INFORMS fORGED Competition Report

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1 Introduction

In this contest we used monthly demand data for hardware to creates a non-clairvoyant, optimal inventory system. Our approach to designing this inventory system involves a two tier method that (1) predicts demand based on history till current time and (2) Utilizes estimated demand to solve a stochastic optimization problem that minimizes operational costs.

Initially, we estimated demand using an Auto Regressive Integrated Moving Average (ARIMA). Then we exploited the asymmetries in the holding versus back order cost to minimize an objective that scaled with the size of the expected positive error (positive difference between the expected demand and current inventory). This objective was lower bounded by the positive error and a scaling factor, α was tuned via line search to minimize costs. Despite some promising performance, we shifted away from this model because we felt that it was highly dependent on the accuracy of our predictions and may be highly susceptible to unexpected changes in the demand distributions. We shifted towards a technique that exploits the structure of our data which we hypothesize will perform better under extreme demand distribution shifts.

In the next sections we cover the selected Method, Algorithm summary and Implementation and conclusions.

2 Methods

In this section, we discuss our mathematical model. We start by defining the notation to be used subsequently. Let

2.1 Notations

- $x^+ = \max(x, 0)$.
- $\mathbb{E}(X)$ is the expectation of a random variable X.
- $L^0(X_1, \dots, X_n)$ is the set of all random variables which is measurable by the σ-field generated by $X_1, \dots, X_n$.

2.2 Problem Formulation

Let

- n be the number of months in the period we are optimizing over.
- $Q \in \mathbb{R}^n_{\geq 0}$ be a sequence of non-negative real numbers represents order quantity at each month.
- $D \in \mathbb{R}^n_{\geq 0}$ be a sequence of demands in each month. We assume D is a stochastic process.
- $I_0 \in \mathbb{R}_{\geq 0}$ be the initial inventory level.

Then, the sequence of inventory level at the beginning of each month is

$$I_i = (Q_i + I_{i-1} - D_i)^+, i = 1, \dots, n,$$

where $x^+ = \max(x, 0)$. Because D is given and I is a function of D and Q, our decision variable is Q. The expected total cost function is

$$\begin{aligned}\mathbb{E}[F(Q)] &= \sum_{i=1}^{n} \mathbb{E}[f(Q_i, I_{i-1}, D_i)], \\ f(x, y, z) &= \left[(x + y - z)^+ + (x + y - z - 90)^+ + 3(z - x - y)^+\right].\end{aligned}$$

Because it is not allowed to use unseen demand value, Q_i should be a function of $D_1, \dots, D_{i-1}$. Hence, our optimization problem is

$$\min \mathbb{E}[F(Q)] \text{ subject to } Q_i \in L^0(D_1, \dots, D_{i-1}), Q_i \geq 0. \tag{$\star$}$$

Our approach to tackle this problem is a greedy way. Namely, we solve the following optimization problem at each month.

$$\min \mathbb{E}[f(q, I_{i-1}, D_i)] \text{ subject to } q \in L^0(D_1, \dots, D_{i-1}), q \geq 0. \tag{$\star\star$}$$

In other words, we optimize the order quantity q for the cost in the current month, not the following total cost, given previous observations, $D_0, \dots, D_{i-1}$ at month i.

Remark 1. Note that the local optimality of order quantities in ($\star\star$) does not imply the global optimality in ($\star$). The following is a brief observation. Assume ($\star$) has an optimal solution $Q^\star$. For contradiction, suppose that there exists $i \in \{1, \dots, n\}$ for which there exists $q \in L^0(D_1, \dots, D_{i-1})$ such that $E[f(Q_i^\star, I_{i-1}, D_i)] > E[f(q, I_{i-1}, D_i)]$. I.e. $Q^\star$ is not optimal at month i. Then, one naive idea to deduce contradiction is to replace $Q_i^\star$ with q. However, if we do so, the beginning inventory in the next month I_i also changes. Let I_i be the beginning inventory of $Q_i^\star$ and $\tilde{I}_i$ be that of q. Suppose we order q at month i. Then, if $I_i - \tilde{I}_i + Q_{i+1}^\star \geq 0$, we can order $I_i - \tilde{I}_i + Q_{i+1}^\star$ to realize the same inventory level as $Q^\star$ at month $i + 1$. By following $Q^\star$ for the following months

$i+2,\dots,n$, we obtain a series of order quantity which has smaller cost. However, $I_i - \tilde{I}_i + Q^\star_{i+1} \geq 0$ is not always true. In the bad scenario $I_i - \tilde{I}_i + Q^\star_{i+1} < 0$, we cannot adjust the inventory level in the next moth. Furthermore, it may not be impossible to adjust in the following months depending on the structure of the demand series. Since the demand structure is unknown at this point, we decided to employ a greedy method as a heuristic.

Remark 2. If the demand process is a Markov process, the problem can be formulated as a reinforcement learning problem. However, an ARIMA model fitted well to the demand process in our first few experiments, hence it seems more natural to assume the demand process depends on the history and it is not a Markov process.

Hence, this problem is almost the same as the news vendor problem [2], but with a slightly different cost function.

Proposition 1. *Consider* $(\star\star)$ *given* $D_1,\dots,D_{i-1}$*. The objective function is convex and the first derivative is*

$$\frac{\partial}{\partial q}\mathbb{E}[f(q, I_{i-1}, D_i)] = 4G(q + I_{i-1}) + G(q + I_{i-1} - 90) - 3,$$

where G *is the cumulative distribution function of* D_i*.*

Proof. Because f is convex in q, so is the expectation. The first derivative is

$$\begin{aligned}
&\frac{\partial}{\partial q}\mathbb{E}[f(q, I_{i-1}, D_i)] \\
&= \frac{\partial}{\partial q}\int (q + I_{i-1} - x)^+ + (q + I_{i-1} - x - 90)^+ + 3(x - q - I_{i-1})\,G(dx) \\
&= \int \mathbf{1}_{x \leq q + I_{i-1}} + \mathbf{1}_{x \leq q + I_{i-1} - 90} - 3 \cdot \mathbf{1}_{x \geq q + I_{i-1}}\,G(dx) \\
&= G(q + I_{i-1}) + G(q + I_{i-1} - 90) - 3(1 - G(q + I_{i-1})) \\
&= 4G(q + I_{i-1}) + G(q + I_{i-1} - 90) - 3.
\end{aligned}$$

□

Now, we establish an optimality condition for $(\star\star)$.

Theorem 1. *If* G *is continuous and strictly increasing,* $(\star\star)$ *has a unique minimizer.*

Proof. We first consider the unconstrained version of $(\star\star)$. Because the objective function is convex, the existence of a minimizer is equivalent to the existence of a point that satisfies the first-order condition

$$4G(q + I_{i-1}) + G(q + I_{i-1} - 90) - 3 = 0.$$

Because

$$\lim_{q \to -\infty} 4G(q + I_{i-1}) + G(q + I_{i-1} - 90) - 3 = -3,$$
$$\lim_{q \to +\infty} 4G(q + I_{i-1}) + G(q + I_{i-1} - 90) - 3 = 2,$$

and G is continuous, there exists q which satisfies the first order condition. Also, since G is strictly increasing, such q is unique. Let q^* be the unique optimizer of the unconstrained problem. Because I_{i-1} is $\sigma(D_1 \ldots, D_{i-1})$-measurable, so is q^*. However, it could be a negative value. Because the objective function is convex, $\max(q^*, 0)$ is the minimizer to $(\star\star)$. $\square$

Remark 3. The theorem still holds for the general distributions, and this theorem is enough for our model because G will be assumed to be a normal distribution later.

2.3 The model of demands

To predict the demand at each month, we employed the ARIMA (autoregressive integrated moving average) model, which is a standard model for time series. The ARIMA model is used in the rolling basis. The rolling basis means the following. At each month t, we fit the model on the observed data up to month $t-1$.

We briefly review the ARIMA model before going into the detail of our model. ARIMA model is a combination of 3 time series models. The AR model is mostly like a linear regression in static estimation. $AR(p)$ is defined as following.

$$X_t = \sum_{i=1}^{p} \alpha_i X_{t-i} + \varepsilon_t,$$

where X is the target time series and ε_t is the noise at time t. Normally, the residuals are assumed to be independent and have normal distribution with mean 0 [1].

$MA(q)$ model is defined as following.

$$X_t = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

Lastly, "integrated" refers the order of differencing before fit the ARMA model. The difference operator Δ takes a time series as the input and returns another time series. It is defined by

$$\Delta X_\bullet = X_{\bullet+1} - X_\bullet.$$

Note that if the initial values is given, the original process can be recovered by taking a cumulative sum.

Thus, ARIMA has 3 parameters one each from AR, order of differencing. In summary, $ARIMA(p, d, q)$ is defined as following. Let $Y = \Delta^d X$. Then,

$$Y_t = \sum_{i=1}^{p} \alpha_i Y_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

For the sake of the simplicity in the modeling (explained later), we keep the MA parameter 0. This means that we fit an AR model after differencing. To determine the differencing order, we applied the augmented Dickey-Fuller (ADF) test. The AR model assumes the process is stationary, and ADF test is a standard statistical test to examine the stationarity of the process. To end this, we first removed the annual seasonality term. And applied ADF test with the relevant critical value 5%. The resulting p-value is about 37%, which means the process is not stationary. The process is differenced once and again the ADF test is applied. The p-value is 0.0001%, which implies that the differenced process is stationary. From this result, the order of differencing d is set to be 1. The AR parameter is tuned by a validation step, which is explained later.

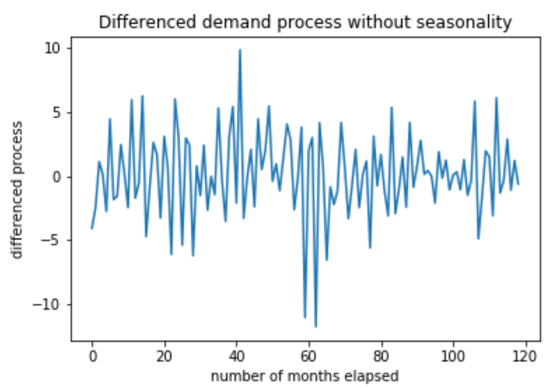

Figure 1: The demand process is differenced after the seasonality is removed.

Let $\mathrm{AR}_p(X_1, \ldots, X_n)$ refer an ARIMA$(p, 1, 0)$ one-time-ahead predictor trained on $X_1, \ldots, X_n$. Namely, it produces a prediction at time t based on the observations up to month $t-1$.

$$\hat{X}_t = \mathrm{AR}_p(X_1, \ldots, X_{t-1}) := \sum_{i=1}^{p} \alpha_i X_{t-i}.$$

Now, our model of demand is

$$D_t = \mathrm{AR}_p(D_1, \ldots, D_{t-1}) + \varepsilon_t,$$

where D is the observed demand process and ε_t is the residual at month t. Now, we impose an assumption on the structure of residuals.

Assumption 1. *$\varepsilon_1, \ldots, \varepsilon_n$ are independent and identically distributed. Furthermore, they have normal distribution with mean 0 and a common variance.*

Remark 4. This looks the same as the assumption of the AR model. However, we cannot deduce it because they are from different AR models.

Under this assumption, the first order condition becomes

$$4\Phi_\sigma(q + I_{i-1} - \hat{D}_i) + \Phi_\sigma(q + I_{i-1} - \hat{D}_i - 90) - 3 = 0,$$

where Φ_σ is the cumulative distribution function of a normal distribution with mean 0 and standard deviation σ. In our implementation, we use the maximum likelihood variance estimator for normal distribution, $\hat{\sigma}$ to replace unknown σ and the first-order condition is solved by a line search.

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2.$$

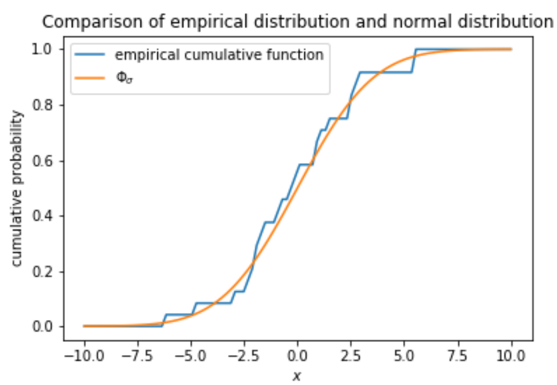

Figure 2: The empirical distribution and a normal distribution with mean 0 and the same standard deviation are compared. It seems assuming the residuals are normally distributed makes sense in practice.

Now we are set to discuss the validation step to determine the AR parameter. We split the given 10-year data into 8-year training data set and 2-year test data set. The last 2 years of the training set is used to compute $\hat{\sigma}$ based on different parameters of the AR models and the order quantity is computed over the period of test set by solving the first order condition. Finally, the total cost over the period of the test data set is computed.

As a result, we picked up 3 for the AR model parameter. The reason is that a big value leads to over-fitting and lose generalizability and also the cost does not change largely even if we pick a large parameter.

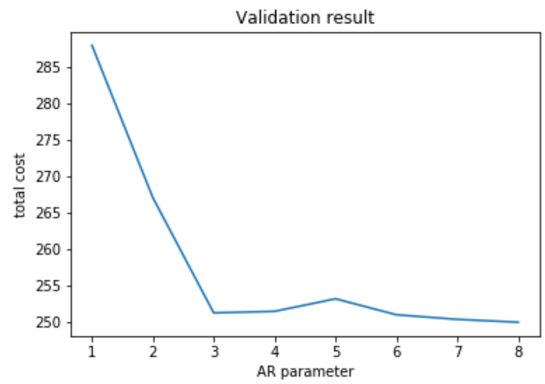

Figure 3: The validation result

2.4 Algorithm summary and implementation

The following is our algorithm to compute order quantity over the period from January 2006 to December 2007. The input is "Ten-Year-Demand.csv", which we call the training data set, and the test data set in the same format. We have two stages to produce order quantities and one stage to evaluate the results.

2.4.1 Training stage

1. The training data set is split into two parts; the first 8 years and the last 2 years.

2. The demands over the last 2 years in the training data set are predicted on the rolling basis.

3. Subtract the predicted demands from the observed demands over the last 2 years in the training data to get the residuals.

4. The sample variance of the residuals computed in the previous step is computed.

2.4.2 Prediction stage

1. At the beginning of each month, the demand in the month is predicted by fitting an ARIMA model on the observed demands up to the previous month.

2. The order quantity is computed by solving the first-order condition. The equation is solved by a simple line search.

2.4.3 Evaluation stage

At last, we compute the monthly cost and the total cost for the order quantities produced in the previous prediction stage.

3 Conclusion

We formulated the problem as a functional optimization problem and developed a greedy heuristic using a modified News-vendor model. We established an optimality condition to the sub-problems and built a numerical scheme allowing the creation of our planning system.

References

[1] Wikipedia contributors. Autoregressive integrated moving average — Wikipedia, the free encyclopedia, 2019. [Online; accessed 11-October-2019].

[2] Wikipedia contributors. Newsvendor model — Wikipedia, the free encyclopedia, 2019. [Online; accessed 11-October-2019].