CS 228 : Logic in Computer Science

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CNF Explosion

Consider the formula $\varphi = (p_1 \wedge p_2 \cdots \wedge p_n) \vee (q_1 \wedge q_2 \ldots q_m)$

- What is the equivalent CNF formula?
- $\bigwedge_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}} (p_i \vee q_j)$ has *mn* clauses
- Distributivity explodes the formula

Tseitin Encoding : The Idea

- Introducing fresh variables, Tseitin encoding can give an equisatisfiable formula without exponential explosion.
- $\varphi = p \vee (q \wedge r)$
- Replace $q \wedge r$ with a fresh variable x and add a clause which asserts that x simulates $q \wedge r$
- $(p \vee x) \wedge (x \leftrightarrow (q \wedge r))$
- It is enough to consider (Why?) $(p \vee x) \wedge (x \rightarrow (q \wedge r))$ which is $(p \vee x) \wedge (\neg x \vee q) \wedge (\neg x \vee r)$

Tseitin Encoding

- Assume the input formula is in NNF (all negations attached only to literals) and has only $\wedge, \vee$
- Replace each $G_1 \wedge \cdots \wedge G_n$ just below a $\vee$ with a fresh variable p and conjunct $(\neg p \vee G_1) \wedge \cdots \wedge (\neg p \vee G_n)$ (same as $p \rightarrow G_1 \wedge \cdots \wedge G_n$).

Tseitin Encoding

- $\varphi = (p_1 \wedge p_2 \cdots \wedge p_n) \vee (q_1 \wedge q_2 \cdots \wedge q_m)$
- Choose fresh variables x, y
- $\psi = (x \vee y) \wedge \bigwedge_{i \in \{1,\ldots,n\}} (\neg x \vee p_i) \wedge \bigwedge_{j \in \{1,\ldots,m\}} (\neg y \vee q_j)$ has $m + n + 1$ clauses
- φ and ψ are equisatisfiable. Prove.

(DPLL)Davis-Putnam-Loveland-Logemann Method

DPLL

- DPLL combines search and deduction to decide CNF satisfiability
- Underlies most modern SAT solvers

Partial Assignments

An assignment is a function $\alpha : V \rightarrow \{0, 1\}$ which maps each variable to true(1) or false (0). A partial assignment α is one under which some variables are unassigned.

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An assignment is a function $\alpha : V \rightarrow \{0, 1\}$ which maps each variable to true(1) or false (0). A partial assignment α is one under which some variables are unassigned.

- Under a partial assignment α, the state of a variable v is true if $\alpha(v) = 1$, false if $\alpha(v) = 0$, and unassigned otherwise.
- Let $V = \{x, y, z\}$ and let $\alpha(x) = 1, \alpha(y) = 0$. Then the state of x under α is true, state of y is false, and the state of z is unassigned.

State of a Clause

Assume we have a formula in CNF. Under a partial assignment α,

- a clause C is true if there exists some literal ℓ in C whose state is true
- a clause C is false if the state of all literals in C is false
- Otherwise, the state of C is unassigned

Consider the partial assignment $\alpha(x) = 0, \alpha(y) = 1$.

- The state of $C = x \vee y \vee z$ is true
- The state of $C = x \vee \neg y \vee z$ is unassigned
- The state of $C = x \vee \neg y$ is false

State of a Formula

Under a partial assignment α,

- A CNF formula F is true if for each $C \in F$, C is true
- A CNF formula F is false if there $C \in F$ such that C is false
- Otherwise, F is unassigned.

Consider the partial assignment $\alpha(x) = 0, \alpha(y) = 1$.

- The state of $F = (x \vee y \vee z) \wedge (x \vee \neg y \vee z)$ is unassigned
- The state of $F = (x \vee y \vee z) \wedge (x \vee \neg y)$ is false

Unit Clause and Unit Literal

Let C be a clause and α a partial assignment. Then

- C is a unit clause under α if there is a literal $\ell \in C$ which is unassigned, and the rest are false.
- Then ℓ is a unit literal under α.

Let $\alpha(x) = 0, \alpha(y) = 1$ be a partial assignment.

- $C = x \vee \neg y \vee \neg z$ is a unit clause and $\neg z$ is a unit literal
- $C = x \vee \neg y \vee \neg z \vee w$ is not a unit clause
- $C = x \vee \neg y$ is not a unit clause

DPLL

DPLL maintains a partial assignment, to begin with the empty assignment.

- Assigns unassigned variables 0 or 1 randomly
- Sometimes, forced to assign 0 or 1 to unit literals

DPLL Actions

- DPLL has 3 actions : decisions, unit propagation and backtracking
- Decisions : Decide an assignment for a variable (random choice)
- Implied assignments or unit propagation : to deal with unit literals
- Backtrack when in a conflict

DPLL Algorithm

- At any time, the state of the algorithm is a pair (F, α) where F is the CNF and α is a partial assignment
- A state (F, α) is successful if α sets some literal in each clause of F to be true
- A conflict state is one where α sets all literals in some clause of F to be false

DPLL Algorithm

- Let $F|\alpha$ denote the set of clauses obtained by deleting from F, any clause containing a true literal from α, and deleting from each remaining clause, all literals false under α. Let $\alpha(x) = 0, \alpha(y) = 1$.
- For $F = (x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z)$, $F|\alpha = \{\neg z\}$
- For $F = (x \vee y) \wedge (\neg x \vee \neg y)$, $F|\alpha = \{\}$.
- For $F = (x \vee \neg y)$, $\perp \in F|\alpha$
- If (F, α) is successful, then $F|\alpha = \{\}$
- If (F, α) is in conflict, then the empty clause $\perp$ is in $F|\alpha$.

The DPLL Algorithm

Input : CNF formula F.

1. Initialise α as the empty assignment
2. While there is a unit clause L in $F|\alpha$, add $L = 1$ to α (unit propagation)
3. If $F|\alpha$ contains no clauses, then stop and output α
4. If $F|\alpha$ contains the empty clause, then apply the learning procedure to add a new clause C to F. If it is the empty clause, output UNSAT. Otherwise, backtrack to the highest level at which C is a unit clause, go to line 2.
5. Decide on a new assignment $p = b$ to be added to α, goto line 2.

DPLL Example

$c_1 = \neg p_1 \vee p_2$

$c_2 = \neg p_1 \vee p_3 \vee p_5$

$c_3 = \neg p_2 \vee p_4$

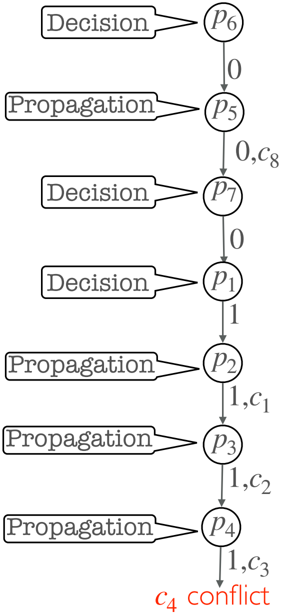
$c_4 = \neg p_3 \vee \neg p_4$

$c_5 = p_1 \vee p_5 \vee \neg p_2$

$c_6 = p_2 \vee p_3$

$c_7 = p_2 \vee \neg p_3 \vee p_7$

$c_8 = p_6 \vee \neg p_5$



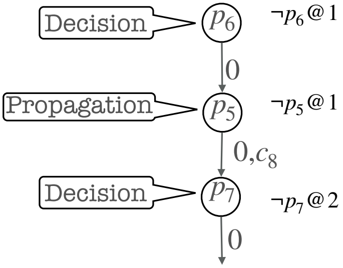
Clause Learning

Run of DPLL

The partial assignment in construction is called a a run of DPLL. In the previous slide, the run ended in a conflict.

Decision Level

During a run, the decision level of a true literal is the number of decisions after which the literal was made true.



Implication Graphs

During a DPLL run, we maintain a data structure called an implication graph.
Under a partial assignment α, the implication graph $G = (V, E)$,

- V is the set of true literals under α, and the conflict node
- $E = \{(\ell_1, \ell_2) \mid \neg\ell_1$ belongs to the clause due to which unit propagation made ℓ_2 true$\}$

Each node is annotated with the decision level.

$c_1 = \neg p_1 \vee p_2$

$c_2 = \neg p_1 \vee p_3 \vee p_5$

$c_3 = \neg p_2 \vee p_4$

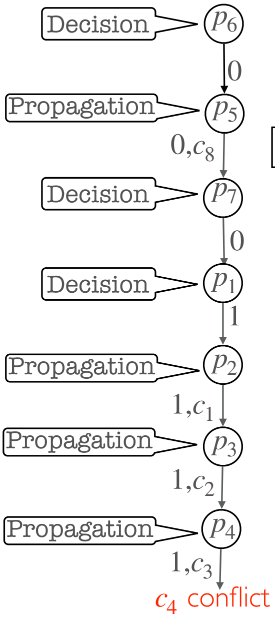
$c_4 = \neg p_3 \vee \neg p_4$

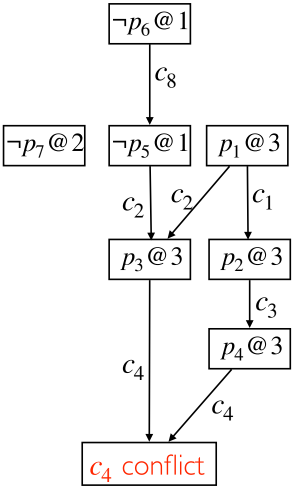
$c_5 = p_1 \vee p_5 \vee \neg p_2$

$c_6 = p_2 \vee p_3$

$c_7 = p_2 \vee \neg p_3 \vee p_7$

$c_8 = p_6 \vee \neg p_5$





Conflict Clause

Traverse the implication graph backwards to find the set of decisions that created a conflict. The negations of the causing decisions is the conflict clause.

$c_1 = \neg p_1 \vee p_2$

$c_2 = \neg p_1 \vee p_3 \vee p_5$

$c_3 = \neg p_2 \vee p_4$

$c_4 = \neg p_3 \vee \neg p_4$

$c_5 = p_1 \vee p_5 \vee \neg p_2$

$c_6 = p_2 \vee p_3$

$c_7 = p_2 \vee \neg p_3 \vee p_7$

$c_8 = p_6 \vee \neg p_5$

$$\begin{tikzcd}
& \boxed{\neg p_6 @ 1} \arrow[d, "c_8"] & \\
\boxed{\neg p_7 @ 2} & \boxed{\neg p_5 @ 1} \arrow[d, "c_2"'] & \boxed{p_1 @ 3} \arrow[dl, "c_2"'] \arrow[d, "c_1"] \\
& \boxed{p_3 @ 3} \arrow[dd, "c_4"'] & \boxed{p_2 @ 3} \arrow[d, "c_3"] \\
& & \boxed{p_4 @ 3} \arrow[dl, "c_4"] \\
& \boxed{c_4 \text{ conflict}} &
\end{tikzcd}$$

Conflict clause : $p_6 \vee \neg p_1$ is added : resolve c_4 with c_3, c_1, c_2, c_8

Clause Learning

- We add the conflict clause to the input set of clauses
- backtrack to the second last conflicting decision, and proceed like DPLL

Adding the conflict clause

- does not affect satisfiability of the original formula (think of resolution)
- ensures that the conflicting partial assignment will not be tried again

$c_1 = \neg p_1 \lor p_2$

$c_2 = \neg p_1 \lor p_3 \lor p_5$

$c_3 = \neg p_2 \lor p_4$

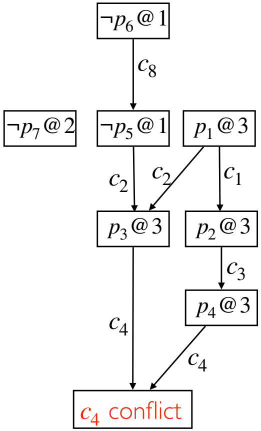
$c_4 = \neg p_3 \lor \neg p_4$

$c_5 = p_1 \lor p_5 \lor \neg p_2$

$c_6 = p_2 \lor p_3$

$c_7 = p_2 \lor \neg p_3 \lor p_7$

$c_8 = p_6 \lor \neg p_5$



$$\begin{tikzcd}
 & \boxed{\neg p_6 @ 1} \arrow[d, "c_8"] & \\
\boxed{\neg p_7 @ 2} & \boxed{\neg p_5 @ 1} \arrow[d, "c_2"'] & \boxed{p_1 @ 3} \arrow[dl, "c_2"'] \arrow[d, "c_1"] \\
 & \boxed{p_3 @ 3} \arrow[dd, "c_4"'] & \boxed{p_2 @ 3} \arrow[d, "c_3"] \\
 & & \boxed{p_4 @ 3} \arrow[dl, "c_4"] \\
 & \boxed{c_4 \text{ conflict}} &
\end{tikzcd}$$

The second last decision is $p_6 = 0$. Unit propagation will force $p_1 = 0$.

The combination $p_6 = 0, p_1 = 1$ will not be tried again.

$c_1 = \neg p_1 \vee p_2$

$c_2 = \neg p_1 \vee p_3 \vee p_5$

$c_3 = \neg p_2 \vee p_4$

$c_4 = \neg p_3 \vee \neg p_4$

$c_5 = p_1 \vee p_5 \vee \neg p_2$

$c_6 = p_2 \vee p_3$

$c_7 = p_2 \vee \neg p_3 \vee p_7$

$c_8 = p_6 \vee \neg p_5$

$c_9 = p_6 \vee \neg p_1$

$$\begin{tikzcd}
 & \boxed{\neg p_6 @ 1} \arrow[d, "c_8"] \arrow[dr, "c_9"] & \\
\boxed{\neg p_7 @ 2} \arrow[ddddr, bend right] & \boxed{\neg p_5 @ 1} \arrow[d, "c_5"'] & \boxed{\neg p_1 @ 1} \arrow[dl, "c_5"] \\
 & \boxed{\neg p_2 @ 1} \arrow[d, "c_6"] \arrow[ddd, bend left=60] & \\
 & \boxed{p_3 @ 1} \arrow[d, "c_4"] \arrow[dd, bend right=60] & \\
 & \boxed{\neg p_4 @ 1} & \\
 & \boxed{c_7 \text{ conflict}} &
\end{tikzcd}$$

Conflict clause : $p_7 \vee p_6$ is added and backtrack

Set $p_7 = 1$ by unit propagation.

$c_1 = \neg p_1 \vee p_2$

$c_2 = \neg p_1 \vee p_3 \vee p_5$

$c_3 = \neg p_2 \vee p_4$

$c_4 = \neg p_3 \vee \neg p_4$

$c_5 = p_1 \vee p_5 \vee \neg p_2$

$c_6 = p_2 \vee p_3$

$c_7 = p_2 \vee \neg p_3 \vee p_7$

$c_8 = p_6 \vee \neg p_5$

$c_9 = p_6 \vee \neg p_1$

$c_{10} = p_6 \vee p_7$

$$\begin{tikzcd}
 & \boxed{\neg p_6 @ 1} \arrow[dl, "c_{10}"'] \arrow[d, "c_8"] \arrow[dr, "c_9"] & \\
\boxed{p_7 @ 1} & \boxed{\neg p_5 @ 1} \arrow[d, "c_5"'] & \boxed{\neg p_1 @ 1} \arrow[dl, "c_5"] \\
 & \boxed{\neg p_2 @ 1} \arrow[d, "c_6"] & \\
 & \boxed{p_3 @ 1} \arrow[d, "c_4"] & \\
 & \boxed{\neg p_4 @ 1} &
\end{tikzcd}$$