DEPARTMENT OF MATHEMATICS
NIT PATNA

Assignment Problem

1. For what value of t is the rank of the following matrix A equal to 3?

$$A = \begin{pmatrix} t & 1 & 1 & 1 \\ 1 & t & 1 & 1 \\ 1 & 1 & t & 1 \\ 1 & 1 & 1 & t \end{pmatrix}$$

2. Answer true or false:
(a) $\{(x, y) : x^2 + y^2 = 0, x, y \in R\}$ is a subspace of $\mathbb{R}^2$.
(b) $\{(x, y) : x^2 + y^2 \leq 1, x, y \in R\}$ is a subspace of $\mathbb{R}^2$.
(c) $\{(x, y) : x^2 + y^2 = 0, x, y \in C\}$ is a subspace of C^2.
(d) $\{(x, y) : x^2 - y^2 = 0, x, y \in R\}$ is a subspace of $\mathbb{R}^2$.
(e) $\{(x, y) : x - y = 0, x, y \in R\}$ is a subspace of $\mathbb{R}^2$.

3. Consider $P_n[x]$ and $P[x]$ be polynomial spaces over R. Answer true or false:
(a) $\{p(x) : p(x) = ax + b, a, b \in R\}$ is a subspace of $P_3[x]$.
(b) $\{p(x) : p(x) = ax^2, a \in R\}$ is a subspace of $P_3[x]$.
(c) $\{p(x) : p(x) = a + x^2, a \in R\}$ is a subspace of $P_3[x]$.
(d) $\{p(x) : p(x) \in P[x]$ has degree $3\}$ is a subspace of $P[x]$.
(e) $\{p(x) : p(0) = 0, p(x) \in P[x]\}$ is a subspace of $P[x]$.

4. Let V be the subspace of $\mathbb{R}^4$ spanned by the 4-tuples, $\alpha_1 = (1, 2, 3, 4)^t$, $\alpha_2 = (2, 3, 4, 5)^t$, $\alpha_3 = (3, 4, 5, 6)^t$, $\alpha_4 = (4, 5, 6, 7)^t$. Find a basis of V and dim V.

5. Show that $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B' = \{\beta_1, \beta_2, \beta_3\}$ are bases for $\mathbb{R}^3$:

$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$
$$\beta_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

Find the matrix that transform from basis B to basis B'; that is, a matrix $M_{BB'}$ such that $[x]_{B'} = M_{BB'}[x]_B$. If a vector $u \in \mathbb{R}^3$ has coordinate $(2, 0, -1)$ under the basis B, what is the coordinate of u under B'?

6. Find a basis and the dimension for each of the following vector spaces:

   1. $M_n(\mathbb{R})$, $n \times n$ real matrices, over $\mathbb{R}$.
   2. $S_n(\mathbb{R})$, $n \times n$ real skew-Hermitian matrices, over $\mathbb{R}$..
   3. $U_n(\mathbb{R})$, $n \times n$ real upper-triangular matrices, over $\mathbb{R}$.
   4. $L_n(\mathbb{R})$, $n \times n$ real lower-triangular matrices, over $\mathbb{R}$.

5. $D_n(\mathbb{R})$, $n \times n$ real diagonal matrices, over $\mathbb{R}$.

7. Does the equation
$$Tp(x) = p(x+2)$$
define a linear transformation from P_5 to P_{10}? If so, what are its domain and its range?

8. Does the equation
$$T(x, y, z) = (0, 0)$$
define a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$? If so, what are its domain and its range?

9. What are the kernels of the linear transformations named below?
(a) The linear transformation T defined by:
$$Tp(x) = \int_{-3}^{x+9} p(t)dt,$$
from P_6 to P_7.
(b) The linear transformation D of differentiation on P_5.
(c) The linear transformation T on $\mathbb{R}^2$ defined by
$$T(x, y) = (2x + 3y, 7x - 5y);$$
(d) The linear transformation T from P_5 to P_{10} defined by the change of variables
$$Tp(x) = p(x^2);$$
(d) The linear transformation T on $\mathbb{R}^2$ defined by
$$T(x, y) = (x, 0).$$

10. Under what conditions on the scalars α and β are the vectors $(1, \alpha)$ and $(1, \beta)$ in C^2 linearly independent?

11. Each system is in echelon form. For each, say whether the system has a unique solution, no solution, or infinitely many solutions.
(a) $\begin{aligned} -3x + 2y &= 0 \\ 2y &= 0 \end{aligned}$ (b) $\begin{aligned} x + y &= 4 \\ y - z &= 0 \end{aligned}$ (c) $\begin{aligned} 3x + 6y + z &= -0.5 \\ -z &= 2.5 \end{aligned}$

12. Which of these subsets of P_3 are linearly dependent and which are independent?
   (a) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + x - 5x^2\}$
   (b) $\{-x^2, 1 + 4x^2\}$
   (c) $\{2 + x + 7x^2, 3 - x + 2x^2, 4 - 3x^2\}$
   (d) $\{8 + 3x + 3x^2, x + 2x^2, 2 + 2x + 2x^2, 8 - 2x + 5x^2\}$

13. (a) Show that if the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then so is the set $\{\vec{u}, \vec{u} + \vec{v}, \vec{u} + \vec{v} + \vec{w}\}$.
   (b) What is the relationship between the linear independence or dependence of $\{\vec{u}, \vec{v}, \vec{w}\}$ and the independence or dependence of $\{\vec{u} - \vec{v}, \vec{v} - \vec{w}, \vec{w} - \vec{u}\}$?

14. Find the dimension of each.
   (a) The space of cubic polynomials $p(x)$ such that $p(7) = 0$.
   (b) The space of cubic polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$.
   (c) The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$
   (d) The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$

15. Solve the following system of equations:

(a) $$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 - 6x_3 &= -8 \\ x_2 + 2x_3 &= 3 \\ 3x_1 + 6x_2 - 2x_3 &= -4 \end{aligned}$$

16. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x+z \\ y+z \end{bmatrix}$$

(a) Show that T is a linear transformation.

(b) Determine the standard matrix for T.

(c) Is T onto?

(d) Is T one-to-one?

17. Find the change-of-basis matrix P from the given ordered basis B to the given ordered basis C of the vector space V.

(a) $V = \mathbb{R}^2$; $B = \{(9,2),(4,-3)\}$; $C = \{(2,1),(-3,1)\}$.

(b) $V = \mathbb{R}^2$; $B = \{(-5,-3),(4,28)\}$; $C = \{(6,2),(1,-1)\}$.

(c) $V = \mathbb{R}^3$; $B = \{(2,-5,0),(3,0,5),(8,-2,-9)\}$; $C = \{(1,-1,1),(2,0,1),(0,1,3)\}$.

(d) $V = \mathbb{R}^3$; $B = \{(-7,4,4),(4,2,-1),(-7,5,0)\}$; $C = \{(1,1,0),(0,1,1),(3,-1,-1)\}$.

(e) $V = P_1$; $B = \{7-4x, 5x\}$; $C = \{1-2x, 2+x\}$.

(f) $V = P_2$; $B = \{-4+x-6x^2, 6+2x^2, -6-2x+4x^2\}$; $C = \{1-x+3x^2, 2, 3+x^2\}$.

(g) $V = P_3$; $B = \{-2+3x+4x^2-x^3, 3x+5x^2+2x^3, -5x^2-5x^3, 4+4x+4x^2\}$; $C = \{1-x^3, 1+x, x+x^2, x^2+x^3\}$.

(h) $V = M_2(R)$; $B = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix} \right\}$;

$C = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

18. Find the inverse of each matrix by the Gauss-Jordan method.

(a) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

19. Consider the system of linear equations $\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$.

For what value(s) of k does this have (i) a unique solution? (ii), no solution? (iii) infinitely many solutions? (Justify your assertions).

20. Consider the system of equations $\begin{aligned} x + y - z &= a \\ x - y + 2z &= b \\ 3x + y &= c \end{aligned}$.

(a) Find the general solution of the homogeneous equation.

(b) If $a = 1$, $b = 2$, and $c = 4$, then find the general solution of these inhomogeneous equations.

(c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have no solution.

(d) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$. Find a basis for $ker(A)$ and $img(A)$.

21. Suppose that the following matrix A is the augmented matrix for a system of linear equations $A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & -2 & a^2 \\ -1 & -7 & -11 & a \end{array}\right]$, where a is a real number. Determine all the values of a so that the corresponding system is consistent.

22. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$$

(a) Find the matrix representation of T (with respect to the standard basis for $\mathbb{R}^2$).

(b) Determine the rank and nullity of T.

23. Define the $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$

(a) Show that T is a linear transformation.

(b) Find a matrix A such that $T(x) = Ax$ for each $x \in \mathbb{R}^2$.

(c) Describe the null space (kernel) and the range of T and give the rank and the nullity of T

24. Let A be a real 7×3 matrix such that its null space is spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Then find the rank of the matrix A.

25. Find the eigenvalues and the corresponding eigenvectors of the following matrices: $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

26. Let A be a symmetric matrix. Show that the eigenvalues of A are real numbers.

27. Using Cayley Hamilton Theorem, find the inverse of the following matrices, whenever they exist: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

28. Find null space, column space, range space and row space of the following matrices and also find basis and dimension in each of the above subspaces.

1. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & -2 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 2 & 4 & 0 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$

29. Find all the solutions of the system $2x + y + 2z = 3$, $3x - y + 4z = 7$ and $4x + 3y + 6z = 5$. Find the Null Space of associated coefficient matrix. Also, find the basis and dimension of the Null Space.

30. Consider $W = \{v \in \mathbb{R}^6 : v_1 + v_2 + v_3 = 0, v_2 + v_3 + v_4 = 0, v_5 + v_6 = 0\}$. Supply a basis for W and extend it to a basis of $\mathbb{R}^4$.

31. Find the condition on real numbers a, b, c, d so that the set $\{(x, y, z) | ax + by + cz = d\}$ is a subspace of $\mathbb{R}^3$.

32. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces W_i, $i = 1, 2$ is contained in the other.

33. Show that $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} = LS\{(1, 0, 0, -1), (0, 1, 0, 1), (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

34. Find all the subspaces of $\mathbb{R}^2$.

35. Determine whether the following sets of vectors are linearly independent or not

    1. $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ of $\mathbb{R}^3$.
    2. $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 2, 0, 0), (1, 1, 1, 1)\}$ of $\mathbb{R}^4$.
    3. $\{(1, 0, 2, 1), (1, 3, 2, 1), (4, 1, 2, 2)\}$ of $\mathbb{R}^4$.

36. Suppose $B = (1, 0, 0)^T, (1, 1, 0)^T, (1, 1, 1)^T$ and $B' = (1, 1, 1)^T, (1, -1, 1)^T, (1, 1, 0)^T$ are two ordered bases of $\mathbb{R}^3$, find the linear transformations T_B and $T_{B'}$ induced by the order bases B and B', respectively. Then, determine $M_{BB'}$, $M_{B'B}$ and verify that $M_{BB'}M_{B'B} = I_3$.

37. Let V and W be finite dimensional vector spaces over $\mathbb{R}$ and let T be the linear transformation from V to W. If $dim(V) = dim(W)$ then prove that the following statements are equivalent.

    1. T is one-one.
    2. Null Space of T, $N_T = \{0\}$.
    3. T is onto.
    4. $dim(R_T) = dim(W) = dim(V)$.