Appendix: Test Functions

§1.Test Functions for Unconstrained Optimization Problems

Problem 1.1 *Rosenbrock function*:

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2, \tag{1.1}$$

$$x_0 = [-1.2, 1]^T,\ x^* = [1, 1]^T,\ f(x^*) = 0.$$

Problem 1.2 *Extended Rosenbrock function*:

$$f(x) = \sum_{i=1}^{n-1} [100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2], \tag{1.2}$$

$$x_0 = [-1.2, 1, \cdots, -1.2, 1]^T,\ x^* = [1, 1, \cdots, 1, 1]^T,\ f(x^*) = 0.$$

Problem 1.3 *Wood function*:

$$\begin{aligned} f(x) \quad = \quad & 100(x_1^2 - x_2)^2 + (x_1 - 1)^2 + (x_3 - 1)^2 + 90(x_3^2 - x_4)^2 \\ & +10.1[(x_2 - 1)^2 + (x_4 - 1)^2] + 19.8(x_2 - 1)(x_4 - 1), \end{aligned} \tag{1.3}$$

$$x_0 = [-3, -1, -3, -1]^T,\ x^* = [1, 1, 1, 1]^T, f(x^*) = 0.$$

Problem 1.4 *Powell singular function*:

$$f(x) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4, \tag{1.4}$$

$$x_0 = [3, -1, 0, 1]^T,\ x^* = [0, 0, 0, 0]^T,\ f(x^*) = 0.$$

Problem 1.5 *Cube function*

$$f(x) = 100(x_2 - x_1^3)^2 + (1 - x_1)^2, \tag{1.5}$$

$$x_0 = [-1.2, -1]^T,\ x^* = [1, 1]^T,\ f(x^*) = 0.$$

Problem 1.6 *Trigonometric function*

$$f(x) = \sum_{i=1}^{n} \left[n + i(1 - \cos x_i) - \sin x_i - \sum_{j=1}^{n} \cos x_j \right]^2, \tag{1.6}$$

$$x_0 = \left[\frac{1}{5n}, \cdots, \frac{1}{5n} \right]^T, \ x^* = [0, \cdots, 0]^T, \ f(x^*) = 0.$$

Problem 1.7 *Helical valley function*

$$f(x) = 100[(x_3 - 10\theta)^2 + (\sqrt{x_1^2 + x_2^2} - 1)^2] + x_3^2, \tag{1.7}$$

where

$$2\pi\theta = \begin{cases} \arctan(x_1/x_2) & \text{if } x_1 > 0, \\ \pi + \arctan(x_2/x_1) & \text{if } x_1 < 0, \end{cases}$$

$$x_0 = [-1, 0, 0]^T, \ x^* = [1, 0, 0]^T, \ f(x^*) = 0.$$

§2. Test Functions for Constrained Optimization Problems

The test functions for constrained optimization are selected from Hock and Schittkowski [176].

Problem 2.1 (No. 14 in [176])
Number of Variables: $n = 2$
Objective Function:

$$f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$$

Constraints:

$$-0.25x_1^2 - x_2^2 + 1 \geq 0,$$
$$x_1 - 2x_2 + 1 = 0.$$

Start: $x_0 = (2, 2), \ f(x_0) = 1.$
Solution: $x^* = (0.5(\sqrt{7} - 1), \ 0.25(\sqrt{7} + 1)),$
$f(x^*) = 9 - 2.875\sqrt{7}.$

Problem 2.2 (No. 22 in [176])

Number of Variables: $n = 2$
Objective Function:

$$f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$$

Constraints:

$$-x_1 - x_2 + 2 \geq 0$$
$$-x_1^2 + x_2 \geq 0$$

Start: $x_0 = (2, 2),\ f(x_0) = 1.$
Solution: $x^* = (1, 1),\ f(x^*) = 1.$

Problem 2.3 (No. 59 in [176])
Number of Variables: $n = 2$
Objective Functions:

$$\begin{aligned} f(x) \quad &= -75.196 + 3.8112x_1 + 0.0020567x_1^3 - 1.0345\text{E-}5x_1^4 \\ &+6.8306x_2 - 0.030234x_1x_2 + 1.28134\text{E-}3x_2x_1^2 \\ &+2.266\text{E-}7x_1^4x_2 - 0.25645x_2^2 + 0.0034604x_2^3 - 1.3514\text{E-}5x_2^4 \\ &+28.106/(x_2 + 1) + 5.2375\text{E-}6x_1^2x_2^2 + 6.3\text{E-}8x_1^3x_2^2 \\ &-7\text{E-}10x_1^3x_2^3 - 3.405\text{E-}4x_1x_2^2 + 1.6638\text{E-}6x_1x_2^3 \\ &+2.8673\exp(0.0005x_1x_2) - 3.5256\text{E-}5x_1^3x_2 \end{aligned}$$

Constraints:

$$\begin{aligned} &x_1x_2 - 700 \geq 0, \\ &x_2 - x_1^2/125 \geq 0, \\ &(x_2 - 50)^2 - 5(x_1 - 55) \geq 0, \\ &0 \leq x_1 \leq 75, \\ &0 \leq x_2 \leq 65. \end{aligned}$$

Start: $x_0 = (90, 10),\ f(x_0) = 86.878639$
Solution: $x^* = (13.55010424,\ 51.66018129),\ f(x^*) = -7.804226324.$

Problem 2.4 (No. 63 in [176])
Number of Variables: $n = 3$
Objective Function:

$$f(x) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3$$

Constraints:

$$\begin{aligned} &8x_1 + 14x_2 + 7x_3 - 56 = 0, \\ &x_1^2 + x_2^2 + x_3^2 - 25 = 0, \\ &0 \le x_i,\ i = 1, 2, 3. \end{aligned}$$

Start: $x_0 = (2, 2, 2),\ f(x_0) = 976$
Solution: $x^* = (3.512118414,\ 0.2169881741,\ 3.552174034),\ f(x^*) = 961.7151721$

Problem 2.5 (No. 25 in [176])
Number of Variables: $n = 3$
Objective Function:

$$f(x) = \sum_{i=1}^{99} (f_i(x))^2$$

where

$$\begin{aligned} f_i(x) &= -0.01i + \exp(-\frac{1}{x_1}(u_i - x_2)^{x_3}) \\ u_i &= 25 + (-50 \ln(0.01i))^{2/3},\ i = 1, \cdots, 99. \end{aligned}$$

Constraints:

$$\begin{aligned} 0.1 \le\ & x_1 \ \le 100 \\ 0 \le\ & x_2 \ \le 25.6 \\ 0 \le\ & x_3 \ \le 5 \end{aligned}$$

Start: $x_0 = (100, 12.5, 3),\ f(x_0) = 32.835$
Solution: $x^* = (50, 25, 1.5),\ f(x^*) = 0$

Problem 2.6 (No. 35 in [176])
Number of Variables: $n = 3$
Objective Function:

$$\begin{aligned} f(x) &= 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 \\ &\quad +2x_1x_2 + 2x_1x_3 \end{aligned}$$

Constraints:

$$\begin{aligned} &3 - x_1 - x_2 - 2x_3 \ge 0 \\ &0 \le x_i,\ i = 1, 2, 3. \end{aligned}$$

Start: $x_0 = (0.5, 0.5, 0.5),\ f(x_0) = 2.25$
Solution: $x^* = (4/3, 7/9, 4/9),\ f(x^*) = 1/9.$

Problem 2.7 (No. 38 in [176])
Number of Variables: $n = 4$
Objective Function:

$$\begin{aligned} f(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 \\ &\quad +10.1((x_2 - 1)^2 + (x_4 - 1)^2) + 19.8(x_2 - 1)(x_4 - 1) \end{aligned}$$

Constraints:

$$-10 \le x_i \le 10,\ i = 1, \cdots, 4$$

Start: $x_0 = (-3, -1, -3, -1),\ f(x_0) = 19192$
Solution: $x^* = (1, 1, 1, 1),\ f(x^*) = 0.$

Problem 2.8 (No. 43 in [176])
Number of Variables: $n = 4$
Objective Function:

$$f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$$

Constraints:

$$\begin{aligned} &8 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 \ge 0 \\ &10 - x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 \ge 0 \\ &5 - 2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 \ge 0 \end{aligned}$$

Start: $x_0 = (0, 0, 0, 0),\ f(x_0) = 0.$
Solution: $x^* = (0, 1, 2, -1),\ f(x^*) = -44$

Problem 2.9 (No. 73 in [176])
Number of Variables: $n = 4$
Objective Function:

$$f(x) = 24.55x_1 + 26.75x_2 + 39x_3 + 40.50x_4$$

Constraints:

$$\begin{aligned} &2.3x_1 + 5.6x_2 + 11.1x_3 + 1.3x_4 - 5 \ge 0 \\ &12x_1 + 11.9x_2 + 41.8x_3 + 52.1x_4 - 21 \\ &-1.645(0.28x_1^2 + 0.19x_2^2 + 20.5x_3^2 + 0.62x_4^2)^{\frac{1}{2}} \ge 0 \\ &x_1 + x_2 + x_3 + x_4 - 1 = 0 \\ &0 \le x_i,\ i = 1, \cdots, 4. \end{aligned}$$

Start: $x_0 = (1, 1, 1, 1),\ f(x_0) = 130.8$
Solution:

$$x^* = (0.6355216, -0.12\text{E-}11, 0.3127019, 0.05177655),$$
$$f(x^*) = 29.894378$$

Problem 2.10 (No. 83 in [176])
Number of Variables: $n = 5$
Objective Function:

$$f(x) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$$

Constraints:

$$92 \geq a_1 + a_2x_2x_5 + a_3x_1x_4 - a_4x_3x_5 \geq 0$$
$$20 \geq a_5 + a_6x_2x_5 + a_7x_1x_2 + a_8x_3^2 - 90 \geq 0$$
$$5 \geq a_9 + a_{10}x_3x_5 + a_{11}x_1x_3 + a_{12}x_3x_4 - 20 \geq 0$$
$$78 \leq x_1 \leq 102$$
$$33 \leq x_2 \leq 45$$
$$27 \leq x_i \leq 45,\ i = 3, 4, 5,$$

where

$$a_1 = 85.334407,\ a_2 = 0.0056858,\ a_3 = 0.0006262,$$
$$a_4 = 0.0022053,\ a_5 = 80.51249,\ a_6 = 0.0071317,$$
$$a_7 = 0.0029955,\ a_8 = 0.0021813,\ a_9 = 9.300961,$$
$$a_{10} = 0.0047026,\ a_{11} = 0.0012547,\ a_{12} = 0.0019085$$

Start: $x_0 = (78, 33, 27, 27, 27),\ f(x_0) = -32217$
Solution: $x^* = (78, 33, 29.99526, 45, 36.77581),\ f(x^*) = -30665.53867$

Problem 2.11 (No. 86 in [176])
Number of Variables: $n = 5$
Objective Function:

$$f(x) = \sum_{j=1}^{5} e_jx_j + \sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij}x_ix_j + \sum_{j=1}^{5} d_jx_j^3$$

Constraints:

$$\sum_{j=1}^{5} a_{ij}x_j - b_i \geq 0,\ i = 1, \cdots, 10,$$
$$0 \leq x_i,\ i = 1, \cdots, 5,$$

where

j	1	2	3	4	5
e_j	-15	-27	-36	-18	-12
c_{1j}	30	-20	-10	32	-10
c_{2j}	-20	39	-6	-31	32
c_{3j}	-10	-6	10	-6	-10
c_{4j}	32	-31	-6	39	-20
c_{5j}	-10	32	-10	-20	30
d_j	4	8	10	6	2
a_{1j}	-16	2	0	1	0
a_{2j}	0	-2	0	4	2
a_{3j}	-3.5	0	2	0	0
a_{4j}	0	-2	0	-4	-1
a_{5j}	0	-9	-2	1	-2.8
b_j	-40	-2	-0.25	-4	-4

Start: $x_0 = (0, 0, 0, 0, 1),\ f(x_0) = 20$
Solution: $x^* = (0.3, 0.33346761, 0.4, 0.42831010, 0.22396487),\ f(x^*) = -32.34867897$

Problem 2.12 (No. 93 in [176])
Number of Variables: $n = 6$
Objective Function:

$$\begin{aligned} f(x) \quad = \quad & 0.0204x_1x_4(x_1 + x_2 + x_3) + 0.0187x_2x_3(x_1 + 1.57x_2 + x_4) \\ & +0.0607x_1x_4x_5^2(x_1 + x_2 + x_3) \\ & +0.0437x_2x_3x_6^2(x_1 + 1.57x_2 + x_4) \end{aligned}$$

Constraints:

$$\begin{aligned} & 0.001x_1x_2x_3x_4x_5x_6 - 2.07 \geq 0, \\ & 1 - 0.00062x_1x_4x_5^2(x_1 + x_2 + x_3), \\ & -0.00058x_2x_3x_6^2(x_1 + 1.57x_2 + x_4) \geq 0, \\ & 0 \leq x_i,\ i = 1, \cdots 6. \end{aligned}$$

Start: $x_0 = (5.54, 4.4, 12.02, 11.82, 0.702, 0.852)$, $f(x_0) = 137.066$

Solution:

$$x^* = (5.332666, 4.656744, 10.43299,$$
$$12.08230, 0.7526074, 0.87865084),$$
$$f(x^*) = 135.075961$$

Problem 2.13 (No. 108 in [176])
Number of Variables: $n = 9$
Objective Function:

$$f(x) = -0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7)$$

Constraints:

$$1 - x_3^2 - x_4^2 \geq 0,$$
$$1 - x_5^2 - x_6^2 \geq 0,$$
$$1 - x_9^2 \geq 0,$$
$$1 - x_1^2 - (x_2 - x_9)^2 \geq 0,$$
$$1 - (x_1 - x_5)^2 - (x_2 - x_6)^2 \geq 0,$$
$$1 - (x_1 - x_7)^2 - (x_2 - x_8)^2 \geq 0,$$
$$1 - (x_3 - x_5)^2 - (x_4 - x_6)^2 \geq 0,$$
$$1 - (x_3 - x_7)^2 - (x_4 - x_8)^2 \geq 0,$$
$$1 - x_7^2 - (x_8 - x_9)^2 \geq 0,$$
$$x_1x_4 - x_2x_3 \geq 0,$$
$$x_3x_9 \geq 0,$$
$$-x_5x_9 \geq 0,$$
$$x_5x_8 - x_6x_7 \geq 0,$$
$$0 \leq x_9.$$

Start:

$$x_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1),$$
$$f(x_0) = 0$$

Solution:

$$x^* = (0.8841292, 0.4672425, 0.03742076, 0.9992996,$$
$$0.8841292, 0.4672424, 0.03742076, 0.9992996,$$
$$0.26\text{E-}19),$$
$$f(x^*) = -0.8660254038$$

Problem 2.14 (No. 110 in [176])
Number of Variables: $n = 10$
Objective Function:

$$f(x) = \sum_{i=1}^{10}[(\ln(x_i - 2))^2 + (\ln(10 - x_i))^2 - (\prod_{i=1}^{10} x_i)^2]$$

Constraints:

$$2.001 \le x_i \le 9.999,\ i = 1, \cdots, 10.$$

Start: $x_0 = (9, \cdots, 9),\ f(x_0) = -43.134337$
Solution: $x^* = (9.35025655, \cdots, 9.35025655),\ f(x^*) = -45.77846971$

Problem 2.15 (No. 111 in [176])
Number of Variables: $n = 10$
Objective Function:

$$f(x) = \sum_{j=1}^{10} \exp(x_j)(c_j + x_j - \ln(\sum_{k=1}^{10} \exp(x_k)))$$

where

$$c_1 = -6.089,\ c_2 = -17.164,\ c_3 = -34.054,$$
$$c_4 = -5.914,\ c_5 = -24.721,\ c_6 = -14.986,$$
$$c_7 = -24.100,\ c_8 = -10.708,\ c_9 = -26.662,\ c_{10} = -22.179$$

Constraints:

$$\exp(x_1) + 2\exp(x_2) + 2\exp(x_3) + \exp(x_6) + \exp(x_{10}) - 2 = 0,$$
$$\exp(x_4) + 2\exp(x_5) + \exp(x_6) + \exp(x_7) - 1 = 0,$$
$$\exp(x_3) + \exp(x_7) + \exp(x_8) + 2\exp(x_9) + \exp(x_{10}) - 1 = 0,$$
$$-100 \le x_i \le 100,\ i = 1, \cdots, 10.$$

Start: $x_0 = (-2.3, \cdots, -2.3),\ f(x_0) = -21.015$
Solution:

$$\begin{aligned} x^* &= (-3.201212, -1.912060, -0.2444413, -6.537489, \\ &\quad -0.7231524, -7.267738, -3.596711, -4.017769, \\ &\quad -3.287462, -2.335582), \\ f(x^*) &= -47.76109026 \end{aligned}$$

Problem 2.16 (No. 112 in [176])
Number of Variables: $n = 10$
Objective Function:

$$f(x) = \sum_{j=1}^{10} x_j (c_j + \ln \frac{x_j}{x_1 + \cdots + x_{10}})$$

where c_j are defined in Problem 2.15.
Constraints:

$$\begin{aligned} & x_1 + 2x_2 + 2x_3 + x_6 + x_{10} - 2 = 0, \\ & x_4 + 2x_5 + x_6 + x_7 - 1 = 0, \\ & x_3 + x_7 + x_8 + 2x_9 + x_{10} = 0, \\ & \text{1.E-6} \le x_i,\ i = 1, \cdots, 10. \end{aligned}$$

Start: $x_0 = (0.1, \cdots, 0.1),\ f(x_0) = -20.961$
Solution:

$$\begin{aligned} x^* &= (0.01773548, 0.08200180, 0.8825646, 0.7233256\text{E-}3, \\ &\quad 0.4907851, 0.4335469\text{E-}3, 0.01727298, \\ &\quad 0.007765639, 0.01984929, 0.05269826), \\ f(x^*) &= -47.707579 \end{aligned}$$

Problem 2.17 (No. 117 in [176])
Number of Variables: $n = 15$
Objective Function:

$$f(x) = -\sum_{j=1}^{10} b_j x_j + \sum_{j=1}^{5} \sum_{k=1}^{5} c_{kj} x_{10+k} x_{10+j} + 2 \sum_{j=1}^{5} d_j x_{10+j}^3$$

Constraints:

$$2\sum_{k=1}^{5} c_{kj}x_{10+k} + 3d_j x_{10+j}^2 + e_j - \sum_{k=1}^{10} a_{kj}x_k \geq 0,\ j = 1, \cdots, 5,$$
$$0 \leq x_i,\ i = 1, \cdots, 15,$$

where

j	1	2	3	4	5
a_{6j}	2	0	-4	0	0
a_{7j}	-1	-1	-1	-1	-1
a_{8j}	-1	-2	-3	-2	-1
a_{9j}	1	2	3	4	5
a_{10j}	1	1	1	1	1
b_{5+j}	-1	-40	-60	5	1

and other parameters are defined as in Problem 2.11.

Start:

$$x_0 = 0.001(1, 1, 1, 1, 1, 1, 60000, 1, 1, 1, 1, 1, 1, 1, 1),$$
$$f(x_0) = 2400.1053$$

Solution:

$$\begin{aligned} x^* &= (0, 0, 5.174136, 0, 3.061093, 11.83968, 0, 0, \\ &\quad 0.1039071, 0, 0.2999929, 0.3334709, 0.3999910, \\ &\quad 0.4283145, 0.2239607) \\ f(x^*) &= 32.348679 \end{aligned}$$