CHAPTER 1

Properties of Matter

1.1 Introduction

The advancement in technology & Engineering is possible through our detailed understanding of properties of matter. The mechanical properties of solids generally deals with deformations due to the external forces impressed on the materials. The property, elasticity is important for many applications in Engineering & Technology. [e.g a bridge used for traffic is subjected to loads or forces of varying amounts. Before a steel bridge is constructed, the steel samples are sent for testing to find whether the steel can withstand the loads likely to be put on them or not].

The development of dynamics begins with the concept of an infinitely small but massive particles for their behaviour under the influence of applied forces. Later the idea of rigid body came into force. The behaviour of ordinary material under the action of forces constitutes the study of elasticity.

1.2 Rigid Body

A *rigid body* is one in which the relative position of its constituent particles does not change under the influence of impressed forces or in other words an ideal rigid body does not change in shape or size under the influence of applied force of any magnitude. When a body is subjected to external forces, it may get deformed. Such a force is called *Deforming Force*. As a result of deforming forces applied to a body reactionary forces come into play internally due to relative displacement of its molecules, tending to balance the load and restore the body to its original condition. The body undergoes a change in its shape or size or both. These reactionary forces tend to restore the body to its original conditions. When the deforming forces are removed, the body tends to recover its original form.

The property of a material body by virtue of which the bodies are restored to their original shape or size or both after the removal of external deforming force is called elasticity. Extent to which the original form of a body is restored, when the deforming forces are removed varies from material to material. Based on their behaviour perfect elastic and perfect plastic bodies are available.

Bodies which can recover completely their original conditions on removal of deforming forces are known as perfectly elastic bodies.	Bodies which do not show any tendency to recover their original condition & retain completely are known as perfectly plastic bodies.
They develop a definite amount of deformation which does not increase when the force is prolonged. (eg) quartz fibre	Partially regain their original form. Putty

Generally no body is perfectly elastic or plastic. Actual bodies behave between these two limit. Concept of perfectly elastic and perfectly plastic bodies is an idealization.

1.3 Stress

When a body is deformed by external force, internal reactionary forces are developed between the molecules of the body to oppose the action of the deforming force, which tend to restore the body to its original condition.

The internal restoring force developed / unit area of the body when subjected to external deforming force is called stress.

Being a disturbed force, it is measured in the same manner as fluid pressure (i.e.,) in terms of load on deforming force applied / unit area of the body, being equal in magnitude but opposite in direction to it, until a permanent change has been brought about in the body.

These forces are self adjusting forces. As deforming force increases restoring force also increases. When deforming force is equal to restoring force, body attains equilibrium. Due to this, at equilibrium, stress can be measured by the deforming force applied on a unit area of the body.

$$\text{Stress} = \frac{\text{Restoring force}}{\text{area}} = \frac{\text{Deforming force}}{\text{area}} = \frac{F}{A}$$

Unit of stress is Newton per metre^2 (N/m^2) and its dimensional formula is $[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$. The stress developed in a body depends upon how the external forces are applied over it. Depending on this three types of stress, are as follows.

Types of stress

(a) Longitudinal stress (or) Tensile stress

If the deforming force acting on a body is along its longitudinal axis and produces a change in its length, then the deforming force / unit area acting normal to the surface is called longitudinal or normal or tensile stress.

If the applied force is a thrust or produces a compression, then the stress is called normal compressive stress.

Thus, Tensile stress = force / unit area = F/A

$$\text{Tensile stress} = F/A$$

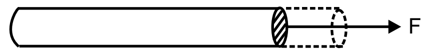


Fig. 1

(b) Volume or Bulk stress

If the equal deforming forces can be applied uniformly on each 6 faces of a cube in outward direction, then the cube suffers an increase in its volume.

Under equilibrium, the applied force/unit area is called volume or bulk stress.

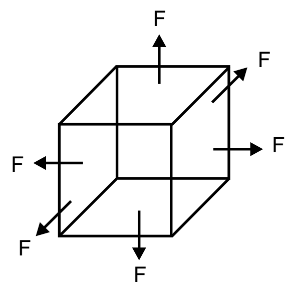


Fig. 2

(c) Shearing stress (or) Tangential stress

If the deforming forces are applied tangentially over the top surface of a cube and bottom surface being kept fixed then, the top face gets displaced towards the direction of applied force.

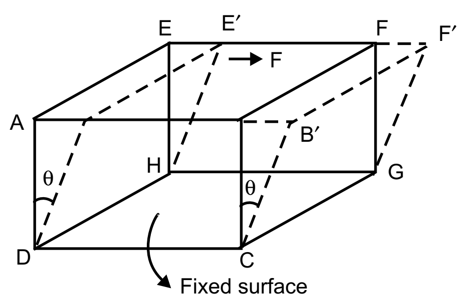


Fig. 3

The tangential force/unit area or the stress which tends to make one part of the body slide across the other part is termed as shearing stress or tangential stress.

1.4 Strain

A body under deforming forces undergoes a change in length, volume or shape. Then the body is said to be under strain. The strain produced in the body is measured in terms of the fractional change produced in the dimensions of a body. Under a system of forces in equilibrium strain is also measured as the ratio of change in dimension of the body to its original dimension. As strain is just a ratio, it is a dimensionless quantity, having no units.

Types of strain

(a) Longitudinal strain or Tensile strain

If the deforming force is of the nature of pull or a tension and acting along the longitudinal axis of a wire of length L, and produces a change in length ΔL without any change in shape, then this fractional change ΔL/L, is called longitudinal strain.

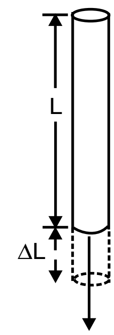


Fig. 4

$$\text{Longitudinal strain} = \frac{\text{change in length}}{\text{original length}} = \frac{\Delta L}{L}$$

If length increases from its natural length, then it is tensile strain. If in case there is decrease in length then it is compressive strain.

(b) Volume strain

When the forces or pressure are applied uniformly and normally inwards (or outwards) over the whole surface of a body of volume V, then its volume gets decreased (or increased) by an amount ΔV without any change in shape. The ratio of this change in volume to its original volume is called volume strain.

$$\text{Volume strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\Delta V}{V}$$

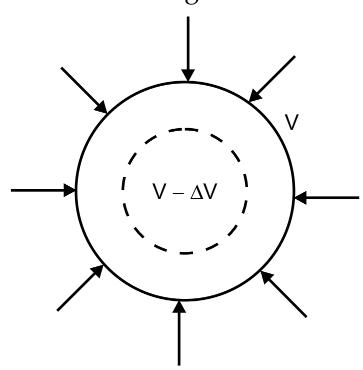


Fig. 5

(c) Shearing strain or Shear

When the deforming forces are applied tangentially over the top surface of the body, it suffers a change in shape without any change in volume or length and is said to be sheared. Shear is numerically equal to the ratio of the displacement of any layer in the direction of applied tangential force to its distance from the fixed surface.

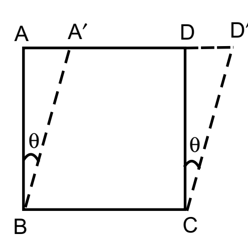


Fig. 6

The strain produced is measured by an angle which a tilted surface makes with original vertical surface.

$$\frac{AA'}{AB} = \theta$$

Strain produced by stretching or compressing force, both volume & shape of the body may alter, while the strain produced by shearing forces only the shape of the body is altered though the volume remains constant.

Elastic limit

The maximum stress which produces maximum amount of recoverable deformation is called Elastic limit. If the stress applied exceeds the elastic limit, then the substance does not return to its original state when the stress is removed. The substance is then said to have acquired permanent set.

1.5 Hooke's Law

If a substance is subjected to a stress below the elastic limit, it recovers completely when the stress is removed or within elastic limit, the stress is directly proportional to strain produced.

The linear relationship between the stresses and deformations produced below elastic limit is called Hooke's law. (i.e.,)

$$\text{Stress} \propto \text{Strain}$$

$$\frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

$$= E \rightarrow \text{Modulus of elasticity (coefficient of elasticity)}$$

The value of modulus of elasticity depends upon the type of stress and strain produced.

Young's modulus

If the strain is longitudinal then the modulus of elasticity is called "Young's Modulus" (Y).

$$Y = \frac{\text{Longitudinal stress}}{\text{Longitudinal strain}} = \frac{F/A}{\Delta L/L}$$

$$= \frac{FL}{A\Delta L}$$

Within elastic limit, the ratio of longitudinal stress to the corresponding longitudinal strain is called Young's modulus of elasticity.

Bulk modulus

When a uniform pressure (normal force) is applied normally over the whole surface of a body of an isotropic material, it suffers a change in its volume though its shape remains unchanged within elastic limit. The ratio of the volume stress to the volume strain is called bulk modulus of elasticity of the material.

$$K = \frac{\text{Volume stress}}{\text{Volume strain}} = \frac{F/A}{\Delta V/V} = \frac{PV}{\Delta V}$$

Reciprocal of K is called compressibility. Unit N/m^2 and its dimensional formula N/m^2 $[\text{ML}^{-1}\text{T}^{-2}$

Modulus of rigidity

When a body is subjected to tangential deforming force, it suffers a change in shape but volume remains unchanged. Then body is said to be sheared. The stress developed in this case is called shearing stress, due to which a shearing strain is developed.

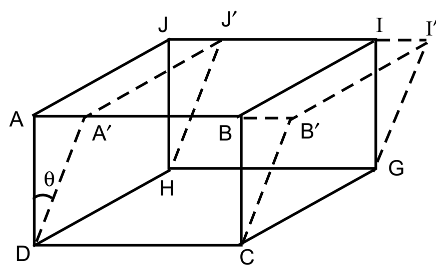


Fig. 7

Within the elastic limit, the ratio of shearing stress or tangential stress to shearing strain is called modulus of rigidity of the material.

If θ is angle of shear, then

$$\theta = \tan\theta = \frac{BB'}{BC}$$

$$= \frac{\text{displacement of top surface}}{\text{distance of top surface from fixed surface}}$$

$$n = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{F/A}{\theta}$$

Unit of n is N/m^2 and dimensional formula is $[\text{ML}^{-1}\ \text{T}^{-2}]$

1.6 Poisson's Ratio

When a wire is pulled, it not only becomes longer but also thinner. If a force produces elongation or extension in its own direction, a contraction also occurs in a direction perpendicular to it, that is in lateral direction or vice versa. The fractional change in the direction of applied force is longitudinal strain, fractional change in perpendicular direction is lateral strain.

Within elastic limit, ratio of lateral strain to longitudinal strain is constant for a given material and is called poisson's ratio (σ).

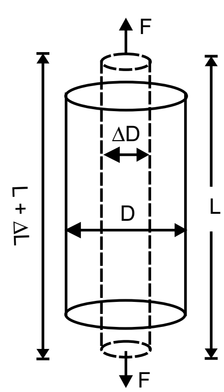


Fig. 8

In the Fig, a wire of original length L and diameter D is acted upon by two equal and opposite force F along the length. Its length increases by ΔL, while its diameter decreases by ΔD.

$$\begin{aligned}
\alpha &= \frac{\Delta L}{L}; \quad \beta = \frac{\Delta D}{D} \\
\sigma &= \frac{\beta}{\alpha} = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{-(\Delta D/D)}{\Delta L/L} \\
&= -\frac{L}{D}\,\frac{\Delta D}{\Delta L}
\end{aligned}$$

σ as a differential coefficient,

$$\sigma = \frac{L}{D}\,\frac{dD}{dL}$$

Minus sign indicates that increase in the direction of force would be accompanied by decrease in the direction perpendicular to the force. σ is dimensionless and has no units.

1.7 Relation between the Three Moduli of Elasticity (Y, K and n)

To arrive at the relation between Young's modulus, bulk modulus and rigidity modulus, the three moduli need to the expressed in terms of longitudinal strain (α) and lateral strain (β) and hence in terms of σ (Poisson's ratio) must be known.

(i) Relation between Young's modulus and $\boldsymbol{\alpha}$

$$Y = \frac{1}{\alpha}$$

(ii) Bulk modulus in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$K = \frac{1}{3(\alpha - 2\beta)}$$

(iii) Rigidity modulus in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$n = \frac{1}{2(\alpha + \beta)}$$

(iv) Relation between $\boldsymbol{Y}$, $\boldsymbol{K}$ and $\boldsymbol{\sigma}$

We know

$$\sigma = \frac{\beta}{\alpha}; \quad Y = \frac{1}{\alpha}$$

$$\text{and} \qquad K = \frac{1}{3(\alpha - 2\beta)}$$

$$\text{now} \quad K = \frac{1}{3\alpha(1-2(\beta/\alpha))} = \frac{1/\alpha}{3(1-2\sigma)} = \frac{Y}{3(1-2\sigma)} \qquad \left(\because\ Y = \frac{1}{\alpha}\right)$$

$$\text{(or)} \quad \boxed{Y = 3K(1-2\sigma)} \tag{1}$$

(v) Relation between n, Y and σ

$$\begin{aligned} n &= \frac{1}{2(\alpha+\beta)} \\ &= \frac{1}{2\alpha(1+\beta/\alpha)} = \frac{1/\alpha}{2(1+\sigma)} = \frac{Y}{2(1+\sigma)} \end{aligned}$$

$$\text{(or)} \quad \boxed{Y = 2n(1+\sigma)} \tag{2}$$

(vi) Relation between Y, K, n and σ and Relation between the three moduli of Elasticity

Now from (1) and (2)

$$Y = 3K(1-2\sigma) \text{ and } Y = 2n(1+\sigma)$$

$$\text{(or)} \qquad \frac{Y}{3K} = 1-2\sigma \tag{3}$$

$$\text{and} \qquad \frac{Y}{n} = 2(1+\sigma) \tag{4}$$

Adding the two equations (3) and (4)

$$\frac{Y}{3K} + \frac{Y}{n} = 3 \quad \text{(or)} \quad \boxed{Y = \frac{9Kn}{n+3K}}$$

$$\text{(or)} \quad \boxed{\frac{9}{Y} = \frac{1}{K} + \frac{3}{n}}$$

Now, Dividing equation (3) by (4)

$$\frac{n}{3K} = \frac{1-2\sigma}{2(1+\sigma)} \quad \text{(or)} \quad 2n(1+\sigma) = 3K(1-2\sigma)$$

$$\text{(or)} \quad 2n + 2n\sigma = 3K - 6K\sigma$$

$$\text{(or)} \quad \boxed{\sigma = \frac{3K-2n}{2n+6K} = \frac{3K-2n}{2(n+3K)}}$$

1.7.1 Limiting values of poisson's ratio

$$\text{We know} \qquad Y = 3K(1-2\sigma) \ \& \ Y = 2n(1+\sigma)$$

$$3K(1-2\sigma) = 2n(1+\sigma) \tag{5}$$

(i) If σ is positive, RHS is positive.

$\therefore$ LHS must be positive.

It will be only when

$$2\sigma < 1$$

$$\sigma < \frac{1}{2}$$

$$\boxed{\sigma < 0.5}$$

(ii) If σ is negative. LHS is positive.

$\therefore$ RHS to be positive.

$$1 + \sigma > 0 \Rightarrow \boxed{\sigma > -1}$$

$$\boxed{-1 < \sigma < \frac{1}{2}}$$

1.8 Behaviour of a Wire under an Increasing Load

Let a wire be clamped at one end & loaded at the other end gradually from zero value until the wire break down.

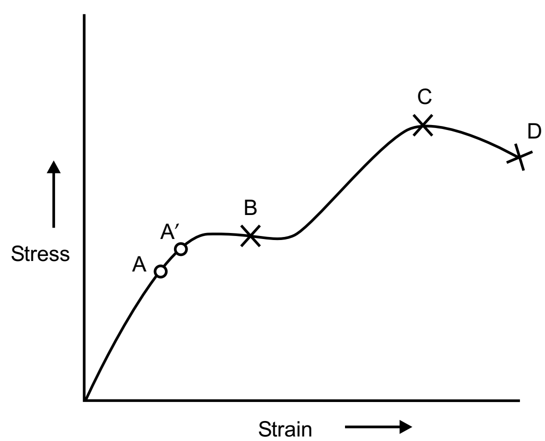


Fig. 9

A–Proportional limit; A'–Elastic limit

B–Yield point; C–Ultimate tensile strength D–Breaking stress

▶ **Solution**

$$\text{stress} = \frac{F}{A}; \quad \text{strain} = \frac{\Delta l}{l}$$

$$Y = \text{Young's modulus} = \frac{\text{stress}}{\text{strain}}$$

$$F = 10 \text{ kg} = 10 \times 1000 \times 980 \text{ dynes}$$

$$A = 1 \text{ sq.mm} = 0.01 \text{ sq.cm}$$

$$\begin{aligned} l &= 1.5 \text{ m} \\ &= 150 \text{ cm} \\ \Delta l &= 1.55 \text{ mm} \\ &= .155 \text{ cm} \end{aligned}$$

$$\text{stress} = \frac{98 \times 10^5}{0.01} = 98 \times 10^7 \text{ dynes/cm}^2$$

$$\text{strain} = \frac{\Delta l}{l} = \frac{0.155}{150} = 10.33 \times 10^{-4}$$

$$Y = \frac{98 \times 10^7}{10.33 \times 10^{-4}} = 9.49 \times 10^{11} \text{ dynes/cm}^2$$

☐ Solved Problem 2

Find the force which must act tangentially over the surface of a body of surface area 18 cm^2 in order to produce a shear of 2°. $n = 9 \times 10^{11}$ dynes/cm^2.

▶ **Solution**

$$n = \frac{F/A}{\phi} \quad \Rightarrow \quad F = nA\phi$$

$$n = 9 \times 10^{11}; \quad A = 18 \text{ cm}^2; \quad \phi = 2^\circ = 2 \times \frac{\pi}{180}$$

$$F = 9 \times 10^{11} \times 18 \times \frac{2\pi}{180} = 56.52 \times 10^{10} \text{ dynes}$$

☐ Solved Problem 3

A brass bar 1 cm square in cross section is supported on 2 knife edges 100 cm apart. A load of 1 kg at the centre of the bar depresses that point by 2.51 mm. What is Young's modulus for the bars?

▶ **Solution**

$$y = \frac{WL^3}{48YI_g} \; ; \quad I_g = \frac{bd^3}{12}$$

$$\begin{aligned} b &= d = 1 \text{ cm} \\ l &= 100 \text{ cm} \\ W &= 1 \text{ kg} = 100 \times 981 \text{ dynes} \end{aligned}$$

$$\begin{aligned} y &= 2.51 \text{ mm} \\ &= 0.251 \text{ cm} \end{aligned}$$

$$Y = \frac{12WL^3}{48ybd^3} = \frac{WL^3}{4yd^3b}$$
$$= \frac{1000 \times 981 \times 100^3}{4 \times 0.251 \times 1} = 9.77 \times 10^{11} \text{ dynes/cm}^2$$

□ Solved Problem 4

A solid cylinder of 2 cm radius weighing 200 g is rigidly connected with its axis vertical to the lower end of the fine wire. The period of oscillation of the cylinder under the influence of the torsion of the wire is 2 sec. Calculate the couple necessary to twist it through 4 complete turns.

► Solution

Period of the cylinder executing torsional vibrations is

$$T = 2\pi\sqrt{\frac{I}{c}}$$
$$I = \frac{MR^2}{2}$$
$$= \frac{200 \times 10^{-3} \times (2 \times 10^{-2})^2}{2} = 400 \times 10^{-7} \text{ kgm}^2$$
$$C = \frac{4\pi^2 I}{T^2}$$
$$= \frac{4 \times \pi^2 \times 400 \times 10^{-7}}{2^2} \text{ Nm}$$

4 complete turns are exactly equal to $4 \times 2\pi$ radian angle of twist.

$$\theta = 8\pi \text{ radian}$$

$\therefore$ required twisting couple $= C\theta$

$$= \frac{4 \times \pi^2 \times 400 \times 10^{-7}}{2^2} \times 8\pi$$
$$= 9.9 \times 10^{-3} \text{ Nm}$$

□ Solved Problem 5

Calculate the density of lead under a pressure 2×10^8 N/m^2. Density of lead is 11.4×10^3 kg/m^3. Bulk modulus of elasticity $= 8 \times 10^9$ Nm^2.

► **Solution**

$$K = \frac{-PV}{dV}$$

$$\text{mass} \quad m = V\rho$$

$$Vd\rho + \rho dV = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V}$$

$$K = \frac{P}{d\rho/\rho}$$

$$\begin{aligned} ds &= \frac{P\rho}{K} \\ &= \frac{2 \times 10^8 \times 11.4 \times 10^3}{8 \times 10^9} \\ &= 0.285 \times 10^3 \text{ kg/m}^3 \end{aligned}$$

Density under applied pressure $= \rho + d\rho$

$$\begin{aligned} &= 11.4 \times 10^3 + 0.285 \times 10^3 \\ &= 11.685 \times 10^3 \text{ kg/m}^3 \end{aligned}$$

☐ Solved Problem 6

Calculate the maximum length of the steel rod that can hang vertically without breaking. The breaking stress for steel is 8×10^6 N/m^2 and $\rho_{\text{steel}} = 8 \times 10^3$ kg/m^3.

► **Solution**

$$\begin{aligned} \text{stress} &= \frac{mg}{\text{area}} = \frac{\text{vol} \times \rho \times g}{\text{area}} \\ &= \frac{\text{area} \times \text{length} \times \rho \times g}{\text{area}} \end{aligned}$$

$$S_{\max} = l_{\max} \times \rho \times g$$

$$l_{\max} = \frac{S_{\max}}{\rho g} = \frac{8 \times 10^6}{8 \times 10^3 \times 9.8} = 102.04 \text{ m}$$

☐ Solved Problem 7

The couple / unit twist for a certain solid cylinder of radius r is 100 Nm. Calculate the contribution to this couple due to the central part up to radius $r/4$ and due to the outer most part between radii $3r/4$ and r.

► **Solution**

$$C = \frac{\pi n r^4}{2l} = 100 \text{ Nm}$$

Couple required / unit twist for an elementary cylindrical shell of radius x and thickness dx is

$$dc = \frac{2\pi n}{l} x^3 dx$$

Couple in central part (ie.) 0 to $\frac{r}{4}$

$$C' = \int dc = \frac{2\pi n}{l} \int_0^{r/4} x^3 dx$$

$$= \frac{2\pi n r^4}{4l(4)^4} = 100 \times \frac{1}{256} = 0.39 \text{ Nm}$$

Similarly for outer most part between $\frac{3r}{4}$ and r

$$C'' = \int dc = \frac{2\pi n}{l} \int_{3r/4}^{r} x^3 dx = \frac{2\pi n}{l} \left(\frac{x^4}{4}\right)^r$$

$$= \frac{2\pi n}{4l} \left(r^4 - \left(\frac{3r}{4}\right)^4\right) = \frac{\pi n r^4}{2l} \left(\frac{175}{256}\right)$$

$$= 68.3 \text{ Nm}$$

□ Solved Problem 8

If the cross section of a cantilever is rectangular with sides of length a and b and if the maximum depressions of the end of the beam for a given load and y_a and y_b respectively, when a and b are vertical show that $y_a/y_b = b^2/a^2$.

► Solution

$$y = \frac{wl^3}{3YI}\ ; \quad y_a = \frac{wl^3}{3YI_a}\ ; \quad y_b = \frac{wl^3}{3YI_b}$$

$$\frac{y_a}{y_b} = \frac{I_b}{I_a}$$

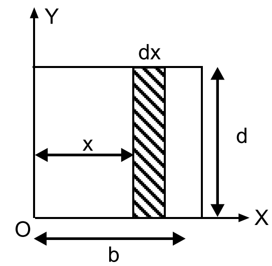
$$I_a = \frac{ba^3}{12}\ ; \quad I_b = \frac{ab^3}{12}$$

$$\frac{y_a}{y_b} = \frac{ab^3}{12} \Big/ \frac{ba^3}{12} = \frac{b^2}{a^2}$$

□ Solved Problem 9

The modulus of rigidity and poisson's ratio of the material of a wire are 2.87×10^{10} N/m and 0.379 respectively. Find the Young's modulus of the material of the wire.

Consider a strip of thickness dx and depth d as shown.



$$dA = ddx$$

$$A = \int_0^b ddx = bd.$$

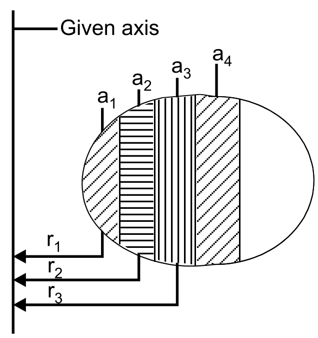
First moment of dA about Y-axis $= xdA = xddx$.

Total first moment of area about Y-axis $= \int_0^b xddx = \frac{db^2}{2}$

$$\therefore \quad \overline{x} = \frac{db^2}{2}/bd = b/2.$$

1.17.3 Moment of inertia

Consider a plane area which is split into small areas $a_1, a_2, a_3, \ldots$ Let the C.G of the small areas from a given axis be at a distance of $r_1, r_2, r_3 \ldots$



Moment of inertia of the plane area about the given axis is given by

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^3 + \ldots$$

$$\text{or} \quad I = \Sigma a r^2$$

The moment of inertia is the sum of the products of the area (or mass) and the square of the distances from the axis of rotation.

1.18 Radius of Gyration

Suppose the body consists of n particles of mass m

$$I = \Sigma m r^2 = m[r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2]$$

$$I = mn\frac{[r_1^2 + r_2^2 + r_3^2 + ...r_n^2]}{n}$$

$$I = MK^2 \text{ or } K = \sqrt{\frac{I}{M}}$$

$$\text{where} \quad K^2 = \frac{r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2}{n}$$

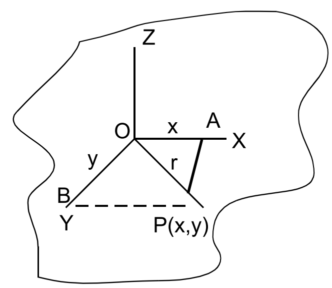
$$\text{or} \quad K = \sqrt{\frac{[r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2]}{n}}$$

K is called the **radius of gyration** and is equal to the root mean square distance of the particles from the axis of rotation.

Suppose the whole mass of the body is concentrated at a single point such that M.I of this concentrated point mass is same as the M.I of the whole body about the axis, the distance of that single point from the axis is called the radius of gyration of the body about the axis.

1.18.1 Theorem of perpendicular axis

This theorem states that the moment of inertia of a plane lamina body about an axis perpendicular to the plane is equal to the sum of moment of inertia about two mutually perpendicular axis in the plane of the lamina such that the three mutually perpendicular axis have a common point of intersection. Consider a plane lamina having the axis OX and OY in the plane of the lamina. The axis OZ passes through O and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles each of mass m. Let a particle of mass m be at P with coordinates (x, y) and situated at a distance r from the point of intersection of the axis.



$$r^2 = x^2 + y^2$$

Moment of inertia of the particle about the axis $OZ = mr^2$.

Moment of inertia of the whole lamina about the axis,

$$OZ = \Sigma mr^2 = I_z$$

Moment of inertia of the whole lamina about the axis OX

$$I_x = \Sigma my^2$$

$$\text{Similarly, } I_y = \Sigma mx^2$$

$$I = I_x + I_y$$

$$I_z = \Sigma mr^2$$

$$= \Sigma m(x^2 + y^2)$$

$$I_z = \Sigma mx^2 + \Sigma my^2$$

$$I_z = I_x + I_y$$

1.18.2 Theorem of parallel axis

It states that the moment of inertia of a plane area about an axis in the plane of area through the C.G of the plane area be represented by IG, then the moment of inertia of the given plane area about a parallel axis AB at a distance h from C-G of the area is given by,

$$I_{AB} = I_G + Ah^2$$

I_{AB} - Moment of inertia of the given area about AB.

I_G - Moment of inertia of the given area about C.G.

A - Area of the section

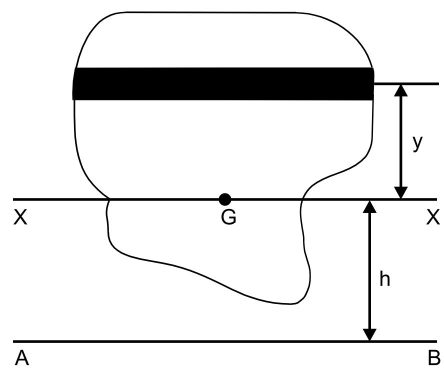
h - Distance between C-G of the section and the axis AB.

Proof A lamina of plane area A is shown in the fig.

X − X The axis in the plane of area A and passing through the C.G of the area.

AB The axis that is parallel to axis X X

Consider a strip parallel to X X at a distance y from the X X axis.



Let the area of the strip = d A

Moment of inertia of the strip about $XX = y^2dA$.

Moment of inertia of the total area about X X

$$I_G = \sigma y^2 dA$$

Moment of inertia of the area dA about AB

$$= dA(h + y)^2$$

Moment of inertia of the total area A about AB

$$\begin{aligned} I_{AB} &= \Sigma dA[h^2 + y^2 + 2hy] \\ &= \Sigma h^2 dA + \Sigma y^2 dA + \sigma 2hydA \end{aligned}$$

As h or h^2 is a constant

$$= h^2 \Sigma dA + \Sigma y^2 dA + 2h\Sigma ydA.$$

But $\Sigma dA = A$ and $\Sigma y^2 dA = I_G$

$$= h^2 A + I_G + 2h\Sigma ydA.$$

Moment of area of the strip about X-X = ydA and ΣydA represents moment of total area about X-X axis. Moment of total area about X-X axis is equal to the product of the total area A and the distance of C.G of the total area from X-X axis. Since distance of C.G of the total area from X-X axis is zero Σ ydA = 0.

$$\therefore \qquad I_{AB} = I_G + Ah^2.$$

The theorem of parallel axis states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis and the product of the area of the body and the square of distance between the two parallel axis.

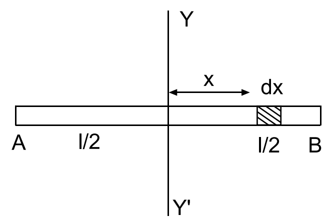
1.19 Moment of Inertia of Rigid bodies

Moment of Inertia of simple bodies can be determined as follows:

1. Take a general element.
2. Write down the expression for mass of the element and its distance from the axis.
3. Integrate the term between suitable limits such that the entire mass of the body is covered.

1.19.1 Moment of inertia of a thin uniform bar (Rod)

Consider a thin uniform bar AB of mass m and length l rotating about an axis passing through its centre and perpendicular to its length (axis YY_1)



Mass of the bar = M

Length of the bar $= l$

Mass per unit length $= \dfrac{M}{\ell}$

Take an element of length dx at a distance x from the axis

Mass of the element $= \left(\dfrac{M}{\ell}\right) dx$

Moment of inertia of the element about $YY' = \left(\dfrac{M}{\ell}\right) dx x^2$

Moment of inertia of the bar AB about the axis YY'

$$I = \int_{-l/2}^{l/2} \left(\frac{M}{l}\right) x^2 dx$$

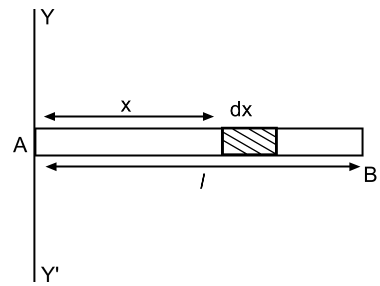
$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$= \frac{M}{l}\left[\frac{l^3}{24} + \frac{l^3}{24}\right]$$

$$= \frac{M}{l}\frac{l^3}{12} = \frac{Ml^2}{12}$$

1.19.2 About an axis at the end of the rod and normal to it

Moment of inertia of the bar about the axis YY'

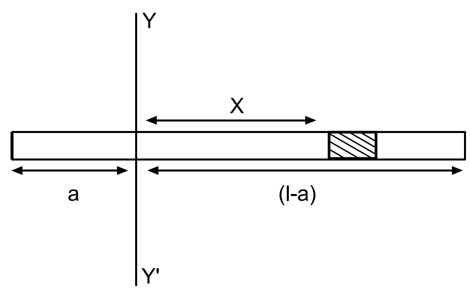


$$I = \int_0^l \left(\frac{M}{l}\right) x2dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{M}{l}\frac{l^3}{3} = \frac{Ml^2}{3}$$

1.19.3 Moment of inertia of a bar about an axis perpendicular to its length at a distance a from one end



$$I = \frac{M}{l}\int_{-a}^{l-a} x^2 dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_{-a}^{l-a}$$

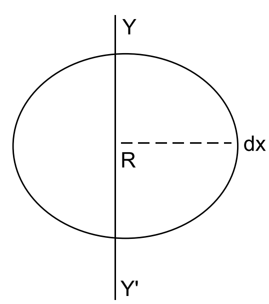
$$= \frac{M}{l}\left[\frac{(l-a)^3}{3} + \frac{a^3}{3}\right]$$

$$= \frac{M}{l}\left[\frac{l^3 - a^3 - 3l^2a + 3la^2}{3} + \frac{a^3}{3}\right]$$

$$= \frac{M}{3}[l^2 3la + 3a^2]$$

1.19.4 Moment of inertia of a ring

Consider a thin uniform ring of mass M and radius R. The ring rotates about an axis YY^1 passing through its centre.



Mass of the ring = M

Length of the ring $= 2\pi R.$

Mass per unit length $= \dfrac{M}{2\pi R}$

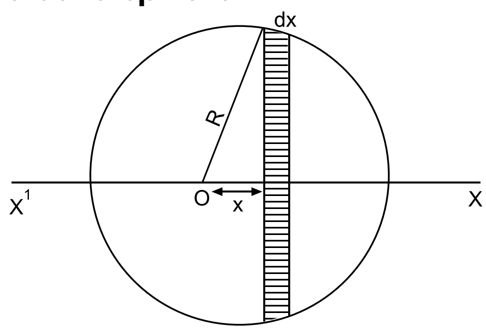
Take an element of length dx. It distance from the axis is R.

Mass of the element $= \left(\dfrac{M}{2\pi R}\right) dx$

Moment of inertia of the element about the axis $= \left(\dfrac{M}{2\pi R}\right) R^2 dx.$

Moment of inertia of the ring $I = \frac{MR}{2\pi} \int_0^{2\pi R} dx = \frac{MR}{2\pi} [x]_0^{2\pi R} = MR^2$

1.19.5 Moment of inertia of a solid sphere



About the diameter

Consider a solid sphere of radius R and mass M.

Volume of the sphere $= \frac{4}{3}\pi R^3$

Mass per unit volume $= \frac{M}{4/3\pi R^3}$

Consider an element of thickness dx at a distance x from the centre.

Radius of the element $r = \sqrt{R^2 - x^2}$

Volume of the element $= \pi r^2 dx$

$$= \pi(R^2 x^2)dx$$

Mass of the element $= \frac{M}{4/3\pi R^3}\pi(R^2 x^2)dx$

$$= \frac{3M}{4R^3}(R^2 x^2)dx$$

Moment of inertia of the element about the axis XX^1

$$= \frac{1}{2} \times \text{ mass } \times \text{ square of radius.}$$

$$= \left[\frac{3M}{4R^3}\left(R^2 - x^2\right)\right]\frac{r^2}{2}dx$$

$$= \left(\frac{3M}{8R^3}\right)(R^2 x^2)(R^2 x^2)dx$$

$$= \frac{3M}{8R^3}(R^2 x^2)^2 dx$$

Moment of inertia of the whole sphere about the axis XX^1.

$$I = 2\int_0^R \frac{3M}{8R^3}(R^2 x^2)^2 dx$$

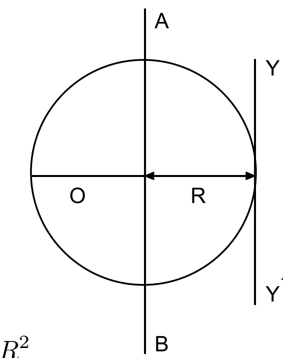
$$= \frac{3}{4}\frac{M}{R^3}\int_0^R (R^4 + x^4 - 2R^2x^2)dx$$

$$= \frac{3M}{4R^3}\left[R^4 x + \frac{x^5}{5} - \frac{2R^2x^3}{3}\right]_0^R$$

$$= \frac{3M}{4R^3}\left[R^5 + \frac{R^5}{5} - \frac{2R^5}{3}\right]$$

$$= \frac{3MR^2}{4} \times \frac{8}{15}$$

$$\frac{2MR^2}{5} = \frac{2}{5}MR^2$$



About a tangent

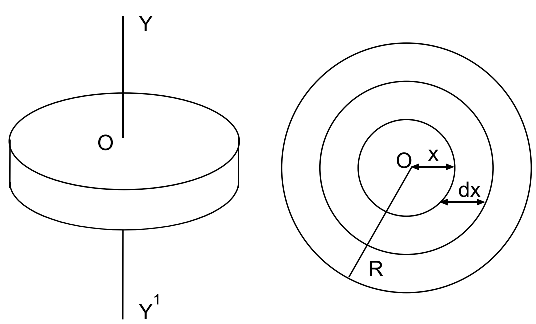
Moment of inertia of a solid sphere about a tangent is given by,

$$I_y = I_{AB} + MR^2$$

$$= \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

1.19.6 Moment of inertia of a uniform circular disc

Consider a uniform circular disc of mass M and radius R rotating about an axis passing through its centre.



Mass of the disc = M

Area of the disc $= \pi R^2$

Mass per unit area $= \dfrac{M}{\pi R^2}$

Consider a thin element of the disc of radius x and radial thickness dx.

Area of the element $= 2\pi x dx.$

Mass of the element $= \dfrac{M}{\pi R^2} 2\pi x dx$

$$= \frac{2M}{R^2} x dx$$

Moment of inertia of the element about the axis of rotation

$$= \text{mass} \times x^2$$
$$= \frac{2M}{R^2} x^3 dx$$

Moment of inertia of the whole disc about the axis of rotation $= \int\limits_0^R \frac{2M}{R^2} x^3 dx$

$$I = \frac{2M}{R^2} \frac{R^4}{4}$$
$$= \frac{1}{2} M R^2$$

Moment of Inertia of a disc about its diameter

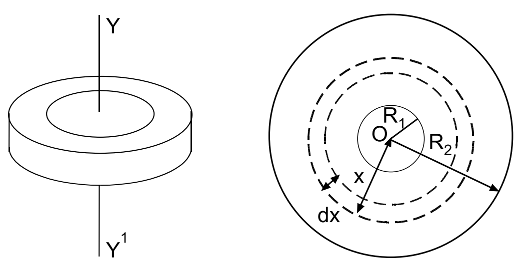
$$I = \frac{MR^2}{4}$$

where $I = I_x + I_y = 2I_x$.

$$\text{or } I_x = \frac{1}{2} \frac{MR^2/2}{2} = \frac{MR^2}{4}$$

1.19.7 Moment of inertia of an annular disc

Consider a uniform circular (annular) disc of inner radius R_1, and outer radius R_2. Let mass of the disc be M.



Area of the disc $= \pi \left(R_2^2 - R_1^2\right)$

Mass per unit area $= \dfrac{M}{\pi \left(R_2^2 - R_1^2\right)}$

Consider an element of radius x and radial thickness dx

Area of the element $= \dfrac{M}{\pi \left(R_2^2 - R_1^2\right)} 2\pi x dx$

$$= \frac{2Mxdx}{R_2^2 - R_1^2}$$

Moment of inertia of the element about an axis passing through its centre and perpendicular to the plane of the disc.

$$= \frac{2Mxdx}{R_2^2 - R_1^2}x^2$$

Moment of inertia of the whole disc about YY'

$$I = \int_{R_1}^{R_2} \frac{2Mx^3dx}{R_2^2 - R_1^2}$$

$$= \frac{2M}{R_2^2 - R_1^2}\left[\frac{x^4}{4}\right]_{R_1}^{R_2}$$

$$= \frac{2M}{(R_2 - R_1)(R_2 + R_1)}\frac{(R_2^4 - R_1^4)}{4}$$

$$= \frac{2M}{R_2^2 - R_1^2}\frac{(R_2^2 - R_1^2)(R_2^2 + R_1^2)}{4}$$

$$I = \frac{M}{2}(R_2^2 + R_1^2)$$

b) About the diameter $I = I_x + I_y = 2I_x\{I_x = I/2\}$

$$= \frac{M}{4}(R_2^2 + R_1^2)$$

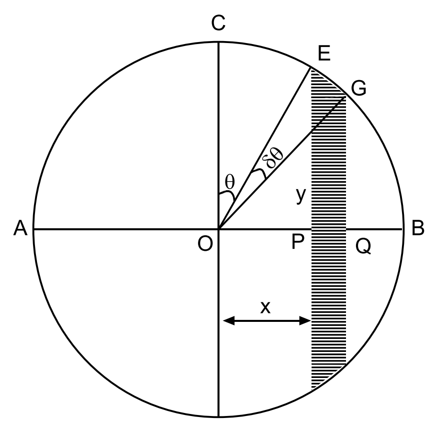
c) About a tangent in the plane of the disc.

$$I = I_X + MR_2^2$$

$$= \frac{M(R_2^2 + R_1^2)}{4} + MR_2^2$$

1.19.8 Moment of inertia of a spherical shell

Consider a spherical shell of radius R and mass M. Consider an element between two planes P and Q. The distance between the two planes is dx.



$\angle GOE = \delta q$

Then EG $= R\delta\theta$.

Radius of thin element

$y = R\cos\theta$

$x = R\sin\theta dx = R\cos\theta d\theta$

Surface area of the element $= 2py(EG)$

$= 2\pi R\cos\theta R d\theta$

$$= 2\pi R^2 \cos\theta d\theta$$
$$= 2\pi R dx$$

Mass per unit area of the shell $= \dfrac{M}{4\pi R^2}$

Mass of the element $= \dfrac{M}{4\pi R^2} 2\pi R dx$

$$= \frac{M dx}{2R}$$

Moment of inertia of the element about the diameter AB

$$\frac{M dx}{2R} y^2 = \frac{M dx}{2R}(R^2 - x^2)$$

Moment of inertia of the whole shell about the diameter

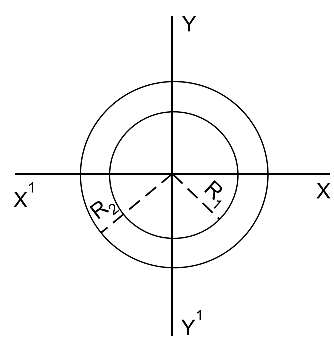
$$= 2\int_O^R \frac{M dx}{2R}(R^2 - x^2)$$
$$= \frac{M}{R}\left[R^2 x - \frac{x^3}{3}\right]_0^R$$
$$= \frac{M}{R}\left[R^3 - \frac{R^3}{3}\right] = \frac{2}{3}MR^2$$

About the tangent

$$I + MR^2 = \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2$$

1.19.9 Moment of inertia of a hollow sphere

Consider a hollow sphere of inner radius R_1 and outer radius R_2



Mass of the sphere = M

Let the density of the material $= \rho$

$$M = \frac{4}{3}\pi(R_2^3 - R_1^3)\rho \qquad (32)$$

Moment of inertia of the hollow sphere

$$I = \frac{2}{5}(M_1 R_2^2 - M_2 R_1^2)$$
$$I = \frac{2}{5}\left(\frac{4}{3}\pi R_2^3 \rho R_2^2 - \frac{4}{3}\pi R_1^3 \rho R_1^2\right)$$
$$= \left(R_2^5 - R_1^5\right)\pi\rho$$
$$\rho = \frac{M}{{}^4\!/_3\pi(R_2^3 - R_1^3)} \text{from equation (1)}$$

$$I = \frac{\frac{2}{5}M(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)}$$

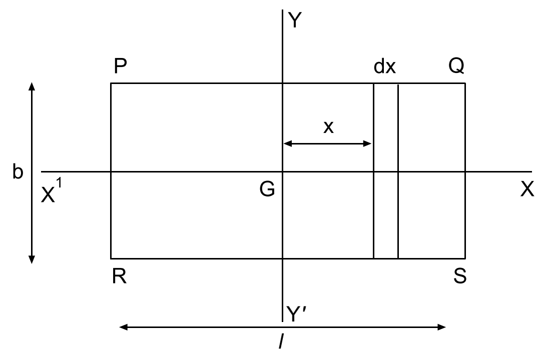
when $R_1 = 0; R_2 = R$

$$I = \frac{2}{5}MR^2$$

1.19.10 Moment of inertia of a rectangular plate

Consider a rectangular plate of uniform thickness, length and breadth be l and b respectively.

Consider an element of length dx at a distance x from YY'.



Mass of the plate = M

Area of the plate $= l \times b$

Mass per unit area $= \dfrac{M}{l \times b}$

Area of the element $= bdx$

Mass of the element $= \dfrac{M}{lb}bdx$

$$= \frac{M}{l}dx$$

Moment of Inertia of the element about the axis $YY' = \dfrac{M}{l}dxx^2$

Moment of inertia of the whole plate about the axis YY'

$$I_y = \frac{M}{l}\int_{-l/2}^{l/2} x^2 dx$$

$$= \frac{M}{l}\left[\frac{l^3}{24} + \frac{l^3}{24}\right] = \frac{Ml^2}{12}$$

Similarly about XX'

$$I_x = \frac{Mb^2}{12}$$

Moment of Inertia of the plate about an axis ZZ' passing through the centre of gravity (perpendicular axis theorem).

$$I_Z = I_x + I_y$$

$$I_z = \frac{M}{12}(l^2 + b^2)$$

Moment of Inertia of the plate about an axis PR

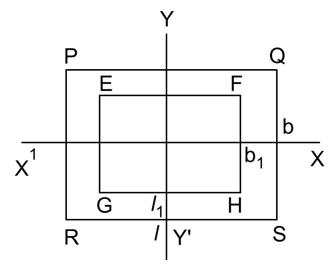
$$I_{PR} = I_y + M\left(\frac{l}{2}\right)^2$$
$$= \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$$

Moment of Inertia of the plate about an axis PQ.

$$I_{PQ} = I_x + M\left(\frac{b}{2}\right)^2$$
$$= \frac{Mb^2}{3}$$

1.19.11 Moment of inertia of hollow rectangular section

Moment of Inertia of solid rectangular about X-X axis



$$I_x = \frac{Mb^2}{12}$$

Moment of Inertia of section EFGH about X-X axis

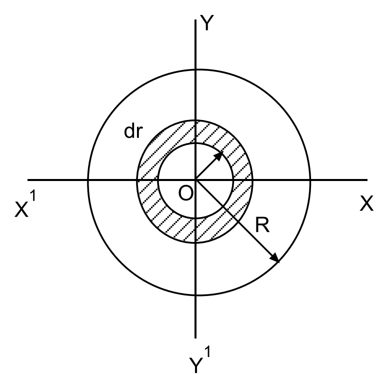
$$= \frac{Mb_1^2}{12}$$

Moment of Inertia of hollow rectangular section about X-X axis

$$= \frac{Mb^3}{12} - \frac{Mb_1^3}{12}$$

1.19.12 Moment of inertia of a circular section

Let us consider a circular section of radius R and O as centre. Consider an elementary circular ring of radius r and thickness dr.



Area of elementary circular ring = 2πr dr

Moment of inertia of the elementary circular ring about an axis passing through O and perpendicular to the plane of the paper

Moment of Inertia $= 2\pi$r dr. r^2

$= 2\pi$ r^3 dr.

Moment of Inertia of the whole circular section about an axis passing through O and perpendicular to the plane of the paper is given by

$$\int_0^R 2\pi r^3\ dr = 2\pi\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2M}{l} \int_0^{l/2} \left[\frac{R^2}{4} + x^2 \right] dx$$

$$= \frac{2M}{\ell} \left[\frac{R^2}{4} x + \frac{x^3}{3} \right]_0^{l/2}$$

$$= \frac{2M}{\ell} \left[\frac{R^2}{4} . \frac{l}{2} + . \frac{l^3}{24} \right]$$

$$= \frac{M}{l} \left[\frac{R^2}{4} l + \frac{l^3}{12} \right]$$

$$= \frac{M}{12l} \left[3R^2 l + l^3 \right]$$

$$= M \left[\frac{R^2}{4} + \frac{l^2}{12} \right]$$

1.20 Oscillations of Rigid Bodies

A mechanical vibration is the motion of a particle or a body which oscillates about a position of equilibrium. A mechanical vibration generally results when a system is displaced from a position of stable equilibrium. The time interval required for the system to complete a full cycle of motion is called the period of vibration. The number of cycles per unit time defines the frequency, and the maximum displacement of the system from its position of equilibrium is called the amplitude of vibration.

When a body vibrates under the action of restoring forces only, the motion is called a free vibration. If resisting forces are also present, the motion is called damped free vibration. If the vibrations are caused by a periodic force applied to the body, the motion is called a forced vibration which may or may not be damped.

1.20.1 Simple pendulum

A simple pendulum displaced slightly from its equilibrium position executes oscillation.

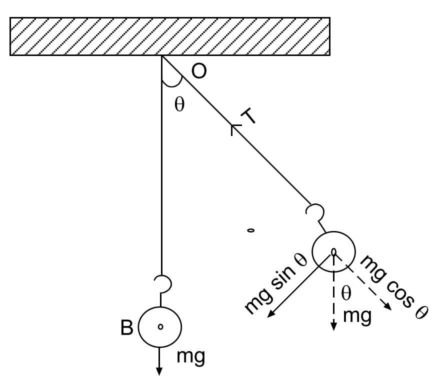


Figure shows a simple pendulum displaced to a position A by a small angle q from its mean position B. At the position A, the weight mg of the bob acts vertically downwards and tension T in the string acts along the string upwards. The weight mg can be reduced into two mutually perpendicular components.

1. The component mg $\cos\theta$, along the string, is balanced by the tension T in the string.

   $\therefore$ mg $\cos\theta$ = T.

2. The component mg $\sin\theta$, along the direction perpendicular to the string is unbalanced. This force on the bob is directed towards the mean position and under the influence of this force the bob moves towards the mean position.

The displacement AB = x = $l\theta$ where l is the effective length of the pendulum and θ is very small.

The restoring force acting on the oscillating particle = mg $\sin\theta$

For small angular displacement θ, $\sin\theta = \theta$.

$\therefore$ Restoring force F = mg θ.

(ve sign indicates restoring force is opposite to the direction of displacement).

Substituting the value for θ

$$F = \frac{-mg}{\ell}x$$

Since the restoring force is directly proportional to the displacement and is directed opposite to the displacement, the oscillations of simple pendulum are simple harmonic.

$$\therefore \text{ Force constant } K = \frac{\text{Force}}{\text{displacement}}$$
$$= \frac{F}{x} = \frac{mg\theta}{\ell\theta} = \frac{mg}{\ell}$$

Time period of the pendulum

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi\sqrt{\frac{m}{mg/\ell}}$$
$$= 2\pi\sqrt{\frac{\ell}{g}}$$

Frequency of oscillation of simple pendulum

$$v = \frac{1}{T}$$
$$= \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

The time period does not depend on the mass of the body but only on the length ℓ of the pendulum.

CHAPTER 1

Oscillations and Waves

The phenomenon of wave motion is prevalent in almost all branches of Physics. Vibrations and wave motions are intimately related topics. Waves have as their source a vibration. In this chapter, we shall discuss certain aspects of vibration followed by the ideas related to travelling waves, wave equation and solution of one and three dimensional problems.

1.1 HARMONIC OSCILLATION—THE BASICS

A motion that is repeated at regular intervals of time is called **periodic motion**. The solutions of the equations of motion of such systems can always be expressed as functions of sines and cosines. Motions described by functions of sines and cosines are often referred to as **harmonic motion**. If the motion is back and forth over the same path, it is called **vibratory or oscillatory**. As oscillation is one round trip of the motion, period T is the time required for one oscillation. The number of oscillations per unit time is the frequency f of oscillation. Hence,

$$T = \frac{1}{f} \tag{1.1}$$

The distance, linear or angular, of the oscillating particle from its equilibrium position is its displacement (linear or angular). The maximum displacement is called the **amplitude** A of the motion.

1.2 SIMPLE HARMONIC MOTION (SHM)

The motion of a particle is said to be **simple harmonic** if it oscillates from an equilibrium position under the influence of a force that is proportional to the distance of the particle from the equilibrium position. Also, the force is such that it directs the particle back to its equilibrium position.

1.2.1 Equation of Motion

A familiar example of a particle executing SHM is the motion of a mass m attached to an elastic spring moving on a horizontal frictionless table. When the spring is stretched a distance x from the unextended position, the restoring force F acting on the mass is given by

$$F = -kx \tag{1.2}$$

where k is the proportionality constant, but in the present case [Eq. (1.2)] it is the spring constant k of the spring. The negative sign indicates that the force is directed against the motion, which is towards the equilibrium position. The acceleration a of the mass is $\left(\dfrac{d^2x}{dt^2}\right)$. Applying Newton's second law, we get

$$-kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{1.3}$$

Equation (1.3) is the differential equation of motion of a particle executing SHM.

1.2.2 Solution of Equation of Motion

We can rewrite Eq. (1.3) as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x \tag{1.4}$$

where
$$\omega^2 = \frac{k}{m} \tag{1.5}$$

Multiplying both sides of Eq. (1.4) by $2\left(\dfrac{dx}{dt}\right)$, we get

$$2\frac{dx}{dt}\,\frac{d^2x}{dt^2} = -\,\omega^2 2x\,\frac{dx}{dt}$$

Integrating with respect to t

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2x^2 + C$$

where C is a constant.

At the maximum displacement position $x = A$ and velocity $\left(\frac{dx}{dt}\right) = 0$. Hence, $C = \omega^2 A^2$ and

$$\frac{dx}{dt} = \omega\sqrt{(A^2 - x^2)} \tag{1.6}$$

It may be noted from Eqs. (1.4) and (1.6) that in harmonic motion, neither the acceleration nor the velocity of the particle is constant. Rewriting Eq. (1.6) as

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt$$

and integrating

$$\sin^{-1}\frac{x}{A} = \omega t + \phi$$

where ϕ is a constant

$$x = A\, \sin(\omega t + \phi) \tag{1.7}$$

where A is the amplitude and ϕ is a constant.

First let us have a physical significance of ω. If time t in Eq. (1.7) is increased by $\frac{2\pi}{\omega}$,

$$\begin{aligned} x &= A\, \sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right] \\ &= A\, \sin(\omega t + 2\pi + \phi) \\ &= A\, \sin(\omega t + \phi) \end{aligned}$$

That is, the function simply repeats itself after a time $\frac{2\pi}{\omega}$. Hence, $\frac{2\pi}{\omega}$ is the period of oscillation T. Since $\omega^2 = \frac{k}{m}$,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{1.8}$$

The period of a simple harmonic motion is, thus, independent of the amplitude of motion. Frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1.9}$$

The quantity ω is often called the **angular frequency**. The quantity $(\omega t + \phi)$ is called the **phase** of the motion. The constant ϕ is called the phase constant. If we start counting time when the particle is in its mean position, that is when $x = 0$, $t = 0$ we have from Eq. (1.7) $A \sin \phi = 0$ or $\phi = 0$.

Hence,

$$x = A \sin \omega t$$

If we start counting time when the particle is in one of its extreme positions, that is when

$$x = A, \quad t = 0 \text{ then,}$$

$$A = A \sin \phi \quad \text{or} \quad \sin \phi = 1 \quad \text{or} \quad \phi = \frac{\pi}{2}$$

Then $$x = A \sin\left(\omega t + \frac{\pi}{2}\right)$$

If we put $\phi = \phi' + \dfrac{\pi}{2}$, we get

$$x = A \sin\left(\omega t + \phi' + \frac{\pi}{2}\right)$$

$$= A \cos(\omega t + \phi')$$

Thus, a SHM may be expressed either in terms of a sine or a cosine function. Only the phase constants will have different values in the two cases. What we discussed above is applicable to any body executing simple harmonic motion.

1.2.3 Energy in SHM

Since force is the negative gradient of potential V

$$\therefore \qquad -kx = -\frac{dV}{dx} \quad \text{or} \quad dV = kx\,dx$$

On integration, we get

$$V = \frac{1}{2}kx^2 + C$$

Taking the equilibrium position as the zero of potential energy, $C = 0$, we obtain

$$V = \frac{1}{2}kx^2 \tag{1.10}$$

Total energy E is given by

$$E = \text{kinetic energy} + \text{potential energy}$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

At the extreme positions, the velocity is zero and $x = A$. Hence,

$$E = \frac{1}{2}kA^2 \tag{1.10a}$$

At the equilibrium position x will be equal to 0 and the velocity will be maximum, say υ_0. Consequently

$$E = \frac{1}{2}m\upsilon_0^2 \tag{1.11}$$

From Eqs. (1.10a) and (1.11) we get

$$\upsilon_0 = A\sqrt{\frac{k}{m}} = \omega A \tag{1.11a}$$

1.3 DAMPED HARMONIC OSCILLATION

A harmonic oscillator in which the motion is damped by the action of an additional force, is said to be a **damped harmonic oscillator**. In most of the cases, the damping force is proportional to its velocity, that is the additional frictional force = $b\left(\frac{dx}{dt}\right)$, where the constant b is called the **damping constant**. Including the damping force, the differential equation of a damped harmonic oscillator can be written as

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.12}$$

Replacing $\frac{k}{m}$ by ω^2 and writing $\frac{b}{m}$ = 2λ, Eq. (1.12) takes the form

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \tag{1.13}$$

Writing $\left(\frac{d^2}{dt^2}\right) = D^2$ and $\left(\frac{d}{dt}\right) = D$, the auxiliary equation of Eq. (1.13) is

$$D^2 + 2\lambda D + \omega^2 = 0 \tag{1.14}$$

The roots of this equation are

$$D = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{(\lambda^2 - \omega^2)} \tag{1.15}$$

Next, let us examine the nature of the solutions for different cases.

Case I $\lambda > \omega$: The roots of the auxiliary equation α_1 and α_2 are then real and distinct.

$$\alpha_1 = -\lambda + \sqrt{(\lambda^2 - \omega^2)} \quad ; \quad \alpha_2 = -\lambda - \sqrt{(\lambda^2 - \omega^2)} \tag{1.16}$$

The solution of Eq. (1.13) is then

$$x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} \tag{1.17}$$

where c_1 and c_2 are constants and α_1 and α_2 are given by Eq. (1.16). To determine c_1 and c_2, let the mass is given a displacement x_0 and then released, so that $x = x_0$, $\left(\frac{dx}{dt}\right) = 0$ at $t = 0$. Hence, from Eq. (1.17), we get

$$x_0 = c_1 + c_2 \qquad \text{and} \qquad c_1\alpha_1 + c_2\alpha_2 = 0$$

Simplifying

$$c_1 = -\frac{x_0\alpha_2}{\alpha_1 - \alpha_2} \qquad \text{and} \qquad c_2 = \frac{x_0\alpha_1}{\alpha_1 - \alpha_2} \tag{1.18}$$

Consequently, Eq. (1.17) reduces to

$$x = \frac{x_0}{\alpha_1 - \alpha_2}(\alpha_1 e^{\alpha_2 t} - \alpha_2 e^{\alpha_1 t}) \tag{1.19}$$

which shows that x is always positive and decreases to zero as $t \to \infty$. The motion is non-oscillatory, and is therefore, referred to as **over damped** or **dead beat** motion.

Case II $\lambda = \omega$: The roots of the auxiliary equation are real and equal, each being $-\lambda$. The general solution of Eq. (1.13) is then

$$x = (c_1 + c_2 t)\, e^{-\lambda t} \tag{1.20}$$

As in Case I, if $x = x_0$ and $\left(\frac{dx}{dt}\right) = 0$ at $t = 0$ the constants $c_1 = x_0$ and $c_2 = \lambda x_0$. Hence, the solution of Eq. (1.13) is

$$x = x_0\,(1 + \lambda t)e^{-\lambda t} \tag{1.21}$$

Again, x is always positive and decreases to zero as $t \to \infty$, The nature of motion is similar to that of Case I and it will be clear that Case II separates the non-oscillatory motion of Case I from the interesting oscillatory motion of Case III. Hence, motion in Case II is termed as **critically damped.**

Case III $\lambda < \omega$: When $\lambda < \omega$, the roots of the auxiliary equation are

$$-\lambda \pm i\sqrt{(\omega^2 - \lambda^2)} = -\lambda \pm i\omega' \tag{1.22}$$

where
$$\omega' = \sqrt{\omega^2 - \lambda^2} \tag{1.23}$$

That is, the roots of the auxiliary equation are imaginary and the solution of Eq. (1.13) is

$$x = e^{-\lambda t}(c_1 \cos\omega' t + c_2 \sin\omega' t) \tag{1.24}$$

As in Case I, $x = x_0$, $\frac{dx}{dt} = 0$ at $t = 0$ leads to $c_1 = x_0$ and $c_2 = \frac{\lambda x_0}{\omega'}$. Consequently,

$$x = x_0 e^{-\lambda t}\left(\cos\omega' t + \frac{\lambda}{\omega'}\sin\omega' t\right) \tag{1.25}$$

$$= x_0\, e^{-\lambda t}(\cos\omega' t\cos\delta + \sin\omega' t\sin\delta) \tag{1.26}$$

where
$$\cos\delta = 1, \quad \sin\delta = \frac{\lambda}{\omega'} \quad \text{or} \quad \tan\delta = \frac{\lambda}{\omega'} \tag{1.27}$$

Now, Eq. (1.26) takes the form

$$x = x_0\, e^{-\lambda t}\cos(\omega' t - \delta) \tag{1.28}$$

where
$$\lambda = \frac{b}{2m}, \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{\lambda}{\omega'}\right) \tag{1.29}$$

The motion is oscillatory with an amplitude $x_0 \exp(-\lambda t)$ which decreases with $t \to 0$. From Eq. (1.29) it is obvious that when damping force is present, ω' is less than ω, the angular frequency of the undamped wave. Hence, the period T' is longer when friction is present. Thus, the effect of damping is to increase the period of oscillation and the plot of the displacement x as a function of the time t is shown in Fig. 1.1. Again the theory we discussed is applicable to any body executing a damped harmonic oscillation.

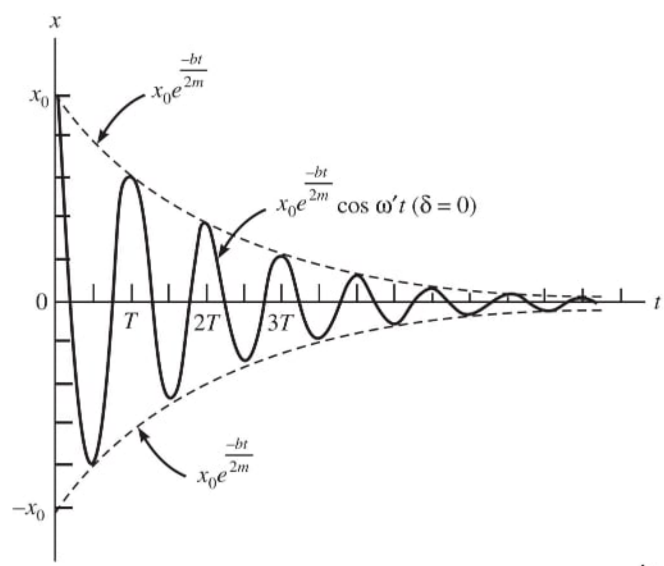


Fig. 1.1 Damped harmonic motion. The amplitude $x_0\, e^{-\lambda t}$, where $\lambda = \frac{b}{2m}$ is also plotted.

1.4 FORCED OSCILLATIONS AND RESONANCE

In the previous sections we have been discussing the natural oscillations that occur when a body of mass m is displaced and released. In the absence of damping force the natural frequency of vibration is ω (Eq. 1.5) and when damping is present it is ω' (Eq. 1.23). We shall now investigate the situation when the body is subjected to an oscillatory external force $F_0 \sin \omega'' t$. The equation of motion of such a forced oscillator is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin \omega'' t \tag{1.30}$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = f_0 \sin \omega'' t \tag{1.31}$$

where $\omega = \left(\sqrt{\frac{k}{m}}\right)$ is the natural frequency of the body, and $2\lambda = \left(\frac{b}{m}\right)$ is the damping constant for unit mass and $f_0 = \frac{F_0}{m}$.

When external periodic force is applied, both the damping and forced terms contribute to the motion of the oscillator and a tussle occurs between the contributions of both. After some initial erratic situations, ultimately, the system reaches a steady state. The oscillations that result are called the **forced oscillations**. These forced oscillations have the frequency of the external force and not that of the natural frequency of the body. Let us consider a solution of the type when the steady state is reached,

$$x = A \sin(\omega'' t - \theta) \tag{1.32}$$

for the equation of motion, Eq. (1.31). Then

$$\frac{dx}{dt} = A\omega'' \cos(\omega'' t - \theta)$$

$$\frac{d^2x}{dt^2} = -A\omega''^2 \sin(\omega'' t - \theta)$$

Substituting these values in Eq. (1.31), we get

$$-A\,\omega''^2 \sin(\omega'' t - \theta) + 2\lambda A\,\omega'' \cos(\omega'' t - \theta) + \omega^2 A \sin(\omega'' t - \theta) = f_0 \sin[(\omega'' t - \theta) + \theta]$$

where we have added and subtracted θ in the right hand side term. Expanding the right side, we obtain

$$-A\omega''^2 \sin(\omega'' t - \theta) + 2\lambda A\omega'' \cos(\omega'' t - \theta) + \omega^2 A \sin(\omega'' t - \theta)$$

$$= f_0 \sin(\omega'' t - \theta)\cos\theta + f_0 \cos(\omega'' t - \theta)\sin\theta$$

$$(-A\omega''^2 - f_0\cos\theta + \omega^2 A)\sin(\omega'' t - \theta) + (2\lambda A\omega'' - f_0 \sin\theta)\cos(\omega'' t - \theta) = 0 \tag{1.33}$$

For this equation to hold good for all values of t, the coefficients of the terms $\sin(\omega''t - \theta)$ and $\cos(\omega''t - \theta)$ must vanish separately.

i.e. $$-A\omega''^2 - f_0\cos\theta + \omega^2 A = 0 \quad \text{and} \quad 2\lambda A\omega'' - f_0\sin\theta = 0 \tag{1.34}$$

$$-A\omega''^2 + \omega^2 A = f_0\cos\theta \quad \text{and} \quad 2\lambda A\omega'' = f_0\sin\theta \tag{1.35}$$

Squaring two Eqs. (1.34) and (1.35) and adding, we get

$$A^2[(-\omega''^2 + \omega^2)^2 + 4\lambda^2\omega''^2] = f_0^2$$

$$A = \frac{f_0}{\sqrt{(\omega^2 - \omega''^2)^2 + 4\lambda^2\omega''^2}} \tag{1.36}$$

which gives the amplitude of the forced oscillation. From Eq. (1.35), we obtain

$$\tan\theta = \frac{2\lambda\omega''}{\omega^2 - \omega''^2} \tag{1.37}$$

This gives the phase difference between the forced oscillator and the applied force. Substituting the value of A in Eq. (1.32), we obtain

$$x = \frac{f_0}{\sqrt{(\omega^2 - \omega''^2)^2 + 4\lambda^2\omega''^2}}\sin(\omega''t - \theta) \tag{1.38}$$

That is, the system is undamped and vibrate with the frequency of the driving force ω'', but with a phase lag of θ given by Eq. (1.37).

The denominator of the expression for amplitude A is large when the frequency of the driving force ω'' is very different from the natural frequency ω of the undamped system. That means that the amplitude of the resulting motion is small. However, as the driving frequency ω'' approaches the natural frequency ω, the amplitude increases and reaches the maximum value when both are nearly equal. The phenomenon is called **resonance** and the frequency ω'' which gives the maximum amplitude is called the **resonant frequency**. The amplitude of the forced vibration depends on the damping force also. The greater the damping force, larger is the denominator in Eq. (1.43), resulting in a smaller amplitude. Figure 1.2 shows five curves giving the amplitude of the forced vibration as a function of $\frac{\omega''}{\omega}$. Each one corresponds to a different value of damping constant λ.

The solutions of forced vibrations is useful in acoustic systems and alternating current circuits as well as in mechanics.

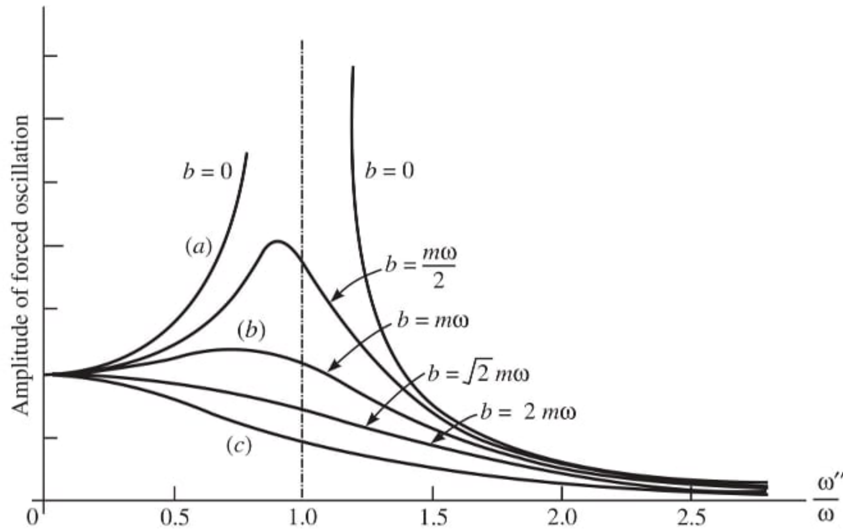


Fig. 1.2 Amplitude of a forced harmonic oscillator versus $\frac{\omega''}{\omega}$ for five values of b.

In Fig. 1.2, curve (a) is for zero damping and, curve (b) is for intermediate damping and curve (c) is for high damping.

1.5 QUALITY FACTOR

Quality factor decides the quality of the oscillator as far as damping is concerned. For the damped harmonic oscillator, the displacement x is given by Eq. (1.28). The velocity of the particle at a particular instant

$$\frac{dx}{dt} = -x_0\, e^{-\lambda t}\, [\lambda \cos(\omega' t - \delta) + \omega' \sin(\omega' t - \delta)]$$

Kinetic energy (K.E.) of the oscillator is given by

$$\text{K.E.} = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2} m\, x_0^2\, e^{-2\lambda t}\, [\lambda \cos(\omega' t - \delta) + \omega' \sin(\omega' t - \delta)]^2 \tag{1.39}$$

When expanded, the expression will have a term in $\cos^2(\omega' t - \delta)$ another term in $\sin^2(\omega' t - \delta)$ and a cross term. The average value of the terms $\cos^2(\omega' t - \delta)$ and $\sin^2(\omega' t - \delta)$ over a period = $\frac{1}{2}$. The contribution from the cross term over a period is zero. Hence, average kinetic energy of the particle over one cycle at the given instant t is

$$\text{K.E.} = \frac{1}{2} m x_0^2\, e^{-2\lambda t} \left(\frac{\lambda^2}{2} + \frac{\omega'^2}{2} \right) \tag{1.40}$$

$$\cong \frac{1}{4} m x_0^2\, e^{-2\lambda t} \omega'^2 \tag{1.41}$$

since $\lambda^2 \ll \omega'^2$. Using Eqs. (1.10) and (1.5), the potential energy (P.E.) of the oscillating particle at a given time t is

$$\text{P.E.} = \frac{1}{2}\, m\omega'^2 x_0^2\, e^{-2\lambda t} \cos^2(\omega' t - \delta) \tag{1.42}$$

Average P.E. over a time period $= \dfrac{1}{4}\, m\omega'^2 x_0^2 e^{-2\lambda t}$ (1.43)

Total energy

$$\text{E} = \text{K.E.} + \text{P.E.} = \frac{1}{2}\, m\omega'^2 x_0^2 e^{-2\lambda t} \tag{1.44}$$

Power dissipation P is rate of loss of total energy with time

$$P = -\frac{dE}{dt} = m x_0^2\, \omega'^2 \lambda\, e^{-2\lambda t}$$

$$= 2\lambda E \tag{1.45}$$

Quality factor Q is defined by

$$Q = \frac{2\pi \times \text{Energy of the oscillator}}{\text{Energy lost per cycle}}$$

$$= \frac{2\pi E}{\left(\dfrac{\text{Energy lost}}{s}\right) \times T} = \frac{2\pi E}{2\lambda E T} = \frac{\pi}{\lambda T}$$

Since $T = \dfrac{2\pi}{\omega'}$

$$Q = \frac{\pi\omega'}{\lambda 2\pi} = \frac{\omega'}{2\lambda} \tag{1.45a}$$

The quality factor Q will be large if the damping coefficient λ is small and Q will be small if λ is large. In other words, quality factor represents the efficiency of the oscillator.

1.6 TYPES OF WAVES

When we throw a stone in a pool of water, waves form and travel outward. When we shout, sound waves are generated and travel in all directions. Light waves are generated and travel outward when an electric bulb is switched on. In all these cases, there is a transfer of energy in the form of waves. Different types of waves exist in the different branches of Physics. We shall discuss some of these in the next sections.

Maximum velocity is when the partcile is in its mean position

Maximum velocity $= \omega A = (20\pi\,\mathrm{s}^{-1})\,4\ \mathrm{cm}$

$= 251.2\ \mathrm{cm\,s}^{-1}$

Example 1.3 A damped vibrating system, starting from rest, reaches the first amptitude of 40 cm which reduces to 4 cm in that direction after 100 oscillations. If the period of each oscillation is 2.5 s, find the damping constant.

Solution: The amplitude of vibration $= x_0\, e^{-\lambda t}$

From the mean position, the time taken to reach the first amplitude is $\frac{T}{4}$. Hence

Amplitude

$$A_1 = x_0\, e^{-\lambda \frac{T}{4}}$$

$$A_2 = x_0\, e^{-\lambda\left(\frac{T}{4}+T\right)}$$

$$A_{n+1} = x_0\, e^{-\lambda\left(\frac{T}{4}+nT\right)}$$

$$\frac{A_1}{A_{n+1}} = e^{n\lambda T}$$

$$\frac{40\,\mathrm{cm}}{4\,\mathrm{cm}} = e^{100\lambda T} \quad \text{or} \quad \ln 10 = 100\,\lambda T$$

$$\lambda = \frac{\ln 10}{100 \times 2.5} = 0.92 \times 10^{-2}$$

Damping constant $\lambda = 0.92 \times 10^{-2}$

Example 1.4 A particle executing SHM along a straight line has velocity 16 cm/s and 12 cm/s when passing through points 3 cm and 4 cm from the mean position, respectively. Find (i) the amplitude and (ii) period of oscillation.

Solution: From Eq. (1.6), we get

$$\upsilon = \omega\sqrt{A^2 - x^2} \tag{i}$$

$$16\,\mathrm{cm\,s}^{-1} = \omega\sqrt{A^2 - 9} \tag{ii}$$

$$12\,\mathrm{cm\,s}^{-1} = \omega\sqrt{A^2 - 16} \tag{iii}$$

Dividing Eq. (ii) by Eq. (iii), we get

$$\frac{4}{3} = \frac{\sqrt{A^2 - 9}}{\sqrt{A^2 - 16}}$$

$$\frac{A^2 - 9}{A^2 - 16} = \frac{16}{9} \quad \text{or} \quad 7\,A^2 = 175$$

$$A = 5 \text{ cm}$$

From Eq. (ii), we get

$$16\,\text{cm s}^{-1} = \omega \times 4\,\text{cm}$$

$$\omega = 4\,\text{s}^{-1}$$

$$\text{Period } T = \frac{2\pi}{\omega} = \frac{2\pi}{4\,\text{s}^{-1}} = 1.571\,\text{s}$$

Example 1.5 A spring stretches 0.15 m when a 0.3 kg mass is hung from it. The spring is then stretched an additional 0.1 m from this equilibrium point and released. Determine (i) spring constant k (ii) the amplitude of the oscillation A (iii) the maximum velocity v_0.

Solution: (i) From Eq. (1.2)

$$k = \frac{F}{x} = \frac{mg}{x} = \frac{(0.3\text{ kg})\,(9.8\text{ m s}^{-2})}{0.15\text{ m}}$$

$$= 19.6 \text{ N m}^{-1}$$

(ii) The spring is stretched 0.1 m from equilibrium and released, the amplitude A = 0.1m.

(iii) The maximum velocity v_0 is obtained when the mass passes the equilibrium position. From Eq. (1.6), we get

$$v_0 = \omega A.$$

From Eq. (1.5)

$$\omega = \sqrt{\frac{k}{m}}.$$

Hence,

$$v_0 = A\sqrt{\frac{k}{m}} = 0.1\text{ m}\sqrt{\frac{19.6\text{ N m}^{-1}}{0.3\text{ kg}}}$$

$$= 0.81 \text{ m s}^{-1}$$

Example 1.6 The visible region of the electromagnetic spectrum is 400 nm to 700 nm. Calculate the frequency equivalent of the visible region in Hz. Velocity of light is 3×10^8 m/s.

Solution: The frequency ν is given by

$$\nu = \frac{\text{Velocity}}{\text{Wavelength}}$$

When electromagnetic spectrum is 400 nm, we get

$$\nu = \frac{3 \times 10^8 \text{ m/s}}{400 \times 10^{-9}\text{ m}} = 7.5 \times 10^{14} \text{ Hz}$$

When electromagnetic spectrum is 700 nm, we get

$$\nu = \frac{3\times 10^8 \text{ m/s}}{700\times 10^{-9} \text{ m}} = 4.29\times 10^{14} \text{ Hz}$$

The frequency equivalent of 400 nm to 700 nm is 7.5×10^{14} Hz to 4.29×10^{14} Hz.

Example 1.7 The displacement of a sound wave is given by

$$u\,(x, t) = 1.5\times 10^{-3} \sin\left(\frac{2\pi x}{8} - 80\,\pi t\right)$$

where x is measured in metres and t in seconds. Evaluate the amplitude, wavelength, frequency and velocity of the wave.

Solution: The standard equation for the displacement is given by

$$u = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

The given equation can be written as

$$u = 1.5\times 10^{-3} \sin 2\pi\left(\frac{x}{8} - 40\,t\right)$$

Comparing the two equations, we get

Amplitude (A) = 1.5×10^{-3} m, Wavelength (λ) = 8 m,

Period $T = \dfrac{1}{40}$ s, Frequency $\nu = 1/T = 40$ Hz

Velocity $\upsilon = \nu\lambda$ = (40 Hz) (8m) = 320 m/s

Example 1.8 Find the equation of a wave of amplitude 2 cm, period 0.5 s and velocity 200 cm/s moving along the x-axis.

Solution: Amplitude (A) = 2 cm, Frequency $= \dfrac{1}{0.5\,\text{s}} = 2\,\text{Hz}$

$$\text{Wavelength } (\lambda) = \frac{\text{Velocity}}{\text{Frequency}} = \frac{200\,\text{cm/s}}{2\,\text{Hz}} = 100\,\text{cm}$$

The general equation of a wave is given by

$$u = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

The equation of the wave having the given parameters is

$$u = 2 \sin 2\pi\left(\frac{x}{100} - \frac{t}{0.5}\right)$$

where x and y are in centimetres and t in seconds.

Example 1.9 A wave of wavelength 0.30 m is travelling down a 300 m long wire whose total mass is 15 kg. If the wire is under tension of 1000 N, what is the velocity and frequency of the wave?

Solution: Velocity of the wave in the string

$$\upsilon = \sqrt{\frac{T}{m}}$$

where T is tension, m is the mass per unit length

$$\upsilon = \sqrt{\frac{1000\,\text{N}}{\dfrac{(15\,\text{kg})}{(300\,\text{m})}}} = \sqrt{\frac{(1000\,\text{N})\ (300\,\text{m})}{15\,\text{kg}}} = 141.4\,\text{m/s}$$

The frequency

$$\nu = \frac{\upsilon}{\lambda} = \frac{141.4\,\text{m/s}}{0.3\,\text{m}} = 471.3\,\text{Hz}$$

REVIEW QUESTIONS

1.1 When do you say that a motion is simple harmonic? Write the equation of motion of a particle of mass m executing simple harmonic motion.

1.2 Is the acceleration of a simple harmonic oscillator ever zero? If so, where?

1.3 If we double the amplitude of a SHM, how does this change the frequency, maximum velocity and total energy?

1.4 Write the differential equation of a damped harmonic oscillator whose damping force is proportional to its velocity. When do you say the system is critically damped?

1.5 Distinguish between critically damped and dead beat motions when a harmonic oscillator is damped by a force proportional to its velocity.

1.6 What do you understand by quality factor? On what factors does it depend?

1.7 Define wavelength and frequency of a periodic motion. State the relation between the two. Write the equation that represents periodic motion.

1.8 With examples, distinguish between transverse and longitudinal waves.

1.9 Explain (i) wave velocity and (ii) particle velocity. State the relation between the two.

1.10 For a wave travelling in the positive x-direction the displacement y is represented by a function of $(x - \upsilon t)$ as $y = f(x - \upsilon t)$. Show that υ is the wave velocity of the wave.

1.11 Explain the difference between the speed of a transverse wave travelling down a rope and the speed of a tiny piece of the rope.

1.12 What kind of waves do you think will travel down a horizontal metal rod if you strike its end (i) vertically from above (ii) horizontally parallel to its length?