$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

let, $\vec{A} = 5x^2\hat{i} + 3y^3x\hat{j} + 4x^2y^2z^2\hat{k}$

$$\vec{\nabla}\cdot\vec{A} = 10x + 9xy^2 + 8x^2y^2z$$

$$\vec{\nabla}\cdot\vec{A}\Big|_{(1,2,1)} = 10 + 9(4) + 8(4)(3)$$
$$= 10 + 36 + 96$$
$$= 142$$

$\underset{\text{vector}}{\vec{\nabla}}\cdot\vec{A} \rightarrow$ divergence of $\vec{A}$ $\rightarrow$ number
(flux of $\vec{A}$)

$\vec{\nabla}(\underset{\text{scalar}}{F}) \rightarrow$ vector $\rightarrow$ gradient (slope)

$\vec{\nabla}\times\vec{A} \rightarrow$ curl $(\vec{A})$

$\vec{A} = 3x^2\hat{i} + 4xy\hat{j} + 7xyz\hat{k}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 4xy & 7xyz \end{vmatrix} = \hat{i}(7xz - \cancel{0}) - \hat{j}(7yz - 0) + \hat{k}(4y - 0)$$
$$= 7xz\,\hat{i} - 7yz\,\hat{j} + 4y\,\hat{k}$$

* if curl of any vector is non-zero, it is rotating.

$\underset{\text{always a rotating vector}}{\vec{\nabla}\times\vec{B} \neq 0}$ $\qquad$ $\underset{\text{irrotating vector}}{\vec{\nabla}\times\vec{B} = 0}$

* if $\vec{\nabla}\cdot\vec{A} = 0$, then $\vec{A}$ is solenoidal vector
(flux through $\vec{A} = 0$)

Cartesian Coordinates : (x, y, z) $\quad \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Spherical " : (r, θ, ϕ) $\quad \bar{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta} + \hat{\phi}\left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial\phi}$

Cylindrical " : (s, ϕ, z) $\quad \bar{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$

(I) Gradient : $\bar{\nabla}F \rightarrow$ slope
(II) Divergence : $\bar{\nabla}\cdot\bar{A} \rightarrow$ number $\rightarrow$ solenoidal character of $\bar{A}$
(III) Curl : $\bar{\nabla}\times\bar{A} \rightarrow$ vector $\rightarrow$ rot" nature of $\bar{A}$

1. Gauss' (Green's) Theorem of Vector Algebra

$$\int_V (\bar{\nabla}\cdot\bar{F})\,dV = \int_S \bar{F}\,ds$$

* divergence of a vector $\bar{F}$ over a volume 'V' is surface integral of $\bar{F}$

2. Stoke's Theorem of Vector Algebra

$$\int_S (\bar{\nabla}\times\bar{F})\,ds = \int_C \bar{F}\,dl$$

* curl of a vector $\bar{F}$ over a surface 'S' is line/curl integral of $\bar{F}$

Q) $\bar{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

Prove (I) Is the vector field A solenoidal?
(II) " " " " A irrotational?

$\bar{\nabla}\cdot\bar{A} = 2x + 2y + 2z \rightarrow \therefore$ Non-solenoidal

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$
$$= 0$$

$\therefore$ Irrotational

$$\bar{B} = \frac{2z^2y}{x^3}\hat{i} + \frac{x^2}{z^2}\hat{j} + \frac{2yz}{x^2}\hat{k}$$

$\bar{\nabla}\cdot\bar{B} = -\frac{6z^2y}{x^4} + 0 + \frac{2y}{x^2} \longrightarrow$ Non-solenoidal

$$\bar{\nabla}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2z^2y}{x^3} & \frac{x^2}{z^2} & \frac{2yz}{x^2} \end{vmatrix} = \hat{i}\left(\frac{2z}{x^2} - \frac{2x^2}{z^3}\right) - \hat{j}\left(-\frac{4yz}{x^3} - \frac{4yz}{x^3}\right) + \hat{k}\left(\frac{2x}{z^2} - \frac{2z^2}{x^3}\right)$$

$\therefore$ Irrotational

Q) $\phi = \frac{e^{-a/r}}{r^2}$

Find gradient of ϕ

grad $(\phi) = \nabla\phi$

$= \frac{\partial}{\partial r}\left(\frac{e^{-a/r}}{r^2}\right)\hat{i}$

$= \frac{\left(e^{-a/r}\cdot\left(\frac{a}{r^2}\right)\cdot r^2 - e^{-a/r}\cdot 2r\right)}{r^4}\hat{i}$

$= \frac{e^{-a/r}(a-2r)}{r^4}\hat{i}$

Q) Find the value of 'a' of $\overline{A} = (x+3y)\hat{i} + (2y+3z)\hat{j} + (x+az)\hat{k}$, if $\overline{A}$ is solenoidal.

$\overline{\nabla}\cdot\overline{A} = 0$

$\Rightarrow 1+2+a=0$

$\Rightarrow \boxed{a=-3}$

<u>Important Entities</u>

(I) $\text{Div}(\text{Grad}\,\phi) = \overline{\nabla}\cdot(\overline{\nabla}\phi) = \nabla^2\phi$

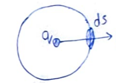
(II) $\text{Curl}(\text{Grad}\,\phi) = \overline{\nabla}\times(\overline{\nabla}\phi) = 0$

(III) $\text{Div}(\text{Curl}\,\overline{A}) = \overline{\nabla}\cdot(\overline{\nabla}\times\overline{A}) = 0$

(IV) $\text{Curl}(\text{Curl}\,\overline{A}) = \overline{\nabla}\times(\overline{\nabla}\times\overline{A})$
$= \overline{\nabla}(\overline{\nabla}\cdot\overline{A}) - \overline{\nabla}^2 A$

(V) $\text{Div}(\overline{A}\times\overline{B}) = \overline{\nabla}\cdot(\overline{A}\times\overline{B})$
$= \overline{B}\cdot(\overline{\nabla}\times\overline{A}) - \overline{A}\cdot(\overline{\nabla}\times\overline{B})$

<u>Gauss Theorem of Electricity</u>



$\phi = \int_S E\cdot ds$

$= \int_S E\,ds\cos\theta$

$\phi_{net} = \int_S E\,ds$

$= \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\int_S ds$

$= \frac{q}{4\pi\epsilon_0 r^2}\times 4\pi r^2$

$= \frac{q}{\epsilon_0}$

$\boxed{\phi = \int_S E\cdot ds = \frac{q_{enclosed}}{\epsilon_0}}$ (integral form)

$\downarrow$

$\int_V \overline{\nabla}\cdot\overline{E}\,dV = \frac{1}{\epsilon_0}\int_V \rho\,dV$

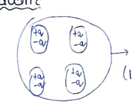
$\boxed{\overline{\nabla}\cdot\overline{E} = \frac{\rho}{\epsilon_0}}$ (differential form)

$\overline{\nabla}\cdot(\epsilon_0\overline{E}) = \rho$

$\boxed{\overline{\nabla}\cdot\overline{D} = \rho}$

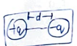
- Charge should be enclosed by surface and static (NOT dynamic)
- Applicable for conductors

<u>Dielectric</u>



$\rightarrow E=0$ (insulator)

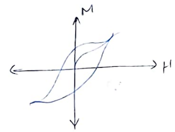
$E = 10\,V$

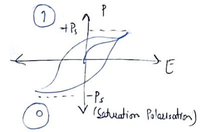


<u>induced dipole</u>

Energy density $= \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\mu H^2$

$\boxed{P = \frac{qd}{V}}$ (polarization)





- More the area, more the energy absorbed by the material

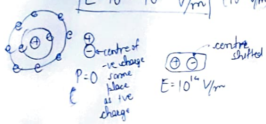
- Mica → least dielectric material (insulator)

Types of Polarisation (P)

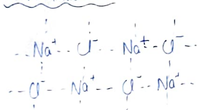
1. Electronic Polarization (P_E)
2. Ionic Polarization (P_I)
3. Orientational Polarization (P_o)
4. Space-charge Polarization (P_s)

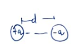
Electronic Polarization

$\boxed{E = 10^{12} - 10^{14}\ V/m}$ (10^{16} V/m)



Ionic Polarization



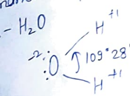


$\boxed{E = 10^{8} - 10^{10}\ V/m}$

Orientational Polarization

- Permanent dipole moment

e.g. - H_2O

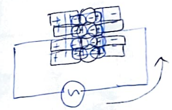


$E = 0 \Rightarrow$ dipole moment $= 0$ (distributed randomly making net dipole moment zero)

$E \neq 0 \Rightarrow$ all oriented in dir of $\bar{E}$

($10^6 - 10^8$ V/m)

Space - Charge Polarization



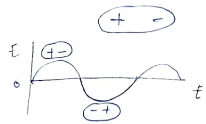
For P_s, $\boxed{E = 10^2 - 10^6\ V/m}$

$\boxed{\vec{P} = P_E + P_I + P_o + P_s}$

vector

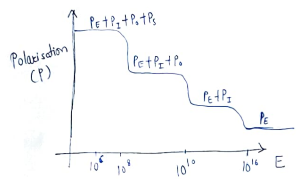
- Those material that have high dielectric constant are used in memory devices
- The system that shows P_E will show P_I, P_o, P_s as well, but vice-versa is not true.

Effect of (AC) electric field on Polarisation or dielectric constant



$\boxed{E = E_0 \sin \omega t}$

- Time required for flipping of dipole moment in applied ext. AC field: relaxation time (electric)

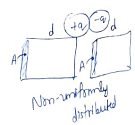


- The material that has P_E can possess all other polarizations.

Dielectric breakdown: when applied field exceeds the critical value of a material, the dipoles break and the dielectric starts to behave as a conductor.

ϵ (dielectric constant) → tensor quantity

$$\boxed{\text{Polarisation} = \frac{\text{dipole moment}}{\text{Volume}}}$$



$$\vec{P} = \frac{p}{A \cdot d} = \frac{q\,d}{A\,d} = \frac{q}{A}$$

$\boxed{\vec{P} = \sigma_b}$ (surface bound-charge density)

$$-Q = \int_S P \cdot ds$$

$$-\int_V \rho\, dV = \int_S P\, ds$$

$$-\int_V \rho\, dV = \int_V \bar{\nabla} \cdot \bar{P}\, dV$$

$$-\int_V \rho_b\, dV = \int_V \bar{\nabla} \cdot \bar{P}\, dV$$

ρ_b → bound volume charge density

$$\boxed{\rho_b = -\bar{\nabla} \cdot \bar{P}}$$

Gauss Theorem: $\bar{\nabla} \cdot \bar{E} = \rho / \epsilon_0$

$$\bar{\nabla} \cdot (\epsilon_0 \bar{E}) = \rho = \rho_{free} + \rho_b$$

$$\Rightarrow \bar{\nabla} \cdot (\epsilon_0 \bar{E}) = \rho_{free} - \bar{\nabla} \cdot \bar{P}$$

$$\Rightarrow \bar{\nabla} (\epsilon_0 \bar{E} + \bar{P}) = \rho_{free}$$

$\bar{\nabla} \cdot \bar{D} = \rho_{free}$ (we know that)

$$\therefore \boxed{\bar{D} = \epsilon_0 \bar{E} + \bar{P}}$$

$\boxed{\bar{\nabla} \cdot \bar{D} = \rho_{free}}$

$\vec{D}$ → displacement vector ($\bar{D} = \epsilon_0 \bar{E}$)

$\bar{P} \propto \bar{E}$

$$\boxed{\bar{P} = \epsilon_0 \chi_e \bar{E}}$$

χ_e → electrical susceptibility — it tells the extent to which a material can be polarized.

ϵ_0 → permittivity of free space

$$\boxed{\epsilon_0 = 8.85 \times 10^{-12}\, C^2/m^2 N}$$

$$\bar{D} = \epsilon_0 \bar{E} + \epsilon_0 \chi_e \bar{E} = \epsilon_0 \bar{E}(1 + \chi_e)$$

$\boxed{\bar{D} = \epsilon_0 \epsilon_r \bar{E}}$

$\boxed{\bar{D} = \epsilon \bar{E}}$

$\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0}}$

$\boxed{\epsilon_r = 1 + \chi_e}$

Q.) An isotropic material of relative permittivity 'ϵ_r' is placed normal to a uniform ext. electric field with an electric displacement vector of mag. $5 \times 10^{-4}\ C/m^2$. If the vol. of the slab is $0.5\ m^3$ and mag. of polarisation $4 \times 10^{-4}\ C/m^2$. Find value of ϵ_r and total dipole moment of slab.

$$P = \frac{\text{dipole moment}}{\text{Volume}}$$

$$\Rightarrow \text{dipole moment} = 4 \times 10^{-4} \times 0.5 \quad C\text{-}m$$

$$= 2 \times 10^{-4}\ C\text{-}m$$

$$\overline{D} = \epsilon_0 \overline{E} + \overline{P}$$

$$\Rightarrow \overline{D} = \epsilon_0 \epsilon_r \overline{E}$$

$$\Rightarrow \epsilon_r = \frac{D}{\epsilon_0 E} = \frac{5 \times \cancel{10^{-4}}}{\cancel{\epsilon_0} \times \frac{\cancel{10^{-4}}}{\cancel{\epsilon_0}}} = 5$$

$$\cancel{= \frac{5 \times 10^{-4}}{\epsilon_0 \frac{\sqrt{41 \times 10^{-8}}}{\epsilon_0}}}$$

$$\cancel{= \frac{5 \times 10^{-4}}{\sqrt{41} \times 10^{-4}}}$$

$$\cancel{= \frac{5}{\sqrt{41}}}$$

$$\cancel{\overline{E} = \frac{\overline{D} + (-\overline{P})}{\epsilon_0}}$$

$$\cancel{= \frac{\sqrt{(5 \times 10^{-4})^2 + (-4 \times 10^{-4})^2}}{\epsilon_0}}$$

$$\cancel{= \frac{\sqrt{41 \times 10^{-8}}}{\epsilon_0}}$$

$$E = \frac{D - P}{\epsilon_0}$$

$$= \frac{10^{-4}}{\epsilon_0}$$

$$\boxed{\overline{\nabla} \cdot \overline{D} = \rho_{free}}$$ ← differential form of Gauss theorem (when polarisation exists)

$$\int_V (\overline{\nabla} \cdot \overline{D})\, dV = \int_V \rho\, dV$$

$$\boxed{\oint_S D \cdot ds = Q_{free}}$$

Q.) Two parallel plates of capacitors having equal and opposite charges are separated by 6 mm thick dielectric material of dielectric constant 2.8. If the electric field strength inside is 10^5 V/m. Determine $\overline{P}$, $\overline{D}$ and energy density in the dielectrics

$$\overline{D} = \epsilon_0 \epsilon_r \overline{E}$$

$$= 8.85 \times 10^{-12} \times 2.8 \times 10^5 \quad C/m^2$$

$$= 24.78 \times 10^{-7}\ C/m^2 = 2.478 \times 10^{-6}\ C/m^2$$

$$\overline{P} = \epsilon_0 \chi_e \overline{E}$$

$$= \epsilon_0 (\epsilon_r - 1) \overline{E}$$

$$= 8.85 \times 10^{-12} \times 1.8 \times 10^5 \quad C/m^2$$

$$= 15.930 \times 10^{-7}\ C/m^2$$

$$= 1.5930 \times 10^{-6}\ C/m^2$$

Energy density $= \frac{1}{2} \epsilon E^2$

$$= \frac{1}{2} \epsilon_0 \epsilon_r E^2$$

$$= \frac{1}{\cancel{2}} \times 8.85 \times 10^{-12} \times \cancel{2.8}^{\,1.4} \times 10^{10}\ J/m^3$$

$$= 12.390 \times 10^{-2}\ J/m^3$$

$$= 1.2390 \times 10^{-1}\ J/m^3$$

8.85 × 2.8: 7080, 1770, 24.780

8.85 × 1.8: 7080, 885, 15.930

885 × 14: 3540, 885, 12390

Q.) Dielectric constant of a gas at N.T.P is 1.00074. Calculate the polarisation density of each atom of gas when it is held in ext. elec. field 3×10^4 V/m

$$\bar{P} = \epsilon_0 \chi_e \bar{E}$$

$$= \epsilon_0 (\epsilon_r - 1) E$$

$$= 8.85\times10^{-12} \times 0.00074 \times 3\times10^4 \ C/m^2$$

$$= 0.01965\times10^{-8} \ C/m^2$$

$$= 1.965\times10^{-10} \ C/m^2$$

885 × 74 = 3540 + 6195 = 65500; 65500 × 3 = 196500

Q.) Two parallel plates having equal and opposite charges separated by 2 cm thick slab of $\epsilon_r = 3$. If E inside 10^6 V/m. Calculate $\bar{D}$ and $\bar{P}$.

$$\bar{P} = \epsilon_0 \chi_e \bar{E}$$

$$= \epsilon_0 (\epsilon_r - 1) E$$

$$= 8.85\times10^{-12} \times 2\times10^6 \ C/m^2$$

$$= 17.7\times10^{-6} \ C/m^2$$

$$= 1.77\times10^{-5} \ C/m^2$$

885 × 2 = 1770

$$\bar{D} = \epsilon_0 \epsilon_r \bar{E}$$

$$= 8.85\times10^{-12} \times 3\times10^6$$

$$= 26.55\times10^{-6} \ C/m^2$$

$$= 2.655\times10^{-5} \ C/m^2$$

885 × 3 = 2655

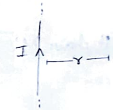
Magnetostatics

$\phi = B.A$

B → magnetic flux density

H → mag field intensity

M → magnetisation = $\frac{\text{mag. moment}}{\text{volume}}$



$$B = \frac{\mu_0}{4\pi}\frac{2I}{r} = \frac{\mu_0 I}{2\pi r}$$

Origin of Magnetism

(i) Orbital motion of e^- → very weak mag moment

(ii) Spin motion of e^-

Types of magnetic material

(i) Diamagnetic → $\chi_M < 0$ → only orbital motion

(ii) Paramagnetic → $\chi_M = 0$ (absence of H), $\chi_M > 0$ (small +ve) (presence of H)

(iii) Ferromagnetic → $\chi_M >> 0$

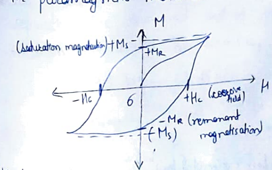
(iv) Anti-ferromagnetic

(v) Ferrimagnetic

$$\boxed{\chi_M = \frac{M}{H}}$$

χ_M → mag. susceptibility

- In the absence of applied mag. field (H), the mag. moment/magnetization = 0 in paramagnetic material.

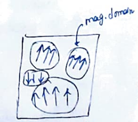


"Hysteresis loop" (ferromagnetic)
↳ energy absorbed by material under mag. field

$3d^5$ → Fe, Co, Ni

↑	↑	↑	↑	↑

(all unpaired e^-)



Ferromagnetic

- each domain has mag. moment even in absence of mag. field but net = 0
- when H is applied, all domains oriented in that dirⁿ giving large value of χ_M.

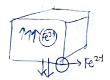
$T_c \to$ curie temp.
(ferromagnetic → paramagnetic)

Anti-ferro → | ↑↓ | ↑↓ | ↑↓ | ↑↓ | ↑↓ | (absence of field H)
$M = 0$

· On applying field, it shows 'M'.

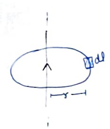
T_N (Néel temp.)
(anti-ferro → para)
e.g.- MnO

Ferri → residual of spin/mag. moment due to structure
· spinal
· hexa ferrite
e.g.- AB_2O_4, $NiFe_2O_4$, $Fe\,Fe_2O_4$, Fe_3O_4
$\downarrow$
$(FeO)(Fe_2O_3)$



· 2 get cancelled
· 1 residual remains

Biot-Savart law



$B = \frac{\mu_0 I}{2\pi r}$

$\oint B \cdot dl = \oint B\, dl \cos\theta$

$\theta = 0$

$= B \oint_c dl$

$= B(2\pi r)$

$= \frac{\mu_0 I}{2\pi r}(2\pi r)$

$\boxed{\oint_c B \cdot dl = \mu_0 I_{enclosed}}$ → "Ampere's circuital law" (integral form)

$\oint_c B \cdot dl = \mu_0 I_{enclosed}$

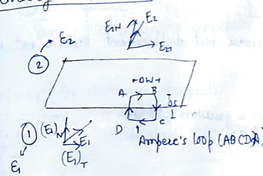
By Stoke's theorem of vector algebra,

$\int_s (\vec{\nabla} \times \vec{B})\, ds = \mu_0 \int J\, ds$

$J = \frac{I}{A}$
$I = \int J \cdot ds$

$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 J}$ → differential form of Ampere's circuital law

Boundary Conditions



Let us consider two mediums of permittivity ε_1 and ε_2 and ABCD be a Ampere's loop through both mediums having width ΔW and thickness ΔS

Let E_1 be electric field in med. ①
E_2 be electric field in med. ②
having tangential and normal components
$E_1 = E_{1N} + E_{1T}$
$E_2 = E_{2N} + E_{2T}$

• The condition should be satisfied by the field at the interface separating the two med. and at the common boundary of these medium, are called as "boundary condition".

For closed loop, $\oint B \cdot dl = 0$

Analogually, $\oint E \cdot dl = 0$ (for closed path ABCDA whose width ΔW and thickness ΔS)

$0 = E_{2T} \cdot \Delta W + \left[-E_{2N}\frac{\Delta S}{2} - E_{1N}\frac{\Delta S}{2}\right] + \left[-E_{1T} \cdot \Delta W\right] + \left[E_{1N}\frac{\Delta S}{2} + E_{2N}\frac{\Delta S}{2}\right]$

$\Rightarrow E_{1T}\Delta W = E_{2T}\Delta W \Rightarrow \boxed{E_{1T} = E_{2T}}$ "boundary condition"

* Tangential component of electric field will be always same

$$\boxed{E_{1T} = E_{2T}} \quad \text{——— (I)}$$

$$D_{2T} = \epsilon_2 E_{2T}$$

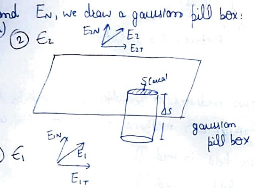
$$\Rightarrow E_{2T} = \frac{D_{2T}}{\epsilon_2}$$

Similarly, $E_{1T} = \dfrac{D_{1T}}{\epsilon_1}$

$$\therefore \boxed{\frac{D_{2T}}{\epsilon_2} = \frac{D_{1T}}{\epsilon_1}} \quad \text{——— (II)}$$

" E_T is always continuous but displacement vector is not in case of different permittivity"

* To understand E_N, we draw a gaussian pill box:
(find condition)



We know that,

$$\oint D \cdot ds = q \quad \text{(Gauss Theorem of electricity)}$$

Under close limit $\Delta S \to 0$

$$D_{2N} S - D_{1N} S = \sigma S$$

$$\boxed{D_{2N} - D_{1N} = \sigma}$$

- Normal component of D is ~~d~~ discontinuous, which amounts to free charge density (σ)

If $\sigma = 0$, $\boxed{D_{2N} = D_{1N}}$ → D_N becomes continuous ——— (III)

$\boxed{\epsilon_2 E_{2N} = \epsilon_1 E_{1N}}$ → E_N is always discontinuous ——— (IV)

(I), (II), (III), (IV) are boundary conditions of $\bar{D}$ and $\bar{E}$ when separated ~~by~~ into mediums of permittivities ϵ_1 and ϵ_2

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I r}{2(\pi r^2)} = \frac{\mu_0 J r}{2}$$

$$B = \frac{\mu_0}{4\pi} \frac{I}{r^2}$$

$$B(\bar{r}) = \frac{\mu_0}{4\pi} \int J(r) \times \frac{\hat{r}}{r^2} \, d\tau$$

$$\boxed{B = \frac{\mu_0}{4\pi} \int \frac{I \, dl \, \hat{r}}{r^2}}$$

$\boxed{\bar{\nabla} \cdot \bar{B} = 0}$ ⇒ magnetic monopoles can't exist

Maxwell's eq^n (for electrostatic/ charges).
magnetostatic
"Steady state" → no time factor

$$\boxed{\nabla E = \frac{\rho}{\epsilon_0}} \equiv \boxed{\bar{\nabla} \cdot \bar{D} = \rho}$$

$$\boxed{\bar{\nabla} \cdot \bar{B} = 0}$$

$$\boxed{\bar{\nabla} \times \bar{E} = 0}$$

$$\boxed{\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}} \equiv \boxed{\bar{\nabla} \times \bar{H} = \bar{J}}$$

Continuity eq^n: $\boxed{\dfrac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \bar{J} = 0}$

$\bar{\nabla}\cdot\bar{B} = 0$

$\Rightarrow \bar{B} = \bar{\nabla}\times\underbrace{\bar{A}}_{\text{vector potential}}$

$\nabla^2\underbrace{\phi}_{\text{scalar potential}} = 0$

$\bar{\nabla}\times\bar{E} = 0$

$\Rightarrow \bar{E} = -\bar{\nabla}\phi$

$\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \mu_0\bar{J}$

$\bar{\nabla}(\underbrace{\bar{\nabla}\cdot\bar{A}}_{0}) - \nabla^2 A = \mu_0 J$

$\bar{\nabla}\cdot\bar{A} = 0$

$\underbrace{\boxed{\nabla^2 A = -\mu_0 J}}_{\text{Poisson eq}^n}$

$\boxed{\nabla^2\phi = \rho/\epsilon_0}$

$\rho = \rho_{free} + \rho_{bound}$

$J = J_{free} + J_{bound}$

$\boxed{J_{bound} = \bar{\nabla}\times\underbrace{\bar{M}}_{\text{magnetisation}}}$

$\bar{\nabla}\times\bar{B} = \mu_0(J_f + J_b)$

$\Rightarrow \bar{\nabla}\times\frac{\bar{B}}{\mu_0} = J_f + (\bar{\nabla}\times\bar{M})$

$\Rightarrow \bar{\nabla}\times\underbrace{\left(\frac{\bar{B}}{\mu_0} - \bar{M}\right)}_{\bar{H}} = J_f$

$\Rightarrow \bar{\nabla}\times\bar{H} = \bar{J}$

$\frac{B}{\mu_0} - M = H$

$\Rightarrow \boxed{\bar{B} = \mu_0(\bar{M} + \bar{H})}$

$\bar{H}$ (In magnetostat) $\equiv \bar{D}$ (In electrostat) — $\bar{D}$: displacement vector

$\boxed{\chi_M = \frac{\bar{M}}{\bar{H}}}$

$\boxed{\bar{M} \propto \bar{H}} \Rightarrow \boxed{\bar{M} = \bar{H}\chi_M}$

$\Rightarrow \boxed{\bar{B} = \mu\bar{H}}$

$\bar{B} = \mu_0 \cancel{(\;)}\underbrace{(1+\chi_M)}_{\mu_r}\bar{H}$

$\bar{E} \equiv \bar{B}$

$\bar{P} \equiv \bar{M}$

$\bar{D} \equiv \bar{H}$

$\boxed{1+\chi_m = \mu_r}$

$\boxed{\mu_0\mu_r = \mu}$

Continuity Eq^n

"It says that the total current flowing out of some volume must be equal to the rate of decrease of the charge enclosed within that volume if charge is neither being created nor being destroyed."

$$I = \frac{-dQ}{dt}$$

$I \rightarrow$ total current flowing out of the volume

$\frac{-dQ}{dt} \rightarrow$ rate of decrease of charge

$$I = \int_S J\cdot dS \qquad Q = \int_V \rho\, dV$$

$$\int_S \bar{J}\cdot dS = \frac{-d}{dt}\int_V \rho\, dV$$

By Gauss Theorem of Vector Algebra

$$\int_V (\bar{\nabla}\cdot\bar{J})\, dV = \int_V -\frac{\partial\rho}{\partial t}\, dV$$

$$\boxed{\frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot\bar{J} = 0} \Rightarrow \boxed{\frac{\partial\rho}{\partial t} = -\bar{\nabla}\cdot\bar{J}}$$

(Continuity eq^n)

Maxwell's fourth eq^n:

$$\bar{\nabla}\times\bar{H} = \bar{J}$$

Taking dot product with $\bar{\nabla}$ on both sides:

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{H}) = \bar{\nabla}\cdot\bar{J}$$

$\boxed{\bar{\nabla}\cdot\bar{J} = 0} \rightarrow$ xxx

Maxwell's correction:

$$\bar{\nabla}\times\bar{H} = \bar{J} + \bar{J}'$$

Faraday's law

$$\mathcal{E}_{emf} = \frac{-\partial \phi_B}{\partial t} \qquad \phi_B = \int_S \bar{B}\cdot ds$$

$$\mathcal{E}_{emf} = \int_C \bar{E}\cdot dl = \frac{-\partial \phi}{\partial t}$$

By stoke's theorem of vector algebra

$$\int_S (\overline{\nabla}\times\bar{E})\, ds = \int_S \left(-\frac{\partial B}{\partial t}\right) ds$$

$$\Rightarrow \boxed{\overline{\nabla}\times\bar{E} = \frac{-\partial B}{\partial t}} \longleftarrow \text{Maxwell's third law (Faraday's law)}$$

Maxwell's Eqn

1. $\overline{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0}$ → Gauss law of electrostatics
2. $\overline{\nabla}\cdot\bar{B} = 0$ → Non-existence of monopoles
3. $\overline{\nabla}\times\bar{E} = \frac{-\partial \bar{B}}{\partial t}$ → Faraday's law
4. $\overline{\nabla}\times\bar{H} = \bar{J} + \bar{J}'$ → modified Ampere's law

Stationary charge

$$\bar{\nabla}\cdot\bar{J} = 0$$

$$\frac{\partial \rho}{\partial t} = 0$$

In general,

$$\overline{\nabla}\cdot\bar{J} = \frac{-\partial \rho}{\partial t} \qquad \bar{J} = \sigma\bar{E}$$

$$\overline{\nabla}\cdot(\sigma\bar{E}) = \frac{-\partial \rho}{\partial t} \qquad \bar{D} = \epsilon\bar{E}$$

$$\overline{\nabla}\cdot\left(\frac{\sigma}{\epsilon}\bar{D}\right) = \frac{-\partial \rho}{\partial t}$$

$$\underbrace{\overline{\nabla}\cdot\bar{D}}_{\rho} = -\frac{\epsilon}{\sigma}\left(\frac{\partial \rho}{\partial t}\right)$$

$$\Rightarrow \rho = -\frac{\epsilon}{\sigma}\left(\frac{\partial \rho}{\partial t}\right) \Rightarrow \boxed{\rho = \rho_0 e^{-t/\tau}}$$

$\tau \rightarrow$ time constant/relaxation time

Maxwell's Fourth eqn

$$\overline{\nabla}\times\bar{H} = \bar{J} + \bar{J}'$$

$$\Rightarrow \overline{\nabla}\cdot(\overline{\nabla}\times\bar{H}) = \overline{\nabla}\cdot\bar{J} + \overline{\nabla}\cdot\bar{J}'$$

$$\Rightarrow \overline{\nabla}\cdot\bar{J}' = -\overline{\nabla}\cdot\bar{J}$$

$$\Rightarrow \overline{\nabla}\cdot\bar{J}' = \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \overline{\nabla}\cdot\bar{J}' = \frac{\partial}{\partial t}(\overline{\nabla}\cdot\bar{D})$$

$$\Rightarrow \overline{\nabla}\cdot\bar{J}' = \overline{\nabla}\cdot\left(\frac{\partial \bar{D}}{\partial t}\right)$$

$$\Rightarrow \boxed{\bar{J}' = \frac{\partial \bar{D}}{\partial t}} \rightarrow \text{displacement current density}$$

$$\boxed{\overline{\nabla}\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}}$$

Maxwell's Eqn

1. $\overline{\nabla}\cdot\bar{D} = \rho$ → Gauss law of electrostatics
2. $\overline{\nabla}\cdot\bar{B} = 0$ → Non-existence of monopoles
3. $\overline{\nabla}\times\bar{E} = \frac{-\partial \bar{B}}{\partial t}$ → Faraday's law
4. $\overline{\nabla}\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$ → modified Ampere's law

$$\overline{\nabla}\cdot(\overline{\nabla}\times\bar{H}) = \overline{\nabla}\cdot\bar{J} + \overline{\nabla}\cdot\frac{\partial \bar{D}}{\partial t}$$

$$\Rightarrow 0 = \overline{\nabla}\cdot\bar{J} + \overline{\nabla}\cdot\frac{\partial \bar{D}}{\partial t}$$

$$\Rightarrow \overline{\nabla}\cdot\bar{J} = -\overline{\nabla}\cdot\frac{\partial \bar{D}}{\partial t}$$

$$\Rightarrow \cancel{\bar{J} = \frac{-\partial \bar{D}}{\partial t}} \Rightarrow \overline{\nabla}\cdot\bar{J} + \frac{\partial}{\partial t}(\overline{\nabla}\cdot\bar{D}) = 0$$

$$\Rightarrow \cancel{\bar{J} + \frac{\partial \bar{D}}{\partial t} = 0} \Rightarrow \overline{\nabla}\cdot\bar{J} + \frac{\partial \rho}{\partial t} = 0 \rightarrow \text{continuity eq}^n$$

(I) Flux of ele. field of a circuit if charge is enclosed

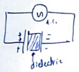
(II) Flux of mag. field is zero

(III) Changing mag flux can induce ele. field

(IV) For steady state, mag. field can be found using $\bar{\nabla} \times \bar{B} = \mu \bar{J}$
(charge not varying with time)

if varying, $\bar{\nabla} \times \bar{H} = \mu(\bar{J} + \epsilon d$ we use Maxwell's correction.

Demonstration of Displacement Current



Maxwell's Eq^n in free space

$\mu \to \mu_0$

$\epsilon \to \epsilon_0$

$\rho = 0 \quad J = 0$ (let)

(I) $\bar{\nabla} \cdot \bar{E} = 0$

(II) $\bar{\nabla} \cdot \bar{B} = 0$

(III) $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

(IV) $\bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$

$\boxed{\bar{B} = \mu_0 \bar{H}}$

$\boxed{\bar{D} = \epsilon_0 \bar{E}}$

(III) $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\left(\bar{\nabla} \times \frac{\partial \bar{B}}{\partial t}\right)$

$\left(\bar{\nabla}(\bar{\nabla} \cdot \bar{E}) - \bar{\nabla}^2 \bar{E}\right) = -\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$

$\downarrow$

$0 - \nabla^2 E = -\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$

$\nabla^2 E = +\frac{\partial}{\partial t} \frac{\mu_0}{\cancel{\mu_0}}\left(\frac{\partial D}{\partial t}\right)$

$\nabla^2 E = \frac{\mu_0}{\cancel{\mu_0}}\left(\frac{\partial^2 D}{\partial t^2}\right)$

$\nabla^2 E = \mu_0 \epsilon_0 \left(\frac{\partial^2 E}{\partial t^2}\right)$

$$\boxed{\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0}$$

(IV) $\bar{\nabla} \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$

$\bar{\nabla} \times (\bar{\nabla} \times \bar{H}) = \bar{\nabla} \times \frac{\partial \bar{D}}{\partial t}$

$\bar{\nabla}(\bar{\nabla} \cdot \bar{H}) - \nabla^2 \bar{H} = \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{D})$

$\bar{\nabla} \cdot \frac{(\bar{\nabla} \cdot \bar{B})}{\mu_0} - \nabla^2 H = \epsilon_0 \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{E})$

$0 - \nabla^2 H = -\epsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}$

$$\boxed{\nabla^2 \bar{B} = \mu_0 \epsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}}$$

$$\boxed{\nabla^2 \bar{B} - \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} = 0}$$

$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0$

$\therefore \frac{1}{v^2} = \mu_0 \epsilon_0$

$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}}$ m/s

$= \frac{1}{\sqrt{4\pi \times 8.85 \times 10^{-19}}}$ m/s

$\approx 3 \times 10^8$ m/s

Plane wave eq^n

$$\bar{E}(r,t) = \bar{E}_0\, e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$$\bar{H}(r,t) = \bar{H}_0\, e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$\frac{\omega}{k} \rightarrow$ phase velocity

$$\bar{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{k}\cdot\bar{r} = k_x x + k_y y + k_z z$$

$$\bar{E}_0 = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} \qquad \bar{H}_0 = H_{0x}\hat{i} + H_{0y}\hat{j} + H_{0z}\hat{k}$$

$$\bar{\nabla}\times\bar{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$$= \left[\hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{j}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\right]\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$$= \hat{i}$$

$$\boxed{\bar{\nabla}\times\bar{E} = i[\bar{k}\times\bar{E}]}$$

$$= -\mu_0 \frac{\partial \bar{H}}{\partial t}$$

$$= -\mu_0(-i\omega\bar{H})$$

$$\boxed{\bar{k}\times\bar{E} = \omega\bar{B}}$$

$$\bar{\nabla}\times\bar{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)} & E_{0y} e^{i(\bar{k}\cdot\bar{r} - \omega t)} & E_{0z}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)} \end{vmatrix}$$

$$= \hat{i}\left(E_{0z}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot i\cdot k_y - E_{0y} e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot ik_z\right) - \hat{j}\left(E_{0z}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot i\cdot k_x - E_{0x}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot i\cdot k_z\right)$$
$$+ \hat{k}\left(E_{0y}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot ik_x - E_{0x}\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot ik_y\right)$$

$$= i\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\cdot\left[(k_y\cdot E_{0z} - k_z E_{0y})\hat{i} - (k_x E_{0z} - k_z E_{0x})\hat{j} + (k_x E_{0y} - k_y E_{0x})\hat{k}\right]$$

$$= i[\bar{k}\times\bar{E}]$$

Similarly,

$$\bar{\nabla}\times\bar{H} = i\cdot e^{i(\bar{k}\cdot\bar{r} - \omega t)}\left[(k_y H_{0z} - k_z H_{0y})\hat{i} - (k_x H_{0z} - k_z H_{0x})\hat{j} + (k_x H_{0y} - k_y H_{0x})\hat{k}\right]$$

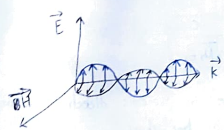
$$= i[\bar{k}\times\bar{H}]$$

$$\bar{\nabla}\times\bar{H} = \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$i[\bar{k}\times\bar{H}] = \epsilon_0(-i\omega E)$$

$$\Rightarrow \boxed{\bar{k}\times\bar{H} = -\omega\bar{D}}$$

- $\bar{E}$ and $\bar{B}$ are normal to each other and EM waves is normal to both.



- $\vec{E}, \vec{H}, \vec{k}$ form a set of orthogonal vectors.

Maxwell's eq^n in isotropic medium

$\epsilon \rightarrow$ permittivity $\qquad \bar{D} = \epsilon \bar{E}$

$\mu \rightarrow$ permeability $\qquad \bar{B} = \mu \bar{H}$

$$\rho = 0, \bar{J} = 0$$

$$\bar{E} = \bar{E}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$$\bar{H} = \bar{H}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

From eq^n (iii),

$$\bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t}$$

Taking $\bar{\nabla} \times$ on both sides,

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$$

$$\Rightarrow \bar{\nabla}(\underbrace{\bar{\nabla}\cdot\bar{E}}_{0}) - \nabla^2 \bar{E} = -\mu\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{H})$$

$$\Rightarrow 0 - \nabla^2 \bar{E} = -\mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\Rightarrow \cancel{\boxed{\frac{\partial \bar{E}}{\partial t} - \frac{1}{\mu\epsilon}\frac{\partial^2 \bar{E}}{\partial t^2} = 0}} \qquad \boxed{\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$$

Similarly,

$$\boxed{\nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2}}$$

$$\therefore \frac{1}{v^2} = \mu\epsilon$$

$$\Rightarrow \boxed{v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}}$$

• ~~Velocity~~ Speed of E.M wave in isotropic dielectric medium is always smaller than that of in free space.

$$\boxed{\text{Refractive index} = \frac{c}{v} = \sqrt{\mu_r \epsilon_r}}$$

• For non-magnetic medium, $\mu_r \approx 1$

$$\boxed{\text{Refractive index} = \sqrt{\epsilon_r}}$$

$$\boxed{\bar{k} \times \bar{E} = \mu_0 \omega \bar{H}} \qquad \boxed{\bar{k} \times \bar{H} = -\epsilon_0 \omega \bar{E}}$$

Taking "$\bar{k}\times$" on both sides,

$$\bar{k} \times (\bar{k} \times \bar{E}) = \mu_0 \omega (\bar{k} \times \bar{H})$$

$$\Rightarrow \bar{k}(\underbrace{\bar{k}\cdot\bar{E}}_{0}) - \bar{E}(\bar{k}\cdot\bar{k}) = \mu_0\omega(-\epsilon_0 \omega \bar{E})$$

$$\Rightarrow -k^2 \bar{E} + \mu_0 \epsilon_0 \omega^2 \bar{E} = 0$$

$$\Rightarrow (\mu_0 \epsilon_0 \omega^2 - k^2) = 0$$

$$\Rightarrow k^2 = \mu_0 \epsilon_0 \omega^2$$

$$\Rightarrow \left(\frac{\omega}{k}\right)^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\Rightarrow \left(\frac{\omega}{k}\right) = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$\therefore$ Phase velocity $\left(\frac{\omega}{k}\right) = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$

$$\approx 3 \times 10^8 \ m/s$$

$$\bar{k} \times \bar{E} = \mu_0 \omega \bar{H}$$

Taking mod on both sides,

$$|\bar{k} \times \bar{E}| = |\mu_0 \omega \bar{H}|$$

$$\Rightarrow |\bar{k}||\bar{E}| = \mu_0 \omega |\bar{H}|$$

$$\Rightarrow \frac{E}{H} = \frac{\mu_0 \omega}{k}$$

$$\Rightarrow \frac{E}{H} = \frac{\mu_0}{\sqrt{\mu_0 \epsilon_0}}$$

$$\Rightarrow \frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 3.77 \times 10^2 \ \frac{N\text{-}m}{A^2\text{-}s}$$

$$\Rightarrow \frac{E}{H} \approx 377 \ \Omega$$

characteristic impedance

("min^m" impedance of a EM wave due to its existence")

"EM wave is transverse in nature"

Proof:

$\vec{\nabla}\cdot\vec{E}=0$

$\vec{\nabla}\cdot\vec{H}=0$

$\vec{E}=(E_{ox}\hat{i}+E_{oy}\hat{j}+E_{oz}\hat{k})\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{H}=(H_{ox}\hat{i}+H_{oy}\hat{j}+H_{oz}\hat{k})\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{k}\cdot\vec{r}=k_x x+k_y y+k_z z$

$\vec{\nabla}=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

$\vec{\nabla}\cdot\vec{E}=i\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\cdot(E_{ox}k_x+E_{oy}k_y+E_{oz}k_z)$

$\vec{\nabla}\cdot\vec{H}=i\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\cdot(H_{ox}k_x+H_{oy}k_y+H_{oz}k_z)$

$\vec{\nabla}\cdot\vec{E}=0$ $\Rightarrow i(\vec{k}\cdot\vec{E})=0$ $\Rightarrow \vec{k}\perp\vec{E}$

$\vec{\nabla}\cdot\vec{H}=0$ $\Rightarrow i(\vec{k}\cdot\vec{H})=0$ $\Rightarrow \vec{k}\perp\vec{H}$

Hence, EM wave is transverse in nature

EM wave in conducting medium

charge on surface ($\rho=0$)

Maxwell Eq^n: $\boxed{\vec{J}=\sigma\vec{E}}$

1. $\vec{\nabla}\cdot\vec{E}=0$
2. $\vec{\nabla}\cdot\vec{H}=0$
3. $\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$
4. $\vec{\nabla}\times\vec{H}=\sigma\vec{E}+\epsilon\frac{\partial\vec{E}}{\partial t}$

Taking cross product with $\vec{\nabla}$ on eq^n (IV),

$\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=(\vec{\nabla}\times\vec{E})\sigma+\epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$

$\Rightarrow \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{H}}_{0})-\vec{\nabla}^2 H=-\sigma\mu\frac{\partial\vec{H}}{\partial t}+(-\epsilon\mu)\frac{\partial^2\vec{H}}{\partial t^2}$

$\Rightarrow \boxed{\vec{\nabla}^2 H=\sigma\mu\frac{\partial\vec{H}}{\partial t}+\epsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}}$

Similarly, taking cross product with $\vec{\nabla}$ on eq^n (III),

$\boxed{\nabla^2\vec{E}=\mu\sigma\frac{\partial}{\partial t}\vec{E}+\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}}$

If $\sigma=0$, $\boxed{\nabla^2\vec{E}=\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}}$ and, $\boxed{\nabla^2\vec{H}=\mu\epsilon\frac{\partial^2\vec{H}}{\partial t^2}}$

If $\mu=\mu_0$ and $\epsilon=\epsilon_0$

$\boxed{\nabla^2\vec{E}=\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$ and, $\boxed{\vec{\nabla}^2\vec{H}=\epsilon_0\mu_0\frac{\partial^2\vec{H}}{\partial t^2}}$

$\left.\begin{array}{l}\cdot\ \mu\sigma\frac{\partial\vec{E}}{\partial t}\\ \cdot\ \mu\sigma\frac{\partial\vec{H}}{\partial t}\end{array}\right\}$ loss factor / dissipating factor

Q) Show that the eq^n of continuity is contained in Maxwell's eq^n.

Q) Show that the electric and magnetic energy densities in a plane travelling wave are equal. Also prove that the total energy density is $U=\epsilon_0 E^2$ or $=\mu_0 H^2$.

Sol: $U_E=\frac{1}{2}\epsilon_0 E^2=\frac{1}{2}\mu_0 H^2$

$U_H=\frac{1}{2}\mu_0 H^2=\frac{1}{2}\epsilon_0 E^2$

$\because U_E=U_H$

$\therefore U=U_E+U_H=\epsilon_0 E^2=\mu_0 H^2$

Q) $\vec{E}_1=k(xy\,\hat{i}+2yz\,\hat{j}+3xz\,\hat{k})$ and $\vec{E}_2=k(y^2\hat{i}+(2xy+z^2)\hat{j}+2yz\,\hat{k})$. Which field is possible and which is not?

$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E}_1=k(y+2z+3x)\\ \vec{\nabla}\cdot\vec{E}_2=k(0+2x+2y)\end{array}\right\}$ Both possible

· If $\vec{\nabla}\cdot\vec{E}=0$, then NOT possible.

Q) $\vec{B}_1=k(3x^2z\,\hat{i}+5xy^2\,\hat{j}+3x^4z^3\,\hat{k})$

$\vec{B}_2=k(y^4\hat{i}+(2x^2y^2+z^2)\hat{j}+2y^3z^4\hat{k})$

$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{B}_1=k(6xz+10xy+9x^4z^2)\\ \vec{\nabla}\cdot\vec{B}_2=k(4x^2y+8y^3z^3)\end{array}\right\}$ NOT possible

Poynting Theorem/Work-Energy Theorem/Power Loss Theorem

- To find the instantaneous power

$$\mathrm{Div}(\bar{A}\times\bar{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\cancel{\vec{\nabla}\cdot(\vec{A}\times\vec{B})=}$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

From Maxwell's eqⁿ (IV),

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\vec{J} + E\cdot\frac{\partial\vec{D}}{\partial t}$$

$\boxed{\vec{J} = \sigma\vec{E}}$

$\boxed{\vec{E}\cdot\vec{J} = \sigma E^2}$

$\boxed{\vec{B} = \mu\vec{H}}$

$\boxed{D = \epsilon\vec{E}}$

From Maxwell's eqⁿ (III),

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\therefore \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \left[\vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right]$$

$$\Rightarrow -\bar{\nabla}\cdot(\bar{E}\times\bar{H}) = \frac{2}{2}\mu\bar{H}\cdot\frac{\partial\bar{H}}{\partial t} + \bar{E}\cdot\bar{J} + \frac{2}{2}\epsilon\bar{E}\cdot\frac{\partial\bar{E}}{\partial t}$$

$$= \frac{1}{2}\mu\frac{\partial H^2}{\partial t} + \bar{E}\cdot\bar{J} + \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t}$$

$$= \bar{E}\cdot\bar{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2\right)$$

$$\Rightarrow \bar{\nabla}\cdot(\bar{E}\times\bar{H}) = -\bar{E}\cdot\bar{J} - \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\mu H^2\right)}_{U_H} - \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\epsilon E^2\right)}_{U_E}$$

$$-\int_V \bar{\nabla}\cdot(\bar{E}\times\bar{H})\,dV = \int_V(\bar{E}\cdot\bar{J})\,dV + \int_V \frac{\partial}{\partial t}(U_E + U_H)\,dV$$

$$\Rightarrow \boxed{-\int_S(\bar{E}\times\bar{H})\,ds = \int_V(\bar{E}\cdot\bar{J})\,dV + \int_V\frac{\partial}{\partial t}(U_E + U_H)\,dV}$$ ← integral form of Poynting Theorem

"Integral form of Poynting Theorem states that ~~internal~~ net inwards flux of the poynting vector through some closed surface is sum of power dissipated in the vol enclosed by the surface and the rate of change of energy stored in the vol enclosed by the surface."

- Poynting Vector $\boxed{\vec{S} = \vec{E}\times\vec{H}}$

$\boxed{\text{Unit: } W/m^2}$

Q) Calculate the value of poynting vector for a 60 W bulb at a distance of 0.5 m from it.

$$\therefore S = \frac{P}{A} = \frac{60}{4\pi(0.5)^2}\ W/m^2$$

$$= \frac{60}{4\pi\times 25}\times 10^2\ W/m^2$$

$$= \frac{240}{4\pi}\ W/m^2$$

$$= \frac{60}{\pi}\ W/m^2$$

Q) Calculate the value of poynting vector at surface of sun if power radiated is 3.8×10^{26} W and 7×10^8 m is its radius.

$$\therefore S = \frac{P}{A} = \frac{3.8\times10^{26}}{4\pi\times7\times7\times10^{16}}$$

$$= \frac{3.8}{4\times22\times7}\times\frac{10^{10}}{10}\ W/m^2$$

Q) A plane wave travelling in free space has an amplitude of electric field $E_y = 100$ V/m and frequency 1 GHz.

(i) Find the phase velocity, wavelength and propagation constant

(ii) Determine the characteristic impedance of free space

(iii) Find mag. and dirⁿ of mag. field intensity

(i) $\text{speed} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c = 3\times10^8\ m/s$

$V = f\lambda$

$\Rightarrow \lambda = \frac{c}{f} = \frac{3\times10^8}{10^9} = 0.3\ m$

$c = \frac{\omega}{k}$

$\Rightarrow k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi f}{\lambda f} = \frac{2\pi}{\lambda} = \frac{2\times3.14}{0.3015} \approx 2.93\ m^{-1}$

(ii) $Z = \frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$

(iii) H (→ A/m) is in z-dirⁿ as E is in y and k is in x.

$B = \frac{E}{c} = \frac{100}{3\times10^8} = 33.33\times10^{-8}\ T$

$= 3.333\times10^{-7}\ T$

$H = \frac{B}{\mu} =$

Q) A plane wave travelling in +ve x-dirⁿ in a lossless unbounded medium having permeability $\mu = \mu_0/3$ and permittivity $\epsilon = 3\epsilon_0$.

(i) Find the velocity of wave

(ii) If $E_y = 10$ V/m and $E_z = 0$ V/m. Find amplitude and dirⁿ of mag. field intensity (H)

(i) $v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\frac{\mu_0}{3}\epsilon_0 \times 3}} = c = 3\times10^8$ m/s

(ii) H is in z-dirⁿ

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{9\epsilon_0}} = \frac{1}{3}\times 377\,\Omega$$

$$\Rightarrow H_z = \frac{10\times3}{377} = \frac{30}{377} = 0.079 \text{ A/m}$$

Q) A plane travelling wave in free space has an avg. poynting vector with a magnitude of $3\ \frac{J}{m^2 s}$. Find the avg. energy density of wave.

$P = 3$ J/m²s

$U = \epsilon_0 E^2 = \mu_0 H^2$

$$\therefore \frac{P}{U} = \frac{EH}{\mu_0 H^2}$$

$$= \frac{1}{\mu_0}\frac{E}{H}$$

$$= \frac{1}{\mu_0}\sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$= c$$

$$\therefore \frac{P}{c} = \frac{3}{3\times10^8} = 10^{-8}\ J/m^3$$

Properties of Matter

Simple Harmonic Motion

- An object oscillates periodically and always tends to its mean position such that total energy is constant.

Types:

(1) Linear SHM $\longrightarrow F \propto (-x)$

(2) Angular SHM $\longrightarrow \tau \propto (-\theta)$

$$F \propto -x$$

$$\boxed{F = -kx}$$

$$F = m\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad -(1), \quad \omega^2 = \frac{k}{m}$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

$$\Rightarrow \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

"A particle of mass 'm' executing S.H.M with displacement from its mean position 'x' along a straight line at any time 't', then the force acting on it is $\boxed{F = -kx}$"

let us assume,

$$x = Be^{\alpha t}$$

$$\Rightarrow \frac{dx}{dt} = \alpha Be^{\alpha t}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \alpha^2 Be^{\alpha t} = \alpha^2 x$$

$$\Rightarrow \frac{d^2x}{dt^2} - \alpha^2 x = 0$$

Putting it in eqⁿ (1),

$$\alpha^2 x + \omega^2 x = 0$$

$$\Rightarrow x(\alpha^2+\omega^2) = 0 \Rightarrow \underbrace{Be^{\alpha t}}_{\text{can't be zero}}(\alpha^2+\omega^2) = 0 \Rightarrow \boxed{\alpha^2 = -\omega^2} \Rightarrow \boxed{\alpha = \pm i\omega}$$

$\therefore \boxed{x = Be^{\pm i\omega t}}$

Linear combination for general solⁿ:

$\boxed{x = B_1 e^{i\omega t} + B_2 e^{-i\omega t}}$, where $B_1, B_2 \rightarrow$ constants

$x = B_1[\cos(\omega t) + i\sin(\omega t)] + B_2[\cos(\omega t) - i\sin(\omega t)]$

$= \underbrace{(B_1+B_2)}_{A\sin\delta}.\cos(\omega t) + \underbrace{(B_1-B_2)i}_{A\cos\delta}.\sin(\omega t)$

$= A\sin\delta.\cos\omega t + A\cos\delta.\sin\omega t$

$\boxed{x = A\sin(\omega t+\delta)} \leftarrow$ solⁿ of 'x'

- It gives displacement of a particle executing SHM at any instant time 't' with 'A' as the maxᵐ (amplitude) displacement of the particle.
- if we put $\boxed{t \rightarrow t + \frac{2\pi}{\omega}}$, we get same eqⁿ.

$x = A\sin[\omega(t+\frac{2\pi}{\omega})+\delta]$

$= A\sin[2\pi + \omega t + \delta]$

$= A\sin[\omega t + \delta]$

$T \rightarrow$ timeperiod $\qquad (\omega t+\delta) \rightarrow$ phase with which the particle executes SHM (constant)

$\boxed{T = \frac{2\pi}{\omega}}$

- Particle executing SHM → "harmonic oscillator"

$x = A\sin(\omega t+\delta)$

$\boxed{V = \frac{dx}{dt} = A\omega\cos(\omega t+\delta)}$

$\boxed{V = \omega\sqrt{A^2-x^2}}$

at $x=0$, $\boxed{V = A\omega} \rightarrow V_{max}$ at mean position

$\boxed{a = \frac{dV}{dt} = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t+\delta) = -\omega^2 x}$ $\qquad \boxed{a_{max} = -\omega^2 A}$

$\boxed{\text{Total Energy} = \frac{1}{2}m\omega^2A^2}$

At mean position, KE → maxᵐ and PE = 0

$KE = \frac{1}{2}mv^2$

$= \frac{1}{2}m\omega^2(A^2-x^2)$

$\because x = 0$

$\therefore KE = \frac{1}{2}m\omega^2A^2$.

$T.E = KE + PE$

$= \frac{1}{2}m\omega^2A^2 + 0$

$= \frac{1}{2}m\omega^2A^2$

At extreme position, KE = 0 ($\because v=0$) and PE → maxᵐ

and $F = -m\omega^2 A$

In general, $\therefore \cancel{PE = -\int F\,dx}$

$F = -kx$

$PE = -\int F\,dx$

$= \frac{1}{2}kx^2$

$KE = \frac{1}{2}mv^2$

$= \frac{1}{2}m\omega^2(A^2-x^2)$

$\therefore T.E = \frac{1}{2}m\omega^2(A^2-x^2) + \frac{1}{2}m\omega^2x^2$

$= \frac{1}{2}m\omega^2A^2$

Q:) Find the max^n velocity and acceleration of SHM of $T = 10\pi$ and amplitude $A = 5\times10^{-2}$ m.

$V = \omega\sqrt{A^2-x^2}$

$V_{max} = \omega A$

$= \frac{2\pi}{T} A$

$= \frac{\cancel{2\pi}}{\cancel{10\pi}_5} \times 5\times10^{-2}$ m/s

$= 10^{-2}$ m/s

$a = -\omega^2 x$

$a_{max} = -\omega^2 A$

$= -\left(\frac{2\pi}{10\pi}\right)^2 \times 5\times10^{-2}$ m/s^2

$= -\frac{1}{5}\times10^{-2}$ m/s^2

Q:) A particle of mass $m = 100$ g is placed in a field of potential $U = 5x^2+10$ erg/gm. Find the frequency.

$U = 5x^2+10$ erg/gm $mU = 500x^2+1000$ erg

$F = -\frac{dU}{dx} = -10x$ dynes

$\Rightarrow 100\frac{d^2x}{dt^2} = -10x$

$\Rightarrow \frac{d^2x}{dt^2} = -\frac{1}{10}x$

$\omega^2 = \frac{1}{10}$

$\omega = \pm\sqrt{\frac{1}{10}}$

$\Rightarrow 2\pi f = \sqrt{\frac{1}{10}}$

$\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{1}{10}}$ Hz

Q:) A mass of 1 kg is attached to a spring of stiffness constant 16 N/m. Find its natural frequency.

$T = 2\pi\sqrt{\frac{m}{K}}$

$f = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{16}{1}}$

$= \frac{4}{2\pi}$

$= \frac{2}{\pi}$ Hz

Q:) A particle executing SHM along a straight line has vel. 16 cm/s and 12 cm/s when passing through points 3 cm and 4 cm from its mean position respectively. Find (i) amplitude (ii) time period

$V = \omega\sqrt{A^2-x^2}$

$16 = \omega\sqrt{A^2-3^2}$ — (i) $12 = \omega\sqrt{A^2-4^2}$ — (ii)

eq^n (i) ÷ (ii),

$\frac{\cancel{16}^4}{\cancel{12}_3} = \frac{\omega\sqrt{A^2-3^2}}{\omega\sqrt{A^2-4^2}}$

$\Rightarrow \frac{A^2-9}{A^2-16} = \frac{16}{9}$

$\Rightarrow 9A^2-81 = 16A^2-256$

$\Rightarrow 7A^2 = 175$

$\Rightarrow A = \sqrt{\frac{175^{25}}{7}}$ cm $= 5$ cm

$256 - 81 = 175$

$\therefore 16 = \omega\sqrt{5^2-3^2}$

$\Rightarrow 16 = 4\omega$

$\Rightarrow \omega = 4$ rad/s

$\Rightarrow \frac{2\pi}{T} = 4$

$\Rightarrow T = \frac{\pi}{2}$ s

$T = \frac{\pi}{2}$ s

Q:) Find the eq^n of a wave of amplitude 2 cm, time period 0.5 s and velocity is 200 cm/s along x-axis.

$y = A\sin(\omega t - kx + \phi)$

$= \frac{2}{100}\sin(4\pi t - 2\pi x)$

$V = \frac{\omega}{K}$

$\Rightarrow K = \frac{\omega}{V} = \frac{2\pi}{TV} = \frac{2\pi}{\left(\frac{1}{2}\right)\times\cancel{200}^{\cancel{100}\,50}}$

$= \frac{\pi}{50}$ cm^{-1}

$= 2\pi$ m^{-1}

$\omega = \frac{2\pi}{T} = \frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$ s^{-1}

Q.) The displacement of a sound wave is given by $u(x,t) = 1.5\times10^{-3} \sin\left(\frac{2\pi}{8}x - 80\pi t\right)$, where $x \to$ metres and $t \to$ seconds. Find out amplitude, wavelength and frequency of wave.

$$u(x,t) = 1.5\times10^{-3} \sin\left(\frac{2\pi}{8}x - 80\pi t\right)$$

$$\boxed{A = 1.5\times10^{-3}}$$

$$\cancel{\omega = \frac{2\pi}{8}}$$

$$\Rightarrow \frac{2\pi}{T} = \cancel{\frac{2\pi}{8}}\ 80\pi \quad 40$$

$$\Rightarrow \cancel{\boxed{T = 8s}} \quad \boxed{T = \frac{1}{40}\,s = 0.025s} \quad \boxed{f = 40\ Hz}$$

$$k = \frac{2\pi}{\lambda} \quad \Rightarrow \lambda = \frac{2\pi}{2\pi}\cdot 8\ m = 8m$$

$$\cancel{\Rightarrow \lambda = \frac{2\pi}{80\pi} = \frac{1}{40}\ m = 0.025\ m}$$

Q.) The visible region of EM spectra is 400 nm – 700 nm. Find equivalent frequency range.

$$v = f\lambda$$

$$\Rightarrow 3\times10^8 = f\times400\times10^{-9}$$

$$\Rightarrow f = \frac{3\times10^8}{400}\times10^9 = 0.75\times10^{15}\ Hz$$

$$v = f\lambda$$

$$\Rightarrow 3\times10^8 = f\times700\times10^{-9}$$

$$\Rightarrow f = \frac{3\times10^8\times10^9}{700} = \frac{3}{7}\times10^{15}\ Hz$$

Q.) A spring stretches 0.15 m when a 0.3 kg mass hangs from it. The spring is then stretched on addition 0.1 m and then released. Determine (i) spring constant (ii) amplitude of oscillation 'A' and (iii) V_{max}

(i) $mg = kx$

$$\Rightarrow 0.3\times10 = k\times0.15$$

$$\Rightarrow k = \frac{3\times100}{0.15}^{20} = 20\ N/m$$

(ii) $A = 0.1\ m$

(iii) $V_{max} = \omega A$

$$= \sqrt{\frac{k}{m}}\cdot A$$

$$= \sqrt{\frac{200}{0.3}}\times0.1\ m/s$$

$$= \frac{1}{10}\sqrt{\frac{200}{3}}\ m/s \approx 0.81\ m/s$$

Damped Oscillation (Damped Harmonic Motion)

spring constant $= k$

x' = displacement of body from the equilibrium state at any inst. of time 't'

$\frac{dx'}{dt}$ = inst. velocity

Two types of forces:

(i) Restoring force $= -kx \to$ act in the opposite dirⁿ

(ii) Damping force $= -q'\frac{dx'}{dt} \to$ opposite dirⁿ to the motion (velocity)

$$m\frac{d^2x'}{dt^2} = -kx' - q'\frac{dx'}{dt}$$

$$\boxed{\frac{d^2x'}{dt^2} + \frac{k}{m}x' + \frac{q'}{m}\frac{dx'}{dt} = 0} \quad \text{—— (i)}$$

"2nd order differential eqⁿ"

$$\boxed{\frac{d^2x'}{dt^2} + 2s\frac{dx'}{dt} + \omega^2 x' = 0} \quad \text{—— (ii)}$$

Assume, $x' = Ae^{\alpha t}$

$$\frac{dx'}{dt} = A\alpha e^{\alpha t} = \alpha x'$$

$$\frac{d^2x'}{dt^2} = \alpha^2 x'$$

Putting it in eqⁿ (ii),

$$\alpha^2 x' + 2s\alpha x' + \omega^2 x' = 0.$$

$$\Rightarrow x'(\alpha^2 + 2s\alpha + \omega^2) = 0$$

$$\Rightarrow \underbrace{Ae^{\alpha t}}_{\text{can't be zero}}(\alpha^2 + 2s\alpha + \omega^2) = 0$$

$$\alpha^2 + 2s\alpha + \omega^2 = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2s \pm \sqrt{4s^2 - 4\omega^2}}{2} = -s \pm \sqrt{s^2 - \omega^2}$$

$$\alpha_1 = -s + \sqrt{s^2 - \omega^2} \qquad \alpha_2 = -s - \sqrt{s^2 - \omega^2}$$

$$\therefore x' = \boxed{A_1 e^{(-s+\sqrt{s^2-\omega^2})t} + A_2 e^{(-s-\sqrt{s^2-\omega^2})t}} \longrightarrow \text{General sol}^n \text{ of eq}^n \text{ (II)}$$

— (III)

$A_1, A_2 \to$ arbitrary constant

$\therefore s, \omega \leftarrow$ sol^n depends on

(I) $s^2 > \omega^2$ (II) $s^2 - \omega^2$ (III) $s^2 < \omega^2$

Case (I): $s^2 > \omega^2$ — "Overdamped"

$\because s^2 > \omega^2$

$\therefore \sqrt{s^2-\omega^2} \to$ real, +ve, less than 's'

- Hence, both components (exponents) in (III) are -ve, and the displacement x' decreases exponentially without performing any oscillation (continuously to zero)

↳ "Overdamped"

Case (II): $s^2 = \omega^2$ — "Critically damped"

$\therefore \sqrt{s^2-\omega^2} = \beta$ (we assume it's not exactly zero instead a very small quantity)

From eq^n (III),

$$\boxed{x' = A_1 e^{(-s+\beta)t} + A_2 e^{(-s-\beta)t}}$$

$$x' = e^{-st}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$$

$$x' = e^{-st}\left[A_1\left(1 + \beta t + \frac{(\beta t)^2}{2!} + \ldots\right) + A_2\left(1 + (-\beta t) + \frac{(-\beta t)^2}{2!} + \ldots\right)\right]$$

$$= e^{-st}\left[(A_1 + A_2) + \beta t (A_1 - A_2)\right]$$

let $P' = A_1 + A_2$

$Q' = (A_1 - A_2)\beta$

$$= e^{-st}(P' + t\, Q')$$

if $t \to \uparrow$, $P' + Q't \to \uparrow$ and $e^{-st} \to \downarrow$ and finally tends to zero

↳ "Critical damped motion"

e.g. – Voltmeter and Ammeter, the pointer moves to ~~current~~ correct position and comes to rest without oscillation

Case (III): $s^2 < \omega^2$ → "Oscillatory Motion"

↳ "Damped harmonic oscillator"

$\therefore \sqrt{s^2-\omega^2} \to$ imaginary

$= i\sqrt{\omega^2 - s^2}$

$= i\beta'$ $\left(\beta' = \sqrt{(\omega^2 - s^2)}\right)$

From eq^n (III),

$$x' = A_1 e^{(-s+i\beta')t} + A_2 e^{(-s-i\beta')t}$$

$$= e^{-st}\left(A_1 e^{i\beta' t} + A_2 e^{-i\beta' t}\right)$$

$$= e^{-st}\left[A_1\left(\cos(\beta' t) + i\sin(\beta' t)\right) + A_2\left(\cos(-\beta' t) + i\sin(-\beta' t)\right)\right]$$

$$= e^{-st}\left[\left(A_1 \cos(\beta' t) + i\sin(\beta' t)\right) + \left(A_2 \cos(\beta' t) - i\sin(\beta' t)\right)\right]$$

$$= e^{-st}\left[\cos(\beta' t)\cdot\underbrace{(A_1 + A_2)}_{A\sin\delta} + \sin(\beta' t)\cdot\underbrace{i(A_1 - A_2)}_{A\cos\delta}\right]$$

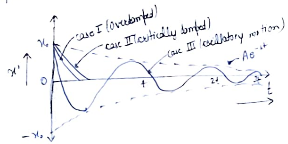
$$= e^{-st}\left[A\sin\delta\cdot\cos\beta' t + A\cos\delta\sin\beta' t\right]$$

$$= Ae^{-st}\sin(\delta + \beta' t)$$

$$\boxed{x' = Ae^{-st}\cdot\sin\left(t\sqrt{\omega^2 - s^2} + \delta\right)}$$

↳ "Oscillatory Motion" → The oscillations are not simple harmonic because amplitude Ae^{-st} is not constant but $\downarrow$ing with time.

Graph:



Forced Oscillation

$$m\frac{d^2x}{dt^2} + \underbrace{b\frac{dx}{dt}}_{\text{Damping force}} + \underbrace{kx}_{\text{Restoring force}} = \underbrace{F_0 \sin \omega'' t}_{\text{Applied force}}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega'' t$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = f_0 \sin \omega'' t$$

where, $\omega^2 = \frac{k}{m}$; $\underbrace{2\lambda}_{\text{Damping constant}} = \frac{b}{m}$; $f_0 = \frac{F_0}{m}$

$\Rightarrow \underbrace{\omega}_{\text{natural frequency}} = \sqrt{\frac{k}{m}}$

Let $x = A\sin(\omega'' t - \theta)$

$$\frac{dx}{dt} = A\omega'' \cos(\omega'' t - \theta)$$

$$\frac{d^2x}{dt^2} = -A(\omega'')^2 \sin(\omega'' t - \theta)$$
$$= -(\omega'')^2 x$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = f_0 \sin(\omega'' t)$$

$\Rightarrow -(\omega'')^2 x + 2\lambda A\omega'' \cos(\omega'' t - \theta) + \omega^2 A \sin(\omega'' t - \theta) = f_0 \sin(\omega'' t)$

$\Rightarrow -(\omega'')^2 A \sin(\omega'' t - \theta) + 2\lambda A\omega'' \cos(\omega'' t - \theta) + \omega^2 A \sin(\omega'' t - \theta) = f_0 \sin(\omega'' t)$

$\Rightarrow A\omega'' \left(-\omega'' \sin(\omega'' t - \theta) + 2\lambda \cos(\omega'' t - \theta) + \frac{\omega^2 A}{\omega''} \sin(\omega'' t - \theta)\right) = f_0 \sin(\omega'' t)$

$= f_0 \sin(\omega'' t - \theta + \theta)$

$= f_0 [\sin(\omega'' t - \theta)\cos\theta + \cos(\omega'' t - \theta)\sin\theta]$

$\Rightarrow -(\omega'')^2 A \sin(\omega'' t - \theta) + \omega^2 A \sin(\omega'' t - \theta) - f_0 \sin(\omega'' t - \theta)\cdot\cos\theta$
$= f_0 \cos(\omega'' t - \theta)\cdot\sin\theta - 2\lambda A\omega'' \cos(\omega'' t - \theta)$

$\Rightarrow \sin(\omega'' t - \theta)\,[-(\omega'')^2 A + \omega^2 A - f_0 \cos\theta] = \cos(\omega'' t - \theta)\,[f_0 \sin\theta - 2\lambda A\omega'']$

$\Rightarrow [-(\omega'')^2 A + \omega^2 A - f_0 \cos\theta]\,\sin(\omega'' t - \theta) = [f_0 \sin\theta - 2\lambda A\omega'']\cos(\omega'' t - \theta)$

$\Rightarrow [-(\omega'')^2 A + \omega^2 A - f_0 \cos\theta]\,\sin(\omega'' t - \theta) - [f_0 \sin\theta - 2\lambda A\omega'']\cos(\omega'' t - \theta) = 0$

$\Rightarrow [-(\omega'')^2 A + \omega^2 A - f_0 \cos\theta]$

$-A(\omega'')^2 - f_0\cos\theta + \omega^2 A = 0$ and $f_0 \sin\theta - 2\lambda A\omega'' = 0$

$\Rightarrow f_0 (\cos\theta) = A\omega^2 - A(\omega'')^2$ — (I) $\qquad \Rightarrow f_0 \sin\theta = 2\lambda A\omega''$ — (II)

$(I)^2 + (II)^2$,

$$\Rightarrow f_0^2 = [A\omega^2 - A(\omega'')^2]^2 + [2\lambda A\omega'']^2$$

$$\Rightarrow f_0^2 = A^2\omega^4 + A^2(\omega'')^4 - 2A^2\omega^2(\omega'')^2 + 4\lambda^2 A^2 (\omega'')^2$$

$$\Rightarrow f_0^2 = A^2[\omega^4 + (\omega'')^4 - 2\cdot\omega^2(\omega'')^2 + 4\lambda^2(\omega'')^2]$$

$$\Rightarrow A^2 = \frac{f_0^2}{\omega^4 + (\omega'')^4 - 2\omega^2(\omega'')^2 + 4\lambda^2(\omega'')^2}$$

$$\Rightarrow A = \pm \frac{f_0}{\sqrt{\omega^4 + (\omega'')^4 - 2\omega^2(\omega'')^2 + 4\lambda^2(\omega'')^2}}$$

$$\Rightarrow \boxed{A = \frac{f_0}{\sqrt{[\omega^2 - (\omega'')^2]^2 + 4\lambda^2(\omega'')^2}}}$$

$$\boxed{\tan\theta = \frac{2\lambda\omega''}{(\omega^2 - (\omega'')^2)}} \qquad (I \div II)$$

$$\therefore \boxed{x' = \frac{f_0}{\sqrt{[\omega^2 - (\omega'')^2]^2 + 4\lambda^2(\omega'')^2}} \sin(\omega'' t - \theta)}$$

x_{max} when $\boxed{\omega = \omega''}$ "Resonant frequency"

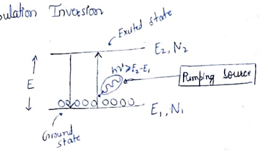
The system is ~~undamped and~~ vibrate with frequency ω'' but will have a phase lag of $\boxed{\theta = \tan^{-1}\frac{2\lambda\omega''}{\omega^2 - \omega''^2}}$

LASER

(Light Amplification by Stimulated Emission of Radiation)

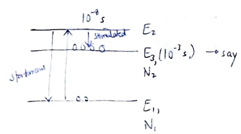
(i) Absorption
(ii) Spontaneous Emission — radiationless
(iii) Stimulated Emission

→ Pumping
→ Population Inversion



$\boxed{h\nu \geq E_2 - E_1}$

- e^- stay in excited state for 10^{-8} s.
- For LASER, lifetime of e^- in excited state should be $> 10^{-8}$ s



- if $\boxed{N_2 > N_1}$, it is called "population inversion", i.e higher energy state has more atoms than ground state.

- Only the stimulated emission atoms contribute to LASER.

(i) Absorption

$P_{12} \propto u(\nu)$
└ number/energy density

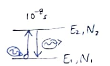
- Probability of occurence of absorption from state 1 to state 2 is proportional to energy density $u(\nu)$ of the radiation

$\boxed{P_{12} = B_{12} \cdot u(\nu)}$

- B_{12} → proportionality constant known as "Einstein's coefficient of absorption of radiation"

(ii) Spontaneous Emission

- Probability of occurence of spontaneous emission — transition from state 2 to 1 depends only on the property of state 2 and 1



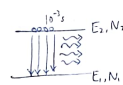
$\boxed{P'_{21} = A_{21}}$

- A_{21} → "Einstein's coefficient of spontaneous emission of radiation"

(iii) Stimulated Emission ($N_2 > N_1$)

- Probability of occurence of stimulated emission transition from upper level to state is proportional to number density $u(\nu)$

$P''_{21} \propto u(\nu)$



$\boxed{P''_{21} = B_{21} \, u(\nu)}$

- B_{21} → "Einstein's coefficient of stimulated emission of radiation"

$\boxed{P_{21} = P'_{21} + P''_{21}}$ (Total probability of emission)

$N_1 \times$ total absorption $= N_2 \times$ total emission

$\boxed{N_1 \cdot P_{12} = N_2 \cdot P_{21}}$ (at thermal equilibrium) $\Rightarrow N_1 \cdot B_{12} \cdot u(\nu) = N_2 [A_{21} + B_{21} u(\nu)]$

$$N_1 \cdot P_{12} = N_2 \cdot P_{21}$$

$$\Rightarrow N_1 \cdot B_{12} \cdot \upsilon(\gamma) = N_2 \left[A_{21} + B_{21} \cdot \upsilon(\gamma)\right]$$

$$\Rightarrow \upsilon(\gamma) \left[N_1 B_{12} - N_2 \cdot B_{21}\right] = N_2 \cdot A_{21}$$

$$\Rightarrow \boxed{\upsilon(\gamma) = \frac{N_2 \cdot A_{21}}{N_1 \cdot B_{12} - N_2 \cdot B_{21}}}$$