Engineering Mathematics II
(Probability and Statistics)

Mathematical Stat.	Applied stat
→ Probability	→ Sampling
→ Measure	
→ St. Distribution	

Probability
↓
Chance

Uncertain	Certain
→ Probability	• $(A+B)^2 = A^2 + 2AB + B^2$
• tossing of coin	• $s = ut + \frac{1}{2}at^2$
• throwing a die	

Apriori/Probability

let us suppose a random experiment 'E' in its sample space 'S' contain a finite number

• $n(S)$ → Sample points

and event 'A' which contains

• $n(A)$

$$\boxed{P(A) = \frac{n(A)}{n(S)}}$$

Statistical Regularity

$$f_n(A) = \frac{n(A)}{A}$$

event A is found to occur on event A

Statistical Regularity Probability/Apriori

$$P(A) = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{n(A)}{A}$$

e.g.- a company manufactures blades

Let A = blade is defective

n = no. of blades taken

$n(A)$ = no. of defective blades

$n(A)$ =	3	2	5	8	9	12	12	15	...
n =	10	15	20	25	30	35	40	45	...

$$P(A) = \frac{n(A)}{n} = \frac{1}{3}$$

Theorem

$$\boxed{0 \leq P(A) \leq 1}$$

$P(A) = 0 \rightarrow$ impossible case

$P(A) = 1 \rightarrow$ certain case

$$\boxed{0 < P(A) < 1}$$

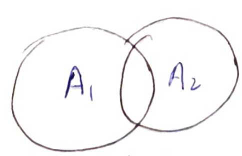
$$\boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2)}$$

- A_1 and A_2 are disjoint (mutually exclusive)

$$\boxed{P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)}$$

Total Probability Theorem

$$\boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)}$$



- not disjoint

Conditional Probability

- Let, A and B are two events with random experiment E
- Make the hypothesis that the event A has occured $n(A)$ times and B occurs simultaneously with A, $n(A \cap B)$ times in n separations
- Then ration $\boxed{\frac{n(A \cap B)}{n(A)}}$ is called "conditional probability".

Conditional Frequency Ratio

$$f_n(B/A) = \frac{n(A\cap B)}{n(A)}$$

If $f_n(B/A)$ tends to finite as $n \to \infty$, then conditional frequency ratio is called conditional probability.

$$\cdot\ P(B/A) = \lim_{n\to\infty} f_n(B/A) = \lim_{n\to\infty} \frac{n(A\cap B)}{n(A)}$$

$$= \frac{P(A\cap B)}{P(A)}, \quad P(A) \neq 0$$

$$\cdot\ P(A/B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) \neq 0$$

e.g.- let one card be drawn from a full pack, A = spade, B = king find the probability of king supposing spade occured

$$\therefore \text{Probability } P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{1}{13}$$

Multiplication Rule of Probability

$$\boxed{P(A\cap B) = P(A).\,P\left(\frac{B}{A}\right) = P(B).\,P\left(\frac{A}{B}\right)}$$

Similarly, let 'n' events $A_1, A_2, \ldots, A_n$,

$$\boxed{P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).\,P\left(\frac{A_2}{A_1}\right).\,P\left(\frac{A_3}{A_1\cap A_2}\right) \ldots P\left(\frac{A_n}{\bigcap_{i=1}^{n-1} A_i}\right)}$$

For independent events,

$$\boxed{P(A\cap B) = P(A).\,P(B)}$$

Bay's Theorem

If $A_1, A_2, \ldots, A_n$ be a given set of 'n' pairwise mutually exclusive and exhaustive events, then for any event 'A',

① $$P(A) = \sum_{i=1}^{n} P(A_i).\,P(A/A_i)$$

② $$P\left(\frac{A_i}{A}\right) = \frac{P(A_i).\,P(A/A_i)}{P(A)}$$

e.g. there are three urns

first urn contains → 3 red + 4 black balls

second " " → 2 red + 6 black "

third " → 5 red + 3 black "

one urn is chosen and a red ball is drawn from the urn.

let A_1 = Ist urn chosen

A_2 = IInd " "

A_3 = IIIrd " "

A → ball is red

$$\therefore P(A) = P(A_1) \cdot P\left(\frac{A}{A_1}\right) + P(A_2) \cdot P\left(\frac{A}{A_2}\right) + P(A_3) \cdot P\left(\frac{A}{A_3}\right)$$

$$= \frac{1}{3}\left(\frac{3}{7} + \frac{2}{8} + \frac{5}{8}\right)$$

$$= \frac{73}{168}$$

Q.) In a bolt factory, machine 'A', 'B', 'C' manufacture respectively 25%, 35% and 40% of the total. Their outputs are 5%, 4%, 2% respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the ~~product~~ probabilities that it was manufactured by machine (i) A (ii) B (iii) C?

Solⁿ: X_A → event that bolt is manufactured by A

X_B → " " " " " " B

X_C → " " " " " " C

X → " " bolt is defective

$P(X_A) = \frac{1}{4}$

$P(X_B) = \frac{7}{20}$

$P(X_C) = \frac{2}{5}$

$$P(X) = P(X_A) \cdot P\left(\frac{X}{X_A}\right) + P(X_B) \cdot P\left(\frac{X}{X_B}\right) + P(X_C) \cdot P\left(\frac{X}{X_C}\right)$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{20}\right) + \left(\frac{7}{20}\right)\left(\frac{1}{25}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{50}\right)$$

$$= \frac{1}{80} + \frac{7}{500} + \frac{1}{125}$$

$$= \frac{69}{2000}$$

(i) $$P\left(\frac{X_A}{X}\right) = \frac{P(X_A \cap X)}{P(X)}$$

$$= \frac{P(X_A) \cdot P\left(\frac{X}{X_A}\right)}{P(X)}$$

$$= \frac{1 \times 200\cancel{0}}{8\cancel{0} \times 69}$$

$$= \frac{25}{69}$$

(ii) $P\left(\frac{X_B}{X}\right) = \frac{28}{69}$

(iii) $P\left(\frac{X_C}{X}\right) = \frac{16}{69}$

Q.) The probability of a college student being a male is $\frac{1}{8}$ and that of being a female is $\frac{7}{8}$. The probability that a male student completes the course is $\frac{2}{3}$ and that of a female student is $\frac{1}{3}$. A student is selected at random and is found to have completed the course. What is the probability that the student is (i) male (ii) female?

Random Variables (R.V)

"STOCHASTIC / CHANCE Variables" "Spectrum"

e.g. - three electronic components are tested and result may be either defective or non-defective

$$S = \{NNN, NND, NDN, \ldots\ldots, DDD\}$$

$$R.V(D) = \{0, 1, 2, 3\}$$

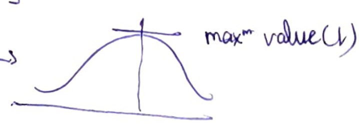
- A r.v. is called a "discrete" variable if the set of possible outcomes is countable.

- A r.v. is called a "continuous" variable if it can take values on a continuous scale.

PMF → Probability Mass Function → discrete
PDF → Probability Density Function → continuous

$F(X=x) = P(X \leqslant x)$, $-\infty < x < \infty$ → Cumulative distribution function

Distribution
- binomial } discrete
- poisson } discrete
- normal (continuous) →



e.g. - a consignment of 8 similar micro-computer to a retail outlet contain 3 defectives.
If a firm makes a random purchase of 2 of these computer.
Find a prob. distribution for no. of ~~objects~~ defectives

$X \rightarrow$ r.v. for two defectives

So, here x takes the value 0, 1, 2

$$P(X=0) = F(X=0) = \frac{{}^3C_0 \cdot {}^5C_2}{{}^8C_2} = \frac{10}{28}$$

$$F(X=1) = \frac{{}^3C_1 \cdot {}^5C_1}{{}^8C_2} = \frac{15}{28}$$

$$F(X=2) = \frac{{}^3C_2 \cdot {}^5C_0}{{}^8C_2} = \frac{2}{28}$$

x	0	1	2
$P(X=x)$	0.36	0.63	0.11

$$\sum F(X=x) = 1$$
$$\sum F(X=x) \geq 0$$

$$0 < F(X=x) < 1$$

Continuous Distribution Function

A function $F(X=x)$ is a probability density function (PDF) for continuous r.v defined:

(i) $F(X=x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} F(X=x)\, dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b F(X=x)\, dx$

e.g.- Suppose that the error in the rxn temp. (°C) for a controlled lab experiment is continuous r.v. 'X' having probability density function

$$F(X=x) = \begin{cases} x^2/3 , & -1 < x < 2 \\ 0 , & \text{otherwise} \end{cases}$$

(a) Verify that the total probability is unity

(b) Find $P(0 \leq x \leq 1)$.

(a) $\int_{-\infty}^{\infty} F(X=x)\, dx = \int_{-1}^{2} \frac{x^2}{3}\, dx = \left[\frac{x^3}{9}\right]_{-1}^{2} = \frac{8}{9} - \frac{(-1)}{9} = \frac{9}{9} = 1$

(b) $P(0 \leq x \leq 1) = \int_0^1 \frac{x^2}{3}\, dx = \left[\frac{x^3}{9}\right]_0^1 = \frac{1}{9}$

Expectation or Mean of R.V.

(i) Let, X be a discrete R.V. whose distribution is given:

$$\begin{array}{llllll} X & : & x_0 & x_1 & x_2 & \ldots \\ P(X=x) & : & f_0 & f_1 & f_2 & \ldots \end{array}$$

Mean $= E(X) = x_0 f_0 + x_1 f_1 + x_2 f_2 + \ldots$
(μ, m)

$$\boxed{E(x) = \sum_{i=0}^{n} x_i f_i}$$

(ii) Let, the PDF of a continuous R.V

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$

e.g. - $f(x) = \begin{cases} 1/2 & , -1 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$

$$E(X) = \int_{-1}^{1} \frac{x}{2}\, dx$$

$$E(2x^3) = \int_{-\infty}^{\infty} 2x^3 \cdot f(x)$$

$$= \frac{x^4}{4}\Big|_{-1}^{1}$$

$$= 0$$

Properties of Expectation

(i) $E(a) = a$, where $a \to$ constant

(ii) $E(ax) = a \cdot E(x)$

(iii) $E(x \pm y) = E(x) \pm E(y)$

(iv) $E(xy) = E(x) \cdot E(y)$

if independent

Q.) A number is chosen at random from the set $\{1, 2, \ldots, 100\}$ and another number is chosen at random from the set $\{1, 2, \ldots, 50\}$. Find the expected value of the product.

Let, $x \rightarrow$ first no.

and, $y \rightarrow$ second no.

for

$X: 1 \quad 2 \quad 3 \quad \ldots \quad 100$

$f_i: \frac{1}{100} \quad \frac{1}{100} \quad \frac{1}{100} \quad \ldots \quad \frac{1}{100}$

$y: 1 \quad 2 \quad \ldots \quad 50$

$g_i: \frac{1}{50} \quad \frac{1}{50} \quad \ldots \quad \frac{1}{50}$

$$E(X) = \sum_{i=1}^{100} x_i f_i$$

$$= \frac{1}{100}(1+2+3+\ldots+100)$$

$$= \frac{\cancel{100} \times 101}{\cancel{100} \times 2}$$

$$= \frac{101}{2}$$

$$= 50.5$$

$$E(y) = \frac{1}{50}(1+2+\ldots+50)$$

$$= \frac{\cancel{50} \times 51}{2 \times \cancel{50}}$$

$$= \frac{51}{2}$$

$$= 25.5$$

$$\therefore E(XY) = E(X).E(Y)$$

$$= \left(\frac{101}{2}\right)\left(\frac{51}{2}\right)$$

Variance and Standard Deviation

The variance of a r.v. denoted by $Var(X)$ and,

$$Var(X) = E[(x-m)^2]$$

$$\boxed{Var(X) = E[(x-E(x))^2]}$$

$\mu = m = E(x) = \text{mean}$

$$S.D = \sqrt{Var(X)}$$

$$\boxed{\sigma_x = \sqrt{[E(x-E(x))^2]}}$$

$$Var(X) = E\{X-m\}^2$$

$$= E(x^2+m^2-2xm)$$

$$= E(x^2) + E(m^2) - E(2xm)$$

$$= E(x^2) - 2mE(x) + m^2$$

$$= E(x^2) - 2m^2 + m^2$$

$$= E(x^2) - m^2$$

$$\boxed{Var(X) = E(x^2) - m^2}$$

$$\boxed{Var(X) = E(x^2) - [E(x)]^2}$$

$Var(\text{constant}) = 0$

(I) $Var(X) = E(x^2) - [E(x)]^2$

(II) $Var(ax+b) = a^2\, Var(X)$

(III) $Var(a) = 0$

(IV)

e.g. - $f(x) = \begin{cases} \frac{1}{2}x & , 0 \le x \le 2 \\ 0 & , \text{elsewhere} \end{cases}$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_0^2 x \cdot \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^2$$

$$= \frac{1}{4}(4)$$

$$= 1$$

$$E(x^2) = \int_0^2 x^2 \frac{1}{2}\,dx$$

$$= \left[\frac{x^3}{2 \cdot 3}\right]_0^2$$

$$= \frac{2^3}{2 \cdot 3}$$

$$= 4/3$$

$$Var(X) = E(x^2) - [E(x)]^2$$

$$= 4/3 - 1$$

$$= 1/3$$

$$S.D(x) = \frac{1}{\sqrt{3}}$$

Q.) A random variable x has following probability mass function

X :	0	1	2	3	4	5	6	7
$P(X=x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Determine the constant 'k'

(ii) Determine the min^m value of x, so that $P(X \le x) > 1/2$

(iii) Obtain the distribution $F(X)$.

(iv) Evaluate $P(X<6)$, $P(X \ge 6)$, $P(0 < X \le 6)$, $P(x < 3 / x \le 6)$

(i) $\sum_{x=0}^{n} P(X=x) = 1$

$$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow 10k^2 + 10k - k - 1 = 0$$

$$\Rightarrow 10k(k+1) - 1(k+1) = 0$$

$$\Rightarrow k = 1/10, -1$$

$$\boxed{k = \frac{1}{10}}$$

(iv) $P(X<6) = 1 - P(X \geq 6)$

$= 1 - \{P(X=6) + P(X=7)\}$

$= 1 - \left\{\frac{2}{100} + \frac{17}{100}\right\}$

$= \frac{81}{100}$

$P(X \geq 6) = \frac{19}{100}$

$P(3 < X \leq 6) = P(X=4) + P(X=5) + P(X=6)$

$= \frac{30}{100} + \frac{1}{100} + \frac{2}{100}$

$= \frac{33}{100}$

$P(X \gtrless 3 \mid X \leq 6) = \dfrac{P\{(X \gtrless 3) \cap (X \leq 6)\}}{P(X \leq 6)}$

$= \dfrac{33/100}{83/100}$

$= \frac{33}{83}$

(11) $P(X \leq k) > \frac{1}{2}$ (iii)

$P(X \leq 1) = P(X=0) + P(X=1)$

$= 0 + \frac{1}{10}$

$= \frac{1}{10}$

$P(X \leq 2) = 0 + \frac{1}{10} + \frac{2}{10}$

$= \frac{3}{10}$

$P(X \leq 3) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10}$

$= \frac{5}{10}$

$= \frac{1}{2}$

$P(X \leq 4) = \frac{1}{2} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$

$\therefore \boxed{k = 4}$

(III) Distribution function:

$$F(x) = \begin{cases} 0 & , -\infty \le x < 1 \\ \frac{1}{10} & , 1 \le x < 2 \\ \frac{3}{10} & , 2 \le x < 3 \\ \frac{1}{2} & , 3 \le x < 4 \\ \frac{4}{5} & , 4 \le x < 5 \\ \frac{81}{100} & , 5 \le x < 6 \\ \frac{83}{100} & , 6 \le x < 7 \\ 1 & , 7 \le x < \infty \end{cases}$$

cumulative frequency

Q.) Find the mean variance of continuous variable having pdf

$$f(x) = \begin{cases} 1 - |1-x| & , 0 < x < 2 \\ 0 & , \text{elsewhere} \end{cases}$$

Also find $E(x^3)$.

$$E(x) = \int_0^2 [1 - |1-x|] dx$$

$$= \int_0^1 [1-(1-x)] dx + \int_1^2 [1-(-1+x)] dx$$

$$= \int_0^1 x \, dx + \int_1^2 (2-x) \, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + 2[x]_1^2 - \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + 2 - \frac{3}{2}$$

$$= 1$$

$$E(x^2) = \int_0^4 [1 - |1-x^2|] dx$$

$$= \int_0^1 [1-(1+x^2)] dx + \int_1^4 \{1 + (1-x^2)\} dx$$

$$= \int_0^1 x^2 dx + \int_1^4 (2-x^2) dx = \frac{1}{3} + 2.3 - \frac{1}{3} \times 63$$

$$= \frac{1}{3} + \frac{18}{3} - \frac{63}{3} = -\frac{44}{3}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{-44}{3} - 1$$

$$= -\frac{47}{3}$$

Theoretical Discrete and Continuous Distribution

(Special Types of Distribution)

↳ 7 types

(I) Binomial Distribution
(II) Poisson Distribution } discrete

(III) Normal Distribution → continuous

↳ sociology, market research, etc.

Binomial Distribution (Bernoulli's dist.)

A discrete r.v. is said to be binomial dist. with parameters P (0<P<1) and n (a positive integer)

$X : 0 \quad 1 \quad 2 \quad 3 \quad ---- \quad n$

$f_i : f_0 \quad f_1 \quad f_2 \quad f_3 \quad ---- \quad f_n$

$$P(X=i) = {}^{n}C_{r}\, \underset{(p)}{p^{i}}\, \underset{(q)}{(1-p)^{n-r}} \quad ; i = 0, 1, 2, --$$

$\boxed{B(p, n)}$

$X \sim B(p, n)$

parameters

Properties:

(I) The experiment consists of 'n' repeated times (trials)

(II) Each trial result's ~~is~~ ~~an~~ outcome classified as successful (p) or unsuccessful (q)

(III) The repeated trials are independent

Limitations:

(I) $n \to$ finite

(II) The value of p and q are around ½ (never very small)

$$P(X=x) = \begin{cases} {}^{n}C_{i}\, p^{i}\, q^{n-i}, & i \in \mathbb{Z}^{+} \cup \{0\} \\ 0 & , \text{otherwise} \end{cases}$$

Tossing of coin

$n=1$, H T

$P = p(H)+p(T) = p(H) \cup p(T) = P(p \cup q)$

$= (p+q)^1$

$n=2$, HH HT TH TT

$P(HH)+P(HT)+P(TH)+P(TT)$

$= P(H).P(H)+2.P(H).P(T)+P(T).P(T)$

$= p^2+2pq+q$

$= (p+q)^2$

$n=3$, HHH, HHT, —— , TTT

$(p+q)^3$

⋮

$n=n$, $(p+q)^n$

$= {}^nC_0 p^0 q^n + {}^nC_1 p^1 q^{n-1} + \cdots + {}^nC_i p^i q^{n-i} + \cdots + {}^nC_n p^n q^0$

Q.) The probability that a certain kind of component will survive a given shock test is $\frac{3}{4}$. Find the probability that exactly 2 of the next 4 components will survive.

Sol^n: $n=4$, $p=\frac{3}{4}$, $q=\frac{1}{4}$

$\therefore X \sim B(n,p)$

$P(x=2) = {}^4C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^2$

$= \frac{4\times3}{2} \times \frac{3\times3}{4\times4} \times \frac{1}{16}$

$= 0.21$

Mean and Variance of Binomial Distribution

$$\text{Mean} = E(X) = \sum_{x=0}^{n} x \cdot P(X=x)$$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \cdot \left(\frac{n}{\cancel{x}}\right) {}^{n-1}C_{x-1} \, p^x q^{n-x}$$

$$= np \sum_{x=0}^{n} {}^{n-1}C_{x-1} \cdot p^{x-1} q^{n-x}$$

$$= np \, (p+q)^{n-1}$$

$$= np \cdot (1)^{n-1}$$

$$= np$$

$$\therefore \boxed{\text{Mean} = np}$$

$$\text{Variance} = \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= npq$$

$$E(x^2) = \sum_{x=0}^{n} x^2 \cdot {}^{n}C_x \, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1) + x\} \, {}^{n}C_x \, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \cdot {}^{n}C_x \, p^x \cdot q^{n-x} + \sum_{x=0}^{n} x \cdot {}^{n}C_x \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \cdot \frac{n(n-1)}{x \cdot (x-1)} \cdot {}^{n-2}C_{x-2} \, p^x q^{n-x} + \sum_{x=0}^{n} x \left(\frac{n}{x}\right) {}^{n-1}C_{x-1} \, p^x q^{n-x}$$

$$= n(n-1) \sum_{x=0}^{n} {}^{n-2}C_{x-2} \, p^x q^{n-x} + \underbrace{n \sum_{x=0}^{n} {}^{n-1}C_{x-1} \, p^x q^{n-x}}_{E(x)}$$

$$= n(n-1)p^2 \sum_{x=0}^{n} {}^{n-2}C_{x-2} \, p^{x-2} \cdot q^{n-x} + np$$

$$= n(n-1)p^2 + np$$

$$\therefore \text{Var}(X) = n(n-1)p^2 + np - (np)^2 = npq$$

$$\boxed{\text{Var}(X) = npq}$$

e.g.- Let x be the no. of 1s obtained in 15 throws of a die. Find its mean and variance.

Sol^n: X is a binomial distribution

$p(\text{getting } 1) = \frac{1}{6}$

$q = 5/6$

$n = 15$

Mean $= np$

$= 15/6 = 2.5$

$Var(X) = npq$

$= 15\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$

$= \frac{75}{36}$

$= 2.09$

Q.) A defective die is thrown TEN times indepently. The prob. that an even will appear 5 times is twice the probability that an even no. appears 4 times. What is the prob. that odd face appears in each of ten times?

Sol^n: let success(p) = evenface

X = no. of evenfaces among 10 throws

$P(X=x) = {}^{10}C_x\, p^x (1-p)^{n-x}$

A/q,

$P(X=5) = 2\,P(X=4)$

${}^{10}C_5\, p^5 (1-p)^5 = 2\cdot {}^{10}C_4\, p^4 (1-p)^4$

$p(1-p) = \frac{2\cdot {}^{10}C_4}{{}^{10}C_5} = \frac{2\times \cancel{10!}\times 5!\,5!}{4!\,6!\times \cancel{10!}} = \frac{2\times 5\times \cancel{4!}\times \cancel{5!}}{\cancel{4!}\times 6\times \cancel{5!}} = \frac{5}{3}$

$3p - 3p^2 = 5$

$3p^2 - 3p + 5 = 0$

$p = 5/8$

$\therefore P(X=0) = {}^{10}C_0\, p^0 (1-p)^{10} = 10\cdot\frac{5}{8}\left(\frac{3}{8}\right)^{10}$

Poisson Distribution

Def^n: A r.v 'X' is said to be follow poisson distribidtion iff it assume non-negative value and its prob mass function is

$$P(x,\lambda) = \begin{cases} P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!} & , x = 0, 1, 2, \ldots \\ 0 & , \text{elsewhere} \end{cases}$$

· λ is a parameter

$\boxed{\lambda > 0}$

$\boxed{\lambda = np}$

$X \sim P(X)$

· Used when prob. of success is very less (small)

Derivation:

Binomial dist. tends to poisson dist. under the following conditions –

(I) Probability of success is very small $(p \to 0)$

(II) Number of trials is very large $(n \to \infty)$

$\boxed{\lambda = np}$

$$P_x = \begin{cases} {}^nC_x \cdot p^x \cdot q^{n-x} & , x = 0, 1, 2, \ldots \\ 0 & , \text{otherwise} \end{cases}$$

$$= \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(x-1))\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \cancel{n^x}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)\cdot\frac{\lambda^x}{\cancel{n^x}}\cdot\left(1-\frac{\lambda}{n}\right)^{n-x}$$

We know that,

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

and, $$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = 1$$

Then,

$$\boxed{P_x = \frac{e^{-\lambda}\cdot\lambda^x}{x!}}$$

Mean and Variance for Poisson Distribution

Mean $= E(X)$

$$= \sum_{x=0}^{\infty} x\cdot\frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty}\frac{e^{-\lambda}\cdot\lambda^x}{(x-1)!}$$

$$= \cancel{\sum_{x=1}^{\infty}}\; e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\cdot\lambda\cdot e^{\lambda}$$

$$= \lambda$$

Mean $= E(X)$

$= \lambda$

$= np$

Variance:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2\cdot\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty}[x(x-1)+x]\frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

$$= \underbrace{\sum_{x=0}^{\infty}\frac{e^{-\lambda}\cdot\lambda^x}{(x-1)!}}_{\lambda} + \sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-2)!}$$

$$= \lambda + \sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-2)!}$$

$$= \lambda^2 e^{-\lambda}\sum_{x=2}\frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2\cdot e^{-\lambda}e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$\therefore Var(x) = (\lambda^2+\lambda) - \lambda^2$$

$= \lambda$

$= np$

Q.) The average no. of radioactive particles through a counter during 1 ms in a ~~lab~~ laboratory experiment is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Solⁿ: $x = 6$ and $\lambda = 4$

$$P(X=6, \lambda=4) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= \frac{e^{-4} \cdot 4^{6}}{6!}$$

$$= \frac{0.0183 \times 4^{6}}{720}$$

$$= 0.1041$$

Q.) In a certain factory turning razor blades, there is a small chance $\frac{1}{5000}$ for any blade to be defective. The blade are in packets of 10. Use Poisson Distribution to calculate the approx. no. of packets containing (i) No defective (ii) 2 defective blades respectively in one consignment of 10000 packets $(e^{-.02} = 0.9802)$

Solⁿ: 1 defective blade in 5000 blades

So, avg. no. of defective blades (in each packet) $= 10 \times \frac{1}{5000}$

$$= \frac{1}{500}$$

$$= 0.002$$

Let, 'X' be no. of defective blades in packet.

$$P_x = P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

(1) $x = 0$

$$P_x = \frac{e^{-\lambda} \cdot \lambda^{0}}{0!} = e^{-\lambda}$$

$$= e^{-0.02}$$

$$= 0.9802$$

∴ No. of pkt. in consignment containing no defectives = 10000 X 0.9802

(ii) $x = 2$

$$P_x = \frac{e^{-\lambda} \cdot \lambda^2}{2!}$$

$$= \frac{e^{-0.02} \times (0.02)^2}{2!}$$

$$= 0.00019604$$

$\therefore$ No. of pkt. in consignments containing two defectives

$$= 0.00019604 \times 10000$$

$$\approx 1.9604$$

$$\approx 2$$

Q) 2% of the items made by a machine are defective. Find the prob. that 3 or more items are defective in a sample of 100 items.
(Given : $e^{-1} = 0.368$, $e^{-2} = 0.135$, $e^{-3} = 0.498$)

Sol^n: $n = 100$

$$p = 2\% \cancel{\text{ of } 100}$$

$$= \frac{2}{100}$$

$$= 0.02$$

$$P_x = \frac{e^{-\lambda} \cdot \lambda^x}{x!} = \frac{e^{-2} \cdot 2^x}{x!}$$

$\therefore$ Required Probability $= P(X \geq 3)$

$$= 1 - P(X < 3)$$

$$= 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\}$$

$$= 1 - \left\{ \frac{e^{-2} \cdot 2^0}{0!} + \frac{e^{-2} \cdot 2^1}{1!} + \frac{e^{-2} \cdot 2^2}{2!} \right\}$$

$$= 1 - e^{-2} \{1 + 2 + 2\}$$

$$= 1 - 5(0.135)$$

$$= 0.325$$

Q.) Red blood cell deficiency may be determined by examining a specimen of blood under microscope. Suppose a certain small fixed volume contained on the average 20 red cells for normal people. Find the prob. that a specimen from a normal person will contain less than 15 red cells.

Sol^n: $\lambda = 20$ (given)

Probability that a normal person contain x red cells

$$= \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \frac{e^{-20} \cdot 20^x}{x!}$$

$$P(X<15) = \sum_{x=0}^{14} \frac{e^{-20} \cdot 20^x}{x!}$$

$$= e^{-20}\left[1 + \frac{20}{1!} + \frac{20^2}{2!} + \dots + \frac{20^{14}}{14!}\right]$$

- Measures of Central Tendency

Mean, Median, Mode, Geometrical Mean $\left\}\begin{array}{l}\text{Grouped data} \\ \text{Raw data}\end{array}\right.$

$$\left.\begin{array}{ll} \text{(I)} & 7, 8, 9, 10, 11 \\ \text{(II)} & 3, 6, 9, 12, 15 \\ \text{(III)} & 1, 5, 9, 13, 17 \end{array}\right\} \text{Mean} = 9$$

- Homogeneous data → data points lie closer to the mean
- Heterogeneous data

- Measures of Dispersion

↳ Skewness, Kurtosis

(I) Range : A (greatest) B (smallest)

$$\boxed{\text{Range} = A - B}$$

(II) Quartile

Q_1 Q_2 Q_3 Q_4

$$\boxed{Q = \frac{Q_1 - Q_3}{2}}$$

(III) Mean Deviation

$$\cancel{MD = \frac{1}{N} f_i}$$

$$\boxed{M.D = \frac{1}{N} \sum f_i \, |x_i - A|}$$

$$\boxed{M.D = \frac{1}{N} \sum f_i \, |x_i - \overline{X}|}$$

$$\boxed{\sum_i f_i = N}$$

(IV) R.M.S — standard deviation

$$\boxed{S^2 = \frac{1}{N} \sum_i f_i (x_i - A)^2}$$

$$\boxed{\sigma^2 = \frac{1}{N} \sum_i f_i (x_i - A)^2}$$

(V) Moment

$$\boxed{\mu_r' = \frac{1}{N} \sum_i f_i (x_i - A)^r}$$

$$\boxed{\mu_r' = \frac{1}{N} \sum f_i \, d_i^r}$$ Here, $\boxed{d_i = x_i - A}$

$$\boxed{\mu_r = \frac{1}{N} \sum (x_i - \overline{X})^r}$$

$$\boxed{\mu_0 = \frac{1}{N} \sum_i f_i = 1}$$

$$\mu_1 = \frac{1}{N} \sum f_i (x_i - \overline{x})^1$$

$$= \frac{1}{\cancel{N}} \sum_i f_i \cancel{x_i} - \frac{1}{N} \overline{x} \cancel{\sum_i f_i}^{1}$$

$$= 1 - 1$$

$$\boxed{\mu_1 = 0}$$

$$\mu_2 = \frac{1}{N} \sum_i f_i (x_i - \overline{x})^2$$

$$\boxed{\mu_2 = \sigma^2}$$

Pearson's β and γ coefficient

$\boxed{\beta_1 = \frac{\mu_3^2}{\mu_2^3}}$ $\boxed{\gamma_1 = \beta_2}$

$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$ $\boxed{\gamma_2 = \beta_2 - 3}$

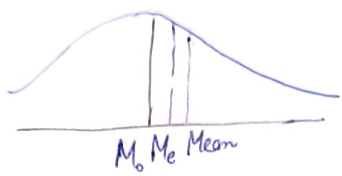
convexity of curve

Skewness

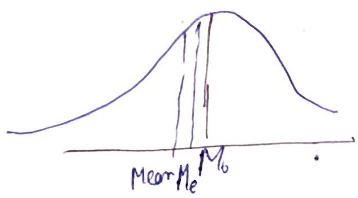
↳ "Lack of symmetry"

Mean
= Median
= Mode

"Symmetric Curve"
↓
"Normal Curve"
↳ N(0, 1)
↓
mean = 0
σ = 1

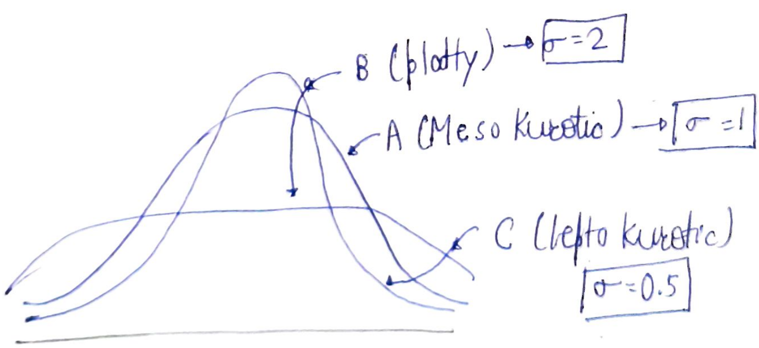


$M_o > M_e > \text{Mean}$



$\text{Mean} > M_e > M_o$

Kurtosis



A → normal curve → meso kurtic

B → platy → more flatter

C → highly peaked → lepto kurtic

$\beta_2 = \frac{\mu_4}{\mu_2^2}$ $\quad \gamma_2 = \beta_2 - 3$

(A) $\rightarrow$ $\beta_2 = 3$; $\gamma_2 = 0$ (normal / mesokurtic curve)

Normal Distribution

- For continuous data (r.v)

- Continuous r.v 'X' is said to have a normal distribution if its probability density function is given by:

$$N(x) \sim f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}} & , -\infty < x < \infty \\ 0 & , \text{otherwise} \end{cases}$$

where μ, σ are two parameters of the distribution

$$X \sim N(\mu, \sigma)$$

- $f(x) \geq 0$

Moreover, $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} . dx$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \quad \left(\text{by } z = \frac{x-\mu}{\sqrt{2}.\sigma}\right)$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u} . u^{\frac{1}{2}} du \quad (z^2 = u)$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

$$= 1$$

$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

(Gamma function)

$\boxed{\Gamma(n+1) = n\Gamma(n)}$

$$\boxed{F(x) = \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}$$

Mean for Normal Distribution

$$\text{Mean} \cdot E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \frac{\mu}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u . e^{-u^2} du + \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \left(\frac{x-\mu}{\sigma\sqrt{2}} = u\right)$$

$$= 0 + \frac{\mu}{\sqrt{\pi}} \cdot 2 \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{0}^{\infty} z^{\frac{1}{2}-1} . e^{-z} dz \quad (u^2 = z)$$

$$= \frac{\mu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\mu}{\sqrt{\pi}} . \sqrt{\pi}$$

$$= \mu$$

$$\boxed{\text{Mean} = \mu}$$

Standard Deviation for Normal Distribution

$$Var(X) = E\{(X-\mu)^2\}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 . e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad E(x^2) = \int x^2 P_x \, dx$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 . e^{-z^2} dz \qquad \left[z = \frac{x-\mu}{\sqrt{2}\,\sigma}\right]$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z^2 . e^{-z^2} dz$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u^{\frac{1}{2}} e^{-u} du \qquad [z^2 = u]$$

$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$ $\qquad$ $\Gamma(n+1) = n\Gamma(n)$

$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$

$= \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi}$

$= \sigma^2$

$$\left.\boxed{\begin{array}{l} Var(X) = \sigma^2 \\ S.D = \sigma \\ Mean = \mu \end{array}}\right\} \text{For normal distribution}$$

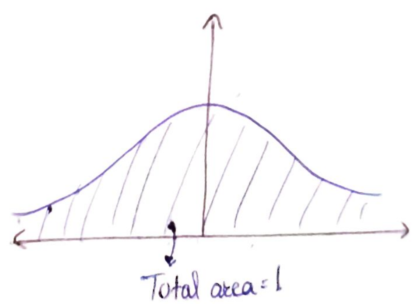
- $n \to$ large
- Prob is moderate $\approx \frac{1}{2}$

(PDF → continuous)

Important Result

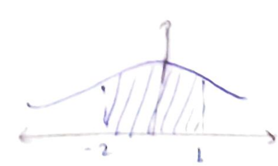
- In continuous random variable X for normal distribution with parameters 'μ' and 'σ', then

$$\boxed{Z = \frac{X-\mu}{\sigma}}$$



Q) To find the no. of days on which power supplied will lie b/w 280 to 310 MW. Suppose $m = 300$, $\sigma = 10$. Find $P(280 < X < 310)$.

$Z = \frac{X - m}{\sigma} = \frac{X - 300}{10}$



$P(280 < X < 310) = P\left(\frac{280 - 300}{10} < \frac{X - m}{10} < \frac{310-300}{10}\right)$

$= P(-2 < Z < 1)$

$= \int_{-2}^{1} \phi(z)\, dz$ $\qquad$ $\phi(z) = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz$

= area enclosed by std. Normal curve and $z = -2$ and $z = 1$

$= 0.4772 + 0.3413$

$= 0.8185 = 81.85\%$

Q.> The length of belt produced by a machine is normally distributed with mean = 4 and S.D = 0.5. A belt is defective if its length doesn't lie in the interval (3.8, 4.3). Find the % of defective belts produced by machine. $\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.6} e^{-t^2/2}\,dt = 0.7257 \text{ and } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+0.4} e^{-t^2/2}\,dt = 0.6564\right]$

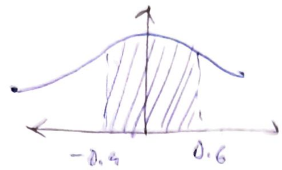
Sol: $x \rightarrow$ length of belt

$m = 4$, $SD = 0.5$

$$P(3.8 < X < 4.3) = P\left(\frac{3.8-4}{0.5} < \frac{X-4}{0.5} < \frac{4.3-4}{0.5}\right)$$

$$= P(-0.4 < Z < 0.6)$$

= area enclosed by std. normal curve and $z = -0.4$ to $z = 0.6$



$$= \int_{-0.4}^{0.6} \phi(t)\,dt$$

$$= \int_{-\infty}^{0.6} \phi(t)\,dt - \int_{-\infty}^{-0.4} \phi(t)\,dt$$

$$= 0.7257 - \int_{0.4}^{\infty} \phi(t)\,dt \quad \text{(Graph is symmetrical)}$$

$$= 0.7257 - \left[1 - \int_{-\infty}^{0.4} \phi(t)\,dt\right]$$

$$= 0.7257 - [1 - 0.6564]$$

$$= 0.3811$$

$\therefore$ Probability that the length of the whole belt lies b/w 3.8 and 4.3

$= 0.3811$

$= 38.11\%$

Probability belt is NOT defective = 0.6189

$= 61.89\%$

Q.) X is normal distribution and mean of X is 12 and SD = 4.

(a) (i) Find out the prob. of the following

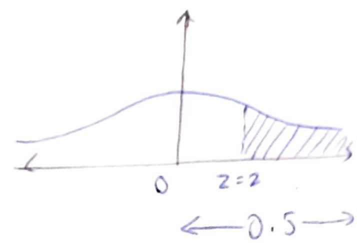
(i) $X \geq 20$ (ii) $X \leq 20$ (iii) $0 \leq X \leq 12$

(b) Find 'x' when $P(X > x) = 0.24$

(c) Find x_0 and x_1 when $P(x_0 < X < x_1) = 0.5$ and $P(X > x_1) = 0.25$

sol^n: $m = 12$, $SD(\sigma) = 4$

(a) (i) $P(X \geq 20)$



$$Z = \frac{X-12}{4} = \frac{20-12}{4} = 2$$

$\therefore P(X \geq 20) = P(Z \geq 2)$

$= 0.5 - P(0 \leq Z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228$

(ii) $P(X \leq 20) = 1 - P(X \geq 20)$

$= 1 - 0.0228$

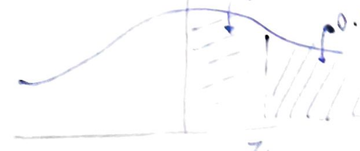
$= 0.9772$

(iii) $P(0 \leq X \leq 12) = P\left(\frac{0-12}{4} \leq \frac{X-12}{4} \leq \frac{12-12}{4}\right)$

$= P(-3 \leq Z \leq 0)$

$= 0.49865$

(b) $P(X > x) = 0.24$



$\Rightarrow P\left(\frac{X-12}{4} > \frac{x-12}{4}\right) = 0.24$

$\Rightarrow P(Z > z_1) = 0.24$

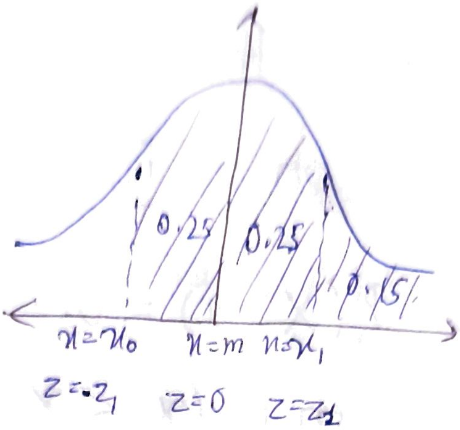
$\Rightarrow P(0 < Z < z_1) = \frac{1}{2} - 0.24 = 0.26$

$z_1 = \frac{x-12}{4} = 0.71$

$\Rightarrow x = 14.84$

(c) $P(x_0 < X < x_1) = 0.5$

$P(X > x_1) = 0.25$



$x = x_1 \Rightarrow z = \frac{x_1 - 12}{4} = z_1$ (say)

$x = x_0 \Rightarrow z = \frac{x_0 - 12}{4} = -z_1$ (say)

$P(z > z_1) = 0.25$

$P(0 < z < z_1) = 0.5 - 0.25 = 0.25$

$\Rightarrow z_1 = 0.67$ (from Table)

$\frac{x_1 - 12}{4} = 0.67$

$\frac{x_0 - 12}{4} = -0.67$

$\therefore x_1 = 14.68$

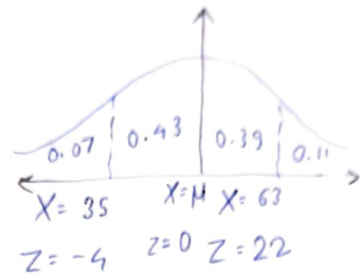
$x_0 = 9.32$

Q. In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What is mean and SD of the distribution?

Sol^n: If $X \sim N(\mu, \sigma^2)$, then we are given

$P(X<63) = 0.89$ $\qquad$ $P(X<35) = 0.07$

$P(X>63) = 0.11$ $\qquad$ $P(X>35) = 0.93$



when, $X = 35$ $\quad Z = \frac{35-\mu}{\sigma} = -Z_1$ (say)

$X = 63$ $\quad Z = \frac{63-\mu}{\sigma} = Z_2$ (say)

$P(0<Z<Z_2) = 0.39 \Rightarrow \frac{63-\mu}{\sigma} = 1.23$ — (I)

$P(0<Z<Z_1) = 0.43 \Rightarrow \frac{35-\mu}{\sigma} = -1.48$ — (II)

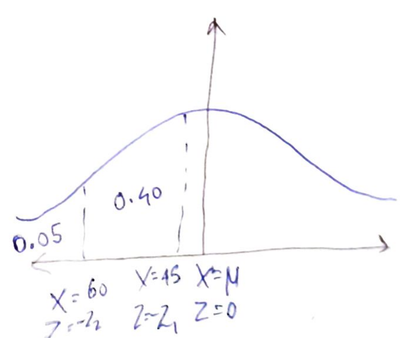
(I) − (II), we get:

$\frac{28}{\sigma} = 2.71 \Rightarrow \sigma = 10.33$

From (I), $\mu = 35 + 1.4(10.33)$

$= 50.3$

Q. If a large group of men, 5% are under 60 inches high and 40% b/w 60 and 45. Assuming normal dist., find mean and SD.



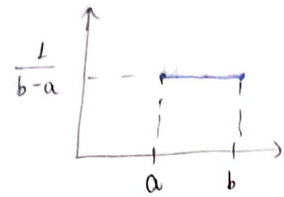
Another Distribution

I. Uniform or Rectangular Distribution

A r.v. 'X' is said to be a uniform dist. on interval $[a,b]$, $-\infty < a < b$

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \leq x \leq b \\ 0 & , \text{otherwise} \end{cases}$$

$$\boxed{X \sim U[a,b]}$$

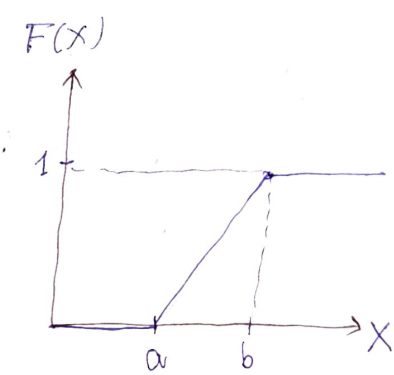


e.g. -

$$F(X) = \int_{-\infty}^{x} f(x)\,dx$$

is given by:

$$F(X) = \begin{cases} 0 & , -\infty < x < a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ 1 & , b \leq x < \infty \end{cases}$$



(I) Mean $= \frac{1}{2}(a+b)$

(II) Variance $= \frac{(a-b)^2}{12}$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \int_{-\infty}^{a} x.0\,dx + \int_{a}^{b} x.\frac{1}{b-a}\,dx + \int_{b}^{\infty} x.0\,dx$$

$$= \frac{(b^2-a^2)}{2(b-a)}$$

$$= \frac{1}{2}(a+b)$$

$\text{Var}(X) = E(X^2) - [E(X)]^2$

II Exponential Distribution

$$f(x) = \begin{cases} \frac{1}{b} e^{\left(\frac{x-a}{b}\right)} & , x \geq a \\ 0 & , \text{otherwise} \end{cases}$$

where a, b $(b > a)$ are two parameters

$$\int_{-\infty}^{\infty} f(x)\, dx \quad \int_{a}^{\infty} \frac{1}{b} e^{\left(\frac{x-a}{b}\right)} dx = 1$$

(i) Mean $= \mu$

(ii) Variance $= \frac{1}{\lambda^2}$ $\quad \left(\lambda = \frac{\mu}{x}\right)$

III Erlang Distribution (General Gamma Distribution)

$$f(x) = \begin{cases} \frac{\lambda^K \cdot x^{K-1} \cdot e^{-\lambda x}}{\Gamma K} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

IV. Weibull Distribution

$$f_x(x) = e^{-y} \alpha\beta \cdot x^{\beta-1}$$

$$= \alpha\beta x^{\beta-1} \cdot e^{-\alpha x^{\beta}}, x > 0$$

V. Normal distribution

(Gaussian dist.)

Next Episode:

(i) Moment-Generating function

(ii) Critical Limit Theorem

Moment Generating Function (MGF)

Def^n: The MGF of a r.v. 'X' is a function in a real variable, given by $M_x(t) = E(e^{tx})$

(i) Discrete

$$M(t) = \sum_i e^{tx_i} \cdot P(X = x_i)$$

(ii) Continuous

$$M(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

Theorem (i): If a r.v. has MGF $M_x(t)$, then about a

$$M_x^{(r)}(t) = E\left((x-a)^r\right) = r^{th} \text{ moment about } a, r = 1, 2, 3, \ldots$$

Theorem (ii): Let $x_1, x_2, \ldots, x_n$ be the independent r.v., then M.G.F of their sum is,

$$M_{x_1+x_2+x_3+\cdots+x_n}(t) = E\left(e^{t(x_1+x_2+\cdots+x_n)}\right)$$

$$= E\left((e^{tx_1}) \cdot (e^{tx_2}) \cdots (e^{tx_n})\right)$$

$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots E(e^{tx_n})$$

$$= M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$$

MGF of Bernoulli's distribution:

P.M.F : $f_i = P(X=i) = {}^nC_i \cdot p^i (1-p)^{n-i}$, $i = 0, 1, 2, \ldots$

$$M(t) = E(e^{tx})$$

$$= \sum_{i=0}^{n} e^{ti} \cdot f_i$$

$$= \sum_{i=0}^{n} {}^nC_i \, (pe^t)^i (1-p)^{n-i}$$

$$= (1 - p + pe^t)^n$$

$$= (q + pe^t)^n \quad \text{where, } q = (1-p)$$

Mean and Variance

$$M^{(1)}(t) = \frac{d}{dt}(M(t)) = \frac{d}{dt}(q+pe^t)^n$$

$$= n(q+pe^t)^{n-1} \cdot pe^t$$

$$= npe^t(q+pe^t)^{n-1}$$

$$M^{(2)}(t) = \frac{d^2}{dt^2}(M(t)) = np\left[e^t.(q+pe^t)^{n-1} + e^t.(n-1)(q+pe^t)^{n-2}.pe^t\right]$$

$$= np\left[e^t(q+pe^t)^{n-1} + npe^{2t}(q+pe^t)^{n-2} - \cancel{np}e^{2t}(q+pe^t)^{n-2}\right]$$

$$= npe^t(q+pe^t)^{n-2}\left\{(q+pe^t)+(n-1)pe^t)\right\}$$

$$\text{Mean} = M^{(1)}(0)$$

$$= np(q+p)^{n-1}$$

$$= np \cdot 1$$

$$= np$$

$$Var(X) = M^{(2)}(0) - \left\{M^{(1)}(0)\right\}^2$$

$$= np(p+q)^{n-2}\left\{q+p+(n-1)p\right\} - (np)^2$$

$$= np(1-p)$$

$$= npq$$

Limit Theorem

Tchebyshoff's Inequality

Theorem: Let 'X' be a r.v. having mean 'μ' and variance 'σ^2', then for any $\varepsilon > 0$,

$$\boxed{P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}}$$

Q.) Show by Tchebyscheff's Inequality that if 2000 throws with a coin the probability that the no. of heads lie b/w 900 and 1100 is at least $\frac{19}{20}$

Solⁿ: Let 'X' denote the no. of heads

This is the case of binomial $B(n, p)$.

$$n = 2000, \quad p = \frac{1}{2}$$

$$\mu = E(X) = np = 1000$$

and $Var(X) = \sigma^2 = npq = 2000 \times \frac{1}{2} \times (1 - \frac{1}{2})$

$$= 500$$

Next, $P(900 < X < 1100) = P(-100 < X - 1000 < 100)$

$$= P(|X - 1000| < 100)$$

$$= 1 - P(|X - 1000| \geq 100) \quad \text{— (1)}$$

Now by T-theorem,

$$P(|X - 1000| \geq 100) \leq \frac{\sigma^2}{100}$$

$$P(|X - 1000| \geq \underbrace{100}_{\varepsilon}) \leq \frac{500}{(100)^2} = \cancel{8}\,\frac{1}{20}$$

From eqⁿ (1), $P(900 < X < 1100) \geq 1 - \frac{1}{20} = \frac{19}{20}$

Central Limit Theorem (C.LT)

Let $X_1, X_2, \ldots, X_n$ be 'n' independent and identically r.v. with common μ and variance (σ^2) respectively.

Also, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

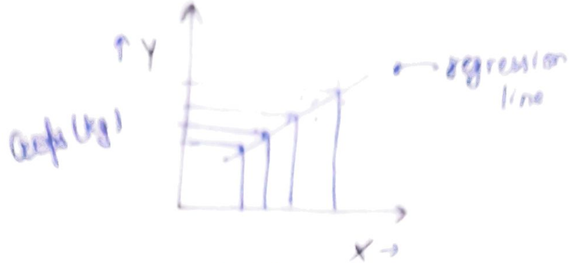
Then, the distribution of $\bar{X}$ converges to the $\underbrace{\text{normal distribution}}_{\text{as } n \to \infty \text{ (error ver)}}$ with μ and variance $\frac{\sigma^2}{n}$,

$$\boxed{Z = \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}}$$

Co-relation

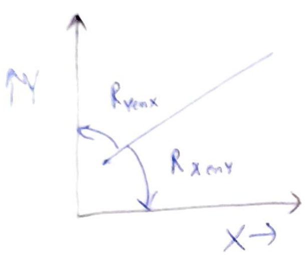
Relation (X, Y)

R (X, Y)

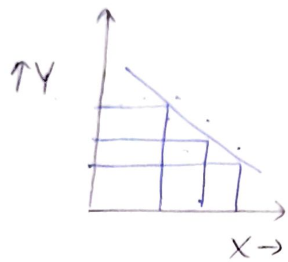


Fertilizer

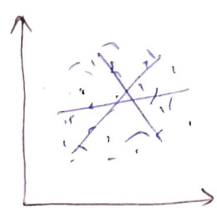
- (+ve) relation



- regression line
- best fit
- average line
- greatest of fit

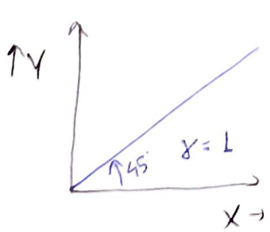


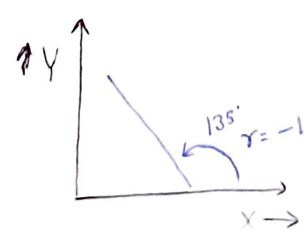
- (-ve) relation

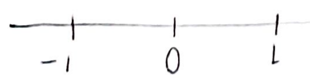


- No relation (most points can't be covered with any line)

Coefficient of co-relation







r = 0

· no reln

Cov

$$\boxed{r = \frac{Cov(x,y)}{\sigma_x \sigma_y}}, |r| \leq 1 \qquad \boxed{Cov(x,y) = \frac{1}{n}\sum_{i=1}^{n}\left[(x_i - \bar{X})(Y_i - \bar{Y})\right]}$$

$\sigma_x = Var(X) = \frac{1}{n}\left(\sum x_i - \bar{X}\right)$; $\sigma_y = Var(Y) = \frac{1}{n}\left(\sum y_i - \bar{Y}\right.$

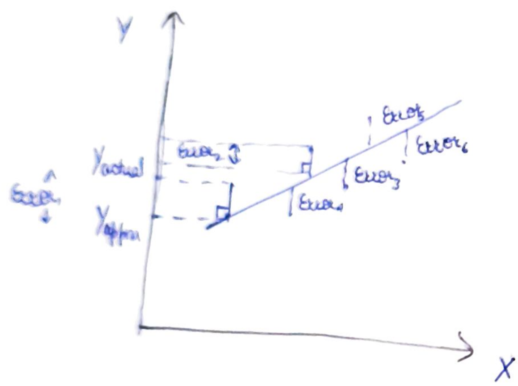
$$\mathrm{Cov}(x,y) = \frac{1}{n}\sum_{i=1}^{n}\left[(x_i-\bar{X})(Y_i-\bar{Y})\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[x_i y_i - x_i\bar{Y} - y_i\bar{X} + \bar{X}\cdot\bar{Y}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{Y}\left(\frac{\sum_i x_i}{n}\right) - \bar{X}\left(\frac{\sum_i y_i}{n}\right) + \bar{X}\bar{Y}\cancel{\frac{\sum_{i}^{n} 1}{n}}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{X}\bar{Y} - \cancel{\bar{X}\bar{Y}} + \cancel{\bar{X}\bar{Y}}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{X}\bar{Y}$$



let $Y = ax + b$ covers most points

$$Y_1 = ax_1 + b + e_1$$
$$Y_2 = ax_2 + b + e_2$$
$$\vdots$$
$$Y_n = ax_n + b + e_n$$

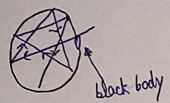
$$e_i = Y_i - (ax_i + b)$$
$$e_i^2 = (Y_i - ax_i - b)^2$$

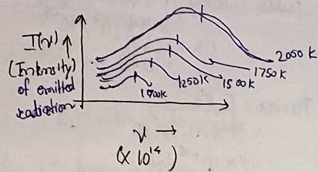
$$\delta = \sum_i (e_1^2 + e_2^2 + \dots + e_n^2)$$
$$= \sum_i (Y_i - (ax_i + b)^2)$$

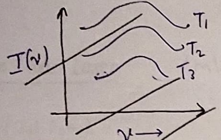
$$\frac{\partial \delta}{\partial a} = 0 \qquad \frac{\partial \delta}{\partial b} \quad 0$$

Quantum Mechanics

Black Body Radiation



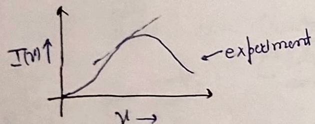




* If temp. ↑es, total amount of emitted radiation ↑es
* The position of maxima shifts towards higher frequency region with temperature.

A/c Rayleigh Jean law,

$$I(\nu) = \frac{8\pi\nu^2}{c^3} kT \, d\nu$$



* Satisfied at lower frequency region, $\underbrace{\text{not at higher frequencies}}_{\text{"UV catastrophe"}}$

A/c Planck's modification,

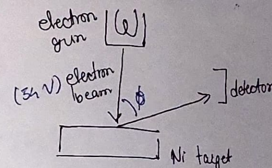
$$I(\nu) = \frac{8\pi h\nu^3}{c^3}\left(\frac{1}{e^{h\nu/kT}-1}\right)$$

Case I : $h\nu \gg kT$
Case II : $h\nu = kT$
Case III : $h\nu < kT$

* It explained black body radiation (partially)

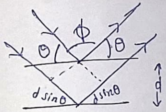
Davisson - Germer Experiment

* Demonstration of matter wave



* Max^m intensity at $\phi = 50°$

Bragg's Law



$\Delta x = 2d\sin\theta$

$\boxed{2d\sin\theta = n\lambda}$

$2\theta + \phi = 180°$

$\Rightarrow 2\theta + 50° = 180°$

$\Rightarrow \boxed{\theta = 65°}$

$\boxed{d_{Ni} = 0.91\ A°}$

Let, $n = 1$

$2d\sin\theta = \lambda$

$\Rightarrow \lambda = 2 \times 0.91 \times \sin 65°\ A°$

$= 1.65\ A°$ (when considered wave)

Generally, $\boxed{\frac{1}{2}mv^2 = eV}$ Also, $\boxed{\lambda = \frac{h}{p}}$

$$\boxed{\lambda = \sqrt{\frac{150}{V}}}$$

$\Rightarrow \lambda = 1.67\ A°$ (when considered wave-particle)

Wave → interference, diffraction, frequency, wavelength, polarisation

Particle → mass, velocity, momentum, blackbody radiation, PE effect

De-Broglie Hypothesis

• Wave-Particle duality

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

wave property

particle property

$\boxed{E = h\nu}$ $\boxed{E = mc^2}$

$$\boxed{\text{Kinetic Energy} = \frac{1}{2}mv^2 = \frac{p^2}{2m}}$$

$$\boxed{p = \sqrt{2mE}}$$

$$\boxed{\lambda = \frac{h}{\sqrt{2mE}}}$$

Velocity of De-Broglie Wave

$$\boxed{\lambda = \sqrt{\frac{150}{V}}}$$

potential

$$eV = \frac{1}{2}mv^2$$

$$\lambda = \frac{h}{mv}$$

$$\Rightarrow \lambda = \frac{h}{\sqrt{2meV}} = \sqrt{\frac{150}{V}} = \frac{12.25}{\sqrt{V}}$$

$$\boxed{\lambda = \frac{12.25}{\sqrt{V}}}$$

$h = 6.63 \times 10^{-34}$ Js

$e = 1.6 \times 10^{-19}$ C

$m_e = 9 \times 10^{-31}$ kg

$V = 100$ V

$\boxed{\lambda_e = 1.226\ Å}$

• Particle heavier ⇒ 'λ' is less

$m_p = 1.67 \times 10^{-27}$ kg

$\boxed{\lambda_p = 0.28\ Å}$

Phase and Group Velocities

$$y = a \sin(\omega t - kx)$$

Here, $\omega t - kx \rightarrow$ constant

$$\boxed{\frac{dx}{dt} = \frac{\omega}{k} = v_p}$$

$\boxed{\lambda = \frac{h}{p}}$ $\boxed{E = h\nu}$ $\boxed{v = \lambda\nu}$

$$v_p = \frac{E}{h} \times \frac{h}{p} = \frac{E}{mv} = \frac{mc^2}{mv}$$

$$\boxed{v_p v = c^2}$$

if $c \gg v$ / $\boxed{v_p > c}$

• The phase vel. of a de-broglie wave associated with particle moving with vel. 'V' is greater than vel. of light 'c'.

Group Velocities

$$y_1 = a \sin(\omega_1 t - k_1 x)$$

$$y_2 = a \sin(\omega_2 t - k_2 x)$$

After superposition,

$$y = y_1 + y_2$$

$$= a\left[2 \sin\left(\frac{(\omega_1+\omega_2)t - (k_1+k_2)x}{2}\right) \cos\left(\frac{(\omega_1-\omega_2)t - (k_1-k_2)x}{2}\right)\right]$$

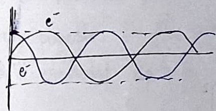
Let, $\left(\omega = \frac{\omega_1+\omega_2}{2}\right)$; $k = \left(\frac{k_1+k_2}{2}\right)$

$$= 2a \cos(\omega t - kx) \cdot \sin\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$$

$$\boxed{v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}}$$

$$v_g = \frac{\partial\omega}{\partial k} = \frac{\partial(2\pi\nu)}{\partial(2\pi/\lambda)} = \lambda^2 \frac{\partial\nu}{\partial\lambda}$$

$$\boxed{v_g = -\frac{\lambda^2}{2\pi}\frac{\partial\omega}{\partial\lambda}}$$



• maxima/minima related to v_g.

Relationship b/w v_g and v_p

$$v_g = \frac{\partial\omega}{\partial k} \qquad v_p = \frac{\omega}{k}$$

$$= \frac{\partial}{\partial k}(v_p k) = v_p + k\frac{dv_p}{dk}$$

$$k = \frac{2\pi}{\lambda} \Rightarrow dk = -\frac{2\pi}{\lambda^2}d\lambda \Rightarrow \frac{k}{dk} = \frac{-\lambda}{d\lambda}$$

$$\boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}}$$

↳ $v_g \ll v_p$

Heisenberg's Uncertainty Principle

- It is impossible to determine the exact position and momentum of a small moving particle (like e^-) simultaneously.

$$\boxed{\Delta x \cdot \Delta p \geq \frac{h}{4\pi}}$$

$$\boxed{\Delta x \cdot \Delta p \approx \hbar/2} \quad ; \hbar = \frac{h}{2\pi}$$

$$\boxed{\Delta E \cdot \Delta t \approx \hbar/2}$$

Applications:

(I) Non-existence of e^- in the nucleus.

$$\Delta p = \frac{h}{2\pi \Delta x}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 10^{-14}}$$

$$= 5.275 \times 10^{-21} \text{ kgm/s}$$

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{(5.275 \times 10^{-21})^2}{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19}} \text{ eV}$$

$$= 9.6 \text{ MeV}$$

- The max^m energy that e^- can possess ~~to exist in nucleus~~ is 4 MeV. but it requires 9.6 MeV to reside inside the nucleus.

(II) Radius of Bohr's 1st orbit

$$\Delta x \cdot \Delta p \approx \hbar$$

$$KE = \Delta T = \frac{(\Delta p)^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{\Delta x}\right)^2$$

$$PE = \frac{1}{4\pi\epsilon_0} \cdot \frac{(Ze)(-e)}{\Delta x}$$

Total energy $\Delta E = KE + PE$

$$= \frac{1}{2m}\left(\frac{\hbar}{\Delta x}\right)^2 - \frac{Ze^2}{4\pi\epsilon_0 \Delta x}$$

$$\boxed{\Delta x = \frac{\epsilon_0 \hbar^2}{\pi m Ze^2}}$$

For max^m/min^m; $\frac{\partial(\Delta E)}{\partial(\Delta x)} = 0$

(III) Energy in a box of ~~a~~ particle infinite potential well (of a particle)

(IV) Ground State Energy of a linear harmonic oscillator.

(III) Energy of a particle in a box (infinite potential well)

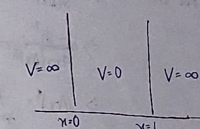
- The particle is confined in a box of length 'L',

$$\therefore \Delta x \approx L$$

Now, $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$ (Uncertainty Principle)

$$\Rightarrow \Delta p \geq \frac{\hbar}{2L}$$

Also potential $V(x) = \begin{cases} 0 & , 0 < x < L \\ \infty & , x < 0 \text{ (or) } x > L \end{cases}$



i.e. PE=0 inside the box

$$KE = \frac{p^2}{2m}$$

Taking $p \approx \Delta p$

$$E = \frac{(\Delta p)^2}{2m}$$

$$\Rightarrow E_{min} \left(\frac{\hbar}{2L}\right)^2 \left(\frac{1}{2m}\right) = \frac{\hbar^2}{8mL}$$

* The existence of non-zero ground state energy $\boxed{E = \frac{\hbar^2}{8mL}}$ is a direct consequence of the uncertainty principle. If this energy were zero, then momentum will be exactly zero which'll contradict $\Delta p \geq \frac{\hbar}{2L}$.

(IV) Ground State Energy of a linear harmonic oscillator

$$\Delta x \cdot \Delta p \approx \frac{\hbar}{2} \Rightarrow \Delta p \approx \frac{\hbar}{2\Delta x}$$

$$KE = \frac{(\Delta p)^2}{2m} = \left(\frac{\hbar}{2\Delta x}\right)^2\left(\frac{1}{2m}\right) = \frac{\hbar^2}{8m(\Delta x)^2}$$

$$PE = \frac{1}{2}k(\Delta x)^2$$

$$E = KE + PE$$

$$= \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}k(\Delta x)^2 \quad \text{——(I)}$$

For min^m energy, $\frac{\partial E}{\partial(\Delta x)} = 0$

$$\Rightarrow \frac{-\hbar^2}{4m(\Delta x)^3} + k(\Delta x) = 0$$

$$\Rightarrow (\Delta x)^4 = \frac{\hbar^2}{4mk}$$

$$\Rightarrow \Delta x = \left(\frac{\hbar^2}{4mk}\right)^{1/4} \quad \text{——(II)}$$

From (I) and (II),

$\boxed{E_{min} = \frac{1}{2}\hbar\omega}$, where $\omega = \sqrt{\frac{K}{m}}$ (angular frequency of oscillator)

$\llcorner$ "Zero-point energy"

- The existence of the zero-point energy is a direct consequence of the uncertainty principle because if this ground state energy were zero, both Δx and Δp would be precisely zero, contradicting the uncertainty principle.

Wave Function

$\psi(x, y, z, t) \rightarrow$ has both characters (particle & wave)

- has no meaning by itself

$|\psi|^2 \rightarrow$ probability of finding a particle,

$$\boxed{\int_{-\infty}^{\infty} |\psi|^2 \, dV = 1} \text{ (Normalization condition)}$$

Properties of $\psi(x, y, z, t)$:

(I) - Continuous
- Finite
- Single-valued

(II) $\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z}$ be continuous, finite, single-valued

(III) ψ should be normalized

* ψ is complex $(a+ib)$ in nature.

Schroedinger's Equation

Time Independent:

$\psi = a+ib$

$\psi^* = a-ib$, (conjugate)

By Newtonian mechanics,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Same analogy,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$v \rightarrow$ velocity of particle associated with wave.

$$\psi(x, y, z, t) = \underbrace{\psi_0(x, y, z)}_{\text{time-independent amplitude}} \cdot e^{-i\omega t} = \psi_0(r) \cdot e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = (-i\omega)(-i\omega)\psi_0 \cdot e^{-i\omega t}$$

$$= -\omega^2 \psi_0 \cdot e^{-i\omega t}$$

$$= -\omega^2 \psi$$

Now, $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{\omega^2}{v^2}\psi=0$, $\omega=2\pi\nu = \frac{2\pi v}{\lambda}$

$\Rightarrow \frac{\omega}{v}=\frac{2\pi}{\lambda}$

$$\nabla^2\psi+\frac{\omega^2}{v^2}\psi=0$$

$$\Rightarrow \boxed{\nabla^2\psi+\frac{4\pi^2}{\lambda^2}\psi=0}$$

Now $\lambda=\frac{h}{p}=\frac{h}{mv}$

$$\Rightarrow \nabla^2\psi+\frac{4\pi^2m^2v^2}{h^2}\psi=0$$

Also, Energy = KE + PE
(E) (V)

$$\Rightarrow (E-V)=\frac{1}{2}mv^2$$

$$\Rightarrow \nabla^2\psi+\frac{8\pi^2m}{h^2}(E-V)\psi=0$$

$$\Rightarrow \boxed{\nabla^2\psi+\frac{2m}{\hbar^2}(E-V)\psi=0}$$

$$\boxed{\nabla^2\psi+\frac{2mE}{\hbar^2}\psi=0}$$ (For free particle $V=0$)

Time-Dependent Schrodinger's eqn

$$\psi=\psi_0 e^{-i\omega t}$$

$E=h\nu$

$\Rightarrow \nu=\frac{E}{h}$

$$\frac{\partial\psi}{\partial t}=-i\omega\psi_0 e^{-i\omega t}$$

$$=-i2\pi\nu\psi$$

$$=-\frac{i2\pi E}{h}\psi$$

$$=-\frac{iE}{\hbar}\psi$$

$$=\frac{E}{i\hbar}\psi$$

$$\frac{\partial \psi}{\partial t} = \frac{E}{i\hbar}\psi$$

$$\Rightarrow E\psi = i\hbar \frac{\partial \psi}{\partial t}$$

Time independent eqⁿ:

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E\psi - V\psi) = 0$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(i\hbar\frac{\partial \psi}{\partial t} - V\psi\right) = 0$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial \psi}{\partial t}}$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial \psi}{\partial t}$$

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2 + V = H \text{ (hamiltonian)}}$$

$$\boxed{H\psi = \cancel{\hbar}E\psi}$$

eigenfunction

$$A\psi = a\psi$$

operator   eigen-value (energy value)

e.g. $A = \frac{\partial^2}{\partial x^2}$ and $\psi = e^{2x}$, find eigen-values

$$A\psi = \frac{\partial^2}{\partial x^2}(e^{2x})$$

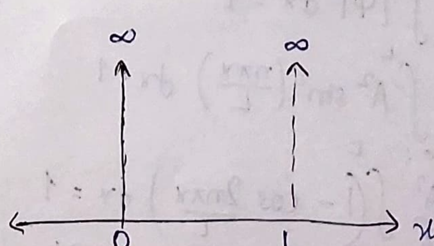
$$= 4e^{2x}$$

$\therefore$ Eigenvalue $(a) = 4$

Particle in box (having infinite potential well).

- Particle is free $(V=0)$

$$V = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & x<0 \text{ and } x>L \end{cases}$$



$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$\because V = 0$

$$\therefore \nabla^2\psi + \frac{2mE}{\hbar^2}\psi = 0$$

$$\Rightarrow \boxed{\nabla^2\psi + k^2\psi = 0} \quad \text{where, } k^2 = \frac{2mE}{\hbar^2}$$

Let, $\psi = A \sin(kx) + B\cos(kx)$

at $x=0,\ \psi=0$
$x=L,\ \psi=0$
(Particle can't exist at walls as position will be exactly known)

at $x=0,\ \psi=0$

$0 = 0 + B\cos k$

$\Rightarrow \boxed{B=0}$

at $x=L,\ \psi=0$

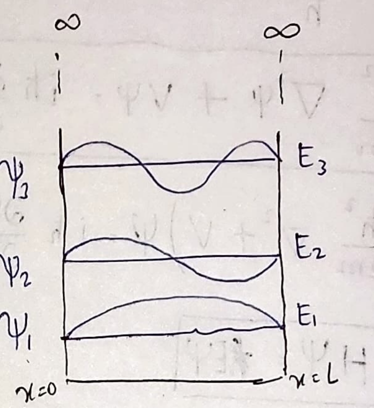
$0 = A\sin(kL)$

$\Rightarrow \boxed{kL = n\pi}$, $n \in \mathbb{Z}$

$\Rightarrow k^2L^2 = n^2\pi^2$

$\Rightarrow \dfrac{2mE}{\hbar^2} = \dfrac{n^2\pi^2}{L^2}$

$\Rightarrow \boxed{E_n = \dfrac{\hbar^2 n^2\pi^2}{2mL^2}}$



$E_1 = \dfrac{\hbar^2\pi^2}{2mL^2}$, $E_2 = 4\dfrac{\hbar^2\pi^2}{2mL^2}$, $E_3 = 9\dfrac{\hbar^2\pi^2}{2mL^2}$, ...

$\therefore$ Energy is quantized (discrete energy levels)

if $\psi \to$ normalized

then, $\int |\psi|^2 dV = 1$

$\psi = A\sin(kx)$

$k = \dfrac{n\pi}{L}$

$\psi = A\sin\left(\dfrac{n\pi x}{L}\right)$

In 1-D,

$\int |\psi|^2 dx = 1$

$\Rightarrow \int_0^L A^2 \sin^2\left(\dfrac{n\pi x}{L}\right) dx = 1$

$\Rightarrow \dfrac{A^2}{2}\int_0^L \left(1 - \cos\dfrac{2n\pi x}{L}\right) dx = 1$

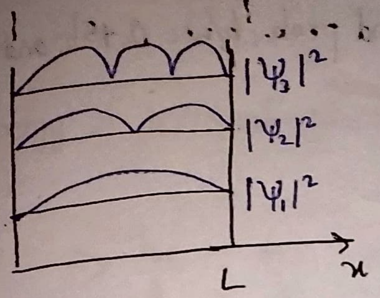
$\Rightarrow \dfrac{A^2}{2}\left[x - \dfrac{\sin\left(\frac{2n\pi x}{L}\right)}{\left(\frac{2n\pi}{L}\right)}\right]_0^L = 1$

$\Rightarrow \dfrac{A^2}{2}\left[L - \dfrac{L}{2n\pi}(0-0)\right] = 1$

$\Rightarrow A = \pm\sqrt{\dfrac{2}{L}}$

$\therefore \boxed{\psi_n = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{n\pi}{L}x\right)}$ ← normalized wave function

$\boxed{E_n = \dfrac{\hbar^2 n^2\pi^2}{2mL^2}}$



$|\psi|^2 \rightarrow$ probability of finding particle

- if $E = E_1$, then max^m probability of finding particle $= L/2$

  $E = E_2$, " " " " " " $= \frac{L}{4}, \frac{3L}{4}$

Q.) Find the probability that particle present in a box 'L' wide ~~can~~ be found b/w 0.45 L and 0.55 L for ground state and first excited state.

Sol^n: For ground state,

$$\psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right)$$

$$\int_{0.45L}^{0.55L} |\psi_1|^2 \, dx = \frac{2}{L} \int_{0.45L}^{0.55L} \sin^2\left(\frac{\pi}{L}x\right) dx$$

$$= \frac{2}{L} \int_{0.45L}^{0.55L} \left(1 - \cos\frac{2\pi x}{L}\right) dx$$

$$= \frac{2}{L} \left[(0.10L) - \frac{L}{2\pi} \left(\sin\left(\frac{2\pi}{\cancel{L}}(0.55)\cancel{L}\right) - \sin\left(\frac{2\pi}{\cancel{L}}(0.45)\cancel{L}\right) \right) \right]$$

$=$

For $n = 2$,

Q.) If $\psi = ax$ and $x=0$ to $x=1$, then find prob. b/w 0.45 and 0.55.

$$\int_0^1 |\psi|^2 dx = 1$$

$$\Rightarrow a^2 \int_0^1 x^2 dx = 1$$

$$\Rightarrow \frac{a^2}{3} = 1$$

$$\Rightarrow a = \pm\sqrt{3}$$

Now,

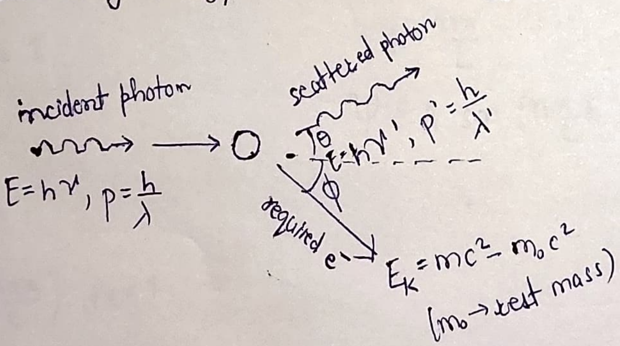
$$\int_{0.45}^{0.55} |\psi|^2 dx = 3\int_{0.45}^{0.55} x^2 dx$$

$$= \frac{3}{3}\left[x^3\right]_{0.45}^{0.55}$$

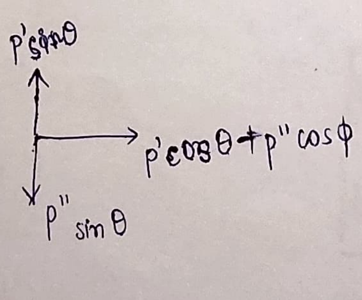
$$=$$

<u>Compton Scattering</u>

- coherent scattering : energy absorbed = energy released (higher wavelengths)
- incoherent scattering: energy absorbed > energy released (lower wavelengths)



$$\boxed{m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}}$$



A/c conservation of energy,

Energy of incident photon = Energy of scattered photon + Energy of required e^-

$$\Rightarrow \frac{hc}{\lambda} = \frac{hc}{\lambda'} + (m - m_0)c^2$$

$$\Rightarrow hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = (m - m_0)c^2$$

$$\Rightarrow h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = (m - m_0)c \quad \text{——— (I)}$$

A/c conservation of momentum,

$$\frac{h}{\lambda'}\sin\theta = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{——— (II)}$$

$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\cos\phi \quad \text{——— (III)}$$

$$\boxed{\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta)} \qquad \boxed{\frac{h}{m_0 c} \approx 0.0242\ A^\circ}$$

$\theta = 0^\circ \Rightarrow \Delta\lambda = 0$

$\theta = 90^\circ \Rightarrow \Delta\lambda = \frac{h}{m_0 c}$ $\qquad$ $\Delta\lambda \rightarrow$ compton wavelength shift

$\theta = 180^\circ \Rightarrow \Delta\lambda = \frac{2h}{m_0 c}$

* % compton shift for $\lambda = 4000\ A^\circ$ (visible region) $= \frac{(\Delta\lambda)_{max}}{\lambda} \times 100\%$

$= 0.001\%$ (Not observed)

- Significant for X-rays ($\lambda \approx 1.5\ A^\circ$)