Algebraic Topology 1

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Learning the Lingo

The following is a transcription of notes for Professor Andrew Blumberg's "Algebraic Topology 1." For consultation regarding the material in the notes, his office is room 607 in the Math department. Although there isn't a strict textbook this course will be using, some choice texts to read ahead of the notes will be

- Peter May's "Concise course in Algebraic Topology"
- Haynes Miller's "Notes on Algebraic Topology"
- Munkres's "Elements of Algebraic Topology"
- Weibel's "Homological Algebra"
- Saunders & Maclane's "Categories for the Working Mathematician" (Although, IMO, Riehl's "Category Theory in Context" is better written.)

Algebraic topology, at its core, answers questions regarding classification, turning geometric problems into ones relying on algebra (which is somehow supposed to be easier). For instance, consider the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$;

Question: Are $\mathbb{R}^2$ and $\mathbb{R}^3$ the same as sets? $\to$ Well, although there is a bijection $\mathbb{R}^3 \cong \mathbb{R}^2$, this is insufficient for the ways we want to think about things in Algebraic topology. As we'll later see, $\mathbb{R}^2 \neq \mathbb{R}^3$ in our senses of "being the same."

1.1 Category Theory for People who aren't Peter May

Definition 1.1.1. A *category* C is a collection of data consisting of objects $\mathrm{Obj}(C)$ and a collection of morphisms between said-objects. Each object $X \in \mathrm{Obj}(C)$ has an identity morphism $1_X : X \to X$. Additionally, any morphisms $f, g \in \mathrm{Mor}(C)$ are associative with respect to composition.

Example 1.1.2. One useful example of a category we may work with is the category Vect, with objects consisting vector spaces with linear maps as morphisms.

Definition 1.1.3. Given two categories C, D, then a *functor* $F : C \to D$ is a map of categories such that:

1. F takes objects in C to objects in D, ie. $x \in \mathrm{Obj}(C) \mapsto F(x) \in \mathrm{Obj}(D)$.
2. For an object $X \in \mathrm{Obj}(C)$, we have functors acting on morphisms $f \in \mathrm{Mor}(C)$, ie. $f(X) \in \mathrm{Obj}(C) \mapsto F(f(X)) \in \mathrm{Obj}(D)$.

One useful aspect of category theory is that we can give definitions that better specialize familiar notions:

Definition 1.1.4. In a category C, we call a morphism $f : X \to Y$ an *isomorphism* if there exists a map $g : Y \to X$ such that $f \circ g \cong \mathrm{id}_Y$ and $g \circ f \cong \mathrm{id}_X$.

Example. For instance, for the category C given by the diagram:

$$\begin{tikzcd}
\bullet_0 \arrow[r] \arrow[d] & \bullet_1 \\
\bullet_2 &
\end{tikzcd}$$

We see that the only isomorphisms are the identity morphisms of each object. In the category Top, the isomorphisms are homeomorphisms. In the category Vect, the isomorphisms are are invertible linear maps of vector spaces.

Addendum. As noted in class, functors of categories $F : C \to D$ preserve isomorphisms, ie. $F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_Y) = \mathrm{id}_{F(X)}$.

Now, if we were to be working in the category Set, then taking $X, Y \in \mathrm{Obj}(\mathsf{Set})$, we define cartesian products as $X \times Y := \{(a, b) \mid a \in X, b \in Y\}$. However, this definition of products only really works when we're in the category Set, and doesn't bode well in general. As such, we have to work to adapt this definition more broadly in categorical language.

Definition 1.1.5. Given maps $f : A \to X$ and $g : A \to Y$, observe that we obtain a unique product map $A \to X \times Y$ given by $x \mapsto (f(x), g(x))$. Using this, we define a categorical *product*, as saying that anytime we have maps $A \to X$ and $A \to Y$, we have a unique map $A \to X \times Y$, giving the diagram:

$$\begin{tikzcd}
A \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d] \\
& Y \arrow[r] & X
\end{tikzcd}$$

Definition 1.1.6. The dual-notion of a product, or *coproduct*, is built up analogously. Given maps $Y \to Z$ and $X \to Z$, we obtain a colimit by "flipping"[1] the morphisms of the above diagram so as to obtain the diagram:

$$\begin{tikzcd}
& Z & \\
X \arrow[ur] \arrow[r, "\pi_X"'] & X \coprod Y \arrow[u, dashed] & Y \arrow[ul] \arrow[l, "\pi_Y"] \\
& \varnothing \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

[1]Slightly imprecise language, we are considering maps from the empty sets into $X, Y \in \mathrm{Obj}(C)$ as well and using those to construct our notion of coproducts

1.2 More Category Theory for People Who Aren't Peter May

As a side-note, I'll be noting some additional things not covered in the lecture since I feel these are important pre-requisites to include for my own full understanding of the material. I'll try my best not to obfuscate Blumberg's exposition of the material.

Definition 1.2.1. For a category C, the *opposite category* C^{op} is a category including the following data:

- The objects of C^{op} are the same as those in C.
- The morphisms $f^{op} \in C^{op}$ switches the domains and codomains for each morphism in $f \in \mathrm{Mor}(C)$, ie.
$$[f^{op} : Y \to X] \in \mathrm{Mor}(C^{op}) \leftrightsquigarrow [f : X \to Y] \in \mathrm{Mor}(C)$$

Lemma 1.2.2. *In a category* C*, the following are equivalent:*

1. $f \in \mathrm{Map}_C(x, y)$ *is an isomorphism.*
2. *For all objects* $z \in \mathrm{Obj}(C)$*, post-composition with* $f : x \to y$ *defines a bijection.*
$$\mathrm{Map}_C(z, x) \xrightarrow{\cong} \mathrm{Map}_C(z, y)$$
3. *For all objects* $d \in \mathrm{Obj}(C)$*, pre-composition with* $f : x \to y$ *defines a bijection*
$$\mathrm{Map}_C(y, d) \xrightarrow{\cong} \mathrm{Map}_C(x, d)$$

Proof. Assuming (1.), then $f : x \to y$ has an inverse $g : y \to x$ where, by associativity and identity laws for composition over the category C, post-composition defines an inverse function
$$g_* : \mathrm{Map}_C(z, y) \to \mathrm{Map}_C(z, x)$$
to the function $f_* : \mathrm{Map}_C(z, x) \to \mathrm{Map}_C(z, y)$. As we can see, $g_* \circ f_* : \mathrm{Map}_C(z, x) \to \mathrm{Map}_C(z, x)$ and $f_* \circ g_* : \mathrm{Map}_C(z, y) \to \mathrm{Map}_C(z, y)$ are both identity functions. Now, assuming (ii.), there must be $g \in \mathrm{Map}_C(y, x)$ whose image under image under $f_* : \mathrm{Map}_C(y, x) \to \mathrm{Map}_C(y, y)$ is the identity 1_y. By construction, $1_y = f \circ g$, but by associativity, the elements of gf, $1_x \in \mathrm{Map}_C(x, x)$ have the common image f under the function $f_* : \mathrm{Map}_C(x, x) \to \mathrm{Map}_C(x, y)$ when $gf = 1_x$. Therefore, (1.) $\iff$ (2.); We obtain (1.) $\iff$ (3.) by duality. $\square$

1.2.1 Sidenote: Enriched Categories

Let us consider the category Top, with which we know $\mathrm{Map}_C(x, y)$ is itself an object of Top. We express an interest in categories C where $\mathrm{Map}_C(x, y)$ is an abelian group. To better capture this notion, we introduce the idea of an enriched category.

We consider enriched categories as categories where hom-sets (ie. $\mathrm{Map}_C(x, y)$) are not just sets and have more structure in an enriched category $\mathcal{V}$.

Addendum. Consider three objects $U, V, W \in \mathrm{Obj}(C)$. Given morphisms $f \in \mathrm{Map}_C(U, V)$ and $g \in \mathrm{Map}_C(V, W)$, we know that there exists a composition map $g \circ f \in \mathrm{Map}_C(U, W)$. Viewing all objects of the form $\mathrm{Map}_C(-, -)$ as objects of an enriched category $\mathcal{V}$, we wish to have a relation of the form
$$\mathrm{Map}_C(U, V) \times \mathrm{Map}_C(V, W) \to \mathrm{Map}_C(U, W)$$
One of the conditions to have this structure on $\mathcal{V}$ is to induce the structure of a monoidal category.

Definition 1.2.3. A *monoidal category* $(\mathcal{V}, \otimes, i)$ consists of the following data:

1. A category $\mathcal{V}$
2. A monoidal product as a bi-functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
3. A monoidal unit i

which satisfies the natural isomorphisms expressing associativity and unitality of the monoidal product, ie.

$$\alpha : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z, \quad \lambda : i \otimes X \xrightarrow{\cong} X, \quad \rho : X \otimes i \xrightarrow{\cong} X$$

Addendum 1.2.4. A *symmetric monoidal category* has the additional condition that the monoidal product is commutative, meaning that there is an additional natural isomorphism $X \otimes Y \cong Y \otimes X$

Example 1.2.5. The triple $(\mathsf{Mod}_R, \otimes_R, R)$ where Mod_R is the category of modules over a commutative ring R is a monoidal category.

Definition 1.2.6. A *category* C *enriched over* $\mathcal{V}$ *(or a* $\mathcal{V}$*-category* $\mathbf{C}$*)* is given by the data of:

1. A collection of objects, denoted $\mathrm{Obj}(C)$.
2. For each ordered pair of objects $X, Y \in \mathrm{Obj}(C)$, there is an object $\mathrm{Map}_C(X, Y) = \mathbf{C}(X, Y)$ in $\mathcal{V}$.
3. For each ordered triple $X, Y, Z \in C$, there is a morphism $\circ : \mathbf{C}(X, Y) \otimes \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$ in $\mathcal{V}$.
4. For each object $X \in \mathrm{Obj}(C)$, there is a morphism $\mathrm{id}_X : i \to \mathbf{C}(X, X)$ in $\mathcal{V}$ such that
    - For each $W, X, Y, Z \in \mathrm{Obj}(C)$, the composition in $\mathbf{C}$ is associative such that the diagram:

$$\begin{tikzcd}
\mathbf{C}(Y,Z) \otimes \mathbf{C}(X,Y) \otimes \mathbf{C}(W,X) \arrow[r, "1\otimes\circ"] \arrow[d, "\circ\otimes 1"] & \mathbf{C}(Y,Z) \otimes \mathbf{C}(W,Y) \arrow[d, "\circ"] \\
\mathbf{C}(X,Z) \otimes \mathbf{C}(W,X) \arrow[r, "\circ"] & \mathbf{C}(W,Z)
\end{tikzcd}$$

    is commuting.
    - For each $X, Y, Z \in \mathrm{Obj}(C)$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}(X,Y) \otimes i \arrow[d, "1\otimes \mathrm{id}_X"] \arrow[dr, "\cong"] & & i \otimes \mathbf{C}(X,Y) \arrow[dl, "\cong"'] \arrow[d, "\mathrm{id}_Y \otimes 1"] \\
\mathbf{C}(X,Y) \otimes \mathbf{C}(X,X) \arrow[r, "\circ"] & \mathbf{C}(X,Y) & \mathbf{C}(Y,Y) \otimes \mathbf{C}(X,Y) \arrow[l, "\circ"]
\end{tikzcd}$$

Example 1.2.7 (Enriched Categories)**.** We list some examples of enriched categories:

- Given the symmetric monoidal category $\mathcal{V} = (\mathsf{Vect}_K, \otimes_K, K)$, we can define Vect_K to be a $\mathcal{V}$-category. For two linear maps $f, g \in \mathsf{Vect}_K(U, W)$, we can easily verify and check that we have associativity and commutativity as in (Definition 1.2.6).
- Consider the category of modules over a fixed commutative ring R, which we denote Mod_R. The category of Abelian groups Ab has a monoidal structure such that Mod_R is enriched over $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

1.2.2 Natural Transformations & (Co)Limits

Definition 1.2.8. Given categories C and D, and functors $F, G : C \rightrightarrows D$, a *natural transformation* $\alpha : F \implies G$ consists of:

1. An arrow $\alpha_C : F(c) \to G(c)$ for each object $c \in \text{Obj}(C)$, the collection of which defines the components of the natural transformation.
2. For each morphism $f : c \to d$ in C, we have a commuting diagram:

$$\begin{array}{ccc} F(c) & \xrightarrow{\alpha_C} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(d) & \xrightarrow{\alpha_D} & G(d) \end{array}$$

Definition 1.2.9. A *natural isomorphism* is a natural transformation $\alpha : F \implies G$ in which every component α_C is an isomorphism.

Definition 1.2.10. For any object $c \in \text{Obj}(C)$ and any category J, the *constant functor* $I_C : J \to C$ is a functor sending every object of J to $c \in C$ and every morphism $f \in \text{Mor}(J)$ to the identity morphism 1_c.

Definition 1.2.11. A *cone* over a diagram $F : J \to C$ with summit $c \in \text{Obj}(C)$ is a natural transformation $\lambda : I_c \implies F$ whose domain is the constant functor at c. The components $\{\lambda_i : c \to F_i\}_{i \in I}$ of the natural transformation are called the *legs of the cone*. More explicity,

- The data of a cone $F : J \to C$ with summit c consists of a collection of morphisms $\lambda_i : c \to F_i$.
- A family of morphisms $\{\lambda_i : c \to F_i\}$ defines a cone over F iff for each morphism $f : x \to y$ in J, the following triangle commutes in C:

$$\begin{tikzcd} & c \arrow[dl, "\lambda_x"'] \arrow[dr, "\lambda_y"] & \\ Fx \arrow[rr, "F(f)"'] & & Fy \end{tikzcd}$$

Dually, a cone under F with nadir $c \in \text{Obj}(C)$ is a natural transformation $\lambda : F \implies I_C$ whose legs are components $\{\lambda_j : F_i \to c\}_{i \in I}$. For each morphism $f : x \to y$ in J, the above triangle with morphisms λ_x, λ_y flipped will commute.

Addendum 1.2.12. A *cone under a diagram* $F : J \to C$ is also called a cocone and is defined analogously to cones over a diagram. A cone under $F : J \to C$ is precisely a cone over $F^{op} : J^{op} \to C^{op}$.

Definition 1.2.13. For any diagram $F : J \to C$, there is a functor,

$$\mathsf{Cone}(-, F) : C^{op} \to \mathsf{Set}$$

which sends $c \in C$ to the set of cones over F with summit c. Using the Yoneda Lemma[2], a *limit* consists of an object $\lim F \in C$ together with a universal cone $\lambda : \lim F \implies F$, called the limit cone, defining a natural isomorphism

$$\mathbf{C}(-, \lim F) \cong \mathsf{Cone}(-, F)$$

[2] Recall that Yoneda's Lemma states: For any functor $F : C \to \mathsf{Set}$ whose domain C is locally small, then for any object $c \in C$, there's a bijection:

$$\text{Hom}(\mathbf{C}(c, -), F) \cong Fc$$

Definition 1.2.14. Dually, there is a functor $\mathsf{Cone}(F, -) : C \to \mathsf{Set}$ that sends $c \in C$ to the set of cones with nadir c. A *colimit* of F is a representation for $\mathsf{Cone}(F, -)$. Again, by Yoneda's Lemma, a colimit consists of an object $\operatorname{Colim} F \in C$ together with a universal conve $\lambda : F \implies \operatorname{Colim} F$, called the colimit cone, defining a natural isomorphism:

$$\mathbf{C}(\operatorname{Colim} F, -) \cong \mathsf{Cone}(F, -)$$

Addendum. We may also equivalently define limits of F as the terminal object in the category of cones over F and colimits as the initial object in the category of cones over F.

Example 1.2.15 (Definition of Product)**.** A product is a limit of a diagram indexed by a discrete category with only identity morphisms. A diagram in C indexed by a discrete category J consists of a collection of objects $F_j \in C$ indexed by $j \in J$.

A cone over this diagram with summit $c \in C$ is a J-indexed family of morphisms $\{\lambda_j : c \to F_j\}_{j \in J}$. This limit is denoted by $\prod_{j \in J} F_j$ and the legs of the cone are maps,

$$\left(\pi_k : \prod_{j \in J} F_j \to F_k\right)_{k \in J}$$

Definition 1.2.16. A category is *complete* if it contains all limits. A category is *cocomplete* if it contains all colimits.

Example 1.2.17. The category Set is complete and cocomplete. The category $\mathrm{Fun}(C^{op}, \mathsf{Set})$ is also complete and cocomplete; Observe that given the map $Y : C \to \mathrm{Fun}(C^{op}, \mathsf{Set})$, which maps $x \mapsto \mathrm{Map}_C(-, x) = \mathbf{C}(-, x)$, that Y is a fully-faithful[3] functor, and so the map

$$\mathrm{Map}_C(x, y) \xrightarrow{\cong} \mathrm{Map}(Y(x), Y(y))$$

is an isomorphism.

Why does this matter? Consider the category Top. When constructing objects like Klein bottles, $\mathbb{R}$ or $\mathbb{C}$ projective space, or any cell complex via attaching maps, we define these as sets equipped with particular topologies.

These topologies can all be uniformly defined by our notions of limits and colimits via universal cones. As we'll observe, constructed topological spaces can be characterized either as a limit or colimit of a specific diagram over the category Top. By mapping things out of standard topological objects (For instance, mapping out of S^n, ie. $\{\mathrm{Map}_C(S^n, -)\}$), we're able to extrapolate more information about these objects.

$\mathrm{Map}_{\mathsf{Top}}(X, Y)$ is a space. Considering the map $H : [0, 1] \to \mathrm{Map}_{\mathsf{Top}}(X, Y)$ with path homotopies given between $f : X \to Y \leftrightsquigarrow H(0)$ and $g : X \to Y \leftrightsquigarrow H(1)$, we'll later see that these are the same. The maps $\{\mathrm{Map}_C(S^n, -)\}$ are the same if there exists a path between them, and so we develop a notion of homotopy via $\mathrm{Map}_{\mathsf{Top}}(I, \mathrm{Map}_{\mathsf{Top}}(X, Y))$.

[3]Recall that for locally small categories C and D, the functor $F : C \to D$ induces a function $F_{X,Y} : \mathrm{Map}_C(x, y) \to \mathrm{Map}_D(F(x), F(y))$. We say F is faithful if $F_{X,Y}$ is injective, and full if $F_{X,Y}$ is surjective.

1.3 Homotopy (And more Category) Theory

[4] An adjunction consists of an opposing pair of functors $F : C \leftrightarrows D : G$ that enjoy special relations to each other.

Definition 1.3.1. An *adjunction* consists of a pair of functors $F : C \to D$ and $G : D \to C$ together with an isomorphism

$$\mathrm{Map}_D(Fc, d) \cong \mathrm{Map}_C(c, Gd)$$

for each $c \in C$ and $d \in D$ that is natural in both variables. Here, F is *left-adjoint* to D and G is *right-adjoint* to C.

Consider a map $H : I \to \mathrm{Map}_{\mathsf{Top}}(X, Y)$. By adjunction, this is the same data as a map $H : X \times I \to Y$. We'll be interested in the homotopy category HoTop; And here, we will consider the quotients: $\mathrm{Map}_{\mathsf{Top}}(X, Y)/\sim$ where " $\sim$ " is the relation of homotopy, which is to say there is a path between morphisms $X \to Y$.

Under these relations we develop a new notion of equivalence between spaces X and Y, which we call homotopy equivalence.

Definition 1.3.2. Consider two morphisms $f : X \to Y$ and $g : Y \to X$. We say that X and Y are *homotopy equivalent* if $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

Recall that our initial goal with algebraic topology is to find functors from Top to some algebraic category, but which functors will we need. Well, as it turns out, we need these functors to respect homotopy equivalence, which is say that $X \sim Y \implies F(X) \cong F(Y)$.

Addendum. Note that $F(X)$ and $F(Y)$ are isomorphic, not necessarily strictly equal. A slightly stronger restriction on our functors we might want is that they factor through HoTop. This is stronger since, whenever f and g are homotopic, the map they induce must also be the same.

Now, we're interested in classifying spaces up to homotopy, but dealing with every space and map in Top would probably kill me. As such, we're going to identify the class of spaces which is "big enough" (in some arcane sense).

- Consider the push-out diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

- We also have the push-out diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

- We also then have the push-out

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

  Here, we have pushouts which do not respect homotopy equivalence, which is not desirable. However, we have (at least) three roughly equivalent tactics of solving this problem:

[4]Some of the notes in the following section are provided by Peyton Chui.

1. Maps out of spheres S^n, these being in the set of based maps[5] $\{\mathrm{Map}_*(S^n, X)/\sim\}$. This turns out to have an algebraic structure, ie. homotopy groups.
2. Spaces built out of simple pieces which are then restricted by attaching spheres along their boundary, ie. CW-complexes.
3. Combinatorial models of space developed out of simplicial complexes.

Consider the space $\mathrm{Map}_*(S^0, X)/\sim$. The data of the based map $S^0 \to X$ is simply picking a point in X, and two maps are homotopic to each other when their images of points are path-connected. As such, we can identify $\mathrm{Map}_*(S^0, X)/\sim$ with path components in X; We call this $\pi_0(X)$, or the 0-th homotopy group. Thinking about $\mathrm{Map}_*(S^1, X)$, or $\pi_1(X)$, it has an algebraic structure given by:

$$\Big(\mathrm{Map}_*(S^1, X)/\sim\Big) \times \Big(\mathrm{Map}_*(S^1, X)/\sim\Big) \to \mathrm{Map}_*(S^1 \vee S^1, X)/\sim \to \mathrm{Map}_*(S^1, X)/\sim$$

Looking at $\pi_n(X)$ for $n \geqslant 2$, we'll later see that these form abelian groups.

Definition 1.3.3. We say that the map $f : X \to Y$ is a *weak equivalence* if the induced map on homotopy groups, $f_* : \pi_n(X) \to \pi_n(Y)$ is an equivalence for all n.

Question: What is the relationship between weak-equivalence and homotopy equivalence?
Well, if $X \simeq Y$, then X is weakly equivalent to Y. This is a fairly-straightforward exercise where we induced a map $\mathrm{Map}_*(S^n, X)/\sim \to \mathrm{Map}_*(S^n, Y)/\sim$ given by homotopy equivalence and do so in the reverse direction as well.

Question: Can we compute $\pi_n(X)$ for reasonable n and X? Well, occasionally.
This will be what leads us to our soon-to-be-defined notion of a CW-complex.

[5]Based maps are maps out of based spaces/pointed spaces, like (X, x_0) with base point x_0 into (Y, y_0), ie. $f : (X, x_0) \to (Y, y_0)$.