CS 228 : Logic in Computer Science

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The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- Intuitively, $p \rightarrow q \vdash \neg p \vee q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

Semantics

- Each propositional variable is assigned values true/false. Truth tables for each of the operators $\vee, \wedge, \neg, \rightarrow$ to determine truth values of complex formulae.
- $\varphi_1, \ldots, \varphi_n \models \psi$ iff whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true, so does ψ. $\models$ is read as semantically entails
- Two formulae φ and ψ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$
- Two formulae φ and ψ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$

Soundness of Propositional Logic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever ψ can be proved from $\varphi_1, \ldots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true

Soundness

- Assume $\varphi_1, \ldots, \varphi_n \vdash \psi$.
- There is some proof (of length k lines) that yields ψ. Induct on k.
- When $k = 1$, there is only one line in the proof, say φ, which is the premise. Then we have $\varphi \vdash \varphi$, since φ is given. But then we also have $\varphi \models \varphi$.
- Assume that whenever $\varphi_1, \ldots, \varphi_n \vdash \psi$ using a proof of length $\leqslant k - 1$, we have $\varphi_1, \ldots, \varphi_n \models \psi$.
- Consider now a proof with k lines.

Soundness : Case $\wedge i$

- How did we arrive at ψ? Which proof rule gave ψ as the last line?
- Assume ψ was obtained using $\wedge i$. Then ψ is of the form $\psi_1 \wedge \psi_2$.
- ψ_1 and ψ_2 were proved earlier, say in lines $k_1, k_2 < k$.
- We have the shorter proofs $\varphi_1, \ldots, \varphi_n \vdash \psi_1$ and $\varphi_1, \ldots, \varphi_n \vdash \psi_2$
- By inductive hypothesis, we have $\varphi_1, \ldots, \varphi_n \models \psi_1$ and $\varphi_1, \ldots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$.

Soundness : Case $\to i$

- Assume ψ was obtained using $\to i$. Then ψ is of the form $\psi_1 \to \psi_2$.
- A box starting with ψ_1 was opened at some line $k_1 < k$.
- The last line in the box was ψ_2.
- The line just after the box was ψ.
- Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \to \psi_2$
- The equivalence of $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \to \psi_2$ and $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness : Other cases

Completeness

$\varphi_1, \ldots, \varphi_n \models \psi \Rightarrow \varphi_1, \ldots, \varphi_n \vdash \psi$

Whenever $\varphi_1, \ldots, \varphi_n$ semantically entail ψ, then ψ can be proved from $\varphi_1, \ldots, \varphi_n$. The proof rules are complete

Completeness : 3 steps

- Given $\varphi_1, \ldots, \varphi_n \models \psi$
- Step 1: Show that $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$
- Step 2: Show that $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$
- Step 3: Show that $\varphi_1, \ldots, \varphi_n \vdash \psi$

Completeness : Step 1

- Assume $\varphi_1, \ldots, \varphi_n \models \psi$. Whenever all of $\varphi_1, \ldots, \varphi_n$ evaluate to true, so does ψ.
- If $\not\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots (\varphi_n \rightarrow \psi) \ldots))$, then ψ evaluates to false when all of $\varphi_1, \ldots, \varphi_n$ evaluate to true, a contradiction.
- Hence, $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots (\varphi_n \rightarrow \psi) \ldots))$.

Completeness : Step 2

- Given $\models \psi$, show that $\vdash \psi$
- Assume $p_1, \ldots, p_n$ are the propositional variables in ψ. We know that all the 2^n assignments of values to $p_1, \ldots, p_n$ make ψ true.
- Using this insight, we have to give a proof of ψ.

Completeness : Step 2

Truth Table to Proof

Let φ be a formula with variables $p_1, \ldots, p_n$. Let $\mathcal{T}$ be the truth table of φ, and let l be a line number in $\mathcal{T}$. Let $\hat{p}_i$ represent p_i if p_i is assigned true in line l, and let it denote $\neg p_i$ if p_i is assigned false in line l. Then

1. $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
2. $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$ if φ evaluates to false in line l