# CS 228 : Logic in Computer Science

Krishna. S

# FO without equality

Let us focus on FO without "=". Recall that "=" is always interpreted as equality.

**Herbrand Theorem**

Let $\Gamma = \{\varphi_1, \varphi_2, \ldots\}$ be a set of equality-free sentences in Skolem Normal Form. Then $\Gamma$ is satisfiable iff $\Gamma$ has a Herbrand model.

If $\Gamma$ has a Herbrand model, clearly $\Gamma$ is satisfiable. The converse needs a proof.

# The converse

Assume $\Gamma$ is satisfiable. Let $\tau_H$ be the Herbrand signature for $\Gamma$.

- Let $\mathcal{A}$ be a $\tau_H$ structure such that $\mathcal{A} \models \Gamma$. ($U^{\mathcal{A}}$ need not be the Herbrand universe)
- Let $\mathcal{B}$ be a Herbrand structure over $\tau_H$. ($U^{\mathcal{B}}$ is the Herbrand universe)
- We need to create a Herbrand model for $\Gamma$
- Try "merging" $\mathcal{A}$ and $\mathcal{B}$ to obtain a Herbrand model $M$ for $\Gamma$.
    - Define $M$ so that its universe is the Herbrand universe over $\tau_H$.
    - Let $M$ interpret functions and constants like $\mathcal{B}$ (both have the same Herbrand universe)
    - How to interpret the relations from $\tau_H$ (same as those of $\tau$)?
    - Let $M$ interpret relations like $\mathcal{A}$ (not obvious, their universes are not the same.)

# Building the Herbrand Model $M$

- Let $R$ be ant $n$-ary relation in $\tau_H$ (hence in $\tau$).
- For each $n$-tuple $(t_1, \ldots, t_n)$ with $t_i$ coming from the Herbrand universe $H(\Gamma)$, we must say whether $(t_1, \ldots, t_n) \in R^M$ or not
- Each $t_i \in H(\Gamma)$ is a ground term in $\tau_H$ (so variable free).
- Since $\mathcal{A}$ is a structure over $\tau_H$, if $t \in H(\Gamma)$ is a ground term from $\tau_H$, $\mathcal{A}$ interprets $t$ as an element of $U^{\mathcal{A}}$.
- For each $n$-tuple $(t_1, \ldots, t_n)$, $t_i \in H(\Gamma)$, we know whether $(t_1, \ldots, t_n) \in R^{\mathcal{A}}$ or not
- Define $R^M = R^{\mathcal{A}}$.
- Prove that if $\mathcal{A} \models \varphi$ for any $\varphi \in \Gamma$, then $M \models \varphi$.
- The proof is by induction on the number of quantifiers in $\varphi$. Recall that each $\varphi$ is in Skolem Normal Form.

# Base case : $\varphi$ has 0 quantifiers

$\mathcal{A} \models \varphi$ iff $M \models \varphi$. Do structural induction on $\varphi$.

- Assume $\varphi$ is an atomic formula. Then $\varphi$ is $R(t_1, \ldots, t_n)$ where $R$ is an $n$-ary relation from $\tau_H$, and $t_1, \ldots, t_n$ are all terms from $H(\Gamma)$.
- By the construction of $M$, $R^M = R^{\mathcal{A}}$.
- Hence $M \models \varphi$ iff $\mathcal{A} \models \varphi$.
- Same reasoning holds for $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ and $\neg\varphi$.
- Hence, $\mathcal{A} \models \varphi$ iff $M \models \varphi$.

# Post Inductive Hypothesis

Assume that for any $\psi \in \Gamma$ with $\leqslant k-1$ quantifiers, if $\mathcal{A} \models \psi$, then $M \models \psi$. Let $\varphi \in \Gamma$ have $k$ quantifiers, $\varphi = \forall x_1 \forall x_2 \ldots \forall x_k\ \zeta(x_1, \ldots, x_k)$ where $\zeta$ is quantifier free.

- Let $\kappa(x_1) = \forall x_2 \ldots \forall x_k\ \zeta(x_1, \ldots, x_k)$, and $\varphi = \forall x_1 \kappa(x_1)$.
- $\mathcal{A} \models \varphi$ implies $\mathcal{A} \models \forall x_1 \kappa(x_1)$. That is, $\mathcal{A} \models \kappa(a)$ for any $a \in U^{\mathcal{A}}$.
- Since $\mathcal{A}$ is a structure over $\tau_H$, if $t \in H(\Gamma)$ is a ground term from $\tau_H$, $\mathcal{A}$ interprets $t$ as an element of $U^{\mathcal{A}}$.
- Thus, $\mathcal{A} \models \kappa(t)$ for any $t \in H(\Gamma)$.
- By induction hypothesis, $M \models \kappa(t)$ for any $t \in H(\Gamma)$.
- Since $H(\Gamma)$ is the universe of $M$, $M \models \forall x_1 \kappa(x_1)$. That is, $M \models \varphi$.

# Equality

$\varphi = \forall x[(f(x) \neq x) \wedge (f(f(x)) = x)]$.

- Herbrand universe over $\tau_H = \{c, f\}$ is $\{c, f(c), f(f(c)), \dots\}$ all distinct
- Then $f(f(c)) \neq c$, and a Herbrand structure cannot satisfy $\varphi$
- However, $\varphi$ is satisfiable. Define a structure $\mathcal{A} = (\{0, 1\}, f^{\mathcal{A}}(0) = 1, f^{\mathcal{A}}(1) = 0)$, $\mathcal{A} \models \varphi$
- For formulae which have equality, Herbrand's Theorem does not apply directly
- If $\varphi$ has equality, convert it to an equisatisfiable sentence without equality and apply Herbrand

Let $\varphi$ be in Skolem normal form with equality. Then $\varphi$ is satisfiable iff there is an equisatisfiable formula $\psi$ in Skolem normal form without equality which has a Herbrand model.