Contents

0.1. Introduction

. Introduction

These notes were taken at Columbia University's MATH 6307 course in Fall 2024, taught by Andrew Blumberg. Please send questions, comments, complaints, and corrections to yie2001@columbia.edu. Alternatively, these notes are hosted on Github at https://github.com/Y-o-e-l/Algebraic-Topology-I-Notes-, and you can submit a pull request.

These notes are an overview of Algebraic Topology in-so-far as some tools in K-theory by the end of Spring 2025. Some of the references for the course include:

- Peter May's "Concise course in Algebraic Topology"
- Haynes Miller's "Notes on Algebraic Topology"
- Munkres's "Elements of Algebraic Topology"
- Weibel's "Homological Algebra"
- Saunders & Maclane's "Categories for the Working Mathematician" (Although, IMO, Riehl's "Category Theory in Context" is better written.)

Algebraic topology, at its core, answers questions regarding classification, turning geometric problems into ones relying on algebra (which is somehow supposed to be easier). For instance, consider the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$;

Question: Are $\mathbb{R}^2$ and $\mathbb{R}^3$ the same as sets? $\to$ Well, although there is a bijection $\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^2$, this is insufficient for the ways we want to think about things in Algebraic topology. As we'll later see, $\mathbb{R}^2 \neq \mathbb{R}^3$ in our senses of "being the same."

Category Theory for People who aren't Peter May. Here, we introduce some of the language to begin talking about our ideas. It's a good idea to play around with some of these things before diving head-first into things.

Definition 0.1.1. A **category** C is a collection of data consisting of objects $\mathsf{Obj}(C)$ and a collection of morphisms between said-objects. Each object $X \in \mathsf{Obj}(C)$ has an identity morphism $1_X : X \to X$. Additionally, any morphisms $f, g \in \mathsf{Mor}(C)$ are associative with respect to composition.

Example 0.1.2. One useful example of a category we may work with is the category Vect, with objects consisting vector spaces with linear maps as morphisms. ◀

Definition 0.1.3. Given two categories C, D, then a **functor** $F : C \to D$ is a map of categories such that:

(1) F takes objects in C to objects in D, ie. $x \in \mathsf{Obj}(C) \mapsto F(x) \in \mathsf{Obj}(D)$.
(2) For an object $X \in \mathsf{Obj}(C)$, we have functors acting on morphisms $f \in \mathsf{Mor}(C)$, ie. $f(X) \in \mathsf{Obj}(C) \mapsto F(f(X)) \in \mathsf{Obj}(D)$.

Example 0.1.4. One useful example of a category we may work with is the category Vect, with objects consisting vector spaces with linear maps as morphisms. ◀

Definition 0.1.5. Given two categories C, D, then a **functor** $F : C \to D$ is a map of categories such that:

(1) F takes objects in C to objects in D, ie. $x \in \mathsf{Obj}(C) \mapsto F(x) \in \mathsf{Obj}(D)$.
(2) For an object $X \in \mathsf{Obj}(C)$, we have functors acting on morphisms $f \in \mathsf{Mor}(C)$, ie. $f(X) \in \mathsf{Obj}(C) \mapsto F(f(X)) \in \mathsf{Obj}(D)$.

One useful aspect of category theory is that we can give definitions that better specialize familiar notions:

Definition 0.1.6. In a category C, we call a morphism $f : X \to Y$ an **isomorphism** if there exists a map $g : Y \to X$ such that $f \circ g \xrightarrow{\cong} \mathrm{id}_Y$ and $g \circ f \xrightarrow{\cong} \mathrm{id}_X$.

Example 0.1.7. For instance, for the category C given by the diagram:

$$\begin{tikzcd} \bullet_0 \arrow[r] \arrow[d] & \bullet_1 \\ \bullet_2 & \end{tikzcd}$$

We see that the only isomorphisms are the identity morphisms of each object. In the category Top, the isomorphisms are homeomorphisms. In the category Vect, the isomorphisms are are invertible linear maps of vector spaces. ◂

REMARK. As noted in class, functors of categories $F : C \to D$ preserve isomorphisms, ie. $F(f) \circ F(g) = F(f \circ g) = F(id_Y) = \mathrm{id}_{F(X)}$.

Now, if we were to be working in the category Set, then taking $X, Y \in \mathsf{Obj}(\mathsf{Set})$, we define cartesian products as $X \times Y := \{(a,b) \mid a \in X, b \in Y\}$. However, this definition of products only really works when we're in the category Set, and doesn't bode well in general. As such, we have to work to adapt this definition more broadly in categorical language.

Definition 0.1.8. Given maps $f : A \to X$ and $g : A \to Y$, observe that we obtain a unique product map $A \to X \times Y$ given by $x \mapsto (f(x), g(x))$. Using this, we define a categorical **product**, as saying that anytime we have maps $A \to X$ and $A \to Y$, we have a unique map $A \to X \times Y$, giving the diagram:

$$\begin{tikzcd}
A \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d] \\
& Y \arrow[r] & X
\end{tikzcd}$$

Definition 0.1.9. The dual-notion of a product, or **coproduct**, is built up analogously. Given maps $Y \to Z$ and $X \to Z$, we obtain a colimit by "flipping"[1] the morphisms of the above diagram so as to obtain the diagram:

$$\begin{tikzcd}
& Z & \\
X \arrow[ur] \arrow[r, "\pi_X"'] & X \coprod Y \arrow[u, dashed] & Y \arrow[ul] \arrow[l, "\pi_Y"] \\
& \varnothing \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Definition 0.1.10. A functor $F : C \to D$ is considered **full** if for each $x, y \in C$, the map $\mathrm{Map}_C(x,y) \xrightarrow{F} \mathrm{Map}_D(Fx, Fy)$ is surjective. We say F is **faithful** if $\mathrm{Map}_C(x,y) \xrightarrow{F} \mathrm{Map}_D(Fx, Fy)$ is injective.

Definition 0.1.11. Given categories C and D, and functors $F, G : C \rightrightarrows D$, a *natural transformation* $\alpha : F \implies G$ consists of:

(1) An arrow $\alpha_C : F(c) \to G(c)$ for each object $c \in \mathsf{Obj}(C)$, the collection of which defines the components of the natural transformation.
(2) For each morphism $f : c \to d$ in C, we have a commuting diagram:

$$\begin{tikzcd}
F(c) \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\
F(d) \arrow[r, "\alpha_D"] & G(d)
\end{tikzcd}$$

Definition 0.1.12. A *natural isomorphism* is a natural transformation $\alpha : F \implies G$ in which every component α_C is an isomorphism.

Definition 0.1.13. For any object $c \in \mathsf{Obj}(C)$ and any category J, the *constant functor* $I_C : J \to C$ is a functor sending every object of J to $c \in C$ and every morphism $f \in \mathsf{Mor}(J)$ to the identity morphism 1_c.

[1] Slightly imprecise language, we are considering maps from the empty sets into $X, Y \in \mathsf{Obj}(C)$ as well and using those to construct our notion of coproducts

Definition 0.1.14. A *cone* over a diagram $F : J \to C$ with summit $c \in \mathsf{Obj}(C)$ is a natural transformation $\lambda : I_c \implies F$ whose domain is the constant functor at c. The components $\{\lambda_j : c \to F_j\}$ of the natural transformation are called the *legs of the cone*. More explicitly,

- The data of a cone $F : J \to C$ with summit c is a collection of morphisms $\lambda_j : c \to F_j$ indexed by objects $j \in J$.
- A family of morphisms $\{\lambda_i : c \to F_i\}$ defines a cone over F iff for each morphism $f : x \to y$ in J, the following triangle commutes in C:

$$\begin{tikzcd} & c \arrow[dl, "\lambda_x"'] \arrow[dr, "\lambda_y"] & \\ Fx \arrow[rr, "F(f)"'] & & Fy \end{tikzcd}$$

Dually, a cone under F with nadir $c \in \mathsf{Obj}(C)$ is a natural transformation $\lambda : F \implies I_C$ whose legs are components $\{\lambda_j : F_i \to c\}_{i \in I}$. For each morphism $f : x \to y$ in J, the above triangle with morphisms λ_x, λ_y flipped will commute.

REMARK. A *cone under a diagram* $F : J \to C$ is also called a cocone and is defined analogously to cones over a diagram. A cone under $F : J \to C$ is precisely a cone over $F |:\rangle\langle:| J |\to\rangle\langle\to| C |$.

Definition 0.1.15. For any diagram $F : J \to C$, there is a functor,

$$\mathsf{Cone}(-, F) : C |\to\rangle\langle\to| \mathsf{Set}$$

which sends $c \in C$ to the set of cones over F with summit c. Using the Yoneda Lemma[2], a *limit* consists of an object $\lim F \in C$ together with a universal cone $\lambda : \lim F \implies F$, called the limit cone, defining a natural isomorphism

$$\mathbf{C}(-, \lim F) \xrightarrow{\cong} \mathsf{Cone}(-, F)$$

Definition 0.1.16. Dually, there is a functor $\mathsf{Cone}(F, -) : C \to \mathsf{Set}$ that sends $c \in C$ to the set of cones with nadir c. A *colimit* of F is a representation for $\mathsf{Cone}(F, -)$. Again, by Yoneda's Lemma, a colimit consists of an object $\operatorname{colim} F \in C$ together with a universal cone $\lambda : F \implies \operatorname{colim} F$, called the colimit cone, defining a natural isomorphism:

$$\mathbf{C}(\operatorname{colim} F, -) \xrightarrow{\cong} \mathsf{Cone}(F, -)$$

REMARK. We may also equivalently define limits of F as the terminal object in the category of cones over F and colimits as the initial object in the category of cones over F.

Example 0.1.17 (Definition of Product)**.** A product is a limit of a diagram indexed by a discrete category with only identity morphisms. A diagram in C indexed by a discrete category J consists of a collection of objects $F_j \in C$ indexed by $j \in J$.

A cone over this diagram with summit $c \in C$ is a J-indexed family of morphisms $\{\lambda_j : c \to F_j\}_{j \in J}$. This limit is denoted by $\Pi_{j \in J} F_j$ and the legs of the cone are maps,

$$\left(\pi_k : \prod_{j \in J} F_j \to F_k \right)_{k \in J}$$

◂

Definition 0.1.18. A category is *complete* if it contains all limits. A category is *cocomplete* if it contains all colimits.

Example 0.1.19. The category Set is complete and cocomplete. The category $\mathsf{Fun}(C |,\rangle\langle,| \mathsf{Set})$ is also complete and cocomplete; Observe that given the map $Y : C \to \mathsf{Fun}(C |,\rangle\langle,| \mathsf{Set})$, which maps $x \mapsto \mathrm{Map}_C(-, x) = \mathbf{C}(-, x)$, that Y is a fully-faithful[3] functor, and so the map

$$\mathrm{Map}_C(x, y) \xrightarrow{\cong} \mathrm{Map}(Y(x), Y(y))$$

[2]Recall that Yoneda's Lemma states: For any functor $F : C \to \mathsf{Set}$ whose domain C is locally small, then for any object $c \in C$, there's a bijection:

$$\mathrm{Hom}(\mathbf{C}(c, -), F) \xrightarrow{\cong} Fc$$

[3]Recall that for locally small categories C and D, the functor $F : C \to D$ induces a function $F_{X,Y} : \mathrm{Map}_C(x, y) \to \mathrm{Map}_D(F(x), F(y))$. We say F is faithful if $F_{X,Y}$ is injective, and full if $F_{X,Y}$ is surjective.

is an isomorphism. $\triangleleft$

0.1.0.1. *Why does this matter?* Consider the category Top. When constructing objects like Klein bottles, $\mathbb{R}$ or $\mathbb{C}$ projective space, or any cell complex via attaching maps, we define these as sets equipped with particular topologies.

These topologies can all be uniformly defined by our notions of limits and colimits via universal cones. As we'll observe, constructed topological spaces can be characterized either as a limit or colimit of a specific diagram over the category Top. By mapping things out of standard topological objects (For instance, mapping out of S^n, ie. $\{\mathrm{Map}_C(S^n, -)\}$), we're able to extrapolate more information about these objects.

$\mathrm{Map}_{\mathsf{Top}}(X, Y)$ is a space. Considering the map $H : [0,1] \to \mathrm{Map}_{\mathsf{Top}}(X, Y)$ with path homotopies given between $f : X \to Y \leftrightsquigarrow H(0)$ and $g : X \to Y \leftrightsquigarrow H(1)$, we'll later see that these are the same. The maps $\{\mathrm{Map}_C(S^n, -)\}$ are the same if there exists a path between them, and so we develop a notion of homotopy via $\mathrm{Map}_{\mathsf{Top}}(I, \mathrm{Map}_{\mathsf{Top}}(X, Y))$.

Sidenote: Categories of topological spaces. A nice category of topological spaces to consider is Top, the category of compactly generated, weak Hausdorff spaces (and continuous maps); we can also consider Top_*, the category of based, compactly generated, weak Hausdorff spaces and continuous, based maps. This is an important and old trick which eliminates some pathological behavior in quotients. It's reasonable to imagine that point-set topology shouldn't be at the heart of foundational issues, but there are various ways to motivate this, e.g. to make Top more resemble a topos or the category of simplicial sets.

Definition 0.1.20. Let X be a topological space.

- A subset $A \subseteq X$ is **compactly closed** if for every compact Hausdorff space Y and $f : Y \to X$, $f^{-1}(A)$ is closed.
- X is **compactly generated** if every compactly closed subset of X is closed.
- X is **weak Hausdorff** if for every compact Hausdorff space Y and continuous map $f : Y \to X$, $f(Y)$ is a closed subset of X.[4]

The intuition behind compact generation is that the topology is determined by compact Hausdorff spaces. The weak Hausdorff condition is strictly stronger than T_1 (points are closed), but strictly weaker than being Hausdorff. Any space you can think of without trying to be pathological will meet these criteria.

There is a functor k from all spaces to compactly generated spaces which adds the necessary closed sets. This has the unfortunate name of **k-ification** or **kaonification**;[5] by putting the compactly generated topology on $X \times X$, we mean taking $k(X \times X)$. There's also a "weak Hausdorffification" functor w which makes a space weakly Hausdorff, which is some kind of quotient.[6]

When computing limits and colimits, it's often desirable to compute them in the category of spaces and then apply k and w to return to Top. This works correctly for limits, but for colimits, w is particularly badly behaved: you cannot compute the colimit in Top by computing it in Set and figuring out the topology. In general, you'll need to take a quotient.

Nonetheless, there are nice theorems which make things work out anyways.

Proposition 0.1.21. *Let $Z = \mathrm{colim}(X_0 \to X_1 \to X_2 \to \dots)$ be a sequential colimit (sometimes called a **telescope**); if each X_i is weak Hausdorff, then so is Z.*

Proposition 0.1.22. *Consider a diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C & \end{tikzcd}$$

where f is a closed inclusion. If A, B, and C are weakly Hausdorff, then $B \amalg_A C$ is weakly Hausdorff.

These are the two kinds of colimits people tend to compute, so this is reassuring.

One reason we require regularity on our topological spaces is the following, which is not true for topological spaces in general.

[4] When X is compactly generated, this is equivalent to being **k-Hausdorff**, i.e. the diagonal map $\Delta : X \to X \times X$ is closed when $X \times X$ has the compactly generated topology. See [**Str09**, **Rez**] for more details.

[5] Kaonification is of course distinct from koanification, the process which makes statements more confusing.

[6] The k functor is right adjoint to the forgetful map, which tells you what it does to limits.

Lemma 0.1.23. *Let X, Y, and Z be in* Top*; then, the natural map*

$$\mathrm{Map}(X \times Y, Z) \longmapsto \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is a homeomorphism.

CHAPTER 1

Build-A-Homology-Workshop

Consider a map $H : I \to \mathrm{Map}_{\mathsf{Top}}(X,Y)$. By adjunction, this is the same data as a map $H : X \times I \to Y$. We'll be interested in the homotopy category HoTop; And here, we will consider the quotients: $\mathrm{Map}_{\mathsf{Top}}(X,Y)/\sim$ where "$\sim$" is the relation of homotopy, which is to say there is a path between morphisms $X \to Y$.

Under these relations we develop a new notion of equivalence between spaces X and Y, which we call homotopy equivalence.

Definition 1.0.1. Consider two morphisms $f : X \to Y$ and $g : Y \to X$. We say that X and Y are **homotopy equivalent** if $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

Recall that our initial goal with algebraic topology is to find functors from Top to some algebraic category, but which functors will we need. Well, as it turns out, we need these functors to respect homotopy equivalence, which is say that $X \sim Y \implies F(X) \xrightarrow{\cong} F(Y)$.

REMARK. Note that $F(X)$ and $F(Y)$ are isomorphic, not necessarily strictly equal. A slightly stronger restriction on our functors we might want is that they factor through HoTop. This is stronger since, whenever f and g are homotopic, the map they induce must also be the same.

Now, we're interested in classifying spaces up to homotopy, but dealing with every space and map in Top would probably kill me. As such, we're going to identify the class of spaces which is "big enough" (in some arcane sense). Now, consider the push-out diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

We also have the push-out diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

We also then have the push-out

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Here, we have pushouts which do not respect homotopy equivalence, which is not desirable. However, we have (at least) three roughly equivalent tactics of solving this problem:

(1) Maps out of spheres S^n, these being in the set of based maps[1] $\{\mathrm{Map}_*(S^n, X)/\sim\}$. This turns out to have an algebraic structure, ie. homotopy groups.
(2) Spaces built out of simple pieces which are then restricted by attaching spheres along their boundary, ie. CW-complexes.
(3) Combinatorial models of space developed out of simplicial complexes.

Consider the space $\mathrm{Map}_*(S^0, X)/\sim$. The data of the based map $S^0 \to X$ is simply picking a point in X, and two maps are homotopic to each other when their images of points are path-connected. As such, we can identify

[1] Based maps are maps out of based spaces/pointed spaces, like (X, x_0) with base point x_0 into (Y, y_0), ie. $f : (X, x_0) \to (Y, y_0)$.

$\text{Map}_*(S^0, X)/\sim$ with path components in X; We call this $\pi_0(X)$, or the 0-th homotopy group. Thinking about $\text{Map}_*(S^1, X)$, or $\pi_1(X)$, it has an algebraic structure given by:

$$\left(\text{Map}_*(S^1, X)/\sim\right) \times \left(\text{Map}_*(S^1, X)/\sim\right) \to \text{Map}_*(S^1 \vee S^1, X)/\sim \to \text{Map}_*(S^1, X)/\sim$$

Looking at $\pi_n(X)$ for $n \geq 2$, we'll later see that these form abelian groups.

Definition 1.0.2. We say that the map $f : X \to Y$ is a **weak equivalence** if the induced map on homotopy groups, $f_* : \pi_n(X) \to \pi_n(Y)$ is an equivalence for all n.

Question: What is the relationship between weak-equivalence and homotopy equivalence?
Well, if $X \simeq Y$, then X is weakly equivalent to Y. This is a fairly-straightforward exercise where we induced a map $\text{Map}_*(S^n, X)/\sim \to \text{Map}_*(S^n, Y)/\sim$ given by homotopy equivalence and do so in the reverse direction as well.

Question: Can we compute $\pi_n(X)$ for reasonable n and X? Well, occasionally.
This will be what leads us to our soon-to-be-defined notion of a CW-complex.

Combinatorial Things & Pushing Out. Recall that last lecture (9/10), we can build spaces and extract topological spaces via looking at maps of S^n. Namely, we built up an algebraic space called the "homotopy group" $\pi_n := \text{Map}_{\mathsf{Top}}(S^n, -)/\sim$. It's an interesting philosophical question as to why we use S^n and not some other space. But ignoring that for now, lets discuss some combinatorial models of space.

For our combinatorial model, we want a construction out of simplices such that: 0-simplices are given by points, 1-simplices are given by edges, 2-simplices are given by triangles, and so on...

Definition 1.0.3. The (topological) **n-simplex** is given by the set

$$\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0 \quad \forall i\}$$

Observe that Δ^2 is homotopic to D^2, with boundary $\partial\Delta^2$ being homotopic to S^1. As such, the inclusion of boundary $\partial\Delta^2 \hookrightarrow \Delta^2$ is homotopic to the inclusion $i : S^1 \hookrightarrow \Delta^2$. This may make it appear like our combinatorial models are similar to maps out of spheres, however, these are actually easier to work with since the data of an n-simplex consists of a set with a collection of subsets S such that S is closed under passage to subsets.

For the one-point space $\{*\}$, consider the push-out given by:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ \{*\} \arrow[r] & X/A \end{tikzcd}$$

Notice that gives us the quotient topology X/A, where A maps into the one-point space, and, as a set, the quotient is given by $X \setminus A$ together with one point corresponding to the set A. This is only well-behaved when the map $A \to X$ is a closed inclusion such that open sets U are closed sets under the image of $A \to X$. This is closely related to the following question:

Suppose that A and A', as well as X and X' are homotopy equivalences. Is the map $X/A \to X'/A'$ a homotopy equivalence then?

$$\begin{tikzcd} & A' \arrow[rr] \arrow[dd] & & X' \arrow[dd] \\ A \arrow[ur] \arrow[rr] \arrow[dd] & & X \arrow[ur] \arrow[dd] & \\ & * \arrow[rr] & & X'/A' \\ * \arrow[ur] \arrow[rr] & & X/A \arrow[ur, dashed] & \end{tikzcd}$$

As it turns out, not in general. This depends on the maps $X \to X'$ and $A \to A'$ themselves.

Construction 1.0.4. Suppose that we have the map $f : A \to X$, the mapping-cylinder is given by the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ A \times I \arrow[r] & M_f \end{tikzcd}$$

where $M_f := (A \times I) \sqcup X / \sim$ with relations $(a, 0) \sim x$ and $(a, 1) \sim f(x)$. The **mapping cylinder** M_f can be visualized means of "homotopically gluing" a subset $A \subset X$ with a projection of the A outside of X. Then, $\tilde{f} : M_f \to X$ and $\tilde{g} : X \to M_f$ is a homotopy equivalence.

REMARK. Via a similar construction taking A to be a cylindrical object, we obtain a **mapping cone** $C_f := M_f/(A \times \{0\})$, which crushes everything down to a point.

Construction 1.0.5. Consider the pushout given by,

$$\begin{tikzcd} F \arrow[r] \arrow[d] & P \arrow[d, "\rho"] \\ \{*\} \arrow[r] & B \end{tikzcd}$$

As we have a map $* \to b \in B$, this push-out describes the fibre $f^{-1}(b)$ of the map $\rho : P \to B$.

1.0.1. CW-Complexes.

Definition 1.0.6. A **CW-complex** is a space given by $X = \text{colim}_i X_i$ of the diagram,

$$X_0 \to X_1 \to X_2 \to \ldots$$

The CW-structure is endowed by the push-out,

$$\begin{tikzcd} \bigvee_i S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \bigvee_i D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

Equivalently, a CW-complex is a space X equipped with a sequence of subspaces

$$\underbrace{\text{sk}_{-1}(X)}_{=\varnothing} \subseteq \underbrace{\text{sk}_0(X)}_{=\{*\}} \subseteq \text{sk}_1(X) \subseteq \ldots \subseteq \text{sk}_n(X) = X$$

REMARK. A space can be endowed with multiple CW-structures, they are not unique for the space.

EXAMPLE 1.0.7. The circle S^1 can be given its CW-structure either by the pushout:

$$\begin{tikzcd} \{\{*_1\}, \{*_2\}\} \arrow[r] \arrow[d] & \{p\} \\ \underbrace{\{*_1, *_2\}}_{\text{forms an edge}} \arrow[r] & S^1 \end{tikzcd}$$

Or, S^1 can be given by the push-out:

$$\begin{tikzcd} S^0 \coprod S^0 \arrow[r] \arrow[d] & * \coprod * \arrow[d] \\ D^1 \coprod D^1 \arrow[r] & S^1 \end{tikzcd}$$

As we now can observe, S^1 has more-than-one CW-structure it can be endowed with. ◅

Definition 1.0.8. For CW-complexes X, Y, letting X_n and Y_n denote the n-skeletons of X and Y respectively, a continuous function $f : X \to Y$ is a **cellular map** if it is a map of n-skeletons so that $f(X_n) \subseteq Y_n$.

- Now, a very natural question to ask in regards to CW-complexes is: "As CW-complexes have a filtration, what exactly is their **associated-graded object**[2]?" Well, as it turns out, this effectively ends up being $X_{n+1}/X_n \xrightarrow{\cong} \bigvee S^{n+1}$.

As we will see, the only information which proves essential in our understanding of a CW-complex, and the category of CW-complexes, the only real data of our push-outs,

$$\begin{tikzcd} \bigvee S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \bigvee D^n \arrow[r] & X_n \end{tikzcd}$$

are the data of maps $\bigvee S^{n-1} \to X_{n-1}$.

Definition 1.0.9. A **CW-decomposition** of a space X is a filtration $X = \mathrm{colim}_i X_i \ni X_0 = \{*\}$ such that we have push-outs,

$$\begin{tikzcd} \coprod_i S^{n-1} \arrow[r, "\phi"] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_i D^n \arrow[r] & X_n \end{tikzcd}$$

REMARK. For the category of CW-complexes and cellular maps, we wish to show if we can take them to inversions of their weak-equivalences, $\mathsf{Top}^{\mathsf{CW}}[w^{-1}]$. However, what exactly does this mean?

Well, more precisely, we want a functor $L : \mathsf{Top}^{\mathsf{CW}} \to \mathsf{Top}^{\mathsf{CW}}[w^{-1}]$ which is initial among all functors $\mathsf{Top}^{\mathsf{CW}} \to C$ that takes weak-equivalences to isomorphisms.

Construction 1.0.10. Now, we develop our notions of Homology. Recall that the definition of a topological n-simplex (Definition 1.3.1.). Starting out with this combinatorial model of space, we familiarize ourselves with the face and degeneracy maps of said-space, respectively given by:

- Face maps are inclusions $\delta_i : \Delta^{n-1} \to \Delta^n$ given by

$$(t_0, \dots, t_{n-1}) \longmapsto (x_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

- Degeneracy maps $\beta_i : \Delta^{n+i} \to \Delta^n$ are given by a sort-of normalizations of higher simplices into lower simplices, mapping

$$(t_0, \dots, t_{n+i}) \longmapsto (t_0, t_i + t_{i+1}, t_{i+2}, \dots, t_{i+n})$$

Now, consider the category of simplicial sets $\mathsf{sSet} := \mathsf{Fun}(\Delta|,\rangle\langle,|\,\mathsf{Set})$, we have a functor $\mathsf{Top} \to \mathsf{sSet}$ which maps topological spaces to singular objects given by $X \mapsto \mathsf{Sin}_\bullet(X) = \{\text{cts } \sigma : \Delta^\bullet \to X\}$. We have a functor $\mathbb{Z}\{-\} : \mathsf{sSet} \to s\mathsf{Ab}$ which gives an abelianization by taking simplices σ and spitting out $\mathbb{Z}$-linear combinations of simplices. As such, we define,

$$C_n^{\mathsf{Sin}}(X) = S_n(X) := \mathbb{Z}[\sigma], \quad \sigma \in \mathrm{Map}(\Delta^n, X)$$

Now, we consider the category of chain-complexes:

Definition 1.0.11. Fix a ring R, we define the **category of chain-complexes over** R, $\mathsf{Ch}(R)$, to be the sub-category of $\mathsf{Fun}(\mathbb{Z}|,\rangle\langle,|\,\mathsf{Mod}_R)$ for which $\partial_n \circ \partial_{n-1} = 0$.[3]

- We have a functor $s\mathsf{Ab} \to \mathsf{Ch}(R)$ given by mapping:

$$[A_{n+1} \xrightarrow{d_i} A_n] \longmapsto \left(\bigcap_{i=1}^{n-1} \ker(A_n \to A_{n-1}) \right)$$

[2]An associated-graded object is the graded-object which, in degree n, is the cokernel of the n-th inclusion which fits into the SES $0 \to X_{n-1} \to X_n \to \text{Ass-Grade}(X) \to 0$, so we define it by the collection of quotients $\{X_i/X_{i-1}\}_1^n$

[3]Recall that the d_i is a map induced by the face-map where $\sigma \mapsto \sigma \circ \delta_i$, and the differential is given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i(\sigma)$$

This differential ∂_n is often also denoted by d.

- As $\partial_n \circ \partial_{n-1} = 0$, we know that $\mathrm{Im}(\partial_n) \subseteq \ker(\partial_{n-1})$, so we're able to define a functor $\mathsf{Ch}(R) \to \mathsf{AbGr}$ given by
$$H_*(C_*) = \frac{\ker[\partial : C_n \to C_{n-1}]}{\mathrm{Im}[\partial : C_{n+1} \to C_n]}$$
- Then, Homology is given by a composition of functors
$$\mathsf{Top} \to \mathsf{sSet} \to s\mathsf{Ab} \to \mathsf{Ch}(R) \to \mathsf{AbGr}$$

REMARK. The construction of our differential ∂ : is built on the notion of obtaining **quasi-isomorphism** in $\mathsf{Ch}(R)$, (ie. for a quasi-isomorphism $f : X_\bullet \to Y_\bullet$, we have an isomorphism $H_n(X) \xrightarrow{\cong} H_n(Y)$ for all n).

Definition 1.0.12. An **n-cycle** in X is an n-chain c with $\mathrm{d}c = 0$. An n-chain is a **boundary** if it is in the image of $\mathrm{d} : S_{n+1}(X) \to S_n(X)$

Definition 1.0.13. Two chain maps $f_*, g_* : C_*(X) \to C_*(Y)$ are **chain-homotopic** if theres a sequence of homomorphisms $h_n : C_n(X) \to C_{n+1}(Y)$ for each $n \in \mathbb{N}$ such that:
$$\partial \circ h_n + h_{n-1} \circ \partial = f_n - g_n$$

QUESTION. However, this begs the question, if the maps $f, g : X \to Y$ are homotopic, does this induce a chain-homotopy $f_*, g_* : C_*(X) \to C_*(Y)$ of their chain maps?

The answer here, as it turns out, is yes. Now, we have to actually wish PROVE that for two maps $f, g : X \to Y$ which are homotopic, then $f_*, g_* : C_*(X) \to C_*(Y)$ are chain-homotopic.

Definition 1.0.14. A space $X \subseteq \mathbb{R}^n$ is a *star-shaped region* if $x \in X$ if for all $z \in X$, we have $tx + (1-t)z \in X$ for $t \in [0,1]$.

Proposition 1.0.15. *A space X being star-shaped implies X is contractible.*

PROOF. This is easy to see. The homotopy given by $H(z, t) = tx + (1-t)z$ gives us the contractability of our space, ie. null-homotopic. ⊠

Proposition 1.0.16. *If X is a star-shaped region, then $C_*(X)$ is chain-homotopic to $C_*(p)$ for a point $p \in X$.*

PROOF. We have obvious maps $\mathrm{id} : X \to X$ and $i : * \hookrightarrow X$ where this induces a map of complexes,

$$\begin{tikzcd}
C_*(X) \arrow[rr] \arrow[dr] & & C_*(X) \\
& C_*(p) \arrow[ur] &
\end{tikzcd}$$

We wish to obtain a map $h_n : C_n(X) \to C_{n+1}(X)$ where we have $\sigma \in C_n(X)$ where σ is a map $\Delta^n \to X$. We consider the map $h_0(\sigma) : \Delta^{n+1} \to X$ given by
$$t_0 c_z + (1 - t_0)\sigma(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0})$$
where c_z is a 0-chain that picks out $z \in X$, ie. a constant map at z. Then, we want to show $h\partial - \partial h = \mathrm{id} - c_z$ where we show this using face-maps. ⊠

QUESTION. If suffices to show that canonical maps $X \to X \times I$ are chain-homotopic to show that if f, g are homotopic, then f_*, g_* are chain-homotopic. Why?

Well, if $f, g : X \to Y$ are homotopic, then there exists a homotopy $H : X \times I \to X$ such that $H(-, 0) = f$ and $H(-, 1) = g$. This homotopy induces a commuting a diagram given by:

$$\begin{tikzcd}
X \times \{0\} \arrow[dr, "i_0"] \arrow[drr, dashed, bend left, "f"] & & \\
& X \times [0,1] \arrow[r, "H"] & Y \\
X \times \{1\} \arrow[ur, "i_1"'] \arrow[urr, dashed, bend right, "g"'] & &
\end{tikzcd}$$

Homotopy invariance tells us that if $f, g : X \to Y$ are homotopic, then the induced maps $f_*, g_* : C_*(X) \to C_*(Y)$ are equal, so the homology groups of homotopy equivalent spaces are equal.

1.0.1.1. *The Cross-Product.* This notion of a cross-product is the final component we need in showing the chain-homotopy of our maps f_*, g_*. This is a map on chains which satisfies:

Theorem 1.0.17 (Cross-product)**.** *There is a cross-product $\times : S_p(X) \times S_q(X) \to S_{p+q}(X)$ which has the properties:*

- *(Naturality.) If $f : X \to X'$ and $g : Y \to Y'$ are continuous maps, then for with $a \in S_p(X)$ and $b \in S_q(X)$, we have:*
$$(f \times g)_*(a \times b) = f_*(a) \times g_*(b)$$
- *(Bilinearity.) It is bilinear in both entries.*
- *(Leibniz Rule.) The differential of the cross-product satisfies the Leibniz rule so that for $a \in S_p(X)$ and $b \in S_q(X)$, we retrieve:*
$$d(a \times b) = d(a) \times b + (-1)^p a \times d(b)$$
- *(Normality.) Let $j_x : Y \to X \times Y$ be a map given by $y \mapsto (x, y)$ for fixed $x \in X$, and let $i_y : X \to X \times Y$ be given by $x \mapsto (x, y)$. Then, we have*
$$\{x\} \times b = (j_x)_*(b), \quad a \times \{y\} = (i_y)_*(a)$$

Proof. We aim to show this cross-product $\times : S_p(X) \times S_q(X) \to S_{p+q}(X)$ on chains exists. Proceed by induction on $p + q$. The Normality/Normalization axiom gives us the base cases $p + q = 0, 1$. More explicitly, if $p = 0$ or $q = 0$, there is no choice, and if $p = 1$ or $q = 1$, this is also the case. Now, assuming we have the above constructed cross-product for p and q simplices satisfying $p + q = n$, we aim to define this for those satisfying $p + q = n + 1$, so we wish to define the cross-product on the generators and extend this linearly. To begin, let $\mathrm{id}_p : \Delta^p \to \Delta^p$ be the identity map on Δ^p Then, $\sigma : \Delta^p \to X$ is given by $\sigma_*(\mathrm{id}_p)$, so if we want naturality to hold, we must have
$$\sigma \times \tau = (\sigma \times \tau)_*(\mathrm{id}_p \times \mathrm{id}_q)$$
for $\tau : \Delta^q \to X$. As such, it suffices to define the homological cross-product for $\mathrm{id}_p \times \mathrm{id}_q$ and use the above formula to define it for other maps. As we have $\mathrm{id}_p \times \mathrm{id}_q \in S_{p+q}(X)$ by assumption for $p + q = n$, the Leibniz condition forces its boundary to be
$$\mathrm{d}\big(\mathrm{id}_p \times \mathrm{id}_q\big) = \mathrm{d}\big(\mathrm{id}_p\big) \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \mathrm{d}\big(\mathrm{id}_q\big)$$
Since $\mathrm{d}^2 = 0$, a necessary condition for this is that the RHS vanishes. Using that the Leibniz rule holds for degree $n - 1$, we have:
$$\begin{aligned} \mathrm{d}\big[\mathrm{d}\big(\mathrm{id}_p \times \mathrm{id}_q\big)\big] &= \mathrm{d}\big[\mathrm{d}\big(\mathrm{id}_p\big) \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \mathrm{d}\big(\mathrm{id}_q\big)\big] \\ &= \mathrm{d}^2(\mathrm{id}_p) \times \mathrm{id}_q + (-1)^{p-1}\, \mathrm{d}\big(\mathrm{id}_p\big) \times \mathrm{d}\big(\mathrm{id}_q\big) + (-1)^p\, \mathrm{d}\big(\mathrm{id}_p\big) \times \mathrm{d}\big(\mathrm{id}_q\big) + \mathrm{id}_p \times \mathrm{d}^2(\mathrm{id}_q) = 0 \end{aligned}$$
Since $\Delta^p \times \Delta^q$ is star-shaped, $H_{p+q-1}(\Delta^p \times \Delta^q) \xrightarrow{\cong} 0$ since $p + q > 1$. As such, $H_{p+q-1}(\Delta^p \times \Delta^q) \xrightarrow{\cong} 0$ tells us that this cycle must be the boundary of some chain in $S_{p+q}(\Delta^p \times \Delta^q)$. Choosing an arbitrary such chain, we use this as the definition of $\mathrm{id}_p \times \mathrm{id}_q \in S_{p+q}(\Delta^p \times \Delta^q)$. Given the pair of simplicial maps $\sigma : \Delta^p \to X$ and $\tau : \Delta^q \to X$, the definition of their cross-product is forced by Naturality to be:
$$(\sigma \times \tau) = \sigma_*(\mathrm{id}_p) \times \tau_*(\mathrm{id}_q) = (\sigma, \tau)_*(\mathrm{id}_p \times \mathrm{id}_q)$$
A bilinear extension concludes that we have the definition of $\times : S_p(X) \times S_q(X) \to S_{p+q}(X)$. Now, to verify Leibniz rule for $p + q = n + 1$, we compute:
$$\begin{gathered} \mathrm{d}(\sigma \times \tau) = \mathrm{d}(\sigma, \tau)_*\,(\mathrm{id}_p \times \mathrm{id}_q) = ((\sigma, \tau)_* \circ \mathrm{d})(\mathrm{id}_p \times \mathrm{id}_q) = (\sigma, \tau)_*(\mathrm{d}\big(\mathrm{id}_p\big) \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \mathrm{d}\big(\mathrm{id}_q\big)) \\ \sigma_*(\mathrm{d}\big(\mathrm{id}_p\big)) \times \tau_*(\mathrm{id}_q) + (-1)^p \sigma_*(\mathrm{id}_p) \times \tau_*(\mathrm{d}\big(\mathrm{id}_q\big)) = \mathrm{d}\big(\sigma_*(\mathrm{id}_p)\big) \times \tau_*(\mathrm{id}_q) + (-1)^p \sigma_*(\mathrm{id}_p) \times \mathrm{d}\big(\tau_*(\mathrm{id}_q)\big) \\ = \mathrm{d}(\sigma) \times \tau + (-1)^p \sigma \times \mathrm{d}(\tau) \end{gathered}$$
Hence, by induction, we have shown such a cross-product to exist. ⊠

1.0.2. Computing Relative Homology. We now have precisely the technology we need to begin exposition on Homology and ACTUALLY computing it.

From the previous section, we know that we have homotopy invariance on Homology using the construction of a chain-level bilinear cross-product

$$\times : S_p(X) \times S_q(X) \to S_{p+q}(X)$$

which satisfies the Leibniz rule. Supposing instead that we had three chain complexes $A_\bullet$, $B_\bullet$, $C_\bullet$, and that we have our chain-level cross-product $\times : A_p \times B_q \to C_{p+q}$, then this induces a desirable relation on homology:

Corollary 1.0.18. *The above data determines a map on homology,*

$$\times : H_p(A) \times H_q(B) \to H_{p+q}(A)$$

which is similarly natural, bilinear, and normalized.

And this will prove useful in a number of ways. Now, Homology can be thought of more intuitively as the "additive" approximation of the "defect" of a complex from being exact. Whats meant by this is captured in two ways:

- If $A \subseteq X$ is a subspace, then $H_\bullet(X)$ is a combination of $H_\bullet(X - A)$ and $H_\bullet(A)$.
- The homology $H_\bullet(X \cup A)$ is similar to $H_\bullet(A) \oplus H_\bullet(B) - H_\bullet(A \cap B)$, where $A \subseteq X$ is a subspace.

We wish to formalize this, so consider Top_2, where this is the category of "pairs of spaces."

Definition 1.0.19. The category Top_2 consists of the data:

- Objects are 2-tuples (X, A) of a topological space X and a subspace A.
- Morphisms $f : (X, A) \to (Y, B)$ is a continuous map of topological spaces where $f(A) \subseteq B$.

There are some obvious functors relating Top and Top_2 given by $X \mapsto (X, \varnothing)$, $X \mapsto (X, X)$, $(X, A) \mapsto X$, and $(X, A) \mapsto A$.

Lemma 1.0.20. *Let $A_\bullet$ be a sub-complex of $B_\bullet$. There is a unique structure of a chain complexes on the quotient graded abelian group of complexes $C_\bullet$ with entries $C_n = B_n/A_n$ such that $B_\bullet \to C_\bullet$ is a chain-map.*

PROOF. To define a differential $\mathrm{d} : C_n \to C_{n-1}$, we want to represent a class in C_n by a representative $b \in B_n$ with $[\mathrm{d}b] \in B_{n-1}/A_{n-1}$ well-defined. If we replace b with $b + a$ for $a \in A_n$, we have

$$\mathrm{d}(b + a) = \mathrm{d}b + \mathrm{d}a \equiv \mathrm{d}b \mod A_{n-1}$$

Then, $\mathrm{d}^2[b] = [\mathrm{d}^2 b] = 0$, and so we obtain that this complex $C_\bullet$ is well-defined. ⊠

Definition 1.0.21. The **relative singular chain complex** of a pair $(X, A) \in \mathsf{Obj}(\mathsf{Top}_2)$ is given by

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A)$$

This is a functor from pairs of spaces to chain-complexes with obvious relations

$$S_*(X, \varnothing) = S_*(X), \; S_*(X, X) = 0$$

Definition 1.0.22. The **relative singular homology** of the pair (X, A) is the homology of the relative singular chain-complex:

$$H_n(X, A) = H_n(S_\bullet(X, A))$$

Borrowing in exposition from Haynes Miller's notes, the nice feature of the chain-group $S_n(X)$ is that it is free as an abelian group, and so it is also the case for the quotient $S_n(X, A)$ as its freely generated by the cosets of the singular n-simplices in X which do not lie entirely in A.

Example 1.0.23. Consider Δ^n relative to its boundary

$$\partial \Delta^n = \sum_i \mathrm{Im}(\mathrm{d}_i) \xrightarrow{\cong} S^{n-1}$$

We have the identity map $\mathrm{id}_n : \Delta^n \to \Delta^n$. It is not a cycle since its boundary $\mathrm{d}\mathrm{id}_n \in S_{n-1}(\Delta^n)$ is given by an alternating sum of face-maps on the n-simplex. Since the image of each of these face maps lie in $\partial \Delta^{n-1}$, we have $\mathrm{d}[\mathrm{id}_n] \in S_{n-1}(\partial \Delta^n)$, generating a **relative cycle** by $[\mathrm{id}_n] \in S_n(\Delta^n, \partial \Delta^n)$.[4] ◀

[4] We will see that the relative homology $H_n(\Delta^n, \partial \Delta^n)$ is infinite-cyclic with generators being each class of $[\mathrm{id}_n]$.

1.0.3. Relative Homology on LES. A pair of spaces (X,A) gives rise to a short exact sequence of chain-complexes

$$0 \to S_*(A) \to S_*(X) \to S_*(X,A) \to 0$$

In computing homology, we want to see what we get upon passing such short-exact sequences through homology.

Theorem 1.0.24 (Homology SES to LES). *Lets consider the short exact sequence (SES) of chain-complexes*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

Then there is a natural homomorphism $\mathrm{d} : H_n(C) \to H_{n-1}(A)$ *such that we obtain the long exact sequence (LES),*

$$\cdots \to H_n(A) \xrightarrow{f_n} H_n(B) \xrightarrow{g_n} H_n(C) \xrightarrow{\mathrm{d}} H_{n-1}(A) \xrightarrow{f_{n-1}} H_{n-1}(B) \to \cdots$$

PROOF. We proceed by constructing the "boundary" map $\mathrm{d} : H_*(C) \to C_{*-1}(A)$. We have an expanded version of the above SES given by:

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "\mathrm{d}"] & B_n \arrow[r, "g"] \arrow[d, "\mathrm{d}"] & C_n \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

Let $c \in C_n$ be a cycle, ie. $\mathrm{d}c = 0$. By exactness, g is surjective, so we can pick $b \in B_n$ such that $g(b) = c \in C_n$, and consider $\mathrm{d}b \in B_{n-1}$. Now, $g(\mathrm{d}b) = \mathrm{d}g(b) = \mathrm{d}c = 0$, so, by exactness, there exists some $a \in A_{n-1}$ such that $f(a) = \mathrm{d}b$. More accurately, this such $a \in A_{n-1}$ is unique since f is injective by exactness. We want a here to be a cycle ($\mathrm{d}a = 0$), so we note that since $\mathrm{d}^2 b = 0$, so $\mathrm{d}f(a) = 0$. However, because f is injective, we get that $\mathrm{d}a = 0$, so a is a cycle, meaning we can try to define $\mathrm{d}[c] = [a]$.

To ensure this is well, defined, we check that $[a]$ doesn't depend on our choice of $b \in B_n$. Pick some other $b' \in B_n$ such that $g(b') = c$, then there is $a' \in A_{n-1}$ such that $f(a') = \mathrm{d}b'$. Now, as $g(b - b') = 0$, by exactness, there is $\overline{a} \in A_n$ such that $f(\overline{a}) = b - b'$, so we have

$$f(\mathrm{d}\overline{a}) = \mathrm{d}f(\overline{a}) = \mathrm{d}\big(b - b'\big) = f(a - a')$$

Since f is injective, $\mathrm{d}(b - b') = f(a - a')$ has it so $\mathrm{d}\overline{a} = a - a'$, meaning that we have $[a] = [a']$ as desired. Upon checking some other things, we can take $\mathrm{d}[c] = [a] \in H_{n-1}(A_*)$. ⊠

1.0.3.1. *Excision.* Ok, now aim to make precise the idea that $H_*(X,A)$ is "$H_*(X) - H_*(A)$." We should think (very carefully) that $H_*(X,A)$ "depends only on $X - A$."

Definition 1.0.25. A triple (U,A,X), where $U \subseteq A \subseteq X$, is **excisive** if the closure of U is contained in the interior of A, ie. $\overline{U} \subseteq \mathrm{Int}(A)$. We call $(X - U, A - U) \subseteq (X,A)$ an **excision**.

Theorem 1.0.26. *An excisive triple* (U,A,X) *with the inclusion of its excision* $i : (X - U, A - U) \hookrightarrow (X,A)$ *induces an isomorphism*

$$H_*(X - U, A - U) \xrightarrow{\cong} H_*(X,A)$$

This theorem tells us that you can remove points from your topological space X which lie in the subspace A as long as there is a little "padding." Essentially, we get the same result when we collapse the subspace to a point, regardless of whether we removed some subset first.

Corollary 1.0.27. *Suppose that there exists some subspace* $B \subset X$ *such that* $A \subset B \subset X$ *is an excisive triple and* B *deformation retracts onto A, then the map* $(X,A) \to (X/A,*)$ *induces an isomorphism,*

$$H_*(X,A) \xrightarrow{\cong} H_*(X/A,*)$$

PROOF. Consider the commutative diagram of pairs:

$$\begin{tikzcd} (X,A) \arrow[r, "i_1"] \arrow[d] & (X,B) \arrow[d] & (X - A, B - A) \arrow[l, "j_1"'] \arrow[d, "k"] \\ (X/A,*) \arrow[r, "i_2"'] & (X/A, B/A) & (X/A - \{*\}, B/A - \{*\}) \arrow[l, "j_2"] \end{tikzcd}$$

We want the left vertical to be a homology isomorphism, and we will show that the rest of the perimeter consists of homology isomorphisms to induce this. The map k is a homeomorphism of pairs, and $(X-A, B-A)$ is an excision by assumption, so j_1 and k induce isomorphisms on relative homology. The map i_1 induces an isomorphism on relative homology since $A \to B$ is a homology equivalence by the five-lemma and the long-exact sequence of a pair. For $I = [0,1] \subset \mathbb{R}$, as I is a compact Hausdorff space, the map $B \times I \to (B/A) \times I$ is a quotient map. As such, the deformation $B \times I \to B$, which restricts to the constant deformation on A, and, since $A \to B$ is a deformation retract by assumption, this descends to show that $* \to B/A$ is a deformation retract. Therefore, i_2 is a homology isomorphism, and since $\overline{\{*\}} \subseteq \mathrm{Int}(B/A)$ in X/A by definition of the quotient topology, j_2 induces a isomorphism by the excision theorem. ⊠

Definition 1.0.28. A **pointed space** is a pair $(X, *)$ in which the subspace is a singleton. The category of pointed spaces is denoted Top_*. The homology theory here is given by the **reduced homology** of a pointed space $\tilde{H}_*(X) = H_*(X, *)$

Since the pair (D^n, S^{n-1}) is excisive and collapsing the boundary of the disk gives a sphere, we have

$$H_*(D^n, S^{n-1}) \xrightarrow{\cong} \tilde{H}_*(D^n/S^{n-1}) = \tilde{H}_*(S^n)$$

Since S^0 contains two points, its reduced homology is given by $\mathbb{Z}$ in degree 0 and 0 in other degrees; Passing from reduced to unreduced, we net an extra $\mathbb{Z}$ in degree 0. This can be described by generators $i_n \in S_n(D^n, S^{n-1})$ for the corresponding relative n-chain.

Proposition 1.0.29 (Homology group of Spheres). *For $n \geq 0$, we let $* \in S^{n-1}$ by any point, then*

$$H_q(S^n) = \begin{cases} \mathbb{Z} = \langle [\mathrm{d}i_{n+1}] \rangle & q = n > 0 \\ \mathbb{Z} = \langle [c_*^0] \rangle & q = 0, n > 0 \\ \mathbb{Z} \oplus \mathbb{Z} = \langle [c_*^0], [\mathrm{d}i_1] \rangle & q = n = 0 \\ 0 & \textit{Otherwise} \end{cases} \qquad H_q(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} = \langle [i_n] & q = n \\ 0 & \textit{Otherwise} \end{cases}$$

Proof. The division into cases for $H_q(S^n)$ can be eased by passing from unreduced to reduced. For $n \geq 0$, we have

$$\mathrm{d} : H_q(D^n, S^{n-1}) \xrightarrow{\cong} \tilde{H}_{q-1}(S^{n-1})$$

It then remains to check that

$$\tilde{H}_q(S^{n-1}) = \begin{cases} \mathbb{Z} & q = n-1 \\ 0 & q \neq n-1 \end{cases}$$

This follows by inductive argument. We have the isomorphisms of pairs,

$$\tilde{H}_{q-1}(S^{n-1}) \xleftarrow{\cong} H_q(D^n, S^{n-1}) \xrightarrow{\cong} H_q(D^n/S^{n-1}, \{*\})$$

Since $D^n/S^{n-1} \xrightarrow{\cong} S^n$, the arrow on the RHS is an isomorphism since S^{n-1} is a deformation retract of a neighborhood in D^n. ⊠

As an application, we obtain invariance of domains between Euclidean space of different dimensions.

Theorem 1.0.30. *If $m \neq n$, then $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$ are not homeomorphic.*

Proof. Towards contradiction, if we have such a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$, then removing a point $x_0 \in \mathbb{R}^n$, this should induce a homeomorphism

$$f^- : \mathbb{R}^n - \{x_0\} \xrightarrow{\cong} \mathbb{R}^m - \{f(x_0)\}$$

and thus both sides must have the same homology. However, the LHS is homotopy equivalent to S^{n-1} and the RHS is homotopy equivalent to S^{m-1}. These have different homology by our above proposition. ⊠

1.0.3.2. *Eilenberg-Steenrod Axioms.* We abstract the properties we've previously shown for homology into the notion of a **homology theory**:

Definition 1.0.31 (ES Axioms)**.** A **homology theory** on Top consists of:

- A sequence of functors $\{h_n : \mathsf{Top}_2 \to \mathsf{Ab}\}$ for all $n \in \mathbb{Z}$
- A sequence of natural transformations $\{\partial_n : h_n(X,A) \to h_{n-1}(A,\varnothing)\}$

which satisfies the following properties:

(1) $h_n : \mathsf{Top}_2 \to \mathsf{Ab}$ is homotopy-invariant for all $n \in \mathbb{Z}$.
(2) Excisions induce isomorphisms, which is to say that for each excisive triad $U \subseteq A \subseteq X$, the inclusions give us the isomorphism:
$$h_n(X/A, A/A) \xrightarrow{\cong} h_n(X,A)$$
(3) For any pair $(X,A) \in \mathsf{Top}_2$, there is a long exact sequence on homology:
$$\cdots \to h_{n+1}(X,A) \xrightarrow{\partial} h_n(A,\varnothing) = h_n(A) \to h_n(X,\varnothing) = h_n(X) \to h_n(X,A) \xrightarrow{\partial} h_{n-1}(A,\varnothing) \to \cdots$$
(4) We satisfy the wedge (or Milnor) axiom: The inclusions into the coproduct $X_i \to \coprod_{i\in I} X_i$ induces an isomorphism on homology
$$\bigoplus_{i\in I} h_n(X_i) \xrightarrow{\cong} h_n\left(\coprod_{i\in I} X_i\right)$$
(5) The homology group of a point $h_n(\{*\})$ is non-zero iff $n = 0$.

Locality & Mayer-Vietoris. A key idea in the computation of homology is the underlying idea that you should be able to compute this by, in some sense, only using simplices that are small with respect to a cover.

Definition 1.0.32. A collection $\mathscr{A}$ of subsets of X is a **cover** if the union of their interiors covers X, ie.
$$\bigcup_{i\in I} \mathrm{Int}(A_i) = X \quad \text{for } \mathscr{A} = \{A_i\}_{i\in I}$$

Definition 1.0.33. Let $\mathscr{A}$ be a cover of X. An n-simplex $\sigma : \Delta^n \to X$ is **$\mathscr{A}$-small** if its image lies in a element of $\mathscr{A}$, ie. $\mathrm{Im}(\sigma) \in A \in \mathscr{A}$.

REMARK. It is evident that for each $\mathscr{A}$-small σ, $d_i(\sigma) = \sigma \circ \delta_i$ is once again $\mathscr{A}$-small. Then, the subgroups $S_n^{\mathscr{A}}(X)$ spanned by $\mathscr{A}$-small simplices form a sub-complex
$$S_*^{\mathscr{A}}(X) \subseteq S_*(X)$$
of $\mathscr{A}$-small singular chains. This remark nets us an isomorphism on homology, and this is precisely the *Locality principle*.

Theorem 1.0.34 (Locality Principle)**.** *The inclusion $S_*^{\mathscr{A}}(X) \hookrightarrow S_*(X)$ induces an isomorphism on homology:*
$$H_*^{\mathscr{A}}(X) \xrightarrow{\cong} H_*(X)$$

I actually won't take the time to prove this, but it should suffice to provide some of the main ideas. The underlying idea is that by "subdivisions" of simplices, we can replace an n-cycle $[a]$ which represents a homology class with a different representative which has "smaller" simplices. If we do this enough times, the representative will be $\mathscr{A}$-small.

Mayer-Vietoris. Mayer-Vietoris ends up being very closely related to the Locality principle and excision as a result.

Theorem 1.0.35 (Mayer-Vietoris)**.** *Suppose a topological space X is covered by two open sets $U, V \subseteq X$. Then there are natural maps $\partial : H_n(X) \to H_{n-1}(U \cap V)$ such that we obtain the long-exact sequence:*
$$\cdots \to H_n(U\cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \xrightarrow{\partial} H_{n-1}(U \cap V) \to H_{n-1}(U) \oplus H_{n-1}(V) \to \cdots$$

PROOF. To begin, note that for $\mathscr{A} = \{U, V\}$, there is a SES of chain-complexes:
$$0 \to S_*(U\cap V) \to S_*(U) \oplus S_*(V) \to S_*^{\mathscr{A}}(X) \to 0$$
where our first map is obtained by mapping $a \mapsto (a, -a)$ and the second map is given by mapping $(a,b) \mapsto a+b$. Passing through homology and applying the Locality principle, we get exactly what we need. ⊠

CHAPTER 2

Homotopy Theory

Already, we've talked a bit about homotopy theory, and to continue further, we should recall some of the important definitions.

Definition! For a topological space X, we consider the **homotopy group** $\pi_n(X) := \mathrm{Map}_{\mathsf{Top}_*}(S^n, X)/\sim$, where this is the hom-set of based-maps out of the sphere into X with homotopy relations.

Definition! We say a map $f : X \to Y$ is a **weak equivalence** if its induced map on homotopy groups $f_* : \pi_n(X) \to \pi_n(Y)$ is a homotopy equivalence for all $n \in \mathbb{N}$.

Further enabling the analysis and application of homotopy relations requires a substantial amount of formal considerations. Following the exposition of Tammo Tom-Dieck [TD08], we should try to deal with:

(1) The construction of auxilliary spaces from the basic "homotopy cylinder" $X \times I$ which provides us mapping cylinders, mapping cones, suspensions, and dual constructions based on the "path space" X^I. These general constructions will enable to later resolve problems in defining homotopy limits and colimits.
(2) Natural group structures arise out of pointed topological spaces in Top_*, these are provided by explicit constructions: suspensions and loop-spaces.
(3) Exact sequences involving homotopy functors based on "exact sequences" among pointed-spaces (which we often think of as "space levels"). These become what we will later define as fibre and co-fibre sequences.

Now, going back to point (1), we want to construct the mapping cylinder and then generalize this to what we call the *double mapping cylinder*. Consider a map $f : X \to Y$. To make this map nicer, we can perform various constructions, namely we consider the mapping cylinder for this reason.

Definition 2.0.1. Let $f : X \to Y$ be a map. We consider the **mapping cylinder** $M_f := (X \times I) \sqcup Y/\sim$ with relations $(x,0) \sim x$ and $(x,1) \sim f(x) = y$. This is given by the pushout:

$$\begin{tikzcd} X \sqcup X \arrow[r, "{(\mathrm{id},f)}"] \arrow[d, "{\langle i_0, i_1\rangle}"'] & X \sqcup Y \arrow[d, "{\langle j,h\rangle}"] \\ X \times I \arrow[r] & M_f \end{tikzcd} \qquad j(y) = y,\ h(x) = (x,0)$$

Here $i_t(x) = (x,t)$. Since $\langle i_0, i_1\rangle$ are closed embeddings into $X \times I$, the maps (j,h) are closed embeddings as well. We also have a projection $q : M_f \to Y$ given by $(x,t) \mapsto f(x)$ and $y \mapsto y$.

Remark (1). These provide us with some relations $q \circ j = f$ and $q \circ h = \mathrm{id}$. The map $j \circ q$ is homotopic to the identity relative to Y. The homotopy is given by the identity on Y and it contracts I relative 1 to 1. As such, the inclusion $Y \hookrightarrow M_f$ is a homotopy equivalence by pushing $X \times I$ onto Y.

Remark (2). If we have the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

Then if this diagram commutes up to homotopy relation[1], this induces a map $M_f \to M_{f'}$ by post-composing α with f' at the top and pre-composing β with f, and "fixing" a homotopy $\Phi : \beta \circ f \simeq f' \circ \alpha$. This gives us a morphism of

[1] A diagram which commutes up to homotopy relation is often called a **homotopy pushout**.

mapping cylinders with a commutative diagram.

$$\begin{tikzcd} X \sqcup Y \arrow[r] \arrow[d, "(\alpha,\beta)"] & M_f \arrow[d, "M(\alpha,\beta,\Phi)"] \\ X' \sqcup Y' \arrow[r] & M_{f'} \end{tikzcd}$$

where $M(\alpha,\beta,\Phi) = \Theta_{\alpha\beta\Phi} : M_f \to M_{f'}$ is given by

$$\Theta_{\alpha\beta\Phi}(y) = \beta(y),\ y \in Y, \quad \Theta_{\beta\alpha\Phi}(x,s) = \begin{cases} (\alpha(x), 2s) & x \in X,\ s \le 1/2 \\ \Phi_{2s-1}(x) & x \in X,\ s \ge 1/2 \end{cases}$$

As a byproduct of the above remark, this nets us the theorem:

Theorem 2.0.2. *Suppose α and β are homotopy equivalence, then the map $M(\alpha,\beta,\Phi) = \Theta_{\alpha\beta\Phi}$ is a homotopy equivalence.*

2.0.0.1. *Double Mapping Cylinder.* We can easily enumerate on our construction of the mapping cylinder to give it additional structure, and this is where we can obtain constructions such as the *double mapping cylinder*.

Definition 2.0.3. Given maps $f : A \to B$ and $g : A \to C$, the **double mapping-cylinder** $M_{f,g} = M(B \xleftarrow{f} A \xrightarrow{g} C)$ is the quotient of $(B \sqcup A \times I \sqcup C)/\sim$ under the relations $f(a) \sim (a,0)$ and $g(a) \sim (a,1)$. It can be defined in this way via the push-out:

$$\begin{tikzcd} A \sqcup A \arrow[r, "(f,g)"] \arrow[d, "(i_0,i_1)"'] & B \sqcup C \arrow[d, "(j_0,j_1)"] \\ A \times I \arrow[r] & M_{f,g} \end{tikzcd}$$

The map (j_0, j_1) is a closed embedding.

Remark. In the case that $f \equiv \mathrm{id}_A$, we can identify $M_{\mathrm{id}_A,g} = M_g$. We can also glue M_f and M_g along the common subspace A and obtain $M_{f,g}$ up to the attaching map $I \cup_{\{0\}} I \simeq I$. The commutative diagram:

$$\begin{tikzcd} B \arrow[d, "\beta"'] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & C \arrow[d, "\gamma"] \\ B' & A' \arrow[l, "f'"] \arrow[r, "g'"'] & C' \end{tikzcd}$$

induces a map $M_{f,g} \to M_{f',g'}$ given by the quotient $(\beta+\alpha, \mathrm{id}+\gamma)$. We can generalize this to a homotopy-commutative diagram as we had with the normal mapping cylinder.

Lemma 2.0.4. *If we have the homotopy-commuting diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & X \end{tikzcd}$$

Then, we can fit the mapping cylinder $M_{f,g}$ into the diagram with an arrow $M_{f,g} \to X$.

Remark (1). This arrow depends on the choice of homotopy.

Remark (2). Here, we call X the homotopy push-out of the diagram in the previous lemma if $M_{f,g} \to X$ is a homotopy equivalence.

Example 2.0.5. For the projections $X \leftarrow X \times Y \to Y$, the homotopy push-out is given by the join

$$X \star Y := (A \times B \times I)/\sim$$

where $(a, b_1, 0) \sim (a, b_2, 0)$ and $(a_1, b, 1) \sim (a_2, b, 1)$. ◀

2.0.1. Fibrations, Cofibrations, & their Properties.

Definition 2.0.6. A map $i : A \to X$ has the **homotopy extension property** (HEP) for the space Y if for each homotopy $h : A \times I \to Y$ and each map $f : X \to Y$ with $(f \circ i)(a) = h(a, 0)$, there exists a homotopy $H : X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(i(a), t) = h(a, t)$. We call H **an extension of h with initial condition f**.

Definition 2.0.7. A **cofibration** $i : A \to X$ is a map which satisfies HEP for all spaces. Equivalently, this is defined if the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y \times I \arrow[d, "pr_1"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd}$$

has it so $h \circ pr_1 = i \circ f$ such that there exists $\tilde{h}$ which makes the diagram commute.

Definition 2.0.8. If a map $p : E \to B$ has the **homotopy lifting property** for the space Y, this is the same as saying: "For each homotopy $k : Y \times I \to B$ and each map $f : Y \to E$ such that $p \circ f(x) = k(x, 0)$, there exists a homotopy $\tilde{h} : Y \times I \to E$ with $p \circ \tilde{h} = k$ and $\tilde{h}(x, 0) = f(x)$." We call $\tilde{h}$ a **lifting of k with initial condition f**. Our map p is a fibration when it satisfies HLP for all spaces.

Definition 2.0.9. A map $p : E \to B$ is called a **fibration** if the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "k"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

admits a lift. Equivalently, this says we have HLP for all spaces, meaning $k \circ i_0 = p \circ f$ so that there exists this lift $\tilde{h}$ which makes the above diagram commute.

However, by these definitions alone, you cant explicitly prove that a map is a cofibration or a fibration. As such, it suffices to test the HEP and HLP properties for a universal space Y:

- Recall the construction of our "*Mapping cylinder*" $M_i := X \cup_i (A \times I)$ for a map $i : A \to X$ is the push-out of $i : A \to X$ and $i_0 : X \to X \times I$, given by the diagram:

  $$\begin{tikzcd} A \arrow[r, "(i{,}\mathrm{id})"] \arrow[d, "i_0^A"'] & X \arrow[d, "b"] \\ A \times I \arrow[r, "\alpha"'] & M_i \end{tikzcd}$$

  For a pair of maps $f : X \to Y$ and $h : A \times I \to Y$, we have $(h \circ i_0^A) = (f \circ i)$, these corresponding to maps

  $$\sigma : M_i \to Y \text{ with } (\sigma \circ b) = f \text{ and } (\sigma \circ \alpha) = h$$

  Applying this to the pair $i_0^X : X \to X \times I$ and $i \times \mathrm{id} : A \times I \to X \times I$, we obtain a map $r : M_i \to X \times I$ with $(r \circ b) = i_0^X$ and $(r \circ \alpha) = i \times \mathrm{id}$. When $i : A \to X$ is a cofibration, we can use HEP for M_i for a lifting with initial condition b and homotopy α. The HEP then provides us a map $s : X \times I \to M_i$ such that $s \circ i_0^X = b$ and $s \circ (i, \mathrm{id}) = \alpha$. We can then observe that we have $(s \circ r \circ b) = (s \circ i_0^X) = b$ and $(s \circ r \circ \alpha) = (s \circ (i, \mathrm{id})) = \alpha$, with push-out property $(s \circ r) = \mathrm{id}(M_i)$. Therefore, we have that r is an embedding into M_i and s is a retraction.

REMARK. These cofibrations and fibrations satisfy the universal properties of pushouts, making the following diagram(s) commute:

$$\begin{tikzcd} & X \arrow[dr, "i_0^X"] \arrow[drr, bend left, "f"] & & \\ A \arrow[ur, "i"] \arrow[dr, "i_0^A"'] & & X \times I \arrow[r, dashed, "H"] & Y \\ & A \times I \arrow[ur, "(i{,}\mathrm{id})"] \arrow[urr, bend right, "h"'] & & \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[dl, "p"'] & & \\ B & & E \times I \arrow[ul, "ev_0^E"'] \arrow[dl, "p^I"] & X \arrow[l, dashed, "H"] \arrow[ull, bend right, "a"'] \arrow[dll, bend left, "h"] \\ & B \times I \arrow[ul, "ev_0^B"] & & \end{tikzcd}$$

where ev_0^E and ev_0^B are the evaluation maps at 0 for E and B respectively.

REMARK. In other words, we can extend diagrams of the form

$$\begin{tikzcd}
A \arrow[rr, "i_0^A = a\times\{0\}"] \arrow[dd, "i"'] & & A\times I \arrow[dl] \arrow[dd] \\
& Y & \\
X \arrow[ur] \arrow[rr, "i_0^X = x\times\{0\}"] & & X\times I \arrow[ul, dashed]
\end{tikzcd}$$

for a universal space Y.

As we can see by the above exposition, we can use the mapping cylinder to decompose an arbitrary map $f : A \to X$ as a composite of a cofibration and a homotopy equivalence. This is, up to homotopy equivalence, any map can be replaced with a cofibration. Capturing what we had said previously in a "cleaner" manner, note that for the mapping cylinder $M_f = (X \cup_(A \times I))$, the map $f : A \to X$ coincides with:

$$A \xrightarrow{j} M_f \xrightarrow{r} X$$

where $j(a) = (a, 1)$, $r(y) = y$ on Y, and $r(x, t) = f(x)$ on $X \times I$. If $i : X \hookrightarrow M_i$ is an inclusion, then $r \circ \mathrm{id} = \mathrm{id}$ and $\mathrm{id} \simeq i \circ r$. *This provides us a deformation* $h : M_i \times I \to M_i$ of M_i into $i(X)$ by setting

$$h(y,t) = y \quad h((x,t),s) = (x,(1-s)t)$$

Then, we have that our map $j : A \to M_f$ satisfies HEP. This gives an important criterion for maps to be cofibrations.

Proposition 2.0.10. *The map $i : A \to X$ is a cofibration iff it satisfies the homotopy extension property (HEP) for M_i.*

PROOF. The forwards direction holds by definition, so it suffices to only prove the backwards direction. Suppose the lift $X \to M_i \times I$ exists. Then, this induces the map, then there exists a map $M_i \to Y$, which we can compose with $X \times I \to M_i$ to get a map

$$X \times I \to M_i \to Y$$

Hence, we satisfy the HEP for all spaces Y, so the map $i : A \to X$ is a cofibration. ⊠

Lemma 2.0.11. *If $p : E \to B$ is a fibration and $g : A \to B$ is any map, then the induced map $A \times_g E \to A$ is a fibration*

As a dual to (Lemma 2.0.11.), we should have that pushouts of cofibrations are cofibrations, in the sense that for cofibration $i : A \to X$ and a map $g : A \to B$, then the pushout $X \cup_f B := (X \sqcup B)/\sim$, with $f(a) \sim a$, will also be a cofibration.

Lemma 2.0.12. *Let $i : A \to X$ be a cofibration and $g : A \to B$ be any map, then the push-out diagram:*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"'] & B \arrow[d, "H"] \\
X \arrow[r, "F"'] & Y
\end{tikzcd}$$

is a homotopy push-out. This is to say, the induced map $B \to X \cup_g B$ is a cofibration.[2]

[2]This is just a diagram chase with an annoying extended diagram of the push-out. I'll hold off on this, but if you REALLY want to, you can have at it and prove it for yourself.

CHAPTER 3

Stable Homotopy Theory

Building Up the Stable Category. Before the stable category, we'll provide some motivation. There's a choice here: you could just say "let's take the category of spectra," but having some motivation for why we're doing what we're doing is important.

There are lots of ways to think about where stabilization comes from.

(1) The Freudenthal suspension theorem says that if X is nondegenerately based (meaning the based inclusion map $* \hookrightarrow X$ is a cofibration) and $n-1$-connected, then $\pi_q(X) \to \pi_{q+1}(\Sigma X)$ is an isomorphism for $q < 2n-1$ and a surjection when $q = 2n-1$.[1] It's easier to see that cohomology groups are stable under suspension, but this tells us that homotopy groups stabilize in a range that increases at about twice the rate that the connectivity of X does. Since $\Sigma^n X$ is at least n-connected, this suggests you could replace X by the sequence $X, \Sigma X, \dots, \Sigma^n X, \dots$, and keep track of that instead, regarding it as a repository for the **stable homotopy groups** $\pi_n^S(X) := \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$. One way to think of this is as formally making homotopy theory into a homology theory (which it isn't *a prioiri*); you end up taking the same kind of colimit.

You could do this equivariantly: we have representation spheres. But it's not entirely clear what to do.

(2) Another perspective is that the stable category is the result of inverting the canonical map[2]

$$\bigvee_{i=1}^{k} X \longrightarrow \prod_{i=1}^{k} X. \tag{3.0.1}$$

Again, this is something you can think about making precise; the stable category is the initial triangulated category constructed from Top in which (3.0.1) is an isomorphism. In particular, this forces the homotopy category to be additive.

Again, we could do this equivariantly.

(3) Suppose we have a functor F from finite CW complexes to Top such that $F(*) = *$, and suppose F commutes with filtered colimits and preserves weak equivalences (e.g. if it's topologically or simplicially enriched, for formal reasons). By taking colimits, we can obtain a functor $\widehat{F}$ from CW complexes to Top. We say F is **excisive** if it takes pushouts to pullbacks; this is an old perspective, which was used to show the Dold-Thom theorem, that the infinite symmetric product SP^∞ is a cohomology theory. In any case, if F is excisive, $\{F(S^n)\}$ represents a cohomology theory. Namely, the homotopy pushout

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & D^{n+1} \arrow[d] \\ D^{n+1} \arrow[r] & S^{n+1} \end{tikzcd}$$

creates suspension, and via F, becomes a pullback creating Ω. Asking what the excisive functors are leads to the stable category, and is also something you could do equivariantly.

(4) Another perspective: what is the Spanier-Whitehead dual of a point? Taking shifts, what's the Spanier-Whitehead dual of S^n? Equivariantly, one wants to know the Spanier-Whitehead duals of G/H_+. This is important for defining Poincaré duality, etc. Nonequivariantly, the best way to answer this is the Pontrjagin-Thom construction, which not only answers this, but provides a deep understanding for what

[1] The map comes from the loop-suspension adjunction, which gives us a unit $X \to \Omega\Sigma X$, hence a map $\Omega^q X \to \Omega^{q+1}\Sigma X$, and the map on homotopy groups is π_0 of that map. This is the based version of the mapping space and Cartesian product adjunction: $\Sigma X := S^1 \wedge X$ and $\Omega X := \mathrm{Map}(S^1, X)$ are adjoint functors.

[2] The existence of this map follows from the universal property of the product.

the stable homotopy groups of the spheres are. In this chapter, we'll do this equivariantly, and it will tell us what the spheres are.

Pursuant to motivating the stable category, this section is about duality. This is a reinvention of the original construction of the stable category — Spanier's original construction of spectra was motivated by answering questions on duality, and we'll proceed similarly, if a bit ahistorically. Since we're in the equivariant setting, the answers will be slightly different.

Alexander duality is a tale as old as time, from what could be called the prehistory of algebraic topology.[3]

Theorem 3.0.2 (Alexander duality [Ale15]). *Let $K \subset S^n$ be compact, locally contractible, and nonempty.*[4] *Then, K and $S^n \setminus K$ are **Alexander dual** in that there is an isomorphism*

$$\widetilde{H}^{n-i-1}(K;\mathbb{Z}) \cong \widetilde{H}_i(S^n \setminus K;\mathbb{Z}).$$

The proof isn't too hard, e.g. Hatcher does it. This is closely related to considering embeddings in Euclidean spaces, after you take the one-point compactification.

The good part of this proof is that it doesn't depend on the embedding. But there are a few drawbacks:

(1) K does not determine the homotopy type of $S^n \setminus K$. Knot theory is full of examples, and they tell you that the issues arise for the fundamental group.
(2) n can vary, and if you embed $S^n \hookrightarrow S^{n+1}$ as the equator, you get different statements.

Motivated by the second issue, Spanier defined the S-category in the 1950s.[5]

Definition 3.0.3. The ***S*-category** S is the category whose objects are the objects in Top and whose morphisms are

$$\mathrm{Map}_{\mathsf{S}}(X,Y) := \operatorname*{colim}_n \mathrm{Map}_{\mathsf{Top}}(\Sigma^n X, \Sigma^n Y).$$

By the Freudenthal suspension theorem, the hom-sets stabilize at some finite n. Spanier then defined ***S*-duality**, which we might call **Spanier-Whitehead duality**, by specifying that X and Y are S-dual if $Y \cong S^n \setminus X$ in the S-category. The S-category has some issues: it's neither complete nor cocomplete. We like gluing stuff together, so this is unfortunate. ⋖ Spanier proved that to every $X \to S^n$, you can assign a dual $D_n X$, that $\Sigma D_n = D_{n+1}$, and $D_{n+1}\Sigma = D_n$. That is, duality commutes with suspension, so in S, every $X \to S^n$ has a unique S-dual: the duals inside S^n and S^{n+1} are the same in the S-category for sufficiently large n.

Spanier and Whitehead then asked one of their graduate students, Elon Lima, to formalize this S-category, leading to the first notion of the category of spectra [Lim58].

An axiomatic setting for duality. The formal setting for duality is a **closed symmetric monoidal category** C. We're not going to spell out the whole definition, but here are some important parts.

- C is symmetric monoidal, meaning there's a tensor product $\wedge\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, which is (up to natural isomorphism) associative and commutative, and has a unit S. Commutativity is ensured by the **flip map** $\tau\colon X \wedge Y Y \wedge X$.
- There is an internal **mapping object** $F(X,Y) \in \mathsf{C}$ for any $X, Y \in \mathsf{C}$.
- The functors $-\wedge X$ and $F(X,-)$ are adjoint (just like the tensor-hom adjunction).

The unit and counit of the tensor-hom adjunction are used to define duality.

Definition 3.0.4. The **evaluation map** is the unit $X \wedge F(X,Y) \to Y$, and the **coevaluation map** is the counit $X \to F(Y, Y \wedge X)$. The **dual** of X is $DX = F(X,S)$.

You also get a natural map $\nu\colon F(X,Y) \wedge Z \to F(X, Y \wedge)$.

Exercise 3.0.5. Check that $X \cong F(S,X)$, which follows directly from the axioms.

The adjoint of ε is a map $X \to DDX$.

There are a few good references for this: Dold-Puppe [DP61] is one, and [LMS86] is another, though it presents a somewhat old way of doing things.

[3] If you read between the lines, you can find it in Poincaré's works, but it's from the 1940s or 50s stated explicitly.

[4] Classically, one works simplicially, picking a triangulation of S^n and letting K be a subpolyhedron.

[5] The name "S-category" is somewhat misleading: in those days, suspension was sometimes denoted S instead of Σ to make typeseting easier, and the S in S-category stood for suspension, not spheres.

Definition 3.0.6. $X \in \mathsf{C}$ is **strongly dualizable** if there exists an $\eta\colon S \to X \wedge DX$ such that the following diagram commutes.

(3.0.7)
$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d] & X \wedge DX \arrow[d, "\tau"] \\ F(X,X) & DX \wedge X. \arrow[l, "\nu"'] \end{tikzcd}$$

Here, the left-hand map comes from an adjunction to the identity $\mathrm{id}_X : X \cong X \wedge S \to X$. The lower map is more explicitly $\nu\colon F(X,S) \wedge X \to F(X,X)$.

Example 3.0.8. Let R be a commutative ring and $\mathsf{C} = \mathsf{Mod}_R$, and let X be a free R-module. If $\{v_i\}$ is a basis for X and $\{f_i\}$ is the dual basis, then the map $\eta\colon R \to X \otimes_R \mathrm{Hom}_R(X,R)$ is the map sending

$$1 \longmapsto \sum v_i \otimes f_i.$$

If you unravel what (3.0.7) is saying, it says that the map

$$x \longmapsto \sum_i f_i(x) v_i$$

must be the identity. Thus, X is strongly dualizable iff X is finitely generated and projective. That is, *X is strongly dualizable iff it's a retract of a finite-rank free module*, which is a perspective that will be useful later. ◀

Another way to think of this is that X is strongly dualizable with dual Y iff there exist maps $\varepsilon : X \wedge Y \to S$ and $\eta\colon S \to X \wedge Y$ such that the compositions

$$X \cong S \wedge X \xrightarrow{\eta \wedge \mathrm{id}_X} X \wedge Y \wedge X \xrightarrow{\mathrm{id}_X \wedge \varepsilon} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{id}_Y \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge \mathrm{id}_Y} S \wedge Y \cong Y$$

are the identity.[6] From this and a diagram chase, you get some nice results.

Proposition 3.0.9. *If X and Y are dual, then there are isomorphisms $Y F(X,S)$ and $X DDX$.*

To paraphrase Lang, the best way to learn this is to prove all the statements without looking at the proofs, like all diagram chases.

If you like string calculus, you can think of these in terms of S- or Z-shaped diagrams. In this form, these results are sometimes known as the **Zorro lemmas**.

Another consequence of this formulation is that $- \wedge DX$ is right adjoint to $- \wedge X$, so by uniqueness of adjoints there's a natural isomorphism $- \wedge DX \cong F(X,-)$.

Atiyah duality. The Whitney theorem tells us that for any manifold M and sufficiently large n, there's an embedding $M \hookrightarrow \mathbb{R}^n$. This means we can compute the Spanier-Whitehead dual of a manifold, which is the setting of Atiyah duality. We'll assume M is compact.

By the tubular neighborhood theorem, there's an $\varepsilon > 0$ and a tubular neighborhood M_ε such that M_ε is the disc bundle of the normal bundle $\nu \to M$.[7]

Brown Representability. Some of our intuitions require the notion of Brown representability. This is originally due to [**Bro62**], and we'll give Neeman's interpretation [**Nee96**]. It works in any triangulated category, and is very close to the small object argument, which already makes it a good thing.

Fix a triangulated category C that has small coproducts. The stable homotopy categories $\mathrm{Ho}(\mathsf{Sp})$ and $\mathrm{Ho}(\mathsf{Sp}^G)$ are the examples to keep in mind.

Definition 3.0.10. An $x \in \mathsf{C}$ is **compact** if for all countable coproducts over $y_i \in \mathsf{C}$,

$$\mathrm{Map}_{\mathsf{C}}\left(x, \coprod_i y_i\right) \cong \coprod_i \mathrm{Map}_{\mathsf{C}}(x, y_i).$$

[6] In some presentations, this is how duality in a symmetric monoidal category is defined; the two approaches are equivalent.

This is not the usual definition of compactness in category theory, which uses filtered colimits, but in the stable setting these are the same as coproducts, motivating our definition. For example, this definition does not characterize compact topological spaces.

Definition 3.0.11.

- A **generating set** of a category C is a set $T \subseteq \mathrm{ob}(\mathsf{C})$ that **detects zero**, i.e. for all $x \in \mathsf{C}$, $x = 0$ iff $\mathrm{Map}_{\mathsf{C}}(z, x) = 0$ for all $z \in T$. If C is triangulated, we additionally require T to be closed under shift.
- C is **compactly generated** if it has a generating set consisting of compact objects.

We introduce these to skate around set-theoretic issues: at some point, we'd like to take a coproduct over all objects in the category, but that's too large. Instead, taking the coproduct over all generators will have the same power, and is actually well-defined.

For example, dualizable spectra (resp. dualizable G-spectra) are a compact generating set for $\mathrm{Ho}(\mathsf{Sp})$ (resp. $\mathrm{Ho}(\mathsf{Sp}^G)$).

Theorem 3.0.12 (Brown representability [Bro62, Nee96])**.** *Let C be a compactly generated triangulated category and $H\colon \mathsf{C}|\!\rightarrow\rangle\langle\!\rightarrow|\mathsf{Ab}$ be a functor such that*

(1)
$$H\left(\coprod_i X_i\right) \cong \prod_i H(X_i)$$
and

(2) *H sends exact triangles $X \to Y \to Z \to X[1]$ to long exact sequences*

$$H(X) \longrightarrow H(Y) \longrightarrow H(Z) \longrightarrow H(X[1]) \longrightarrow H(Y[1]) \longrightarrow H(Z[1]) \longrightarrow H(X[2]) \longrightarrow \cdots$$

*Then, H is **representable**, i.e. there's an $X \in \mathsf{C}$ and a natural isomorphism $\mathrm{Hom}_{\mathsf{C}}(-, X) \to H$.*

Proof. We're going to build X inductively. Fix a generating set T for C of compact objects.

In the base case, let
$$U_0 := \coprod_{x \in T} H(x) \qquad \text{and} \qquad X_0 := \coprod_{\substack{(\alpha,t)\in U_0\\ \alpha\in H(t)}} t.$$
Thus,
$$H(X_0) = H\left(\coprod_{(\alpha,t)\in U_0} t\right) \cong \prod_{(\alpha,t)\in U_0} H(t),$$
and in particular there is a distinguished element $\alpha_0 \in H(X_0)$ which is α at the (α, t) factor. By the Yoneda lemma, α_0 specifies a natural transformation $\theta_0\colon \mathrm{Map}_{\mathsf{C}}(-, X_0) \to H$, and by construction, θ_0 is surjective for each $t \in T$.

Now we induct: assume we have X_i and α_i specifying a natural transformation $\theta_i\colon \mathrm{Map}_{\mathsf{C}}(-, X_i) \to H$. Then, define
$$U_{i+1} := \coprod_{t\in T} \ker(\theta_i(t)\colon \mathrm{Map}_{\mathsf{C}}(t, X_i) \to H(t)). \qquad \text{and} \qquad K_{i+1} := \coprod_{(f,t)\in U_{i+1}} t.$$
There's a natural map $K_{i+1} \to X_i$ which applies f; let X_{i+1} be its cofiber. Applying H, this produces a map
$$H(X_{i+1}) \longrightarrow H(X_i) \longrightarrow H(K_{i+1}),$$
and by construction, $\alpha_i \mapsto 0$, so by exactness, we can lift to $\alpha_{i+1} \in H(X_{i+1})$. This α_{i+1} specifies a natural transformation $\theta_{i+1}\colon \mathrm{Map}_{\mathsf{C}}(-, X_{i+1}) \to H$, and the following diagram commutes:
$$\begin{tikzcd} \mathrm{Map}_{\mathsf{C}}(-, X_i) \arrow[r, "\theta_i"] \arrow[d] & H \\ \mathrm{Map}_{\mathsf{C}}(-, X_{i+1}). \arrow[ur, "\theta_{i+1}"'] & \end{tikzcd}$$
Thus we have a tower $\mathbb{N} \to \mathsf{C}$ sending $i \mapsto X_i$, and we can define
$$X := \operatorname*{hocolim}_i X_i,$$
which we'll show represents H. The **homotopy colimit** in a triangulated category is TODO.

REMARK. Though triangulated categories are a good language to know, as they're historically interesting and often useful, they have drawbacks: the octahedral axiom is awkward, for example, and sometimes the triangulated structure works against you.

For this reason, it can be useful to remember that triangulated categories arise as the homotopy categories of stable ∞-categories, where there's additional versatility simplifying some arguments. For this reason, we use words such as "cofiber" and "homotopy colimit," because they're secretly the same thing.

Exercise 3.0.13. The homotopy colimit is also called the **telescope**. Relate this to the usual definition of a telescope.

Anyways, we'll construct a natural transformation $\theta\colon \mathrm{Map}_{\mathsf{C}}(-,X) \to H$ as follows. Since H sends cofiber sequences to long exact sequences, applying it to

$$\coprod_i X_i \longrightarrow \coprod_i X_i \longrightarrow X$$

produces a long exact sequence

$$\cdots \longrightarrow H(X) \longrightarrow \prod_i H(X_i) \longrightarrow \prod_i H(X_i) \longrightarrow \cdots,$$

so in particular we can lift the $\alpha_i \in X_i$ to an $\alpha \in X$, which defines θ as before, and there is a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathsf{C}}(-,X_0) \arrow[r, "\theta_0"] \arrow[d] & H. \\ \mathrm{Map}_{\mathsf{C}}(-,X) \arrow[ur, "\theta"'] & \end{tikzcd}$$

In particular, θ is surjective on T (namely, $\mathrm{Map}_{\mathsf{C}}(t,X) \to H(t)$ is surjective for $t \in T$).

To see that θ is injective on T, we use compactness. Let $f \in \mathrm{Map}_{\mathsf{C}}(t,X)$ be such that $\theta(f) = 0$; we wish to show that $f = 0$. Since t is compact, f lies in some $\mathrm{Map}_{\mathsf{C}}(t,X_i)$, i.e. $f \in \ker(\mathrm{Map}_{\mathsf{C}}(t,X_i) \to H(t))$. Therefore, $f \in U_{i+1}$, so it's killed in X_{i+1}, and thus is 0 in the colimit. Therefore θ is injective on T, hence and isomorphism.

There exists a largest full triangulated subcategory C' of C that's closed under small coproducts and such that $\theta|_{\mathsf{C}'}$ is a natural isomorphism; we'll show that $\mathsf{C}' = \mathsf{C}$.[8]

Running the whole argument again,[9] we obtain a $Z \in \mathsf{C}'$ and a natural transformation $\widetilde{\theta}\colon \mathrm{Map}_{\mathsf{C}'}(-,Z) \to H$. We know $\mathrm{Map}_{\mathsf{C}'}$ and $\mathrm{Map}_{\mathsf{C}}$ are isomorphic, and proved that θ and $\widetilde{\theta}$ are isomorphisms on the generating set, we can consider the cofiber of

$$\mathrm{Map}_{\mathsf{C}}(-,Z) \longrightarrow \mathrm{Map}_{\mathsf{C}}(-,X).$$

By the Yoneda lemma, such natural transformations are naturally identified with $\mathrm{Map}_{\mathsf{C}}(Z,X)$, and you can compute the cofiber of $Z \to X$ in terms of these maps. This is 0, so $Z \cong X$.

So we've proven that H is representable when restricted to the objects in the generating set T in C. Furthermore, both H and $\mathrm{Hom}_{\mathsf{C}}(-,X)$ take coproducts to products and take cofiber sequences to long exact sequences. As a consequence, we can conclude that the full subcategory C' of C on which the natural transformation $\theta\colon \mathrm{Hom}_{\mathsf{C}}(-,X) \to H$ is an isomorphism is a full triangulated subcategory of C which is closed under small coproducts (in C) and contains T.[10]

Therefore, it suffices to show that for any compactly generated triangulated category C, a full triangulated subcategory C' that contains T and has small coproducts must in fact be all of C. We show this as follows. Without loss of generality assume that C' is the smallest full triangulated subcategory containing T and small coproducts; that is, take C' to be the smallest localizing subcategory containing T.

Now fix an object $c \in \mathsf{C}$ and consider the functor $\mathrm{Hom}_{\mathsf{C}}(-,c)$. Applying the construction from above, we obtain an object z and a natural transformation $\mathrm{Hom}_{\mathsf{C}}(-,z) \to \mathrm{Hom}_{\mathsf{C}}(-,c)$. By construction, the object $z \in \mathsf{C}'$ — each X_i is in C', and so the homotopy colimit is too. Moreover, the map $\mathrm{Hom}_{\mathsf{C}}(t,z) \to \mathrm{Hom}_{\mathsf{C}}(t,c)$ is an isomorphism for all $t \in T$.

8 Dissecting what "largest full triangulated subcategory" means requires a little care, but such a C' exists.

9 It's important that we use the same generating set T for this.

10 Terminology: a triangulated subcategory that is closed under small coproducts (taken in the ambient category) is called **localizing**.

The Yoneda lemma now implies that we have a map $z \to c$ which corresponds to the natural transformation $\mathrm{Hom}_{\mathsf{C}}(-,z) \to \mathrm{Hom}_{\mathsf{C}}(-,c)$. Consider the cofiber c/z. By the previous paragraph and the fact that $\mathrm{Hom}_{\mathsf{C}}(t,-)$ takes triangles to long exact sequences, $\mathrm{Hom}_{\mathsf{C}}(t,c/z) = 0$ for all $t \in T$, and so $c/z = 0$. Therefore c and z are isomorphic. Since z is in C', $c \in \mathsf{C}'$. As c was chosen arbitrarily, we conclude that C and C' coincide. ⊠

This section is motivated by [MMSS01], an excellent paper that constructs the point-set stable category using diagram spectra. You should absolutely read this paper; it's a masterwork of exposition and making things look simple and clear in retrospect.

The approach of diagram spectra is different from, but equivalent to, the approaches taken in [LMS86, May96]. Our goal is to define a complete, cocomplete, symmetric monoidal category Sp such that

- the S-category is a full subcategory of Sp, and
- there is a symmetric monoidal functor $\Sigma^\infty \colon \mathsf{Top} \to \mathsf{Sp}$, which is left adjoint to a right adjoint $\Omega^\infty \colon \mathsf{Sp} \to \mathsf{Top}$.

There's a sense in which Sp is the smallest category satisfying these hypotheses, or that you get it by adding limits and colimits to S. In particular, we are constructing a stable analogue of the category of topological spaces, *not* its homotopy category. Historically, [CITE ME: Boardman] constructed the stable homotopy category as a formal completion of the S-category. Then, people tried to find "point-set models," stable model categories whose homotopy categories are isomorphic to Boardman's category. There are several options, but explicit proofs that their homotopy categories are equivalent to Boardman's are rare in the literature. ◄

Definition 3.0.14. By a **diagram** D we mean a small category, which we assume is enriched in Top and symmetric monoidal. The category of D**-spaces** is the category $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$ of enriched functors.

The category of D-spaces is symmetric monoidal under **Day convolution**. The idea is to build a symmetric monoidal product via left Kan extension: if F and G are D-spaces, the functor $F \,\overline{\wedge}\, G \colon (d_1,d_2) \mapsto F(d_1) \wedge G(d_2)$ is a $(\mathsf{D}\times\mathsf{D})$-space. To produce a D-space from this, let $\boxtimes \colon \mathsf{D} \to \mathsf{D}$ be the monoidal product on D, and consider the left Kan extension

$$\begin{tikzcd} \mathsf{D}\times\mathsf{D} \arrow[r, "F\overline{\wedge}G"] \arrow[d, "\boxtimes"'] & \mathsf{Top} \\ \mathsf{D} \arrow[ur, dashed, "F\wedge G"'] & \end{tikzcd} \tag{3.0.15}$$

This is our symmetric monoidal product $F \wedge G$. More explicitly,

$$\begin{aligned}(F\wedge G)(z) &= \operatorname*{colim}_{x\boxtimes y = z} F(x)\wedge G(y) \\ &:= \int^{x,y\in\mathsf{D}} F(x)\wedge G(y)\wedge \mathsf{D}(x\boxtimes y, z).\end{aligned}$$

This looks like the usual convolution, and the analogy with harmonic analysis can be taken further, e.g. in an unpublished paper of Isaksen-Behrens.

For any $d \in \mathsf{D}$, there's an **evaluation** functor $\mathrm{Ev}_d \colon \mathsf{Fun}(\mathsf{D},\mathsf{Top}_*) \to \mathsf{Top}_*$ sending $X \mapsto X(d)$. It's adjoint to $F_d \colon \mathsf{Top}_* \to \mathsf{Fun}(\mathsf{D},\mathsf{Top}_*)$ defined by

$$(F_d A)(e) := \mathrm{Map}_{\mathsf{D}}(d,e)\wedge A_+.$$

The unit for the symmetric monoidal structure on $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$ is $F_0 S^0$.

Let R be a **commutative monoid object** in $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$, which approximately means there are maps $F_0 S^0 \to R$, $S^0 \to R(0)$, and a unital, associative, commutative map $R(d)\wedge R(e) \to R(d\boxtimes e)$. For example, the unital condition is that the composition

$$R(d)\wedge S^0 \longrightarrow R(d)\wedge R(0) \longrightarrow R(d\boxtimes 0) \cong R(d)$$

must be the identity.

In this case, we can define the category Mod_R of R**-modules** in $\mathsf{Fun}(\mathsf{D},\mathsf{Top})$, those D-spaces M with an action map $\mu \colon R\wedge M \to M$ (satisfying the usual conditions). This is also a symmetric monoidal category (this requires R to be commutative), defined in the same way as the tensor product of modules over a ring: $M\wedge_R N$ is the coequalizer

$$M\wedge R\wedge N \rightrightarrows M\wedge N \longrightarrow M\wedge_R N.$$

Example 3.0.16 (Prespectra)**.** Let $\mathsf{D} = \mathbb{N}$, with only the identity maps. This is symmetric monoidal under addition: $[m] \boxtimes [n] := [m+n]$. The assignment $S_{\mathbb{N}} \colon [n] \mapsto S^n$ is a monoid in $\mathbb{N}$-spaces, and the category of $S_{\mathbb{N}}$-modules is classically called **prespectra**; the monoidal structure is the identification of $S^m \wedge S^n \cong S^{m+n}$.

Warning! $S_{\mathbb{N}}$ is *not* a commutative monoid! $S^n \wedge S^m \ncong S^m \wedge S^n$.

This was the cause of thirty years of pain and suffering in the community — they didn't know they were unhappy. People knew what the smash product should be on the homotopy category, and wanted a point-set model that's symmetric monoidal, unlike this example. ◁

Symmetric spectra are one answer, which we won't use in these notes. If all of this had been stated in terms of the Day convolution from the get-go, people probably would have figured out symmetric spectra as early as the 1960s, but hindsight is always clearer, and here we are. Symmetric spectra were introduced in [**HSS00**]; see [**Sch12**] for a detailed introduction.

Example 3.0.17 (Orthogonal spectra [**May80**])**.** Let $\mathscr{I}$ denote the category whose objects are finite-dimensional real inner product spaces V, and whose morphisms $\mathscr{I}(V,W)$ are the linear isometric isomorphisms $V \to W$.

In this category, $V \oplus W$ and $W \oplus V$ aren't equal, but are isomorphic, and the isomorphism between them is reflected in the flip between $S^n \wedge S^m$ and $S^m \wedge S^n$. In particular, the assignment $S_{\mathscr{I}} \colon V \mapsto S^V$ (the one-point compactification of V) is a *commutative* monoid, so the category of $S_{\mathscr{I}}$-modules is a symmetric monoidal category, called the category of **orthogonal spectra**. This is the model of the stable category that we will use. ◁

Example 3.0.18 ($\mathscr{W}$-spaces [**And74**])**.** Let $\mathscr{W}$ be the category of finite CW complexes (with either all maps or cellular maps; it doesn't really matter). $\mathscr{W}$-spaces are already like spectra, in a sense, in that they're modules over the identity functor $i \colon \mathscr{W} \hookrightarrow \mathsf{Top}$. There's a map $\varphi \colon A \to \mathrm{Map}(B, A \wedge B)$ sending $a \mapsto (b \mapsto a \wedge b)$, so if F is a $\mathscr{W}$-space, we have a sequence of maps

$$A \xrightarrow{\varphi} \mathrm{Map}(B, A \wedge B) \longrightarrow \mathrm{Map}(F(B), F(A \wedge B)).$$

Taking its adjoint defines a map

$$A \wedge F(B) \longrightarrow F(A \wedge B),$$

so F is a module over i. ◁

The assignment $n \mapsto \mathbb{R}^n$ defines a functor $\mathbb{N} \to \mathscr{I}$, and therefore a functor from prespectra to orthogonal spectra; with the right model structures, this induces an equivalence of their homotopy categories. Similarly, the assignment $V \mapsto S^V$ defines a functor $\mathscr{I} \to \mathscr{W}$, hence a functor from orthogonal spectra to $\mathscr{W}$-spaces, and this also will induce a homotopy equivalence.

Example 3.0.19 (Γ-spaces)**.** Let D be the category of finite based sets and based maps, e.g. $n_+ = \{0, 1, \dots, n\}$ with 0 as the basepoint. D-spaces are called **Γ-spaces**, and agree with Segal's notion of Γ-spaces [**Seg74**], which are defined differently. The multiplication comes from the map $\psi \colon 2_+ \to 1_+$ sending $1, 2 \mapsto 1$.

Let $d_i : n_+ \to 1_+$ send $j \mapsto \delta_{ij}$ (i.e. 1 if $i = j$, and 0 otherwise). A Γ-space is **special** if the induced map

$$X(n_+) \xrightarrow{\varphi_n} \prod_n X(1_+)$$

is a weak equivalence; it's **very special** if in addition the composition

$$X(1_+) \times X(1_+) \underset{\simeq}{\xleftarrow{\varphi_2}} X(2_+) \xrightarrow{\psi_*} X(1_+)$$

induces a commutative monoid structure on $\pi_0 X(1_+)$.

Kan extension defines a functor from D to the category of finite CW complexes, and working with π_*-equivalences of these, one obtains a model structure on the category of Γ-spaces. This is Quillen equivalent to the category of **connective spectra**, i.e. those whose negative homotopy groups vanish. ◁

In the equivariant case, there's even more structure, and notions of "extra special" Γ-spaces, as we will see in §**??**.

Definition 3.0.20. A prespectrum is an **Ω-prespectrum** if for all n, $X_n \xrightarrow{\cong} \Omega^m X_{m+n}$.

Definition 3.0.21. If X is a prespectrum and $q \in \mathbb{Z}$, the q^{th} **homotopy group** of X is

$$\pi_q(X) := \operatorname*{colim}_n \pi_{n+q} X(n).$$

A π_***-isomorphism** of prespectra is a map that induces an isomorphism on all homotopy groups.

Notice that negative homotopy groups exist, and may be nontrivial. This is one approach to defining the stable category, and is not the only one. In [**Ada74**] (which is an excellent book), Adams uses a more naïve viewpoint of "cells first, maps later" which doesn't require such abstraction, but it would be a huge mess to prove that his model is complete or cocomplete. The diagram spectra approach rigidly separates point-set techniques (easy, but not as useful) from operations on the homotopy category (more useful, but harder), and this separation is often useful. The ∞-categorical perspective mashes it all together, which can be confusing, but is the only setting in which you can prove things such as the stable category being initial. $\lessdot$ We'll reintroduce G-actions soon, and this is pretty slick using orthogonal spectra: we can replace $\mathscr{I}$ with the category of finite-dimensional G-representations with invariant inner products. Orthogonal spectra also have a really nice homotopy theory relative to symmetric spectra (which have other advantages that don't apply as much to us).

Because diagram categories are presheaves on nice categories, they inherit some good properties from Top_*; in particular, they are complete and cocomplete, and limits and colimits may be taken pointwise. This is also true for categories of modules over monoids in D-spaces, though it requires more work to prove: computing colimits is a bit harder, just like how the free product of groups is more complicated than the direct product. Categories of rings are *not* bicomplete, though.

We'll use prespectra and orthogonal spectra to define Quillen equivalent models for the stable homotopy category. As such, familiar constructions from stable homotopy theory can be constructed as prespectra and orthogonal spectra.

Example 3.0.22 (Suspension spectra)**.** Let $X \in \mathsf{Top}_*$. The **suspension spectrum** of X, denoted $\Sigma^\infty X$, is the stable homotopy type corresponding to the homotopy type of X.[11]

- In prespectra, the suspension spectrum of X is $\Sigma^\infty X \colon [n] \mapsto S^n \wedge X$. The $S_{\mathbb{N}}$-module structure is the data of the structure maps $S^m \wedge (S^n \wedge X) \to S^{m+n} \wedge X$.
- To define an orthogonal spectrum E, one must define for each finite-dimensional real inner product space V a pointed space $E(V)$ with an $\mathrm{O}(V)$-action and for each pair of such inner product spaces V and W, a structure map $S^V \wedge E(W) \to E(V \oplus W)$ that's $\mathrm{O}(V) \times \mathrm{O}(W)$-equivariant.

  Wanting to do this for every space forces our hand: we have to use the trivial action. The suspension spectrum of X sends $V \mapsto S^V \wedge X$, where the $\mathrm{O}(V)$-action is the usual action in the first component and the trivial action in the second component. The structure maps $S^V \wedge S^W \wedge X \to S^{V \oplus W} \wedge X$ are $\mathrm{O}(V) \times \mathrm{O}(W)$-equivariant, as desired.

The suspension spectrum of S^0 is the sphere spectrum $S_{\mathbb{N}}$ or $S_{\mathscr{I}}$. $\lessdot$

Example 3.0.23 (Eilenberg-Mac Lane spectra)**.** If A is an abelian group, the **Eilenberg-Mac Lane spectrum** HA is the spectrum that represents cohomology with coefficients in A. If in addition A is a commutative ring, HA is a commutative monoid in spectra, which defines the ring structure on A-cohomology.

- There is an identification $K(A,n) \xrightarrow{\cong} \Omega K(A, n+1)$; let $i_n \colon \Sigma K(A,n) \to K(A,n+1)$ be its adjoint. As an $\mathbb{N}$-space, $HA\colon [n] \mapsto K(A,n)$; the map i_n makes it into a prespectrum. If A is also a commutative ring, one obtains maps $K(A,m) \wedge K(A,n) \to K(A, m+n)$, and these satisfy the axioms to ensure that HA is a commutative ring spectrum.
- For orthogonal spectra, the construction is more complicated, since we must choose a model for $K(A,n)$ with an O_n-action on it. In particular, we must assume A is countable. Given an inner product space V, the A**-linearization** $A[S^V]$ of S^V is, as a set, A tensored with the reduced free abelian group on K (so the basepoint maps to zero), topologized as the quotient

$$\coprod_{k=0}^{\infty} A^k \times (S^V)^k \twoheadrightarrow A[S^V], \qquad (a_1, \dots, a_k, x_1, \dots, x_k) \longmapsto \sum_{j=1}^{k} a_j \cdot x_j.$$

[11] For a space X without basepoint, X_+ denotes the based space $X \amalg *$, with the extra point as a basepoint. Then, one considers $\Sigma^\infty X_+$.

This is a model for $K(A, \dim V)$, but has an $\mathrm{O}(V)$-action induced from the $\mathrm{O}(V)$-action on S^V. The structure map $S^V \wedge A[S^W] \to A[S^{V\oplus W}]$ sends

$$v \wedge \left(\sum_j a_j \cdot w_j\right) \longmapsto \sum_j a_j \cdot (v \wedge w_j), \tag{3.0.24}$$

and this is $\mathrm{O}(V) \times \mathrm{O}(W)$-equivariant, so defines an orthogonal spectrum. If A is a commutative ring, the ring spectrum structure on HA is defined by the multiplication maps $\mu \colon A[S^V] \wedge A[S^W] \to A[S^{V\oplus W}]$ sending

$$\mu \colon \left(\sum_i a_i x_i\right), \left(\sum_j b_j y_j\right) \longmapsto \sum_{i,j} (a_i b_j)(x_i \wedge y_j),$$

and the unit maps $e \colon S^V \to A[S^V]$ send $x \mapsto 1 \cdot x$. This construction is discussed in more detail in [**Sch12**, Example I.1.14] and [**Sch17**, Example V.1.9]. ◀

Example 3.0.25 (Thom spectra). TODO ◀

Definition 3.0.26. Let $f : X \to Y$ be a map of D-spaces (or prespectra or orthogonal spectra).

- f is a **level equivalence** if for all $d \in \mathsf{D}$, $f(d) \colon X(d) \xrightarrow{\simeq} Y(d)$ is a weak equivalence.
- f is a **level fibration** if for all $d \in \mathsf{D}$, $f(d)$ is a fibration.

That is, f is a natural transformation, and it acts through weak equivalences (resp. fibrations).

Theorem 3.0.27. *The category of* D*-spaces has a model structure, called the* ***level model structure****, in which the weak equivalences are the level equivalences and the fibrations are level fibrations. Moreover, this model category is cofibrantly generated.*

Exercise 3.0.28. Starting with the usual model structure on Top_*, construct the level model structure.

The "cofibrantly generated" part means that cofibrant objects behave like CW complexes, and in particular there is a theory of cellular objects. If $F_d \colon \mathsf{Top}_* \to \mathsf{Fun}(\mathsf{D}, \mathsf{Top}_*)$ is the left adjoint to Ev_d we constructed above, then the **generating cofibrations** are the maps $F_d(S^{n-1}_+ \to D^n_+)$ for each $n \geq 1$ and $d \in \mathsf{D}$, and the **acyclic cofibrations** are $F_d(D^n_+ \to (D_n \times I)_+)$ for each $n \geq 1$ and $d \in \mathsf{D}$.

Since all spaces are fibrant, all D-spaces are fibrant in the level model structure. The cofibrant objects are the retracts of **cellular objects**, which are built by iterated pushouts

$$\begin{tikzcd} \bigvee F_d S^{n-1}_+ \arrow[r] \arrow[d] & X_n \arrow[d] \\ \bigvee F_d D^n_+ \arrow[r] & X_{n+1}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

While this is all nice, it's not what we're looking for, as it contains no information about stable phenomena. It's like the category of spaces, just with more of them. For example, it's not even true that $X \to \Omega\Sigma X$ is a weak equivalence, which is important if you want Ω and Σ to be homotopy inverses. We'll define the correct model structure in the next section.

Recall that we defined a π_*-isomorphism of prespectra to be a map $f : X \to Y$ such that $\pi_q f : \pi_q X \to \pi_q Y$ is an isomorphism for all q. We'll extend this to orthogonal spectra: let $U \colon \mathsf{Sp}^{\mathscr{I}} \to \mathsf{Sp}^{\mathbb{N}}$ be pullback by the map $[n] \mapsto \mathbb{R}^n$, i.e. $UX([n]) = X(\mathbb{R}^n)$. U is right adjoint to a left Kan extension $P \colon \mathsf{Sp}^{\mathbb{N}} \to \mathsf{Sp}^{\mathscr{I}}$.

Definition 3.0.29. A map of orthogonal spectra $f : X \to Y$ is a π_***-isomorphism** if $Uf : UX \to UY$ is a π_*-isomorphism of prespectra.

We also defined an Ω-spectrum in prespectra, or an Ω-prespectrum, to be a prespectrum where the adjoints to the structure maps $X_n \xrightarrow{\cong} \Omega^m X_{n+m}$ are homeomorphisms. This is a pretty rigid condition, and so Ω-prespectra have nice properties.

Definition 3.0.30. Similarly, we define an Ω**-spectrum** in orthogonal spectra to be an orthogonal spectrum X such that the adjoints to the structure maps $X(U) \xrightarrow{\cong} \Omega^V X(U \oplus V)$ are homeomorphisms.

Classically, there were prespectra and then there were spectra (or Ω-spectra), and you would use some "spectrification" functor that took a prespectrum and produced a spectrum of the same homotopy type. Turning the adjoint maps into homeomorphisms looks difficult and is, as it involves some categorical and point-set wizardry. If you like this stuff, check out the appendix of [**LMS86**]. The first point-set symmetric monoidal model for the stable category [**EKMM97**] relies on this and even more magic, both clever and surprising.

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