SMOOTHING GRAPH SIGNALS VIA RANDOM SPANNING FORESTS

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Introduction

We discuss an inspiring use of the links between random processes on graphs and Laplacian-based numerical algebra:

- Based on *random spanning forests*, we propose two novel and efficient Monte Carlo estimators for smoothing graph signals.
- Moreover, we provide a theoretical analysis on bias and variance of our estimators.
- On the empirical side, these estimators are illustrated in two well-known applications.

Problem Definition

Regularized Regression on Graphs

$$\hat{\mathbf{x}} = \arg\min_{z\in\mathbb{R}^n} q\underbrace{||\mathbf{y}-\mathbf{z}||^2}_{\text{Fidelity}} + \underbrace{\mathbf{z}^T\mathsf{L}\mathbf{z}}_{\text{Regularization}}, \quad q>0$$

where $\mathbf{y} = (y_1, y_2, ..., y_n)$ is a graph signal and y_i is the measurement from node i. L denotes the graph Laplacian of the given graph and q sets the trade-off between data-fidelity and regularization.

The explicit solution to this problem is:

$$\hat{\mathbf{x}} = \mathsf{K}\mathbf{y} \text{ with } \mathsf{K} = (\mathsf{L}+q\mathsf{I})^{-1}q\mathsf{I}$$

where I is the identity matrix.

- Direct computation of K requires $\mathcal{O}(n^3)$ elementary operations due to the inverse.
- For large n, iterative methods and polynomial approximations are the state-of-the-art. Both compute $\hat{\mathbf{x}}$ in linear time in the number of edges $|\mathcal{E}|$.

Random Spanning Forests on Graphs

For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{W})$:

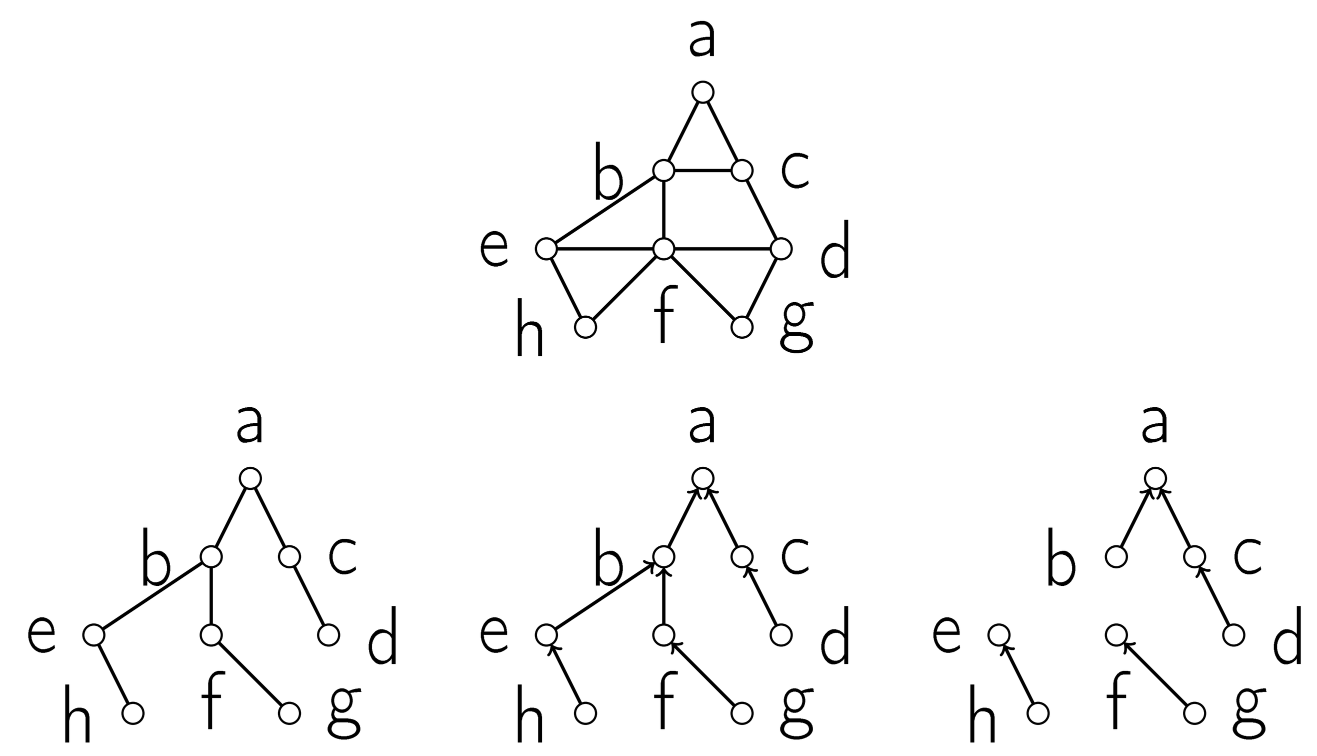

Fig. 1: Original graph, a spanning tree, a rooted spanning tree and a rooted spanning forest

Random Spanning Forests (RSF)

Consider the following parametric distribution over rooted spanning forests:

$$P(\Phi_q = \phi) \propto q^{|\rho(\phi)|}\prod_{\tau\in\phi}\prod_{(i,j)\in\tau}\mathsf{W}_{i,j}$$

where q is a parameter and $\rho(\phi)$ denotes the set of roots in the forest ϕ. One can sample from this distribution by a variant of Wilson's algorithm in time $\mathcal{O}(|\mathcal{E}|/q)$ [1].

A very surprising connection between RSFs and linear algebra appears in the following simple equation:

$$P(r_{\Phi_q}(i) = j) = \mathsf{K}_{i,j} \text{ with } \mathsf{K} = (\mathsf{L}+q\mathsf{I})^{-1}q\mathsf{I}$$

where $r_{\Phi_q}(.)$ returns the root of node i in the spanning forest Φ_q.

RSF based Estimators

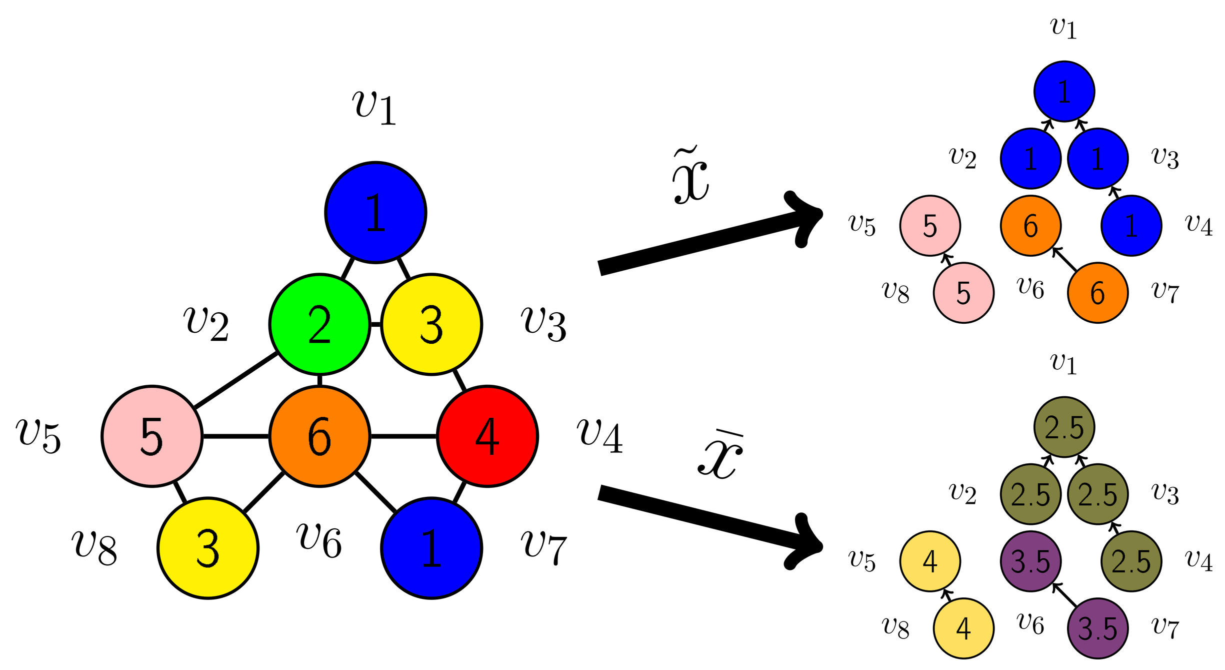

Fig. 2: An illustration for the estimators

RSF Estimator	Formula	Expectation	Variance
$\tilde{x}(i)$	$y\left[r_{\Phi_q}(i)\right]$	$\mathbb{E}[\tilde{x}(i)] = \delta_i\mathsf{K}\mathbf{y}$	$\sum_{i\in\mathcal{V}}\mathrm{Var}(\tilde{x}(i)) = \mathbf{y}^T(\mathsf{I}-\mathsf{K}^2)\mathbf{y}$
$\bar{x}(i)$	$\frac{1}{\lvert\mathcal{V}_{t(i)}\rvert}\sum_{j\in\mathcal{V}_{t(i)}} y_j$	$\mathbb{E}[\bar{x}(i)] = \delta_i\mathsf{K}\mathbf{y}$	$\sum_{i\in\mathcal{V}}\mathrm{Var}(\bar{x}(i)) = \mathbf{y}^T(\mathsf{K}-\mathsf{K}^2)\mathbf{y}$

Experiments

Tikhonov denoising of graph signals. We consider an image as a noisy graph signal where the underlying graph is 2-D grid:

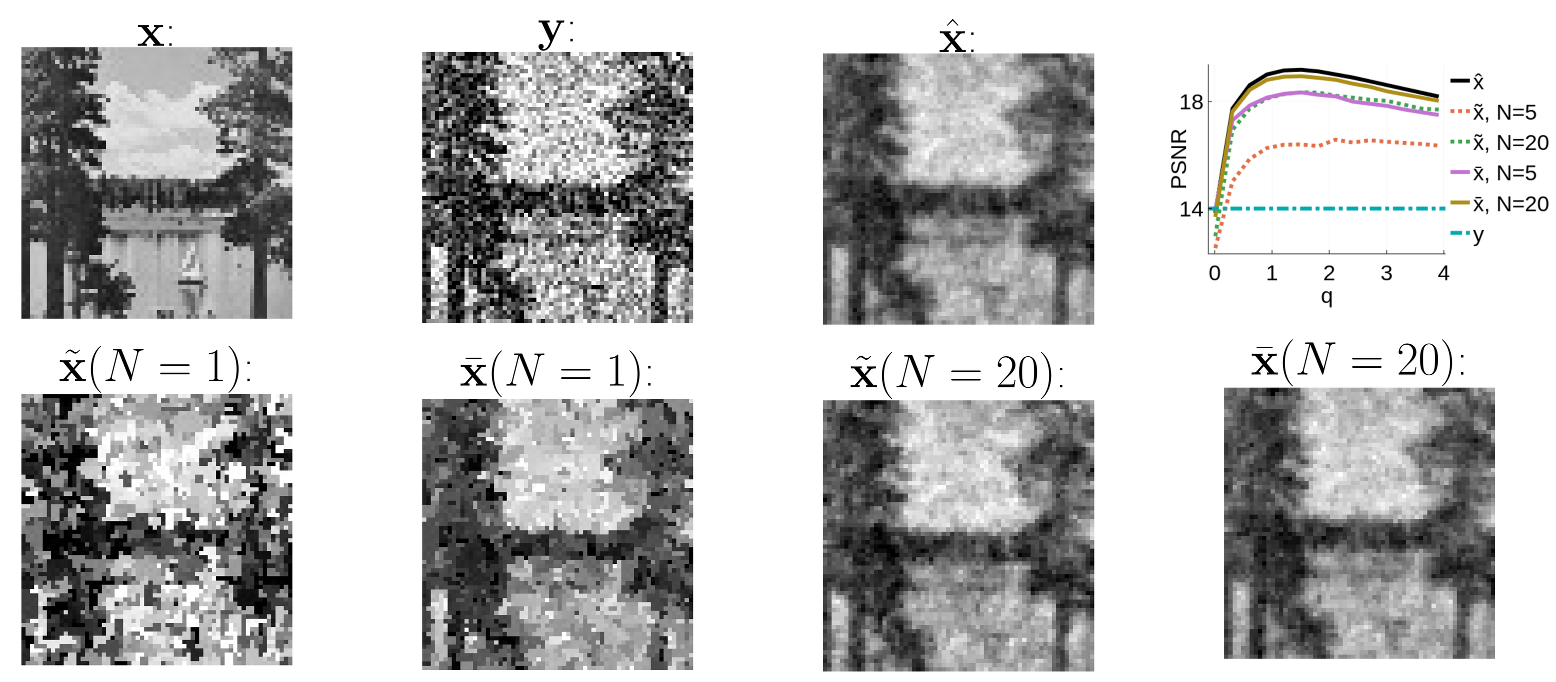

Fig. 3: Illustration on an image.

SSL for node classification on graphs aims at recovering labels of all nodes with a few available labels as *a priori* knowledge. Given labels in $\mathbf{y}_l$ for class l, the solution [2] under a smoothness prior regulated by $\mu > 0$ is the classification function $\mathbf{f}_l = \frac{\mu}{2+\mu}\left(\mathsf{I}-\frac{2}{2+\mu}\mathsf{D}^{-\sigma}\mathsf{W}\mathsf{D}^{\sigma-1}\right)^{-1}\mathbf{y}_l = \mathsf{D}^{1-\sigma}\mathsf{K}\mathsf{D}^{\sigma-1}\mathbf{y}_l$ where $\mathsf{K} = (\mathsf{Q}+\mathsf{L})^{-1}\mathsf{Q}$ and $\mathsf{Q} = \frac{\mu}{2}\mathsf{D}$. We run our methods to estimate $\mathbf{f}_l$'s on a graph with $|\mathcal{V}| = 3000$ and two communities.

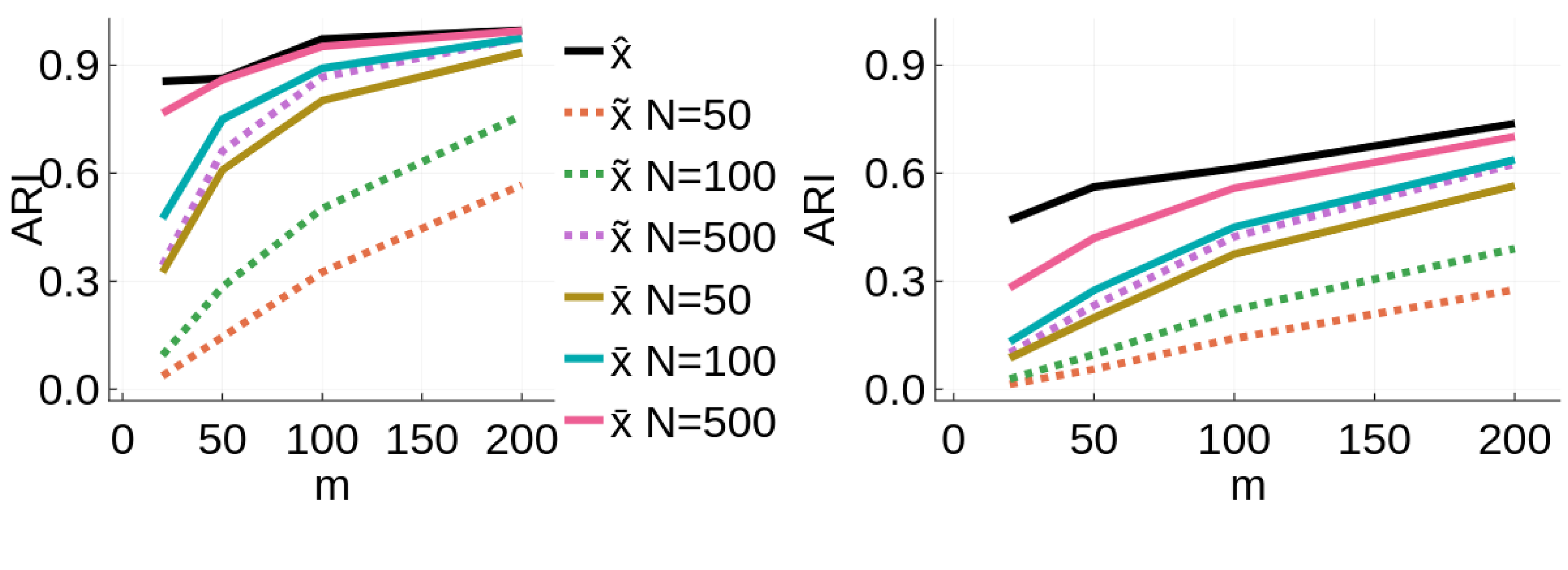

Fig. 4: Adjusted Rand Index (ARI) compared on strong (left) and fuzzy (right) communities

References

[1] L. Avena and A. Gaudillière. Random spanning forests, Markov matrix spectra and well distributed points. *arXiv:1310.1723 [math]*, Oct. 2013. arXiv: 1310.1723.

[2] K. Avrachenkov, A. Mishenin, P. Gonçalves, and M. Sokol. Generalized Optimization Framework for Graph-based Semi-supervised Learning. In *Proceedings of the 2012 SIAM International Conference on Data Mining*, pages 966–974.

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