Logic in CS Fall 2025

Problem Sheet 2

S. Krishna

1. An **adequate set of connectives** is a set such that for every formula, there is an equivalent formula with only connectives from that set. For example, $\{\neg, \vee\}$ is adequate for propositional logic since any occurrence of $\wedge$ and $\rightarrow$ can be removed using the equivalences:

$$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$
$$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$$

   (a) Show that $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, and $\{\rightarrow, \perp\}$ are adequate sets of connectives. ($\perp$ treated as a nullary connective).

   (b) Show that if $C \subseteq \{\neg, \wedge, \vee, \rightarrow, \perp\}$ is adequate, then $\neg \in C$ or $\perp \in C$.

2. The binary connective **xor**, $F \oplus G$, is defined by the truth table corresponding to $(\neg F \wedge G) \vee (F \wedge \neg G)$. Show that xor is not complete—that is, it cannot express all binary Boolean connectives.

3. Suppose F is an inconsistent set of formulae. For each $G \in F$, let F_G be the set obtained by removing G from F.

   (a) Prove that for any $G \in F$, $F_G \vdash \neg G$.

   (b) Prove this using a formal proof.

4. In the class, we have discussed about two normal forms, namely, CNF and DNF. In this question, we introduce another one called **Algebraic Normal Form (ANF)**. Informally, ANFs are expressions involving $\oplus$ (xor) and $\wedge$ (conjunction) connectives. For example, $y = (x_1 \wedge x_2) \oplus x_3$ is in ANF. More formally, a well formed formula ϕ over propositional variables $x_1, x_2, \ldots, x_n$ in ANF form is written as:

$$\phi(x_1, x_2, \ldots, x_n) = c_0 \oplus \bigoplus_{1\le i\le n} (c_i \wedge x_i) \oplus \bigoplus_{1\le i,j\le n} (c_{ij} \wedge x_i \wedge x_j) \oplus \cdots \oplus (c_{1\ldots n} \wedge x_1 \wedge x_2 \cdots \wedge x_n),$$

   where each constant literal $c_t \in \{\perp, \top\}$. It can be proven that every wff of n variables can be uniquely represented in this form.

   Convert the following Boolean function into its equivalent ANF form:

$$\phi(x_0, x_1, x_2) = (\neg x_0 \wedge \neg x_1 \wedge \neg x_2) \vee (\neg x_0 \wedge \neg x_1 \wedge x_2) \vee (x_0 \wedge \neg x_1 \wedge x_2)$$

5. Consider a formula φ which is of the form $C_1 \wedge C_2 \wedge \cdots \wedge C_n$ where each clause C_i is of the form $(\top \rightarrow \alpha)$, $(\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \beta)$, or $(\gamma \rightarrow \perp)$, where $\alpha, \alpha_i, \beta, \gamma$ are literals. A logician wishes to apply HornSAT to this formula φ by renaming negative literals (if any) with fresh positive literals. Thus, if any $\alpha, \alpha_i, \beta, \gamma$ was of the form $\neg p$, the logician will replace $\neg p$ with a fresh variable p'.

   The logician claims that he can check satisfiability of φ correctly by applying HornSAT on the new formula (call it φ') in the following way: φ is satisfiable iff HornSAT concludes that φ' is satisfiable, and φ is unsatisfiable iff HornSAT concludes that φ' is unsatisfiable. Do you agree with the logician?

6. Show that the satisfiability of any 2-CNF formula can be checked in polynomial time.

7. Call a set of formulae **minimal unsatisfiable** iff it is unsatisfiable, but every proper subset is satisfiable. Show that there exist minimal unsatisfiable sets of formulae of size n for each $n \geq 1$.