CHAPTER

1

Vector Analysis

1.1 INTRODUCTION

The physical quantities which we usually come across in physics and also our daily lives are broadly classified into two types: (i) scalars, and (ii) vectors. A scalar is a quantity which is completely specified by its magnitude and has no reference to the idea of direction. For example—mass, length, time, temperature, work, speed, electric charge, etc., are scalars and are added according to the ordinary rules of algebra. The physical quantities which have both magnitude and direction and which can be added according to the triangle rule are called vector quantities. Acceleration, momentum, velocity, weight, electric field intensity, magnetic field intensity etc., are vector quantities as each involve magnitude and direction and follow the triangle rule. If any physical quantity has both magnitude and direction but does not add up according to the triangle rule, it will not be called a vector quantity. Electric current in a wire has magnitude as well as direction but there is no meaning of triangle rule there. Thus, electric current is not a vector quantity.

1.2 REPRESENTATION OF A VECTOR

A vector is represented analytically by bold-faced letter such as **A** or putting an arrow on the top of it such as $\vec{A}$ in Fig. 1.1.

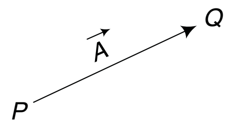


Fig. 1.1 Vector representation.

1.3 SOME IMPORTANT DEFINITIONS ABOUT VECTORS

Equality of vectors Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have the same magnitude and direction regardless of their initial points [Fig 1.2].

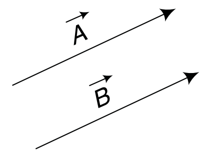


Fig. 1.2 Equality of two vectors.

Negative vector A vector having the same magnitude but direction opposite to that of the given vector is called a negative vector relative to that vector. In Fig. 1.3, $\vec{A}$ and $\vec{B}$ are opposite vectors.

Unit vector A vector of unit magnitude is called a unit vector. If any vector $\vec{A}$ is of magnitude $|\vec{A}|$ then the unit vector is represented by $\frac{\vec{A}}{|\vec{A}|}$.

Null or zero vector It is a vector having zero magnitude. It has no definite direction. It is denoted by $\vec{O}$.

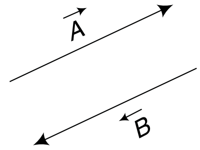


Fig. 1.3 Opposite vectors.

Coplanar vectors If a system of vectors lie in the same plane then they are called coplanar vectors. In Fig. 1.4, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar vectors.

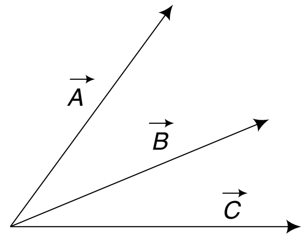


Fig. 1.4 Coplanar vectors.

Collinear or parallel vectors Vectors which have the same line of action or having lines of action parallel to the same line are called collinear vectors.

Reciprocal vectors A vector $\vec{A}^{-1}$ having the same direction as that of a given vector $\vec{A}$ but magnitude as the reciprocal of $\vec{A}$ is known as the reciprocal vector of $\vec{A}$. Thus $\vec{A}^{-1} = \frac{1}{|\vec{A}|}\hat{A}$.

Polar vectors A vector which has a linear motion in a particular direction and changes its sign under inversion or reflection is called a polar vector. Linear velocity, linear momentum, force, etc., are examples of polar vectors.

Axial vector A vector corresponding to the rotation about a certain axis is called an axial vector. Angular velocity, angular momentum, torque, etc., are axial vectors.

1.4 RESOLUTION OF A VECTOR INTO COMPONENTS

Any vector can be resolved into component vectors along three axes of the cartesian coordinate system. Here, the three axes *OX*, *OY* and *OZ* are mutually perpendicular to each other.

If $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along the *x*, *y* and *z* axes and A_x, A_y and A_z be the vector intercepts of *A* along the *x*, *y* and *z* axes respectively, then we may write $\vec{A} = \hat{i}\,A_x + \hat{j}\,A_y + \hat{k}\,A_z$.

By geometry, the magnitude of the vector $\vec{A}$ is

$$A = |\vec{A}| = \sqrt{OP^2 + OQ^2 + OR^2}$$

or,
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \ldots(1.1)$$

The unit vector along $\vec{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{A} = \frac{A_x}{A}\hat{i} + \frac{A_y}{A}\hat{j} + \frac{A_z}{A}\hat{k} \qquad \ldots(1.2)$$

Here,
$$\frac{A_x}{A} = \cos\alpha,\ \frac{A_y}{A} = \cos\beta,\ \frac{A_z}{A} = \cos\gamma \qquad \ldots(1.3)$$

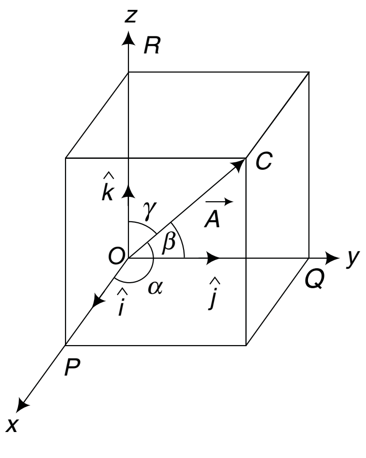


Fig. 1.5 Resolution of a vector into components.

are called the direction cosines of vector $\vec{A}$. Generally they are represented by the letters *l*, *m* and *n* respectively and α, β, γ are the angles as shown in Fig. 1.5.

If we put the values of A_x, A_y and A_z from Eq. (1.3) in Eq. (1.1), we get

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \qquad \ldots(1.4)$$

or,
$$l^2 + m^2 + n^2 = 1 \qquad \ldots(1.5)$$

Thus, the sum of squares of the direction cosines is equal to unity.

1.5 PRODUCT OF TWO VECTORS

The product of two vectors is classified as (i) scalar product or dot product, and (ii) vector product or cross product.

1.5.1 Scalar or Dot Product

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is equal to the product of the magnitudes of these vectors multiplied by the cosine of the angle between them as shown in Fig. 1.6.

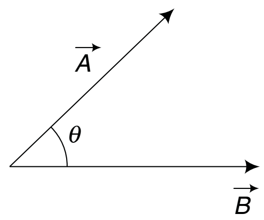


Fig. 1.6 Scalar product of two vectors $\vec{A}$ and $\vec{B}$.

Hence,
$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos \theta$$

where θ is the angle between directions of A and B.

Note that $\vec{A} \cdot \vec{B}$ is a scalar and not a vector.

1.5.2 Properties and Other Results of Scalar Product

(i) Scalar product of two vectors obeys *commutative* law

i.e., $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

(ii) If two vectors $\vec{A}$ and $\vec{B}$ are mutually perpendicular, $\cos \theta = 0$, $\vec{A} \cdot \vec{B} = 0$. This condition is known as the *orthogonality* condition of two vectors. If this property is applied to a unit vector, then

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

(iii) Scalar product of two vectors is *distributive*

i.e., $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

(iv) Scalar multiplication is *associative*. If m, n are two scalars and A, B are two vectors, then

$$m\vec{A} \cdot n\vec{B} = mn\,(\vec{A} \cdot \vec{B}) = (n\vec{A}) \cdot (m\vec{B}) = \vec{A} \cdot mn\,\vec{B}$$

1.5.3 Physical Applications of Scalar Product

(i) Work done Work done by a force $\vec{F}$ causing a displacement $\vec{d}$ is given by Fig. 1.7.

$$W = F \cos \theta\, d = \vec{F} \cdot \vec{d}$$

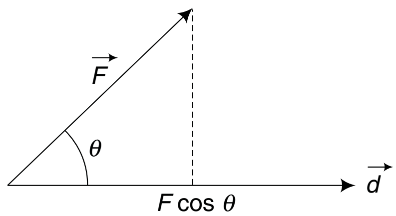


Fig. 1.7 Work done by a force.

(ii) Power The rate of doing work is power. So, power is

$$p = \frac{dW}{dt} = \frac{d}{dt}(\vec{F} \cdot \vec{d}) = \vec{F} \cdot \vec{v}$$

where $\vec{v}$ is the velocity of the body.

(iii) Electric flux Let us consider an elementary area $\vec{ds}$ in an electric field $\vec{E}$ [Fig. 1.8]. The normal electric flux coming out of the area $= \vec{E}.\,\hat{n}\,ds$ where $\hat{n}$ is the outward unit normal to the surface.

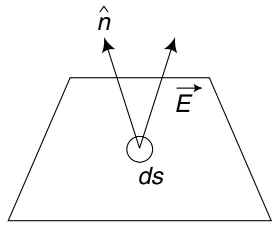


Fig. 1.8 Normal component of electric flux.

(iv) Magnetic flux (φ) The magnetic flux of a magnetic field of flux density $\vec{B}$ passing normally through an area $\vec{S}$ is

$$\varphi = \vec{B}.\,\vec{S}$$

1.5.4 Vector or Cross Product

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as a vector whose magnitude is equal to the product of the magnitudes of the vectors and the sine of the angle between their directions and its direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. The vector product is represented as

$$\vec{C} = \vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta\,\hat{n}$$

The direction of $\vec{C}$ is determined by the screw rule [Fig. 1.9] or any right-hand rule.

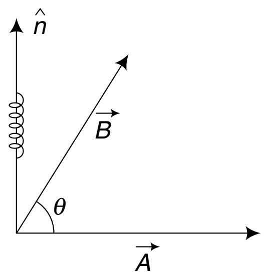


Fig. 1.9 Representation of cross product of two vectors $\vec{A}$ and $\vec{B}$.

1.5.5 Properties and Other Results of Vector Product

(i) Vector product does not obey *commutative* law.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(ii) The *distributive law* for cross products holds good.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) The vector product is *associative.*

$$\vec{A} \times (m\vec{B}) = (m\vec{A}) \times \vec{B}$$

(iv) For the orthonormal vector triad $\hat{i}, \hat{j}, \hat{k}$,

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k};\ \ \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \text{ and } \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

and $$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

(v) If $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ and $\vec{B} = B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}$, then

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1.5.6 Physical Application of Vector Product

(i) Moment of a force or torque ($\vec{\tau}$) The moment of a force (or torque) about a fixed point is the vector $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of the particle and $\vec{F}$ is the applied force.

(ii) Angular momentum ($\vec{L}$) Angular momentum of a particle is defined as the moment of linear momentum, i.e., $\vec{L} = \vec{r} \times \vec{p}$ where p is the momentum of the particle.

(iii) Force on a moving charge in a magnetic field When a charged particle moves in a magnetic field, a force acts on it. If a charge q moves with a velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$, then the force experienced by the charge is

$$\vec{F} = q\,(\vec{v} \times \vec{B}) \qquad \text{…(1.6)}$$

1.6 TRIPLE PRODUCT

Suppose we have three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. If the vector product of two vectors $\vec{B}$ and $\vec{C}$ is a vector, this may be multiplied scalarly or vectorially with the first vector $\vec{A}$. This is known as triple product. There are two types of triple product

(i) $\vec{A}.(\vec{B}\times\vec{C})$ is known as scalar triple product.

(ii) $\vec{A}\times(\vec{B}\times\vec{C})$ is known as vector triple product.

1.6.1 Scalar Triple Product

The scalar triple product of three vectors is a scalar and is represented as

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$$

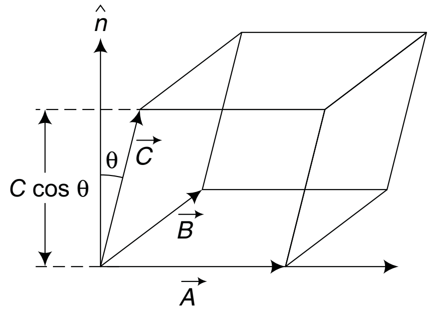


Fig. 1.10 Volume of a parallelepiped.

Properties of scalar triple product The scalar triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ represents the volume of a parallelepiped whose three adjacent sides are $\vec{A}$, $\vec{B}$ and $\vec{C}$.

The volume of the parallelepiped with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$ [Fig 1.10]

$$= (\text{Height of the parallelepiped}) \times \text{Area of the base}$$
$$= C\cos\theta\,|(\vec{A}\times\vec{B})|$$
$$= \vec{C}.(\vec{A}\times\vec{B}) = (\vec{A}\times\vec{B}).\vec{C}$$

The sign of $\vec{C}.(\vec{A}\times\vec{B})$ can be either positive or negative according as $\vec{C}$, $\vec{A}$ and $\vec{B}$ do or do not form a right-handed system. The scalar quantity $(\vec{A}\times\vec{B}).\vec{C}$ denotes the volume of the parallelepiped.

Any face of the parallelepiped may be taken as base. Hence, its volume can be represented by any of the three expressions

$$\vec{A}.(\vec{B}\times\vec{C}),\ \vec{B}.(\vec{C}\times\vec{A}),\ \vec{C}.(\vec{A}\times\vec{B})$$

Scalar triple product of $\vec{A}$, $\vec{B}$ and $\vec{C}$ is given by $\vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

1.6.2 Vector Triple Product

The vector triple product of three vectors is a vector product of one vector with the vector product of the other two vectors. The vector triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is $\vec{A}\times(\vec{B}\times\vec{C})$.

Vector triple product can be written as the sum of two terms

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}) \qquad \ldots(1.7)$$

In general,

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

The vector $\vec{A}\times(\vec{B}\times\vec{C})$ represents a vector coplanar with $\vec{B}$ and $\vec{C}$, but $(\vec{A}\times\vec{B})\times\vec{C}$ represents a vector coplanar with $\vec{A}$ and $\vec{B}$. So, the product does not represent the same vector.

1.7 SCALAR AND VECTOR FIELDS

A mathematical function, or a graphical sketch, constructed so as to describe the variation of a quantity in a given region is said to represent the *field* of that quantity associated with that region. According to the nature of the physical quantity, there are two main kinds of fields.

(a) Scalar fields (b) Vector fields

A field is a spatial distribution of quantity, which may or may not be a function of time. The *scalar field* is for a scalar quantity and *vector field* for a vector quantity. The distribution of temperature in a medium, distribution of electrostatics or gravitational potential, etc., are examples of scalar fields. A scalar field is represented by a function in space as $\psi(x, y, z)$. If the scalar quantity is also varying with time then the function is $\psi(x, y, z, t)$. The distribution of magnetic and electric field intensity, the distribution of velocity in a fluid, the force field are examples of vector fields. The vector field $\vec{F}$ in the cartesian coordinate system can be written as $\vec{F}(x, y, z, t) = F_x(x, y, z, t)\,\hat{i} + F_y(x, y, z, t)\,\hat{j} + F_z(x, y, z, t)\,\hat{k}$

1.8 GRADIENT OF SCALAR FIELD

If $\psi(x, y, z)$ be a scalar function of position of coordinates (x, y, z) in space, then ψ can be differentiated with respect to x keeping the other two coordinates y and z constant. This type of differentiation is known as partial derivatives. The partial derivatives along the three orthogonal axes are $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial z}$.

The gradient of a scalar point function $\psi(x, y, z)$ is defined as $\vec{\nabla}\psi$ and given by

$$\text{grad } \psi = \vec{\nabla}\psi = \hat{i}\frac{\partial \psi}{\partial x} + \hat{j}\frac{\partial \psi}{\partial y} + \hat{k}\frac{\partial \psi}{\partial t} \qquad \ldots(1.8)$$

where vector differential operator ∇ (del) is defined as

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The 'del' operator is a vector operator, when it operates on a scalar point function; it converts the scalar function into a vector function, grad $\psi\,(\vec{\nabla}\psi)$ is a vector quantity.

According to the theory of partial derivatives,

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial z}dz \qquad \ldots(1.9)$$

This shows how ψ varies as we go a small distance (dx, dy, dz), away from the point (x, y, z). The above relation can be written as

$$d\psi = \left(\hat{i}\frac{\partial \psi}{\partial x} + \hat{j}\frac{\partial \psi}{\partial y} + \hat{k}\frac{\partial \psi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{\nabla}\psi.\, d\vec{r} \qquad \ldots(1.10)$$

where, $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$ is the infinitesimal displacement vector.

1.8.1 Geometrical Interpretation of the Gradient

Like an ordinary vector, a gradient has both magnitude and direction. From Eq. (1.10), we have

$$d\psi = \vec{\nabla}\psi.\, d\vec{r} = |\vec{\nabla}\psi|\, |d\vec{r}| \cos\theta \qquad \ldots(1.11)$$

where θ is the angle between the vector $\vec{\nabla}\psi$ and $d\vec{r}$. For a fixed value of $|d\vec{r}|$, the maximum charge of ψ occurs when $\theta = 0$. So, for fixed distance $|d\vec{r}|$, $d\psi$ is greatest when one moves in the same direction as $\vec{\nabla}\psi$.

Putting cos $\theta = 1$ in Eq. (1.11) we get maximum rate of increase of the function,

$$\left.\frac{d\psi}{dr}\right|_{max} = |\vec{\nabla}\psi| \qquad \ldots(1.12)$$

The gradient of a scalar field is a vector whose magnitude is equal to the maximum rate of change of scalar field and direction is along that change.

Example: The electric field intensity $\vec{E} = -\vec{\nabla}V$ where V is the electric potential.

1.8.2 Directional Derivative

Suppose $\hat{e}$ represents a unit vector along a specific direction. The component of $\vec{\nabla}\psi$ along $\hat{e}$ is $\vec{\nabla}\psi.\ \hat{e}$. This is known as the directional derivative of ψ along $\hat{e}$. The directional derivative of ψ in the direction $\hat{e}$ is the component of $\vec{\nabla}\psi$ along $\hat{e}$. Since $\vec{\nabla}\psi.\ \hat{e} \le |\vec{\nabla}\psi|$, $\vec{\nabla}\psi$ is equal to the largest directional derivative of ψ.

1.9 DIVERGENCE OF VECTOR FIELD

The divergence of a vector field at any point is defined as the net outflow or flux of that field per unit volume.

The divergence of a vector point function $\vec{A}$ is denoted by div $\vec{A}$ and can be written as

$$\text{div}\,\vec{A} = \vec{\nabla}.\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\,(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \ldots(1.13)$$

Physical meaning $\vec{\nabla}.\vec{A}$ is the measure of how much the vector $\vec{A}$ spreads out (i.e., diverges) from the point in question.

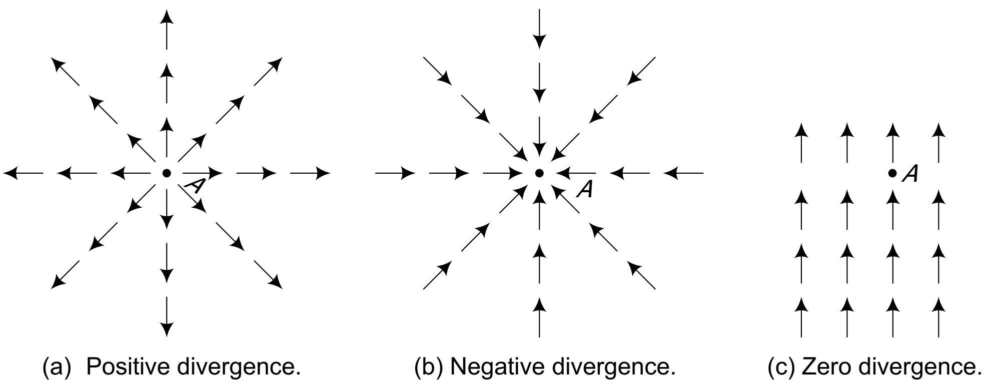


Fig. 1.11 Divergence of a vector field.

For example, the vector function in Fig. 1.11(a) has a large positive divergence at the point A. It indicates a net outflow, while a negative value of divergence [Fig. 1.11(b)] represents a net inflow and the function in Fig. 1.11(c) has zero divergence at A; it is not spreading out at all. In this case, $\vec{\nabla}.\vec{A} = 0$ and it implies that there is no inflow or outflow.

A vector $\vec{A}$, which satisfies the condition div $\vec{A} = 0$ is called a *solenoidal vector*. For example, the magnetic field vector $\vec{B}$ is a solenoidal vector.

The points at which the divergence of $\vec{A}$ is greater than zero are called *sources* and the points at which divergence of $\vec{A}$ is less than zero are called sinks. The divergence of a vector field is a scalar quantity. The divergence of a vector $\vec{A}$ may be written as

$$\vec{\nabla}.\vec{A} = \underset{\Delta V \to 0}{\text{Lt}} \frac{\iint_S \vec{A}.\vec{dS}}{\Delta V} \quad \ldots(1.14)$$

where dS is the surface enclosing the volume ΔV.

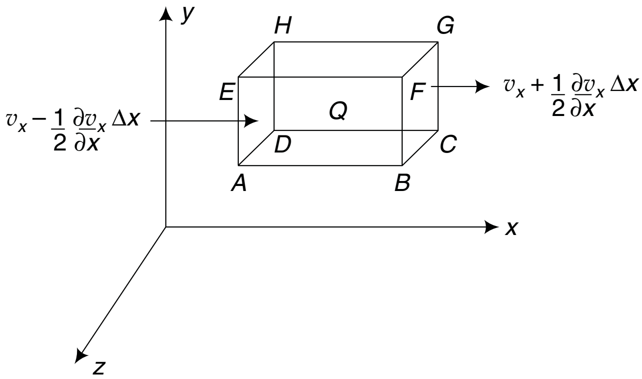


Fig. 1.12 Representation of the divergence of $\vec{v}$ in cartesian coordinates.

1.9.1 Divergence of a Fluid

Let us consider an elementary parallelepiped of volume $\Delta x\ \Delta y\ \Delta z$ within a fluid as shown in Fig. 1.12, where Δx, Δy and Δz are length, breadth and height of the parallelepiped. Suppose $\vec{v}$ represents a vector point at the centre point Q of the parallelepiped.

Here $$\vec{v} = \hat{i}v_x + \hat{j}\ v_y + \hat{k}\ v_z$$

The x component of $\vec{v}$ at the face $ADHE = v_x - \frac{1}{2}\frac{\partial v_x}{\partial x} \times \Delta x$

The x component of $\vec{v}$ at the face $BCGF = v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x$

Per unit time, the volume of the fluid crossing $ADHE$

$$= \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\ \Delta z \quad \ldots(1.15)$$

Per unit time, the volume of the fluid crossing $BCGF$

$$= \left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\ \Delta z \quad \ldots(1.16)$$

Hence in the x direction, the gain in volume of the fluid per unit time

$$= \left[\left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right) - \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\right]\Delta y\ \Delta z$$

$$= \frac{\partial v_x}{\partial x}\Delta x\ \Delta y\ \Delta z \quad \ldots(1.17)$$

Similarly, the gain in volume of the fluid per unit time along the y direction

$$= \frac{\partial v_y}{\partial y}\Delta x\ \Delta y\ \Delta z \quad \ldots(1.18)$$

and along the z direction

$$= \frac{\partial v_z}{\partial z}\Delta x\ \Delta y\ \Delta z \quad \ldots(1.19)$$

So, the total flux or gain of fluid per unit volume per unit time

$$= \frac{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x\ \Delta y\ \Delta z}{\Delta x\ \Delta y\ \Delta z}$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right).\,(\hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z)$$

$$= \vec{\nabla}.\,\vec{v} \qquad \ldots(1.20)$$

If there is no gain of fluid anywhere, then $\vec{\nabla}.\,\vec{v} = 0$. This is called the continuity equation for an incompressible fluid.

1.10 CURL OF A VECTOR FIELD

The curl of a vector field at any point measures the rate of rotation of that vector. The curl of a vector field is also known as circulation or rotation. The curl of a continuously differentiable vector point function $\vec{A}\,(x, y, z)$ is defined by the equation

$$\text{Curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\times(\hat{i}\,A_x + \hat{j}\,A_y + \hat{k}\,A_z)$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad \ldots(1.21)$$

1.10.1 Physical Meaning

Curl is a measure of how much the vector '*curl around*' the point in question. For example, the existence of curl of $\vec{v}$, the velocity at a point in a space indicates **circulation *or* vorticity** at that point of the liquid flow. If curl $\vec{v} = 0$, it means that if a wheel is placed in the liquid, it will not rotate. But if curl $\vec{v} \neq 0$, the wheel will rotate.

If a free magnetic pole is placed near a current-carrying conductor, the pole rotates around the conductor, which means $\oint_c \vec{H}.\,\vec{dl} \neq 0$, so curl $\vec{H} \neq 0$. But in the case of an electrostatic field, $\oint_c \vec{E}.\,\vec{dl} = 0$, so curl $\vec{E} = 0$.

The curl of a vector field at a point is defined as the amount of maximum line integral at any point in a vector field per unit area around a closed curve and is directed along the normal to the plane of the area.

Thus,
$$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \underset{\Delta s\to 0}{\text{Lt}}\,\frac{\left[\oint_c \vec{A}\cdot\vec{dl}\right]_{\max}}{\Delta s}\,\hat{n} \qquad \ldots(1.22)$$

Irrotational vector If the curl of a vector field $\vec{A}$ is zero then the vector field $\vec{A}$ is called an *irrotational vector*. Gravitational field, electrostatic fields, etc., are irrotational fields.

1.11 CURL IN THE CONTEXT OF ROTATIONAL MOTION

Consider a rigid body R rotating about an axis passing through O [Fig 1.13] with an angular velocity $\vec{\omega}$. If $\vec{r}$ be the position vector of a point P on the rigid body, then its linear velocity

$$\vec{v} = \vec{\omega}\times\vec{r} \qquad \ldots(1.23)$$

$$= (\hat{i}\omega_1 + \hat{j}\omega_2 + \hat{k}\omega_3)\times(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(\omega_2 z - \omega_3 y) - \hat{j}(\omega_1 z - \omega_3 x) + \hat{k}(\omega_1 y - \omega_2 x)$$

So, $$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

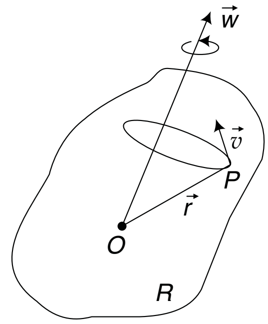


Fig. 1.13 Rotational motion of a rigid body.

$$= \hat{i}(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$$

$$= 2\,\vec{\omega} \qquad \text{...(1.24)}$$

Thus, we see that the curl of a vector field $\vec{v}$ is associated with the rotational properties of the vector field and shows that the angular velocity of a uniformly rotating body is one half the curl of the linear velocity.

1.11.1 Some Important Rules on Gradient, Divergence and Curl

(i) $\vec{\nabla}(\varphi + \psi) = \vec{\nabla}\varphi + \vec{\nabla}\psi$

(ii) $\vec{\nabla}\cdot(\vec{P} + \vec{Q}) = \vec{\nabla}\cdot\vec{P} + \vec{\nabla}\cdot\vec{Q}$

(iii) $\vec{\nabla} \times (\vec{P} + \vec{Q}) = \vec{\nabla}\times\vec{P} + \vec{\nabla}\times\vec{Q}$

(iv) $\vec{\nabla}\cdot(\psi\vec{P}) = \vec{\nabla}\psi\cdot\vec{P} + \psi\vec{\nabla}\cdot\vec{P}$

(v) $\vec{\nabla} \times (\psi\vec{P}) = \vec{\nabla}\psi\times\vec{P} + \psi\vec{\nabla}\times\vec{P}$

(vi) $\vec{\nabla}\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\vec{\nabla}\times\vec{P}) - \vec{P}\cdot(\vec{\nabla}\times\vec{Q})$

(vii) $\vec{\nabla}\times(\vec{P}\times\vec{Q}) = (\vec{Q}\cdot\vec{\nabla})\vec{P} - \vec{Q}(\vec{\nabla}\cdot\vec{P}) - (\vec{P}\cdot\vec{\nabla})\,\vec{Q} + \vec{P}\,(\vec{\nabla}\cdot\vec{Q})$

(viii) $\vec{\nabla}\;(\vec{P}\cdot\vec{Q}) = (\vec{Q}\,.\,\vec{\nabla})\vec{P} + (\vec{P}\cdot\vec{\nabla})\vec{Q} + \vec{Q}\times(\vec{\nabla}\times\vec{P}) + \vec{P}\times(\vec{\nabla}\times\vec{Q})$

(ix) $\vec{\nabla}\cdot(\vec{\nabla}\psi) = \nabla^2\psi = \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} + \dfrac{\partial^2\psi}{\partial z^2}$

(x) $\vec{\nabla} \times (\vec{\nabla}\psi) = 0$

(xi) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{P}) = 0$

(xii) $\vec{\nabla}\times(\vec{\nabla}\times\vec{P}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{P}) - \nabla^2\vec{P}$

Here $\vec{P}$ and $\vec{Q}$ are differentiable vector functions and φ and ψ are differentiable scalar functions.

1.12 VECTOR INTEGRATION

In vector analysis, the integrals that generally come are the line integral, the surface integral and the volume integral.

1.12.1 Line Integration

Let $\vec{dl}$ be an element of length on a smooth curve PQ and $\vec{A}$ be continuous vector point function. The scalar product of $\vec{A}$ with the line element $\vec{dl}$ is called the line integral of the vector $\vec{A}$ and for an extended path it will be equal to the integral

$$\int_P^Q \vec{A}.\vec{dl} = \int_P^Q A\, dl \cos\theta \qquad \text{...(1.25)}$$

It is defined as the line integral of the vector $\vec{A}$ along the curve PQ [Fig. 1.14], where θ is the angle between $\vec{A}$ and elementary length $\vec{dl}$. In terms of the components of $\vec{A}$ along three cartesian coordinates, we have

$$\int_P^Q \vec{A}.\vec{dl} = \int_P^Q (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_P^Q (A_x\, dx + A_y\, dy + A_z\, dz) \qquad \text{...(1.26)}$$

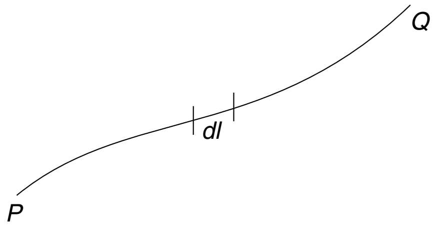


Fig. 1.14 Elementary length *dl* along the curve *PQ*.

If the path of integration is a closed curve, then we write $\oint$ instead of $\int$.

If the value of line integral depends only on the initial and final points in the vector field and independence of the path then the vector field is called ***conservative field***. All central force fields such as gravitational field and electrostatic field are conservative fields. For a conservative force field,

$$\oint_c \vec{A}\cdot\vec{dl} = 0 \qquad \text{...(1.27)}$$

If the line integral over a closed path in a vector field $\vec{A}$ is zero, then $\vec{A}$ will be the gradient of a scalar function i.e, $\vec{A} = \vec{\nabla}\psi$, where ψ is the scalar point function. If $\vec{A}$ is conservative then $\vec{\nabla}\times\vec{A}$ will be zero.

1.12.2 Surface Integral

The surface integral of a vector field $\vec{F}$ over a piecewise smooth surface S in space is defined as the integral of the normal component of $\vec{F}$ across the surface and can be written as

$$\iint_S \vec{F}.\vec{dS} \quad \text{or} \quad \iint_S \vec{F}\cdot\hat{n}\, dS$$

where ds is the elementary surface on S and $\hat{n}$ is a unit vector along the outward drawn normal to the surface.

If $\qquad \vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ and $\vec{ds} = \hat{i}dy\,dz + \hat{j}dx\,dz + \hat{k}dx\,dy$

Then $\qquad \iint_S \vec{F}.\vec{dS} = \iint_S (F_x dy\,dz + F_y dx\,dz + F_z dx\,dy)$

The surface integral can also be written as

$$\int_S \vec{F}.\vec{dS} = \int_S \vec{F}\cdot\hat{n}\,dS$$

The notation $\oiint_S$ is used for a closed surface S. Sometimes $\oint_S$ may also be used.

1.12.3 Volume Integral

Let $\vec{F}$ be a single-valued continuous vector function in volume V. Suppose the volume is divided into a large number of small volume elements (dV). The volume integral of $\vec{F}$ is the sum of the products of values of vector fields and the volume for all elements and for infinite large volume elements, volume integral

$$= \iiint_V \vec{F}\, dV$$

If $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ then volume integral

$$= \iiint_V (\hat{i}F_x + \hat{j}F_y + \hat{k}F_z)\, dV$$

$$= \hat{i} \iiint_V F_x\, dV + \hat{j}\iiint_V F_y\, dV + \hat{k}\iiint_V F_z\, dV$$

For scalar point function $\psi(x, y, z)$, volume integration will be

$$= \iiint_V \psi\, dV = \iiint_V \psi dxdydz$$

It is a scalar quantity.

The notations $\int_V \vec{F}dV$ or $\int_V \psi dV$ are also used to indicate volume integration for the respective vector and scalar fields.

1.13 INTEGRAL THEOREMS

There are mainly three fundament integral theorems in relation to the integration of vector fields: (i) Gauss' Divergence Theorem (ii) Stoke's Theorem, and (iii) Theorem for Gradient. The first one is a correlation between a closed surface and its enclosed volume. The second one is the theorem for curls, which correlates a closed line to its enclosed surface. The third one is the theorem for gradient, which correlates closed points to a line.

1.13.1 Gauss' Divergence Theorem

Statement **The theorem states that the surface integral of the normal component of a vector field taken around a closed surface is equal to the integral of the divergence of the vector taken over the volume enclosed by the surface.**

Let us consider a volume V enclosed by a closed surface S. If a vector function $\vec{F}$ is continuously differentiable throughout V then from Gauss' divergence theorem.

$$\oiint_S \vec{F}.\, \vec{dS} = \iiint_V (\vec{\nabla}\cdot \vec{F})\, dV$$

or,

$$\oiint_S \vec{F}.\, \hat{n}dS = \iiint_V (\vec{\nabla}\cdot \vec{F})\, dV \qquad \ldots(1.28)$$

where $\hat{n}$ is the outward drawn unit normal to the surface.

Significance If $\vec{F}$ represents the velocity of a fluid, then the flux of $\vec{F}$ is the total amount of fluid passing through the surface per unit time. A point of positive divergence acts as a surface of the fluid measuring the amount of fluid it is producing but a point of negative divergence measures the amount of fluid it is absorbing. The divergence theorem is basically a statement of incompressibility of fluid. For incompressible fluid, the volume integral of the divergence measures the net amount of fluid that is produced inside the region, and the surface integral of $\vec{F}$ (velocity flux) over the closed surface $\boldsymbol{S}$ gives the net amount of fluid flowing out through the surface per unit time. Hence, both are equal from the law of conservation of mass. So, from divergence theorem

Amount of fluid produced inside the volume = Net amount of fluid flowing out through the surface

1.13.2 Stoke's Theorem

Statement **The line integral of the tangential component of a vector taken around a closed path is equal to the surface integral of the normal component of the curl of the vector taken over any surface having the path as the boundary.**

Mathematically, the theorem states that if $\vec{F}$ is a continuously differentiable vector point function in a region of space and S is an open two-sided surface bounded by a closed, non-intersecting curve C, then

$$\oint_C \vec{F}.\vec{dr} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{dS}$$

or,
$$\oint_C \vec{F}.\vec{dr} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS \quad \text{...(1.29)}$$

where $\hat{n}$ is the outward drawn unit normal to the surface S. Stoke's theorem relates surface integral with line integral.

Significance Curl measures the amount of twist or rotation of the vector $\vec{F}$ at each point, the integral of the curl through a surface or flux of the curl can be determined if we go all around the edge and find how much of the fluid is flowing out of the boundary. There is no restriction on the shape of the surface S. The only condition is that the boundary of the surface S must coincide with the curve C.

1.14 COORDINATE SYSTEM

For solving three-dimensional problems, we require a coordinate system. There are mainly three types of coordinate systems (i) cartesian coordinate system, (ii) cylindrical coordinate system, and (iii) spherical coordinate system.

1.14.1 Cartesian Coordinate System (*x*, *y*, *z*)

In the cartesian coordinate system, let P be the position vector with respect to origin O given by [Fig 1.15] by $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, where $\hat{i}$, $\hat{j}$, $\hat{k}$, are unit vectors along the X, Y and Z respectively. Here unit vectors are not the functions of the coordinates.

Now elementary displacement

$$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz \quad \text{...(1.30)}$$

Elementary surface area in XY plane is

$$dS = dx\,dy$$

and the corresponding volume element

$$dV = dx\,dy\,dz$$

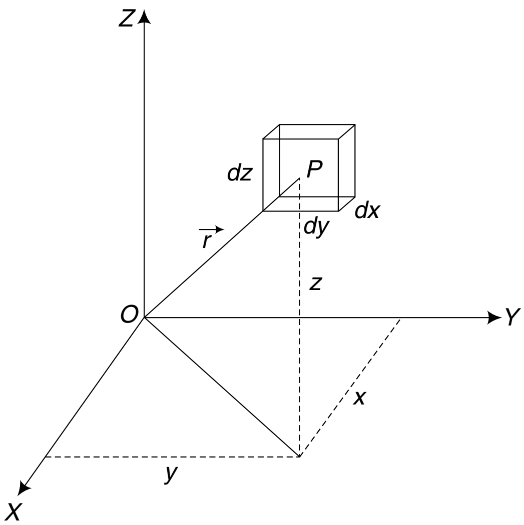
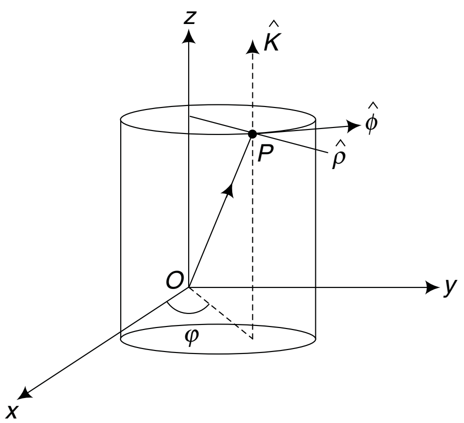


Fig. 1.15 The elementary volume element *dV* in the cartesian coordinate system.

1.14.2 Cylindrical Polar Coordinate System (ρ, φ, z)

In the cylindrical coordinate system, the position of the point P is $P(\rho, \varphi, z)$ [Fig. 1.16(a,b)]. The unit vector $\hat{\rho}$ at P is directed radially outward from the z axis. The unit vector $\hat{\varphi}$ is directed along the direction of increasing of φ and the unit vector $\hat{k}$ is along the direction of the z axis.



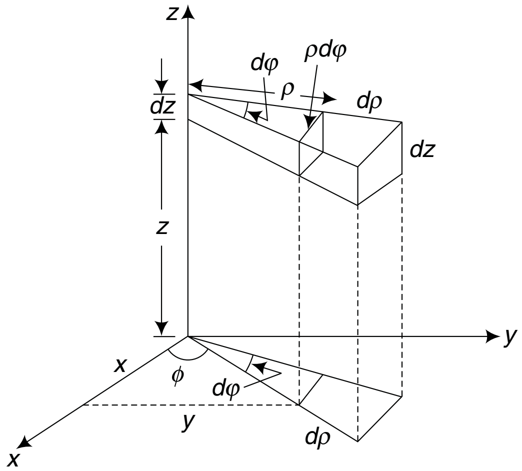


Fig. 1.16 (a) A point in cylindrical coordinate system. (b) The elementary volume element *dV* in cylindrical coordinate system.

From Fig. 1.16(b)

$$x = \rho \cos \varphi,\ y = \rho \sin \varphi \text{ and } z = z$$

Again in the cartesian coordinates, the position vector $\vec{r}$ can be written as

$$\begin{aligned}\vec{r} &= \hat{i}x + \hat{j}y + \hat{k}z \\ &= \hat{i}\rho \cos \varphi + \hat{j}\rho \sin \varphi + \hat{k}z \qquad \text{...(1.31)}\end{aligned}$$

The elementary curve length $(dr)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + \rho^2(d\varphi)^2 + (dz)^2$ ...(1.32)

and $$d\vec{r} = \hat{\rho}d\rho + \hat{\varphi}(\rho d\varphi) + \hat{k}\,dz \qquad \text{...(1.33)}$$

The volume element in cylindrical coordinate can be expressed as [from Fig 1.16(b)]

$$dV = d\rho \times \rho d\varphi \times dz = \rho d\rho\, d\varphi\, dz \qquad \text{...(1.34)}$$

The unit vectors in the cylindrical coordinates system are:

$$\hat{\rho} = \frac{\partial \vec{r}}{\partial \rho}\Bigg/\left|\frac{\partial \vec{r}}{\partial \rho}\right| = \hat{i} \cos \varphi + \hat{j} \sin \varphi$$

$$\hat{\varphi} = \frac{\partial \vec{r}}{\partial \varphi}\Bigg/\left|\frac{\partial \vec{r}}{\partial \varphi}\right| = -\hat{i} \sin \varphi + \hat{j} \cos \varphi$$

$$\hat{k} = \frac{\partial \vec{r}}{\partial z}\Bigg/\left|\frac{\partial \vec{r}}{\partial z}\right| = \hat{k}$$

1.14.3 Spherical Polar Coordinate System (r, θ, φ)

In the spherical coordinate system, the position of the point A is A (r, θ, φ) [Fig 1.17]. The unit vector $\hat{r}$ is directed radially outward. The unit vector $\hat{\theta}$ is normal to the conical surface and it is directed to the directional in which θ increases.

The unit vector $\hat{\varphi}$ is along the direction in which φ increases. The relation between cartesian and spherical polar coordinates is

$$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi \text{ and } z = r\cos\theta$$

So, the position vector

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$
$$= \hat{i}r\sin\theta\cos\varphi + \hat{j}r\sin\theta\sin\varphi + \hat{k}r\cos\theta \quad \ldots(1.35)$$

If dl is the elementary length then in spherical coordinates

$$\vec{dl} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\varphi}r\sin\theta d\varphi \quad \ldots(1.36)$$

and $dl^2 = (dr)^2 + r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\varphi)^2 \quad \ldots(1.37)$

The surface element ds in spherical coordinates is obtained from Fig. 1.17.

$$dS = AB \times AC = r\sin\theta d\varphi \times rd\theta$$
$$= r^2\sin\theta\,d\theta\,d\varphi \quad \ldots(1.38)$$

and volume element $dV = dS \times$ height (dr)

$$= r^2\sin\theta\,d\theta\,d\varphi\,dr \quad \ldots(1.39)$$

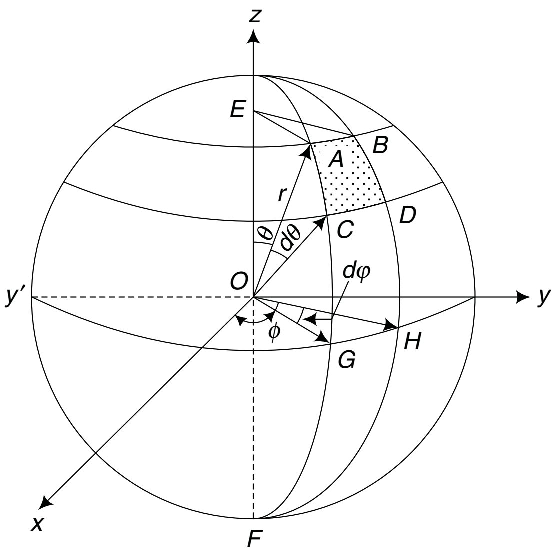


Fig. 1.17 Surface element in spherical polar coordinate system.

The unit vectors in the spherical polar coordinate system are

$$\hat{r} = \frac{\partial\vec{r}}{\partial r}\Big/\left|\frac{\partial\vec{r}}{\partial r}\right| = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\hat{\theta} = \frac{\partial\vec{r}}{\partial\theta}\Big/\left|\frac{\partial\vec{r}}{\partial\theta}\right| = \hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi - \hat{k}\sin\theta$$

and
$$\hat{\varphi} = \frac{\partial\vec{r}}{\partial\varphi}\Big/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = -\hat{i}\sin\varphi + \hat{j}\cos\varphi$$

1.14.4 Gradient, Divergence, Curl and Laplacian in Cartesian, Cylindrical and Spherical Coordinate System

(i) In cartesian coordinate system

(a) Gradient of a scalar function $\psi(x, y, z)$

$$\vec{\nabla}\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$$

(b) Divergence of vector field $\vec{A}\,(x, y, z)$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(c) Curl of vector field $\vec{A}(x, y, z)$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

(d) Laplacian of $\psi(x, y, z)$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

(ii) In cylindrical coordinate system

(a) Gradient of scalar function $\psi(\rho, \varphi, z)$

$$\vec{\nabla}\psi = \hat{\rho}\frac{\partial \psi}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial \psi}{\partial \varphi} + \hat{k}\frac{\partial \psi}{\partial z}$$

(b) Divergence of vector field $\vec{A}$ (ρ, φ, z)

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{\rho}A_\rho + \hat{\varphi}A_\varphi + \hat{k}A_z)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

(c) Curl of vector field $\vec{A}(\rho, \varphi, z)$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & A_\varphi & A_z \end{vmatrix}$$

(d) Laplacian of $\psi(\rho, \varphi, z)$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}.$$

(iii) In spherical coordinate system

(a) Gradient of a scalar function $\psi(r, \theta, \varphi)$

$$\vec{\nabla}\psi = \hat{r}\frac{\partial \psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \varphi}$$

(b) Divergence of vector field $\vec{A}$ (r, θ, φ)

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

(c) Curl of vector field $\vec{A}(r, \theta, \varphi)$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

(d) Laplacian of $\psi(r, \theta, \varphi)$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}.$$

Worked Out Problems

Example 1.1 If $\vec{A} = 4\hat{i} + 3\hat{j} + \hat{k}$, $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$ find a unit vector perpendicular to vectors $\vec{A}$ and $\vec{B}$. Find also the angle between the vector $\vec{A}$ and $\vec{B}$.

Sol. $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\hat{i} - 6\hat{j} - 10\hat{k}$$

and $|\vec{A} \times \vec{B}| = \sqrt{7^2 + (-6)^2 + (-10)^2} = \sqrt{185}$

The unit normal $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \dfrac{7\hat{i} - 6\hat{j} - 10\hat{k}}{\sqrt{185}}$

If θ is the angle between $\vec{A}$ and $\vec{B}$, then

$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$, here $|\vec{A}| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$

$|\vec{B}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$$\therefore \quad \sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}| \times |\vec{B}|} = \frac{\sqrt{185}}{3\sqrt{26}}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{185}}{3\sqrt{26}}\right) = 62.77^\circ$$

Example 1.2 Show that the vectors $\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ from the sides of a right-angled triangle.

Sol. In our problem we find that

$$\vec{B} + \vec{C} = (\hat{i} - 3\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 4\hat{k})$$
$$= 3\hat{i} - 2\hat{j} + \hat{k} = \vec{A}$$

$\therefore$ The given vectors form a triangle.

Again $|\vec{A}| = \sqrt{9 + 4 + 1} = \sqrt{14}$

$|\vec{B}| = \sqrt{1 + 9 + 25} = \sqrt{35}$

$|\vec{C}| = \sqrt{4 + 1 + 16} = \sqrt{21}$

$|\vec{A}|^2 + |\vec{C}|^2 = 14 + 21 = 35 = |\vec{B}|^2.$

$\therefore$ $\vec{A}, \vec{B}, \vec{C}$ form a right-angled triangle.

Example 1.3 If $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, then show that $\vec{A}$ and $\vec{B}$ are perpendicular.

Sol. $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

or, $|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2$

or, $2\,\vec{A} \cdot \vec{B} = -2\,\vec{A} \cdot \vec{B}$

or, $4\,\vec{A} \cdot \vec{B} = 0$, So $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

Example 1.4 The three adjacent sides of a parallelepiped are represented by $\hat{i} + 2\hat{j}$, $4\hat{j}$ and $\hat{j} + 3\hat{k}$. Calculate its volume.

Sol. Let $\vec{A} = \hat{i} + 2\hat{j}$, $\vec{B} = 4\hat{j}$ and $\vec{C} = \hat{j} + 3\hat{k}$

The volume of the parallelepiped is $V = \vec{A} \cdot (\vec{B} \times \vec{C})$

Here $$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 12\hat{i}$$

$$\therefore \quad \vec{A} \cdot (\vec{B} \times \vec{C}) = (\hat{i} + 2\hat{j}) \cdot 12\hat{i} = 12$$

So, volume $V = 12$ units

Example 1.5 For what value of m are the following three vectors coplanar?

$$\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k},\ \vec{B} = 3\hat{i} + 4\hat{j} + 5\hat{k} \text{ and } \vec{C} = \hat{i} + \hat{j} - m\hat{k}$$

Sol. Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ will be coplanar if their scalar triple product is zero, i.e., $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

So, $$\begin{vmatrix} 3 & 2 & 1 \\ 3 & 4 & 5 \\ 1 & 1 & -m \end{vmatrix} = 0$$

or, $3(-4m - 5) + 2(5 + 3m) + 1(3 - 4) = 0$

or, $m = -1$

Example 1.6 Find the area of the parallelogram determined by the vectors $\vec{A} = 3\hat{i} + 2\hat{j}$ and $\vec{B} = 2\hat{j} - 4\hat{k}$.

Sol. The area of the parallelogram is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 0 & 2 & -4 \end{vmatrix} = -8\hat{i} + 12\hat{j} + 6\hat{k}$$

The magnitude of the area of the parallelogram is $\sqrt{64 + 144 + 36} = 2\sqrt{61}$ units.

Example 1.7 If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that $2 \sin \frac{\theta}{2} = |\hat{a} - \hat{b}|$.

[WBUT 2012]

Sol.
$$\begin{aligned} |\hat{a} - \hat{b}|^2 &= (\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b}) \\ &= |\hat{a}|^2 + |\hat{b}|^2 - 2\hat{a} \cdot \hat{b} && |\hat{a}| = 1 \\ &= 1 + 1 - |\hat{a}|\,|\hat{b}| \cos \theta && |\hat{b}| = 1 \\ &= 2(1 - \cos \theta) \\ &= 4 \sin^2 \frac{\theta}{2} \end{aligned}$$

$$\therefore \quad |\hat{a} - \hat{b}| = 2 \sin \frac{\theta}{2}$$

Example 1.8 Find the torque about the point 0 (2, – 1, 3) of *a* force $\vec{F}(3, 2, -4)$ passing through the point $P(1, -1, 2)$.

Sol. The position vector of P relative to O is

$$\begin{aligned} \vec{r} &= (\hat{i} - \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) \\ &= -\hat{i} - \hat{k} \end{aligned}$$

Again force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

The required moment or torque is $\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2\hat{i} - 7\hat{j} - 2\hat{k}$

Example 1.9 If $\vec{a}, \vec{b}, \vec{c}$ satisfy the relation $\vec{a} + \vec{b} + \vec{c} = 0$ show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

Sol. We have $\vec{a} + \vec{b} = -\vec{c}$

$\therefore \quad b \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c}$

or, $\vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c}$

or, $\vec{b} \times \vec{a} = -\vec{b} \times \vec{c} = \vec{c} \times \vec{b}$

or, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

Similarly, $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$\therefore \quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Example 1.10 Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$

Sol. We have $\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c}$

$= 0$

Example 1.11 *A* particle is acted upon by constant forces $(5\hat{i} + 2\hat{j} + \hat{k})$ and $(2\hat{i} - \hat{j} - 3\hat{k})$ and is displaced from the origin to the point $(4\hat{i} + \hat{j} - 3\hat{k})$. Show that the total work done by the forces is 35 units.

Sol. The displacement produced in moving from origin to the point $(4\hat{i} + \hat{j} - 3\hat{k})$ is $(4\hat{i} + \hat{j} - 3\hat{k})$.

So, the total work done

$$= (5\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} - 3\hat{k}) + (2\hat{i} - \hat{j} - 3\hat{k}) \cdot (4\hat{i} + \hat{j} - 3\hat{k})$$

$$= 20 + 2 - 3 + 8 - 1 + 9$$

$$= 35 \text{ units}$$

Example 1.12 *A* charge of 2.0 μC moves with a speed of 2.0×10^6 m/s along the positive x axis. *A* magnetic field B of strength $(0.20\,\hat{j} + 0.40\,\hat{k})$ tesla exists in space. Find the magnetic force acting on the charge.

Sol. The force on the charge $= q(\vec{v} \times \vec{B})$

$= 2 \times 10^{-6} \times [2 \times 10^6\,\hat{i} \times (0.2\hat{j} + 0.4\hat{k})]$

$= (0.8\hat{k} - 1.6\,\hat{j})$ Newton

Example 1.13 If $\vec{A}(t)$ has a constant magnitude then show that $\dfrac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

Sol. We know that $\vec{A} \cdot \vec{A} = |A|^2 = \text{constant}$

$$\therefore \quad \vec{A} \cdot \frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{A} = 0 \text{ or, } 2\vec{A} \cdot \frac{d\vec{A}}{dt} = 0$$

So, $\dfrac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

Example 1.14 If $\varphi(x, y, z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\varphi$ at the point $(1, -2, -1)$. [WBUT 2004, 2006]

Sol. Here $\varphi(x, y, z) = 3x^2y - y^3 z^2$

Again $\vec{\nabla}\varphi = \hat{i}\dfrac{\partial\varphi}{\partial x} + \hat{j}\dfrac{\partial\varphi}{\partial y} + \hat{k}\dfrac{\partial\varphi}{\partial z}$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3z^2) = 6xy$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3z^2) = 3x^2 - 3y^2z^2$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial z}(3x^2y - y^3z^2) = -2y^3z$$

$\therefore$ $\vec{\nabla}\varphi = \hat{i}6xy + \hat{j}(3x^2 - 3y^2z^2) - \hat{k}\,2y^3z$

Now at (1, – 2, – 1), $\vec{\nabla}\varphi|_{1,-2,-1} = \hat{i}[(6\times(1)(-2)] + \hat{j}[3 - 3\times(-2)^2\times(-1)^2] - \hat{k}[2\times(-2)^3(-1)]$
$= -12\hat{i} - 9\hat{j} - 16\hat{k}$

Example 1.15 Find the directional derivative of $\psi(x, y, z) = xy^2z + 4x^2z$ at (–1, 1, 2) in the direction $(2\hat{i} + \hat{j} - 2\hat{k})$.

Sol. Here $\vec{\nabla}\psi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(xy^2z + 4x^2z)$

$$= (y^2z + 8xz)\,\hat{i} + 2xyz\hat{j} + (xy^2 + 4x^2)\hat{k}$$

at (–1, 1, 2) $\vec{\nabla}\psi = -14\hat{i} - 4\hat{j} + 3\hat{k}$

The unit vector in the direction of $2\hat{i} + \hat{j} - 2\hat{k}$ is

$$\hat{n} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Hence the directional derivative is $\vec{\nabla}\psi\cdot\hat{n} = (-14\hat{i} - 4\hat{j} + 3\hat{k})\cdot\left(\dfrac{2}{3}\hat{i} + \dfrac{1}{3}\hat{j} - \dfrac{2}{3}\hat{k}\right) = -38/3$

Example 1.16 Show that $\vec{\nabla}\psi$ is a vector perpendicular to the surface $\psi(x, z, y) = c$, where c is a constant.

Sol. Let the position vector $\vec{r} = \hat{i}\,x + \hat{j}y + \hat{k}z$ belong to the surface $\psi(x, z, y)$ = constant. Then $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$, represents a vector which is tangent to the surface ψ.

Again $d\psi = \dfrac{\partial\psi}{\partial x}dx + \dfrac{\partial\psi}{\partial y}dy + \dfrac{\partial\psi}{\partial z}dz = 0$

$$= \left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{\nabla}\psi\cdot d\vec{r} = 0$$

So, $\vec{\nabla}\psi$ is perpendicular to $d\vec{r}$ and therefore perpendicular to the surface.

Example 1.17 Find a unit normal to the surface $z^2 = x^2 - y^2$ at the point (1, 0, – 1).

Sol. Let $\psi(x, y, z) = x^2 - y^2 - z^2$

$\therefore$ $\vec{\nabla}\psi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(x^2 - y^2 - z^2)$

$$= 2x\hat{i} - 2y\hat{j} - 2z\hat{k}$$

$$\vec{\nabla}\psi|_{1,0,-1} = 2\hat{i} + 2\hat{k} = 2(\hat{i} + \hat{k})$$

Now unit normal $\hat{n} = \left.\dfrac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|}\right|_{1,0,-1} = \pm 2\dfrac{(\hat{i} + \hat{k})}{\sqrt{2^2 + 2^2}} = \pm\left[\dfrac{2(\hat{i} + \hat{k})}{2\sqrt{2}}\right] = \pm\left[\dfrac{\hat{i}}{\sqrt{2}} + \dfrac{\hat{j}}{\sqrt{2}}\right] = \pm\dfrac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

Positive and negative signs imply that the unit normal is either outward or inward.

Example 1.18 In what direction from the point (1, 3, 2) is the directional derivative of $\psi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

Sol. Here $\vec{\nabla}\psi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(2xz - y^2)$

$$= 2z\hat{i} - 2y\hat{j} + 2x\hat{k}$$

$$\vec{\nabla}\psi|_{1,3,2} = 4\hat{i} - 6\hat{j} + 2\hat{k}$$

The directional derivative is maximum in the direction $\vec{\nabla}\psi = 4\hat{i} - 6\hat{j} + 2\hat{k}$

The magnitude of this maximum is $|\vec{\nabla}\psi| = \sqrt{(4)^2 + (-6)^2 + (2)^2} = \sqrt{56} = 2\sqrt{14}$

Example 1.19 Find the values of the constants a, b, c so that the directional derivative of $\psi = axy^2 + byz + cz^2x^3$ at (1, 2, – 1) has a maximum of magnitude 64 in a direction parallel to the z axis.

Sol. Here $\vec{\nabla}\psi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(axy^2 + byz + cz^2x^3)$

$$= (ay^2 + 3x^2cz^2)\hat{i} + (2\,axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

Now $\vec{\nabla}\psi|_{1,2,-1} = (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$

Since ψ has a maximum magnitude along the z axis, so

$\vec{\nabla}\psi|_{1,2,-1}\cdot\hat{k} = 64$ or, $(2b - 2c) = 64$

and $4a + 3c = 0$ Now solving these equations,

$4a - b = 0$ we have $a = 6$

$b = 24$

$c = -8$

Example 1.20 Show that $\vec{\nabla}r^n = nr^{n-2}\vec{r}$

Sol. $\vec{\nabla}r^n = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{n/2}$

$$= \hat{i}\left\{\frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}\cdot 2x\right\} + \hat{j}\left\{\frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}\cdot 2y\right\} + \hat{k}\left\{\frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}\cdot 2z\right\}$$

$$= n(x^2 + y^2 + z^2)^{n/2-1}(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= n(r^2)^{n/2-1}\vec{r} = nr^{n-2}\vec{r}$$

Example 1.21 Find the angle between the surfaces $x^2 + y^2 = 9 - z^2$ and $x^2 + y^2 = (z + 3)$ at the point (2, – 1, 2).

Sol. Let $\psi_1 = x^2 + y^2 + z^2 - 9$ and $\psi_2 = x^2 + y^2 - z - 3$

These two surfaces are constant.

Now $\vec{\nabla}\psi_1 = \vec{\nabla}(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

and $\vec{\nabla}\psi_1\big|_{2,-1,2} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Again $\vec{\nabla}\psi_2 = \vec{\nabla}(x^2 + y^2 - z - 3) = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\vec{\nabla}\psi_2\big|_{2,-1,2} = 4\hat{i} - 2\hat{j} - \hat{k}$

Since $\vec{\nabla}\psi_1$ and $\vec{\nabla}\psi_2$ are normal to the surfaces ψ_1 and ψ_2, the angle between the surfaces is the angle between the normal to the surfaces. Let θ be the angle between the surfaces.

So, $\vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_2 = |\vec{\nabla}\psi_1|\,|\vec{\nabla}\psi_2| \cos\theta$

or, $$\cos\theta = \frac{\vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_2}{|\vec{\nabla}\psi_1|\,|\vec{\nabla}\psi_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k})\cdot(4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{4^2 + (-2)^2 + 4^2}\,\sqrt{4^2 + (-2)^2 + (-1)^2}} = \frac{16}{6\sqrt{21}}$$

or, $$\theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right) = 54.41^\circ$$

Example 1.22 Find the unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3). [WBUT 2007]

Sol. Here $\psi = x^2 + y^2 - z^2 - 100$

or, $\vec{\nabla}\psi = \vec{\nabla}(x^2 + y^2 - z^2 - 100) = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

or $\vec{\nabla}\psi|_{1,2,3} = 2\hat{i} + 4\hat{j} - 6\hat{k}$

Now unit normal $$\hat{n} = \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|} = \frac{2\hat{i} + 4\hat{j} - 6\hat{k}}{\sqrt{2^2 + 4^2 + (-6)^2}} = \pm\frac{\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{14}}$$

Example 1.23 Show that $\vec{\nabla}\cdot\vec{r} = 3$

Sol. Here $$\vec{\nabla}\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$
$$= 1 + 1 + 1 = 3$$

Example 1.24 Show that $\vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.

Sol. Here $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z,\ \vec{a}\cdot\vec{r} = (\hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3)\cdot(\hat{i}x + \hat{j}y + \hat{k}z) = a_1x + a_2y + a_3z$

Now $$\vec{\nabla}(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$
$$= \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3 = \vec{a}$$

Example 1.25 Find $\vec{\nabla}\cdot\vec{F}$ if $\vec{F} = 2x^2z\hat{i} - xy^2z\hat{j} + 3yz^2\hat{k}$

Sol. We have $$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x^2z\hat{i} - xy^2z\hat{j} + 3yz^2\hat{k})$$
$$= 4xz - 2xyz + 6yz$$

Example 1.26 Determine the constant a so that the vector field
$\vec{F} = (2x + 3y)\hat{i} + (3y - 2z)\hat{j} + (y + az)\hat{k}$ is solenoidal.

Sol. If vector $\vec{F}$ is solenoidal then $\vec{\nabla}\cdot\vec{F} = 0$

Here $\qquad \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(2x+3y) + \frac{\partial}{\partial y}(3y-2z) + \frac{\partial}{\partial z}(y+az) = 0$

$= 2 + 3 + a = 0$

or, $\qquad a = -5$

Example 1.27 Show that the vector field $\vec{F} = \frac{\vec{i}\,x + \vec{j}\,y}{\sqrt{x^2+y^2}}$ is a "source" field.

Sol. For a source field, $\vec{\nabla}\cdot\vec{F}$ should be positive and for a sink field, $\vec{\nabla}\cdot\vec{F}$ should be negative.

Here $\qquad \vec{F} = \frac{\hat{i}x + \hat{j}y}{\sqrt{x^2+y^2}}$

$$\therefore \qquad \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\hat{i}x+\hat{j}y}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= -\frac{x^2}{(x^2+y^2)^{3/2}} + \frac{1}{(x^2+y^2)^{1/2}} - \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{1}{(x^2+y^2)^{1/2}}$$

$$= -\frac{(x^2+y^2)}{(x^2+y^2)^{3/2}} + \frac{2}{(x^2+y^3)^{1/2}}$$

$$= -\frac{1}{(x^2+y^2)^{1/2}} + \frac{2}{(x^2+y^2)^{1/2}} = \frac{1}{\sqrt{x^2+y^2}}$$

$\therefore \quad \vec{\nabla}\cdot\vec{F}$ is +ve, so $\vec{F}$ is a source field.

Example 1.28 Evaluate $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)$

Sol. We know that $\vec{\nabla}\cdot(\psi\vec{A}) = \vec{\nabla}\psi\cdot\vec{A} + \psi\vec{\nabla}\cdot\vec{A}$

Here we consider $\psi = \frac{1}{r^3}$ and $\vec{A} = \vec{r}$

So, $\qquad \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = \vec{\nabla}\frac{1}{r^3}\cdot\vec{r} + \frac{1}{r^3}\vec{\nabla}\cdot\vec{r}$

$$= -3r^{-5}\,\vec{r}\cdot\vec{r} + \frac{3}{r^3} \qquad [\because \quad \vec{\nabla}r^n = nr^{n-2}\,\vec{r} \text{ and } \vec{\nabla}\cdot r = 3]$$

$$= -3r^{-3} + \frac{3}{r^3} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

Example 1.29 Find $\vec{\nabla}\cdot\vec{F}$ and $\vec{\nabla}\times\vec{F}$ where $\vec{F} = \vec{\nabla}(x^3+y^3+z^3-3xyz)$. [WBUT 2001]

Sol. Here, $\vec{F} = \vec{\nabla}(x^3+y^3+z^3-3xyz)$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

Now $\qquad \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot[(3x^2-3yz)\,\hat{i} + (3y^2-3xz)\,\hat{j} + (3z^2-3xy)\,\hat{k}]$

$$= 6x + 6y + 6z = 6\,(x + y + z)$$

Again $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(3x^2 - 3yz) - \frac{\partial}{\partial x}(3z^2 - 3xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]$$

$$= (-3x + 3x)\,\hat{i} + (-3y + 3y)\,\hat{j} + (-3z + 3z)\,\hat{k} = 0$$

Example 1.30 Prove that $\nabla^2 \ln r = \dfrac{1}{r^2}$ [WBUT 2007]

Sol. Here $$\ln r = \ln (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2} \ln (x^2 + y^2 + z^2)$$

$$\nabla^2 \ln r = \vec{\nabla} \cdot \vec{\nabla} \ln r = \vec{\nabla} \cdot \vec{\nabla}\, \frac{1}{2} \ln (x^2 + y^2 + z^2)$$

$$= \vec{\nabla} \cdot \left[\frac{1}{2}\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\} \ln (x^2 + y^2 + z^2)\right]$$

$$= \frac{1}{2}\vec{\nabla} \cdot 2\left(\frac{\hat{i}x + \hat{j}y + \hat{k}z}{x^2 + y^2 + z^2}\right) = \vec{\nabla} \cdot \frac{\vec{r}}{r^2}$$

So, $$\nabla^2 \ln r = \vec{\nabla} \cdot \frac{\vec{r}}{r^2}$$

We know that $\vec{\nabla} \cdot (\psi \vec{A}) = \vec{\nabla}\psi \cdot \vec{A} + \psi \vec{\nabla} \cdot \vec{A}$

So, $$\nabla^2 \ln r = \vec{\nabla}\frac{1}{r^2} \cdot \vec{r} + \frac{1}{r^2}\vec{\nabla} \cdot \vec{r}$$

$$= -\,2r^{-4}\,(\vec{r} \cdot \vec{r}) + \frac{3}{r^2} = -\frac{2}{r^2} + \frac{3}{r^2} = \frac{1}{r^2}$$

Example 1.31 Show that $\vec{\nabla} \times \vec{r} = 0$

Sol. $$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) + \hat{j}\left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right) + \hat{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = 0$$

Example 1.32 Show that the vector $\vec{F} = (4xy - z^3)\,\hat{i} + 2x^2\,\hat{j} - 3xz^2\,\hat{k}$ is irrotational.

Sol. If $\vec{F}$ is an irrotational vector then $\vec{\nabla} \times \vec{F} = 0$

Now $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(4xy - z^3) - \frac{\partial}{\partial x}(-3xz^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right]$$

$$= 0 + \hat{j}(-3z^2 + 3z^2) + \hat{k}(4x - 4x) = 0$$

Example 1.33 Show that $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$

Sol. Let $\vec{F} = \hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3$

Now $$\vec{\nabla}\times\vec{F} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Again $$\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}\right) = 0$$

Example 1.34 Show that $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. [WBUT 2004, 2006]

Sol. Here $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}x^2\right) + \hat{j}\left[\frac{\partial}{\partial z}(2xy + z^3) - \frac{\partial}{\partial x}3xz^2\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= 0 + \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

For a conservative force field, $\vec{\nabla}\times\vec{F} = 0$

Example 1.35 If the vectors $\vec{A}$ and $\vec{B}$ are irrotational then show that the vector $\vec{A}\times\vec{B}$ is solenoidal. [WBUT 2006]

Sol. If $\vec{A}$ and $\vec{B}$ are irrotational then

$$\vec{\nabla}\times\vec{A} = 0 \text{ and } \vec{\nabla}\times\vec{B} = 0$$

Now $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - (\vec{\nabla}\times\vec{B})\cdot\vec{A} = 0 - 0 = 0$

Hence $(\vec{A}\times\vec{B})$ is solenoidal.

Example 1.36 Show that $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

Sol. Let $\vec{\nabla} = \vec{P}$

So that $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{P}\times(\vec{P}\times\vec{A})$

Again $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\,\vec{c}$

So, $\vec{P}\times(\vec{P}\times\vec{A}) = \vec{P}\,(\vec{P}\cdot\vec{A}) - (\vec{P}\cdot\vec{P})\,\vec{A}$

or, $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\,(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

Example 1.37 If $\vec{\omega}$ is a constant vector, $\vec{r}$ is the position vector and $\vec{v} = \vec{\omega}\times\vec{r}$. Prove that $\vec{\nabla}\cdot\vec{v} = 0$

Sol. $\vec{\nabla}.\vec{v} = \vec{\nabla}\cdot(\vec{\omega}\times\vec{r})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot [\hat{i}\,(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

$$= 0 + 0 + 0 = 0$$

Example 1.38 Show that $\vec{\nabla}\times\vec{\nabla}\psi = 0$

Sol. $$\vec{\nabla}\times\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix} = \hat{i}\left[\frac{\partial^2\psi}{\partial y\,\partial z} - \frac{\partial^2\psi}{\partial z\,\partial y}\right] + \hat{j}\left[\frac{\partial^2\psi}{\partial z\,\partial x} - \frac{\partial^2\psi}{\partial x\,\partial z}\right] + \hat{k}\left[\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial y\,\partial x}\right] = 0$$

Example 1.39 If $\vec{\nabla}\cdot\vec{E} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial\vec{t}}$, $\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t}$, then show that $\vec{E}$ and $\vec{B}$ satisfy $\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2}$.

Sol. We have $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \nabla\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{\partial t}\left(\frac{\partial\vec{E}}{\partial t}\right) = -\frac{\partial^2\vec{E}}{\partial t^2}$

But $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$

So, $-\nabla^2\vec{E} = -\frac{\partial^2\vec{E}}{\partial t^2}$ $\quad\therefore\ \nabla^2\vec{E} = \frac{\partial^2\vec{E}}{\partial t^2}$

Similarly, $\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial^2\vec{B}}{\partial t^2}$

But $\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}\,(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} = -\nabla^2\vec{B}$

So, $\nabla^2\vec{B} = \frac{\partial^2\vec{B}}{\partial t^2}$

i.e., $\vec{E}$ and $\vec{B}$ satisfy the equation $\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2}$

Example 1.40 Show that $\nabla^2\left(\frac{1}{r}\right) = 0$ [WBUT 2011]

Sol. $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{\sqrt{x^2+y^2+z^2}}$$

Now $$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-1/2} = \frac{\partial}{\partial x}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\right]$$

$$= -\frac{\partial}{\partial x}[x\,(x^2+y^2+z^2)^{-3/2}]$$

$$= -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2}$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly, $\frac{\partial^2}{\partial y^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2y^2 - z^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}$

and $\frac{\partial^2}{\partial z^2}(x^2 + y^2 + z^2)^{-½} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$

So, $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 + y^2 + z^2)^{-½}$$

$$= \frac{2x^2 - y^2 - z^2 + 2y^2 - z^2 - x^2 + 2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$$

Example 1.41 Show that $\vec{A} = (6xy + z^3)\,\hat{i} + (3x^2 - z)\,\hat{j} + (3xz^2 - y)\,\hat{k}$ is irrotational. Find ψ such that $\vec{A} = \vec{\nabla}\psi$. [WBUT 2012]

Sol. A vector $\vec{A}$ is called irrotational if $\vec{\nabla} \times \vec{A} = 0$

Here $$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(6xy + z^3) - \frac{\partial}{\partial x}(3xz^2 - y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]$$

$$= \hat{i}\,(-1 + 1) + \hat{j}\,(3z^2 - 3z^2) + \hat{k}\,(6x - 6x)$$
$$= 0$$

So, $\vec{A}$ is irrotational.

Here $$\vec{A} = \vec{\nabla}\psi = \hat{i}\,\frac{\partial\psi}{\partial x} + \hat{j}\,\frac{\partial\psi}{\partial y} + \hat{k}\,\frac{\partial\psi}{\partial z} = (6xy + z^3)\,\hat{i} + (3x^2 - z)\,\hat{j} + (3xz^2 - \mathrm{y})\,\hat{k}$$

Then $$\frac{\partial\psi}{\partial x} = 6xy + z^3,\quad \frac{\partial\psi}{\partial y} = 3x^2 - z,\quad \frac{\partial\psi}{\partial z} = 3xz^2 - y$$

Integrating first with respect to x, keeping y and z constant,

$$\psi = 6\,\frac{x^2}{2}y + xz^3 + C_1(y, z) = 3x^2y + xz^3 + C_1(y, z)$$

Integrating second with respect to y, keeping x, and z constant,

$$\psi = 3x^2y - yz + C_2\,(x, z)$$

Integrating third with respect to z, keeping x and y constant,

$$\psi = xz^3 - yz + C_3\,(x, y)$$

Comparison of all equations in ψ, shows that there will be a common value of ψ if we choose

$$C_1\,(x, z) = -\,yz, \quad C_2\,(x, z) = xz^3, \quad C_3\,(x, y) = 3x^2y$$

So that $\quad \psi = 3x^2y + xz^3 - yz + C \quad$ [where C is pure constant]

Example 1.42 Show that $\vec{E} = \dfrac{\vec{r}}{r^2}$ is irrotational. Find ψ such that $\vec{E} = -\vec{\nabla}\psi$ and such that $\psi(a) = 0$ where $a > 0$.

Sol.
$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \frac{\vec{r}}{r^2} = \frac{1}{r^2}\vec{\nabla} \times \vec{r} + \vec{\nabla}\frac{1}{r^2} \times \vec{r}$$

$$= 0 + \left(-\frac{2}{r^4}\vec{r} \times \vec{r}\right) = 0$$

So E is irrotational.

Since E is irrotational, so $\quad \vec{E} = -\vec{\nabla}\,\psi = \dfrac{\vec{r}}{r^2} = \dfrac{\hat{r}}{r}$, or, $-\hat{r}\,\dfrac{\partial\psi}{\partial r} = \dfrac{\hat{r}}{r}$

or,
$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}$$

Now integrating on both sides, $\psi = -\ln r + C$ where C is constant

Applying boundary condition, $\psi(a) = 0$, so $C = \ln a$

$\therefore \quad \psi = -\ln r + \ln a = \ln\left(\frac{a}{r}\right)$

Example 1.43 Show that $\nabla^2 f(r) = \dfrac{d^2 f(r)}{dr^2} + \dfrac{2}{r}\dfrac{df(r)}{dr}$

Sol. We know that $\quad \nabla^2 f(r) = \vec{\nabla}\,.\,\vec{\nabla} f(r)$

$$= \vec{\nabla}\,.\,\hat{r}\,\frac{df(r)}{dr} = \vec{\nabla}\,.\,\frac{\vec{r}}{r}\frac{df(r)}{dr}$$

$$= \frac{1}{r}\frac{df(r)}{dr}(\vec{\nabla}\,.\,\vec{r}) + \vec{\nabla}\left(\frac{1}{r}\frac{df(r)}{dr}\right).\,\vec{r}$$

[Here we have used $\quad \vec{\nabla}.(\psi\vec{A}) = \psi\;\vec{\nabla}.\vec{A} + \vec{\nabla}\psi\,.\,\vec{A}$]

$$= \frac{3}{r}\frac{df(r)}{dr} + \hat{r}\,\frac{d}{dr}\left(\frac{1}{r}\frac{df(r)}{dr}\right).\,\vec{r}$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \hat{r}\left[\frac{1}{r}\frac{d^2 f(r)}{dr^2} - \frac{1}{r^2}\frac{df(r)}{dr}\right].\,\vec{r}$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2} - \frac{1}{r}\frac{df(r)}{dr} \qquad [\because\;\; \hat{r}\,.\,\vec{r} = r]$$

$$= \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$$

Example 1.44 Find $f(r)$ such that $\nabla^2 f(r) = 0$

Sol. In the previous problem, we have seen that

$$\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$$

But, here $\nabla^2 f(r) = 0$ so $\frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} = 0$

or, $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df(r)}{dr}\right) = 0$$

Now integrating $$r^2 \frac{df(r)}{dr} = A \text{ (constant)}$$

or, $$\frac{df(r)}{dr} = \frac{A}{r^2}$$

Again integrating $$f(r) = -\frac{A}{r} + B \quad \text{(where } B \text{ is constant)}$$

So, $$f(r) = B - \frac{A}{r} = B + \frac{A'}{r} \quad \text{(where } A = -A'\text{)}$$

Example 1.45 Calculate the amount of work done in moving a particle in a force field $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$ along the curve $\vec{r} = \hat{i}\,t + \hat{j}\,t^2 + \hat{k}\,t^3$, where t is varying from – 1 to 1.

Sol. We know that position vector $r = \hat{i}x + \hat{j}y + \hat{k}z$

$$= \hat{i}\,t + \hat{j}\,t^2 + \hat{k}\,t^3$$

So $x = t$, $y = t^2$ and $z = t^3$, the parametric equation of the curve. Here, $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k} = t\,(t^2)\,\hat{i} + t^2\,(t^3)\,\hat{j} + t\,(t^3)\,\hat{k} = t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k}$

and $$\frac{dr}{dt} = \hat{i} + 2t\hat{j} + 3t^2\,\hat{k}$$

Now work done $$W = \int_c \vec{F}.\,d\vec{r} = \int_{-1}^{+1} (t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k})\,.\,(\hat{i} + 2t\,\hat{j} + 3t^2\hat{k})\,dt$$

$$= \int_{-1}^{+1} (t^3 + 2t^6 + 3t^6)\,dt = \int_{-1}^{+1} (t^3 + 5t^6)\,dt$$

$$= \left|\frac{t^4}{4} + 5\,\frac{t^7}{7}\right|_{-1}^{+1} = \frac{10}{7}$$

Example 1.46 Find the work done in moving a particle in the force field $\vec{F} = 3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\hat{k}$ along the straight line from (0, 0, 0) to (2, 1, 3).

Sol. Work done $$W = \int_c \vec{F}.\vec{dr} = \int_c [3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\hat{k}]\,.\,(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c [3x^2\,dx + (2xz - y)\,dy + zdz]$$

We know that the equation of a straight line

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t \text{ (say)}$$

or, $$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

So, $x = 2t, \quad y = t \quad \text{and} \quad z = 3t$

The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$

So, work done $$W = \int_c [3x^2\, dx + (2xz - y)\, dy + zdz]$$

$$= \int_0^1 [12t^2\,(2dt) + (12t^2 - t)\, dt + 9\,t\, dt]$$

$$= \int_0^1 (36\,t^2 + 8t)\, dt = 16.$$

Example 1.47 Show that the force $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ is conservative. If this force acting on a particle displaces it from the point (0, 1, 2) to (4, 2, 3), calculate the work done.

Sol. For the first part, see Example 1.34.

For the second part, $$W = \int_A^B \vec{F}.\vec{dr} = \int_A^B F_x\, dx + F_y\, dy + F_z\, dz)$$

$$= \int (2xy + z^3)\, dx + x^2 dy + 3xz^2\, dz$$

$$= \int_A^B (2xy\, dx + x^2\, dy) + (z^3\, dx + 3xz^2\, dz)$$

$$= \int_{(0,\,1,\,2)}^{(4,\,2,\,3)} d\,(x^2 y) + d(xz^3) = \int_{(0,\,1,\,2)}^{(4,\,2,\,3)} d\,(x^2 y + xz^3)$$

$$= x^2 y + xz^3 \Big|_{0,\,1,\,2}^{4,\,2,\,3} = 16 \times 2 + 4 \times 27 - 0$$

$$= 140$$

Example 1.48 Show that for a conservative force field, work done in moving a particle from one point P_1 (x_1, y_1, z_1) in this field to another point P_2 (x_2, y_2, z_2) is independent of the path joining the two points.

Sol. For a conservative force field, $\vec{F} = \vec{\nabla}\psi$

Now work done $dW = \vec{F}.\, d\vec{r}$

Total work done $$W = \int dW = \int_{P_1}^{P_2} \vec{F}.\, d\vec{r} = \int_{P_1}^{P_2} \vec{\nabla}\psi\, .\, d\vec{r}$$

$$= \int_{P_1}^{P_2} \left(\hat{i}\,\frac{\partial \psi}{\partial x} + \hat{j}\,\frac{\partial \psi}{\partial y} + \hat{k}\,\frac{\partial \psi}{\partial z}\right).\,(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz)$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy + \frac{\partial \psi}{\partial z}\, dz\right)$$

$$= \int_{P_1}^{P_2} d\psi = \psi\,(P_2) - \psi\,(P_1)$$

Thus we see that work done depends only on the initial and final points but is independent of the path joining them.

Example 1.49 Show that the work done on a particle in moving it from A to B equals its change in kinetic energies at these points whether the force field is conservative or not.

Sol. Let a force acting on a particle displace it from the point A to another point B. Then work done

$$W = \int_A^B \vec{F} . \vec{dr}$$

We know that $\vec{F} = m\,\dfrac{d\vec{v}}{dt}$ where $\vec{v}$ is the velocity of the body.

So,
$$W = \int_A^B m\,\frac{d\vec{v}}{dt} . d\vec{r} = \int_A^B m\,\frac{dv}{dt} . \vec{v}\,dt$$

$$= m\int_A^B d\vec{v} . \vec{v} = m\int_A^B v\,dv = \frac{1}{2}\,m\,(v_B^2 - v_A^2)$$

So, work done on a particle in moving it from A to B equals its change in kinetic energies at these points and does not depend on the nature of the force.

Example 1.50 Evaluate $\oint_c \vec{F} . \vec{dr}$ where $\vec{F} = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$ and C is the closed curve in the xy plane, $x = 3\cos t$, $y = 3\sin t$ from $t = 0$ to 2π.

Sol. In the plane xy, $z = 0$, $\vec{F} = (2x - y)\,\hat{i} + (x + y)\,\hat{j} + (3x - 2y)\,\hat{k}$ and $\vec{dr} = \hat{i}\,dx + \hat{j}\,dy$

So, $\vec{F} . d\vec{r} = (2x - y)\,dx + (x + y)\,dy$

Again $x = 3\cos t$ and $y = 3\sin t$

or, $dx = -\,3\sin t\,dt$ and $dy = 3\cos t\,dt$

So, the total work done

$$W = \oint_c \vec{F} . d\vec{r} = \int_0^{2\pi} (6\cos t - 3\sin t)\,(-\,3\sin t\,dt) + (3\cos t + 3\sin t)\,(3\cos t\,dt)$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt = 9\int_0^{2\pi}\left(1 - \frac{1}{2}\sin 2t\right)dt$$

$$= 9\left[t + \frac{1}{4}\cos 2\,t\right]_0^{2\pi} = 9\left[2\pi + \frac{1}{4} - \frac{1}{4}\right] = 18\,\pi$$

Example 1.51 Evaluate $\iint_S \vec{r} . \hat{n}\,dS$ over the surface S of the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$, $z = 1$

Sol. For any surface S, the projection of that surface on the plane xy (See Appendix A) will be $\iint_S \vec{A} . \hat{n}\,dS = \iint_R \vec{A} . \hat{n}\,\dfrac{dxdy}{|\hat{n} . \hat{k}|}$ where R is the region on the plane xy. Similarly, on the other planes, yz and zx

will be $\iint_R \vec{A}\cdot\hat{n}\frac{dydz}{|\hat{n}.\hat{i}|}$ and $\iint_R \vec{A}\cdot\hat{n}\frac{dzdx}{|\hat{n}.\hat{j}|}$ From Fig. 1.1W, over the surface *ABEF*, unit normal $\hat{n}=\hat{i}$ and $x = 1$

So, $$\vec{r}\cdot\hat{n} = (\hat{i}x + \hat{j}y + \hat{k}z)\cdot\hat{i} = x = 1$$

$$\therefore \iint_{ABEF} \vec{r}\cdot\hat{n}\, dS = \int_0^1\int_0^1 \frac{dydz}{|\hat{n}.\hat{i}|} = \int_0^1\int_0^1 dydz = 1$$

Over the surface *OCDG*, unit normal

$$\hat{n} = -\hat{i} \quad \text{and} \quad x = 0$$

So, $$\vec{r}\cdot\hat{n} = -x = 0$$

$$\therefore \iint_{OCDG} \vec{r}\cdot\hat{n}\, dS = \int_0^1\int_0^2 0\, dydz = 0$$

Over the surface *BCDE*, unit normal

$$\hat{n} = \hat{j} \quad \text{and} \quad y = 1$$

So, $$\vec{r}\cdot\hat{n} = (\hat{i}x + \hat{j}y + \hat{k}z)\cdot\hat{j} = y = 1$$

$$\therefore \iint_{BCDE} \vec{r}\cdot\hat{n}\, dS = \int_0^1\int_0^1 \frac{dxdz}{|\hat{n}.\hat{j}|} = \int_0^1\int_0^1 dxdz = 1$$

Over the surface *AOGF*, unit normal

$$\hat{n} = -\hat{j} \quad \text{and} \quad y = 0$$

So, $$\vec{r}\cdot\hat{n} = -y = 0$$

$$\therefore \iint_{AOGF} \vec{r}\cdot\hat{n}\, dS = \int_0^1\int_0^1 0\, dxdz = 0$$

Over the surface *EDGF*, unit normal

$$\hat{n} = \hat{k} \quad \text{and} \quad z = 1$$

So, $$\vec{r}\cdot\hat{n} = (\hat{i}x + \hat{j}y + \hat{k}z)\cdot\hat{k} = z = 1$$

$$\therefore \iint_{EDGF} \vec{r}\cdot\hat{n}\, dS = \int_0^1\int_0^1 1\frac{dxdy}{|\hat{n}.\hat{k}|} = \int_0^1\int_0^1 dxdy = 1$$

Over the surface *OABC*, unit normal

$$\hat{n} = -\hat{k} \quad \text{and} \quad z = 0$$

So, $$\vec{r}\cdot\hat{n} = (\hat{i}x + \hat{j}y + \hat{k}z)\cdot(-\hat{k}) = -z = 0$$

$$\iint_{OABC} \vec{r}\cdot\hat{n}\, dS = \int_0^1\int_0^1 0\, dxdy = 0$$

So, $$\iint_S \vec{r}\cdot\hat{n}\, dS = 1 + 0 + 1 + 0 + 1 + 0 = 3$$

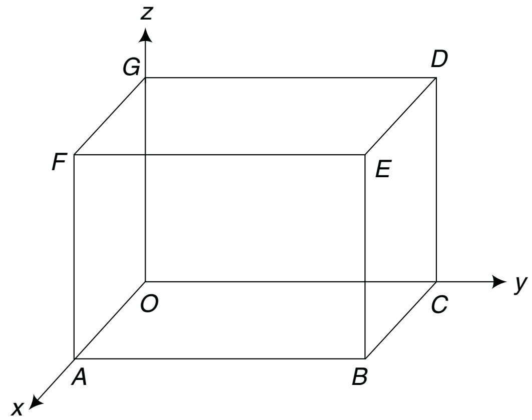


Fig. 1.1W Calculation of surface area of a unit cube bounded by coordinate planes and planes $x = 1$, $y = 1$ and $z = 1$.

[Note: By applying divergence theorem it can be easily shown that

$$\iint_S \vec{r}\,.\,\hat{n}\,dS = \int_V (\vec{\nabla}.\vec{r})\,dV = \int_0^1\int_0^1\int_0^1 3\,dxdydz = |3xyz|_{0,0,0}^{1,1,1} = 3]$$

Example 1.52 Evaluate $\iiint_V (2x + y)\,dV$, where V is the enclosed volume bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$

Sol. Here $\iiint_V (2x + y)\,dV = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x + y)\,dx\,dy\,dz$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\left[\int_{z=0}^{4-x^2} (2x + y)\,dz\right] dxdy = \int_{x=0}^{2}\int_{y=0}^{2} [2x(4 - x^2) + y(4 - x^2)]\,dxdy$$

$$= \int_{x=0}^{2}\left[2x\,(4 - x^2)\,y + \frac{y^2}{2}(4 - x^2)\right]_{y=0}^{2} dx = \int_{x=0}^{2} (16x - 4x^3 + 8 - 2x^2)\,dx$$

$$= 8x^2 - x^4 + 8x - \frac{2x^3}{3}\Big|_0^2 = \frac{80}{3}$$

Example 1.53 If $\vec{H} = \vec{\nabla} \times A$, prove that $\iint_S \vec{H}.\,\hat{n}\,dS = 0$ for any closed surface S.

Sol. Here, $\vec{H} = \vec{\nabla} \times \vec{A}$

Now from divergence theorem, $\iint_S \vec{H}.\,\hat{n}\,dS = \iiint_V (\vec{\nabla}\cdot\vec{H})dV$

But we have $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$ [since divergence of any curl is zero]

$\therefore \quad \iint_S \vec{H}\cdot\hat{n}\,dS = 0$

Example 1.54 If $\hat{n}$ is the unit outward drawn normal to any closed surface of area S, show that $\iiint$ div $\hat{n}\,dV = S$.

Sol. From divergence theorem $\iiint_V \text{div}\,\hat{n}\,dV = \iint_S \hat{n}\cdot\hat{n}\,dS = \iint_S dS = S$

Example 1.55 If S is any closed surface enclosing a volume V and $\vec{A} = ax\,\hat{i} + by\,\hat{j} + cz\,\hat{k}$, prove that $\iint_S \vec{A}\cdot\hat{n}\,dS = (a + b + c)\,V$.

Sol. Applying divergence theorem

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iiint_V (\vec{\nabla}\cdot\vec{A})\,dV$$

Here
$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(ax\,\hat{i} + by\,\hat{j} + cz\,\hat{k})$$
$$= a + b + c$$

So $\iiint_V (\vec{\nabla}\cdot\vec{A})\,dV = \iiint_V (a + b + c)\,dv = (a + b + c)\,V$

Example 1.56 Evaluate $\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS$, where $\vec{A} = (y - z + 2)\,\hat{i} + (yz + 4)\,\hat{j} - xz\,\hat{k}$ and S is the surface of the cube $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 2$, $z = 2$ above the xy plane.

Sol. Here $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz+4 & -xz \end{vmatrix} = \hat{i}\,(0 - y) + \hat{j}\,(-1 + z) + \hat{k}\,(0 - 1) = -y\,\hat{i} + (z - 1)\,\hat{j} - \hat{k}$

$$\text{Now } \iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

above the xy plane $\hat{n} = \hat{k}$, so $\iint_R ((\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{k}\, \dfrac{dxdy}{\hat{k} \cdot \hat{k}|}$

Here $(\vec{\nabla} \times \vec{A}) \cdot \hat{k} = -1$, so $\iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{k}\, dx\, dy = -\int_0^2 \int_0^2 dxdy = -4$

Example 1.57 Evaluate $\int \vec{A} \cdot \hat{n}\, dS$ where $\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$

Sol. By applying divergence theorem

$$\iint_S \vec{A}.\, \hat{n}\, dS = \iiint_V (\vec{\nabla} \cdot \vec{A})\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\, dx\, dy\, dz = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (4z - y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 2z^2 - yz|_0^1\, dxdy = \int_{x=0}^{1} \int_{y=0}^{1} (2 - y)\, dxdy = \int_{x=0}^{1} \left(2y - \frac{y^2}{2}\right)\Bigg|_0^1 dx.$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx = \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}$$

Example 1.58 Evaluate $\iint_S r \cdot \hat{n}\, dS$ over the surface of a sphere of radius a with centre at (0, 0, 0).

Sol. By applying divergence theorem

$$\iint_S \vec{r} \cdot \hat{n}\, dS = \iiint_V (\vec{\nabla} \cdot \vec{r})\, dV = \iiint_V 3\, dV = 3V$$

The volume of the surface of a sphere of radius a is

$$V = \frac{4}{3}\pi a^3 \quad \text{So } \iint_S \vec{r} \cdot \hat{n}\, dS = 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3$$

Example 1.59 Verify Stoke's theorem for $\vec{A} = (x + 3yz)\,\hat{j} + xy\,\hat{k}$ for the square surface as shown in Fig. 1.2W.

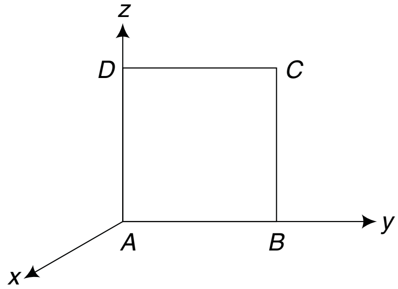


Fig. 1.2W Verification of Stoke's theorem.

Sol. Here
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x+3yz & xy \end{vmatrix}$$
$$= \hat{i}\,(x-3y) + \hat{j}\,(-y) + \hat{k}\,(1)$$

Since $x = 0$, for this surface,
$$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,dS = \int_0^1\int_0^1 [\hat{i}\,(x-3y) - \hat{j}\,y + \hat{k}]\cdot\hat{i}\,\frac{dydz}{|\hat{i}\cdot\hat{i}|}$$
$$= \int_0^1\int_0^1 -3y\,dy\,dz \quad [\text{Here, } x = 0]$$
$$= -\frac{3}{2}.$$

Now $$\int \vec{A}\cdot d\vec{r} = \int_{AB}\vec{A}\cdot d\vec{r} + \int_{BC}\vec{A}\cdot d\vec{r} + \int_{CD}\vec{A}\cdot d\vec{r} + \int_{DA}\vec{A}\cdot \vec{dr}$$

For the first part $\int_{AB}\vec{A}\cdot d\vec{r} = \int\vec{A}\cdot\hat{j}\,dy = \int 0\,dy = 0$ [Here, $z = 0$ $x = 0$]

For the second part $\int_{BC}\vec{A}\cdot d\vec{r} = \int\vec{A}\cdot\hat{k}\,dz = \int 0\,dz = 0$ [Here, $x = 0$]

For the third part $\int_{CD}\vec{A}\cdot d\vec{r} = \int\vec{A}\cdot\hat{j}\,dy = \int_1^0 3y\,dy$ [Here, $z = 1$]
$$= -\frac{3}{2}$$

For the fourth part $\int_{DA}\vec{A}\cdot d\vec{r} = \int\vec{A}\cdot\hat{k}dz = \int 0\,dz = 0$

So, total $$\int\vec{A}\cdot\vec{dr} = 0 + 0 - \frac{3}{2} + 0 = -\frac{3}{2}$$

So, Stoke's theorem is verified.

Review Exercises

Part 1: Multiple Choice Questions

1. If $\hat{n}$ is the unit vector in the direction $\vec{A}$ then [WBUT 2004]

   (a) $\hat{n} = \frac{\vec{A}}{|\vec{A}|}$ (b) $\hat{n} = \vec{A}|\vec{A}|$ (c) $\hat{n} = \frac{|\vec{A}|}{\vec{A}}$ (d) None of these

2. Two vectors $\vec{A}$ and $\vec{B}$ are parallel when [WBUT 2004]

   (a) $\vec{A}\times\vec{B} = 0$ (b) $\vec{A}\cdot\vec{B} = 0$ (c) $\vec{A}\cdot\vec{B} = 1$ (d) None of these

3. The angle between $\hat{i}$ and $(2\hat{i} + \hat{j})$ is [WBUT 2005]

   (a) $\cos^{-1}\frac{2}{5}$ (b) $\cos^{-1}\frac{2}{\sqrt{5}}$ (c) $\cos^{-1}\frac{2}{3}$ (d) None of these

4. Condition of coplanarity of three vectors $(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ is [WBUT 2006]
   (a) $\vec{\alpha}\cdot(\vec{\beta}+\vec{\gamma})=0$ (b) $\vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma})$ (c) $\vec{\alpha}\times(\vec{\beta}\times\vec{\gamma})$ (d) $\vec{\alpha}\cdot(\vec{\beta}-\vec{\gamma})=0$
5. The unit vector along the direction $2\hat{i}+3\hat{j}+4\hat{k}$ is
   (a) $\frac{1}{\sqrt{19}}(2\hat{i}+3\hat{j}+4\hat{k})$ (b) $\frac{1}{\sqrt{29}}(2\hat{i}+3\hat{j}+4\hat{k})$
   (c) $\frac{1}{\sqrt{29}}(2\hat{i}-3\hat{j}+2\hat{k})$ (d) $\frac{1}{\sqrt{19}}(2\hat{i}-3\hat{j}+4\hat{k})$
6. When the magnitude of $\vec{A}$ is constant which one of the following is true? [WBUT 2008]
   (a) $\frac{d\vec{A}}{dt}=0$ (b) $\vec{A}\cdot\frac{d\vec{A}}{dt}=0$ (c) $\vec{A}\times\frac{d\vec{A}}{dt}=0$ (d) $\left|\frac{d\vec{A}}{dt}\right|=0$
7. The angle between $\vec{\nabla}\varphi$ and the surface $\varphi=$ constant is [WBUT 2006]
   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) zero
8. For arbitrary scalar and vector fields φ and $\vec{A}$, which of the following is always correct? [WBUT 2008]
   (a) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=0$ and $\vec{\nabla}\times(\vec{\nabla}\varphi)=0$ (b) $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})=0$ and $\vec{\nabla}\cdot(\vec{\nabla}\varphi)=0$
   (c) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ and $\vec{\nabla}\times(\vec{\nabla}\varphi)=0$ (d) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ and $\vec{\nabla}\cdot(\vec{\nabla}\varphi)=0$
9. If $\vec{A}$ and $\vec{B}$ are irrotational then $\vec{A}\times\vec{B}$ is
   (a) solenoidal (b) irrotational (c) rotational (d) None of these
10. $\vec{\nabla}\cdot\vec{r}$ is equal to [WBUT 2004]
   (a) 2 (b) zero (c) 1 (d) 3
11. A fluid with velocity $\vec{v}$ is said to be incompressible when
   (a) $\vec{\nabla}\cdot\vec{v}=0$ (b) $\vec{\nabla}\times\vec{v}=0$ (c) $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})=0$ (d) None of these
12. The value of $\oint_S \vec{r}\cdot\hat{n}\,dV$ is
   (a) V (b) $2V$ (c) zero (d) $3V$
13. A solenoidal field is one for which
   (a) grad $\vec{A}=0$ (b) div $\vec{A}=0$ (c) curl $\vec{A}=0$ (d) None of these
14. The value of a for which $\vec{A}=\hat{i}\,2ax+\hat{j}\,2y+\hat{k}\,4z$ is solenoidal is equal to
   (a) 2 (b) 3 (c) -3 (d) 1
15. For conservative field, a vector field $\vec{A}$ can be written as
   (a) $\vec{A}=\nabla^2\psi$ (b) $\vec{A}=\vec{\nabla}\cdot\vec{\nabla}\,\psi$ (c) $\vec{A}=-\vec{\nabla}\,\psi$ (d) None of these
16. For irrotational vector field $\vec{A}$
   (a) $\vec{\nabla}\times\vec{A}=0$ (b) $\vec{\nabla}\cdot\vec{A}=0$ (c) grad $\vec{A}=0$ (d) None of these
17. The values of $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}$ is
   (a) 1 (b) 2 (c) zero (d) -1

18. Gradient of ψ represents the rate of change which is
    (a) minimum (b) maximum
    (c) neither maximum nor minimum (d) None of these

[**Ans.** 1 (a), 2 (a), 3(b), 4 (b), 5 (b), 6(b), 7 (a), 8 (c), 9 (a), 10 (d), 11 (a), 12 (d), 13 (b), 14 (c), 15 (c), 16 (a), 17 (c), 18 (b)]

Short Questions with Answers

1. Define scalar field and vector field.

Ans. See section 1.7

2. What is solenoidal vector?

Ans. A vector field is said to be solenoidal if the divergence of that vector field is zero. So for solenoidal vector field ($\vec{A}$)

$$\vec{\nabla}\cdot\vec{A} = 0$$

3. What is irrotational or lamellar vector field?

Ans. A vector field is said to be irrotational if the curl of that vector field is zero. So for irrotational or lamellar vector field ($\vec{A}$)

$$\vec{\nabla}\times\vec{A} = 0$$

4. Define conservative field.

Ans. If the line integral of the vector field depends on the initial and final points but is independent of the path of the integral then the vector field is called a conservative field.

For conservative field, $\oint \vec{A}\cdot\vec{dl} = 0$ then $\vec{A}$ will be the gradient of a scalar function, i.e., $\vec{A} = \vec{\nabla}\psi$.

5. Prove that $\int_V \vec{\nabla}\psi\, dV = \int_S \psi\,\hat{n}\, dS$

Ans. Let $\vec{P} = \psi\,\vec{c}$ where $\vec{c}$ is a constant vector.

Then from divergence theorem $\int_V \vec{\nabla}\cdot(\psi\,\vec{c})\, dV = \int_S \psi\,\vec{c}\cdot\hat{n}\, dS$

Again we know that $\vec{\nabla}\cdot(\psi\,\vec{c}) = \vec{\nabla}\psi\cdot\vec{c} = \vec{c}\cdot\vec{\nabla}\psi$ and $\psi\,\vec{c}\cdot\hat{n} = \vec{c}\cdot(\psi\,\hat{n})$

So, $$\int_V \vec{c}\cdot\vec{\nabla}\,\psi\, dV = \int_S \vec{c}\cdot(\psi\,\hat{n})\, dS$$

Now taking $\vec{c}$ outside of the integration

$$\vec{c}\cdot\int_V \vec{\nabla}\,\psi\, dV = \vec{c}\cdot\int_S \psi\,\hat{n}\, dS$$

So, $$\int_V \vec{\nabla}\,\psi\, dV = \int_S \psi\,\hat{n}\, dS$$

6. Define line integral.

Ans. The integral of a point function along a curve is called line integral.

Let $\vec{r} = \vec{r}\,(x, y, z)$ be the equation of a curve. If ψ and $\vec{A}$ are scalar and vector fields respectively, and $\vec{dr}$ is the displacement then the integral may be $\int_C \psi\, dr$, $\int \vec{A}\cdot\vec{dr}$ and $\int \vec{A}\times\vec{dr}$

Each integral is called the line integral along the curve C.

7. What is gradient?

Ans. Gradient is the maximum variation of a scalar considering all directions. It is the maximum rate of growth of scalar ψ. Gradient is a vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar.

8. Define source and sink.

Ans. If the divergence of a vector field is positive, it indicates a net outward flow from the point. The given point is a 'source point'. But if divergence of a vector field is negative, it indicates a net flow towards the point. The given point is a 'sink point'.

9. State Stoke's theorem.

Ans. See Section 1.13.2

10. What is the physical meaning of divergence of a vector?

Ans. The physical meaning of the divergence of a vector is the limiting value of the net outward flow to some physical quantity like a fluid or electric flux through the surface area of unit volume as the volume tends to approach zero.

Part 2: Descriptive Questions

1. What are scalar and vector fields? Give examples.
2. (a) Define vector field. Give an example.
   (b) What do you mean by a scalar field? If $\psi(x, y, z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\psi$ at the point (1, –2, 1). [WBUT 2004]
3. Define divergence of a vector point function. What is its physical significance?
4. Explain the terms surface integral and volume integral.
5. Show that (i) curl grad $v = 0$ (ii) div curl $\vec{A} = 0$.
6. Define curl of a vector point function. What is its physical significance?
7. State Gauss' divergence theorem. Explain the physical significance of this theorem.
8. Show that $\vec{\nabla}\psi$ is perpendicular to the surface over which ψ is constant.
9. State Stoke's theorem. Define line integral of a vector.
10. If $\vec{A} = \vec{\nabla} \times \vec{B}$, show that $\iint_S \vec{A} \cdot \hat{n}\, dS = 0$ for any closed surface 'S'.
11. Show that $\int_C \varphi \vec{\nabla}\psi \cdot \vec{dr} + \int_C \psi \vec{\nabla}\varphi \cdot \vec{dr} = 0$; φ and ψ are scalar fields.
12. If a rigid body rotates about an axis passing through the origin with angular velocity $\vec{\omega}$ and with linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{v})$.
13. Prove that $\nabla^2 f(r) = \dfrac{d^2 f(r)}{dr^2} + \dfrac{2}{r}\dfrac{df(r)}{dr}$.
14. Prove that $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$. [WBUT 2002]
15. If the vector functions $\vec{F}$ and $\vec{G}$ are irrotational, show that $\vec{F} \times \vec{G}$ is solenoidal. [WBUT 2006]
16. The gradient of a scalar quantity is a vector quantity. Explain. [WBUT 2002]
17. Show that $\nabla^2 (\log_e r) = \dfrac{1}{r^2}$. [WBUT 2007]

18. Given $\vec{F} = f(r)\,\vec{r}$, show that $\vec{\nabla} \times \vec{F} = 0$ and hence show that $\oint_C \vec{F} \cdot \vec{dr} = 0$ where C is a simple closed curve. [WBUT 2008]

19. Show that $\oint_S \vec{B} \cdot \vec{dS} = 0$ where $\vec{B}$ is the magnetic field and S is a closed surface. State the theorem that you have used. [WBUT 2006]

Part 3: Numerical Problems

1. Find the directional derivative of $\psi(x, y, z) = xy^2 z + 4x^2 z$ at $(-1, 1, 2)$ in the direction $(2\hat{i} + \hat{j} - \hat{k})$. $\left[\textbf{Ans. } -\dfrac{38}{3}\right]$

2. Find the torque about the point O (3, –1, 3) of a force $\vec{F}$ (4, 2, 1) passing through the point A (5, 2, 4). [**Ans.** $\hat{i} + 2\hat{j} - 8\hat{k}$]

3. A fluid motion is given by $\vec{V} = (y + z)\,\hat{i} + (z + x)\hat{j} + (x + y)\,\hat{k}$. Show that the motion is irrotational.

4. Find the constant a so that $\vec{V}$ is a conservative vector field where
$$\overline{V} = (axy - z^2)\,\hat{i} + (a - z)\,x^2\,\hat{j} + (1 - a)\,az^2\,\hat{k}.$$ [**Ans.** $a = 4$]

5. Find the work done in moving an object in the field $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ from the point (1, – 2, 1) to (3, 1, 4) independent of the path. [WBUT 2007]

6. Find the work done to move an object along a vector $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$. [WBUT 2005]

7. Find the unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3). $\left[\textbf{Ans. } \dfrac{\pm\,\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{14}}\right]$ [WBUT 2007]

8. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. [**Ans.** – 2] [WBUT 2002]

9. Evaluate $\iint_R \sqrt{x^2 + y^2}\; dx\,dy$ over the region R in the xy plane bounded by $x^2 + y^2 = 36$. [**Ans.** $144\,\pi$]

10. Show that the force field $\vec{F}$ defined by $\vec{F} = (y^2z^3 - 6\,xz^2)\,\hat{i} + 2xyz^3\,\hat{j} + (3xy^2z^2 - 6x^2z)\,\hat{k}$ is conservative.

11. For what value of a is the vector $\vec{A} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + az)\,\hat{k}$ solenoidal? [**Ans.** $a = -2$]

12. Evaluate $\iint_S \vec{r} \cdot \hat{n}\, dS$. [**Ans.** $3V$]

13. Find the unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3). $\left[\textbf{Ans. } \dfrac{1}{\sqrt{14}}\,\hat{i} + \dfrac{2}{\sqrt{14}}\,\hat{j} - \dfrac{3}{\sqrt{14}}\,\hat{k}\right]$ [WBUT 2007]

14. If $\psi(x, y, z) = 3xy^2 - 5x^2z + 2z^2$ find $\nabla^2\psi$. [**Ans.** $6x - 10z + 4$]

15. Evaluate $\iint_S \vec{A} \cdot \hat{n}\, dS$, where $\vec{A} = (2x + 3z)\,\hat{i} - (xz + y)\,\hat{j} + (y^2 + 2z)\,\hat{k}$ and S is the surface of the sphere having centre at (3, – 1, 2) and radius 3. [**Ans.** $108\,\pi$]

CHAPTER

2

Electrostatics

2.1 INTRODUCTION

Electrostatics is an important branch of physics which deals with electric charge at rest. The electric force between two electrons is the same as the electric force between two protons placed at the same distance. The amount of charge on an electron is the same as that on a proton. The charge on a proton is positive and that on an electron is negative. The net charge on the electron–proton system is zero. Stationary charges produce an electric field that is constant with time, hence the term electrostatics. Both electrostatics and magnetostatics can be explained by using vector calculus.

2.2 QUANTIZATION OF CHARGE

In 1911, Millikan successfully showed that charges in tiny oil drops are exact multiples of elementary charges. The magnitude of charge on a proton or an electron ($e = 1.6 \times 10^{-19}\ c$) is called elementary charge. Quantization of charge means that all observable charges are integral multiple of elementary charge $e(= 1.6 \times 10^{-19}c)$.

2.3 CONSERVATION OF CHARGE

The law of conservation of charge states that for an isolated system, the net charge always remains constant. In β-decay a neutron converts itself into a proton and creates an electron. The net charge remains zero before and after the decay. In nuclear fission, the total charge is always conserved.

$$_0n^1 + {}_{92}U^{235} \rightarrow {}_{56}Ba^{141} + {}_{36}K^{92} + 3{}_0n^1$$

Before collision total charge = + 92 e and total charge after collision = (56 + 36) e = 92 e. So total charge is conserved.

2.4 COULOMB'S LAW

Statement

The force between two small charged bodies separated by a distance in air is

(a) directly proportional to the magnitude of each charge

(b) inversely proportional to the square of the distance between them

(c) directed along the line joining the charges

The distance between charges must be large compared to their linear dimension.

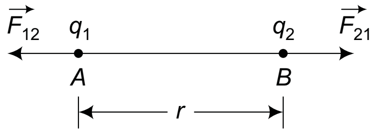


Fig. 2.1 Force between two charges.

In mathematical form, if q_1 and q_2 be two like charges and r is the distance between them [Fig. 2.1] then the force exerted on q_1 due to the charge q_2 is

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{21} \qquad ...(2.1)$$

Here, $\hat{r}_{21}$ is unit vector pointing from q_2 to q_1 and ε_0 is the permittivity of free space. Experimentally, measured value of $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ Nm^2/C^2

Similarly, the force exerted on q_2 due to the charge q_1 is

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12} \qquad ...(2.2)$$

Here, $\hat{r}_{12}$ is a unit vector pointing from q_1 to q_2. So $\vec{F}_{12} = -\vec{F}_{21}$. For two unlike charges, $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12}$ and $\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{21}$ are attractive

Relative permittivity (ε_r) The relative permittivity or dielectric constant of a medium is defined as the ratio of the force between two charges placed at a distance in vacuum (or air) to the force between the same charges placed at the same separation in that medium.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}\left[F_{\text{vacuum}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \text{ and } F_{\text{medium}} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}\right] \qquad ...(2.3)$$

2.5 PRINCIPLE OF SUPERPOSITION

The principle states that when a number of charges are interacting, the total force on a given charge is the vector sum of the individual forces exerted by all other charges on the given charge.

If q_1, q_2, q_3, q_4, ... are the charges situated at A, B, C, D, ..., as shown in Fig. 2.2.

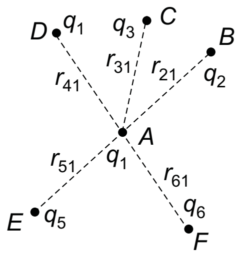


Fig. 2.2 Principle of superposition.

The total force on q_1 due to all other charges is

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} + \frac{q_1 q_3}{r_{31}^2}\hat{r}_{31} + ... + ...\right]$$

$$= \vec{F}_{12} + \vec{F}_{13} + ...$$

The total force on q_2 due to all other charges is

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_2 q_1}{r_{12}^2}\hat{r}_{12} + \frac{q_2 q_3}{r_{32}^2}\hat{r}_{32} + ...\right]$$

If there is a test charge q_0, then total force on the test charge q_0 due to all other charges is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + ... = \sum_{i=1}^{N}\vec{F}_i$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_0 q_i}{r_i^2} \hat{r}_i \qquad ...(2.4)$$

2.6 ELECTRIC FIELD

The electric field due to a charge is the space around the charge in which any other charge is acted upon by an electrostatic force.

If we have many charges $q_1, q_2, ... q_n$ at distances $r_1\ r_2\ ...\ r_n$ respectively from a test charge q_0 then from the principle of superposition, the total force on q_0 is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_0 q_i}{r_i^2} \hat{r}_i$$

$$= \frac{q_0}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i$$

The electric field intensity at the point is

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i \qquad ...(2.5)$$

Thus, the electric field intensity at a point in the electric field is the force on a unit test charge placed at the point concerned.

2.7 CONTINUOUS CHARGE DISTRIBUTIONS

On a uniform charge body, there are three types of distribution of charge:

(i) Line charge distribution If q is the total charge over a conducting wire of length l and infinitesimally small thickness, then charge per unit length λ (line charge density) is

$$\lambda = \frac{q}{l} \text{ coulomb/m}$$

For non-uniform distribution of charge

$$\lambda = \underset{\Delta l \to 0}{\mathrm{Lt}} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

where Δl is a small element of length which carries a charge Δq.

So, total charge over the whole length

$$q = \int_l \lambda\, dl \qquad ...(2.6)$$

(ii) Surface charge distribution If q is the charge uniformly distributed over the conducting surface S, then surface density of charge (charge per unit area) σ is

$$\sigma = \frac{q}{S} \text{ coulomb/m}^2$$

If Δq be the charge contained by a small element ΔS then surface charge density,

$$\sigma = \underset{\Delta S \to 0}{\mathrm{Lt}} \frac{\Delta q}{\Delta S} = \frac{dq}{dS}$$

So, total charge on the surface $q = \int_S \sigma\, dS$ ...(2.7)

(iii) Volume charge distribution If q is the charge uniformly distributed over the volume V then the volume density of charge (charge per unit volume)

$$\rho = \frac{q}{V} \text{ coulomb/m}^3$$

If charge distribution is not uniform then

$$\rho = \underset{\Delta V \to 0}{\text{Lt}} \frac{\Delta q}{\Delta V} = \frac{dq}{dV}$$

where ΔV is a small volume element which carries a charge Δq.

So, total charge over the whole volume

$$q = \int_V \rho \, dV \qquad \text{...(2.8)}$$

2.8 ELECTRIC POTENTIAL

The concept of potential is based on energy consideration. The electric potential at a point in an electric field near a charged conductor is defined as the amount of work done in bringing a unit positive charge from infinity to that point against the electrostatic force. A positively charged body always tends to move from higher potential to lower potential.

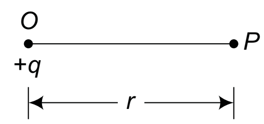


Fig. 2.3 Potential at point *P* due to charge +*q* at *O*.

Potential at a point due to a point charge

The potential at P [Fig. 2.3] is given by

$$V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dr}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} dr = \frac{q}{4\pi\varepsilon_0 r} \qquad \text{...(2.9)}$$

Electric field Intensity as a gradient of potential

Let the electric field at a point $\vec{r}$ due to a charge distribution be $\vec{E}$ and electric potential at the same point be V. Suppose a test charge q_0 is displaced slightly from $\vec{r}$ to $\vec{r} + d\vec{r}$. Then the force on the test charge q_0 is $\vec{F} = q_0\vec{E}$ and work done for small displacement $\vec{dr}$ is $dw = \vec{F} \cdot d\vec{r} = q_0 \vec{E} \cdot d\vec{r}$

The change in potential energy $= -\, d\omega = -\, q_0\vec{E} \cdot d\vec{r}$

So the change in potential $dV = -\,\vec{E} \cdot d\vec{r}$

Again $\qquad dV = \vec{\nabla}V \cdot d\vec{r} = -\,\vec{E} \cdot d\vec{r}$

or, $\qquad \vec{E} = -\,\vec{\nabla}\, V \qquad$ ...(2.10)

Hence, the electric field at a point is equal to the negative gradient of the electrostatic potential at the point.

2.9 ELECTRIC POTENTIAL ENERGY

We define electric potential energy of a system of point charges as the work required to assemble this system of charges by bringing them from infinite distances.

If two point charges q_1 and q_2 are separated by a distance r_{12} then potential energy of the system q_1 and q_2 is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

If another charge q_3 is at a distance r_{13} from q_1 and distance r_{23} from q_2 then potential energy of the system $(q_1 + q_2 + q_3)$ is

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

Generally, the potential energy for a system of n point charges is

$$U = \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{q_i q_j}{r_{ij}} \qquad ...(2.11)$$

In summation, each term is counted twice, so half factor is included here to avoid double counting each term for calculation of potential energy.

2.10 ELECTRIC FLUX

We know that any area element dS is a vector $\overrightarrow{dS}$. If $\hat{n}$ is the unit normal to the area element then

$$\overrightarrow{dS} = \hat{n}\, dS$$

The total number of lines of force passing through a surface placed in an electric field is known as electric flux (φ_E).

Consider a surface of area S inside an electric field $\vec{E}$ [Fig. 2.4]. The surface S is divided into a number of elementary areas $\overrightarrow{dS}$ (known as area vector). The component of the electric field along the area vector $\overrightarrow{dS}$ is given by

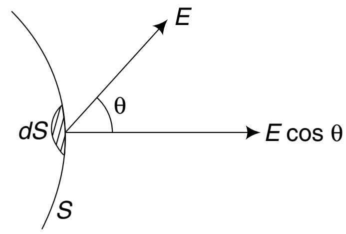


Fig. 2.4 Component of electric field $\vec{E}$ along $\overrightarrow{dS}$.

$$E_n = E \cos\theta$$

So, electric flux

$$d\varphi_E = E_n\, dS = E\cos\theta\, dS$$
$$= \vec{E}\cdot\overrightarrow{dS} = \vec{E}\cdot\hat{n}\, dS \qquad ...(2.12)$$

The total electric flux through S is

$$\varphi_E = \int_s \vec{E}\cdot\hat{n}\, dS \qquad ...(2.13)$$

2.11 SOLID ANGLE

In Fig. 2.5, the solid angle subtended by any surface dS at a point O, distance r away, is given by

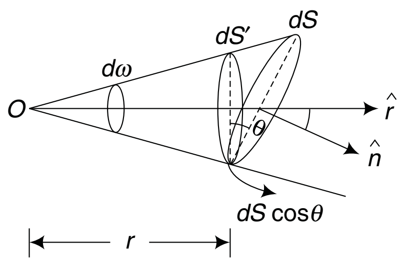


Fig. 2.5 Solid angle at point *O*.

$$d\omega = \frac{dS'}{r^2} = \frac{dS\cos\theta}{r^2}$$

where $dS' = dS\cos\theta$ is the projection of the surface dS.

For sphere $dS' = 4\pi r^2$ and $\theta = 0$, so solid angle subtended by the sphere at its center is

$$\omega = \int_s \frac{dS\cos\theta}{r^2} = 4\pi$$

2.12 GAUSS' LAW

Gauss' law states that the total electric flux through a closed surface in an electric field is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface, where ε_0 is the permittivity of free space.

Mathematically, $\oint_s \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$ when S encloses q

$= 0$ when S does not enclose q ...(2.14)

Proof of Gauss' law

We consider a spherically symmetric closed surface. Suppose a charge q is placed at the center of a sphere a radius r and $\vec{E}$ is the electric field intensity at a point R on the surface [Fig. 2.6].

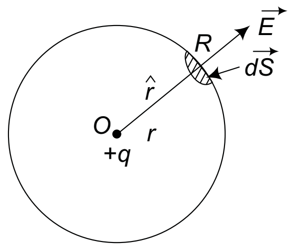


Fig. 2.6 Proof of Gauss' law.

From Coulomb's law,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

The flux of electric field through $\vec{dS}$ is

$$d\varphi_E = \vec{E} \cdot \vec{dS} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \cdot \vec{dS} = \frac{q}{4\pi\varepsilon_0 r^2} dS$$

The total flux of the electric field due to the internal charge q through the closed surface

$$\varphi_E = \int d\varphi_E = \frac{q}{4\pi\varepsilon_0}\int_s \frac{dS}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{4\pi r^2}{r^2} = \frac{q}{\varepsilon_0}$$

which is Gauss' law in electrostatics.

2.12.1 Differential Form of Gauss' Law

The integral form of Gauss' law is

$$\oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} \qquad ...(2.15)$$

If ρ be the volume charge density over a small volume element dV within the closed surface S, then

$$q = \int_V \rho \, dV \qquad ...(2.16)$$

Again from Gauss' divergence theorem

$$\oint_S \vec{E} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{E})\, dV \qquad ...(2.17)$$

Now from Eqs (2.17) and (2.16) into Eq. (2.15)

$$\int_V (\vec{\nabla} \cdot \vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V \rho \, dV \qquad ...(2.18)$$

or $$\int_V \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0$$

This is true for any arbitrary volume V.

$\therefore$ $$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad ...(2.19)$$

This is the differential form of Gauss' law.

In vacuum, electric displacement vector $\vec{D} = \varepsilon_0 \vec{E}$

So $$\vec{\nabla} \cdot \vec{D} = \rho \qquad ...(2.20)$$

This is also the differential form of Gauss' law in terms of electric displacement vector.

2.12.2 Coulomb's Law from Gauss' Law

Here we would like to deduce Coulomb's law from Gauss' law. Here, S is the spherical gaussian surface. In Fig. 2.7, $\vec{E}$ and $\vec{dS}$ on the gaussian surface are directed radially outward. So $\vec{E} \cdot \vec{dS} = E\, dS$.

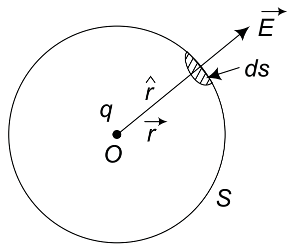


Fig. 2.7 Derivation of Coulomb's law from Gauss' law.

Now from Gauss' law

$$\oint_s \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$$

or, $$E \oint_s dS = E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \quad [E \text{ is constant}]$$

or, $$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

If another point charge q_0 be placed at the point at which $\vec{E}$ is calculated then the force on q_0 due to q is

$$\vec{F} = q_0 \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_0 q}{r^2} \hat{r} \qquad ...(2.21)$$

where $\hat{r}$ is the unit vector.

Equation (2.21) is Coulomb's law.

2.12.3 Application of Gauss' Law

For calculating electric field due to a charge distribution, Gauss' law provides the easiest way. Here, we consider some important applications of Gauss' law.

(i) Electric field due to uniformly charged sphere

Case I Field at a point outside the charged sphere

Let P be a point situated outside the charged sphere having charge q uniformly distributed throughout the volume of the sphere of radius R [Fig. 2.8]. In order to find the electric field intensity at P, a concentric sphere of radius r is drawn as gaussian surface, over which the electric field intensity is directed normal to every point of this surface. If $\vec{E}$ is the electric field intensity at the point P then the total flux through the gaussian surface is

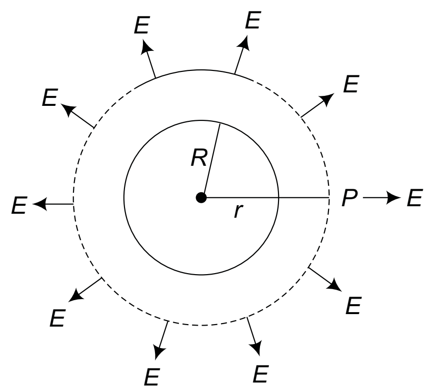


Fig. 2.8 Electric field outside of a charged sphere.

$$\varphi_E = \oint_s \vec{E} \cdot \vec{dS} = E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

or, $$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

For continuous charge distribution of density ρ within the sphere

$$q = \frac{4}{3}\pi R^3 \rho$$

So,
$$E = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3\rho}{r^2} = \frac{R^3\rho}{3\varepsilon_0 r^2} \quad ...(2.22)$$

Case II Field at a point on the surface of the sphere

For any point on the surface of the sphere $r = R$, from Eq. (2.22), we have

$$E = \frac{R^3\rho}{3\varepsilon_0 R^2} = \frac{R\rho}{3\varepsilon_0} \quad ...(2.23)$$

Case III Field at a point inside the charged sphere

We want to find the electric field at a point P inside the sphere at a distance r from the center [Fig. 2.9]. The total charge inside the gaussian surface of radius r

$$q = \frac{4}{3}\pi r^3\rho$$

The total electric flux over the gaussian surface is given by

$$\varphi_E = \oint_s \vec{E}\cdot d\vec{S} = E\times 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{4\pi r^3\rho}{3\varepsilon_0}$$

or,
$$E = \frac{r\rho}{3\varepsilon_0} \quad ...(2.24)$$

The variation of electric field intensity in different cases as discussed is shown in Fig. 2.10.

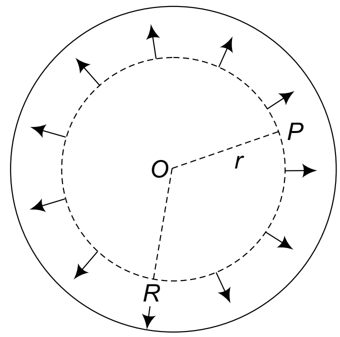


Fig. 2.9 Electric field inside a charged sphere.

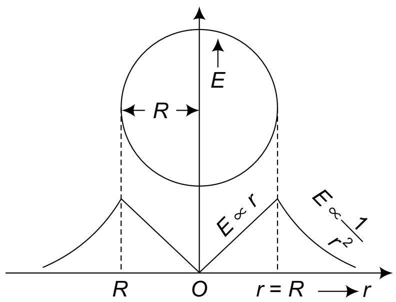


Fig. 2.10 Variation $\vec{E}$ with distance from the center of a charged sphere.

(ii) Electric field due to a charged spherical shell

Case I At a point outside the charged shell

Let P be a point outside the shell at a distance r [Fig. 2.11]. Here $r > R$

The total flux over the gaussian surface

$$\varphi_E = \oint_s \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

or,
$$E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2} \quad ...(2.25)$$

Case II At a point on the surface of the charged shell

Here, $r = R$ and the sphere itself behaves as gaussian surface. The electric field intensity on the surface is then

$$E = \frac{q}{4\pi\varepsilon_0 R^2}$$

Again for spherical shell the surface density of charge

$$\sigma = \frac{q}{4\pi R^2}$$

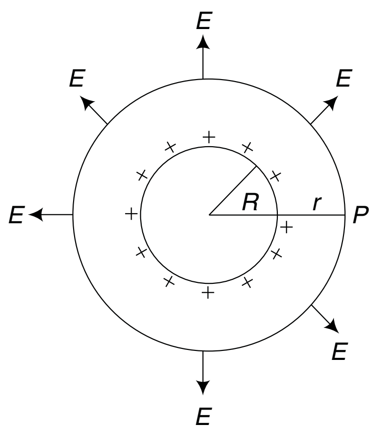


Fig. 2.11 Electric field due to a charged spherical shell.

$$\therefore \qquad E = \frac{\sigma}{\varepsilon_0} \qquad ...(2.26)$$

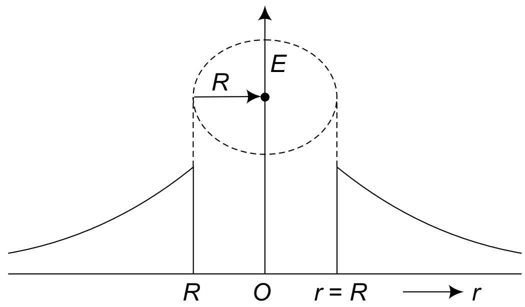


Fig. 2.12 Electric field at a point inside a spherical charged shell.

Case III At a point inside the shell

Here $r < R$, the total charge is situated on the surface of the shell of radius R and no charge is therefore enclosed by the gaussian surface, therefore,

$$\oint_s \vec{E} \cdot d\vec{S} = 0$$

$$\therefore \qquad E = 0 \qquad ...(2.27)$$

The variation of the electric field intensity with distance from the centre is shown in Fig. 2.12 with the help of Eqs (2.25), (2.26) and (2.27).

(iii) Electric field intensity due to long uniformly charged cylinder

Case I At a point outside of the cylinder

Let P_1 be the point at a distance r from the axis of the cylinder of radius R [Fig. 2.13]. Imagine a cylindrical gaussian surface of length l through P_1. The field will have a cylindrical symmetry and the total flux over the gaussian surface

$$\varphi_E = \oint_s \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0},$$

where λ is the line charge density.

or, $$E \times 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\therefore \qquad E = \frac{\lambda}{2\pi r \varepsilon_0} \qquad ...(2.28)$$

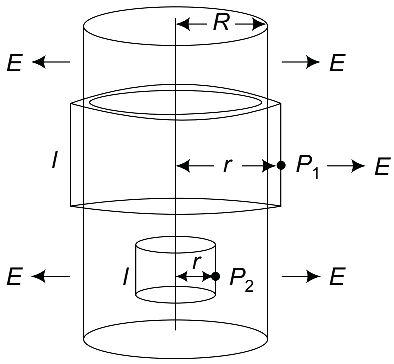


Fig. 2.13 Electric field at a point from the axis of a uniformly charged cylinder.

Case II At a point on the surface of the cylinder

Here $r = R$, the cylinder itself is the gaussian surface. The electric filed at the surface is obtained by replacing r by R in Eq. (2.28)

$$E = \frac{\lambda}{2\pi R \varepsilon_0} \qquad ...(2.29)$$

Case III At a point inside the cylinder

Let P_2 be the point at a distance r ($r < R$) from the axis of the cylinder [Fig. 2.13]. A coaxial cylinder of radius r and length l is constructed as gaussian surface such that the point P_2 lies on the curved surface of the cylinder.

The total charge enclosed by the gaussian surface

$$q' = \pi r^2 l \rho$$

[where, ρ is the volume charge density]

Again $(\pi R^2 l)\rho = \lambda l$

or, $$\rho = \frac{\lambda}{\pi R^2},$$

From Gauss' law, the electric flux

$$\varphi_E = \oint_s \vec{E} \cdot \overrightarrow{dS} = \frac{q'}{\varepsilon_0} = \frac{\pi r^2 l \rho}{\varepsilon_0}$$

or, $$E \times (2\pi r l) = \frac{\pi r^2 l \rho}{\varepsilon_0}$$

or, $$E = \frac{r\rho}{2\varepsilon_0} = \frac{r}{2\varepsilon_0}\left(\frac{\lambda}{\pi R^2}\right) = \frac{\lambda r}{2\pi R^2 \varepsilon_0} \qquad ...(2.30)$$

The variation of electric field intensity with distance r from the axis of a charged cylinder is shown in Fig. 2.14.

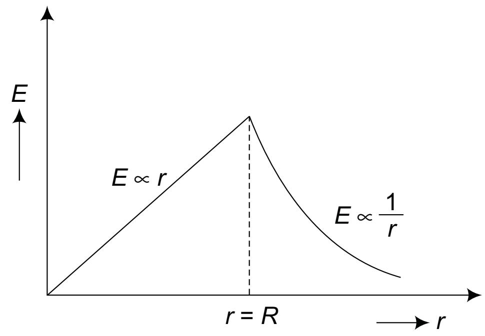


Fig. 2.14 Variation of $\vec{E}$ with distance r from the axis of a charged cylinder.

[**Note: For a hollow charged cylinder, the charge inside the cylinder is zero and so the electric flux inside the cylinder, $\varphi_E = 0$ which gives $E = 0$.**]

(iv) Electric field due to an infinite plane charge sheet

Here we consider an infinite thin plane charge sheet of positive charge having surface density of charge σ [Fig. 2.15]. To find the electric field at a point P_1, let us consider another point P_2 on the other side of the sheet so that two points P_1 and P_2 are equidistant from the sheet. We construct a cylindrical gaussian surface normally through the plane which extends equally on two sides of the plane.

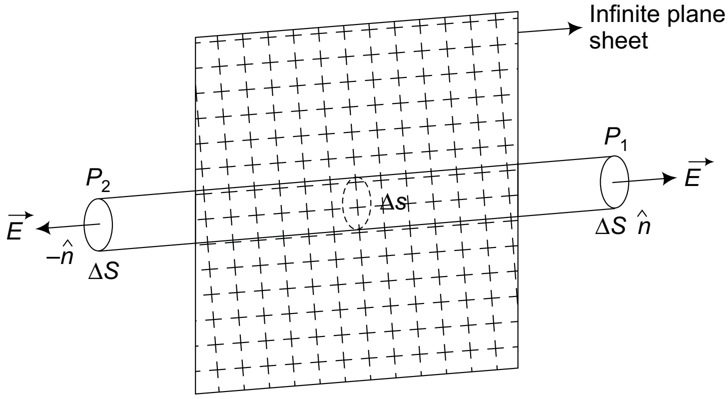


Fig. 2.15 Electric field due to an infinite charged sheet.

The flux of the electric field crossing through the gaussian surface

$$\varphi_E = E\,\Delta s + E\,\Delta S$$

$= 2E\,\Delta S$ [where ΔS is the area of cross section of each end face]

$$= \frac{\sigma \Delta S}{\varepsilon_0}$$

or, $$E = \frac{\sigma}{2\varepsilon_0} \qquad \ldots(2.31)$$

We see that electric field is uniform and does not depend on the distance from the charge sheet.

(v) Electric field near a charged conducting surface

Here we consider a plane conducting sheet. All the charges of the conductor lie on the surface, so the electric field inside the conductor is zero. We have to find the electric field at a point P which is near but outside the conductor. To find the electric field, we construct a gaussian surface as follows [Fig. 2.16]. The total flux through the gaussian surface is

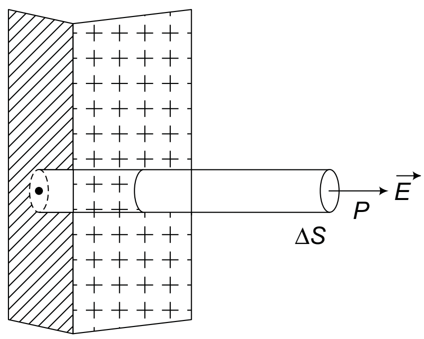


Fig. 2.16 Electric field at a point near a charged conducting surface.

$$\varphi_E = E\,\Delta S$$

and charge enclosed inside the closed surface is $\sigma\,\Delta S$. So from Gauss' law

$$E\,\Delta S = \frac{\sigma\,\Delta S}{\varepsilon_0}$$

or, $$E = \frac{\sigma}{\varepsilon_0} \qquad \ldots(2.32)$$

The electric field near a plane charged conductor is twice the electric field due to a non-conducting plane charge sheet. This is also known as *Coulomb's theorem*. The theorem states that the electric field at any point very close to the surface of a charged conductor is equal to charge density of the surface divided by free space permittivity.

2.13 POISSON'S AND LAPLACE'S EQUATIONS

The differential form of Gauss' law is

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[From Eq. (2.19)]}$$

Again, the electric field ($\vec{E}$) at any point is equal to the negative gradient of the potential V

i.e., $$E = -\vec{\nabla} V \quad \text{[From Eq. (2.10)]}$$

Now combining these two equations, we have

$$\vec{\nabla} \cdot (-\vec{\nabla} V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

or, $$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \ldots(2.33)$$

This is known as Poisson's equation.

Now in a charge-free region ($\rho = 0$), the Poisson's equation becomes

$$\nabla^2 V = 0 \qquad \ldots(2.34)$$

This equation is known as Laplace's equation and is valid only in the charge-free region.

The laplacian operator ∇^2 is a scalar operator and its form in three coordinate systems are:

(a) Cartesian system (x, y, z): $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

(b) Cylindrical coordinate system (ρ, ϕ, z):

$$\nabla^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(c) Spherical polar coordinate system (r, θ, φ)

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

2.13.1 Application of Laplace's Equation (Effective 1D Problem)

(i) Potential between the plates of a parallel-plate capacitor

Let us consider a parallel-plate condenser (capacitor) having two plates, one at $z = 0$ and other at $z = d$ [Fig. 2.17]. The potential at the upper plate is V_A and potential at the lower plate is zero. So, the potential exists only along the z direction.

The Laplace's equation in the cartesian system is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Here $$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

Since the potential exists only along the z direction

So, $$\frac{\partial^2 V}{\partial z^2} = 0 \quad \text{or} \quad \frac{\partial V}{\partial z} = C \text{ (constant)}$$

or, $$V = Cz + D \quad (D \text{ is another constant}) \qquad \ldots(2.35)$$

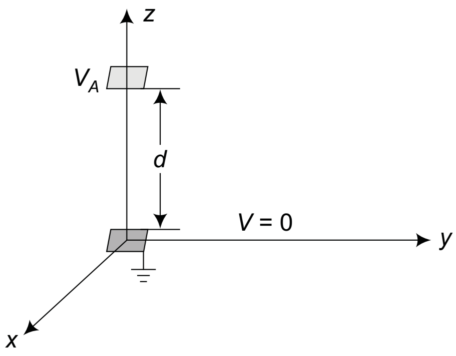


Fig. 2.17 Potential difference between the plates of a parallel-plate capacitor.

Now applying boundary conditions, i.e.,

$z = 0$, $V = 0$

and $z = d$, $V = V_A$

The first condition gives $D = 0$

From the second condition at $z = d$, $V = V_A$

$$V_A = Cd$$

or $$C = \frac{V_A}{d}$$

So, from Eq. (2.35), $$V = \frac{V_A z}{d} \qquad \ldots(2.36)$$

Equation (2.36) is the solution of Laplace's equation, which gives the potential between the plates.

(ii) Potential of a coaxial cylindrical capacitor

Here, we consider a cylindrical capacitor of inner and outer radii a and b ($b > a$) as shown in Fig. 2.18. The potential of the inner cylinder is V_A and the potential of the outer cylinder is zero.

Since the variation of potential exists only along the radial direction.

then $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \left[\because \frac{\partial^2 V}{\partial \varphi^2} = \frac{\partial^2 V}{\partial z^2} = 0\right] \qquad \ldots(2.37)$$

Integrating Eq. (2.37) twice with respect to r, the potential at an arbitrary distance r is

$$V = C \ln r + D \qquad \ldots(2.38)$$

[where C and D are constants of integration]

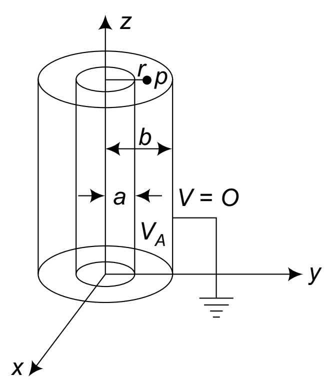


Fig. 2.18 Potential of a cylindrical capacitor.

Now applying boundary conditions, i.e.,

$$r = b,\ V = 0$$

and $$r = a,\ V = V_A$$

we have, from Eq. (2.38), $D = -C \ln b$ and $V_A = C \ln \frac{a}{b}$

so, $$C = \frac{V_A}{\ln \frac{a}{b}} \quad \text{and} \quad D = -V_A \frac{\ln b}{\ln \frac{a}{b}}$$

Now from Eq. (2.38), we have

$$V = \frac{V_A}{\ln \frac{a}{b}} \ln r - V_A \frac{\ln b}{\ln \frac{a}{b}}$$

$$= V_A \frac{\ln \frac{r}{b}}{\ln \frac{a}{b}} \qquad \text{....(2.39)}$$

Equation (2.39) is the solution of Laplace's equation in cylindrical coordinate, which gives the potential inside a cylindrical capacitor.

(iii) Potential of a concentric spherical capacitor

Let us consider a spherical capacitor of inner and outer radii a and b ($b > a$) as shown in Fig. 2.19. The potential of the inner sphere is V_A and the outer sphere is zero.

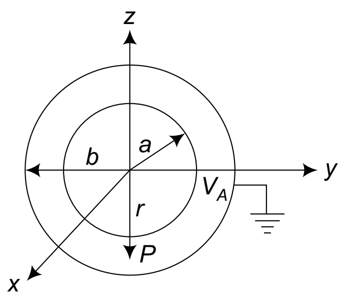


Fig. 2.19 Potential of concentric spherical capacitor.

Since the variation of potential exists only along the radial direction then from Laplace's equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \quad \left[\text{Here, } \frac{\partial V}{\partial \theta} = \frac{\partial^2 V}{\partial \varphi} = 0\right] \qquad \text{...(2.40)}$$

Integrating Eq. (2.40) twice with respect to r, the potential at P will be

$$V = -\frac{C}{r} + D \qquad \text{...(2.41)}$$

[where C and D are integrating constants]

Now applying boundary conditions, i.e.,

$$r = b,\ V = 0$$

and $$r = a,\ V = V_A$$

we have from Eq. (2.41) $D = \frac{C}{b}$

So, $$V = -\frac{C}{r} + \frac{C}{b} = C\left(\frac{1}{b} - \frac{1}{r}\right) \qquad \text{...(2.42)}$$

and from second boundary condition ($r = a$, $V = V_A$), Eq. (2.41) gives $V_A = C\left(\frac{1}{b} - \frac{1}{a}\right)$

or, $$C = \frac{V_A}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

Now putting the values of c in Eq. (2.42)
we have

$$V = \frac{V_A}{\left(\frac{1}{b} - \frac{1}{a}\right)}\left(\frac{1}{b} - \frac{1}{r}\right) \qquad \ldots(2.43)$$

Equation (2.43) is the solution of Laplace's equation in spherical polar coordinate, which gives the potential inside a spherical capacitor.

Worked Out Problems

Example 2.1 Compare the electrostatics force and gravitational force between a proton and electron in a hydrogen atom. Given $e = 1.6 \times 10^{-19}$ C, $m_e = 9.1 \times 10^{-31}$ kg, $m_p = 1.7 \times 10^{-27}$ kg and $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}

Sol. The electrostatic force between a proton and electron is

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = 9 \times 10^9 \times \frac{(1.6 \times 10^{-19})^2}{r^2} \qquad \left[\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ Nm}^2/\text{C}^2\right]$$

The gravitational force between a proton and electron

$$F_g = G\frac{m_p m_e}{r^2} = 6.67 \times 10^{-11}\,\frac{1.7 \times 10^{-27} \times 9.1 \times 10^{-31}}{r^2}$$

or $$\frac{F_e}{F_g} = \frac{9 \times 10^9}{6.67 \times 10^{-11}} \times \frac{(1.6 \times 10^{-19})^2}{9.1 \times 1.7 \times 10^{-58}} = 2.2 \times 10^{34}$$

So, electrostatics force between electron and proton is much greater than gravitational force.

Example 2.2 Two particles P and Q having charges 8.0×10^{-6} C and -2.0×10^{-6} C respectively are held fixed with a separation of 20 cm. Where should a third charged particle be placed so that it does not experience a net electric force?

Sol. Since the particles are charged with opposite sign, the point R where net electric force is zero, can't be between P and R.

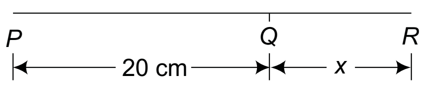


Fig. 2.1W

Suppose $QR = x$ metre and charge on R is q.

The force at R due to P is $= \dfrac{8.0 \times 10^{-6} \times q}{4\pi\varepsilon_0 (x + 0.2)^2}$

The force at R due to Q is $= \dfrac{2.0 \times 10^{-6} \times q}{4\pi\varepsilon_0 x^2}$

They are oppositely directed and the resultant is zero.

So, $$\frac{8 \times 10^{-6}}{(0.2 + x)^2} = \frac{2 \times 10^{-6}}{x^2}$$

or, $$0.2 + x = 2x$$

or, $$x = 0.2 \text{ m}$$

Example 2.3 Four equal charges + q are placed at the corner of a square. Find the point charge at the center of the square so that the system will remain in equilibrium.

Sol. The charge Q at the center (O) must be negative [Fig. 2.2W]. If the net force on a charge at D is zero, then by symmetry it follows that the net force experienced by charges at other points will also be zero.

The resultant force (F_R) at D due to all other charges at different corner will be

$$F_R = F_B + F_C \cos 45° + F_A \cos 45°$$

$$= \frac{1}{4\pi\varepsilon_o}\left[\frac{q^2}{(DO)^2} + \frac{q^2}{a^2}\frac{1}{\sqrt{2}} + \frac{q^2}{a^2\sqrt{2}}\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{(\sqrt{2}a)^2} + \frac{2}{\sqrt{2}a^2}\right]$$

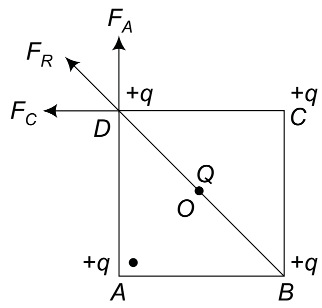


Fig. 2.2W Equilibrium of four equal charges at corners of a square when a charge is placed at the center.

That force must be equal to $\dfrac{Qq}{4\pi\varepsilon_0\left(\dfrac{1}{\sqrt{2}}a\right)^2}$

So $$\frac{Qq \times 2}{4\pi\varepsilon_0 a^2} = \frac{q^2}{4\pi\varepsilon_0 a^2}\left(\frac{1}{2} + \frac{2}{\sqrt{2}}\right)$$

or, $$Q = q\,\frac{(1 + 2\sqrt{2})}{2}$$

So, the charge at the center will be $-\dfrac{q(1 + 2\sqrt{2})}{2}$.

Example 2.4 Two similar balls of mass m are hung from silk threads of length l and carry same charges. Prove that for a small angle φ, the separation of the charges, will be

$$x = \left(\frac{q^2 l}{2\pi\varepsilon_o mg}\right)^{\frac{1}{3}}$$

Sol. In Fig. 2.3W, each ball of charge + q are suspended from O by silk threads. Here $\theta = \dfrac{\varphi}{2}$

The restoring force = $mg \sin\theta$ and electrostatic repulsive force between the balls is = $\dfrac{qq}{4\pi\varepsilon_0 x^2}$

In equilibrium, $mg \sin\theta = \dfrac{q^2}{4\pi\varepsilon_0 x^2}$ where x is the separation of the balls.

From the figure for small θ, $\sin\theta = \dfrac{x/2}{l} = \dfrac{x}{2l}$

$\therefore$ $$mg\,\frac{x}{2l} = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

or, $$x^3 = \frac{q^2 l}{2\pi\varepsilon_0 mg} \quad \text{or,} \quad x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{\frac{1}{3}}$$

Hence proved.

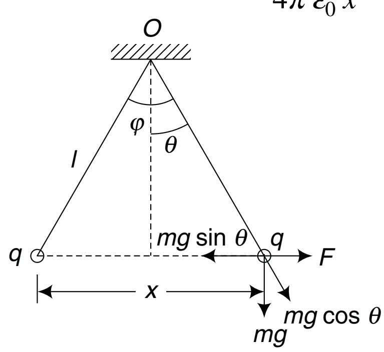


Fig. 2.3W Separation of two like and equal charges suspended from a point.

Example 2.5 An amount of charge Q is divided into two particles. Find the charge on each particle so that the effective force between them will be maximum.

Sol. Suppose the charge on one particle be q, then charge on the other is $(Q - q)$. If they are separated by a distance r then force

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q(Q-q)}{r^2}$$

For maximum F, $\frac{dF}{dq} = 0$

So, $\frac{d}{dq}\left[\frac{1}{4\pi\varepsilon_0} \frac{q(Q-q)}{r^2}\right] = 0$

or, $\frac{d}{dq}(qQ - q^2) = 0$

or, $Q - 2q = 0$ or, $q = \frac{Q}{2}$

So, for maximum F, Q is to be equally divided in the particles.

Example 2.6 Find out electric field intensity at any point on the axis of the uniformly charged rod.

Sol. In Fig. 2.4W, let L is the length of the rod AB uniformly charged (q). If λ be the linear charge density then $\lambda = \frac{q}{L}$ and charge on an elementary length dx is $dq = \lambda\, dx$. The electric field at P due to dx is

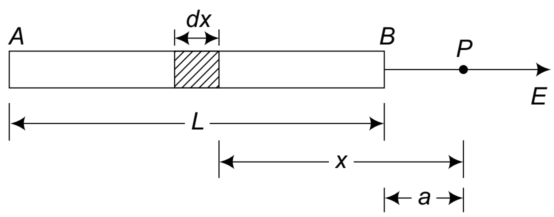


Fig. 2.4W Electric field at a point *P* on the axis of a uniformly charged rod *AB*.

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 x^2}\hat{x} \text{ along } \overrightarrow{BP}$$

For the entire charged rod, electric field acts in the same direction and total field at P is

$$E = \int_a^{a+L} \frac{\lambda}{4\pi\varepsilon_0} \frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a+L}\right] = \frac{\lambda L}{4\pi\varepsilon_0\, a\,(a+L)}$$

If $a >> L$, $E = \frac{1}{4\pi\varepsilon_o} \frac{q}{a^2}$.

So under this condition, a charged rod behaves as a point charge.

Example 2.7 Find out the electric field intensity at any point on the perpendicular bisector of a uniformly charged rod.

Sol. Consider an elementary length dx at a distance x from the center of the rod of length L [Fig. 2.5W].

The charge on this element dx is $dq = \frac{q}{L}dx = \lambda\, dx$, where $\lambda = \frac{q}{L}$ the linear charge density.

The electric field at P due to this element is

$$dE = \frac{dq}{4\pi\varepsilon_0 (AP)^2} = \frac{dq}{4\pi\varepsilon_0 (a^2 + x^2)}$$

Again, the component of dE along OP is $dE \cos\theta$.

The resultant field at P due to the whole charged rod is

$$E = \int dE \cos\theta = \frac{1}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{q\,a\,dx}{L(a^2+x^2)^{3/2}}$$

$$\left[\because \cos\theta = \frac{a}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{q\,a}{4\pi\varepsilon_0 L} \int_{-L/2}^{L/2} \frac{dx}{(a^2+x^2)^{3/2}}$$

We have $x = a\tan\theta$ or $dx = a\sec^2\theta\, d\theta$

or, $$E = \frac{q\,a}{4\pi\varepsilon_0 L}\int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{qa^2}{4\pi\varepsilon_0 a^3 L}\int \cos\theta\, d\theta$$

$$= \frac{q}{4\pi\varepsilon_0 La}\sin\theta = \frac{q}{4\pi\varepsilon_0 La}\left[\frac{x}{(a^2+x^2)^{1/2}}\right]_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{2q}{a\sqrt{L^2+4a^2}} = \frac{q}{2\pi\varepsilon_0 a\sqrt{L^2+4a^2}}$$

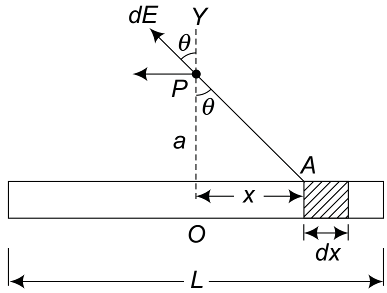


Fig. 2.5W Electric field at a point on the perpendicular bisector of a uniformly charged rod.

Example 2.8 Find out the electric field intensity at a point on the axis of a uniformly charged ring.

Sol. Consider an elementary length dl of the ring (Fig. 2.6W). The charge of the ring $dq = \lambda\, dl$. The field at P due to dq of dl is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} \text{ along } AP$$

The component of dE along the x axis is $dE_x = dE\cos\theta$

$$\therefore \quad dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dl}{r^2}\cos\theta$$

Total field intensity at P due to the whole ring

$$E = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{dl}{r^2}\cos\theta = \frac{\lambda}{4\pi\varepsilon_0 r^2}\int\frac{x}{r}\,dl$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{x}{r^3}\times 2\pi a$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi a x}{(a^2+x^2)^{3/2}} = \frac{qx}{4\pi\varepsilon_0 (a^2+x^2)^{3/2}} \text{ along the } x \text{ axis}$$

When $x >> a$, $\quad E = \dfrac{q}{4\pi\varepsilon_0 x^2}$

So, at large distance the ring behaves like a point charge.

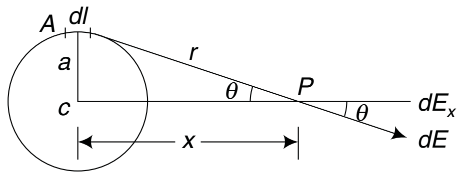


Fig. 2.6W Electric field intensity at a point on the axis of a charged ring.

Example 2.9 Find out the electric field intensity at any point on the axis of a uniformly charged disc.

Sol. In Fig. 2.7W, Let R be the radius of the disc and x be the distance of the field point P from the center O. We consider a concentric ring within radii r and $r + dr$. If σ is the surface charge density then total charge on a surface element dS is $\sigma\, dS$. The field at P due to ds is given by $d\vec{E} = \dfrac{\sigma\, dS}{4\pi\varepsilon_0 a^2}\hat{a}$, here $a^2 = x^2 + r^2$.

The component along the axis of the disc

$$dE_1 = \frac{1}{4\pi\varepsilon_o}\frac{\sigma\, dS}{(r^2 + x^2)}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dS}{(r^2 + x^2)}\frac{x}{(x^2 + r^2)^{1/2}}$$

$$= \frac{\sigma\, dS\, x}{4\pi\varepsilon_0 (r^2 + x^2)^{3/2}}$$

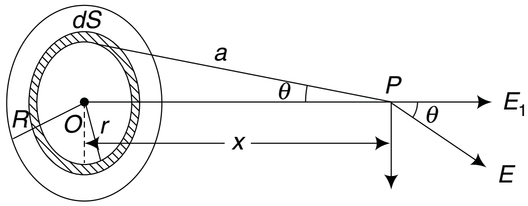


Fig. 2.7W Electric field at a point *P* on the axis of a charged disc.

Again $ds = 2\pi r\, dr$

or, $$dE_1 = \frac{\sigma \times 2\pi r\, dr\, x}{4\pi\varepsilon_0 (r^2 + x^2)^{3/2}}$$

Total intensity at P is $E = \int dE_1 = \dfrac{2\pi x\sigma}{4\pi\varepsilon_0}\displaystyle\int_0^R \frac{r\, dr}{(r^2 + x^2)^{3/2}}$

$$E = \frac{x\sigma}{2\varepsilon_0}\left[\frac{1}{x} - \frac{1}{(R^2 + x^2)^{1/2}}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{(R^2 + x^2)^{1/2}}\right]$$

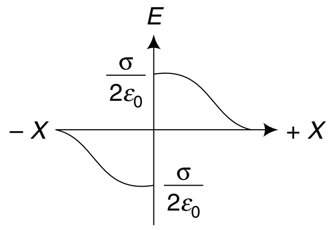


Fig. 2.8W Variation of field intensity with distance from the center of a charged disc.

For infinite charge sheet $R \to \alpha$

$\therefore$ $$E = \frac{\sigma x}{2E_0\, |x|}$$

For positive σ, E is positive when x is positive and it is negative when x is negative. The magnitude of the field is independent of the distance x and is given by $\dfrac{\sigma}{2\varepsilon_0}$. The variation of field intensity with distance from the center of a uniformly charged disc is shown in Fig. 2.8W.

Example 2.10 Five equal charges of 40 nC each are placed at five vertices of a regular hexagon of 6 cm side. The sixth vertex is free. Determine the electric field at the center of the hexagon due to the distribution.

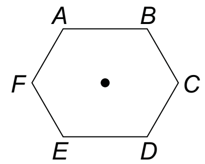


Fig. 2.9W Electric field at center due to five equal charges placed at five corners of a regular hexagon.

Sol. The field at the center due to the charges located at two opposite vertices is zero. Since there is no charge at F (free vertex) [Fig. 2.9W] so the resultant field will be due to charge q located at C. The field is directed from the center to the vacant corner and its magnitude is $\dfrac{q}{4\pi\varepsilon_0 a^2}$, where a is the distance of the center from each of the vertices. So, the field is $\dfrac{9 \times 10^9 \times 40 \times 10^{-9}}{(6 \times 10^{-2})^2}$ N/C or, 10^5 N/C

Example 2.11 Infinite number of positive charges, each of magnitude q is placed on the x axis at the point $x = 1, 2, 4, 8, \ldots$ What will be intensity of electric field at $x = 0$? Also calculate the electric field if alternative charges are of opposite signs.

Sol. The resultant intensity at $x = 0$ [Fig. 2.10W] is

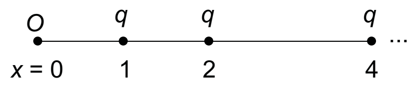


Fig. 2.10W Electric field at $x = 0$ due to equal charge at $x = 1, 2, 4, 8, \ldots$

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1^2} + \frac{q}{2^2} + \frac{q}{4^2} + \frac{q}{8^2} + \ldots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1 - \frac{1}{4}\right)}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{4}{3}\text{ units}$$

If alternate charges are of opposite signs then electric intensity at $x = 0$ is

$$E_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1^2} - \frac{q}{2^2} + \frac{q}{4^2} - \frac{q}{8^2} + \ldots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1 + \frac{1}{4}\right)} = \frac{q}{4\pi\varepsilon_0}\frac{4}{5}\text{ units}$$

Example 2.12 Three charges $q_1\ q_2$ and q_3 are at the vertice of an equilateral triangle of 1 m side. The charges are $q_1 = -2\,\mu C$, $q_2 = 6\,\mu C$ and $q_3 = 4.5\,\mu C$. Find the total potential energy of this charge distribution.

Sol. The total potential energy, $$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right]$$

$$= 9 \times 10^9 \times \frac{1}{a}(q_1 q_2 + q_2 q_3 + q_1 q_3) \quad [r_{12} = r_{23} = r_{31} = a \text{ (Say)}]$$

$$= 9 \times 10^9 \times [-2 \times 6 + 6 \times 4.5 - 2 \times 4.5] \times 10^{-12}$$

$$= 0.054 \text{ Joule}$$

Example 2.13 Three charges q, $2q$, and $4q$ are placed along a straight line of 6 cm length. Where should the charges be placed so that potential energy of the system is minimum. Find out the distance of the charges.

Sol. Let $2q$ be placed between the other two charges [Fig. 2.11W]. Suppose, the distance between q and $2q$ is x m. so, the distance between $2q$ and $4q$ is $(0.06 - x)$ m.

The potential energy of the whole system,

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q \times 2q}{x} + \frac{2q \times 4q}{0.06 - x} + \frac{q \times 4q}{0.06}\right]$$

For minimum value of U $\quad \frac{dU}{dx} = 0$

or, $$-\frac{1}{x^2} + \frac{4}{(0.06 - x)^2} = 0$$

or, $$x = 0.02 \text{ m}$$

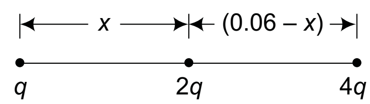


Fig. 2.11W Minimum PE due to charges q, $2q$, $4q$ placed on a line of 6 cm length.

So, the distance between q and $2q$ is 0.02 m and the distance between $2q$ and $4q$ is 0.04 m.

Example 2.14 If the electric field is given by $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$, calculate the electric flux through a surface of area of 20 units laying in the yz plane.

Sol. Since the surface area lies in the yz plane, the area vector $\vec{S}$ is directed along the x direction. So $\vec{S} = 20\hat{i}$.

Here $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore$ electric flux through the surface is $\varphi_E = \vec{E} \cdot \vec{S}$

$$= (6\hat{i} + 3\hat{j} + 4\hat{k}) \cdot 20\hat{i}$$
$$= 120 \text{ units}$$

Example 2.15 Find out electric field intensity at regions I, II , III due to two infinite plane parallel sheets of charge [Fig. 2.12W].

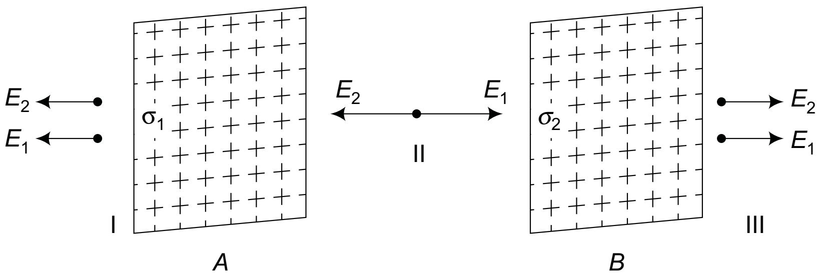


Fig. 2.12W Electric field at three points I, II and III due to two infinite plate charge sheets.

Sol. Let A and B be two infinite plane parallel charge sheets and σ_1, σ_2 be uniform surface densities of charge on A and B respectively. Here $\sigma_1 > \sigma_2$ (say).

In region I, the net electric field

$$E_{\text{I}} = E_1 + E_2 = \left(\frac{-\sigma_1}{2\varepsilon_0}\right) + \left(\frac{-\sigma_2}{2\varepsilon_0}\right) = -\frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$$

In region II, $$E_{\text{II}} = E_1 - E_2 = \frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2)$$

In region III, $$E_{\text{III}} = E_1 + E_2 = \frac{\sigma_1}{2\varepsilon_0} + \frac{\sigma_2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$$

Now if $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$ then $E_{\text{I}} = 0$ $E_{\text{III}} = 0$, and $E_{\text{II}} = \dfrac{\sigma}{\varepsilon_0}$

Example 2.16 The electric field components in Fig. 2.13W are $E_x = \alpha x^{1/2}$, $E_y = E_z = 0$ in which $\alpha = 800$ NC^{-1} m$^{-1/2}$. Calculate electric flux through the cube and the charge within the cube. Assume that $a = 0.1$ m.

Sol. The electric flux is zero for each face of the cube except the two faces $ABCD$ and $EFGH$. The magnitude of electric field at the face $ABCD$, $E_1 = \alpha x^{1/2} = \alpha a^{1/2}$ and at the face $EFGH$, $E_2 = \alpha x^{1/2} = \alpha (2a)^{1/2}$.

So, flux $\varphi_1 = \vec{E}.\overrightarrow{dS} = E_1 S \cos 180°$

$= \alpha\, a^{1/2} \times a^2\,(-1) = -\,a^{5/2}\,\alpha$

and flux $\varphi_2 = \vec{E}.\overrightarrow{dS} = E_2 S \cos 0°$

$= \alpha\,(2a)^{1/2}\,a^2 = 2^{1/2}\,a^{5/2}\,\alpha$

So, net flux $\varphi = (E_2 - E_1) = 2^{1/2}\,a^{5/2}\,\alpha - a^{5/2}\,\alpha$

$= a^{5/2}\,\alpha\,(2^{1/2} - 1)$

Now, putting the value $a = 0.1$ m and

$\alpha = 800$ NC^{-1} m$^{-1/2}$,

we have net flux = 1.05 Nm^2 C^{-1} and

charge $q = \varepsilon_0 \varphi$

So, $q = 8.85\ 10^{-12} \times 1.05$

$= 9.3 \times 10^{-12}$ C

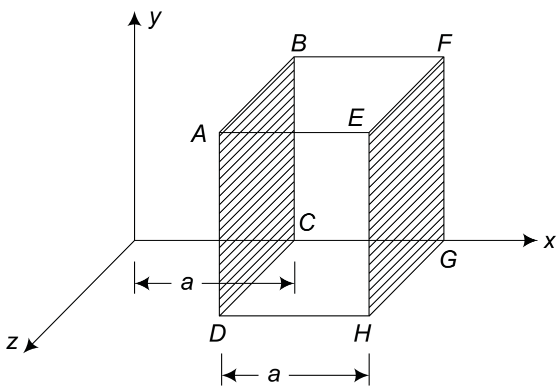


Fig. 2.13W Electric flux through a cube and charge inside it.

Example 2.17 Using Gauss' law in integral form, obtain the electric field due to the following charge distribution in spherical coordinates, [WBUT 2012]

$$\rho\,(r, \theta, \varphi) = \rho_0 \left(1 - \frac{r^2}{a^2}\right) \qquad 0 < r < a$$

$$= 0 \qquad a < r < \infty$$

Sol. Consider region $0 < r < a$, in spherical coordinate volume element $dv = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

Now, from Gauss' law $\int_S \vec{E} \cdot \overrightarrow{dS} = \dfrac{q}{\varepsilon_0}$

or, $$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} \int_{r=0}^{r} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, d\theta\, dr\, d\varphi$$

$$E \times 4\pi r^2 = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right) 4\pi$$

or, $$E = \frac{\rho_0\,(5a^2 r^3 - 3r^5)}{15a^2\, r^2\, \varepsilon_0} \quad \text{for } 0 < r < a$$

For region $a < r < \infty$, apply Gauss' law

$$\oint_s \vec{E} \cdot \overrightarrow{dS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\varphi$$ [Charge enclosed up to $r = a$]

or, $$E \times 4\pi r^2 = \rho_0 \left(\frac{a^3}{3} - \frac{a^5}{5a^2}\right) 4\pi$$

or, $$E = \frac{2\rho_0\, a^3}{15\varepsilon_0\, r^2} \quad a < r < \infty$$

and at $r = a$ $$E = \frac{2}{15}\,\frac{\rho_0\, a}{\varepsilon_0}$$

Example 2.18 If the potential in the region of space near the point (– 2 m, 4 m, 6 m) is $V = 80x^2 + 60y^2$ volt, what are the three components of the electric field at that point?

Sol.
$$\vec{E} = -\vec{\nabla} V = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

or,
$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}$$

So,
$$E_x = -\frac{\partial}{\partial x}(80x^2 + 60y^2) = -160x$$

or, E_x at (– 2 m) = – 160 × (– 2) = 320 Vm^{-1}

$$E_y = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}(80x^2 + 60y^2) = -120\,y$$

E_y at (4 m) = – 120 × 4 = – 480 Vm^{-1}

$$E_z = -\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(80x^2 + 60y^2) = 0$$

Example 2.19 If $V\left(= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$ is the potential, at *a* distance *r*, due to *a* point charge *q*, then determine the electric field due to point charge *q*, at *a* distance *r*.

Sol.
$$\vec{E} = -\vec{\nabla} V = -\hat{r}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$$
$$= -\hat{r}\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r^2}\right) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

Example 2.20 Three point charges *q*, 2*q* and 8*q* are placed on a 9 cm long straight line Fig. 2.14 W. Determine the position where the charges should be placed such that the potential energy of this system is minimum.

Sol. Let *q* charge be placed at a distance *x* from the charge 2*q*.

Now potential energy,
$$U = \frac{2q^2}{x} + \frac{8q^2}{9 - x}$$

For minimum *U*,
$$\frac{dU}{dx} = 0 = -\frac{2q^2}{x^2} + \frac{8q^2}{(9-x)^2}$$

or, $(9 - x)^2 = 4x^2$

or, $9 - x = \pm 2x$

or, $x = 3, -9$ cm

But $x = -9$ is not possible so $x = 3$ cm.

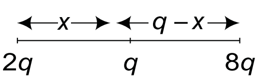


Fig. 2.14 W Placing of three charges *q*, 2*q*, 8*q* on a line of 9 cm length to make PE minimum.

Example 2.21 Show that the potential function $V = V_0(x^2 - 2y^2 + z^2)$ satisfies Laplace's equation, where V_0 is a constant. [WBUT 2004]

Sol. Here
$$V = V_0(x^2 - 2y^2 + z^2)$$

or, $$\vec{\nabla}V = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)[V_0\,(x^2 - 2y^2 + z^2)]$$
$$= V_0\,(2x\,\hat{i} - 4y\hat{j} + 2z\,\hat{k})$$

Again $$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}\,V$$
$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right).\,[V_0\,(2x\hat{i} - 4y\hat{j} + 2z\,\hat{k})]$$
$$= V_0\,(2 - 4 + 2) = 0$$

or, $$\nabla^2 V = 0$$

So, the potential function V satisfies Laplace's equation

Example 2.22 A very long cylindrical object carries charge distribution proportional to the distance from the axis (r). If the cylinder is of radius a, then find the electric field both at $r > a$ and $r < a$ by the application of Gauss' law in electrostatics. [WBUT 2007]

Sol. Let A_1 and A_2 be the points [Fig. 2.15W] at a distance r such that (i) $r < a$ (ii) $r > a$

(i) ***Inside*** ($r < a$)

Here we consider a coaxial cylinder of radius $r < a$ and length l.

Total flux through the cylindrical surface

$$\oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0}$$ (q is the total charge enclosed within the cylinder).

Here, if $\rho\,(r)$ be the charge density then $\rho\,(r) = \lambda r$ where λ is constant.

Then total charge $$q = \int_0^r 2\pi rl\,dr\,\rho$$
$$= \int 2\pi rl\,dr\,\lambda r = 2\pi l\lambda \int_0^r r^2\,dr$$
$$= \frac{2}{3}\,\pi l\lambda\,r^3$$

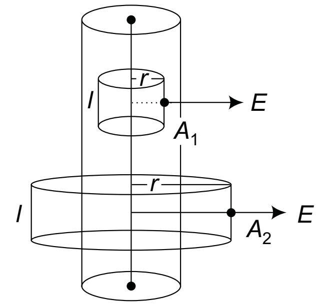


Fig. 2.15W Electric field at points where $r < a$ and $r > a$ due to long cylindrical charged body.

Now, from Gauss' law $$\oint_S \vec{E}\cdot\vec{dS} = \frac{2\pi l\lambda\,r^3}{3\varepsilon_0}$$

or, $$E \times 2\pi rl = \frac{2\pi l\lambda\,r^3}{3\varepsilon_0}$$

or, $$E = \frac{\lambda r^2}{3\varepsilon_0}$$

(ii) ***Outside*** ($r > a$)

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

But $$q = \int_0^a 2\pi rl\,dr\,\rho = 2\pi l \int_0^a r\,dr\,\lambda r = 2\pi\lambda\,l \int_0^a r^2\,dr$$

Now $$\oint_S \vec{E}\cdot\vec{dS} = \frac{2\pi\lambda l}{\varepsilon_0}\int_a^0 r^2\,dr$$

or, $$E \times 2\pi r l = \frac{2}{3}\frac{\pi a^3}{\varepsilon_0}\lambda l$$

or, $$E = \frac{1}{3}\frac{a^3 \lambda}{\varepsilon_0 r}$$

Example 2.23 The potential field at any point in free space is given by $V = 5x^2y + 3yz^2 + 6xz$ volt, where x, y, z are in meters. Calculate the volume charge density at point (2, 5, 3) m.

Sol. From Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Here $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

So, $$\frac{\partial^2 V}{\partial^2 x^2} = \frac{\partial^2}{\partial x^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial x}(10xy + 6z) = 10y$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2}{\partial y^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial y}(5x^2 + 3z^2) = 0$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial z^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial z}(6yz + 6x) = 6y$$

or, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 10y + 6y = 16y$$

Now at point (2, 5, 3), $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 80$$

or, $$\nabla^2 V = 80 = -\frac{\rho}{\varepsilon_0}$$

or, $$\rho = -80\,\varepsilon_0$$
$$= -80 \times 8.854 \times 10^{-12}\ \text{C/m}^3$$

Example 2.24 In a field $\vec{E} = -50y\,\hat{i} - 50x\,\hat{j} + 30\,\hat{k}$ V/m, calculate the differential amount of work done in moving $2\mu C$ charge a distance $5\,\mu$m from P_1 (1, 2, 3,) to P_2 (2, 4, 1).

Sol. We know that work done $dw = -q\vec{E}.\vec{dl}$

Here $$\vec{dl} = 5\hat{e}_{P_1P_2}$$

$$\hat{e}_{P_1P_2} = \frac{(2-1)\,\hat{i} + (4-2)\,\hat{j} + (1-3)\,\hat{k}}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{1}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$$

or, $$\vec{dl} = \frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k})\ \mu\text{m}.$$

or, $$dw = -q\vec{E}\cdot\vec{dl} = -2 \times 10^{-6}(-50y\,\hat{i} - 50x\hat{j} + 30\,\hat{k}).\frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k}) \times 10^{-6}$$

At initial point (1, 2, 3)

$$dw = -2\times10^{-6}(-100\,\hat{i} - 50\,\hat{j} + 30\,\hat{k}).\frac{5}{3}\times10^{-6}\,(\hat{i} + 2\hat{j} - 2\hat{k})$$

$$= -\frac{10}{3}\times10^{-12}\,(-100 - 100 - 60)$$

$$= -\frac{10}{3}\times260\times10^{-12}\text{ Joule} = 8.66\times10^{-10}\text{ Joule}$$

Example 2.25 Show that $V = \frac{1}{r}$ satisfies Laplace's equation.

Sol. The position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Magnitude of $\vec{r}$, $\quad |\vec{r}| = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore \quad \nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 + y^2 + z^2)^{-1/2}$$

Now, $$\frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{\partial}{\partial x}\{-x(x^2 + y^2 + z^2)^{-3/2}\}$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly, $$\frac{\partial^2}{\partial y^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

and $$\frac{\partial^2}{\partial z^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\therefore \quad \nabla^2\left(\frac{1}{r}\right) = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 - 2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$$

So, $V = \frac{1}{r}$ satisfies Laplace's equation.

Example 2.26 The potential in a medium is given by $\varphi(r) = \dfrac{qe^{-r/\lambda}}{4\pi\varepsilon_0 r}$

(i) Obtain the corresponding electric field.

(ii) Find the charge density that may produce the potential mentioned above. [WBUT 2008]

Sol. Here $\varphi(r)$ is purely a function of r. So, electric field

(i) $$\vec{E} = -\vec{\nabla}\varphi = -\hat{r}\frac{\partial\varphi}{\partial r} = -\hat{r}\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right)$$

$$= -\hat{r}\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{\lambda r}e^{-r/\lambda} - \frac{1}{r^2}e^{-r/\lambda}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\hat{r}\left(\frac{1}{\lambda r} + \frac{1}{r^2}\right)e^{-r/\lambda}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}}{\lambda r^2} + \frac{\vec{r}}{r^3}\right)e^{-r/\lambda} \qquad \left[\text{where } \hat{r} = \frac{\vec{r}}{r}\right]$$

(ii) We know that $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

or, $$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = \varepsilon_0 \vec{\nabla} \cdot \left[\frac{q}{4\pi\varepsilon_0} \left(\frac{\vec{r}}{\lambda r^2} + \frac{\vec{r}}{r^3} \right) e^{-r/\lambda} \right]$$

$$= \frac{q}{4\pi} \left[\vec{\nabla} \cdot \left(\frac{\vec{r}}{\lambda r^2} \right) e^{-r/\lambda} + \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) e^{-r/\lambda} \right]$$

By applying $\vec{\nabla} \cdot (\varphi \vec{A}) = \vec{\nabla}\varphi \cdot \vec{A} + \varphi \vec{\nabla} \cdot \vec{A}$

We have $$\rho = \frac{q}{4\pi} \left[\frac{1}{\lambda} \vec{\nabla} \left(e^{-\frac{r}{\lambda}} \right) \cdot \frac{\vec{r}}{r^2} + \frac{1}{\lambda} e^{-r/\lambda} \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^2} \right) + \vec{\nabla} e^{-r/\lambda} \cdot \frac{\vec{r}}{r^3} + e^{-r/\lambda} \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^3} \right) \right]$$

Again $\vec{\nabla}(e^{-r/\lambda}) = -\frac{1}{\lambda} e^{-\frac{r}{\lambda}} \hat{r}$

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) = 0 \quad \text{and} \quad \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^2} \right) = \frac{1}{r^2}$$

So, $$\rho = \frac{q}{4\pi} \left[-\frac{1}{\lambda^2} e^{-\frac{r}{\lambda}} \frac{1}{r} + \frac{1}{\lambda} e^{-\frac{r}{\lambda}} \frac{1}{r^2} - \frac{1}{\lambda} e^{-\frac{r}{\lambda}} \frac{1}{r^2} \right]$$

$$= -\frac{q}{4\pi \lambda^2 r} e^{-r/\lambda}.$$

Example 2.27 Show that $\vec{F} = (2xy + z^3)\,\hat{i} + x^2 \hat{j} + 3xz^2\,\hat{k}$ is a conservative field. Find also the scalar potential. [WBUT 2003]

Sol. We know that for conservative force field $\vec{\nabla} \times \vec{F} = 0$

Here $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$

$$\therefore \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^3) & x^2 & 3xz^2 \end{vmatrix} = (0 - 0)\,\hat{i} + (3z^2 - 3z^2)\,\hat{j} + (2x - 2x)\,\hat{k} = 0$$

So, $\vec{F}$ is a conservative field.

Again, we know that for a conservative field $\vec{F} = \vec{\nabla} V$, where V is the scalar potential.

$$\therefore \quad \vec{F} \cdot \vec{dr} = \vec{\nabla} V . \vec{dr} = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = dV$$

$\therefore$ here $\vec{F} \cdot \vec{dr} = (2xy + z^3)dx + x^2 dy + 3xz^2\, dz$

So, $$dV = \vec{F} . \vec{dr} = (2xy + z^2)\, dx + x^2 dy + 3xz^2\, dz$$
$$= 2xy\, dx + x^2 dy + z^3\, dx + 3xz^2\, dz$$
$$= d\,(x^2 y) + d\,(xz^3)$$
$$= d\,(x^2 y + xz^3)$$

$$\therefore \quad V = \int dV = \int d\,(xy^2 + xz^3) = x^2 y + xz^3 + \text{constant}.$$

Example 2.28 Find the electric field due to the following electric potential.

$$V = \frac{\sin\theta\cos\varphi}{r^2}$$

Sol. We know $\vec{E} = -\vec{\nabla} V$. Here, V is the function of r, θ, φ. So in spherical polar coordinates

$$\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}\hat{e}_\varphi\right]$$

$$\frac{\partial V}{\partial r} = -\frac{2\sin\theta\cos\varphi}{r^3}, \frac{\partial V}{\partial\theta} = \frac{\cos\theta\cos\varphi}{r^2}, \frac{\partial V}{\partial\varphi} = -\frac{\sin\theta\sin\varphi}{r^2}$$

So, $$\vec{E} = \frac{2\sin\theta\cos\varphi}{r^3}\hat{e}_r - \frac{1}{r^3}\cos\theta\cos\varphi\,\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\sin\theta\sin\varphi}{r^2}\hat{e}_\varphi$$

$$= \frac{1}{r^3}[2\sin\theta\cos\varphi\,\hat{e}_r - \cos\theta\cos\varphi\,\hat{e}_\theta + \sin\theta\,\hat{e}_\varphi] \text{ units}$$

Example 2.29 Is it possible for the electric potential in a charge-free space to be given by (a) $V = x^2 + y^2 - y^2$ (b) $x^2 + y^2 - 2z^2$. If not, find the charge density.

Sol. (a) $$V = x^2 + y^2 - z^2$$

or, $$\vec{\nabla}V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z^2)$$

$$= 2x\,\hat{i} + 2y\,\hat{j} + -2z\,\hat{k}$$

and $$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x\,\hat{i} + 2y\,\hat{j} - 2z\,\hat{k})$$

$$= 2 + 2 - 2 = 2.$$

Now by using Poisson's equation, $\nabla^2 V = \frac{\rho}{\varepsilon_0}$

$\therefore$ $$\rho = -2\,\varepsilon_0$$

The space is not charge-free.

(b) $$V = x^2 + y^2 - 2z^2$$

or, $$\vec{\nabla}V = 2x\,\hat{i} + 2y\,\hat{j} - 4z\,\hat{k}$$

$$\nabla^2 V = 2 + 2 - 4 = 0$$

By Poisson's equation, $\nabla^2 V = -\frac{\rho}{\varepsilon_0} = 0$ or, $\rho = 0$

The space is a charge-free region.

Example 2.30 Region between the two coaxial cones is shown in Fig. 2.16W. A potential V_a exists at θ_1 and $V = 0$ at θ_2. The cone vertices are insulated at $r = 0$. Solve Laplace's equation to get potential at a cone at any angle θ.

Sol. The potential is constant with respect to r and φ. So, in spherical polar coordinates, Laplace's equation takes from

$$\frac{1}{r^2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Now after integration with respect to θ

$$\sin\theta\frac{\partial V}{\partial\theta} = \text{constant} = C_1 \text{ (say)}$$

Again integrating $$V = C_1 \ln\left(\tan\frac{\theta}{2}\right) + C_2 \text{ (constant)}$$

Here, boundary constants are $\theta = \theta_1,\ V = V_a$

$\theta = \theta_2,\ V = 0$

Now using boundary conditions

$$V_a = C_1 \ln\left(\tan\frac{\theta_1}{2}\right) + C_2$$

$$0 = C_1 \ln\left(\tan\frac{\theta_2}{2}\right) + C_2$$

After simplification $$V = V_a \frac{\ln\left(\tan\frac{\theta}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}$$

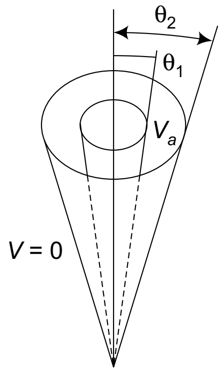


Fig. 2.16W Potential at a cone of angle θ due to two other equipotential cones at angles θ_1 and θ_2.

Review Exercises

Part 1: Multiple Choice Questions

1. Which of the following statements is not correct regarding electrostatic field vector E ? [WBUT 2008]

   (a) $\oint_C \vec{E}\cdot\vec{dr} = 0$, where c is a simple closed curve.

   (b) $\int_a^b \vec{E}\cdot\vec{dr}$ is independent of the path for given end points a and b.

   (c) $\vec{E} = \vec{\nabla}\times\vec{A}$ for some vector potential $\vec{A}$.

   (d) $\vec{E} = \vec{\nabla}\varphi$, for some scalar field φ.

2. Flux of the electric field for a point charge (q) at origin through a spherical surface centered at the origin is [WBUT 2006]

   (a) $\frac{2q}{\varepsilon_0}$ (b) $\frac{q}{\varepsilon_0}$ (c) $\frac{q}{4\pi\varepsilon_0}$ (d) zero

3. Two concentric spheres of radii r_1 and r_2 carry charges q_1 and q_2 respectively. If the surface charge density (σ) is the same for both spheres, the electric potential at the common center will be

   (a) $\frac{\sigma}{\varepsilon_0}\frac{r_1}{r_2}$ (b) $\frac{\sigma}{\varepsilon_0}\frac{r_2}{r_1}$ (c) $\frac{\sigma}{\varepsilon_0}(r_1 - r_2)$ (d) $\frac{\sigma}{\varepsilon_0}(r_1 + r_2)$

4. Six charges, each equal to + q, are placed at the corners of a regular hexagon of side a. The electric potential at the point where the diagonals of the hexagon intersect will be given by

(a) zero (b) $\frac{1}{4\pi\varepsilon_0}\frac{q}{a}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{6q}{a}$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{\sqrt{3q}}{2a}$

5. In free space Poisson's equation is [WBUT 2005]

(a) $\nabla^2 V = 0$ (b) $\nabla^2 V = \frac{\varepsilon_0}{\rho}$ (c) $\nabla^2 V = \alpha$ (d) None of these

6. The electric flux through each of the faces of a cube of 1 m side if a charge q coulomb is placed at its centre is [WBUT 2007]

(a) $\frac{q}{4\varepsilon_0}$ (b) $4\varepsilon_0 q$ (c) $\frac{q}{6\varepsilon_0}$ (d) $\frac{\varepsilon_0}{6q}$

7. Let (r, θ, φ) represent the spherical polar coordinates of a point in a region where the electrostatics potential V is given by $V = K\varphi^2$. Then the charge density in that region [WBUT 2007]

(a) is also a function of φ only (b) is constant in that region
(c) is a function of all the coordinates (r, θ, φ) (d) is a function of (r, θ) only

8. The electrostatic potential energy of a system of two charges q_1 and q_2 separated by a distance r is

(a) $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ (b) $\frac{\varepsilon_0}{4\pi}\frac{q_1 q_2}{r}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{q^2_1 q_2}{r^2}$

9. The magnitude of electric field $\vec{E}$ in the annular region of a charged cylindrical capacitor

(a) same anywhere (b) varies as $\frac{1}{r}$ (c) varies as $\frac{1}{r^2}$ (d) None of these

10. Electric field and potential inside a hollow charged conducting sphere are respectively

(a) $0, 4\pi\varepsilon_0\frac{q}{r}$ (b) $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, 0$ (c) $0, \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ (d) $\frac{q}{4\pi\varepsilon_0 r^2}, \frac{q}{4\pi\varepsilon_o r}$

11. For a closed surface which does not include any charge, the Gauss's law will be

(a) $\oint_S \vec{E}.\vec{ds} = 0$ (b) $\oint_S \vec{E}.\vec{ds} = \frac{q}{\varepsilon_o}$ (c) $\oint_S \vec{E}.\vec{ds} = \frac{q}{4\pi\varepsilon_0}$ (d) None of these

12. Electrostatic field is

(a) conservative (b) non-conservative (c) rotational (d) None of these

13. Laplace's equation for an electrostatic field is

(a) $\nabla^2 V = \frac{\rho}{\varepsilon_0}$ (b) $\nabla^2 V = 0$ (c) $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (d) $\vec{\nabla} V = \frac{\rho}{\varepsilon_0}\hat{r}$

14. Electric field intensity at any point distant r from a plane charged conducting sheet varies as

(a) r^{-1} (b) r^0 (c) r^{-2} (d) r

15. In a region of space, if the electrostatic potential is constant, then the electric field at that region is

(a) zero (b) infinite (c) constant (d) None of these

16. If the flux of the electric field through *a* closed surface is zero,
    (a) the charge inside the surface must be zero
    (b) the electric field must be zero everywhere on the surface
    (c) the charge in the vicinity of the surface must be zero
    (d) None of these

    [**Ans.** 1 (c), 2 (b), 3 (d), 4 (c), 5 (a), 6 (c), 7 (d), 8 (c), 9 (b), 10 (c), 11 (a), 12 (a), 13 (b), 14 (b), 15 (a), 16 (a)]

Short Questions with Answers

1. What is an electric line of force? What is its importance?

Ans. An electric line of force is an imaginary straight or curved path along which *a* positive test charge is supposed to move. The lines of force originate from a single positive charge and converge at an isolated negative charge.

The relative closeness of electric line of force in *a* certain region provides an estimate of the electric field strength in that region.

2. Sketch the electric lines of force due to point charges (i) $q > 0$ (ii) $q < 0$.

Ans.

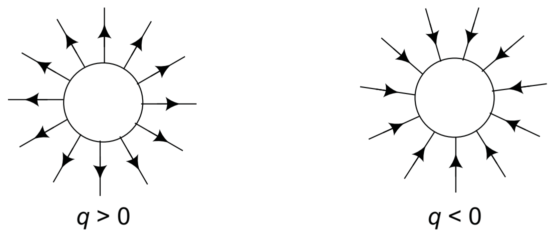


Fig. 2.17W

3. Electrostatic force are much stronger than gravitational force. Give an example.

Ans. A charged glass rod can lift a piece of paper. This shows that the electrostatic force of attraction between the glass rod and paper is much stronger than the gravitational force of attraction between them.

4. State the principle of superposition of electric forces.

Ans. See Section 2.5.

5. State the importance of Gauss' law.

Ans. By applying Gauss' law one can calculate in a simple manner the field intensity due to many different symmetrical configurations of charge. Gauss' law is also important to gain information about the properties of conductors.

6. Obtain Coulomb's theorem from lines of force concept.

Ans. See Section 2.12.3(v).

7. Why is electrostatic field called conservative field?

Ans. A field is conservative when the work done is independent of the path followed and depends only on the initial and final position. For *a* close path work done is zero. In electric field, work done to bring a charge from one point to another point depends on initial and final points. So, the electric field is conservative.

8. Show that electric field is always perpendicular to the equipotential surface.

Ans. In Fig. 2.18W, S is an equipotential surface. A and B are two very close points on the surface. Let electric field $\vec{E}$ make an angle θ with the equipotential surface. The work done for moving a charge q from A to B along the surface is

$$W = qE \cos \theta \times AB$$

Again work done $\quad W = q (V_A - V_B)$

So, $\quad qE \cos \theta \times AB = q (V_A - V_B)$

But $\quad V_A = V_B$ (equipotential surface)

$\therefore \quad qE \cos \theta (AB) = 0$

$\therefore \quad \cos \theta = 0 \quad$ or, $\quad \theta = 90°$

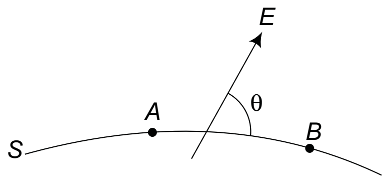


Fig. 2.18W Electric field due to equipotential surface.

9. If Coulomb's law involved $\frac{1}{r^3}$ dependence instead of $\frac{1}{r^2}$, would Gauss' law be still true?

Ans. No, Gauss' law would not hold good.

10. Can electric potential at any point in space be zero while intensity of electric field at that point is not zero?

Ans. Yes, at a point midway between two equal and opposite charges, electric potential is zero but electric field is not zero.

11. No two equipotential surfaces intersect each other. Why?

Ans. We know that two electric lines of force can't intersect, therefore two equipotential surfaces also can't intersect. This is because the electric lines of force and the equipotential surface are mutually perpendicular.

12. Define positive and negative electric flux.

Ans. The electric flux linked with a surface is said to be positive if the electric field vector appears to be leaving the surface.

The electric flux linked with a surface is said to be negative if the electric field vector appears to be entering the surface.

13. Show that Coulomb's law can be derived from Gauss' law.

Ans. See Section 2.12.2.

14. The electric potential is constant in *a* region. What can you say about the electric field there?

Ans. We know that $\quad E = -\dfrac{dV}{dr}$

If V is constant then $\dfrac{dV}{dr} = 0 \quad \therefore E = 0$

So, electric field is zero.

15. Is it possible for a metal sphere of 1 cm radius to hold a charge of one coulomb?

Ans. $V = 9 \times 10^9 \times \dfrac{1}{10^{-2}} = 9 \times 10^{11}$ volt

This is so high a potential that there will be an electrical breakdown of air. On account of ionization of air, the charge on the sphere will leak away.

Part 2: Descriptive Questions

1. What do you mean by conservation of charge? Explain.
   Find out the relation between electric field intensity and potential. What is equipotential surface?
2. State and explain Gauss' law in electrostatics. Obtain its differential form. [WBUT 2002]
3. Derive Coulomb's law from Gauss' law in electrostatics. [WBUT 2007]
4. (a) State Gauss' law in electrostatics and hence obtain Poisson's equation.
   (b) Derive Coulomb's law from Gauss' law. [WBUT 2008]
5. State Gauss' law. Use Gauss' law to find electric field intensity outside, inside and on the surface of a solid sphere.
6. Write down Laplace's equation in cylindrical coordinates and find the solution.
7. If in the region of space electric field is always in the x direction then prove that the potential is independent of y and z coordinates. If the field is constant there is no free charge in that region. [WBUT 2007]
8. (a) State and Prove Gauss' law in electrostatics.
   (b) Using Gauss' law, obtain an expression for the electric field around a charged hollow cylinder. [WBUT 2004]
9. Show that the potential $V = V_0\,(x^2 - 2y^2 + z^2)$ satisfies Laplace's function where V_0 is a constant [WBUT 2004]
10. Write down Laplace's equation in spherical coordinate system and hence find the solution.
11. (a) State Gauss' law of electrostatics.
    (b) Use this law to calculate the electric field between two infinite extent parallel-plate capacitors carrying charge density σ and mutual separation d. Draw the necessary diagram. [WBUT 2006]
12. State Gauss' theorem in electrostatics. Using this theorem, derive an expression for the electric field intensity due to an infinite plane sheet of charge density σ coulomb/m^2.
13. Using Gauss' law, determine the electric field intensity due to a long thin wire of uniform linear charge density.
14. Derive Poisson's and Laplace's equations from fundamentals.

Part 3: Numerical Problems

1. Two point charges Q and q are placed at distance x and $\frac{x}{2}$ respectively from a third charge $4q$. All the three charges are on the same straight line. Calculate Q in terms of q such that the net force on q is zero. [**Ans.** $Q = 4q$]
2. Charge is distributed along the x axis from $x = 0$ to $x = L = 50.0$ cm in such a way that its linear charge density is given by $\lambda = ax^2$ where $a = 18.0\ \mu$ cm^{-3}. Calculate the total charge in the region $0 \le x \le 1$.
   [**Ans.** 0.75 μc] $\left[\textbf{Hints: } q = \int_0^L \lambda\, dx\right]$
3. Consider a uniform electric field $\vec{E} = 3 \times 10^{3}\, \hat{i}$ NC^{-1}. What is the flux of this field through a square of 10 cm on a side whose plane is parallel to the yz plane? [**Ans.** $\varphi_E = 30$ Nm^2 C^{-1}]

4. A point charge of 2.0 μ C is at the centre of a cubic gaussian surface, 9.0 cm on edge. What is the net electric flux through the surface? [**Ans.** 2.26×10^5 Nm^2 C^{-1}]
5. The electric potential $V(x)$ in a region along the x axis varies with distance x (in meter) according to the relation $V(x) = 4x^2$. Calculate the force experienced by 1 μ C charge placed at point $x = 1$ m. [**Ans.** $F = 8 \times 10^{-6}$ N]
6. In a region of space, the electric field is given by $\vec{E} = 8\hat{i} + 4\hat{j} + 3\,\hat{k}$. Calculate the flux through a surface of 1000 units area in the xy plane. [**Ans.** 300 units]
7. Show that the potential function $V = x^2 + z - y^2$ satisfies Laplace's equation [WBUT question bank]
8. *A* circular wire of radius R has a linear charge density $\lambda = \lambda_0 \cos^2\theta$, where θ is the angle with respect to a fixed radius. Calculate total charge. [**Ans.** $\pi R\lambda_0$]
9. n charged spherical water drops, each having a radius r and charge q, coalesce into a single big drop. What is the potential of the big spherical drop? $\left[\textbf{Ans.}\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{n^{2/3}q}{r}\right]$
10. An infinite line charge produces a field of 9×10^4 N C^{-1} at a distance of 2 cm. Calculate the linear charge density. [**Ans.** 10^{-7} cm^{-1}]
11. Determine the charge distribution at $r \neq 0$ which gives a spherically symmetrical potential $V(r) = \dfrac{e^{-\lambda r}}{r}$ where λ is a constant.
12. The volume charge density of a spherical body of radius a centered at the origin is given by

$$\rho(r, \theta, \varphi) = \frac{\rho_0}{r} \quad \text{where } \rho_0 \text{ is constant.}$$

Calculate the total charge in the sphere. [**Ans.** $\varphi = 2\pi\,\rho_0\,a^2$]
13. Is is possible for the electric potential in a charge-free region to be given by (i) $V = x^2 + y^2 - z^2$? (ii) $V = x^2 + y^2 + z^2$? If not find the charge density. [**Ans.** (i) $-4\varepsilon_0$ (ii) $-6\,\varepsilon_0$] [WBUT Question Bank]
14. Two concentric spheres of radii a and b are kept in potential V_a and V_b. If the intervening space is vacuum then write the appropriate differential equation that the electrostatics potential satisfies. Solve this equation to find out the potential in any point between the spheres and also for a point outside the sphere. Calculate the total charge on the outer sphere. [WBUT 2007]
15. A uniformly charged conducting sphere of 2.4 m diameter has a surface charge density of 80.0 μC/m^2. (i) Find the charge on the sphere. (ii) What is the total electric flux leaving the surface of the sphere? [**Ans.** (i) 1.45 mC (ii) 1.6×10^8 Nm^2 C^{-1}]

CHAPTER

3

Dielectrics

3.1 INTRODUCTION

A dielectric is an insulating material in which all the electrons are tightly bound to the nuclei of the atom and there are no free electrons available for the conduction of current. The difference in the name between dielectric and insulator lies in the application for which these materials are used. When these materials are used to prevent the flow of electricity through them or the application of potential difference, then they are called insulators or passive dielectrics. On the other hand, if they are used for charge storage, they are called dielectrics or active dielectrics. Materials such as glass, rubber, mica, porcelain and polymers are examples of dielectrics.

3.2 POLARIZATION

Polarization is defined as the process of creating or inducing dipoles in a dielectric material by an external electric field.

In an atom, there is a positively charged nucleus at the center surrounded by orbiting electrons which are negatively charged. In the absence of an electric field an isolated atom does not have any dipole moment, since the centroids of positive and negative charge coincide [Fig. 3.1(a)]. Suppose now the atom is placed

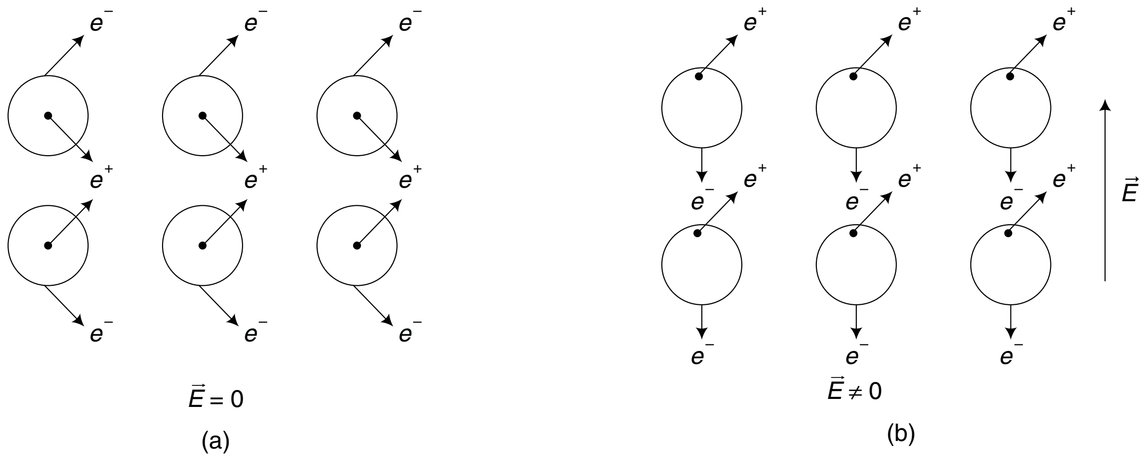


Fig. 3.1 (a) No polarization. (b) Polarization.

in an external electric field. The field will push the positively charged nucleus slightly in the direction of the field and the negatively charged electrons in the opposite direction [Fig. 3.1(b)]. The centroids of the positive and negative charges now no longer coincide and as a result an electric dipole is induced in the atom. The amount of dipole moment induced is proportional to the field because a large field displaces charges more than a smaller field. We say that the atoms are polarized under the influence of the external field.

3.3 TYPES OF DIELECTRICS

On the basis of the polarization concept, dielectrics are the materials that have either permanent dipoles or induced dipoles in the presence of an applied electric field. They are classified into two categories, namely, polar and non-polar dielectrics.

3.3.1 Non-polar Dielectrics

A dielectric material in which, there is no permanent dipole existence in the absence of an external field is called 'non-polar' dielectrics.

For non-polar dielectrics, the center of gravity of the positive and negative charges of the molecules coincide. So such molecules do not have any permanent dipole moment [Fig. 3.2(a)].

Examples O_2, H_2, N_2, CO_2, H_2O_2.

Fig. 3.2 (a) Non-polar dielectrics. (b) Polar dielectrics.

3.3.2 Polar Dielectrics

A dielectric material in which there is an existence of permanent dipole even in the absence of an external field is called polar dielectrics.

For non-polar dielectrics, the center of gravity of the positive charges is separated by finite distance from that of the negative charges of the molecules. So such molecules possess permanent electric dipole [Fig. 3.2(b)].

Examples H_2O, NaCl, HCl, CO.

3.3.3 Dielectric Constant

A capacitor consisting of two parallel conducting plates of area A, separated by a distance d by a vacuum space [Right side of Fig. 3.3], has a capacitance of

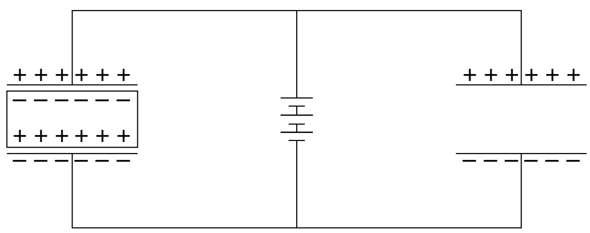


Fig. 3.3 Two identical capacitors: one evacuated and other filled with dielectric.

$$C_0 = \frac{\varepsilon_0 A}{d} \quad ...(3.1)$$

where ε_0 is the permittivity of free space. If C is the capacitance when the space is filled with dielectric material [Left side of Fig. 3.3], then

$$C = \frac{\varepsilon A}{d} \quad ...(3.2)$$

where ε is the permittivity of the dielectric. Now the dielectric constant of the material

$$K = \frac{C}{C_0} = \frac{\varepsilon}{\varepsilon_0} \quad ...(3.3)$$

The dielectric constant of a material is the ratio of the capacitance of a given capacitor completely filled with that material to the capacitance of the same capacitor in vacuum. In other words, the ratio of permittivity of medium to that of the vacuum is also known as dielectric constant or relative permittivity (ε_r).

$$\varepsilon_r = K = \frac{\varepsilon}{\varepsilon_0} \quad ...(3.4)$$

ε_r is a dimensionless quantity and varies widely from material to material. ε_r has a value unity for vacuum and for all other dielectrics ε_r is always greater than 1. For most materials the value of ε_r varies between 1 to 10.

3.3.4 Polarization Vector or Polarization Density

There are two kinds of dipoles in materials—those that are induced and those that are permanent and both cause polarization or charge separation. A dipole moment μ is defined as $\mu = qd$, where q is the magnitude of the charge and d is the distance separating the pair of opposite charges. Dipole moment is a vector, pointing from negative towards positive charges. Polarization vector measures the extent of polarization in a unit volume of dielectric matter. It is defined as the induced dipole moment per unit volume of the dielectric. If N is the number of molecules per unit volume, then the polarization vector or polarization density

$$P = N\mu \quad ...(3.5)$$

The direction of P is along the direction of the applied field.

If a dielectric slab of thickness d and volume V is kept between the two plates of a capacitor [left-hand side of Fig. 3.3], then the dipole moment is $\mu = qd$, where $+q$ and $-q$ are induced charges on the respective faces of the slab.

The polarization is given

$$P = \frac{qd}{V} = \frac{qd}{Ad} \quad [\because V = Ad, \text{ where } A \text{ is the area of the slab}]$$

$$\therefore \quad P = \frac{q}{A} = \sigma_P \text{ (surface charge density of the slab)} \quad ...(3.6)$$

So, **polarization is also defined as the induced surface charge per unit area**. The unit of polarization is coulomb/m^2. Thus polarization density is equal to surface charge density on the dielectric slab. In general, if the polarization vector makes an angle θ with $\hat{n}$, the outward vector of the surface, the surface charge density

$$\sigma_P = \vec{P} \cdot \hat{n} = P \cos\theta \quad ...(3.6a)$$

3.3.5 Susceptibility

The strength of polarization (P) is directly proportional to the applied electric field (E) for dielectrics and is given by

$$P = \varepsilon_0 \chi_e E \quad ...(3.7)$$

The constant of proportionality is usually written as $\varepsilon_0 \chi_e$, where χ_e is known as electric susceptibility of the medium. χ_e is a dimensionless parameter.

Now
$$\chi_e = \frac{P}{\varepsilon_0 E} \quad ...(3.8)$$

Thus, **susceptibility is the ratio of polarization to the net electric field $\varepsilon_0 E$ as modified by the induced charges on the surface of the dielectric.**

3.4 RELATION BETWEEN DIELECTRIC CONSTANT AND ELECTRICAL SUSCEPTIBILITY

We consider a parallel-plate capacitor which has vacuum between its plates. When it is charged with a battery, the electric field of strength E_0 is set up between the plates of the capacitor [Fig. 3.4(a)]. If σ and $-\sigma$ are the surface charge densities of the two plates of the capacitor, then the electric field developed between the plates is given by

$$E_0 = \frac{\sigma}{\varepsilon_0} \quad ...(3.9)$$

Fig. 3.4 (a) Capacitor with vacuum space. (b) Capacitor filled with dielectric.

If a dielectric slab is placed between the plates of the capacitor; then due to polarization charges, appear on the two faces of the slab and establish another field E_p within the dielectric [Fig. 3.4(b)]. This field will be in a direction opposite to the E_0. Under this situation, the net electric field in the dielectric is given by

$$E = E_0 - E_P \quad ...(3.10)$$

If σ_s is the surface charge density on the slab, then by following Eq. (3.9), we can write

$$E_p = \frac{\sigma_P}{\varepsilon_0} \quad ...(3.11)$$

Now, from Eqs. (3.9), (3.10) and (3.11),

$$E = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_P}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\sigma - \sigma_P)$$

or, $$\varepsilon_0 E = \sigma - \sigma_P = \sigma - P \qquad [\because P = \sigma_P]$$

or, $$\sigma = \varepsilon_0 E + P \qquad \text{...(3.12)}$$

Again, by Gauss' law, electric flux density or electric displacement vector D is given by

$$D = \sigma \qquad \text{...(3.13)}$$

Now, from Eq. (3.12)

$$D = \varepsilon_0 E + P \qquad \text{...(3.14)}$$

Again, from Eq. (3.7)

$$P = \varepsilon_0 \chi_e E \qquad \text{...(3.15)}$$

and from electrostatics we know

$$D = \varepsilon E = \varepsilon_0 \varepsilon_r E \qquad \text{...(3.16)}$$

Therefore, from Eqs. (3.15), (3.16) and (3.14)

$$\varepsilon_0 \varepsilon_r E = \varepsilon_0 E + \varepsilon_0 \chi_e E$$

or, $$\varepsilon_r = 1 + \chi_e$$

or $$\chi_e = \varepsilon_r - 1 \qquad \text{...(3.17)}$$

3.5 POLARIZABILITY

Let us consider an individual atom in a dielectric material and the material be subjected to an electric field E. The strength of the dipole induced in an atom is proportional to the actual field acting on the particle, and is given by

$$\mu = \alpha E \qquad \text{...(3.18)}$$

where α is the proportionality constant called polarizability. Its unit is Fm^2.

Note: In the case of gases, the molecules, for most of the time are far apart, so that local electric field (E_{loc}) is the same as the macroscopic field E.

If N be the number of atoms in a unit volume then polarization vector is

$$P = N\mu$$

$$= N \alpha E = \varepsilon_0 \chi_e E \qquad \text{[From Eq. (3.8)]}$$

Therefore, $$\alpha = \frac{\varepsilon_0 \chi_e}{N} \qquad \text{...(3.19)}$$

Polarizability measures the resistance of the particle to the displacement of its electron cloud.

3.6 TYPES OF POLARIZATION

Three basic types of polarization that contribute to the total magnitude of polarization in a material have been identified.

(i) Electronic polarization
(ii) Ionic polarization
(iii) Orientation polarization

Taking into account the three contributions, the total electrical dipole moment

$$\mu = (\alpha_e + \alpha_i + \alpha_0)\, E$$

or, polarization

$$P = N\mu = N(\alpha_e + \alpha_i + \alpha_0)\, E \quad ...(3.20)$$

$$= N\, \alpha\, E$$

where α is total polarizability

α_e is electronic polarizability

α_i is ionic polarizability

α_0 is orientation polarizability

(i) Electronic polarization

Electronic polarization occurs due to the displacement of the positively charged nucleus and negatively-charged electron cloud in opposite directions within a dielectric material upon applying an external electric field E [Fig. 3.5a, b]. The dipole moment (μ_e) induced is proportional to the applied field and the proportionality constant is called electronic polarizability (α_e).

$$\mu_e = \alpha_e E \quad ...(3.21)$$

If a material has N such atoms per unit volume, subjected to homogeneous field E, then the electronic polarization is

$$P = N\,\alpha_e E \quad ...(3.22)$$

The electronic polarizability for a rare gas atom is given by

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N} \quad ...(3.23)$$

The electronic polarization can persist to extremely high field frequencies because electronic standing waves within atoms have very high natural frequencies.

Fig. 3.5 (a) No field applied. (b) Applied electric field.

(ii) Ionic polarization

This type of polarization occurs in ionic materials. In an ionic bond when two different atoms join together, there is transfer of electrons from an atom to another atom, like HCl shows in Fig. 3.6. Even in the absence of an applied field, an HCl molecule has a permanent dipole moment $e \times d$, where d is the distance of separation of ions. In the presence of an applied electric field, the resultant torque lines up the dipoles parallel to the field at absolute zero temperature. The distance between the ions increases from d to $d + x$. The field has induced an additional dipole moment $e \times x$ in the molecule. The induced dipole moment is proportional to the applied electric field and is given by

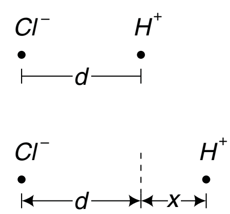


Fig. 3.6 Ionic Polarization.

$$\mu_i = \alpha_i E \qquad ...(3.24)$$

where α_i is the ionic polarizability. Now if N is number of dipole per unit volume, then ionic polarization

$$P_i = N\alpha_i E \qquad ...(3.25)$$

(iii) Orientation polarization

Orientation polarization occurs in dielectric materials which possess molecules with permanent dipole moment, for example, H_2O molecule (polar molecule). In the absence of an external electric field, the permanent dipoles are oriented randomly such that they cancel the effects of each other [Fig. 3.7(a)]. But under the influence of an external applied electric field, each of the dipoles undergo rotation so as to reorient along the direction of the field as shown in Fig. 3.7(b). Thus, the material itself develops electric polarization. This is known as orientation polarization, which depends upon temperature.

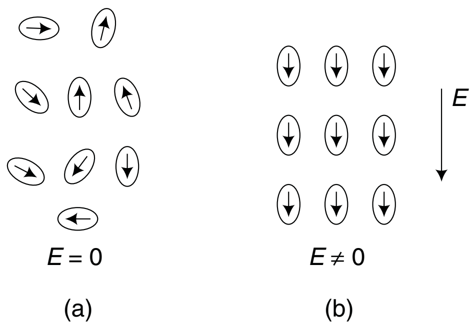


Fig. 3.7 Orientation polarization.

The orientation polarization P_0 is given by Langevin function (1905)

$$P_0 = N\mu L(x) \qquad ...(3.26)$$

where $L(x)$ is known as Langevin function.

Here $x = \dfrac{\mu E}{k_B T}$, k_B is the Boltzmann constant and T is the temperature in Kelvin.

The value of $L(x)$ is $\coth x - \dfrac{1}{x} = \left[\dfrac{e^x + e^{-x}}{e^x - e^{-x}} - \dfrac{1}{x}\right]$

For $\quad x >> 1, P_0 \rightarrow 1$

complete alignment, but this does not occur in gases.

For most practical cases (x is small, i.e., at high temperature)

$$x << 1, L(x) = \frac{x}{3}$$

Or,

$$P_0 = N\mu\frac{x}{3} = \frac{N\mu^2 E}{3k_B T} \qquad ...(3.27)$$

The orientation plorizability α_0 is given by

$$\alpha_0 = \frac{\mu^2}{3k_B T} \qquad ...(3.28)$$

3.7 POLARIZATION IN MONOATOMIC GASES

Let us consider one of constituent atom of a dielectric material (rare gases, such as helium and argon) in the absence of an electric field. Let the radius of the atom be a and its atoms number be Z as shown in Fig. 3.8(a). Here positive nucleus $+Ze$ is surrounded by an electronic cloud of charge $-Ze$. Also nucleus is point charge and electron cloud of charge $-Ze$ distributed homogeneously throughout a sphere of radius a. Therefore, the charge density for electron cloud is given by

$$\rho = \frac{-Ze}{\frac{4}{3}\pi a^3} = -\frac{3}{4}\left(\frac{Ze}{\pi a^3}\right) \qquad ...(3.29)$$

Fig. 3.8 (a) Atom placed in free space. (b) Atom placed into external field.

When the atom is placed into an external electric field E, the nucleus and electron cloud move in the opposite direction [Fig. 3.8(b)], and experience a Lorentz force of magnitude ZeE. Equilibrium is established with nucleus shifted slightly relative to the center of the electron cloud by a distance d.

The nucleus experiences a force (F_N) in the direction of the electric field,

$$F_N = ZeE \qquad \text{...(3.30)}$$

and opposing force F_G due to the electric field of the charge located within the sphere of radius d and concentrated at the center of the electron cloud. By Gauss' law, the electric field (E_G) at edge location of the nucleus due to electrons within the sphere of radius d is

$$E_G \times 4\pi d^2 = \text{Total charge } (q) \text{ enclosed in a sphere of radius } d.$$

or,
$$E_G \times 4\pi d^2 = \frac{4}{3}\pi d^3 \rho/\varepsilon_0$$

$$= \frac{4}{3}\pi d^3 \times \left(-\frac{3Ze}{4\pi a^3}\right)\Big/\varepsilon_0 \qquad \text{[From Eq. (3.29)]}$$

$$= -\frac{Ze\left(\dfrac{d^3}{a^3}\right)}{\varepsilon_0}$$

Now
$$|F_G| = ZeE_G = \frac{Z^2e^2d}{4\pi\varepsilon_0 a^3}$$

But
$$|F_G| = |F_N|$$

So,
$$d = \frac{4\pi\varepsilon_0 a^3}{Ze}E \qquad \text{...(3.31)}$$

Thus, the displacement distance d is proportional to the external electric field E. Due to this displacement, the atom acts as a dipole.

For the single atom, the electronic polarizability of a monoatomic gas can be obtained from induced dipole moment

$$\mu = \alpha_e E = (Ze)\, d = (Ze)\,\frac{4\pi\varepsilon_0 a^3}{Ze}E$$

or
$$\alpha_e = 4\pi\varepsilon_0 a^3 \qquad \text{...(3.32)}$$

Thus, the electronic polarizability is proportional to the volume of the atom and is independent of temperature.

The polarization vector $P = N\mu$

$$\therefore \quad P = N\alpha_e E \qquad ...(3.33)$$

But we know that $P = \varepsilon_0 E(\varepsilon_r - 1)$ ...(3.34)

Now from Eqs. (3.33) and (3.34)

$$N\alpha_e E = \varepsilon_0 E(\varepsilon_r - 1)$$

or,

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N} \qquad ...(3.35)$$

For He, the value of α_e is 0.18×10^{-40} Fm^2, and for Ne, the value of α_e is 0.35×10^{-40} Fm^2. So, bigger the size of the atom, the value of α_e is larger.

3.8 POLARIZATION IN POLYATOMIC GASES

Let us consider a gas containing N molecules per m^3. Assume that each molecule has a permanent electric dipole moment μ.

The polarization is due to the electronic polarization P_e (nucleus shifted slightly relative to the center of the electron cloud), the ionic polarization P_i (ionic nature of bond between atoms) and the orientation polarization P_0 (due to rotation and alignment of the polar molecules in the external electric field). The total polarization of a polyatomic gas is given by

$$P = P_e + P_i + P_0$$

$$= N\alpha_e E + N\alpha_i E + N\frac{\mu^2}{3KT}E$$

$$= N\left(\alpha_e + \alpha_i + \frac{\mu^2}{3KT}\right)E \qquad ...(3.36)$$

Again $P = \varepsilon_0 \chi_e E = (\varepsilon_r - 1)\varepsilon_0 E$ ...(3.37)

So,

$$(\varepsilon_r - 1)\varepsilon_0 = N\left(\alpha_e + \alpha_i + \frac{\mu^2}{3KT}\right) \qquad ...(3.38)$$

P_e and P_i are essentially independent of the temperature but P_0 is temperature dependent.

At a temperature T and zero external electric field, the molecules will be randomly oriented, so, zero polarization.

When there is an external electric field, the molecules will try to align with the field. Each polar molecule can be considered to be a simple dipole. The force on the dipole provides the torque to rotate the molecule so that they will be in the lowest state where they are parallel to the field. If there was no thermal motion, all dipoles would line up along the external field direction.

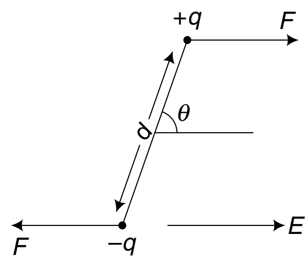


Fig. 3.9 Electric dipole placed in external electric field.

The electric force on the dipole produces a couple [Fig. 3.9] and the torque acting to rotate the dipole is

$$\tau = qEd\sin\theta = \mu E\sin\theta$$

or, $$\vec{\tau} = \vec{\mu} \times \vec{E} \qquad ...(3.39)$$

The potential energy of the dipole for an arbitrary angle θ is given by

$$U(\theta) = -\mu E \cos\theta = -\vec{\mu} \cdot \vec{E} \qquad ...(3.40)$$

The dipole has the lowest potential energy when the dipole is parallel to the electric field and the highest potential energy when antiparallel to the field. For no thermal motion, all dipoles would line along the direction of the external electric field. But at a greater temperature, the thermal motion will be greater and there will be small alignment of the dipoles with the field.

Worked Out Problems

Example 3.1 Two parallel plates have equal and opposite charges. When the space between them is evacuated, the electric field intensity is 3×10^5 V/m and when the space is filled with dielectric, the electric intensity is 1.0×10^5 V/m. What is the induced charge density on the surface of the dielectric?

Sol. Given $E_0 = 3 \times 10^5$ V/m and $E = 1.0 \times 10^5$ V/m

We know that $E = E_0 - \dfrac{P}{\varepsilon_0}$

or,
$$P = \varepsilon_0(E_0 - E)$$
$$= 8.85 \times 10^{-12} (3 - 1) \times 10^5$$
$$= 1.77 \times 10^{-6} \text{ C/m}^2$$

Again $P = \sigma_P$

So, $\sigma_P = 1.77 \times 10^{-6}$ C/m^2

Example 3.2 Calculate the polarizability and relative permittivity in hydrogen gas with a density of 9.8×10^{26} atoms/m^3. [Given the radius of the hydrogen atom to be 0.50 Å].

Sol. Given $N = 9.8 \times 10^{26}$ atoms/m^3

$a = 0.50 \times 10^{-10}$ m

We know that
$$\alpha_e = 4\pi\varepsilon_0 a^3$$
$$= 4 \times 3.14 \times 8.85 \times 10^{-12} \times (0.50 \times 10^{-3})^3$$
$$= 1.38 \times 10^{-41} \text{ Fm}^2$$

So, the polarizability $\alpha_e = 1.38 \times 10^{-41}$ Fm^2

Again
$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N}$$

or,
$$\varepsilon_r = \frac{N\alpha_e}{\varepsilon_0} + 1$$

$$= \frac{9.8 \times 10^{26} \times 1.38 \times 10^{-41}}{8.85 \times 10^{-12}} + 1$$

$$= 1.001$$

So relative permittivity $\varepsilon_r = 1.001$

Example 3.3 In a dielectric material, $E_x = 5$ V/m and $\vec{P} = \frac{1}{10\pi}(3\hat{i} - \hat{j} + 4\hat{k})$ nC/m^2. Calculate (i) χ_e (ii) $\vec{E}$ (iii) $\vec{D}$

Sol. The polarization is given by

$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

or,

$$\chi_e = \frac{\vec{P}}{\varepsilon_0 \vec{E}}$$

Here we consider only the x component

So, (i)

$$\chi_e = \frac{P}{\varepsilon_0 E_x} = \frac{3 \times 10^{-9}}{10\pi} \times \frac{36\pi}{10^{-9} \times 5} = 2.16$$

(ii)

$$\vec{E} = \frac{\vec{P}}{\varepsilon_0 \chi_e} = \frac{1}{10\pi}(3\hat{i} - \hat{j} + 4\hat{k}) \times \frac{36\,\pi}{10^{-9} \times 2.16}$$

$$= 5\hat{i} - \frac{5}{3}\hat{j} + \frac{20}{3}\hat{k} \text{ V/m}$$

(iii)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \frac{10^{-9}}{36\pi}\left(5\hat{i} - \frac{5}{3}\hat{j} + \frac{20}{3}\hat{k}\right) + \frac{10^{-9}}{10\pi}(3\hat{i} - \hat{j} + 4\hat{k})$$

$$= (139.7\,\hat{i} - 46.6\,\hat{j} + 186.3\,\hat{k}) \text{ pC/m}^2$$

Example 3.4 Calculate the dipole moment μ of a molecule of carbon tetrachloride (CCl_4) in a field 10^7 Vm^{-1}. [Given: Density = 1.60 gm/cm^3, Molecular weight = 156, Relative permittivity $\varepsilon_r = 2.24$].

Sol. Molecular density $N = \frac{\text{Avogadro's number}}{\text{Molecular weight}} \times \text{Density}$

$$= \frac{6.02 \times 10^{23}}{156} \times 1.60$$

$$= 6.17 \times 10^{21} \text{ molecules/cm}^3$$

The dipole moment of a single molecule μ is

$$\mu = \frac{\varepsilon_0(\varepsilon_r - 1)}{N} E$$

$$= \frac{8.85 \times 10^{-12} \times 1.24 \times 10^7}{6.17 \times 10^{27}}$$

$$= 1.77 \times 10^{-32} \text{ C/m}$$

There are 74 electrons in each CCl_4 molecule

So, $\mu = 74\,ed$

or, $d = \dfrac{\mu}{74e} = \dfrac{1.77 \times 10^{-32}}{74 \times 1.6 \times 10^{-19}}$

$= 1.5 \times 10^{-15}$ m

Example 3.5 Dielectric constant of a gas at N.T.P is 1.00074. Calculate the dipole moment of each atom of the gas when it is held in an external field of 3×10^4 V/m.

Sol. Given $E = 3 \times 10^4$ V/m $= 3 \times 10^4$ N/C

and $K = \varepsilon_r = 1.00074$

We know $\varepsilon_r = 1 + \chi_e$

or, $\chi_e = \varepsilon_r - 1 = 1.00074 - 1 = 0.00074$

Polarization density $P = \varepsilon_0 \chi_e E$

$= 8.85 \times 10^{-12} \times 0.74 \times 10^{-3} \times 3 \times 10^4$

$= 1.96 \times 10^{-10}$ C/m

No. of atoms of gas per cubic meter

$$N = \frac{6.06 \times 10^{23}}{22.4 \times 10^{-3}} = 2.7 \times 10^{25}$$

So induced dipole moment of each atom

$$\mu = \frac{P}{N} = \frac{1.96 \times 10^{-10}}{2.7 \times 10^{25}}$$

$= 7.27 \times 10^{-36}$ C/m

Example 3.6 A dielectric cube of side L and center at the origin has a polarization vector given as $\vec{P} = \hat{i}x + \hat{j}y + \hat{k}z$. Find the volume and surface bound charge densities and show that the total bound charge vanishes in this case.

Sol. The bound surface charge density is $\sigma_b = \vec{P} \cdot \hat{n}$. For each of the six sides of the cube, there exists a surface charge density. For the side located at $x = L/2$, the surface charge density

$$\sigma_b^1 = \vec{P} \cdot \hat{i}\big|_{L/2} = (\hat{i}x + \hat{j}y + \hat{k}z) \cdot \hat{i}\big|_{L/2} = x\big|_{L/2} = L/2$$

$\therefore$ the total bound surface charge

$$q_{bs} = \int_s \sigma_b \, ds = 6 \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \sigma_b \, dy\, dz = 3L^3$$

The bound volume charge density is

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -(1 + 1 + 1) = -3$$

$\therefore$ the total bound volume charge is

$$q_{bV} = \int \rho_b \, dV = -3 \int dV = -3L^3$$

Hence, the total bound charge within the cube,

$$q = q_{bs} + q_{bv} = 3L^3 - 3L^3 = 0$$

So, total bound charge vanishes.

Example 3.7 The two plates of a parallel-plate capacitor are identical and carry equal amount of opposite charges. The separation between the plates is 5 mm and the space between the plates is filled with a dielectric of dielectric constant 3. The electric field within the dielectric is 10^6 V/m. Calculate (i) polarization vector $\vec{P}$, and (ii) displacement vector $\vec{D}$.

Sol. (i) The magnitude of the polarization vector is

$$\begin{aligned} P &= \varepsilon_0 (k - 1) E \\ &= 8.85 \times 10^{-12} (3 - 1) \times 10^6 \\ &= 17.7 \times 10^{-6} \text{ C/m}^2 \\ &= 17.7 \,\mu \text{ C/m}^2 \end{aligned}$$

(ii) The magnitude of the displacement vector is

$$\begin{aligned} D &= k\varepsilon_0 E \\ &= 3 \times 8.85 \times 10^{-12} \times 10^6 \\ &= 2.65 \times 10^{-7} \text{ C/m}^2 \\ &= 26.5 \,\mu \text{ C/m}^2 \end{aligned}$$

Example 3.8 A dielectric cube of side L, centered at the origin, carries a "frozen-in" polarization $\vec{P} = k\vec{r}$, where k is a constant. Find all the bound charges and check that they add up to zero.

Sol. The bound volume charge density ρ_s is equal to

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, kr) = -3k$$

Since the bound volume charge density is constant, the total bound volume charge in the cube is equal to product of the charge density and the volume

$$q_{bv} = -3k\, L^3$$

The surface charge density σ_s is equal to

$$\sigma_s = \vec{P} \cdot \hat{n} = k\vec{r} \cdot \hat{n}$$

The scalar product between $\vec{r}$ and $\hat{n}$ can be evaluated easily (see Fig. 3.1W) and is equal to

$$\vec{r} \,.\, \hat{n} = r \cos\theta = \frac{1}{2} L$$

Therefore the surface charge density is equal to

$$\sigma_s = k\vec{r} \,.\, \hat{n} = \frac{1}{2} k \cdot L$$

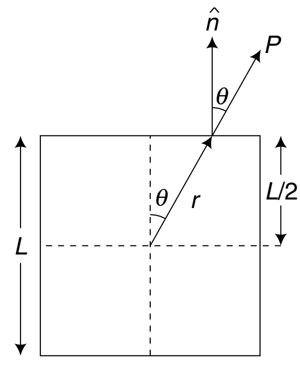


Fig. 3.1W Dielectric cube of side *L*.

The total surface charge density is equal to the product of the surface charge density and the total surface area of the cube

$$q_{bs} = \frac{1}{2} kL \times 6L^2 = 3\ kL^3$$

$\therefore$ the total bound charge on the cube is equal to

$$q = q_{bv} + q_{bs} = -3\ kL^3 + 3\ kL^3$$
$$= 0$$

Example 3.9 The space between the plates of a parallel-plate capacitor [Fig. 3.2W] is filled with two slabs of linear dielectric material. Each slab has thickness S, so that the total distance between the plates is $2S$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

(i) Find the electric displacement $\vec{D}$ in each slab.

(ii) Find the electric field $\vec{E}$ in each slab.

(iii) Find the polarization $\vec{P}$ in each slab.

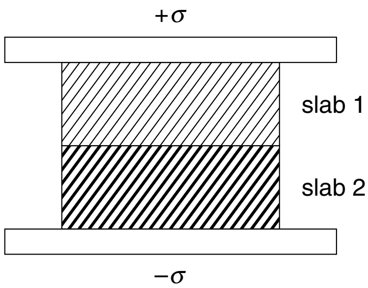


Fig. 3.2W Parallel-plate capacitor filled with two slabs of linear dielectric material.

Sol. (i) The electric displacement $\vec{D}_1$ in slab 1 can be calculated using Gauss' law. Consider a cylinder with cross-sectional area A and axis parallel to the z axis, being used as a gaussian surface. The top of the cylinder is located inside the top metal plate (where electric displacement is zero) and the bottom of the cylinder is located inside the dielectric of slab 1. The electric displacement is directed parallel to the z axis and pointed downwards. So, the displacement flux through this surface is equal to

$$\phi_D = D_1 A$$

The free charge enclosed by this surface is equal to

$$q_{\text{free}} = \sigma A$$

Combining these two we obtain

$$D_1 = \frac{\phi_D}{A} = \frac{q_{\text{free}}}{A} = \sigma$$

In vector notation $\vec{D}_1 = -\sigma\hat{k}$

Similarly for slab 2 $D_2 = -\sigma\hat{k}$

(ii) The electric field $\vec{E}_1$ in slab 1 is equal to

$$\vec{E}_1 = \frac{1}{k_1\varepsilon_0}\vec{D} = -\frac{\sigma}{k_1\varepsilon_0}\hat{k} = -\frac{\sigma}{2\varepsilon_0}\hat{k}$$

The electric field $\vec{E}_2$ in slab 2 is equal to

$$\vec{E}_2 = \frac{1}{k_2\varepsilon_0}\vec{D}_2 = -\frac{\sigma}{k_2\varepsilon_0}\hat{k} = -\frac{2\sigma}{3\varepsilon_0}\hat{k}$$

(iii) The polarization $\vec{P}_1$ of slab 1 is equal to

$$\begin{aligned}\vec{P}_1 &= \vec{D}_1 - \varepsilon_0 \vec{E}_1 \\ &= -\sigma\hat{k} + \frac{\sigma}{2}\hat{k} \\ &= -\frac{\sigma}{2}\hat{k}\end{aligned}$$

The polarization $\vec{P}_2$ of slab 2 is equal to

$$\begin{aligned}\vec{P}_2 &= \vec{D}_2 - \varepsilon_0 \vec{E}_2 \\ &= -\sigma\hat{k} + \frac{2\sigma}{3}\hat{k} = -\frac{\sigma}{3}\hat{k}\end{aligned}$$

Example 3.10 The polarizability of a gas is 0.35×10^{-40} Fm^2. If the gas contains 2.7×10^{25} atoms/m^3 at 0°C and one atmospheric pressure, calculate its relative permittivity.

[Given $\alpha = 0.35 \times 10^{-40}$ Fm^2, $N = 2.7 \times 10^{25}$].

Sol. We know

$$\begin{aligned}\varepsilon_r &= 1 + \frac{N\alpha}{\varepsilon_0} \\ &= \frac{1 + 2.7 \times 10^{25} \times 0.35 \times 10^{-40}}{8.854 \times 10^{-12}} \\ &= 1 + 0.1067 \times 10^{-3} \\ &= 1.000107\end{aligned}$$

So, the relative permittivity is 1.000107.

Example 3.11 A capacitor uses a dielectric material of relative permittivity $\varepsilon_r = 8$. It has an effective surface area of 0.036 m^2 with a capacitance of 6 μF. Calculate the field strength and dipole moment per unit volume if a potential difference of 15 V exists across the capacitor.

Sol. Field strength $E = \frac{V}{d}$ where $d = \frac{\varepsilon_0 \varepsilon_r A}{C}$

$$\begin{aligned}d &= \frac{8.85 \times 10^{-12} \times 8 \times 0.036}{6 \times 10^{-6}} \\ &= 0.42 \times 10^{-6} \text{ m}\end{aligned}$$

or, field strength $$E = \frac{V}{d} = \frac{15}{0.42 \times 10^{-6}} = 35.3 \times 10^6 \text{ V/m}.$$

Dipole moment/unit volume
$$\begin{aligned}&= \varepsilon_0 (\varepsilon_r - 1) E \\ &= 8.85 \times 10^{-12} (8 - 1) \times 3.5 \times 10^6 \\ &= 2.1 \times 10^{-5} \text{ C/m}^2\end{aligned}$$

Review Exercises

Part 1: Multiple Choice Questions

1. In vacuum, electric susceptibility is
   (a) greater than 1 (b) zero (c) less than 1 (d) None of these
2. The relation between three electric vectors E, D and P is
   (a) $\vec{D} = \varepsilon_0 (\vec{E} + \vec{P})$ (b) $\vec{D} = \vec{E} + \varepsilon\vec{P}$ (c) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ (d) $\vec{D} = \frac{1}{\varepsilon_0} (\vec{E} + \vec{P})$
3. The relation between electrical susceptibility and dielectric constant is
   (a) $\chi_e = \varepsilon_0 k$ (b) $\chi_e = k - 1$ (c) $\chi_e = k + 1$ (d) $\chi_e = \frac{k}{\varepsilon_0} - 1$
4. The dimension of polarizability in SI unit is
   (a) Fm^2 (b) Fm (c) Fm^{-1} (d) Fm^{-2}
5. Dielectrics are the substances which are
   (a) semiconductor (b) conductors (c) insulators (d) None of these
6. The electronic polarizability for a rare gas atom is
   (a) $\alpha_e = \frac{(\varepsilon_r - 1)}{\varepsilon_0 N}$ (b) $\alpha_e = N(\varepsilon_r - 1)$ (c) $\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N}$ (d) $\alpha_e = \frac{N}{\varepsilon_r - 1}$
7. A medium behaves like dielectric when the
   (a) displacement current is much greater than the conduction current
   (b) displacement current is zero
   (c) conduction current is almost zero
   (d) displacement current is equal to the conduction current
8. The total polarization of a polyatomic gas is
   (a) $P = N(\alpha_e + \alpha_i)$ (b) $P = N\left(\alpha_e + \alpha_i + \frac{\mu}{KT}\right) E$
   (c) $P = N\left(\alpha_e + \alpha_i + \frac{\mu^2}{3KT}\right) E$ (d) $P = \frac{N\mu E}{KT}$
9. The potential energy of the dipole for an arbitrary angle θ
   (a) $\cup(\theta) = -\vec{\mu} \times \vec{E}$ (b) $\cup(\theta) = -\vec{\mu} \cdot \vec{E}$ (c) $\cup(\theta) = \vec{E} \times \vec{\mu}$ (d) None of these
10. The relation between electronic polarizability and atomic radius for monatomic gases is
    (a) $\alpha_e = a^3$ (b) $\alpha_e = 4\pi\varepsilon_0 a^3$ (c) $\alpha_e = 4\pi\varepsilon_0 a^2$ (d) $\alpha_e = 4\pi\varepsilon_0 a$
11. For polar dielectrics, the orientation polarizability α_0 is given by
    (a) $\alpha_0 = \frac{3KT}{\mu^2}$ (b) $\alpha_0 = \frac{\mu^2}{3KT}$ (c) $\alpha_0 = \mu KT$ (d) None of these
12. Electrical susceptibility χ_e is
    (a) $\chi_e = \frac{P}{\varepsilon_0 E}$ (b) $\chi_e = \frac{P}{3\varepsilon_0 E}$ (c) $\chi_e = \varepsilon_0 EP$ (d) $\chi_e = \frac{3\varepsilon_0 E}{P}$

13. Generally, the dielectrics are
    (a) metallic materials of low specific resistance and have negative temperature coefficient of resistance
    (b) metallic materials of high specific resistance and have negative temperature coefficient of resistance.
    (c) metallic materials of high specific resistance and have positive temperature coefficient of resistance.
    (d) None of these
14. The ionic polarizability is
    (a) independent of temperature
    (b) depends on temperature
    (c) depends on square of the temperature
    (d) None of these

[**Ans.** 1 (b), 2 (c), 3 (b), 4 (a), 5 (c), 6 (c), 7 (a), 8 (c), 9 (b), 10 (b), 11 (b), 12 (a), 13 (b), 14 (a)]

Short Questions with Answers

1. Define polarization.

Ans. Polarization is defined as the process of creating or inducing dipoles in a dielectric material by an external electric field.

2. Define electrical susceptibility.

Ans. The electrical susceptibility is the ratio of polarization (P) to the net electric field ($\varepsilon_0 E$) as modified by the induced charges on the surface of the dielectric.

3. What are non-polar and polar dielectrics?

Ans. A dielectric, in the atoms and molecules of which, the center of gravity of positive and negative charges coincides, is called non-polar dielectric.

A dielectric, in the atoms and molecules of which, the center of gravity of positive and negative charges does not coincide, is called polar dielectric.

4. Define dielectric strength.

Ans. The dielectric strength of a dielectric is defined as the maximum value of the electric field that can be applied to the dielectric without its electric breakdown.

5. What do we mean by 'dielectric constant of glass is 8.5'?

Ans. Dielectric constant of a glass is 8.5 means that the ratio of the capacitance of a capacitor with glass as dielectric to the capacitance of the capacitor with air as dielectric.

6. Define polarizability.

Ans. Polarizability is the ability of an atom or a molecule to become polarized in the presence of an electric field.

7. What is electronic polarization?

Ans. Under the action of an external field, the electron clouds of atoms are displaced with respect to heavy fixed nuclei to a distance less than the dimensions of the atom. This is known as electronic polarization.

Part 2: Descriptive Questions

1. What are polar and non-polar dielectrics? What is meant by polarization of dielectric?
2. Show that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.
3. Derive a relation between electric susceptibility and atomic polarizability on the basis of microscopic description of matters at atomic level.
4. Define the following terms: (i) dipole moment, (ii) electrical susceptibility, (iii) relative dielectric constant, and (iv) polarization.
5. Explain the phenomenon of polarization of dielectric medium and show that $K = 1 + \chi_e$, where the symbols have their usual meanings.
6. Show that electronic polarizability α_e is

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N}, \quad \text{where the symbols have their usual meanings.}$$

7. Derive an expression for electronic polarization of a dielectric medium.
8. What are non-polar and polar dielectrics? Find out the relation between dielectric constant and electrical susceptibility.
9. Define polarization. Show that electronic polarizability is proportional to the volume of the atom and is independent of temperature.
10. Explain polarization in polyatomic gases.

Part 3: Numerical Problems

1. Copper is a FCC crystal with a lattice constant 3.6 Å and atomic number 29. If the average displacement of the electrons relative to the nucleus is 1×10^{-18} m. Applying an electric field, calculate the electronic polarization. [**Ans.** $P = 3.94 \times 10^{-7}$ C/m^2]
2. A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$ where k is constant and $\vec{r}$ is the vector from the center.
   (a) Calculate the bound charges σ_b and ρ_s.
   (b) Find the field inside and outside the sphere. [**Ans.** $\sigma_b = KR$, $\rho_b = -3k$, 0]
3. Two parallel plates of a capacitor having equal and opposite charges are separated by 6.0 mm thick dielectric materials of dielectric constant 2.8. If the electric field strength inside be 10^5 V/m, determine polarization vector and displacement vector. [**Ans.** $P = 1.6 \times 10^{-6}$ C/m^2, $D = 2.5 \times 10^{-6}$ C/m^2]
4. Determine the electric susceptibility at 0°C for a gas whose dielectric constant at 0°C is 1.000041. [**Ans.** $\chi_e = 4.1 \times 10^{-5}$]
5. The electronic polarizability of argon atom is 1.75×10^{-40} Fm^2. What is the static dielectric constant of solid argon, if its density is 1.8×10^3 kg/m^3 (Given atomic weight of $A_r = 39.95$ and $N = 6.025 \times 10^{26}$/K mole). [**Ans.** $\varepsilon_r = 1.5367$]
6. The dielectric constant of helium at 0°C is 1.0000684. If the gas contains 2.7×10^{25} atoms/m^3, find the radius of the electron cloud. [**Ans.** 0.6×10^{-10} m]
7. A gas containing 2.7×10^{25} atoms per m^3 has polarizability of 0.2×10^{-40} Fm^2. Calculate the relative permittivity of the gas. [**Ans.** 1.000061]

CHAPTER

4

Magnetostatics

4.1 INTRODUCTION

Moving charges, or current, are the sources of magnetic fields in the same way as static charges are the sources of electric fields. By using Biot–Savart law and Ampere's law we can calculate magnetic fields due to different current distribution. The magnetic field, like the electric field, is a vector field.

4.2 ELECTRIC CURRENT

Electric charge in motion produces electric current, and the current-carrying medium may be called a conductor. Electric current is simply a flow of charge. If a charge ΔQ crosses an area in time Δt, then average electric current

$$I_{av} = \frac{\Delta Q}{\Delta t}$$

The current at time t is

$$I = \underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} \text{ coulomb/s} \quad ...(4.1)$$

If one coulomb of charge crosses an area in one second, the current is one ampere. The SI unit of current is ampere (A).

4.3 CURRENT DENSITY

A charged particle placed is an electric field $\vec{E}$ experiences a force $\vec{F}$. If the electric field $\vec{E}$ is constant, then the particle will have an average velocity and the average velocity of a charged particle is called drift velocity, $\vec{v}_d$.

Now we define a vector quantity known as *electric current density at a point*. To define current density, we consider a medium of uniform area of cross section S and volume charge density ρ. Then current I at a given point becomes

$$I = v_d \rho S \qquad \ldots(4.2)$$

For uniformly distributed current, the magnitude of current density

$$J = \frac{I}{S} = v_d \rho \qquad \ldots(4.3)$$

But if current density is not uniform, then we define it as

$$J = \underset{\Delta S \to 0}{\mathrm{Lt}} \frac{\Delta I}{\Delta S}$$

Surface ΔS is normal to current direction. The total current through the entire surface S is

$$I = \int_S \vec{J} \cdot \vec{dS} \qquad \ldots(4.4)$$

For many materials, it is found that the current density in the steady state is linearly proportional to the applied electric field intensity. Therefore

$$\vec{J} \propto \vec{E} \quad \text{or,} \quad \vec{J} = \sigma \vec{E}\,^* \qquad \ldots(4.5)$$

The constant of proportionality is known as the conductivity of the medium at a given temperature.

The drift velocity is directed along the direction of electric field and is related to by a constant called the mobility μ,

$$\vec{v}_d = \mu \vec{E} \qquad \ldots(4.6)$$

Mobility (μ) is defined as the drift velocity per unit electric field.

4.4 EQUATION OF CONTINUITY FOR CURRENT

Let us consider a volume V of the conductor enclosed by a surface S. If ρ is the volume charge density then the total charge (Q) within the volume is given by

$$Q = \int_V \rho \, dV$$

From conservation of charge (charge can neither be created nor destroyed), the amount of incoming flow of charge $\left(I = \oint_S \vec{J} \cdot \vec{dS}\right)$ must be equal to the rate of decrease of the total charge $\left(-\frac{dQ}{dt}\right)$ inside the volume.

i.e.
$$\oint_S \vec{J} \cdot \vec{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

By applying Gauss' divergence theorem

$$\int_V (\vec{\nabla} \cdot \vec{J}) \, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

Therefore,

$$\int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

[***Note**: From Eqs (4.2) and (4.6), we have $I = v_d \rho S = \mu \rho S E$

So, current density $J = \frac{I}{S} = \mu \rho E = \sigma E$ where $\sigma = \rho \mu$ is called electrical conductivity.

If $\rho = ne$ then $\sigma = ne\mu$ where n is the number of electrons per unit volume.]

For any arbitrary volume V, the integral must be zero

So, $$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \ldots(4.7)$$

This is known as the ***equation of continuity*** and represents the mathematical statement of local charge conservation.

If the region does not contain a source or sink of charge then $\frac{\partial \rho}{\partial t} = 0$ [for steady current] and Eq. (4.7) reduces to

$$\vec{\nabla} \cdot \vec{J} = 0 \qquad \ldots(4.8)$$

Equation (4.7) represents the condition of steady current flow.

4.5 FORCE ON A MOVING CHARGE IN A STATIC MAGNETIC FIELD

If a charged particle moves across a magnetic field, it is accelerated at right angles to its direction of motion.

The particle experiences a force at right angles to its velocity, with a magnitude proportional to the component of velocity, charge and magnetic field.

So, we can write the infinitesimal magnetic force $\overrightarrow{dF}$ on an infinitesimal charge dq moving with a velocity $\vec{v}$ in a steady magnetic field as

$$d\vec{F} = dq\,(\vec{v} \times \vec{B}) \qquad \ldots(4.9)$$

Since the electric force on an infinitesimal charge dq in an electric field is $dq\,\vec{E}$, so the total electromagnetic force on an infinitesimal charge is

$$d\vec{F} = dq\,(\vec{E} + \vec{v} \times \vec{B}) \qquad \ldots(4.10)$$

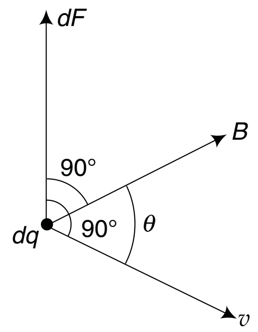


Fig. 4.1 Force on a moving charge in a constant magnetic field.

This is known as ***Lorentz force***.

Now for a single particle of charge e, the Lorentz force will be

$$\vec{F} = e\,(\vec{E} + \vec{v} \times \vec{B}) \qquad \ldots(4.11)$$

In the absence of an electric field ($\vec{E}$), Lorentz force (magnetic force) for a single particle of charge e is

$$\vec{F} = e\,(\vec{v} \times \vec{B}) \qquad \ldots(4.12)$$

The magnitude of the Lorentz force is

$$F = evB \sin\theta \qquad \ldots(4.13)$$

where θ is the angle between $\vec{v}$ and $\vec{B}$ [Fig. 4.1].

No workforce If infinitesimal charge dq moves through a small amount dl then $\vec{dl} = \vec{v}\,dt$, the work done is [from Eq. (4.9)]

$$dW = d\vec{F} \cdot \vec{dl} = dq\,(\vec{v} \times \vec{B}) \cdot \vec{dl} = dq\,(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt$$
$$= 0$$

Since, $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, so $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$

So, magnetic force does no work on a charged particle to move with a velocity v in a static magnetic field $\vec{B}$.

4.6 FORCE ON CURRENT ELEMENT PLACED IN A STATIC MAGNETIC FIELD

We know from Lorentz force [Eq. (4.9)] that

$$d\vec{F} = dq\,(\vec{v} \times \vec{B})$$

where $\overrightarrow{dF}$ is the infinitesimal force on an infinitesimal charge dq moving with a velocity v in a steady magnetic field B [Fig. 4.2]. Again, if dq is the amount of charge flow through the cross section of a conductor in time dt, then electric current

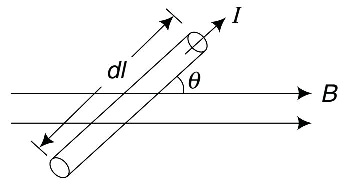


Fig. 4.2 Force on a current element which is in static magnetic field.

$$I = \frac{dq}{dt}$$

So,
$$d\vec{F} = I\,dt\,(\vec{v} \times \vec{B})$$
$$= I\,(dt\,\vec{v} \times \vec{B})$$

$v\,dt$, a segment of length (dl) gives the indication of the distance travelled by a particle in time dt, then

$$d\vec{F} = I\,(\vec{dl} \times \vec{B}) \quad \ldots(4.14)$$

For a finite length of the conductor, the magnetic force

$$\vec{F} = I \int (\vec{dl} \times \vec{B}) \quad \ldots(4.15)$$

Considering the current as the vector along the length dl, the magnetic force per unit length

$$\vec{F} = \vec{I} \times \vec{B} \quad \ldots(4.16)$$

Thus magnetic force depends only on the total current and applied magnetic field and is independent of the amount of charge carried by each particle. The direction of current is perpendicular to the plane containing $\vec{B}$ and $\vec{I}$.

4.7 BIOT–SAVART LAW

Steady currents produce magnetic fields which are constant in time. The right-hand thumb rule, Fig. 4.3(a, b) gives the direction of the magnetic field. According *to* **the thumb rule, if the current flows in the thumb's direction, right-handed fingers curl around in the direction of the magnetic field.** The symbol ⊙ gives the direction of the magnetic field perpendicular to the plane of the paper and ⊗ gives the direction of the magnetic field into the plane of the paper. If the fingers are curled along the current, the stretched thumb will point towards the magnetic field [Fig. 4.3(c)].

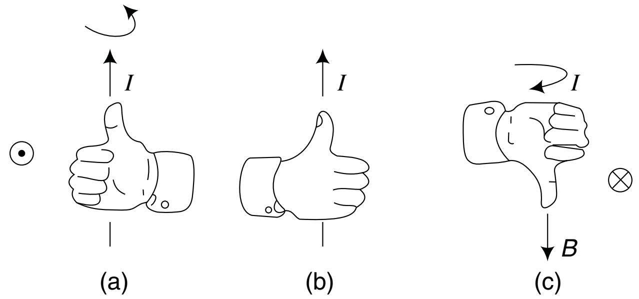


Fig. 4.3 Direction magnetic field by using the right-hand thumb rule.

The Biot–Savart law states that the magnetic field $\overrightarrow{dB}$ due to a current element $I\,\overrightarrow{dl}$ [Fig. 4.4(a, b)] is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{dl} \times \hat{r}}{r^2} \qquad \ldots(4.17)$$

where $\hat{r}$ is the unit vector from the point of interest $I\,\vec{dl}$ towards the point of interest and r is the distance between the current element $I\,\vec{dl}$ and the point of observation.

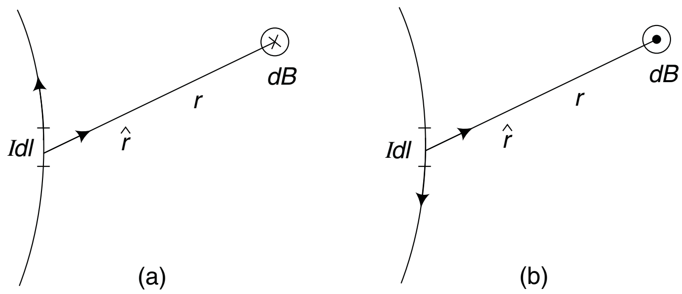


Fig. 4.4 Biot–Savart law for current element *Idl*.

The total field B due to the whole conductor can be obtained after taking the integration

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,\vec{dl} \times \hat{r}}{r^2} \qquad \ldots(4.18)$$

or,
$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J} \times \hat{r}}{r^2}\,dV \qquad \ldots(4.19)$$

where $\quad I\,\vec{dl} = \vec{J}\,dV$

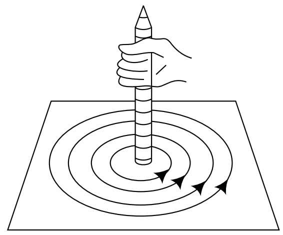


Fig. 4.5 Maxwell's cork-screw rule.

The constant μ_0 is called the permeability of free space and its value

$$\mu_0 = 4\pi \times 10^{-7}\ \text{N/A}^2.$$

Equation (4.18) is the integral form of the ***Biot–Savart law***.

The direction of magnetic field can also be obtained by **Maxwell's cork-screw** rule. Maxwell's cork-screw rule points that **if the direction of the current through a conductor is represented by the linear motion of the cork-screw motion then the direction of the magnetic field can be represented by the direction of rotation of the cork** [Fig. 4.5].

4.8 APPLICATIONS OF BIOT–SAVART LAW

(i) Magnetic field due to a long straight wire

In the diagram [Fig. 4.6] ($\vec{dl} \times \hat{r}$) points into (X) the paper. From Biot–Savart law, dB at P is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{dl} \times \hat{r}}{r^2}$$

where Idl is the small current element at a distance l from O. The magnitude of the magnetic field dB at the point P at a distance x from the wire due to the current element Idl is

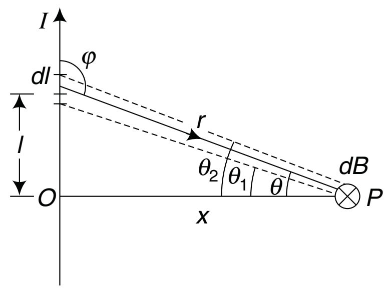


Fig. 4.6 Application of Biot–Savart law in case of a straight wire.

$$dB = \frac{\mu_0}{4\pi}\left|\frac{I\,\vec{dl}\times\hat{r}}{r^2}\right|$$

$$= \frac{\mu_0}{4\pi}\,\frac{I\,dl\sin\varphi}{r^2} \qquad \ldots(4.20)$$

Also, $l = x\tan\theta$

$$dl = x\sec^2\theta\,d\theta \text{ and } \frac{x}{r} = \cos\theta, \text{ so } \frac{1}{r^2} = \frac{\cos^2\theta}{x^2}.$$

and $\sin\varphi = \sin(90° - \theta) = \cos\theta$

Thus, from Eq. (4.20), we have

$$dB = \frac{\mu_0}{4\pi}\,\frac{I(x\sec^2\theta\,d\theta)\cos\theta}{(x\sec\theta)^2} = \frac{\mu_0 I}{4\pi x}\cos\theta\,d\theta$$

Now, total magnetic field B at P in terms of the initial and final angles θ_1 and θ_2 is

$$\therefore \qquad B = \int dB = \frac{\mu_0 I}{4\pi x}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi x}(\sin\theta_2 - \sin\theta_1) \qquad \ldots(4.21)$$

Now for infinite wire $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$

Magnetic field $B = \frac{\mu_0 I}{4\pi x}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right)$

$$= \frac{\mu_0}{4\pi}\left(\frac{2I}{x}\right) \qquad \ldots(4.22)$$

Equation (4.21) shows that magnetic field due to a straight infinite wire is inversely proportional to the distance from the wire.

(ii) Magnetic field at a point on the axis of a circular loop

Here, we consider the center of the loop to be at the origin and its axis is along the x direction [Fig. 4.7].

Now according to Biot–Savart law, the magnetic field at P due to the current element Idl of the loop is given by

$$dB = \frac{\mu_0}{4\pi}\,\frac{I\,dl}{r^2}$$

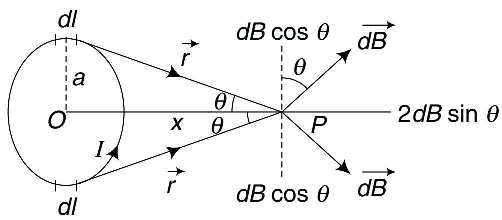


Fig. 4.7 Magnetic field on at a point on the axis of a circular current loop.

The total magnetic field at P is due to the current element $I\,dl$. On both sides of the loop is $dB' = 2\,dB\sin\theta$. Perpendicular component $dB\cos\theta$ due to current elements of both sides of the loop cancels out which is shown in Fig. 4.7.

So, the total magnetic field at P due to current element Idl on both sides is

$$dB' = 2\,dB\sin\theta = 2\times\frac{\mu_0}{4\pi}\,\frac{I\,dl}{r^2}\sin\theta$$

Again $\quad r^2 = x^2 + a^2 \quad$ and $\quad \sin\theta = \frac{a}{r} = \frac{a}{(x^2+a^2)^{1/2}}$

So, $$dB' = 2\frac{\mu_0}{4\pi}\frac{I\,dl\,a}{(x^2+a^2)^{3/2}}$$

Hence, total magnetic field at P is

$$B' = \int dB' = \frac{2\mu_0}{4\pi}\frac{Ia}{(x^2+a^2)^{3/2}}\int_0^{\pi a} dl$$

$$= \frac{\mu_0}{4\pi}\frac{2\pi I a^2}{(x^2+a^2)^{3/2}} \qquad \ldots(4.23)$$

Now for n number of turns, the total magnetic field will be

$$B' = \frac{\mu_0}{4\pi}\frac{2\pi n I a^2}{(x^2+a^2)^{3/2}} \qquad \ldots(4.24)$$

At the center of the loop $x = 0$ so,

$$B'_{\max} = \frac{\mu_0}{4\pi}\left(\frac{2\pi n I}{a}\right) \qquad \ldots(4.25)$$

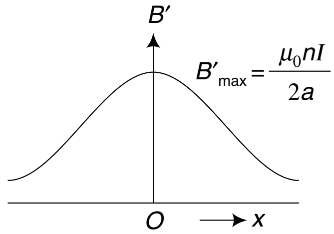


Fig. 4.8 Variation of magnetic field on the axis of a circular were.

The variation of magnetic field (B') with the axis of the coil is shown in Fig. 4.8. The graph shows that the magnetic field is maximum at the center of the coil.

(iii) Magnetic field along the axis of a solenoid

A solenoid is a wire wound closely in the form of a helix around a right circular cylinder. Generally, the length of the solenoid is large as compared to the transverse dimension. To find out the magnetic field $\vec{B}$ at an axial point P at a distance l from O of the solenoid of radius a and carrying a current I, we consider an elementary length dx at a distance x from O [Fig. 4.9]. The current in the section dx of the coil is $nIdx$, where n is the number of turns (N) per unit length, i.e., $\frac{N}{L}$.

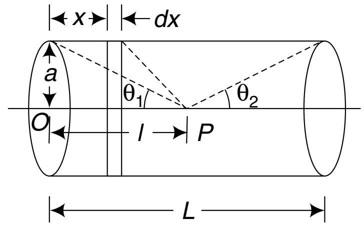


Fig. 4.9

The field at P due to the element dx is

$$dB = \frac{\mu_0(ndx)\,Ia^2}{2[(l-x)^2+a^2]^{3/2}} \qquad \ldots[4.26]$$

The total magnetic field B at P due to the entire solenoid is

$$B = \int dB = \int_0^L \frac{\mu_0 n I a^2}{2}\frac{dx}{[(l-x)^2+a^2]^{3/2}}$$

$$= \frac{\mu_0 nI}{2}\left[\frac{x-l}{\sqrt{(l-x)^2+a^2}}\right]_0^L$$

$$= \frac{\mu_0 nI}{2}\left[\frac{l}{\sqrt{l^2+a^2}} + \frac{L-l}{\sqrt{(L-l)^2+a^2}}\right] \qquad \ldots.(4.27)$$

Again, from Fig. 4.9, $\cos\theta_1 = \frac{l}{\sqrt{l^2+a^2}}$ and $\cos\theta_2 = \frac{L-l}{\sqrt{(L-l)^2+a^2}}$

So from Eq. (4.27), total magnetic field

$$B = \frac{\mu_0 nI}{2}(\cos\theta_1 + \cos\theta_2) \qquad \ldots(4.28)$$

Case I For an infinite solenoid $\theta_1 = \theta_2 = 0$; then $B = \mu_0 nI = \mu_0 \dfrac{NI}{L}$ ...(4.29)

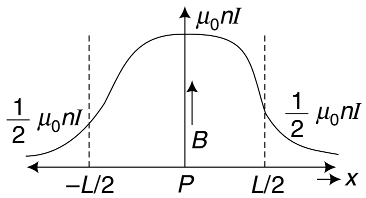


Fig. 4.10 Variation *B* along the axis of a solenoid.

Case II If P is at the right end ($\theta_1 = 0$, $\theta_2 = 90°$) or at the left end ($\theta_1 = 90°$, $\theta_2 = 0$) of the solenoid then

$$B = \frac{\mu_0 nI}{2} = \frac{\mu_0 N}{2L} I \quad ...(4.30)$$

The variation of the magnetic field along the axis of the solenoid is shown in Fig. 4.10. Figure 4.10 shows that for a long solenoid, the magnetic field is maximum at center (P) and just half at the ends of the solenoid.

4.9 FORCE BETWEEN TWO STRAIGHT PARALLEL WIRES

Let C_1 and C_2 be two long parallel wires carrying currents I_1 and I_2 respectively in the same direction [Fig. 4.11(a)]. The separation between the wires is d. The magnetic field at dl, a small element of the wire C_2 due to the current I_1 in C_1 is

$$B = \frac{\mu_0}{4\pi}\left(\frac{2I_1}{d}\right) \quad ...(4.31)$$

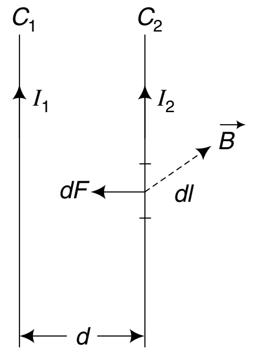


Fig. 4.11 (a) Magnetic force between the parallel wires carrying current.

The direction of B is perpendicular to C_2. The magnetic force at the element dl due to B is

$$d\vec{F} = I_2\, d\vec{l} \times \vec{B}$$

or,

$$|d\vec{F}| = I_2\, dl \frac{\mu_0}{4\pi}\left(\frac{2I_1}{d}\right)$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d}\, dl \quad ...(4.32)$$

The vector product $(d\vec{l} \times \vec{B})$ has a direction towards the wire C_1. So, the direction of the force $d\vec{F}$ is towards the wire C_1. The force per unit length of the wire C_2 due to the wire C_1 is

$$F = \frac{\mu_0 I_1 I_2}{2\pi d} \quad ...(4.33)$$

If the parallel wires carrying currents are in opposite directions, the force will be repulsive in nature.

4.10 MAGNETIC FORCE BETWEEN TWO FINITE ELEMENTS OF CURRENT

From Fig. 4.11(b), the magnetic field due to current I_1 of the conductor A on dl_2 at a distance r is

$$dB = \frac{\mu_0}{4\pi} \frac{I_1 d\vec{l}_1 \times \hat{r}}{r^2}$$

where $I_1 \vec{dl}_1$ is the current element of the conductor A.

Now, the force on current I_2, due to current I_1 is

$$d\vec{F} = I_2\vec{dl}_2 \times d\vec{B} = \frac{\mu_0}{4\pi}\, I_1 I_2\, \frac{\vec{dl}_2 \times (\vec{dl}_1 \times \hat{r})}{r^2} \qquad \ldots(4.34)$$

where $I_2 \vec{dl}_2$ is the current elements of the conductor B and $\hat{r}$ is the unit vector in the direction of r.

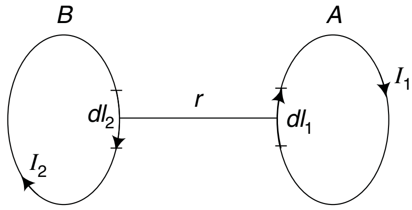


Fig. 4.11 (b) Magnetic field between two finite current-carrying elements.

4.11 DIVERGENCE OF MAGNETIC FIELD

We know from Biot–Savart law, the magnetic field at P [Fig. 4.12] is

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \hat{r}}{r^2}\, dV \qquad \ldots(4.35)$$

where $dV\,(dx'dy'dz')$ is the volume element or source element of current. The position vectors of the source and field points are suppose $\vec{r}'\,(\hat{i}x' + \hat{j}y' + \hat{k}z')$ and $\vec{r}\,(\hat{i}x + \hat{j}y + \hat{k}z)$. Again current density $\vec{J}$ is the function of (x', y', z') and magnetic field $\vec{B}$ is the function of (x, y, z).

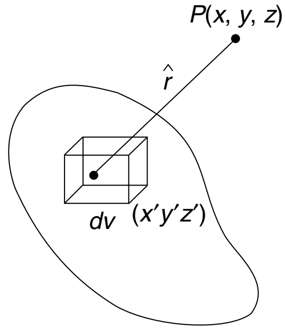


Fig. 4.12 Divergence of magnetic field.

Now taking divergence of equation

$$\vec{\nabla} \cdot \vec{B}\,(r) = \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \cdot \left(\vec{J}(r') \times \frac{\hat{r}}{r^2}\right) dV$$

Now using vector identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B}$$

we have
$$\vec{\nabla} \cdot \vec{B}\,(r) = \frac{\mu_0}{4\pi} \int_V \left[\frac{\hat{r}}{r^2} \cdot (\vec{\nabla} \times J(r') - \vec{J}\,(r') \cdot \vec{\nabla} \times \frac{\hat{r}}{r^2}\right] dV$$

But $\vec{\nabla} \times \vec{J}(r') = 0$ because $\vec{\nabla}$ operator derivatives with respect to the field point $(\vec{r})$ while $\vec{J}$ is the function of the source point$(\vec{r}')$ only.

Again $\vec{\nabla} \times \dfrac{\hat{r}}{r^2} = 0$ from vector calculus.

Hence, $$\vec{\nabla} \cdot \vec{B} = 0 \qquad \ldots(4.36)$$

Thus, the magnetic field is solenoidal.

Physical significance Since magnetic lines of force are continuous, the magnetic flux entering any region of volume is equal to the magnetic flux leaving the volume. Hence the net flux over the volume is equal to zero. Divergence of magnetic field B is defined as the flux of B through the surface enclosing per unit volume. Since, net flux per unit volume is zero, so mathematically

$$\vec{\nabla} \cdot \vec{B} = 0$$

which is known as the differential form of Gauss' law in magnetostatics. Comparing it with Gauss' law in electrostatics $\left(\vec{\nabla} \cdot E = \dfrac{\rho}{\varepsilon_0}\right)$, we may conclude that monopole does not exist.

The magnetic flux $\varphi_B = \oint_S \vec{B} \cdot \vec{dS}$

By applying the divergence theorem,

$$\oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B})\, dV$$

$$= 0 \qquad \ldots(4.36a)$$

Equation (4.36a) states that there are no magnetic flux sources, and magnetic flux lines always close upon themselves. So, there is no source or sink of magnetic flux (law of conservation of magnetic flux), i.e., magnetic monopole does not exist.

Equations (4.36) and (4.36a) are differential and integral forms of Gauss' law in magnetostatics.

4.12 AMPERE'S CIRCUITAL LAW

Ampere's circuital law states that the line integral of the magnetic field $\vec{B}$ around any closed path is equal to μ_0 times the net current enclosed by the path.

Mathematically, $$\int_c \vec{B} \cdot \vec{dl} = \mu_0 I \qquad \ldots(4.37)$$

Amperian loop To explain net current we consider a loop known as Amperian loop that encloses four wires of currents i_1, i_2, i_3 and i_4 but i_5 and i_6 is outside the loop [Fig. 4.13]. Since the direction of the loop is clockwise, the positive side of the plane (lower plane) is away from the viewer, i.e., into the plane of the paper. So i_1 and i_3 are positive and i_2 and i_4 are negative. Hence, total current is $i_1 + i_3 - (i_2 + i_4)$. Any current outside the loop is not included in writing the right-hand side of Eq. (4.37).

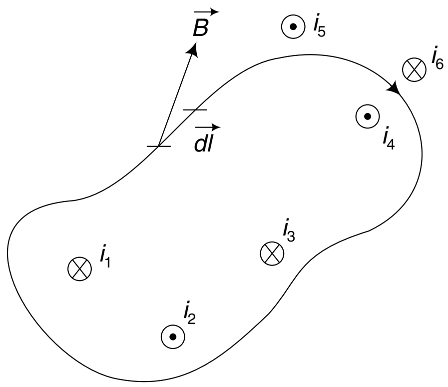


Fig. 4.13 Amperian loop.

Ampere's law is valid for a closed path of any shape. If the path does not include the current then

$$\int_c \vec{B} \cdot \vec{dl} = 0$$

To find magnetic field, there must be two conditions: (i) At each point on the closed path, $\vec{B}$ is either tangential or normal to the path. (ii) If $\vec{B}$ is tangential then at all points of the path, $\vec{B}$ must have the same value.

Ampere's law plays the same role in magnetostatics as Gauss' law plays in electrostatics and is very helpful in determining the magnetic field around a conductor for symmetrical distribution.

4.12.1 Differential Form of Ampere's Law

The total current (I) enclosed by a path enclosing a surfaces S is

$$I = \iint_S \vec{J} \cdot \vec{dS}$$

where $\vec{J}$ is the current density in an element $\vec{dS}$ of the surface bounded by the closed path.

Now, Ampere's law in terms of current density $\vec{J}$ is

$$\int_c \vec{B} \cdot \vec{dl} = \mu_0 \iint_S \vec{J} \cdot \vec{dS} \qquad \ldots(4.38)$$

which is the integral form of the Ampere's circuital law.

Now applying Stoke's law

$$\int_c \vec{B} \cdot \vec{dl} = \iint_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} \qquad \ldots(4.39)$$

Now from Eqs. (4.38) and (4.39), we get

$$\iint_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} = \mu_0 \iint_S \vec{J} \cdot \vec{dS}$$

or,
$$\iint_S [\vec{\nabla} \times \vec{B} - \mu_0 \vec{J}] \cdot \vec{dS} = 0$$

Since the surface element $\vec{dS}$ is arbitrary, so

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \ldots(4.40)$$

which is the differential form of Ampere's law.

In electrostatics $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}, \vec{\nabla} \times \vec{E} = 0$

In magnetostatics $\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

4.13 APPLICATIONS OF AMPERE'S CIRCUITAL LAW

Ampere's circuital law is applicable for line currents, sheet currents or volume currents.

4.13.1 Long Straight Cylindrical Wire

Let us consider an infinitely long conducting wire of radius R, carrying current I as shown in Fig. 4.14. Suppose the current distribution is uniform throughout the cross section of the wire. Now applying Eq. (4.37) to an amperian loop at A_1 [Fig. 4.14] of radius r is

$$\oint \vec{B} . \vec{dl} = \mu_0 I_1 \qquad \ldots(4.41)$$

where
$$I_1 = \frac{I}{\pi R^2} \times \pi r^2 = I \frac{r^2}{R^2}$$

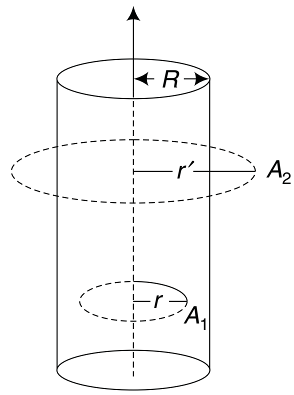


Fig. 4.14 Magnetic field due to a long straight wire of radius *R*.

Now, from Eq. (4.40)

$$\oint \vec{B} \cdot \vec{dl} = I \frac{r^2}{R^2} \mu_0$$

or,
$$B \times 2\pi r = \mu_0 \frac{I r^2}{R^2}$$

so that
$$B = \mu_0 \frac{Ir}{2\pi R^2} \qquad \ldots(4.42)$$

within the wire. Now, outside the wire, applying Eq. (4.37) to an amperian loop at A_2 [Fig. 4.14] of radius $r' > R$ is

$$\int_c \vec{B} \cdot \vec{dl} = \mu_0 I$$

or,
$$B \times 2\pi r' = \mu_0 I$$

or,
$$B = \frac{\mu_0 I}{2\pi r'} \qquad \ldots(4.43)$$

At the surface of the wire, $r' = R$, $B = \dfrac{\mu_0 I}{2\pi R}$ $\qquad \ldots(4.44)$

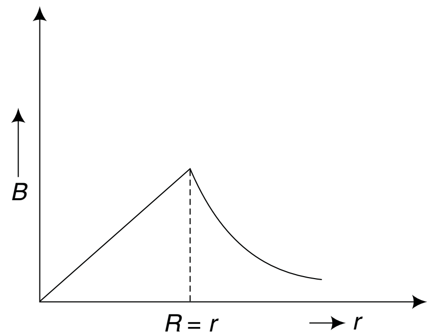


Fig. 4.15 Variation of magnetic field with distance of a current carrying cylindrical wire.

The variation of the magnetic field with distance from the axis of the cylinder is shown in Fig. 4.15.

Note:

(i) For long straight conducting wire of infinite length carrying current I, the magnetic field at any point at a perpendicular distance r from the wire will be $B = \frac{\mu_0 I}{2\pi r}$.

(ii) For a hollow cylinder, since the current I exists only on the surface of the cylinder and inside the cylinder current is zero, so magnetic field inside the cylinder will be zero.

4.13.2 Magnetic Field Inside a Long Solenoid

When a current (I) carrying wire is wound tightly on the surface of a cylindrical tube, we get a solenoid. Generally, the length (L) of the solenoid is large as compared to the transverse dimension. If N is the total number of turns over a length L, we get $\frac{N}{L} = n$ as the number of turns per unit length of the solenoid. Keeping the product nI fixed, if we make n very large and corresponding I very small, then we obtain a surface current of value nI over the curved surface of the cylinder. It turns out that the magnetic field inside a closely wound solenoid is almost uniform over its cross section and can be taken to be negligible outside the volume of the solenoid. Ampere's law thus can easily applied to find out the value of $\vec{B}$ inside the solenoid.

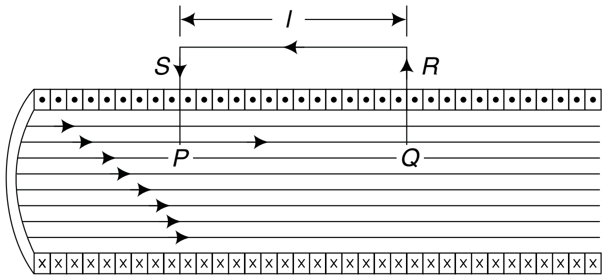


Fig. 4.16 Magnetic field inside a solenoid.

In Fig. 4.16, we draw a rectangle $PQRS$ of length l. The line PQ is parallel to the solenoid axis and hence parallel to the field $\vec{B}$ inside the solenoid. Thus,

$$\int_P^Q \vec{B} \cdot \vec{dl} = Bl$$

Along QR, RS and SP, $\vec{B}.\vec{dl}$ is zero everywhere as $\vec{B}$ is either zero (outside the solenoid) or perpendicular to $\vec{dl}$ (inside the solenoid).

Thus, from $PQRSP$,

$$\oint \vec{B} \cdot \vec{dl} = Bl \qquad \text{...(4.45)}$$

If n be the number of turns per unit length, then total of nl turns cross the rectangle $PQRS$. Each turn carries a current I. Hence net current passes through the area $PQRS$ is nlI.

Now, from Eq. (4.45) we have from Ampere's law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \, nlI$$

or, $$Bl = \mu_0 \, nlI$$

or, $$B = \mu_0 \, nI \qquad \text{...(4.46)}$$

The above equations gives the magnetic field inside a long closely wound solenoid. The relation does not depend on the diameter or the length of the solenoid and magnetic field B is constant over the solenoid cross section.

4.13.3 Magnetic Field Due to Toroid

An endless solenoid in the form of circular shape is called toroid. The magnetic field in such a toroid can be obtained by using Ampere's law.

Let P be a point on the concentric circular path at which magnetic field $\vec{B}$ is to be calculated. By symmetry, the field will have equal magnitude at all points of this circle [Fig. 4.17]. Let the distance of P from the center be r. The field B is tangential at every point of the circle. Hence,

$$\oint \vec{B} \cdot \vec{dl} = B \int dl = B \times 2\pi r$$

$$= \mu_0 \, NI \qquad \text{...(4.47)}$$

where N is the total number of turns and the current crossing the area bounded by the circle is NI.

So, $$B = \frac{\mu_0 \, NI}{2\pi r} \qquad \text{...(4.48)}$$

Thus B is inversely proportional to r. If the cross section of the toroid is very small, the variation in r can be neglected and $\frac{N}{2\pi r}$ can be written as n, the number of turns per unit length. So

$$B = \mu_0 nI \qquad \text{...(4.49)}$$

The field at an external point (P') of the toroid, from Ampere's law

$$\oint B \cdot dl = 0 \text{ or, } B = 0 \qquad \text{...(4.50)}$$

Thus the field outside the toroid is zero.

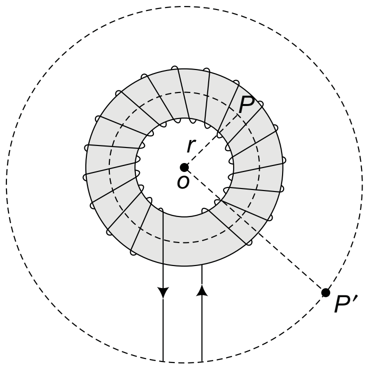


Fig. 4.17 Magnetic field due to a toroid.

4.14 MAGNETIC POTENTIALS

4.14.1 Magnetic Scalar Potential

In electrostatics, scalar potential V plays an important role to find electric field intensity $\vec{E}$. Since $\vec{\nabla} \times \vec{E} = 0$, $\vec{E}$ can be expressed as the gradient of a scalar quantity V. The two powerful relations are:

$$\vec{E} = -\vec{\nabla} \, V \text{ and} \qquad \text{...(4.51)}$$

$$\nabla^2 V = 0 \qquad \text{...(4.52)}$$

If in some region of space, current density $J = 0$, then from Ampere's circuital law in magnetostatics, $\vec{\nabla} \times \vec{B} = 0$. We may therefore express $\vec{B}$ as a gradient of scalar quantity V_m

i.e. $$\vec{B} = -\vec{\nabla} V_m \qquad \ldots(4.53)$$

where V_m is called the magnetic scalar potential.

Since $\vec{\nabla} \cdot \vec{B} = 0$ so $\vec{\nabla} \cdot (-\vec{\nabla} V_m) = 0$ or, $\nabla^2 V_m = 0$ …(4.54)

We see that V_m satisfies *Laplace's equation* in homogeneous magnetic materials, it is not defined in any region where the current density exists.

The magnetic scalar potential may be defined as a scalar whose negative gradient at any point gives the magnetic induction at that point due to a close loop of carrying current.

The magnetic scalar potential is useful in describing the magnetic field around a current source

4.14.2 Magnetic Vector Potential

Gauss' law in magnetostatics state that always $\vec{\nabla} \cdot \vec{B} = 0$. Again we know that divergence of any curl = 0, i.e., $\vec{\nabla} . \vec{\nabla} \times \vec{A} = 0$. Since the curl of $\vec{B}$ is not necessarily zero (only if $\vec{J} = 0$, $\vec{\nabla} \times \vec{B} = 0$), so $\vec{B}$ can't be the gradient of a scalar potential in general but as the curl of a vector field, in the form

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \ldots(4.55)$$

The vector function $\vec{A}$ which satisfies Eq. (4.54) is known as *vector potential.* The vector potential $\vec{A}$ is as important in magnetostatics as the scalar potential function V in electrostatics. The vector potential $\vec{A}$ does not have any physical significance. It can help to determine $\vec{B}$ at a given point, since $\vec{B}$ is the space derivative of $\vec{A}$. The magnetic vector potential may be defined as a vector, the curl of which gives the magnetic induction produced at any point by a closed-loop carrying current.

We know that $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \ldots(4.56)$$

Again $$\vec{B} = \vec{\nabla} \times \vec{A}$$

So, $$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J} \qquad \ldots(4.57)$$

but, $$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} . \vec{A}) - \nabla^2 \vec{A}$$

For steady current, we take $\vec{\nabla} . \vec{A} = 0$

So, $$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\nabla^2 \vec{A}$$

Now from Eq. (4.57) for dc current only, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ …(4.58)

Equation (4.58) is the same as Poisson's equation in electrostatics,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \ldots(4.59)$$

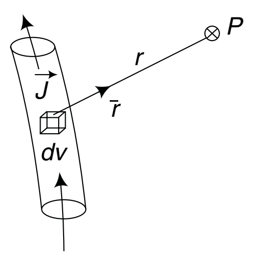


Fig. 4.18 Magnetic vector potential at a distance *r* from a current element.

where V is the electrostatics potential and satisfies

$$V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho \, dV}{r} \qquad \ldots(4.60)$$

Similarly for Eq. (4.58) we have the general solution

$$\vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}}{r} dV \qquad \ldots(4.61)$$

The magnetic vector potential is useful for studying radiation in transmission lines, wave guides, antennas.

Here r is the distance from the current element to the point at which the magnetic vector potential is being calculated [Fig. 4.18]. Thus the field $\vec{B}$ produced by a current can be calculated by first determining $\vec{A}$ using Eq. (4.61) and substituting this in Eq. (4.55).

Worked Out Problems

Example 4.1 How many electrons pass through a wire in 1 minute if the current passing through the wire is 200 mA?

Sol. We know $$I = \frac{q}{t} = \frac{ne}{t}$$

or, $$n = \frac{It}{e} = \frac{200 \times 10^{-3} \times 60}{1.6 \times 10^{-19}} = 7.5 \times 10^{19}.$$

Example 4.2 What is the drift velocity of electrons in a Cu conductor having a cross-sectional area of 5×10^{-6} m^2 if the current is 10 A? Assume that there are 8×10^{28} electrons/m^3.

Sol. Here, area of cross section $A = 5 \times 10^{-6}$ m^2

Current $I = 10$ A

Number density of free electrons, $n = 8 \times 10^{28}$ electrons/m^3.

We know $$v_d = \frac{I}{neA} = \frac{10}{8 \times 10^{28} \times 1.6 \times 10^{-19} \times 5 \times 10^{-6}} = 1.56 \times 10^{-4}\ \text{ms}^{-1}.$$

Example 4.3 Calculate the magnetic field at the center of a regular hexagon [Fig. 4.1W] of side a meter and carrying a current I A.

Sol. The magnetic field at O due to part AB of the hexagon is

$$B' = \frac{\mu_0}{4\pi}\frac{I}{r}(\sin\theta_1 + \sin\theta_2)$$

Here $r = \frac{\sqrt{3}}{2}a$ and $\theta_1 = \theta_2 = 30°$, $\frac{\mu_0}{4\pi} = 10^{-7}$

So $$B' = 10^{-7}\frac{I}{\frac{\sqrt{3}}{2}a}(\sin 30° + \sin 30°)$$

$$= 10^{-7} \times \frac{2I}{\sqrt{3}a}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= 10^{-7} \times \frac{2I}{\sqrt{3}a}$$

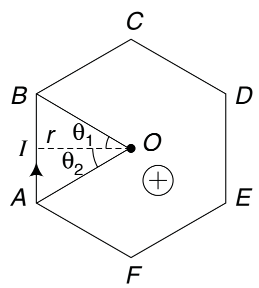


Fig. 4.1W Magnetic field at the center of a hexagon.

So, total magnetic field at O of the hexagon is

$$B = 6B' = 6 \times 10^{-7} \times \frac{2I}{\sqrt{3}a} = \frac{4\sqrt{3}}{a} \times 10^{-7}\ \text{Tesla}$$

Example 4.4 A circular segment QR of a wire $PQRS$ [Fig. 4.2W] of 0.1 m radius subtends an angle of 60° at its center. A current of 6 amperes is flowing through it. Find the magnitude and direction of the magnetic field at the center of the segment.

Sol. Magnetic field due to current in a circular segment making an angle θ at the center is

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

Here, $dl = r\,d\theta$ So, $dB = \dfrac{\mu_0}{4\pi}\dfrac{Ird\theta}{r} = \dfrac{\mu_0}{4\pi}\dfrac{Id\theta}{r}$

So $B = \dfrac{\mu_0}{4\pi}\dfrac{I\theta}{r}$

$$= 10^{-7} \times 6 \times \frac{\pi}{3} \times \frac{1}{0.1} \text{ Tesla}$$

$$= 6.28 \times 10^{-6} \text{ Tesla}$$

$$= 6.28\ \mu \text{ Tesla}$$

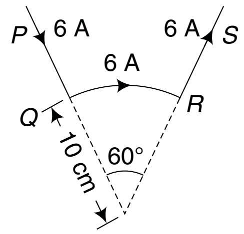


Fig. 4.2W Magnetic field at the center of a circular element of a current-carrying wire.

Example 4.5 A magnetic field $4 \times 10^{-3}\,\hat{k}$ tesla exerts a force of $(4\hat{i} + 3\hat{j}) \times 10^{-10}$ N or a particle having change of 1×10^{-9} C and moving in the *xy* plane. Calculate the velocity of the particle.

Sol. Lorentz force $\vec{F} = q\,(\vec{v} \times \vec{B})$

Here $(4\hat{i} + 3\hat{j}) \times 10^{-10} = 1 \times 10^{-9}\,[(v_x\hat{i} + v_y\hat{j})] \times 4 \times 10^{-3}\,\hat{k}$

or, $(4\hat{i} + 3\hat{j}) \times 10^{-10} = 4 \times 10^{-12}\,[(v_x(-\hat{j}) + v_y(\hat{i})]$

or, $v_x = -\dfrac{3 \times 10^{-10}}{4 \times 10^{-12}} = -\dfrac{3}{4} \times 10^2 = -75$

$$v_y = \frac{4 \times 10^{-10}}{4 \times 10^{-12}} = 100$$

So, $v = -75\,\hat{i} + 100\,\hat{j}$ ms^{-1}

Example 4.6 A test charge having a charge of 0.4 C is moving with a velocity of $4\hat{i} - \hat{j} + 2\hat{k}$ m/s through an electric field of intensity $10\,\hat{i} + 10\,\hat{k}$ and a magnetic field $2\hat{i} - 6\hat{j} - 6\hat{k}$. Determine the magnitude and direction of the Lorentz force acting on the test charge. [WBUT 2007]

Sol. Total Lorentz force $= q\vec{E} + q\,(\vec{v} \times \vec{B})$

Here electric force $= q\vec{E} = 0.4\,(10\,\hat{i} + 10\,\hat{k})$

and magnetic force $= q\,(\vec{v} \times \vec{B}) = 0.4 \begin{vmatrix} i & j & k \\ 4 & -1 & 2 \\ 2 & -6 & -6 \end{vmatrix}$

$$= 0.4\,[(6 + 12)\,\hat{i} + (4 + 24)\,\hat{j} + (-24 + 2)\,\hat{k}]$$

$$= 0.4\,(18\,\hat{i} + 28\,\hat{j} - 22\,\hat{k})$$

Now total Lorentz force, $\vec{F} = q\vec{E} + q\,(\vec{v} \times \vec{B})$

$$= 0.4\,(10\,\hat{i} + 10\,\hat{k}) + 0.4\,(18\,\hat{i} + 28\,\hat{j} - 22\,\hat{k})$$

$$= 0.4\,(28\,\hat{i} + 28\,\hat{j} - 12\,\hat{k})$$

The magnitude of the force $= 0.4\sqrt{(28)^2 + (28)^2 + (-12)^2} = 16.6$ N

Suppose, the total force makes an θ with the x axis, then

$$\vec{F}.\,\hat{i} = 0.4\,(28\,\hat{i} + 28\,\hat{j} - 12\,\hat{k}).\,\hat{i} = 0.4 \times 28$$

or, $\quad (0.4)\sqrt{(28)^2 + (28)^2 + (-12)^2}\cos\theta = 0.4 \times 28$

or, $$\cos\theta = \frac{28}{\sqrt{(28)^2 + (28)^2 + (-12)^2}}$$

or, $$\cos\theta = \frac{7}{\sqrt{107}}$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{7}{\sqrt{107}}\right) = 47.41^\circ$$

Example 4.7 A straight wire carrying a current of 10 A is bent into a semicircular arc of π cm radius as shown in Fig. 4.3W. What is the magnetic field and direction of the magnetic field at center O of the arc?

Sol. The magnetic field at center O due to each straight portion of the wire is O. The magnetic field at center O is only due to half the circular loop. The magnitude of magnetic field

$$B = \frac{\mu_0}{4\pi}\left(\frac{\pi I}{a}\right) = \frac{\mu_0 I}{4a} = \frac{4\pi \times 10^{-7} \times 10}{4 \times \pi \times 10^{-2}} = 10^{-4}\text{ T}$$

The current in the loop is anticlockwise and the direction, of the field is perpendicular to the paper.

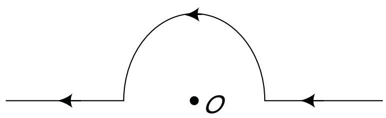


Fig. 4.3W Magnetic field at the center of a semicircular wire carrying current.

Example 4.8 The wire loop $ABCDA$ formed by joining two semicircular wires of radii R_1 and R_2 carries a current I as shown in Fig. 4.4W. Find out the magnetic field at center O.

Sol. The magnetic field due to a semicircular loop of radius R_1 is

$$B_1 = \frac{\mu_o}{4\pi}\left(\frac{\pi I}{R_1}\right)$$

[Here, direction of current is anticlockwise]

and direction of the field is normal to the plane of the loop, directed upward. For a bigger loop, direction of the current is clockwise. The value of the magnetic field due to semi-circular loop of radius R_2 is

$$B_2 = \frac{\mu_0}{4\pi}\left(\frac{\pi I}{R_2}\right)$$

Here direction of the magnetic field is into the plane of the paper. So, net magnetic field

$$B = B_1 - B_2 = \frac{\mu_0}{4\pi}\pi I\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \frac{\mu_0 I}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

and direction is perpendicular to the plane of the paper, hence directed upward.

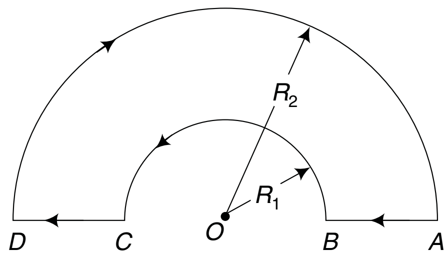


Fig. 4.4W Magnetic field at the center of two concentric semicircular wires carrying current.

Example 4.9 A current-carrying straight wire cannot move but a current-carying square loop adjacent to it can move under the influence of a magnetic force. Show that the square loop in Fig. 4.5W will move towards the wire.

Sol. In Fig. 4.5W, we see that the force acting on arms *AB* and *DC* are equal and opposite. But the force on arm *AD* is given by

$$F_1 = \frac{\mu_0}{4\pi}\left(\frac{2Ii}{a}\right)$$

which is directed towards the wire. The force on arm *BC* is given by

$$F_2 = \frac{\mu_0}{4\pi}\left(\frac{2Ii}{b}\right)$$

which is directed away from the wire. Here, $b > a$ hence $F_1 > F_2$. So the loop will move towards the wire.

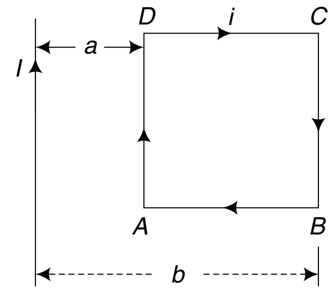


Fig. 4.5W Current-carrying square loop moves under the influence of magnetic force due to the current-carrying wire.

Example 4.10 A long solenoid of 40 cm length has 300 turns. If the solenoid carries a current of 3.5 A, calculate (i) magnetic field at the center of the solenoid, and (ii) magnetic field of the axis at one end of the solenoid.

Sol. (i) The magnetic field at the center of the solenoid is

$$B = \mu_0 \, n \, I = \mu_0 \frac{N}{l} I = 4\pi \times 10^{-7} \times \frac{300}{0.4} \times 3.5 = 3.3 \times 10^{-3} \text{ Tesla}$$

(ii) The magnetic field at one end of the solenoid

$$B = \frac{1}{2}\mu_0 \, n \, I = \frac{1}{2}\mu_0 \frac{N}{l} I = \frac{1}{2} \times \frac{300}{0.4} \times 3.5$$

$$= 1.65 \times 10^{-3} \text{ Tesla.}$$

Example 4.11 In Fig. 4.6W, in between two rails, a metal wire of mass *m* slides without friction. The distance of separation between the two rails is *l*. The break lies in a vertical uniform magnetic field *B* (perpendicular to the paper). Find the velocity of the wire as a function of time *t*, if a constant current *I* flows along one rail and then through the wire and back down the other rail.

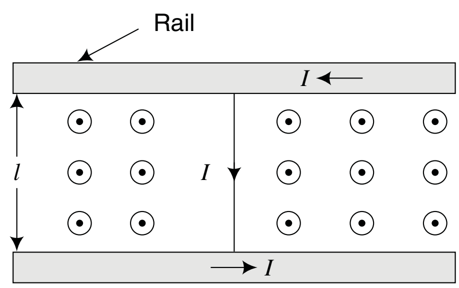


Fig. 4.6W A metal wire slides without friction between two rails carrying current in an opposite direction.

Sol. The force exerted on the wire of length *l* is

$$F = B\,I\,l \sin 90° = B\,I\,l$$

The direction of the force will be to the left according to Fleming's left-hand rule.

The acceleration of the wire

$$a = \frac{F}{m} = \frac{B\,I\,l}{m}$$

Let initial velocity of wire $u = 0$, then velocity at any time *t* is $v = at = \frac{B\,I\,l}{m} t$.

Example 4.12 Two straight wires, each 2 m long, are parallel to one another and are separated by a distance of 2 cm. If each carries a current of 8 *A*, calculate the force experienced by either of the wires.

Sol. The force per unit length experienced by each wire carrying currents I_1, I_2 separated by a distance *d* is given by

$$F = \frac{\mu_0 I_1 I_2}{2\pi a}$$

Here $\quad I_1 = I_2 = 8$ A $d = 2$ cm $= 0.02$ m

so, $\quad F = \dfrac{4\pi \times 10^{-7} \times 8 \times 8}{2\pi \times 0.02} = 64 \times 10^{-5}$ N/m

The total force on either of the wires is

$$F = 2 \times 64 \times 10^{-5} \text{ N} = 128 \times 10^{-5} \text{ N}$$

Example 4.13 In the Bohr model of a hydrogen atom, an electron is revolving in a circular path of 0.4 Å radius with a speed of 10^6 m/s. What is the value of magnetic field at the center of the orbit?

Sol. Here $r = 0.4$ Å $= 0.4 \times 10^{-10}$ m, $v = 10^6$ m/s.

We know that time period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 0.4 \times 10^{-10}}{10^6} = 8\pi \times 10^{-17}$ s

Again, current $I = \dfrac{e}{T} = \dfrac{1.6 \times 10^{-19}}{8\pi \times 10^{-17}} = \dfrac{2}{\pi} \times 10^{-3}$ A

The magnetic field at the center of the orbit

$$B = \frac{\mu_o}{4\pi}\left(\frac{2\pi I}{r}\right) = 10^{-7} \times \frac{2 \times \pi}{0.4 \times 10^{-10}} \times \frac{2}{\pi} = 10 \text{ Tesla.}$$

Example 4.14 The volume current density distribution in cylindrical coordinates is

$$J(r, \varphi, z) = 0 \qquad 0 < r < a$$

$$= J_0\left(\frac{r}{a}\right)\hat{e}_z \qquad a < r < b$$

$$= 0 \qquad b < r < \infty$$

Find the magnetic field in various regions [Fig. 4.7W].

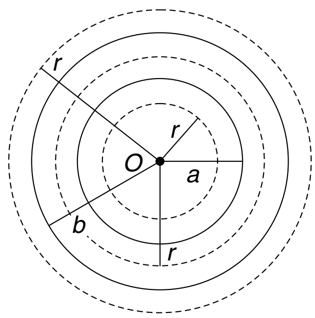


Fig. 4.7W

Sol. From Ampere's circuital law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_S J \cdot d\vec{S}$$

For region $0 < r < a$, $\quad J = 0$

So, $\quad \oint \vec{B} \cdot \vec{dl} = 0 \quad$ or, $\quad B = 0$

For region $a < r < b$ $\quad J(r, \varphi, z) = J_0\left(\dfrac{r}{a}\right)\hat{e}_z$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_{\varphi=0}^{2\pi} \int_a^r J_0\left(\frac{r}{a}\right) \cdot r\, dr\, d\varphi$$

or, $\quad B \times 2\pi r = \mu_0 \dfrac{J_0}{a}\left[\dfrac{r^3}{3}\right]_a^r [\varphi]_0^{2\pi} = \mu_0 \dfrac{2\pi}{3a} J_0 (r^3 - a^3)$

or, $\quad B = \dfrac{\mu_0 J_0}{3ar}(r^3 - a^3)$

Now at $r = b$, $\quad B = \dfrac{\mu_0 J_0}{3ab}(b^3 - a^3)$

For region $b < r < \alpha$, $J = 0$

we have $\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_{\varphi=0}^{2\pi} \int_{r=a}^{b} \frac{J_0 r}{a} r \, dr \, d\varphi$

or, $$B = \frac{\mu_0 J_0}{3ar} (b^3 - a^3)$$

Example 4.15 Show that $\oint_S \vec{B} \cdot \vec{dS} = 0$ where $\vec{B}$ is the magnetic field and S is a closed surface.

[WBUT 2006]

Sol. We have by applying divergence theorem

$$\oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B}) \, dV$$

But $\vec{\nabla} \cdot \vec{B} = 0$ [$\because \vec{B}$ is solenoidal field]

$$\therefore \quad \oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B}) \, dV = 0$$

$$\therefore \quad \oint_S \vec{B} \cdot \vec{dS} = 0$$

which shows that the lines of induction are continuous, meaning, it has no sources or sinks.

Example 4.16 If the vector potential $\vec{A} = (x^2 + y^2 - z^2)\,\hat{j}$ at position (x, y, z), find the magnetic field at (1, 1, 1).

[WBUT 2007]

Sol. Here, vector potential $\vec{A} = (x^2 + y^2 - z^2)\,\hat{j}$

We know that $$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (x^2 + y^2 - z^2) & 0 \end{vmatrix}$$

$$= \hat{i}\left[-\frac{\partial}{\partial z}(x^2 + y^2 - z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2 + y^2 - z^2)\right] = 2z\,\hat{i} + 2x\,\hat{k}$$

At (1, 1, 1) $\quad \vec{B} = 2\hat{i} + 2\hat{k}$

Example 4.17 If the vector potential $\vec{A} = \frac{1}{2}(\vec{a} \times \vec{r})$, where $\vec{a}$ is a constant vector, find the associated magnetic field.

Sol. Let $\quad \vec{a} = \hat{i}\,a_1 + \hat{j}\,a_2 + \hat{k}\,a_3$

So $\quad \vec{a} \times \vec{r} = \hat{i}\,(z\,a_2 - y\,a_3) + \hat{j}\,(x\,a_3 - z\,a_1) + \hat{k}\,(y\,a_1 - x\,a_2)$

We know that $$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\,a_2 - y\,a_3 & x\,a_3 - z\,a_1 & y\,a_1 - x\,a_2 \end{vmatrix}$$

$$= \frac{\hat{i}}{2}\left[\frac{\partial}{\partial y}(y\,a_1 - x\,a_2) - \frac{\partial}{\partial z}(x\,a_3 - z\,a_1)\right] + \frac{\hat{j}}{2}\left[\frac{\partial}{\partial z}(z\,a_2 - y\,a_3) - \frac{\partial}{\partial x}(y\,a_1 - x\,a_2)\right]$$

$$+ \frac{\hat{k}}{2}\left[\frac{\partial}{\partial x}(x\,a_3 - z\,a_1) - \frac{\partial}{\partial y}(z\,a_2 - y\,a_3)\right]$$

$$= a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k} = \vec{a}$$

Example 4.18 Two circular coils having identical turns and radii in the ratio 1:3 are joined in series. Find the ratio of the magnetic fields at the center of the coils.

Sol. Here, $B = \frac{\mu_0 N I}{2R}$. Since the coils are connected in series, therefore I is constant. N is also given to be constant.

So, $B \propto \frac{1}{R}$ $\therefore \frac{B_1}{B_2} = \frac{3}{1}$ or, $B_1 : B_2 = 3:1$

Review Exercises

Part 1: Multiple Choice Questions

1. A wire of length L carrying a current I is bent into a circle. The magnitude of the magnetic field at the center of the circle is

   (a) $\frac{\pi \mu_0 I}{L}$ (b) $\frac{\mu_0 I}{2\pi L}$ (c) $\frac{\mu_0 I}{2L}$ (d) $\frac{2\pi \mu_0 I}{L}$

2. In the region around a moving charge, there is

   (a) electric field only
   (b) magnetic field only
   (c) neither electric field nor magnetic field
   (d) electric as well as magnetic field

3. The magnetic field at the origin due to a current element $i\,\vec{dl}$ placed at a position $\vec{r}$ is

   (a) $\frac{\mu_0}{4\pi}\frac{i\,\vec{dl} \times \vec{r}}{r^2}$ (b) $\frac{\mu_0}{4\pi}\frac{i\,\vec{dl} \times \vec{r}}{r^3}$ (c) $\frac{\mu_0}{4\pi}\frac{i\,\vec{dl} \times \hat{r}}{r^3}$ (d) zero

4. A current-carrying straight wire is kept along axis of a circular loop carrying current. The straight wire

   (a) will exert an inward force on the circular loop
   (b) will exert an outward force on the circular loop
   (c) will not exert any force on the circular loop
   (d) None of these

5. A moving charge produces

   (a) electric field only (b) magnetic field only
   (c) Both of them (d) None of these

6. Which of the following statements is not characteristic of a static magnetic field? [WBUT 2006]
   (a) It is solenoid. (b) It is conservative.
   (c) Magnetic flux lines are always closed.
   (d) It has no sink or source.
7. A current-carrying straight wire cannot move, but a current-carrying square loop adjacent to it can move under the influence of a magnetic force [Fig. 4.8W]. [WBUT 2008]
   The square loop will
   (a) remain stationary (b) move towards the wire
   (c) move away from the wire (d) None of these

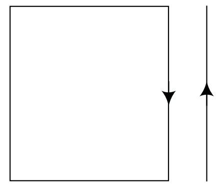


Fig. 4.8W Moving of current-carrying square loop.

8. The direction of magnetic induction due to a straight infinitely long current carrying wire is [WBUT 2008]
   (a) perpendicular to the wire (b) parallel to the wire
   (c) at an inclination of 30° to the wire (d) None of these
9. The equation of continuity in a steady charge distribution is
   (a) $\vec{\nabla} \cdot \vec{J} = 0$ (b) $\vec{\nabla} \times \vec{J} = 0$ (c) $\vec{\nabla} \cdot \vec{J} = \rho$ (d) $\vec{\nabla} \cdot \vec{J} = \dfrac{\rho}{\varepsilon_0}$
10. The work done by the Lorentz force $\vec{F}$ on a charged particle is
   (a) $\vec{F} \cdot d\vec{r}$ (b) zero (c) $\dfrac{q}{\varepsilon_0}$ (d) $q\,F$
11. $\vec{\nabla} \cdot \vec{J} + \dfrac{\partial \rho}{\partial t} = 0$
   (a) integral form of law of conservation of charge
   (b) differential form of law of conservation of charge
   (c) Poisson's equation
   (d) None of these
12. A conduction loop carrying a current I is placed in a uniform magnetic field pointing into the plane of the paper as shown in Fig. 4.9W. The loop will have a tendency to [WBUT 2007]
   (a) contract
   (b) expand
   (c) move towards positive the x axis
   (d) move towards negative the x axis

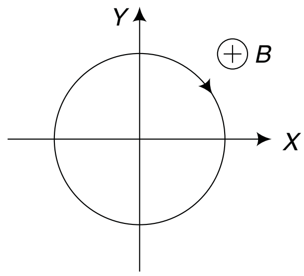


Fig. 4.9W A current-carrying loop in a magnetic field.

13. A copper wire is bent in the form of a sine wave of wavelength λ and peak-to-peak value as shown in Fig. 4.10W. A magnetic field of flux density B tesla acts perpendicular to the plane of the figure in the entire region. If the wire carries a steady current I ampere, the magnetic force on the wire is [WBUT 2007]
   (a) $I\sqrt{(a^2 + \lambda^2)}\,B$ (b) $I a B$
   (c) $I(a + \lambda)\,B$ (d) $I \lambda B$

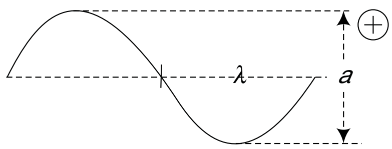


Fig. 4.10W Current-carrying sinusoidal wire in a magnetic field.

14. If the current density $\vec{J} = k\,\hat{r}$ where $\hat{r}$ is a unit vector along $x\,\hat{i} + y\,\hat{j}$, the current through the surface $x^2 + y^2 = a^2$, bounded by $z = 0$ and $z = h$ is [WBUT 2007]

(a) $\pi a^2\, hk$ (b) zero (c) $2\pi ahk$ (d) $\frac{a^3 k}{\pi h}$

15. If $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{B}$ and $\vec{A}$ are any vectors then [WBUT 2005]

(a) $\vec{\nabla} \cdot \vec{B} = 0$ (b) $\vec{\nabla} \cdot \vec{B} = +1$ (c) $\vec{\nabla} \cdot \vec{B} = -1$ (d) None of these

16. Magnetic field due to an infinitely long straight conductor carrying current I is

(a) $\frac{\mu_0}{4\pi}\left(\frac{2\pi I}{a}\right)$ (b) $\frac{\mu_0}{4\pi}\left(\frac{2I}{a}\right)$ (c) $\frac{1}{4\pi\,\mu_0}\frac{I}{a}$ (d) zero

17. Two thin, long parallel wires, separated by a distance 'd' carry a current of I A, in the same direction. They will

(a) attract each other with a force of $\frac{\mu_0 I^2}{2\pi d^2}$ (b) repel each other with a force of $\frac{\mu_0 I^2}{2\pi d^2}$

(c) attract each other with a force of $\frac{\mu_0 I^2}{2\pi d}$ (d) None of these

18. A long wire carries a steady current. It is bent into a circle of one turn and the magnetic field at the center of the coil is B. It is then bent into a circular loop of n turns. The magnetic field at the center of the coil will be

(a) $n^2 B$ (b) $n^3 B$ (c) $n\,B$ (d) $2\,n\,B$

19. A 1.5 m long solenoid 0.4 cm in diameter possesses 10 turns per cm length. A current of 5 A flows through it. The magnetic field at the axis inside the solenoid is

(a) $4\pi \times 10^{-4}$ Tesla (b) $2\pi \times 10^{-3}$ Tesla

(c) $2\pi \times 10^{-6}$ Tesla (d) None of these

20. A long straight wire along the z axis carries a current I in the negative z direction. The magnetic vector field $\vec{B}$ at a point having coordinates (x, y) in the $z = 0$ plane is

(a) $\frac{\mu_0 I}{4\pi}\left(\frac{x\,\hat{i} - y\,\hat{j}}{x^2 + y^2}\right)$ (b) $\frac{\mu_0 I}{2\pi}\left(\frac{y\,\hat{i} - x\,\hat{j}}{x^2 + y^2}\right)$

(c) $\frac{\mu_0 I}{2\pi}\left(\frac{y\,\hat{i} + x\,\hat{j}}{2x^2 + y^2}\right)$ (d) None of these

[**Ans.** 1 (a), 2 (d), 3 (b), 4 (c), 5 (c), 6 (b), 7 (c), 8 (a), 9 (a), 10 (b), 11 (b), 12 (b), 13 (d), 14 (b), 15 (a), 16 (b), 17 (c), 18 (a), 19 (b), 20 (b)]

Short Questions with Answers

1. The net charge on a current-carrying conductor is zero. Then, why does it experience a force in a magnetic field?

Ans. In a conductor, positive ions are stationary. So, they do not experience any force. But the free electrons drift towards the positive end of the conductor with some drift velocity and experience a magnetic field.

2. Why does a solenoid contract when current is passed through it?

Ans. When current is passed through a solenoid, the currents in the different turns of the solenoid flow in the same direction. Again we know that when currents in two parallel conductors flow in the same direction, the conductors attract each other. So, the solenoid contracts.

3. Define current density.

Ans. Current density is defined as the current through an infinitesimal area at any point inside a conductor, the area held perpendicular to the direction of flowing positive charge.

4. What is Lorentz force? Show that Lorentz force does not work on a charged particle.

Ans. See Section 4.5.

5. State Ampere's law both in integral and differential form.

Ans. See Section 4.12.

6. Compare between Lorentz electric force and Lorentz magnetic force.

Ans. Lorentz electric force $F_e = qE$, direction along the field does not depend on velocity and work is done. Lorentz magnetic force $F_m = Bq\,v \sin\theta$, direction perpendicular to plane containing B and v and depends on velocity of the charge, no workforce.

7. Define magnetic scalar potential and magnetic vector potential.

Ans. See Section 4.14.

8. A proton moving through a magnetic field region experiences maximum force. When does this occur?

Ans. When the proton moves perpendicular to the magnetic field, $\theta = 90°$, $\vec{v} \times \vec{B}$ will be maximum. So $\vec{F}$ is maximum.

9. Write the one condition under which an electric charge does not experience a force in a magnetic field.

Ans. Either the electric charge is at rest or it is moving parallel to the direction of the magnetic field.

10. Define an ampere in terms of the force between current-carrying conductors.

Ans. One ampere is that current which if passed in each of the two parallel conductors of infinite length and 1 m apart in vacuum, causes each conductor to experience a force of 2×10^{-7} Nm^{-1} length of conductor.

11. Apply Ampere's law qualitatively to the three parts as shown in Fig. 4.11W.

Ans. For paths I and III, $\oint B.\,\vec{dl} = \mu_0 I$

For path II, the net current is zero, i.e., for path II, $\oint \vec{B} \cdot \vec{dl} = 0$

12. A charge 4*C* is moving with a velocity $\vec{v} = (2\hat{j} + 3\hat{k})$ in a magnetic field $B = (2\hat{j} + 3\hat{k})$ Wbm^{-2}. Find the force acting on the charge.

Ans. Here, $\vec{v}$ and $\vec{B}$ are parallel vectors.

So $\vec{v} \times \vec{B} = 0 \quad \therefore \vec{F} = q(\vec{v} \times \vec{B}) = 0$

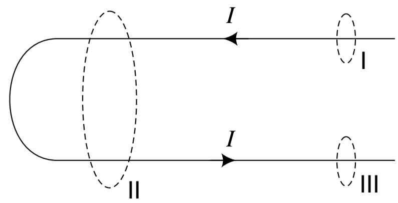


Fig. 4.11W

Part 2: Descriptive Questions

1. State Biot–Savart law. Using Biot–Savart law, calculate the field at the center of a circular current-carrying coil. [WBUT 2002]
2. (a) State Biot–Savart law.
   (b) Using Biot–Savart law, obtain an expression for the magnetic flux intensity at the center of a long current-carrying solenoid
   (c) Show that the field at the end of such a solenoid is half of that at the center. [WBUT 2003]
3. (a) State Biot–Savart law in magnetostatics. Find the magnetic field of an infinitely long straight wire at a transverse distance of d from the expression of $\vec{B}$ found in Biot–Savart law.
   (b) Express Biot–Savart law in terms of current density and hence show that the magnetic field is solenoidal.
   (c) Express Ampere's circuital law in terms of vector potential. (You may use $\vec{\nabla} \cdot \vec{A} = 0$, where $\vec{A}$ is the vector potential.) [WBUT 2008]
4. Find the magnetic induction $\vec{B}$ at a point on the axis of an infinitely long solenoid carrying a current I, number of turns per unit length being n. [WBUT 2007]
5. Find the magnetic field of a circular loop carrying field due to a long solenoid at a point. [WBUT 2007]
6. (a) State Ampere's circuital law.
   (b) By applying Ampere's circuital law, find out magnetic field due to a long solenoid at a point (i) inside the solenoid, and (ii) outside the solenoid.
7. What is Lorentz force? Show that Lorentz force does not work on a charged particle.
8. What do you mean by magnetic vector potential? Why is it called so? [WBUT 2002]
9. Show that $\oint \vec{B} \cdot \vec{dS} = 0$, when $\vec{B}$ is the magnetic field and S is a closed surface. State the theorem that you have used. [WBUT 2006]
10. Show that the equation of continuity is given by $\vec{\nabla} \cdot \vec{J} + \dfrac{\partial \rho}{\partial t} = 0$, where $\vec{J}$ and ρ have their usual meaning. [WBUT 2005]
11. (a) Give an example of an electrical circuit carrying a non-steady current when Ampere's circuital law is not applicable.
    (b) Write the expression of the magnetic field due to a current-carrying conductor. Draw a diagram necessary to explain the symbols. Show that this field is solenoidal. [WBUT 2006]
12. Starting from the definition of current density, derive the equation of continuity in current electricity. [WBUT 2007]
13. State Ampere's law in magnetostatics in integral form and from that deduce its differential form. [WBUT 2007]
14. Write down the condition of steady-state current. Show that Ampere's law implies that the current is in the steady state. [WBUT 2007]
15. Prove that the magnetic field inside a toroid having n numbers of turns per unit length and carrying a current I is $\mu_0 nI$.
16. Find the force per unit length of a current-carrying conductor placed in a uniform magnetic field. Hence find the force between the straight conductors carrying currents.

Part 3: Numerical Problems

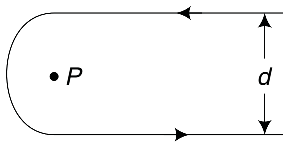


Fig. 4.12W

1. Calculate the drift speed of the electrons when 1 A of current exists in a copper wire of 2 mm^2 cross section. The number of free electrons in 1 cm^3 of copper is 8.5×10^{22}. [**Ans.** 36×10^{-3}]
2. Find the magnetic field at the point P in Fig. 4.12W. The curve portion is a semicircle and the straight wires are long. $\left[\textbf{Ans. } \frac{\mu_o}{2d}\left(\frac{2}{\pi}+\frac{1}{d}\right)\right]$
3. Consider a coaxial cable which consists of an inner wire of radius 'a' surrounded by an outer shell of inner and outer radii 'b' and 'c' respectively. The inner wire carries an equal current in opposite direction. Find the magnetic field at a distance 'r' from the axis, where
   (a) $r < a$ (b) $a < r < b$ (c) $b < r < c$ (d) $r > c$.
   Assume that the current density is uniform in the inner wire and also uniform in the outer shell.
   [WBUT (Question Bank)] $\left[\textbf{Ans. } \text{(a) } B = \frac{\mu_0 Ir}{2\pi a^2} \text{ (b) } \frac{\mu_0 I}{2\pi r} \text{ (c) } \frac{\mu_0 Ir}{2\pi(c^2-b^2)} \text{ (d) } B = 0\right]$
4. Find the magnetic field B due to a semicircular wire of 10.00 cm radius carrying a current of 5.0 A at its center of curvature. [**Ans.** 1.6×10^{-4} Tesla]
5. Two parallel wires carry equal currents of 10 A along the same direction and are separated by a distance of 2.0 cm. Find the magnetic field at a point which is 2.0 cm away from each of these wires. [**Ans.** 1.7×10^{-9} Tesla]
6. An infinite wire carrying a current I is bent in the form of a parabola. Find the magnetic field at the focus of the parabola. $\left[\textbf{Ans. } B = \frac{\mu_0 I}{4a}\right]$
7. If the magnetic scalar potential $\varphi_m = x^2 + y^2 - z^2$ at any point (x, y, z) in current free space, then find the magnetic field at the point (1, 2, 2). [**Ans.** $-2\,\hat{i} - 4\hat{j} + 4\,\hat{k}$] [**Hints:** Let $\vec{B} = -\vec{\nabla}\,\varphi_m$]
8. If the vector potential $\vec{A} = (2z + 5)\,\hat{i} + (3x - 2)\,\hat{j} + (4x - 1)\,\hat{k}$, find the magnetic field. [**Ans.** $B = -2\hat{i} + 3\,\hat{k}$]
9. A solenoid has 4 layers of 1200 turns each. Its length and mean radius are 3 m and 0.25 m respectively. Find the magnetic field at the center if a current of 2.5 A flows through it. [**Ans.** $B = 5.02 \times 10^{-3}$ Tesla]
10. Figure 4.13W shows a current-carrying system of straight wire and loop. Determine the magnetic field at the center O of the loop. Given R is the radius of the loop and I is the current flowing in the system. $\left[\textbf{Ans. } B = \frac{\mu_0 I}{2R}\left(1-\frac{1}{\pi}\right)\right]$

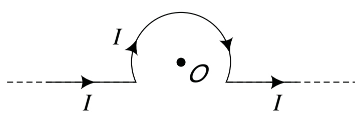


Fig. 4.13W

11. A proton enters a magnetic field of 4 Tesla with a velocity of 2.5×10^6 ms^{-1} at an angle of 30° with the direction of the field. Find the magnitude of the force acting on the proton. [**Ans.** $F = 8 \times 10^{-13}$ N]
12. A particle of charge q moves with a velocity $v = a\,\hat{i}$ in a magnetic field $B = b\hat{j} + c\,\hat{k}$ where a, b and c are constants. Find the magnitude of the force experienced by the particle. [**Ans.** $qa\,(b^2 + c^2)^{1/2}$]

CHAPTER

5

Electromagnetic Field Theory

5.1 INTRODUCTION

In previous chapters we have discussed the fundamentals of electrostatics and magnetostatics. The static electric and magnetic fields are produced by charges at rest and steady current, respectively, and they are independent of each other.

In the present chapter, we discuss the time-varying fields. A time-varying electric field produces a magnetic field and a time-varying magnetic field produces an electric field. Michael Faraday gave the fundamental postulate for electromagnetic induction that relates the time-varying magnetic field with an electric field.

In this chapter, we deal with the interaction between electric and magnetic fields and obtain the four Maxwell's equations. Maxwell provided a mathematical theory that showed a close relationship between all electrical and magnetic phenomena, and form the foundation of electromagnetic theory. The combined Maxwell's equations yield wave equations and predict the existence of electromagnetic waves.

5.2 MAGNETIC FLUX

The magnetic flux linked with a surface held in a magnetic field is defined as the number of magnetic field lines crossing the surface normally.

The magnetic flux linked with a surface ds [Fig. 5.1] held inside a magnetic field is given by

$$d\varphi = B_n\, ds \qquad ...(5.1)$$

where B_n is the normal component of the magnetic field B along the direction of a surface element ds.

Again from Fig. 5.1, $B_n = B \cos \theta$

so,
$$d\varphi = B \cos \theta\, ds$$
$$= \vec{B} \cdot \overrightarrow{ds} \qquad ...(5.2)$$

If the magnetic field B is uniform over the surface area S, then total flux

$$\varphi = \vec{B} \cdot \vec{S} \qquad ...(5.3)$$

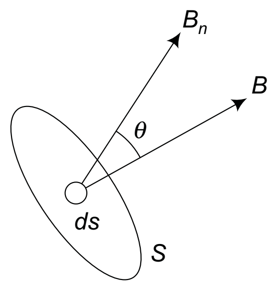


Fig. 5.1 Magnetic flux associated with a surface *S*.

If the normal is drawn in the direction of the magnetic field then the flux is taken as positive and if the normal is drawn opposite to the direction of the field then magnetic flux is taken as negative. The SI unit of magnetic flux is Weber (Wb) or Tesla m^2. Magnetic flux density of magnetic field induction is the magnetic flux per unit area, i.e., $B = \dfrac{\varphi}{A}$.

The SI unit of B is Tesla (T).

5.3 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Faraday observed that if a bar magnet moved either away or toward the axis of a conducting loop with no battery, then a current is produced in the loop. The current exists as long as the magnet is moving. The current flowed in the circuit when the flux through the circuit is altered. Faraday called this phenomenon *electromagnetic induction*. *Electromagnetic induction* is the process in which an emf is induced in a circuit placed in a magnetic field when the magnetic flux linked with the circuit changes. **Faraday's law tells us whenever the flux (φ) of magnetic field through the area bounded by a close conducting loop changes, an emf (ε) is produced in the loop.** Mathematically,

$$|\varepsilon| \propto \frac{d\varphi}{dt} \qquad ...(5.4)$$

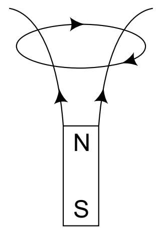


Fig. 5.2 The direction of induced emf according to Lenz's law.

The direction of the induced emf is provided by *Lenz's law*. This law states that *the direction of the induced emf is such that the magnetic flux associated with the current generated by it opposes the original change of flux causing emf.* To explain Lenz's law, we consider a magnet in the direction as shown in Fig. 5.2, i.e., towards the loop. As the magnet gets closer to the loop, the magnetic field increases and hence, the flux of the magnet field through the area of the loop increases. Thus increasing the magnetic flux through the coil, the induced current will flow in the direction shown, so that its own flux opposes the increase in the flux of the magnet. The induced current produces an induced emf. The induced emf is often called the back emf.

So, $$\varepsilon = -\frac{d\varphi}{dt} \qquad ...(5.5)$$

The direction of the current that produces a field towards the magnet can easily be obtained by using the right-hand thumb rule.

Lenz's Law and Conservation of Energy

According to Lenz's law, induced emf opposes the change that produces it. For change in magnetic flux, we have to perform mechanical work. So mechanical energy is converted into electrical energy. Thus, Lenz's law is in accordance with the law of conservation of energy.

5.4 ELECTROMOTIVE FORCE

In Fig. 5.3, we see a rod of length L sliding on a pair of parallel conducting tracks AB and DC. The arrangement is kept in a uniform magnetic field B which is normally out of the plane paper. Suppose, the rod moves parallel to the track with a velocity v making an angle θ with the magnetic flux (φ) link with the loop will change with time and we get an induced emf. Now applying Faraday's law, in unit time the area of the loop increases by the area of the parallelogram with sides L and v, the rate of change of flux is

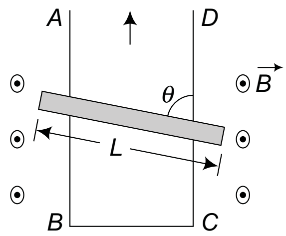


Fig. 5.3 Motional emf.

$$\frac{\partial\varphi}{\partial t} = BLv \sin\theta \qquad ...(5.6)$$

If the resistance of the loop is R, the current will be

$$I = \frac{Blv \sin\theta}{R} \text{ in the clockwise direction.}$$

The value of the induced emf will be $(\varepsilon) = BLv \sin\theta$ ...(5.7)

Note: *Fleming's right-hand rule* The direction of motional emf is given by either Lenz's law or Fleming's right-hand rule. **Fleming's right-hand rule states that if the thumb and the first two fingers of your right-hand are spread out in mutually perpendicular directions then the first finger points in the direction of the magnetic field and the thumb in the direction of motion of the conductor, and the central fingers points in the direction of the induced emf and thus the induced current.**

5.5 INTEGRAL FORM OF FARADAY'S LAW

If at any time t, the flux linked with the closed coil is φ then according to Faraday's law, the induced emf in the coil

$$\varepsilon = -\frac{d\varphi}{dt} \qquad \ldots(5.8)$$

If $\vec{E}$ be the field induced in space then the induced emf ε around the closed path C is given by integration of $\vec{E}$ and can be written as

$$\varepsilon = \oint_C \vec{E}\cdot\vec{dl} \qquad \ldots(5.9)$$

$$\therefore \qquad \oint_C \vec{E}\cdot\vec{dl} = -\frac{d\varphi}{dt} \qquad \ldots(5.10)$$

The total flux through the circuit is equal to the integral of normal component of flux density $\vec{B}$ over the surface bounded by the circuit.

$$\varphi = \int_S \vec{B}\cdot\overrightarrow{dS} \qquad \ldots(5.11)$$

$$\therefore \qquad \oint_C \vec{E}\cdot\vec{dl} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \qquad \ldots(5.12)$$

This is the integral form of Faraday's law of electromagnetic induction. Here, we use $\frac{\overrightarrow{\partial B}}{\partial t}$ instead of $\frac{\overrightarrow{dB}}{dt}$, because $\vec{B}$ is a function of both position and time.

5.5.1 Differential Form of Faraday's Law

Using Stoke's theorem,

$$\oint_C \vec{E}\cdot\vec{dl} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} \qquad \ldots(5.13)$$

Now, from Eq. (5.12), we have

$$\int_S (\vec{\nabla}\times\vec{E})\cdot\overrightarrow{dS} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\overrightarrow{dS} \qquad \ldots(5.14)$$

or, $$\int_S \left(\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}\right)\cdot\overrightarrow{dS}=0 \qquad \text{...(5.15)}$$

Eq. (5.15) must hold for any arbitrary surface S

so, $$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \text{...(5.16)}$$

This is the differential form of Faraday's law of electromagnetic induction.

The sources of the electromagnetic field are of two kinds. First one is the electrostatics field in which energy is conserved during a cyclic process and such a field has no curl. The second one is the magnetic field in which energy is transferred in a cyclic process and such a field is specified by the curl sources and has no divergence.

Taking divergence of Eq. (5.16), we have

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) \qquad \text{... (5.17)}$$

Since divergence of any curl is always zero, this is possible if

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \text{... (5.18)}$$

So, $\vec{B}$ is solenoidal.

Thus, Faraday's law gives two important results:

(a) The electric field is no longer a conservative field when the magnetic field varies with time.

(b) Magnetic free pole does not exist. All magnetic poles occur in pairs.

The time-varying electric and magnetic fields are thus interrelated and these two fields combine to form a single field known as *electromagnetic field*.

According to **Helmholtz theorem, any vector field is uniquely determined if its divergence and curl sources are given.** Electromagnetic fields have both types of sources.

5.6 DISPLACEMENT CURRENT

The concept of displacement current was introduced by Maxwell to account for production of magnetic field in empty space. The current for production of magnetic field is called *displacement current*. The current carried by conductors due to flow of charges is called *conduction current*. In empty space, conduction current is zero.

The displacement current is different from the conduction current in the sense that the former exists only when the electric field varies with time. For steady electric field in a conducting wire, the displacement current is zero. The current arising due to time-varying electric field between the plates of a capacitor is known as the displacement current.

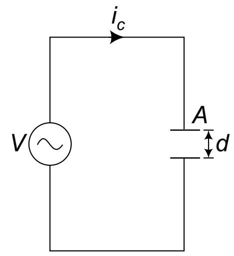


Fig. 5.4 Charging of a capacitor.

Figure 5.4 shows a circuit connecting a time-varying voltage source V to a pure capacitor (C). The current through a capacitor is called displacement current. Actually the displacement current does not flow through the capacitor, the displacement is only an apparent current representing the rate of transport of charge from one plate to another. When a voltage is applied to a capacitor the current through it is

$$I=\frac{dQ}{dt}=C\frac{dV}{dt} \qquad \text{... (5.19)}$$

where Q is the instantaneous charge, equal to CV. Again we know that for a parallel-plate capacitor

$$C = \frac{\varepsilon_0 A}{d} \qquad \text{... (5.20)}$$

where A is the area of the parallel plate, d is the separation between the plate and ε_0 is the free space permittivity.

So,
$$I = \frac{\varepsilon_0 A}{d}\frac{dV}{dt} \qquad \text{... (5.21)}$$

The relation between the electric field (ε_0) in the capacitor with potential

$$E = \frac{V}{d} \qquad \text{... (5.22)}$$

Now from Eq. (5.21)

$$I = \varepsilon_0 A \frac{dE}{dt} \qquad \text{... (5.23)}$$

or,
$$\frac{I}{A} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial D}{\partial t} \qquad \text{... (5.24)}$$

where $D = \varepsilon_0 E$ is known as electric displacement.

Now $\frac{I}{A}$ gives the current density (J_d)

So,
$$J_d = \frac{\partial D}{\partial t} \qquad \text{... (5.25)}$$

J_d is called the displacement current density. The displacement current $\frac{\partial D}{\partial t}$ is zero outside the plates but has a definite value between the plates. This definite value is exactly equal to the value of conduction current outside the plates.

5.7 MODIFIED AMPERE'S LAW

The differential form of Ampere's circuital law is

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} \qquad \text{... (5.26)}$$

which is applied to a steady magnetic field only. Since the divergence of any curl is zero, we have from Eq. (5.26)

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_o \vec{\nabla} \cdot \vec{J} = 0$$

or,
$$\vec{\nabla} \cdot \vec{J} = 0 \qquad \text{... (5.27)}$$

However, the equation of continuity

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad \text{... (5.28)}$$

shows that Eq. (5.28) is true only if

$$\frac{\partial \rho}{\partial t} = 0, \ \rho = \text{constant of time.}$$

So, Ampere's circuital law is valid only for static charge density. Ampere's circuital law in case of time-varying field does not hold good. Maxwell added another term to Ampere's law and ensured that it is valid for a time-varying field.

Let us add an unknown M to Eq. (5.26)

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{M}) \quad ...(5.29)$$

Taking divergence on both sides of Eq. (5.29), we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \cdot (\vec{J} + \vec{M}) = 0$$

or,
$$\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{M} = 0$$

or,
$$\vec{\nabla} \cdot \vec{M} = -\vec{\nabla} \cdot \vec{J} = \frac{\partial \rho}{\partial t} \quad \text{from Eq. (5.28)} \quad ... (5.30)$$

Again, from Gauss' law in electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

or,
$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \quad ... (5.31)$$

Now from Eqs. (5.30) and (5.31),

$$\vec{\nabla} \cdot \vec{M} = \varepsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E})$$

we get
$$\vec{M} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{D}}{\partial t} \quad ...(5.32)$$

After modification, Ampere's law becomes

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \quad ...(5.33)$$

$$= \mu_0 (\vec{J} + \vec{J}_d) \quad ...(5.34)$$

The term $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is known as the displacement current density.

Thus, displacement current density is entirely different from conduction current density. Displacement current is taken as a current only because it produces a magnetic field. Even in perfect vacuum, displacement current exists although there is no charge of any type. The presence of the term $\vec{J}_d$ enabled Maxwell to predict that an electromagnetic field should propagate through space in form of waves.

5.8 CONTINUITY PROPERTY OF CURRENT

From the conservation of charge, it is necessary that the total current density $(\vec{J} + \vec{J}_d)$ should obey the continuity equation. The individual term may or may not be continuous. As an example, in the case of charging of a parallel-plate capacitor, there is a conduction current in the region outside the plates of the capacitor. In empty space between the plates of the capacitor, conduction current is zero, but displacement current has a definite value between the plates. Thus, the individual terms are discontinuous but the sum of conduction current and displacement has the same value both inside and outside the plates.

5.9 MAXWELL'S EQUATIONS

5.9.1 Maxwell's Equations in Differential Form

Maxwell's equations represent the four basic laws of electricity and magnetism. These four laws are (i) Gauss' law in electrostatics, (ii) Gauss' law in magnetostatics, (iii) Faraday's law of electromagnetic induction, and (iv) Ampere's law with Maxwell's correction. All the four Maxwell's equations along with their salient features are being discussed here.

(i) Maxwell's first equation

The first equation represents Gauss' law in electrostatics, which may be written as

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad ...(5.35)$$

This is a time-independent steady-state equation which relates the spatial variation or divergence of an electric field with charge density. This relation is true both for stationary and moving charges.

(ii) Maxwell's second equation

The second Maxwell's equation represents Gauss' law in magnetostatics. Mathematically,

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad ...(5.36)$$

Equation (5.36) states that an isolated magnetic pole does not exist. This is also a time-independent or steady-state equation which gives the spatial variation or divergence of magnetic induction.

(iii) Maxwell's third equation

The third Maxwell's equation represents Faraday's law of electromagnetic induction. Mathematically,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ...(5.37)$$

This is a time-dependent equation. Equation (5.37) shows that a time-varying magnetic field acts as a source of electric field. It relates the spatial variation of electric field with time variation of a magnetic field.

(iv) Maxwell's fourth equation

The fourth Maxwell's equation represents modified Ampere's (Ampere's law with Maxwell's correction). Mathematically,

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad ....(5.38)$$

$$= \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

This is a time-independent equation. Equation (5.38) shows that a time-varying electric field acts as a source of magnetic field. The equation relates the spatial variation of a magnetic field with conduction current density and displacement current density.

Maxwell's equations are the basic equations for electromagnetism.

5.9.2 Maxwell's Equation in Integral Form

The four Maxwell's equations (5.35, 5.36, 5.37, 5.38) can be converted into an integral form.

(i) Maxwell's first equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

By taking the volume integral of both sides of $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ over a volume V with a closed surface S and using divergence theorem,

$$\left.\begin{aligned} &\int_V (\vec{\nabla} \cdot \vec{E})\, dV = \frac{1}{\varepsilon_0} \int_V \rho\, dV \\ \text{or,} \quad &\oiint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} \text{ where } q = \int_V \rho\, dV \end{aligned}\right\} \qquad ...(5.39)$$

We can infer from this equation that the electric lines of force do not constitute continuous close path.

(ii) Maxwell's second equation

$$\vec{\nabla} \cdot \vec{B} = 0$$

By taking the volume integral of both sides of $\vec{\nabla} \cdot \vec{B} = 0$ over a volume V with a closed surface S and using divergence theorem,

$$\left.\begin{aligned} &\int_V (\vec{\nabla} \cdot \vec{B})\, dV = 0 \\ \text{or,} \quad &\oiint_S \vec{B} \cdot \vec{dS} = 0 \end{aligned}\right\} \qquad ...(5.40)$$

We can infer from this equation that there is no magnetic flux sources, and magnetic flux lines always close upon themselves. It is the law of conservation of magnetic flux, i.e., magnetic monopole does not exist.

(iii) Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

By taking the surface integral of both sides of $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ over an open surface S with a contour C and applying Stoke's theorem

$$\left.\begin{aligned} &\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{dS} \\ \text{or,} \quad &\oint_C \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \vec{dS} \end{aligned}\right\} \qquad ...(5.41)$$

Equation (5.41) is the expression of Faraday's law, which states that a changing magnetic field $\vec{B}$ produces an electric field $\vec{E}$ such that line integral of $\vec{E}$ around a closed curve equals the negative rate of change of magnetic flux of a surface bounded by C.

(iv) Maxwell's fourth equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

By taking the surface integral of both sides of $\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$ over an open surface S with a contour C and applying Stoke's theorem

$$\left.\begin{aligned} &\int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} = \mu_0 \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{dS} \\ \text{or,} \quad &\oint_C \vec{B} \cdot \vec{dl} = \mu_0 \left[\int_S \vec{J} \cdot \vec{dS} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot \vec{dS} \right] \end{aligned}\right\} \qquad ...(5.42)$$

Equation (5.42) is the expression of modified Ampere's circuital law which states that the circulation of the magnetic field intensity around any closed path is equal to μ_o times the sum of conduction current and displacement current.

5.9.3 Physical Significance of Maxwell's Equations

(i) The first equation

The first Maxwell's equation known as Gauss' law in electrostatics, states that "The total electric flux through any closed surface is equal to the total charge enclosed by the surface divided by free space permittivity." Mathematically,

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \frac{q}{\varepsilon_0} \quad \text{[integral form]}$$

or,
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[differential form]}$$

If a closed surface does not include any charge, then obviously the total flux over the surface is zero. The equation represents that the electric lines of force are not closed lines. The electric field lines start on positive charges (sources) and end on negative charges (sink). Gauss' law is valid not only for static charges but also for charges in motion.

(ii) The second equation

The second Maxwell's equation is known as Gauss' law in magnetostatics and states that there are no magnetic flux sources and magnetic flux lines always close upon themselves. Mathematically,

$$\oint_S \vec{B} \cdot \overrightarrow{dS} = 0 \quad \text{[integral form]}$$

or,
$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{[differential form]}$$

Magnetic field is solenoidal means, it has no sources or sinks. The total magnetic flux through a closed surface is equal to zero, i.e., the magnetic flux entering into the volume is equal to the magnetic flux leaving the volume. The magnetic monopole does not exist, magnetic poles exist in pairs.

(iii) The third equation

The third Maxwell's equation is known as Faraday's law of electromagnetic induction and states that the line integral of $\vec{E}$ around a closed circuit is equal to the negative rate of change of magnetic flux linking the circuit. Mathematically,

$$\oint_C \vec{E} \cdot \overrightarrow{dl} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \overrightarrow{dS} \qquad \text{[integral form]}$$

or,
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{[differential form]}$$

It shows that the time variation of a magnetic field generates the electric field. So Faraday's law of electromagnetic induction shows how the electric and magnetic fields are interrelated. The time-varying magnetic fields acts as a source of electric field.

(iv) The fourth equation

The fourth Maxwell's equation is known as Ampere's law with Maxwell's correction. It relates the spatial variation of magnetic field with conduction current density and displacement current density. Mathematically,

$$\oint_C \vec{B} \cdot \overrightarrow{dl} = \mu_0 \left[\int_S \vec{J} \cdot \overrightarrow{dS} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot \overrightarrow{dS} \right] \quad \text{[integral form]}$$

or, $\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right]$ [differential form]

We see that both the conduction current density and displacement current density are two possible sources of magnetic field. The term $\frac{\partial \vec{D}}{\partial t}$ which arises from the variation of electric displacement with time is known as displacement current density and its introduction in $\vec{\nabla} \times \vec{B}$ equation was one of the major contributions of Maxwell.

Thus we see that the interrelation between electric and magnetic field generates a single field known as electromagnetic field which gives rise to propagation of electromagnetic waves.

5.9.4 Maxwell's Equations in Free Space

In free space, the following physical conditions are satisfied

$$\rho = 0, J = \sigma E = 0 \text{ as conductivity } \sigma = 0$$

Under these conditions, Maxwell's equations take the following form:

(i) $\vec{\nabla} \cdot \vec{E} = 0$ ...(5.43)

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ ...(5.44)

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ...(5.45)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$ ...(5.46)

5.10 WAVE EQUATIONS IN FREE SPACE

The time-varying electric and magnetic fields give rise to the phenomenon of electromagnetic wave propagation. Here we deduce the relevant wave equation.

In free space, Maxwell's equations are:

(i) $\vec{\nabla} \cdot \vec{E} = 0$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$

$= \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ $(\because \vec{D} = \varepsilon_0 \vec{E})$

- *For Electric Field*

Take curl on both sides of Eq. (iii)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

or, $\vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$ ...(5.47)

But $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

So, from Eq. (5.47)

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or, $$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad ...(5.48)$$

This is the three-dimensional wave equation for the vector field $\vec{E}$ in free space.

- *For Magnetic Field*

Taking curl on both sides of Eq. (iv)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t}\right)$$

or, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \qquad ...(5.49)$$

But $$\vec{\nabla} \cdot \vec{B} = 0 \text{ and } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

So, from Eq. (5.49)

$$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

or, $$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad ...(5.50)$$

This is the three-dimensional wave equation for the vector field $\vec{B}$ in free space.

Thus the fields satisfy the same formal partial differential equations for waves

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \qquad ...(5.51)$$

The three-dimensional wave function ψ depends on x, y, z, t and c is the velocity of the wave.

Thus, we conclude that the field vectors $\vec{E}$ and $\vec{B}$ are propagated in free space as waves whose speed is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad [\text{here, } \mu_0 = 4\pi \times 10^{-7} \text{ weber/amp, } \varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{Nm}^2]$$

$$= 2.9978 \times 10^8 \text{m/s}$$

which is the velocity of light in free space. So, we may conclude that light waves are electromagnetic waves.

From Eqs (5.49) and (5.50), we see that the electric field and the magnetic field satisfy the same wave equation, so they oscillate exactly in the same phase.

5.11 TRANSVERSE NATURE OF ELECTROMAGNETIC WAVE

The electromagnetic wave equations in free space are:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \text{ and } \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

We assume that the plane wave fields are of the form,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad ...(5.52)$$

and

$$\vec{B}(\vec{r}, t) = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad ...(5.53)$$

where $\vec{E}_0$ and $\vec{B}_0$ are vector constant in time and $\vec{k}$ is the propagation vector.

From Maxwell's first equation in free space

$$\vec{\nabla} \cdot \vec{E} = 0$$

or, $$\vec{\nabla} \cdot \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0$$

which gives the relation

$$\vec{k} \cdot \vec{E} = 0 \quad ...(5.54)$$

where $\vec{\nabla}$ is the operator, after operation on $\vec{E}$ its value (ik). Similarly, from Maxwell's second equation in free space

$$\vec{\nabla} . \vec{B} = 0$$

or, $$\vec{\nabla} \cdot \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

which gives $$\vec{k} \cdot \vec{B} = 0 \quad ...(5.55)$$

Equations (5.54) and (5.55) show that both $\vec{E}$ and $\vec{B}$ are perpendicular to the propagation vector $\vec{k}$.

Again from Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

or, $$\vec{\nabla} \times \vec{E}_0 \, e^{i(\vec{k}.\vec{r} - \omega t)} = -\frac{\partial}{\partial t} \vec{B}_0 \, e^{i(\vec{k}.\vec{r} - \omega t)}$$

or, $$i \, \vec{k} \times \vec{E} = -(-i\omega) \, \vec{B}$$

Therefore, $$\vec{k} \times \vec{E} = \omega \vec{B} \quad ...(5.56)$$

So, $\vec{B}$ is perpendicular to both $\vec{k}$ and $\vec{E}$. Therefore, from Eqs (5.54), (5.55) and (5.56), $\vec{E}$, $\vec{B}$ and $\vec{k}$ are mutually perpendicular to each other.

Now, considering only the magnitude of E, B and k, we have

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \quad ...(5.57)$$

where c is the velocity of light.

Thus, we may conclude that (i) electromagnetic waves travel with the speed of light, and (ii) electromagnetic waves are transverse waves. The ratio of electric to magnetic field in electromagnetic waves equal to the speed of light.

From Fig. 5.5 (a, b) we see that both electric field and magnetic field are perpendicular to the direction of motion of the wave. Thus an electromagnetic wave is a transverse wave.

Figure 5.5 (b) is a graphical representation of the sinusoidal wave showing $\vec{E}$ and $\vec{B}$ vectors. At any fixed point, the electrical and magnetic field vectors vary sinusoidally with time. The energy flow in the +ve x direction (i.e., $\vec{E} \times \vec{B}$). Radio waves, light waves, x-rays, γ-rays are examples of electromagnetic waves.

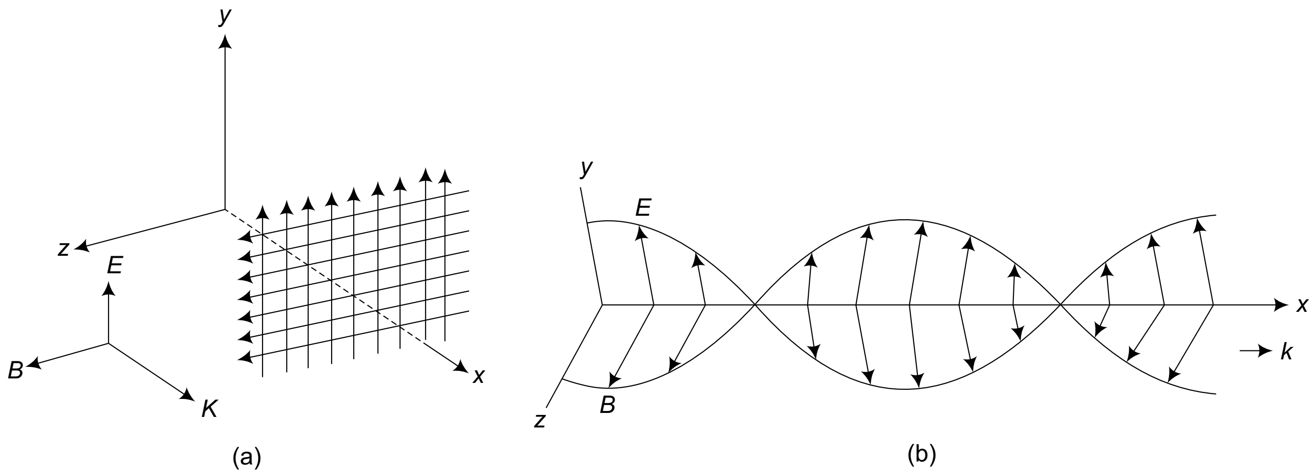


Fig. 5.5 (a) Transverse nature of electromagnetic waves. (b) Representation of the sinusoidal wave showing $\vec{E}$ and $\vec{B}$ vectors.

5.12 POTENTIALS OF ELECTROMAGNETIC FIELD

We know that magnetic field is solenoidal, i.e., $\vec{\nabla} \cdot \vec{B} = 0$, so $\vec{B}$, in terms of vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$.

Again from Faraday's laws of electromagnetic induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \quad [\text{Taking } \vec{B} = \vec{\nabla} \times \vec{A}]$$

Therefore, $$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \qquad ...(5.58)$$

Since the curl of the gradient of a scalar function is zero, so,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi \qquad ...(5.59)$$

or, $$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \qquad ...(5.60)$$

Here, $\vec{A}$ and φ are magnetic vector potential and scalar potential. The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be found if we determine $\vec{A}$ and φ.

5.13 ELECTROMAGNETIC WAVES IN A CHARGE-FREE CONDUCTING MEDIA AND SKIN DEPTH

Inside the conductor $\rho = 0$. Because there is no permanent charge inside the conductor, it can only be redistributed on the surface of the conductor. The propagation of EM waves through conducting, homogeneous, isotropic medium of permittivity ε, permeability μ and conductivity σ hold the relations

$$\left.\begin{aligned} \vec{D} &= \varepsilon\vec{E} \\ \vec{B} &= \mu\vec{H} \\ \vec{J} &= \sigma\vec{E} \end{aligned}\right\} \qquad ...(5.61)$$

Now Maxwell's equations are:

$$\left.\begin{array}{ll}\text{(i)} & \vec{\nabla}\cdot\vec{E}=0\\ \text{(ii)} & \vec{\nabla}\cdot\vec{H}=0\\ \text{(iii)} & \vec{\nabla}\times\vec{E}=-\mu\dfrac{\partial\vec{H}}{\partial t}\\ \text{(iv)} & \vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\dfrac{\partial\vec{E}}{\partial t}\end{array}\right\} \quad \text{...(5.62)}$$

Taking curl on both sides of Eq. 5.62 (iii)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}=-\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

or,
$$-\nabla^2\vec{E}=-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{[Using Eqs. (i) and (iv)]}$$

After arranging, we get

$$\nabla^2\vec{E}-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0 \quad \text{...(5.63)}$$

Similarly taking curl of Eq. (5.62) (iv), we get

$$\nabla^2\vec{H}-\mu\sigma\frac{\partial\vec{H}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=0 \quad \text{...(5.64)}$$

Both Eqs. (5.63) and (5.64) are known as Helmholtz equations for electric field and magnetic field.

Let us now find the plane wave solutions of Maxwell's equations for a conducting medium. We assume that the field vector $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ vary harmonically with time,

$$\left.\begin{array}{ll}\text{i.e.,} & E(\vec{r}, t)=E_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\ \text{and} & H(\vec{r}, t)=H_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\end{array}\right\} \quad \text{...(5.65)}$$

Substituting Eq. (5.65) in Eq. (5.63)

$$-k^2E(r, t)+\varepsilon\mu\omega^2\vec{E}(r, t)+i\sigma\mu\omega\vec{E}(r, t)=0$$

i.e.,
$$k^2=\varepsilon\mu\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right) \quad \text{...(5.66)}$$

The propagation vector is complex, and may be expressed as

$$k=\alpha+i\beta$$

$$=\left[\varepsilon\mu\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right)\right]^{1/2} \quad \text{...(5.67)}$$

From Eq. (5.67)

$$\left.\begin{aligned}\alpha^2-\beta^2&=\varepsilon\mu\omega^2\\ \text{and}\qquad 2\alpha\beta&=\sigma\mu\omega\end{aligned}\right\}\qquad ...(5.68)$$

Solving these equations,

$$\left.\begin{aligned}\alpha&=\omega\sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]}\\ \text{and}\qquad \beta&=\omega\sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]}\end{aligned}\right\}\qquad ...(5.69)$$

For good conductor, if the frequency is not too high, $\frac{\sigma}{\varepsilon\omega} >> 1$.

$\therefore$ propagation vector,

$$k=\sqrt{\mu\sigma\omega i}=\sqrt{\mu\sigma\omega}\,(\cos 45^\circ+i\sin 45^\circ)$$

$$=\sqrt{\mu\sigma\omega}\left(\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}\right)$$

$$=\alpha+i\beta$$

So

$$\alpha=\beta=\sqrt{\frac{\mu\sigma\omega}{2}}\qquad ...(5.70)$$

$$=\frac{1}{\delta}\quad\text{where}\quad \delta=\sqrt{\frac{2}{\mu\sigma\omega}}$$

Since $k=\alpha+i\beta$, Eq. (5.65) can be written as

$$\left.\begin{aligned}\vec{E}&=\vec{E}_0\,e^{-\beta r}\,e^{i(\alpha r-\omega t)}\\ \vec{H}&=\vec{H}_0\,e^{-\beta r}\,e^{i(\alpha r-\omega t)}\end{aligned}\right\}\qquad ...(5.71)$$

These equation indicate that a plane wave cannot propagate in a conducting medium without attenuation.

5.14 SKIN DEPTH OR DEPTH OF PENETRATION (δ)

From Eq. (5.71)

$$\vec{E}=\vec{E}_0\,e^{-r/\delta}\,e^{i\left(\frac{r}{\delta}-\omega t\right)}$$

At $r=\delta$ the amplitude decreases in magnitude to $\frac{1}{e}$ times its value at the surface which is called **skin depth** or **penetration depth** [Fig. 5.6].

The phenomenon that the alternating fields and hence currents are confined within a small region of a conducting medium inside the surface is known as the ***skin effect*** and the small distance from the surface of the conductor is known as ***skin depth***.

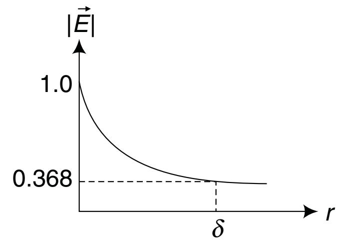


Fig. 5.6 Variation of intensity with distance in a change free conducting medium.

Significance of skin depth The attenuation of electromagnetic waves in a conduction medium is due to the conversion of the electromagnetic energy of the wave into Joule's heat because the electric field of the wave induces currents in a conducting medium which produce the heat. The energy in the form of electromagnetic waves carried by a current propagates in the space surrounding the conductors that partially penetrates the conductor surface to maintain the motion of the electrons. So, the current is maintained in the parts of the conductor which receive electromagnetic energy from the surrounding space. This energy can penetrate the conductor only by such small distance, called the skin depth (δ) and current may exist near the surface of the conductor only within the limits of this depth. The skin depth in copper for 1 mm microwaves is 10^{-4} m and for visible light 10^{-6} m. A poor conductor can be made a good conductor with a thin coating of copper or silver. A silver coating on a piece of glass may be an excellent conductor at microwave frequency. The performance of gold and gold-coated brass waveguides will be same if the coating thickness is equal to skin depth. This method is useful to reduce the cost of the material.

5.14.1 Electromagnetic Shielding

We may enclose a volume with a thin layer of good conductor to act as an electromagnetic shield. Depending on the application, the electromagnetic shield may be necessary to prevent waves from radiating out of the shielded volume or to prevent waves from penetrating into the shielded volume.

5.14.2 Phase Velocity

The phase velocity in the conducting medium is given by

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad \text{...(5.72)}$$

In the conductor, α and β are large. The wave attenuates greatly as it progresses and the phase shift per unit length is also large. The phase velocity of the wave is small. Phase velocity depends on frequency, so dispersion takes place in the conducting medium.

5.15 ELECTROMAGNETIC ENERGY FLOW AND POYNTING VECTOR

The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields. As electromagnetic waves propagate through the space from the source to the receiver, there exists a simple and direct relationship between the rate of energy transfer and the amplitude of electric and magnetic field strengths. The relation may be obtained from Maxwell's equations

$$\left.\begin{aligned} &\text{(i)}\quad \vec{\nabla}\cdot\vec{D} = 0 \\ &\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = 0 \\ &\text{(iii)}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{and} \\ &\text{(iv)}\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \end{aligned}\right\} \qquad \text{...(5.73)}$$

Taking dot product of Eqs. (5.73) (iii) and (iv) with $\vec{H}$ and $\vec{E}$ respectively, we have

$$\vec{H}\cdot\vec{\nabla}\times\vec{E} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \qquad ...(5.74)$$

and
$$\vec{E}\cdot\vec{\nabla}\times\vec{H} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \qquad ...(5.75)$$

Form Eqs. (5.74) and (5.75)

$$\vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{H} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{J} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \qquad ...(5.76)$$

$$= -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{E}\cdot\vec{J}$$

or,
$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{E}\cdot\vec{J} \qquad ...(5.77)$$

$$[\because \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}]$$

From relations $\vec{B} = \mu\vec{H}$ and $\vec{D} = \varepsilon\vec{E}$, we have

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\left[\vec{H}\cdot\frac{\partial}{\partial t}(\mu\vec{H}) + \vec{E}\cdot\frac{\partial}{\partial t}(\varepsilon\vec{E})\right] - \vec{E}\cdot\vec{J}$$

$$= -\left[\frac{1}{2}\mu\frac{\partial}{\partial t}(H^2) + \frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E^2)\right] - \vec{E}\cdot\vec{J}$$

$$= -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right)\right] - \vec{E}\cdot\vec{J}$$

or,
$$\vec{E}\cdot\vec{J} = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\right] - \vec{\nabla}\cdot(\vec{E}\times\vec{H}) \qquad ...(5.78)$$

Integrating over the volume V

$$\int_V (\vec{E}\cdot\vec{J})\,dV = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\right\} dV - \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV$$

or,
$$\int_V (\vec{E}\cdot\vec{J})\,dV = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\right\} dV - \oint_s (\vec{E}\times\vec{H})\cdot d\vec{S} \qquad ...(5.79)$$

$$\left[\because \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = \oint_s (\vec{E}\times\vec{H})\cdot\overrightarrow{dS}\right]$$

or,
$$\oint (\vec{E}\times\vec{H})\cdot\overrightarrow{dS} = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)\right\} dV - \int_V (\vec{E}\cdot\vec{J})\,dV$$

$$= -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}\left(\mu H^2 + \varepsilon\vec{E}^2\right)\right\} dV - \int_V \left(\vec{E}\cdot\vec{J}\right) dV \qquad ...(5.80)$$

Equation (5.80) is known as **Poynting theorem**. This is also known as the **work-energy theorem** of electrodynamics.

Interpretation of each term

(a) $\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\mu H^2 + \varepsilon E^2)\, dV\right.$: The terms $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum will be equal to the total energy stored in electromagnetic field. This expression represents the rate of decrease of energy stored within volume V due to electric and magnetic fields.

(b) $\int_V (\vec{E}\cdot\vec{J})\, dV$ or $\int \sigma E^2 dV$: This term represents the total ohmic power dissipated within the volume. This is a generalisation of Joule's law.

(c) $\oint_S (\vec{E}\times\vec{H})\cdot\vec{dS}$: This term represents the rate at which electromagnetic energy is leaving the volume V through the closed surface S.

The vector $(\vec{E}\times\vec{H})$ is known as the Poynting vector $\vec{P}$ or $\vec{P} = (\vec{E}\times\vec{H})$.

Poynting Vector: The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e., the rate of energy transport per unit area, is called the Poynting vector.

Poynting Theorem: It states that the vector product $\vec{P} = (\vec{E}\times\vec{H})$ at any point is a measure of the rate of energy flow per unit area at that point. The direction of energy flow is in the direction of the vector represented by the product $(\vec{E}\times\vec{H})$ and is perpendicular to both $\vec{E}$ and $\vec{H}$.

5.16 AVERAGE POWER CALCULATION USING POYNTING VECTOR

The Poynting vector $\vec{P} = \vec{E}\times\vec{H}$ gives the instantaneous rate of energy flow. Since the vector $\vec{E}$ and $\vec{H}$ vary harmonically with time, the average flow can be found by taking the average of $\vec{P} = \vec{E}\times\vec{H}$ over a complete period, i.e.,

$$\langle P\rangle = \langle R_e\, E \times R_e\, H\rangle \quad \text{...(5.81)}$$

where R_e stands for the real part.

Here E and H are complex quantities, so

$$\left.\begin{aligned} E &= (E_1 + iE_2)\, e^{-i\omega t} \\ \text{and} \quad H &= (H_1 + iH_2)\, e^{-i\omega t} \end{aligned}\right\} \quad \text{...(5.82)}$$

where E_1, E_2, H_1 and H_2 are real

Now $$R_eE = E_1 \cos\omega t + E_2 \sin\omega t$$

and $$R_eH = H_1 \cos\omega t + H_2 \sin\omega t$$

So $$R_eE \times R_eH = (E_1\times H_1)\cos^2\omega t + (E_1\times H_2)\cos\omega t \sin\omega t + (E_2\times H_1)\sin\omega t\cos\omega t + (E_2\times H_2)\sin^2\omega t$$

Now, over a complete period of oscillation

$$\langle\cos^2\omega t\rangle = \langle\sin^2\omega t\rangle = \frac{1}{2}$$

and $$\langle \sin \omega t \cos \omega t \rangle = 0$$

Therefore,
$$\langle R_eE \times R_eH \rangle = \frac{1}{2}\left[(E_1 \times H_1) + (E_2 \times H_2)\right] \qquad ...(5.83)$$

Let as now compute $R_e(E \times H^*)$

$$E = (E_1 + iE_2)\, e^{-i\omega t} = (E_1 + iE_2)\,(\cos \omega t - i \sin \omega t)$$

$$H = (H_1 + iH_2)\, e^{-i\omega t} = (H_1 + iH_2)\,(\cos \omega t - i \sin \omega t)$$

or, $$H^* = (H_1 - iH_2)\,(\cos \omega t + i \sin \omega t)$$

Therefore,
$$\begin{aligned} R_e\,(E \times H^*) &= (E_1 \times H_1) \cos^2 \omega t + (E_1 \times H_2) \cos \omega t \sin \omega t \\ &\quad + (E_2 \times H_2) \cos^2 \omega t - (E_2 \times H_1) \cos \omega t \sin \omega t \\ &\quad - (E_1 \times H_2) \cos \omega t \sin \omega t + (E_1 \times H_1) \sin^2 \omega t \\ &\quad + (E_2 \times H_1) \cos \omega t \sin \omega t + (E_2 \times H_2) \sin^2 \omega t \end{aligned}$$

$$R_e\,(E \times H^*) = (E_1 \times H_1) + (E_2 \times H_2) \qquad ...(5.84)$$

Hence, $$\langle R_eE \times R_eH \rangle = \frac{1}{2} R_e\,(E \times H^*) \qquad ...(5.85)$$

So, average Poynting vector

$$\langle P \rangle = \frac{1}{2} R_e\,(E \times H^*)$$

The average power $$\langle P \rangle = \frac{1}{2} R_e\,(E \times H^*) \qquad ...(5.86)$$

Worked Out Problems

Example 5.1 A rectangular loop of sides 8 cm and 2 cm having a resistance of 1.6 Ω is placed in a magnetic field of 0.3 Tesla directed normal to the loop. The magnetic field is gradually reduced at the end of 0.02 Ts^{-1}. Find out the induced current.

Sol. Induced emf $e = \dfrac{d\varphi}{dt} = \dfrac{d}{dt}(BA) = A\,\dfrac{dB}{dt}$

$$= (8 \times 2 \times 10^{-4}) \times 0.02$$

$$= 3.2 \times 10^{-5}\ \text{V}$$

Now induced current $I = \dfrac{e}{R} = \dfrac{3.2 \times 10^{-5}}{1.6} = 2.0 \times 10^{-5}$ A

Example 5.2 A metal bar slides without friction on two parallel conducting rails at distance l apart. A resistor R is connected across the rails and a uniform magnetic field B, pointing into this plane fills the entire region. If the bar moves to the right at a constant speed v then what is the current in the resistor?

[WBUT 2008]

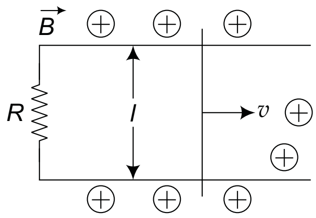


Fig. 5.1W

Sol. Induced emf $|e| = \frac{d\varphi}{dt} = \frac{d}{dt}(BA) = B\frac{dA}{dt}$

If the wire of length l moves a distance dx in time dt then $A = l\,dx$

or, $$|e| = B\frac{d}{dt}(l\,dx) = Bl\frac{dx}{dt} = Blv$$

and induced current $$i = \frac{e}{R} = \frac{Blv}{R}$$

Example 5.3 Flux φ (in Weber) in a closed circuit of resistance 10 Ω varies with time t (in seconds) according to the equation

$$\varphi = 6\,t^2 - 5t + 1$$

Find induced current at $t = 0.25$ second

Sol. The induced emf $e = -\frac{d\phi}{dt} = -\frac{d}{dt}(6t^2 - 5t + 1) = -12t + 5$

At $t = 0.25$ s, $e = -12 \times 0.25 + 5 = -3 + 5 = 2$V

Now induced current $I = \frac{e}{R} = \frac{2}{10} = 0.2$ A

Example 5.4 A metallic wheel with 6 metallic spokes, each 0.5 m long is rotating at a speed of 120 revolutions per minute in a plane perpendicular to a magnetic field of strength 0.2×10^{-4} Tesla. Find the magnitude of the induced emf between the axle and rim of the wheel.

Sol. If a conductor of length l is rotating perpendicularly to a magnetic field (B) about the fixed point with a constant angular velocity ω, then we can easily calculate the induced emf in the conductor.

Let dl be a small element of the conductor and its velocity be v. Then induced emf in the element is

$$de = Bv\,dl$$

Now, total emf induced in the conductor of length l is

$$e = \int de = \int_0^l Bv\,dl = \int_0^l Bl\omega\,dl = B\omega\frac{l^2}{2} \quad [v = \omega l]$$

$$= \frac{1}{2}B\omega l^2$$

In our problem $\omega = 2\pi \times \frac{120}{60} = 4\pi$ rad/s

and induced emf $e = \frac{1}{2}Bl^2\omega$

$$= \frac{1}{2} \times 0.2 \times 10^{-4} \times (0.5)^2 \times 4\pi = 3.14 \times 10^5 \text{ V.}$$

Example 5.5 An ac voltage source $V = V_0 \sin \omega t$ is connected across a parallel-plate capacitor C. Verify that the displacement current in the capacitor is the same as the conduction current in the wire.

Sol. For a parallel-plate capacitor, if A is the area and d is the separation between the plates then

$$C = \frac{\varepsilon_0 A}{d}$$

Again electric field $E = \dfrac{V}{d}$

So, $D = \varepsilon_0 E = \dfrac{\varepsilon_0 V}{d} = \dfrac{\varepsilon_0 V_0 \sin \omega t}{d}$

The displacement current

$$I_d = \int_s \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{ds} = \varepsilon_0 \frac{A}{d} V_0\, \omega \cos \omega t$$

$$= CV_0\, \omega \cos \omega t$$

Again conduction current $I = \dfrac{dQ}{dt} = \dfrac{d}{dt}(CV) = C\dfrac{dV}{dt}$

$$= C\frac{d}{dt}(V_0 \sin \omega t)$$

$$= CV_0\, \omega \cos \omega t$$

So, both currents are same.

Example 5.6 A parallel-plate capacitor with circular plates of 10 cm radius separated by 5 mm is being charged by an external source. The charging current is 0.2 A. Find (i) the rate of change of potential difference between the plates, and (ii) obtain the displacement current.

Sol. Here capacitance $C = \dfrac{\varepsilon_0 A}{d} = \dfrac{8.85 \times 10^{-12} \times \pi (0.1)^2}{5 \times 10^{-3}}$

$$= 5.56 \times 10^{-11}\ F$$

Given $I = \dfrac{dQ}{dt} = C\dfrac{dV}{dt} = 0.2\ A$

So, $\dfrac{dV}{dt} = \dfrac{0.2}{C} = \dfrac{0.2}{5.56 \times 10^{-11}} = 3.6 \times 10^{11}\ V/s.$

Again displacement current $I_d = \varepsilon_0 \dfrac{d\varphi}{dt} = \varepsilon_o A\dfrac{dE}{dt} = \dfrac{dQ}{dt} = I$

So displacement current is equal to 0.2 A.

Example 5.7 Show that $\dfrac{\sigma}{\varepsilon}\rho + \dfrac{\partial \rho}{\partial t} = 0$, where σ is the electric conductivity and ε is the electric permittivity of the medium.

Sol. From equation of continuity, $\vec{\nabla}\cdot\vec{J} + \dfrac{\partial \rho}{\partial t} = 0$ and from Ohm's law, $\vec{J} = \sigma\vec{E}$.

Now $\vec{\nabla}\cdot\vec{J} + \dfrac{\partial \rho}{\partial t} = 0$

or, $$\vec{\nabla}\cdot(\sigma\vec{E})+\frac{\partial\rho}{\partial t}=0$$

or, $$\sigma\vec{\nabla}\cdot\vec{E}+\frac{\partial\rho}{\partial t}=0$$

Again we know $$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon}\quad \text{so, } \frac{\sigma\rho}{\varepsilon}+\frac{\partial\rho}{\partial t}=0 \quad \text{(Proved)}$$

Example 5.8 Given $\vec{E}=\hat{i}E_0\cos\omega\left(\frac{z}{c}-t\right)+\hat{j}E_0\sin\omega\left(\frac{z}{c}-t\right)$, determine the magnetic field $\vec{B}$.

Sol. We know $$\vec{B}=\frac{\vec{k}\times\vec{E}}{\omega}=\frac{k}{\omega}(\hat{k}\times\vec{E})\quad [\because \vec{k}=k\,\hat{k}]$$

So, $$\vec{B}=\frac{k}{\omega}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ E_0\cos\omega\left(\frac{z}{c}-t\right) & E_0\sin\omega\left(\frac{z}{c}-t\right) & 0\end{vmatrix}$$

$$=\frac{k}{\omega}\left[-\hat{i}\,E_0\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,E_0\cos\omega\left(\frac{z}{c}-t\right)\right]$$

$$=-\hat{i}\,\frac{E_0}{c}\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,\frac{E_0}{c}\cos\omega\left(\frac{z}{c}-t\right)\quad\left[\because \frac{k}{\omega}=c\right]$$

Hence, magnetic field $$\vec{B}=-\hat{i}\,\frac{E_0}{c}\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,\frac{E_0}{c}\cos\omega\left(\frac{z}{c}-t\right)$$

Example 5.9 A wave has a wavelength of 4 mm and the electric field associated with it has an amplitude of 40 V/m. Determine the amplitude and frequency of oscillations of the magnetic field.

Sol. The relation between electric and magnetic field

$$B_0=\frac{E_0}{c}=\frac{40}{3\times10^8}=13.3\times10^{-8}\text{ Tesla}$$

Frequency of oscillation $$f=\frac{c}{\lambda}=\frac{3\times10^8}{4\times10^{-3}}=0.75\times10^{11}\text{ Hz}$$

Example 5.10 Calculate the skin depth for radio waves of 3 m wavelength (in free space) in copper, the electrical conductivity of which is 6×10^7 S/m. [Given permeability of free space $\mu_0=4\pi\times10^{-7}$ H/m]

Sol. Given $\lambda=3$ m, $\sigma=6\times10^7$ S/m, $\mu_0=4\pi\times10^{-7}$ H/m

Skin depth $$\delta=\sqrt{\frac{2}{\omega\sigma\mu}}$$

$$\omega=2\pi f=2\pi\frac{c}{\lambda}=\frac{2\pi\times3\times10^8}{3}=2\pi\times10^8\text{ rad/s}$$

Now, skin depth $$\delta=\sqrt{\frac{2}{\omega\mu\sigma}}=\sqrt{\frac{2}{2\pi\times10^8\times6\times10^7\times4\pi\times10^{-7}}}$$

$$=6.5\times10^{-6}\text{ m.}$$

Example 5.11 The earth is considered to be a good conductor when $\frac{\omega\varepsilon}{\sigma} << 1$. Calculate the highest frequencies for which the earth can be considered a good conductor if << 1 means less than 0.1.

[Assume $\sigma = 5 \times 10^{-3}$ mho/m, $\varepsilon = 10\ \varepsilon_0$]

Sol. Here $\frac{\omega\varepsilon}{\sigma} < 0.1$

or,
$$\omega < \frac{0.1\sigma}{\varepsilon} < \frac{0.1 \times 5 \times 10^{-3}}{8.854 \times 10^{-12} \times 10}$$

$\therefore\ \omega < 5.65 \times 10^6$

Highest frequency for which the earth can be considered as a good conductor is

$$f = \frac{\omega}{2\pi} = \frac{5.65 \times 10^6}{2\pi} = 0.9 \text{ MHz.}$$

Example 5.12 Find the skin depth δ at a frequency of 1.6 MHz in *Al*, where $\sigma = 38.2$ Ms/m and $\mu_r = 1$. Also find the propagation constant and wave velocity.

Sol.
$$\sigma = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 1.6 \times 10^6 \times 4\pi \times 10^{-7} \times 38.2 \times 10^6}}$$

$$= 6.43 \times 10^{-2} \text{ mm}$$

The propagation constant $k = \alpha + i\beta$

But
$$\alpha = \beta = \frac{1}{\delta} = 15.53 \times 10^3 \text{ m}^{-1}$$

$\therefore$
$$k = 15.53 \times 10^3 + i\ 15.53 \times 10^3$$

$$= 21.96 \times 10^3 < 45° \text{ m}^{-1}$$

Wave velocity
$$v = \frac{\omega}{\beta} = \omega\delta = 2\pi \times 1.6 \times 10^6 \times 6.43 \times 10^{-5} \text{ m/s}$$

$$= 647.2 \text{ m/s}$$

Example 5.13 Calculate the value of Poynting vector at the surface of the sun if the power radiated by the sun is 3.8×10^{26} W and its radius is 7×10^8 m.

Sol. Here, Power = 3.8×10^{26} W and $r = 7 \times 10^8$ m

If P is the average Poynting vector at the surface of the sun then

$$P = \frac{\text{Power}}{4\pi r^2} = \frac{3.8 \times 10^{26}}{4 \times 3.14 \times (7 \times 10^8)^2}$$

$$= 6.17 \times 10^7 \text{ W/m}^2$$

Example 5.14 Calculate the strength of the electric and magnetic field of radiation if the earth's surface receives sunlight of energy per unit time per unit area is 3 cal/min cm^2.

Sol. Here, solar energy which the earth receives is 3 cal/min cm^2

i.e., $I = 3 \text{ cal/(min cm}^2)$

or, $$I = \frac{3 \times 4.2 \times 10^4}{60} = 2100 \text{ J/m}^2\text{s}$$

$\therefore$ the poynting vector, $$\vec{P} = \vec{E} \times \vec{H}$$
$$= EH \sin 90°$$
$$= 2100 \text{ J/m}^2\text{s}$$

Again $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 377$$

$\because$ $$EH \times \frac{E}{H} = 2100 \times 377$$
$$E^2 = 7917 \times 10^2$$

or, $$E = 890 \text{ V/m}$$

and $$H^2 = \frac{2100}{377} = 5.57$$

or, $$H = 2.36 \text{ A/m}$$

Example 5.15 Find the magnetic field B and Poynting vector P of electromagnetic waves in free space if the components of the electric fields are $E_x = E_y = 0$ and $E_z = E_0 \cos kx \sin \omega t$.

Sol. From Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

But $$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} = \hat{i}\,\frac{\partial E_z}{\partial y} - \hat{j}\,\frac{\partial E_z}{\partial x}.$$

Now $E_z = E_0 \cos kx \sin \omega t$

So $$\frac{\partial E_z}{\partial y} = 0 \text{ and } \frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x}(E_0 \cos kx \sin \omega t) = -E_0 k \sin kx \sin \omega t$$

$\therefore$ $$\vec{\nabla} \times \vec{E} = +E_0\, k \sin kx \sin \omega t\, \hat{j} = -\frac{\partial \vec{B}}{\partial t}$$

$\therefore$ $$\vec{B} = +\frac{E_0 k}{\omega} \sin kx \cos \omega t\, \hat{j}$$

Now Poynting vector $$\vec{P} = \vec{E} \times \vec{H}$$
$$= E_0 \cos kx \sin \omega t\, \hat{k} \times \frac{E_0 k}{\mu_0 \omega} \sin kx \cos \omega t\, \hat{j}$$
$$= \frac{E_0^2 k}{\mu_0 \omega} \times \frac{1}{4} \sin 2kx \sin 2\omega t\, (\hat{k} \times \hat{j})$$

Example 5.16 Consider a monochromatic plane wave, where the electric field is given by

$$\vec{E} = E_0\, e^{i(kz-\omega t)}\, \hat{i}$$

where E_0 is an arbitrary constant vector.

(i) Show that the electric field vector lies in a direction perpendicular to the propagation.

(ii) Determine the corresponding magnetic field.

Sol. From Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

or
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\mu_0 \left[\hat{i}\,\frac{\partial H_x}{\partial t} + \hat{j}\,\frac{\partial H_y}{\partial t} + \hat{k}\,\frac{\partial H_z}{\partial t}\right]$$

Comparing both sides

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z} \quad \text{and} \quad H_x = 0, H_z = 0$$

$\therefore$
$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial}{\partial z}\left[E_0\, e^{i(kz-\omega t)}\right]$$

$$= -\frac{ikE_0}{\mu_0}\, e^{i(kz-\omega t)}$$

$\therefore$
$$H_y = -i\,\frac{k}{\mu_0}\, E_0 \int e^{i(kz-\omega t)}\, dt = \frac{E_0 k}{\mu_0 \omega}\, e^{i(kz-\omega t)}$$

Here, the electric field propagates in the x direction and the magnetic field propagates in the y direction, whereas the wave propagates in the z direction. So, we can say that the electric field vector lies in a direction perpendicular to the propagation.

(ii) The corresponding magnetic field

$$\vec{H} = \frac{E_0 k}{\mu_0 \omega}\, e^{i(kz-\omega t)}\, \hat{j}$$

Example 5.17 Show that for frequency $\leq 10^9$ Hz, a sample of silicon will act like a good conductor. For silicon, one may assume $\dfrac{\varepsilon}{\varepsilon_0} = 12$ and $\sigma = 2$ mho/cm. Also calculate the penetration depth for this sample at frequency 10^6 Hz.

Sol. A material will be good conductor if $\dfrac{\sigma}{\omega\varepsilon} >> 1$

Here $\sigma = 2$ mhos/cm = 200 mhos/m

$\omega = 2\pi f = 2\pi \times 10^9$

$\varepsilon = 12\, \varepsilon_0$

Now $$\frac{\sigma}{\omega\varepsilon} = \frac{200 \times 2}{2 \times \pi \times 10^9 \times 12\,\varepsilon_0} = \frac{400 \times 9 \times 10^9}{12 \times 10^9} = 300$$

So $\frac{\sigma}{\omega\varepsilon} >> 1$; a sample of silicon will act like a conductor at frequency $\leq 10^9$ Hz.

The penetration depth for good conductor

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$= \sqrt{\frac{2}{2\pi \times 10^6 \times 4\pi \times 10^{-7} \times 200}} = 3.6 \times 10^{-2} \text{ m}$$

$$= 3.6 \text{ cm}$$

Example 5.18 Calculate the skin depth for a frequency 10^{10} Hz for silver.

Given $\sigma = 2 \times 10^7$ Sm^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}

Sol. Here $\omega = 2\pi f = 2\pi \times 10^{10}$, $\sigma = 2 \times 10^7$ Sm^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}

Now skin depth $$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

$$= \left(\frac{2}{2\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 2 \times 10^7}\right)^{1/2}$$

$$= 1.12 \times 10^{-6} \text{ m}$$

Example 5.19 Discuss the behavior of copper to electromagnetic waves of frequency 0.5×10^{16} Hz and 7.0×10^{20} Hz. Given the conductivity of copper $\sigma = 5.8 \times 10^7$ mho m^{-1} and permittivity $\varepsilon = 9 \times 10^{-12}$ C^2/Nm^2

Sol. Here $$\omega = 2\pi \times 0.5 \times 10^{16}$$

$$\frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 0.5 \times 10^{16} \times 9 \times 10^{-12}} = 205$$

Since $\frac{\sigma}{\omega\varepsilon} > 100$, the conduction current dominates. Hence for frequency 0.5×10^{16} Hz copper is a conductor.

Now for frequency $f = 7 \times 10^{20}$ Hz, $\omega = 2\pi \times 7 \times 10^{20}$

$$\therefore \quad \frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 7 \times 10^{20} \times 9 \times 10^{-12}} = 1.47 \times 10^{-3}$$

Since $\frac{\sigma}{\omega\varepsilon} < 100$, the displacement current dominates. Hence for frequency 7×10^{20} Hz, copper is a dielectric.

Example 5.20 Calculate the value of Poynting vector for a 60 W lamp at a distance of 0.5 m from it.

Sol. Total average power emitted by the lamp = 60 W

The light emitted by the lamp will spread out in the form of a sphere, the radius of which is equal to the distance from it.

∴ radius of the sphere $R = 0.5$ m

Let P be the average Poynting vector over the surface of the sphere, then

$$P = \frac{\text{Power}}{4\pi R} = \frac{60}{4\pi(0.5)^2} = 19.1 \text{ W/m}^2.$$

Review Exercises

Part 1: Multiple Choice Questions

1. The magnetic flux linked with a coil at any instant 't' is given by $\varphi_t = 5t^3 - 100\,t + 200$, the emf induced in the coil at $t = 2$ seconds is
   (a) 200 V (b) 40 V (c) 20 V (d) –20 V
2. A cylindrical conducting rod is kept with its axis along the positive z axis, where a uniform magnetic field exists parallel to the z axis. The current induced in the cylinder is
   (a) clockwise as seen from the + z axis (b) zero
   (c) anticlockwise as seen from the – z axis (d) None of these
3. In an electromagnetic wave in a free space, the electric and magnetic fields are [WBUT 2008]
   (a) parallel to each other (b) perpendicular to each other
   (c) inclined at an acute angle (d) inclined at an obtuse angle
4. If $\vec{E}$ and $\vec{B}$ are the electric field and the magnetic field of electromagnetic waves traveling in vacuum with propagation vector then [WBUT 2006]
   (a) $\vec{k} \cdot \vec{E} = 0$ (b) $\vec{k} \times \vec{E} = 0$ (c) $\vec{B} \times \vec{E} = 0$ (d) $\vec{k} \times \vec{E} = -\vec{B}$
5. The velocity of a plane electromagnetic wave is given by
   (a) $c = \dfrac{1}{\sqrt{\mu_0\,\varepsilon_0}}$ (b) $c = \dfrac{1}{\mu_0\,\varepsilon_0}$ (c) $c = \mu_0\,\varepsilon_0$ (d) $\dfrac{\varepsilon_0}{\mu_0}$
6. $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ represent
   (a) Ampere's law (b) Laplace's equation
   (c) Gauss' law in electrostatics (d) Faraday's law of electromagnetic induction
7. The differential form of Faraday's law of electromagnetic induction is
   (a) $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ (b) $\vec{\nabla} \times \vec{E} = \dfrac{\partial \vec{B}}{\partial t}$
   (c) $\vec{\nabla} \times \vec{B} = \dfrac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} \times \vec{B} = -\dfrac{\partial \vec{E}}{\partial t}$

8. Maxwell's electromagnetic wave equations in terms of an electric field vector $\vec{E}$ in free space is [WBUT 2005]

   (a) $\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ (b) $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

   (c) $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} E = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

9. Displacement current arises due to [WBUT 2004, 2007]
   (a) positive charge only (b) negative charge only
   (c) time-varying electric field (d) None of these

10. In electromagnetic induction
    (a) mechanical energy is converted into magnetic energy
    (b) mechanical energy is converted into electrical energy
    (c) magnetic energy is converted into mechanical energy
    (d) magnetic energy is converted into electrical energy

11. Waves originating from a point source and traveling in an isotropic medium are described as [WBUT 2007]

    (a) $\varphi = \varphi_0 \exp i\,(kr - \omega t)$ (b) $\varphi = \varphi_0 \exp i\,(kr - \omega t)/r$

    (c) $\varphi = \varphi_0 \exp i\,(kr - \omega t)/r^2$ (d) $\varphi = \varphi_0 \exp i\,(kr + \omega t)/r$

12. Electromagnetic wave is propagated through a region of vacuum, which does not contain any charge or current. If the electric vector is given by $\vec{E} = \vec{E_o} \exp i\,(kx - \omega t)\,\hat{j}$ then the magnetic vector is [WBUT 2007]

    (a) in the x direction (b) in the y direction
    (c) in the z direction (d) rotating uniformly in the xy plane

13. Steady current produces
    (a) magnetostatic field (b) electrostatic field
    (c) time varying electric field (d) time-varying magnetic field

14. A conducting rod is moved with a constant velocity v in a magnetic field. A potential difference appears across the two ends
    (a) if $\vec{v} \parallel \vec{l}$ (b) if $\vec{v} \parallel \vec{B}$ (c) if $\vec{l} \parallel \vec{B}$ (d) None of these

15. A bar magnet is released from rest along the axis of a very long vertical copper tube. After some time, the magnet
    (a) will stop in the tube (b) will move with almost constant speed
    (c) will move with an acceleration g (d) will oscillate

16. The dimension of $\mu_0 \varepsilon_0$ is

    (a) $L^{-2} T^{-2}$ (b) $L^{-2} T^{2}$ (c) LT^{-1} (d) $L^{-1} T^{-1}$

17. The modified Ampere's circuital law is

    (a) $\oint \vec{B} \cdot \vec{dl} = \mu_0 (I + I_d)$ (b) $\oint \vec{B} \cdot \vec{dl} = \mu_0 I$

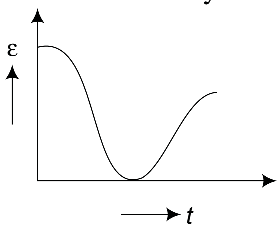
    (c) $\oint \vec{B} \cdot \vec{dl} = \varepsilon_0 I$ (d) $\oint \vec{B} \cdot \vec{dl} = \varepsilon_0 (I + I_d)$

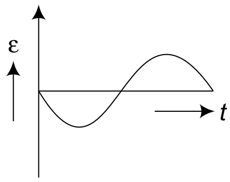
18. The electromagnetic wave is called transverse wave because [WBUT 2008]
    (a) the electric field and magnetic field are perpendicular to each other
    (b) the electric field is perpendicular to the direction of propagation
    (c) the magnetic field is perpendicular to the direction of propagation
    (d) both the electric field and magnetic field are perpendicular to the direction of propagation

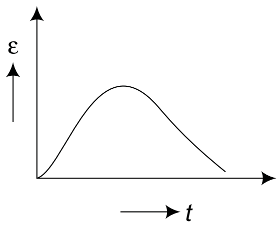
19. The solution of a plane electromagnetic wave $\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{B}}{\partial t^2}$ is
    (a) $\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{r})}$ (b) $\vec{B} = \vec{B}_0 e^{i(\omega t + \vec{k}\cdot\vec{r})}$
    (c) $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ (d) $B = \vec{B}_0 e^{-i(\omega t + \vec{k}\cdot\vec{r})}$

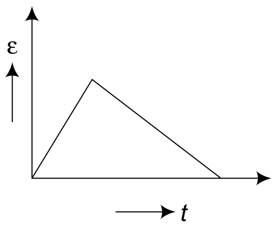
20. When a magnet is being moved towards a coil, the induced emf does not depend upon
    (a) the number of turns of the coil (b) the motion of the magnet
    (c) the magnetic moment of the magnet (d) the resistance of the coil

21. The variation of induced emf (ε) with time (t) in a coil if a short bar magnet is moved along its axis with a constant velocity is best represented as

(a) 

(b) 

(c) 

(d) 

22. The SI unit of Poynting vector is
    (a) Wm (b) Wm^{-1} (c) Wm^2 (d) Wm^{-2}

23. The Poynting vector is given by the expression
    (a) $\vec{E} \times \vec{H}$ (b) $\vec{H} \times \vec{E}$ (c) $\vec{E} \cdot \vec{H}$ (d) None of these

24. Skin depth for a conductor in reference to electromagnetic wave varies
    (a) inversely as frequency (b) directly as frequency
    (c) inversely as square root of frequency (d) directly as square of frequency

25. The ratio of the phase velocity and velocity of light is
    (a) one (b) less than one (c) greater than one (d) zero

26. The value of skin depth (δ) in a conducting medium is
    (a) $\delta = \sqrt{\dfrac{2\sigma}{\mu\omega}}$ (b) $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$ (c) $\delta = \sqrt{\dfrac{1}{\mu\omega\sigma}}$ (d) $\delta = \sqrt{\dfrac{2\mu}{\omega\sigma}}$

27. Skin depth is proportional to

(a) ω (b) μ (c) $\frac{1}{\sqrt{\sigma}}$ (d) $\sqrt{\sigma}$

[**Ans.** 1 (b), 2 (b), 3 (b), 4 (a), 5 (a), 6 (c), 7 (a), 8 (b), 9 (c), 10 (d), 11 (a), 12 (a), 13 (a), 14 (d), 15 (b), 16 (b), 17(a), 18 (d), 19 (c), 20 (d), 21 (b), 22 (d), 23 (a), 24 (c), 25 (b), 26 (b), 27 (c)]

Short Questions with Answers

1 State Faraday's laws of electromagnetic induction.

Ans. (i) Whenever there is a change in the magnetic flux linked with a coil an emf is set up in it and stays as long as the magnetic flux linked with it is changing.

(ii) The magnitude of the induced emf is proportional to the rate of change of magnetic flux linked with the coil, i.e.,

$$\varepsilon \,\alpha\, \frac{d\varphi}{dt}$$

2 What is the difference between conduction current and the displacement current?

Ans.

Conduction Current	*Displacement Current*
(i) It does obey Ohm's law.	(i) It does not obey Ohm's law.
(ii) Conduction current is due to the actual flow of charge in a conductor.	(ii) Displacement current is due to the time-varying electric field in a dielectric.
(iii) Conduction current density $\vec{J} = \sigma \vec{E}$.	(iii) Displacement current density $\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

3. An electron moves along the line *AB* which lies in the same plane as a circular loop of conduction wire, as shown in Fig. 5.2W. Find out the direction of the induced current in the loop.

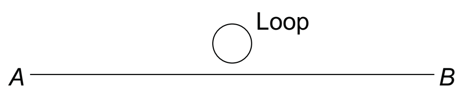


Fig. 5.2W

Ans. The electron moves from *A* to *B* so the direction of the current will be from *B* to *A*. The magnetic field generated in the loop due to the motion of the current will be directed into the plane of the paper. To oppose this, the current in the coil must be anticlockwise, in accordance with Lenz's law.

4. Why is electromagnetic wave called transverse wave?

Ans. An electromagnetic wave is called a transverse wave because by the direction of the propagation, the electric field and magnetic field are mutually perpendicular to each other.

5. What is displacement current?

Ans. See Section 5.6.

6. Why do birds fly off a high-tension wire when current is switched on?

Ans. When current begins to increase from zero to maximum value, a current is induced in the body of the bird. This produces a repulsive force and the bird flies off.

7. Two similar circular coaxial loops carry equal currents in the same direction. If the loops be brought nearer, what will happen to the currents in them?

Ans. When the loops are brought closer, there is an increase of magnetic flux. An induced emf is produced. According to Lenz's law, the induced emf has to oppose the change of magnetic flux. So, the current in each loop will decrease.

8. Two coils are being moved out of a magnetic field. One coil is moved rapidly and the other slowly. In which case is more work done and why?

Ans. More work will be done in the case of a rapidly moving coil. This is because the induced emf will be more in this coil as compared to slow moving coil.

9. Define skin depth.

Ans. Skin depth is defined as the distance in the conductor over which the electric field vector of the wave propagating in the medium decays to $1/e$ times its value at the surface.

10. What is Poynting vector?

Ans. The cross product of the electric vector $\vec{E}$ and the magnetic field vector $\vec{H}$ is known as a Poynting vector. Mathematically Poynting vector $\vec{P} = \vec{E} \times \vec{H}$

11. What is the effect of frequency on skin depth?

Ans. We know that skin depth

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

Thus skin depth is inversely proportional to the square root of the frequency. So, skin depth decreases with increase in frequency.

12. How is skin depth useful in practical situation?

Ans. A poor conductor can be made a good conductor with a thin coating of copper or silver. A silver coating on a piece of glass may be an excellent conductor at microwave frequency. The performance of gold and gold-coated brass wave guides will be the same if the coating thickness is equal to skin depth. So in a practical situation, skin depth method is useful to reduce the cost of the material.

Part 2: Descriptive Questions

1. (a) Write down Faraday's law of electromagnetic induction. [WBUT 2004]
   (b) Express it in differential form. [WBUT 2006]
2. (a) Write down Maxwell's equations in differential form and explain the physical significance of each equation. [WBUT 2002, 2004]
   (b) Show that the wave equation in free space for electric field $\vec{E}$ is given by $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$ [WBUT 2004]
3. (a) State Maxwell's equations. From these equations, derive the wave equations for an electromagnetic wave. What is the velocity of this wave?
   (b) Assuming a plane wave solution, establish the relation between the propagation vector ($\vec{k}$), electric field ($\vec{E}$) and magnetic field ($\vec{B}$). [WBUT 2008]
4. (a) Distinguish between the conduction current and displacement current.
   (b) Write down Faraday's law of electromagnetic induction.

5. (a) Write down Maxwell's field equations, explaining the term used. Show that in vacuum, both electric and magnetic vectors obey wave equation. Assuming a plane wave solution show that magnetic field is always orthogonal to the electric field.
   (b) Find the displacement current within a parallel-plate capacitor in series with a resistor which carries current I. Area of the capacitor plates are A and the dielectric is vacuum. [WBUT 2006]
6. (a) Write and explain differential and integral forms of Maxwell's equations.
   (b) Explain the significance of displacement current.
7. (a) Write down Maxwell's field equations.
   (b) From those equations identify Gauss' law, Ampere's law and Faraday's law.
   (c) How does velocity of light depend on the properties of vacuum? [WBUT 2005]
8. Use Faraday's law of electromagnetic induction and the fact that magnetic induction $\vec{B}$ can be derived from a vector potential $\vec{A}$. Show that the electric field can be expressed as

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \quad \text{where } \varphi \text{ is the scalar potential}$$ [WBUT 2007]

9. (a) What is displacement current? Distinguish between conduction and displacement current.
   (b) Show that $\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$ and $\vec{B} = \vec{B}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$ are possible solutions of Maxwell's electromagnetic wave equations in terms of electric and magnetic field.
10. Starting from Maxwell's equations in free space, show that the magnetic field $\vec{B}$ and the electric field $\vec{E}$ in an electromagnetic wave travel with the same speed.
11. Write down Maxwell's equations in integral form and explain the physical significance of each equation.
12. Show that $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right]$
13. Show that in free space, the electric field $\vec{E}$, magnetic field $\vec{B}$ and propagation vector $\vec{k}$ are perpendicular to each other.
14. The electric field associated with an electromagnetic wave is $\vec{E} = \hat{i}\, E_0 \cos(kz - \omega t) + \hat{j}\, E_0 \sin(kz - \omega t)$, where E_0 is a constant. Find the corresponding magnetic field $\vec{B}$.
15. (a) State Lenz's law. Explain it from the principle of conservation of energy.
    (b) A wire is rotated about one of its ends at right angles to a magnetic field. Deduce the expression for induced emf.
16. Define skin depth. Show that in case of a semi-infinite solid conductor, the skin depth δ is given by

$$\delta = \frac{1}{\sqrt{\omega\mu\sigma}}$$

    where symbols have their usual meanings.
17. What is Poynting vector? Show that Poynting vector measures the flow of energy per unit area per second in an electromagnetic wave.
18. State and prove Poynting theorem.
19. Show that average power $<P> = \frac{1}{2} R_e(E \times H^*)$

20. What is Poynting vector? Find the expression of Poynting vector. What is the physical interpretation of this vector?
21. A plane electromagnetic wave is incident normally on a metal of electrical conductivity σ. Show that the electromagnetic wave is damped inside the conductor and find the skin depth.

Part 3: Numerical Problems

1. A parallel-plate capacitor with plate area A and separation d between the plates is charged by a constant current I. Consider a plane surface of area $\frac{A}{4}$ parallel to the plates and drawn symmetrically between the plates. Calculate the displacement current through this area. $\left[\textbf{Ans. } I_D = \frac{I}{4}\right]$
2. A parallel-plate capacitor with circular plates of radius $a = 5.5$ cm is being charged at a uniform rate so the electric field between the plates changes at a constant rate $\frac{dE}{dt} = 1.5 \times 10^{12}$ V/ms. Find the displacement current for the capacitor. [**Ans.** $I_D = 0.13$A]
3. The magnetic field in a plane electromagnetic wave is given by $B_y = 2 \times 10^{-7} \sin (0.5 \times 10^3 x + 1.5 \times 10^{11} t)$ Tesla. (a) What is the wavelength of the wave? (b) Write an expression for the electric field. [**Ans.** $\lambda = 1.26$ cm, $E_0 = 60$ Vm^{-1}]
4. Capacitance of a parallel-plate capacitor is $2\mu F$. Calculate the rate at which the potential difference between the two plates must charge to get a displacement current of 0.4 A. [**Ans.** $\frac{dV}{dt} = 2 \times 10^5$ V/s]
5. A current of 5 A is passed through a solenoid of 50 cm length, 3.0 cm radius and having 200 turns. When the switch is open, the current becomes zero within 10^{-3} s. Calculate the emf induced across the switch. [**Ans.** $e = 1.42$ volt] $\left[\textbf{Hints: } e = N\frac{d\varphi}{dt}, \varphi = BA = \mu_0 ni\, A, n = \frac{N}{l}\right]$
6. A rectangular loop of 8 cm side and 2 cm having a resistance of 1.6 Ω is placed in a magnetic field and gradually reduced at the rate of 0.02 Tesla/s. Find out the induced current. [**Ans.** 2×10^{-5} A]
7. A 50 cm long bar PQ is moved with a speed of 4 ms^{-1} in a magnetic field $B = 0.01$ Tesla as shown in Fig. 5.3W. Find out the induced emf. [**Ans.** 0.02 V]

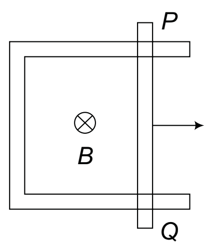


Fig. 5.3W

8. Calculate the skin depth for a frequency 10^{10} Hz for silver. Given $\sigma = 2 \times 10^7$ Sm^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}. [**Ans.** $\delta = 1.12 \times 10^{-6}$ m]
9. Find the skin depth δ at a frequency of 1.6 MHz in aluminium, where $\sigma = 38.2$ mS/m and $\mu_r = 1$. Also find wave velocity. [**Ans.** $\delta = 6.4 \times 10^{-5}$ m, $v = 6.47 \times 10^2$ m/s]
10. A laser beam has a diameter of 2 mm, what is the amplitude of the electric and magnetic field in the beam in vacuum if the power of the laser is 1.5 mW? [Given $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m]. [**Ans.** $E_0 = 600$ V/m $H_0 = 1.59$ amp/m]
11. Find the depth of penetration of a megacycle wave into copper which has conductivity of $\sigma = 5.8 \times 10^7$ mho/m and a permeability equal to that of free space. [**Ans.** $\delta = 6.6 \times 10^{-5}$ m]