investigating quadratic graphs

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1 Abstract

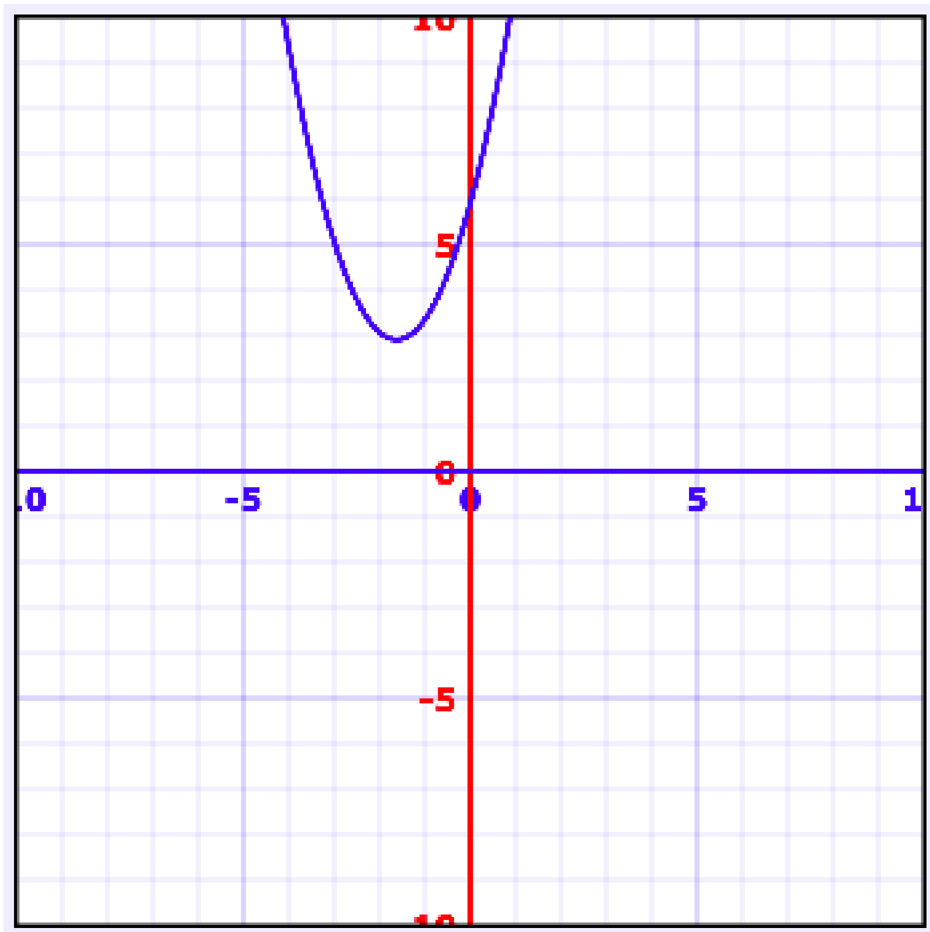
In this document, I intend to explore quadratics and quadratic graphs, with the research question: In what ways are quadratic graphs related to the different forms of a quadratic? We will be analysing features and patterns found. We will start with examples of various quadratic graphs, written in their expanded and factorised forms. Then we will provide an explanation of what we notice, further examples to verify, and lastly, an explanation of why these connections happen. The fundamental form to be explored will be the quadratic in its general form: ax^2+bx+c. We will be referring to a as the coefficient of x^2, b as the coefficient of x, and c as the constant term.

2 Quadratic graphs

The following is a table of the quadratics to be explored in this research, and then, the graphs of each.

number	equation in expanded form	equation in factorised form
1	$y = 1.14x^2 + 3.7x + 5.9$	no real solution
2	$y = 0.41x^2 + 0.4x$	$0.41(x + 0.97561)(x + 0)$
3	$y = 1.26x^2 + 3.7x - (4)$	$1.26(x + 3.77701)(x - 0.840506)$
4	$y = 1.26x^2 + (-4.1x) - (4)$	$1.26(x + 0.785832)(x - 4.0398)$
5	$y = 1.2x^2 + (-4.3x) - (6.1)$	$1.2(x + 1.08816)(x - 4.67149)$
6	$y = 1.2x^2 + (-4x) + 4.2$	no real solution
7	$y = 0.44x^2 + (-4x) + 1.6$	$0.44(x - 0.419343)(x - 8.67157)$
8	$y = 0.82x^2 + (-4x) + 1.6$	$-0.82(x - 0.37168)(x + 5.24973)$
9	$y = 0.82x^2 + 3.9x - (6.3)$	no real solution
10	$y = 1.17x^2 + (-2.4x) + 5.6$	$-1,17(x - 1.39061)(x + 3.44189)$

Table 1: quadratics



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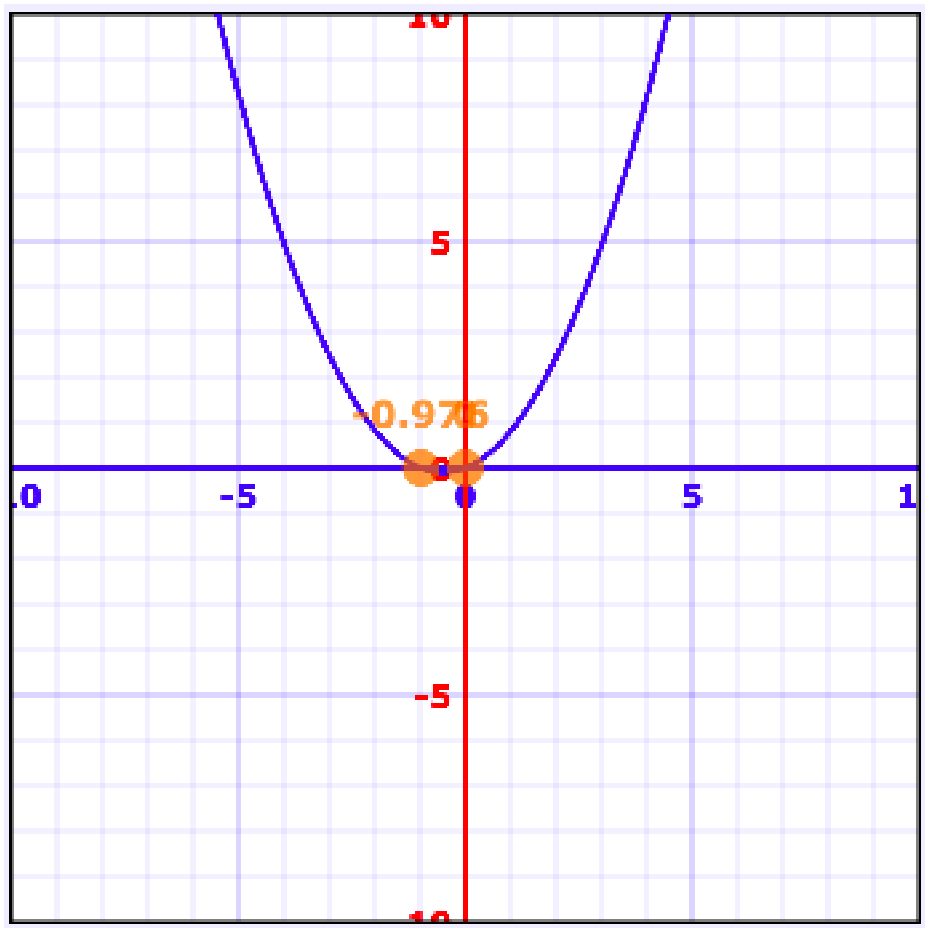

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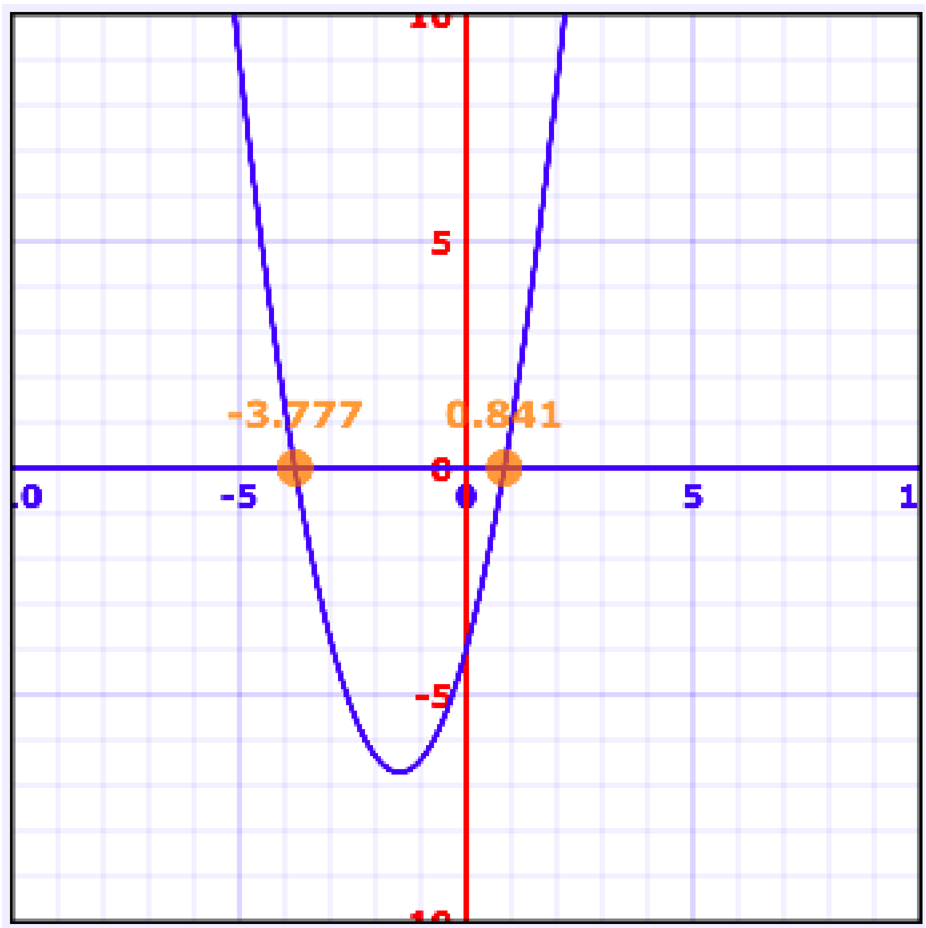

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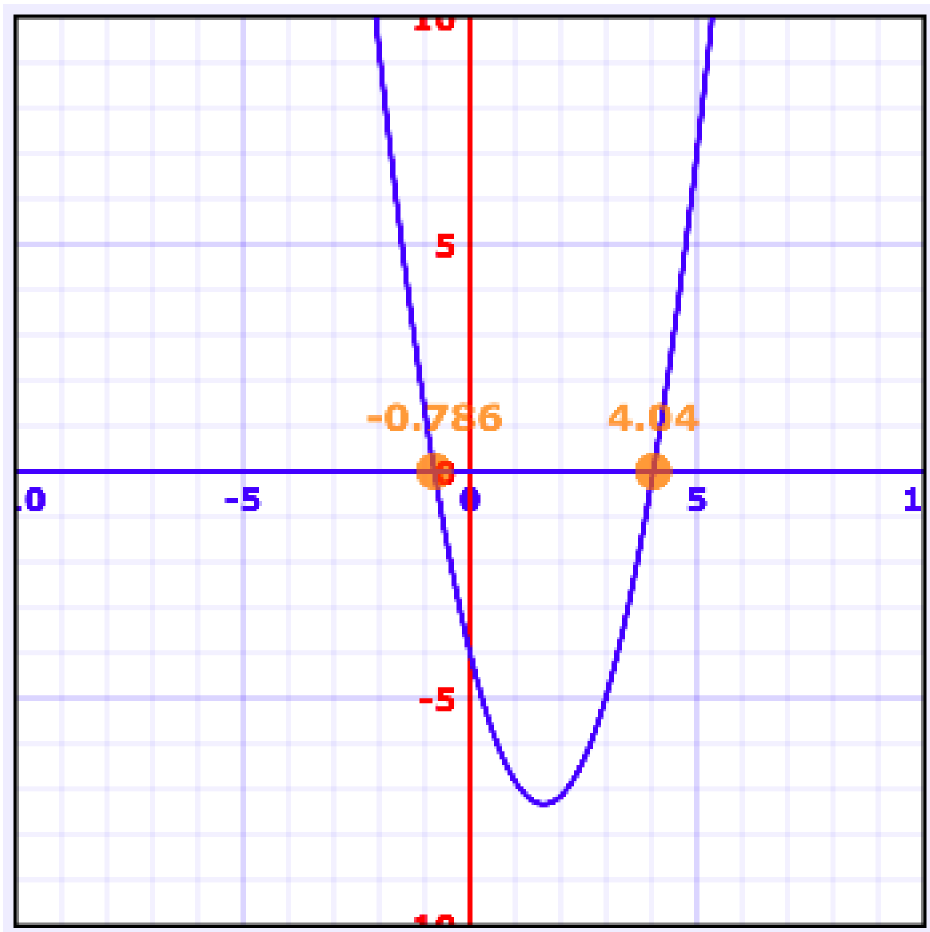

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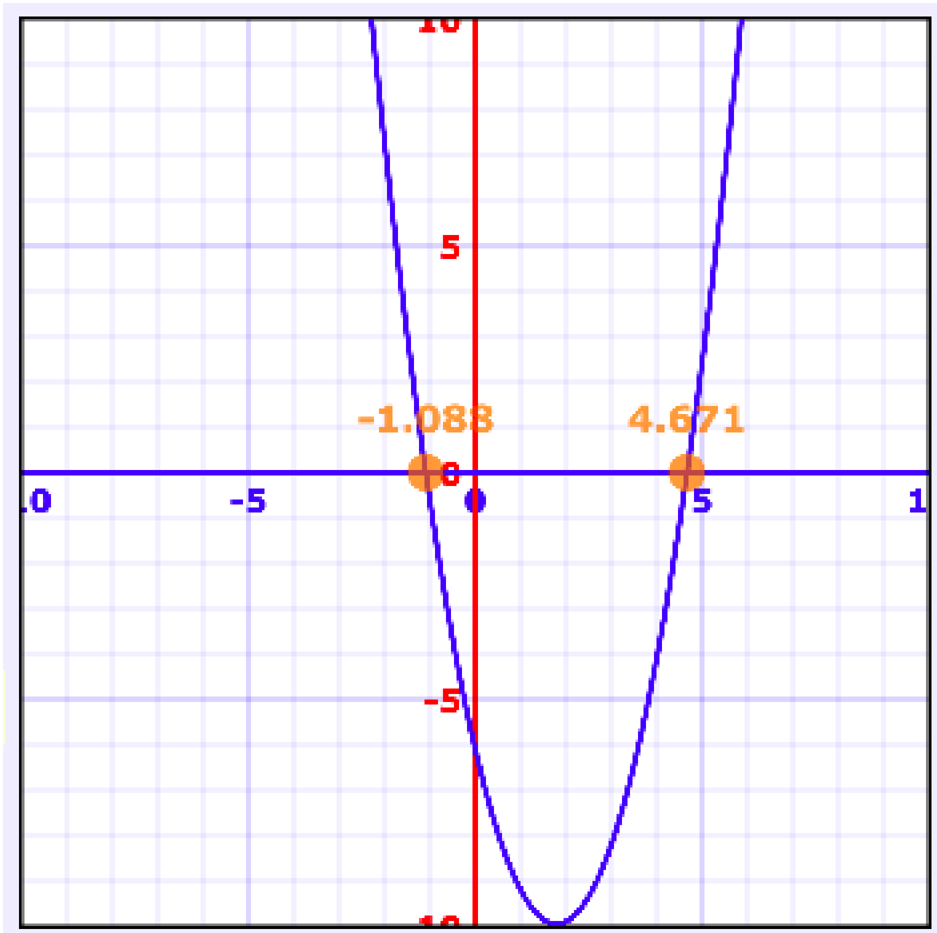

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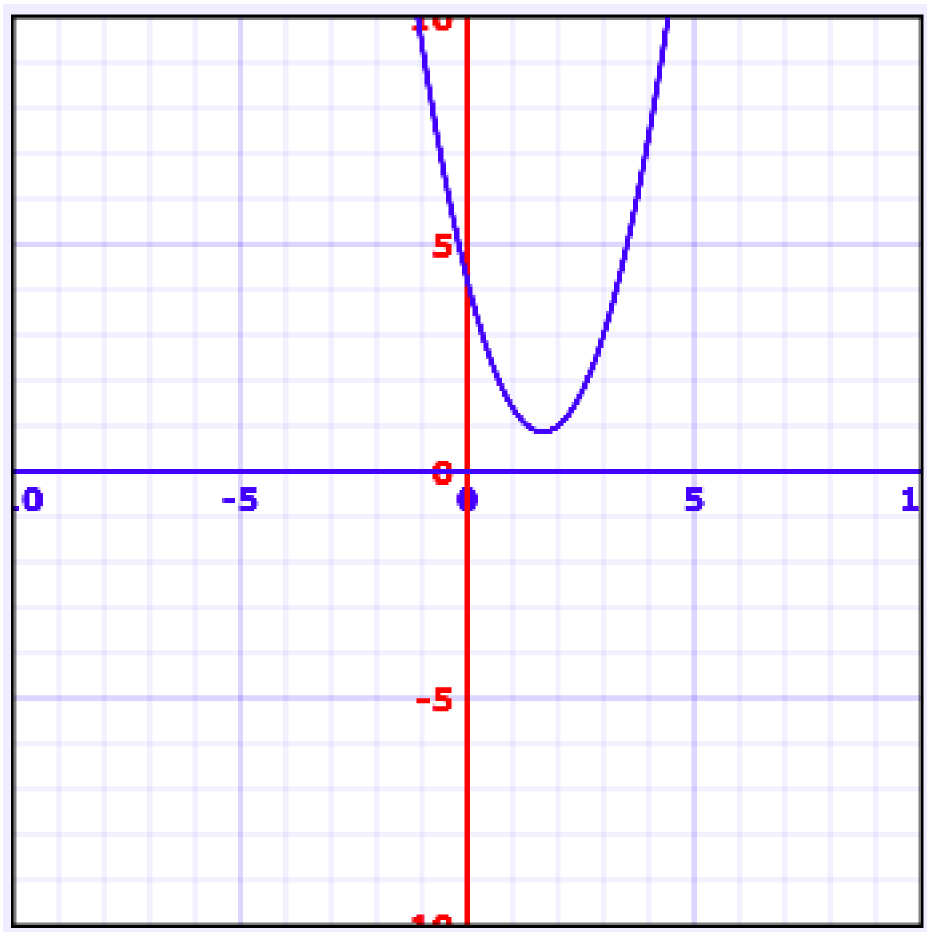

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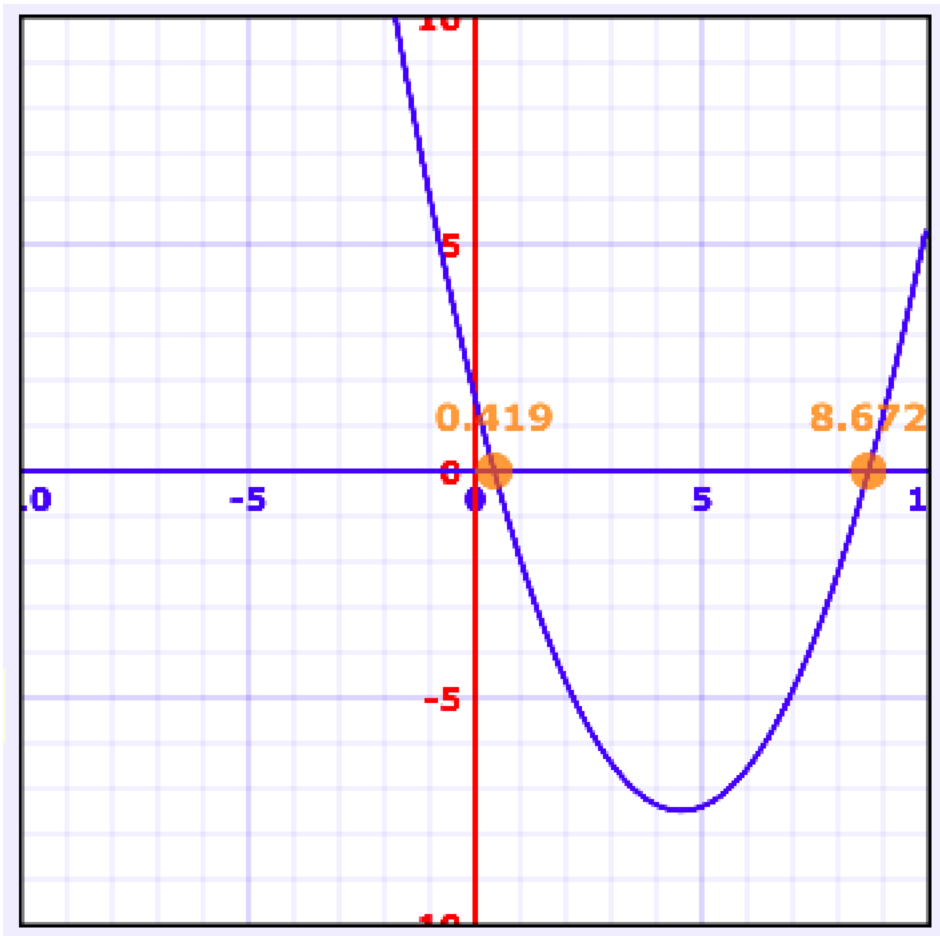

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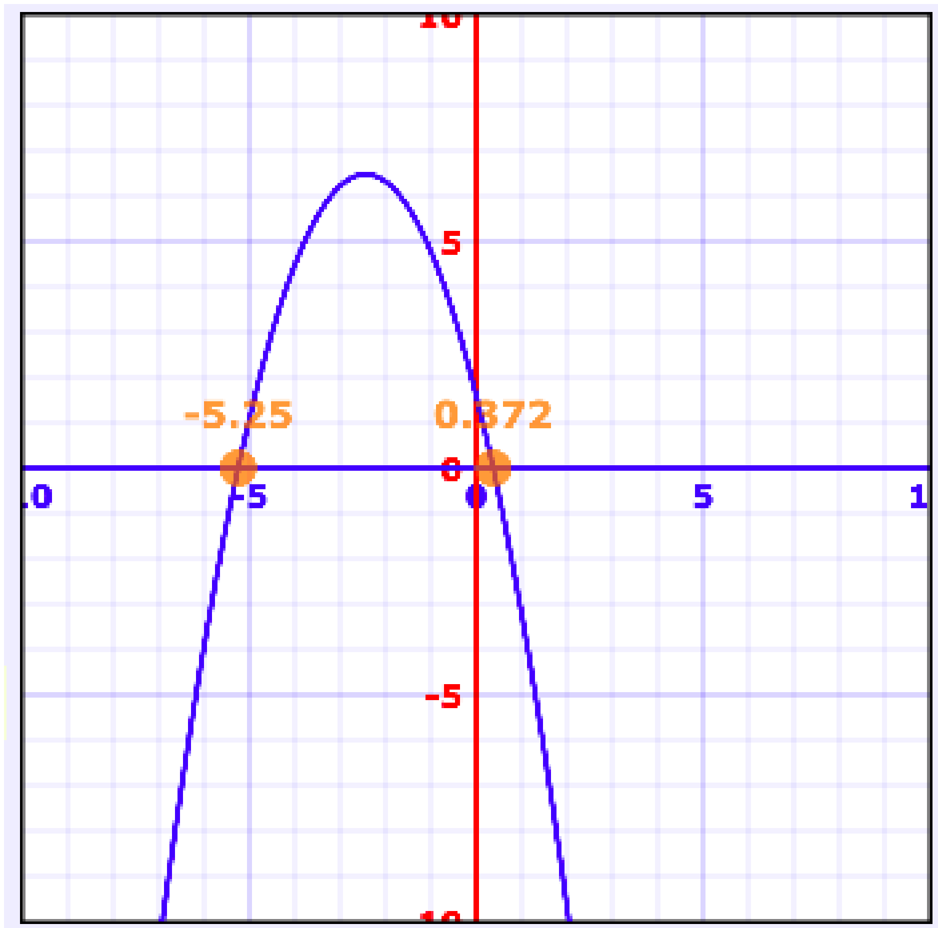

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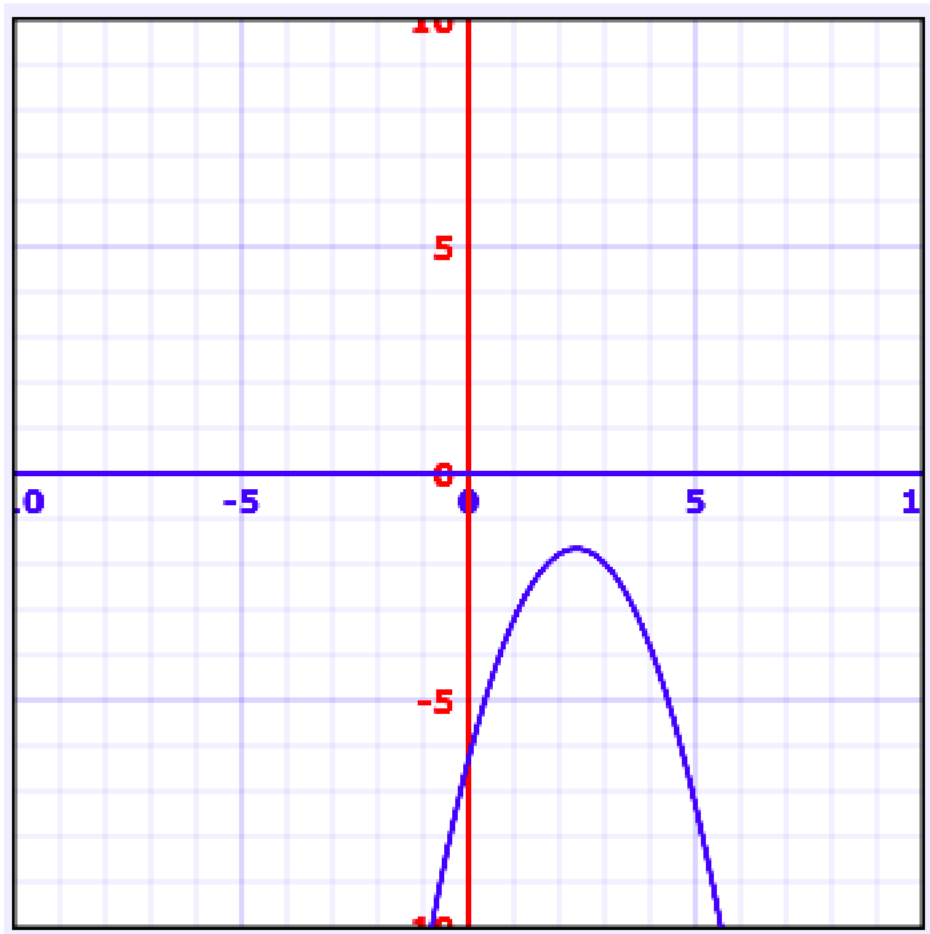

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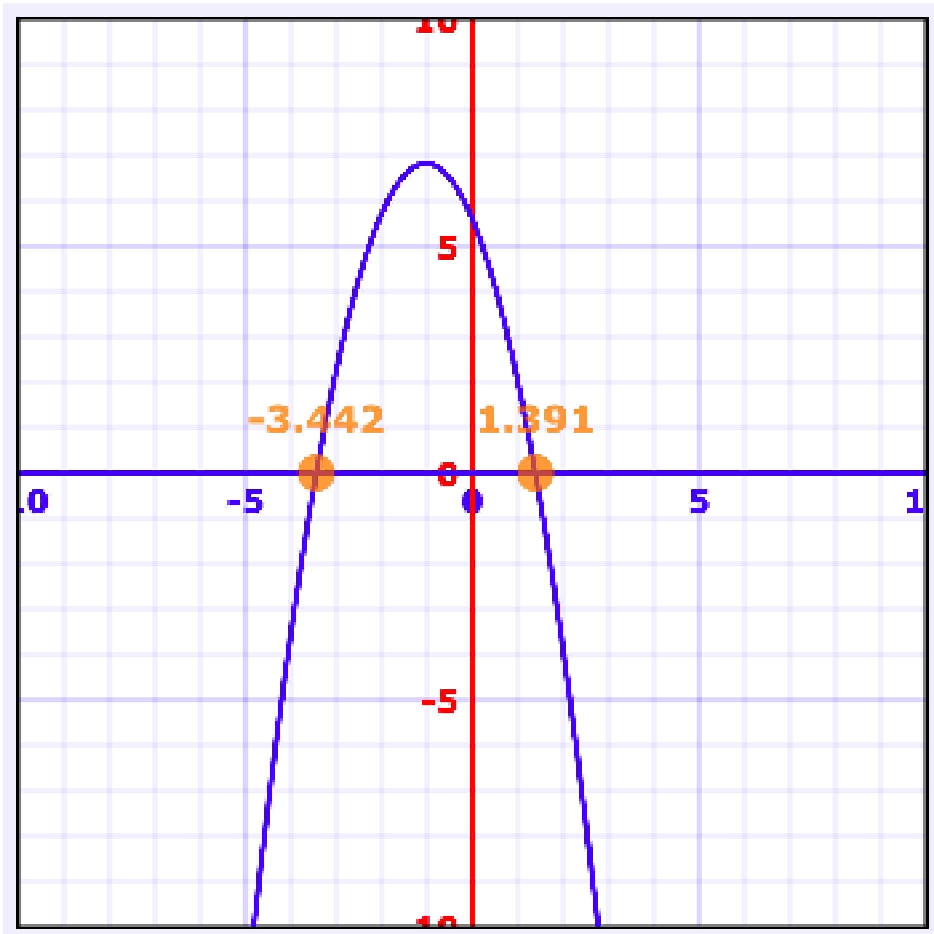

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3 What is noticed

3.1 y-intercepts

The y-intercept, that is, the point of the parabola in contact with the y-axis's central line, is repeatedly the c value/constant term when the equation is written in the form $ax^2 + bx + c$, as we see on table 2

number	c value	y-intercept on the graph
1	5.9	5.9
2	0	0
3	-4	-4
4	-4	-4
5	-6.1	-6.1
6	4.2	4.2
7	1.6	1.6
8	1.6	1.6
9	-6.3	-6.3
10	5.6	5.6

Table 2: c values with y-intercepts

number	a value	graph direction
1	1.14	upsidedown
2	0.41	upsidedown
3	1.26	upsidedown
4	1.26	upsidedown
5	1.2	upsidedown
6	1.2	upsidedown
7	0.44	upsidedown
8	-0.82	up
9	-0.82	up
10	-1.17	up

Table 3: a values with graph direction

3.2 curve direction

the parabola's direction, which may either be up or down, is the direction in which the parabola finishes at, either a maximum or minimum point. As we can see by the table 3 whenever the a value/x^2 coefficient's is positive, the parabola points upside down, having a minimum value but no maximum value, and whenever the a/x^2 coefficient's value is negative, the parabola points normally, having a maximum value but no minimum value.

3.3 minimum/maximum points

the minimum/maximum points, are the points highest or lowest in the parabola, informally the 'tip' of the parabola, as it sits on the curved edge of the parabola. the graphs' minimum/maximum points are always:

$$(-b/2a, a(-b/2a)^2 + (-b/2a)b + c)$$

as we see for the equations 1-10, these are in table 4:

number	x value of min/max point	y value of min/max point	approximate point from graph
1	-1.62280	2.89780	(-1.6, 2.9)
2	-0.97560	-0.000003	(-1,0)
3	-1.46825	-6.71626	(-1.5,-6.8)
4	1.62698	-7.33531	(1.6,-7)
5	3.58333	-6.10002	(3.6,-6.1)
6	3.33333	4.19999	(3.3,4.2)
7	9.09090	1.59996	(9,1.6)
8	4.87804	1.59996	(5,1.6)
9	-4.75609	-6.30002	(-4.8,-6.3)
10	2.05128	5.59999	(2,5.6)

Table 4: min-max points

number	x-intercepts from the factored form	x-intercepts from the graphs
1	no real solution	no real solution
2	0.97561, 0	0.97561, 0
3	3.77701, 0.840506	3.77701, 0.840506
4	0.785832, 4.0398	0.785832, 4.0398
5	1.08816, 4.67149	1.08816, 4.67149
6	no real solution	no real solution
7	0.419343, 8.67157	0.419343, 8.67157
8	0.37168, 5.24973	0.37168, 5.24973
9	no real solution	no real solution
10	1.39061, 3.44189	1.39061, 3.44189

Table 5: x-intercepts

3.4 x-intercepts

the x-intercept, that is, the point/none/points of the parabola in contact with the x-axis's central line, is shown to be the two values p, q on the quadratic equation in factored form:

$$a(x-p)(x-q)$$

this is clear to see as in the graphs, the x-intercepts are highlighted with an orange color. check table 5 for the data.

3.5 symmetry

the symmetry of a parabola refers to a defined x value in which each side of the parabola is the side of a reflection of the parabola on the y-axis in which the x value is. This x value's y-axis is called the axis of symmetry. The axis of symmetry is always located on the x value of the minimum/maximum point.

equation	y-intercept through method	y-intercept through graph
$6.7x^2 + 9x + 3$	3	3
$2x^2 + 7x + (-4)$	-4	-4
$-5x^2 + 3x + (-29)$	-29	-29

Table 6: y-intercept test

equation	curve's direction through method	curve's direction thorugh graph
$6.7x^2 + 9x + 3$	upsidedown	upsidedown
$2x^2 + 7x + (-4)$	upsidedown	upsidedown
$-5x^2 + 3x + (-29)$	up	up

Table 7: curve direction test

3.6 existence of solutions

the existence of a solution is whether or not there exist real values for the x-intercept(s). As we see for graphs 1,6,9, these don't exist, as a result of this, the equation in factored form for these graphs doesn't either exist.

4 Further examples

4.1 y-intercepts

further examples for the y-intercept method are on table 6.

4.2 curve direction

the curve's direction further examples are on table 7

4.3 minimum/maximum points

the min/max points further examples are on table 8

4.4 x-intercepts

the x-intercepts test is on table 9

equation	minimum/maximum through method	minimum/maximum thorugh graph
$6.7x^2 + 9x + 3$	(-0.67164, -0.02238)	(-0.67164, -0.02238)
$2x^2 + 7x + (-4)$	(-1.75, -10.125)	(-1.75, -10.125)
$-5x^2 + 3x + (-29)$	(0.3, -27.65)	(0.3, -27.65)

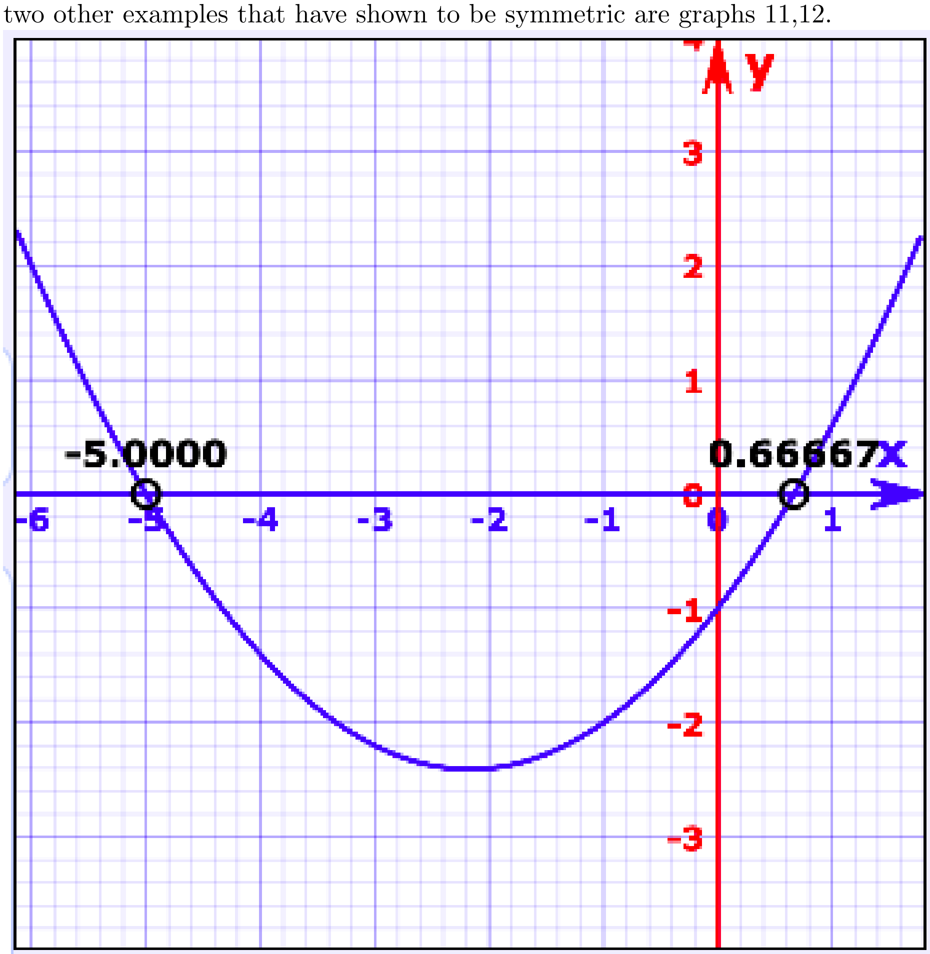
Table 8: min/max test

equation	x-intercepts through method	x-intercepts thorugh graph
$6.7x^2 + 9x + 3$	0.613836, 0.729448	0.613836, 0.729448
$2x^2 + 7x + (-4)$	0.5, -4	0.5, -4
$-5x^2 + 3x + (-29)$	no real solution	no real solution

Table 9: x-intecepts test

4.5 symmetry

two other examples that have shown to be symmetric are graphs 11,12.



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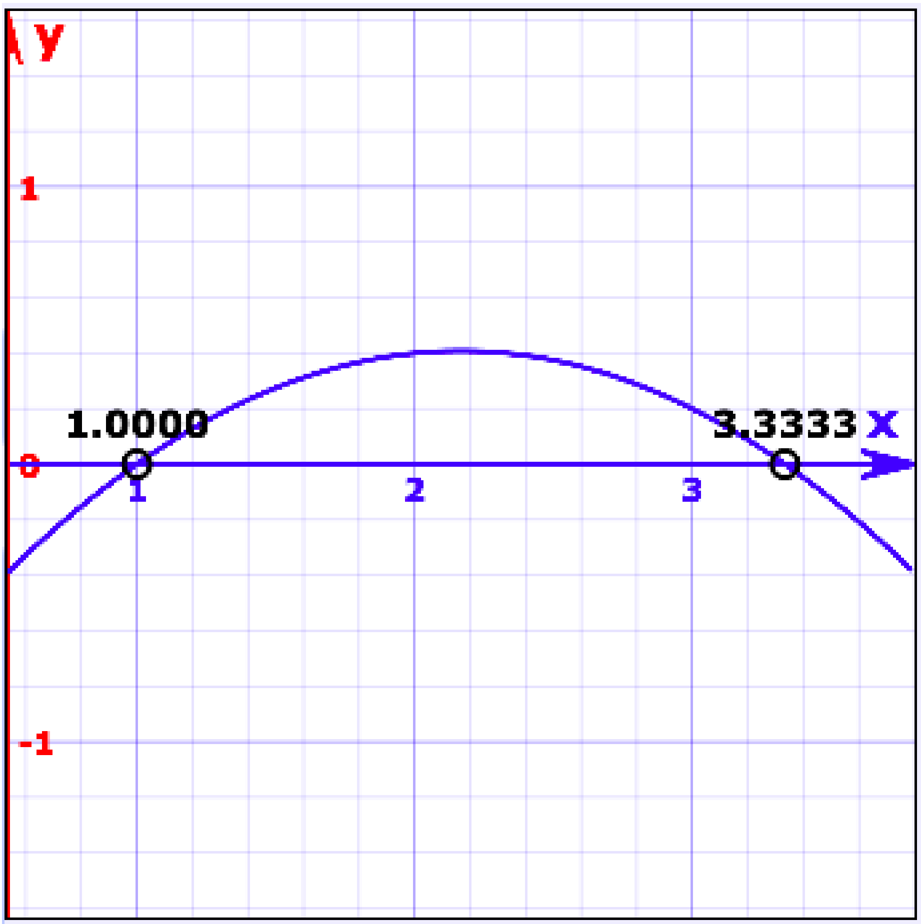

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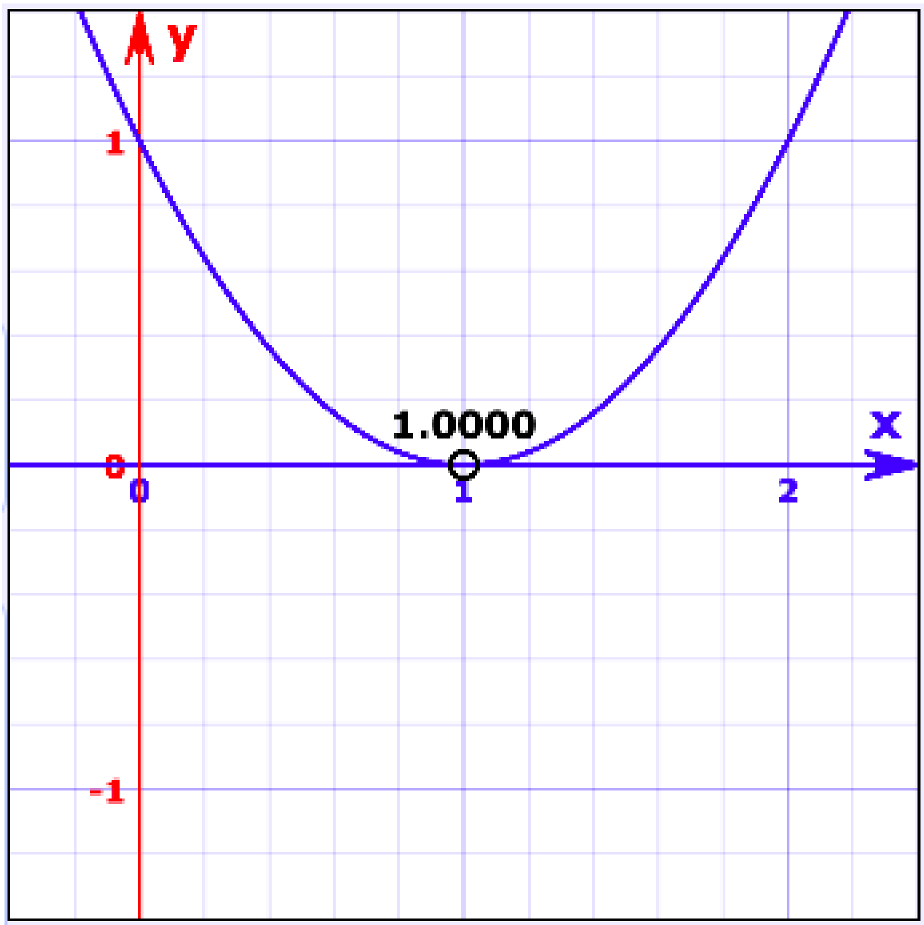
4.6 existence of solutions

graphs 11, 12 show to have solutions and show to cross the positive and negative y-values

4.7 extra



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4.8 example test

the example tests on this section have all shown to be according to the rules of the method used

5 Analytical explanation

5.1 y-intercepts

the y-intercepts always are the c value, simply because of the simple identity that when $x = 0$, the quadratic becomes:

$$x = 0 \Rightarrow a0^2 + b0 + c = c$$

5.2 curve direction

the direction of the curve is related with the sign of a because, as the function approaches values higher than 1, $x^2 > x$, and x^2 is always positive due to the simple identity that all values times another with the same sign equal positive, and thus the minus sign in the a value forces x^2 to be negative, and as it is bigger than x, the quadratic tends to increase to negative values, thus bringing the parabola 'downwards', the inverse situation where a is positive thus brings the parabola to high positive values of the quadratic for high inputs, thus shaping the parabola 'upside down'.

5.3 minimum/maximum points

the minimum/maximum points always are

$$(-b/2a, a(-b/2a)^2 + (-b/2a)b + c)$$

since: removing the c value from the quadratic wont have any effect on the quadratic's x value of the vertex, since it only changes the position of the parabola in the y-axis, thus we can write quadratics without their c value as:

$$ax^2 + bx$$

which factoring gives:

$$x(ax + b)$$

thus, the x-intercepts for this equation are:

$$x = 0 \Rightarrow 0(0 + b) = 0$$

$$x = -b/a \Rightarrow (-b/a)(a(-b/a) + b) = (-b/a)(-b + b) = (-b/a)(0) = 0$$

since a quadratic is always symmetric (check 5.5), the minimum/maximum point must lie half way between $0, -b/a$, which in turn is: $-b/2a$, meaning that the minimum/maximum point is:

$$(-b/2a, a(-b/2a)^2 + (-b/2a)b + c)$$

5.4 x-intercepts

the x-intercepts always are the p, q values since:

$$a(p - p)(p - q) = a(0)(p - q) = 0$$

$$a(q - p)(q - q) = a(q - p)(0) = 0$$

5.5 symmetry

quadratic graphs are always symmetric on the x value of a min/max point, because: quadratics can be written in vertex form. This form is: $y = a(x-h)^2 + k$, (h,k) being the minimum/maximum point. This thus means that whenever you plug $x, -x$ to the vertex form:

$$a(x-h)^2 + k = a(w)^2 + k$$

$$a(-x-h)^2 + k = a(-w)^2 + k$$

$$-w^2 = w^2 = (x-k)^2 = (-k-x)^2$$

this also makes sense as squaring $(x-k)$ simply gives the squared distance of x from k, independent of the x's sign.

5.6 existence of solutions

the existence of solutions for the x-intercept is caused by whether or not the function reaches negative and positive values (since the function is continuous, there thus must be an x-intercept between, this is also proven by the intermediate value theorem), thus, if the function tends to only return values of one sign, there will be no x-intercepts. The factorised form of the quadratic equation best expresses this as:

$$a(p-p)(p-q) = a(0)(p-q) = 0$$

$$a(q-p)(q-q) = a(q-p)(0) = 0$$

since whenever p or q are plugged, the quadratic goes to zero. This thus also explains why there may not be a quadratic in factored form, as there will be no p, q values that exist and fit the equation.

6 Can a quadratic graph be "upside-down"?

This extra question can be answered fairly simply as we see in graphs 1 to 7 that all are upside down.

7 Is it possible to have a quadratic graph with (i) one x-intercept (ii) no x-intercept

as the subsection 3.6 shows, graphs 1, 6 and 9 have no x-intercepts, and graph 13 of the further examples section only has 1 x-intercept.

8 Do all quadratic graphs have a y-intercept?

This extra question is much more theoretical, as we are dealing with the existence of a defined value for when $x = 0$. Axioms of algebra have no rule stating that a quadratic may not have 0 as its inputs, so unless it is stated in the function that the input $x = 0$ is not allowed, all quadratics have 1 unique y-intercept:

$$x = 0 \Rightarrow a0^2 + b0 + c = c$$

thus, all quadratics have 1 unique y-intercept defined by c.