CS 228 : Logic in Computer Science

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Rules for Natural Deduction

Another implies elimination rule or Modus Tollens MT

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.	$\neg p$	MT 1,3

More Rules

- Thanks to MT, we have $p \rightarrow q, \neg q \vdash \neg p$.
- Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- So far, no proof rule that can do this.
- Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.	$(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$	$\rightarrow i$ 2-10

Transforming Proofs

- $(q \to r), (\neg q \to \neg p), p \vdash r$
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots))$ by adding n lines of the rule $\to i$

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \land q$	assumption
3.	p	$\land e_1$ 2
4.	q	$\land e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.	$p \land q \to r$	$\to i$ 2-6

More Rules

The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi \vee \psi}$$

The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

More Rules

The or elimination rule $\vee e$

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e$ (2)
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.	$(p \vee q) \to (p \vee r)$	$\to i$ 2-8

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee e$ (2)
10.	$q \vee r$	$\vee i_2$ 9
11.	$p \vee (q \vee r)$	$\vee i_2$ 10
12.	$p \vee (q \vee r)$	$\vee e$ 1, 2-8, 9-11

Basic Rules So Far

- $\wedge$i, $\wedge e_1$, $\wedge e_2$ (and introduction and elimination)
- $\neg\neg e$, $\neg\neg i$ (double negation elimination and introduction)
- MP (Modus Ponens)
- $\rightarrow i$ (Implies Introduction : remember opening boxes)
- $\vee i_1, \vee i_2, \vee e$ (Or introduction and elimination)

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
2.	p	assumption
3.	q	assumption
4.	p	copy 2
5.	$q \rightarrow p$	$\rightarrow i$ 3-4
6.	$p \rightarrow (q \rightarrow p)$	$\rightarrow i$ 2-5

The Rules of Single Negation

- We have seen $\neg\neg e$ and $\neg\neg i$, the elimination and introduction of double negation.
- How about introducing and eliminating single negations?
- We use the notion of contradictions, an expression of the form $\varphi \wedge \neg\varphi$, where φ is any propositional logic formula.
- Any two contradictions are equivalent : $p \wedge \neg p$ is equivalent to $\neg r \wedge r$. Contradictions denoted by $\perp$.
- $\perp \to \varphi$ for any formula φ.

Rules with $\perp$

The $\perp$ elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg\varphi}{\perp}$$

An Example

- $\neg p \lor q \vdash p \to q$

1.	$\neg p \lor q$	premise
2.	$\neg p$	$\lor e$ (1)
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.	$p \to q$	$\to i$ 3-5
7.	q	$\lor e$ (2)
8.	p	assumption
9.	q	copy 7
10.	$p \to q$	$\to i$ 8-9
11.	$p \to q$	$\lor e$ 1, 2-6, 7-10

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain $\perp$ in the box, then we conclude $\neg\varphi$
- This rule is denoted $\neg i$ and is read as $\neg$ introduction.
- Also known as Proof By Contradiction

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.	$\bot$	$\bot i$ 2,3
5.	$\neg p$	$\neg i$ 2-4

The Last One

Law of the Excluded Middle (LEM)

$$\frac{}{\varphi \vee \neg\varphi}$$

Summary of Basic Rules

- $\wedge i$, $\wedge e_1$, $\wedge e_2$,
- $\neg\neg e$
- MP
- $\rightarrow i$
- $\vee i_1, \vee i_2, \vee e$
- Copy, $\neg i$ or PBC
- $\bot e, \bot i$

Derived Rules

- MT (derive using MP, $\bot i$ and $\neg i$)
- $\neg\neg i$ (derive using $\bot i$ and $\neg i$)
- LEM (derive using $\vee i_1, \bot i, \neg i, \vee i_2, \neg\neg e$)

The Proofs So Far

- So far, the “proof” we have seen is purely syntactic, no true/false meanings were attached
- Intuitively, $p \rightarrow q \vdash \neg p \vee q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

Semantics

- Each propositional variable is assigned values true/false. Truth tables for each of the operators $\vee, \wedge, \neg, \rightarrow$ to determine truth values of complex formulae.
- $\varphi_1, \ldots, \varphi_n \models \psi$ iff whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true, so does ψ. $\models$ is read as semantically entails
  - Recall $\vdash$, and compare with $\models$
- Formulae φ and ψ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$
- Formulae φ and ψ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$