



CS 228 : Logic in Computer Science

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Resolution

- ▶ Given a propositional logic formula φ , is it unsatisfiable?
- ▶ How does a solver do it?
- ▶ Assume it is in CNF

Resolution

- ▶ Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p . Then the clause $R = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ is a **resolvent** of C_1 and C_2 .
- ▶ Let $C_1 = \{p_1, \neg p_2, p_3\}$ and $C_2 = \{p_2, \neg p_3, p_4\}$. As $p_3 \in C_1$ and $\neg p_3 \in C_2$, we can find the resolvent. The resolvent is $\{p_1, p_2, \neg p_2, p_4\}$.
- ▶ Resolvent not unique : $\{p_1, p_3, \neg p_3, p_4\}$ is also a resolvent.

3 rules in Resolution

- ▶ Let G be any formula. Let F be the CNF formula resulting from the CNF algorithm applied to G . Then $G \vdash F$ (Prove!)
- ▶ Let F be a formula in CNF, and let C be a clause in F . Then $F \vdash C$ (Prove!)
- ▶ Let F be a formula in CNF. Let R be a resolvent of two clauses of F . Then $F \vdash R$ (Prove!)

Completeness of Resolution

Show that resolution can be used to determine whether any given formula is unsatisfiable.

- ▶ Given F in CNF, let $Res^0(F) = \{C \mid C \text{ is a clause in } F\}$.
- ▶ $Res^n(F) = Res^{n-1}(F) \cup \{R \mid R \text{ is a resolvent of two clauses in } Res^{n-1}(F)\}$
- ▶ $Res^0(F) = F$, there are finitely many clauses that can be derived from F .
- ▶ There is some $m \geq 0$ such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

Example

Let $F = \{\{p_1, p_2, \neg p_3\}, \{\neg p_2, p_3\}\}$.

- ▶ $Res^0(F) = F$
- ▶ $Res^1(F) = F \cup \{p_1, p_2, \neg p_2\} \cup \{p_1, \neg p_3, p_3\}$.
- ▶ $Res^2(F) = Res^1(F) \cup \{p_1, p_2, \neg p_3\} \cup \{p_1, p_3, \neg p_2\}$

Resolution

Let F be a formula in CNF. If $\emptyset \in \text{Res}^*(F)$, then F is unsatisfiable.

- ▶ If $\emptyset \in \text{Res}^*(F)$. Then $\emptyset \in \text{Res}^n(F)$ for some n .
- ▶ Since $\emptyset \notin \text{Res}^0(F)$ (\emptyset is not a clause), there is an $m > 0$ such that $\emptyset \notin \text{Res}^m(F)$ and $\emptyset \in \text{Res}^{m+1}(F)$.
- ▶ Then $\{p\}, \{\neg p\} \in \text{Res}^m(F)$. By the rules of resolution, we have $F \vdash p, \neg p$, and thus $F \vdash \perp$. Hence, F is unsatisfiable.

Resolution

Prove the converse: F is unsatisfiable implies $\emptyset \in Res^*(F)$.

(Discuss substitution before the proof)

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in \text{Res}^*(F)$.

- ▶ Let F have k clauses C_1, \dots, C_k .
- ▶ wlg, assume that no C_i has both p and $\neg p$
- ▶ Induct on the number n of propositional variables that occur in F .
- ▶ If $n = 1$, then the possible clauses are p , $\neg p$ and $p \vee \neg p$. The third one is ruled out, by assumption.
- ▶ If $F = \{\{p\}\}$ or $F = \{\{\neg p\}\}$, F is satisfiable.
- ▶ Hence, $F = \{\{p\}, \{\neg p\}\}$. Clearly, $\emptyset \in \text{Res}(F)$.

Resolution

- ▶ Inductive hypothesis : If F has $\leq n$ variables and is unsat, then $\emptyset \in Res^*(F)$.
- ▶ Let F have $n + 1$ variables p_1, \dots, p_{n+1} .

- ▶ Let G_0 be the conjunction of all C_i in F such that $\neg p_{n+1} \notin C_i$.
- ▶ Let G_1 be the conjunction of all C_i in F such that $p_{n+1} \notin C_i$.

- ▶ Clauses in $F =$ Clauses in $G_0 \cup$ Clauses in G_1

- ▶ Let $F_0 = \{C_i - \{p_{n+1}\} \mid C_i \in G_0\}$
- ▶ Let $F_1 = \{C_i - \{\neg p_{n+1}\} \mid C_i \in G_1\}$

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- ▶ $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- ▶ $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$
- ▶ If $p_{n+1} = \text{false}$ in F , then F is equisatisfiable with F_0
- ▶ If $p_{n+1} = \text{true}$ in F , then F is equisatisfiable with F_1
- ▶ Hence F is satisfiable iff $F_0 \vee F_1$ is.
- ▶ As F is unsatisfiable, F_0 and F_1 are both unsatisfiable.

Resolution

- ▶ By induction hypothesis, $\emptyset \in Res^*(F_0)$ and $\emptyset \in Res^*(F_1)$.
- ▶ Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- ▶ If $\emptyset \in Res^*(G_0)$ or $\emptyset \in Res^*(G_1)$, then $\emptyset \in Res^*(F)$.
- ▶ Else, $\{p_{n+1}\} \in Res^*(G_0)$ and $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- ▶ Hence $\emptyset \in Res^*(F)$.

Resolution Summary

Given a formula ψ , convert it into CNF, say ζ . ψ is satisfiable iff $\emptyset \notin Res^*(\zeta)$.

- ▶ If ψ is unsat, we might get \emptyset before reaching $Res^*(\zeta)$.
- ▶ If ψ is sat, then truth tables are faster : stop when some row evaluates to 1.