \frac{m}{m^2+1} \cdot dm$$

$$I(m) = \frac{1}{2}\log(m^2+1) + c \quad \text{— (2)}$$

put $m=0$, $\Rightarrow I(0)=0$ from (1)

(2) $\Rightarrow$ $0 = \frac{1}{2}\log(0+1) + c$

$c = 0$

$\therefore$ (2) $\Rightarrow$ $\boxed{I = \frac{1}{2}\log(m^2+1)}$ #

Example 10 : Show that $\int_0^\infty \frac{[\tan^{-1}(\frac{x}{a})-\tan^{-1}(\frac{x}{b})]}{x} = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$, $a > 0, b > a$

$\rightarrow$ Let $I(a) = \int_0^\infty \frac{1}{x}\left[\tan^{-1}\left(\frac{x}{a}\right) - \tan^{-1}\left(\frac{x}{b}\right)\right]dx$ — ①

$$\frac{dI}{da} = \int_0^\infty \frac{1}{x}\cdot\frac{\partial}{\partial a}\left[\tan^{-1}\frac{x}{a} - \tan^{-1}\frac{x}{b}\right]dx$$

$$= \int_0^\infty \frac{1}{x}\cdot\left[\frac{1}{\left[1+\frac{x^2}{a^2}\right]}\frac{\partial}{\partial a}\left(\frac{x}{a}\right) - 0\right]dx$$

$$= \int_0^\infty \frac{1}{\cancel{x}}\cdot\frac{-\cancel{x}/a^{\cancel{2}}}{\left[\frac{a^2+x^2}{\cancel{a^2}}\right]}\cdot dx$$

$$= -\int_0^\infty \frac{dx}{x^2+a^2}$$

$$\frac{dI}{da} = \left[-\frac{1}{a}\cdot\tan^{-1}(x/a)\right]_0^\infty$$

$$= -\frac{1}{a}\left[\frac{\pi}{2} - 0\right]$$

$$\frac{dI}{da} = -\frac{\pi}{2a}$$

Int. w.r.t. a.

$$I(a) = -\frac{\pi}{2}\int\frac{1}{a}\cdot da$$

$$I(a) = -\frac{\pi}{2}\cdot\log a + c \quad \text{— ②}$$

To find c, put $a = b$

$\Rightarrow I(b) = 0$ from ①

② $\Rightarrow I(b) = -\frac{\pi}{2}\log b + c$

$$0 = -\frac{\pi}{2}\log b + c$$

$\therefore c = \frac{\pi}{2}\log b$

putting in ②

$$I = \frac{\pi}{2}\log b - \frac{\pi}{2}\log a$$

$$I = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$$

#

$$\frac{\partial}{\partial x}\left(\tan^{-1}(2x)\right) = \frac{1}{1+4x^2} \quad (2)$$

Example 11 : Prove that $\int_0^\infty \frac{\cos\lambda x}{x}\left(e^{-ax} - e^{-bx}\right)dx = \frac{1}{2}\log\left[\frac{[(b^2+\lambda^2)]}{a^2+\lambda^2}\right]$, a > 0, b > 0

HomeWork
{solved similar example earlier}