The Foundations for Probability Limiting Theory by Zhengyan Lin & Chuanrong Lu & Zhonggen Su

Lecture Notes for Probability Limiting Theory Seminar

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Lecture 1

Preliminary

1.1 Basis for Martingale

Let $(\Omega, \mathscr{A}, P)$ is a probability space and X is an integrable random variable, $\mathscr{G}$ is a subalgebra of $\mathscr{A}$.

Definition 1.1.1. For an integrable random variable X, if there exist a random varibale Y such that

- Y is $\mathscr{G}-$ measurable;
- for any $A \in \mathscr{G}$ we have

$$\int_A Y dP = \int_A X dP,$$

then we say Y is the conditional expectation of X given $\sigma-$algebra $\mathscr{G}$. We can write it as $\mathsf{E}[X|\mathscr{G}]$.

Remark 1.1.2. The conditional expectation is unique in the sense of $a.s.$, see Jacod and Protter [2012].

Remark 1.1.3.

- For any real number c_1 and c_2, we have

$$\mathsf{E}\,(c_1 X_1 + c_2 X_2|\mathscr{G}) = c_1 \mathsf{E}\,(X_1|\mathscr{G}) + c_2 \mathsf{E}\,(X_2|\mathscr{G}) \quad a.s.$$

- Let Y is $\mathscr{G}-$measurable, $\mathsf{E}\,|XY| < \infty$, $\mathsf{E}\,|X| < \infty$, then we have

$$E(XY|\mathscr{G}) = YE(X|\mathscr{F}) \quad a.s.$$

- Let $\mathscr{G}_1 \subset \mathscr{G}_2 \subset \mathscr{A}$, then we have

$$E\left[E\left(X|\mathscr{G}_2\right)|\mathscr{G}_1\right] = E\left(X|\mathscr{G}_1\right) = E\left[E\left(X|\mathscr{G}_1\right)|\mathscr{G}_2\right] \quad a.s.$$

  Note that when $\mathscr{G}_1 = \emptyset$ the first equality will turn to the normal Adam's law.

Let $\{X_n\}$ is a sequence of independent random variables with zero mean and $S_n = \sum_{j=1}^n X_j$. Then we have

$$\begin{aligned} E\left(S_{n+1}|X_1, \cdots, X_n\right) &= E\left(S_n + X_{n+1}|X_1, \cdots, X_n\right) \\ &= S_n + EX_{n+1} = S_n \quad a.s., \end{aligned}$$

which indicates that the conditional expectation of S_{n+1} only relates to the previous n random variables.

Definition 1.1.4. Let $\{\mathscr{A}_n\}$ is an increasing sequence of sub $\sigma-$algebra of $\mathscr{A}$. We say a seqence of random variables $\{S_n\}$ is a **martingle** if

(i) S_n is $\mathscr{A}_n-$measurable;

(ii) $\mathsf{E}\,|S_n| < \infty$;

(iii) for $m < n$, $\mathsf{E}\,(S_n|\mathscr{A}_m) = S_m \quad$ a.s.,

if the equality in (iii) turns out to be $\leq$ $(\geq)$ then we say $\{S_n\}$ is a supermartingale (submartingale).

Example 1.1.1. See pp.24.

Example 1.1.2. See pp.24.

Lemma 1.1.5. (i) $\{S_n\}$ is a submartingale (supermartingale), Φ is a non-decreasing convex (concave) real function. if $\mathsf{E}\,|\Phi(S_n)| < \infty$, then $\{\Phi(S_n)\}$ is also a submartingale (supermartingale).

(ii) $\{S_n\}$ is a martingale, Φ is a real convex function. if $\mathsf{E}\,|\Phi(S_n)| < \infty$, then $\{\Phi(S_n)\}$ is a submartingale.

Proof. We only prove the submartingale case in (i).

- (Measurability.) Since Φ is a convex function on real line, we can see that $\{x : f(x) < a\}$ will be either an empty set or an inteval for any $a \in \mathbb{R}$. And the mixture of measurable function will be mearable function. Thus $\{\Phi(S_n)\}$ is $\mathscr{A}_n-$measurable.
- (Inequality.) By Jensen's inequality, we have

$$\mathsf{E}\,[\Phi\,(S_n)\,|\mathscr{A}_{n-1}] \geqslant \Phi\,[\mathsf{E}\,(S_n|\mathscr{A}_{n-1})] \geqslant \Phi(S_{n-1}) \quad a.s..$$

□

Example 1.1.3. $\Phi(x) = x^+$ and $\Phi(x) = |x|^p$.

Let $\{\mathscr{A}_n\}$ is an increasing sequence of sub $\sigma-$algebra of $\mathscr{A}$ and $N_\infty = \{1, 2, \ldots, \infty\}$.

Definition 1.1.6. Let α is a random variable:

$$\alpha : \Omega \to N_\infty.$$

If α satisfies

$$\{\alpha = n\} \in \mathscr{A}_n, \quad n \geqslant 1$$

or equivlently

$$\{\alpha \leqslant n\} \in \mathscr{A}_n, \quad n \geqslant 1 \tag{1.1.1}$$

then we say α is a **stopping time**.

Remark 1.1.7. It means that the "decision" of whether to stop at time n must be based only on the information present at time n, not on any future info.

Definition 1.1.8. Let α is a stopping time and $\mathscr{A}_\infty = \sigma\left(\{\mathscr{A}_n, n \geqslant 1\}\right)$. We define

$$\mathscr{A}_\alpha = \left\{E : E \in \mathscr{A}_\infty,\ E \bigcap \{\alpha = n\} \in \mathscr{A}_n, n \geqslant 1\right\}$$

as the stopping time $\sigma-$algebra $\mathscr{A}_\alpha$.

By checking $\mathscr{A}_\alpha$ is closed under complement of sets and union of countable sets we can see that this definition is reasonable.

Lemma 1.1.9 (**Doob's**)**.** Let $\{S_n\}$ is a martingale (submartingale) and α, β are two bounded stopping time. If $\alpha \leqslant \beta$, then we have

$$E\left(S_\beta | \mathscr{A}_\alpha\right) = S_\alpha \,(\geqslant S_\alpha) \quad a.s.$$

Proof. We only prove the case of submartingale. To compelte our proof, we need to verify

- integrability of S_β;
- S_α is $\mathscr{A}_\alpha-$measurable;
- Inequality.

Assume α and β can be bounded by m. We have $S_\beta = \sum_{n=1}^m S_n \mathbb{1}\{\beta = n\}$ and $\{S_n\}$ is a martingale. It follows that S_β is integrable. Let B is a Borel set on real line. Since

$$\{S_\alpha \in B\} = \bigcup_{k=1}^m \left(\{S_k \in B\} \bigcap \{\alpha = k\}\right) \in \mathscr{A}_\infty$$

$$\{S_\alpha \in B\} \bigcup \{\alpha = n\} = \{S_n \in B\} \bigcup \{\alpha = n\} \in \mathscr{A}_n.$$

Thus, S_α is $\mathscr{A}_\alpha-$measurable. To prove the inequality we only need to prove that for any $\Lambda \in \mathscr{A}_\alpha$

$$\int_\Lambda S_\alpha dP \leqslant \int_\Lambda \mathsf{E}\left(S_\beta | \mathscr{A}_\alpha\right) dP = \int_\Lambda S_\beta dP. \tag{1.1.2}$$

What we want to do is to fix the index in (1.1.2) and change problem into fixed α and fixed β. Thus we define $\Lambda_n = \Lambda \cap \{\alpha = n\}$ to fix α and $\Lambda_j \cap \{\beta > k\}$ to fixed β, here $k \geqslant j$. The details of proof can be found in pp.25. □

Definition 1.1.10. (**Continuous case.**) See pp.26.

Lecture 2

Infinite Divisible Distribution Function and General CLT

2.1 Infinite Divisible Distribution

2.1.1 Definition of Infinite Divisible Distribution Function

Definition 2.1.1. (**From the perspective of c.f.**) We say a characteristic function $f(t)$ is **infinite divisible** (i.d.) iff. $\forall n \in \mathbb{Z}^*$, there exist some characteristic function $f_n(t)$ such that

$$f(t) = [f_n(t)]^n.$$

And the corresponding distribution function F is called infinite divisible distribution function.

Definition 2.1.2. (**From the perspective of r.v.**) Let X is a random variable, we say **the distribution of X is infinite divisible** iff. $\forall n \in \mathbb{Z}$, there exist an i.i.d random variable sequence $\{X_k^{(n)}\}_{n \geqslant k \geqslant 1}$ such that

$$X = X_1^{(n)} + X_2^{(n)} + \dots X_n^{(n)}.$$

Definition 2.1.3. (**From the perspective of d.f.**) We say a distribution function F is a **infinite divisible distribution function** iff. $\forall n \in \mathbb{Z}$, there exist another distribution function F_n such that

$$F = F_n^{*n}.$$

Remark 2.1.4. From Definition 2.1.1 we have

$$[f(t)]^{\frac{1}{n}} = f_n(t).$$

Thus, for i.d.c.f. $f(t)$, $[f(t)]^{\frac{1}{n}}$ will always be a c.f.

Example 2.1.1. See pp.30.

2.1.2 Properties of i.d.c.f.

Theorem 2.1.5. (i) I.d.c.f. are closed under finite multiplications; (ii) if f is a i.d.c.f., so does $|f|$.

Theorem 2.1.6. If $f(t)$ is a i.d.c.f., then $f(t) \neq 0$.

Proof. It's hard to consider the zeros of a complex function, thus we consider $g = |f|^2$. We assme $f(t) = [f_n(t)]^n$, then $g = |f|^2 = |f_n|^{2n} \triangleq g_n^n$, where $g_n = |f_n|^2$ is also a c.f. Here we transform a problem of complex function into a problem of real function. We define

$$h(t) \triangleq \lim_{n\to\infty} g_n = \begin{cases} 1, & if\ g(t) > 0, \\ 0, & if\ g(t) = 0. \end{cases}$$

we only need to show $h \equiv 1$. Note that $g(0) = 1$ which indicates that $h(0) = 1$. Since g_ns are continous, thus $h(t)$ will also be continous. Thus, h is nowhere 0.

□

Theorem 2.1.7. Let $\{f^{(m)}(t)\}$ is a sequence of i.d.c.f. If there exist a c.f. $f(t)$ such that

$$f^{(m)}(t) \to f(t),$$

then $f(t)$ is a i.d.c.f.

Proof. Note that $f(t) = \lim_{m\to\infty} f^{(m)}(t) = \left[\lim_m f_n^{(m)}(t)\right]^n$. By Continuity Theorm we obtain the conclusion. □

Remark 2.1.8. Note that this theorem does not tell us that i.d.c.f. are closed in the sense of limitation because we need the limiting function is a c.f.

Definition 2.1.9. If c.f. $f(t)$ satisfies

$$f(t) = \exp\{i\alpha t + \lambda(e^{i\beta t} - 1)\},$$

where $\lambda \geq 0$ and α, β are real number, then we say $f(t)$ is a **poisson type c.f.**

Remark 2.1.10. If X is a **Poisson type r.v.** i.e. there exist $\beta \in \mathbb{N}$, $\lambda > 0$ and real number α such that for $k \in \mathbb{Z}*$

$$\Pr(X = \alpha + k\beta) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Then corresponding c.f. of X is $f(t) = \exp\left[i\alpha t + \lambda(e^{i\beta t} - 1)\right]$.

Theorem 2.1.11. $f(t)$ is an i.d.c.f. iff. it can be written as the limitation of the product of finite poisson type c.f.

Proof. For the sufficient part we need to prove $f(t)$ is a c.f. and it is i.d. We assume

$$f(t) = \lim_{\triangle} \prod_{k=1}^{n} \exp\{\mathrm{i}\alpha_k t + \lambda_k(e^{\mathrm{i}\beta_k t} - 1)\}.$$

Note that the finite multiplication of c.f. is also a c.f. and $f(t)$ is continous at 0, we conlude that $f(t)$ is a c.f. And it's easy to see $f(t)$ is i.d. The proof of necessary condition can be found at pp.31. □

Remark 2.1.12. This theorem implies that the limitation of a sequenc of i.d.c.f. must be a c.f. Keep this result in mind, and combine it with Theorem 2.1.7 indicate that the i.d.c.f. class is closed under limitation.

Example 2.1.2. See pp.32.

2.1.3 Levy-Khinchine Representation

Let γ is a real constant and $G(x)$ is a non-decreasing bounded and left continous (in this book, distribution function is defined as $\Pr(X < x)$) function. We define

$$\psi(t) = i\gamma t + \int_{-\infty}^{\infty} g(t,x)\mathrm{d}G(x),$$

where

$$g(t,x) = \begin{cases} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}, & if\ x \neq 0, \\ -\frac{t^2}{2}, & if\ x = 0. \end{cases}$$

Thus it's easy to see $g(t,x)$ is continous and bounded on $\mathbb{R}^2$.(Check the limit of $g(t,x)$ at $x = 0$.)

Theorem 2.1.13. $e^{\psi(t)}$ is a i.d.c.f.

Proof. Note that $g(t,x)$ has different form at 0 and other points, we seperate the integral into three differenc parts. For any $1 > \epsilon > 0$, let

$$\epsilon = x_0 < x_1 < \cdots < x_n = 1/\epsilon,\ x_k \leq \xi_k < x_{k+1}\ (k = 0, 1, \ldots, n-1).$$

We consider

$$\begin{aligned}
&\int_{\varepsilon}^{1/\epsilon} g(t,x)\mathrm{d}G(x) \\
&= \int_{\varepsilon}^{1/\epsilon} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}dG(x) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\mathrm{e}^{it\xi_k} - 1 - \frac{it\xi_k}{1+\xi_k^2}\right)\frac{1+\xi_k^2}{\xi_k^2}\left[G\left(x_{k+1}\right) - G\left(x_k\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}(\mathrm{i}\frac{\xi_k(1+\xi_k^2)[G(x_{k+1}) - G(x_k)]}{(1+\xi_k^2)\xi_k^2}t + \frac{1+\xi_k^2}{\xi_k^2}\left[G\left(x_{k+1}\right) - G\left(x_k\right)\right](\mathrm{e}^{\mathrm{i}\xi_k t} - 1)).
\end{aligned}$$

Thus, by Theorem 2.1.11 we see that

$$f_1^{(\epsilon)}(t) = \exp\{\int_{\varepsilon}^{1/\epsilon} g(t,x)\mathrm{d}G(x)\}$$

is an i.d.c.f. By Lebesgue's dominated convergence theorem (LDCT) we got

$$f_1^{(\epsilon)}(t) \to I_1(t) = \exp\{\int_{x>0} g(t,x)\mathrm{d}G(x)\},$$

again, by Theorem 2.1.7 or 2.1.11 we can see that $I_1(t)$ is also a i.d.c.f. Similarly, if we consider the integral of $g(t,x)$ on $(-\epsilon, -\frac{1}{\epsilon})$ we can prove that

$$I_2(t) = \exp\{\int_{x<0} g(t,x)\mathrm{d}G(x)\}$$

is also an i.d.c.f. Now we write

$$e^{\psi(t)} = I_1 \times I_2 \times e^{\mathrm{i}\gamma t} \times (-\frac{t^2}{2})[G(+0) - G(-0)].$$

① Take $0<b\leq 1$ such that for any $|y|\leq b$, $|A(y)-\frac{1}{3!}|<\frac{1}{4!}$, then $A(y)>\frac{1}{3!}-\frac{1}{4!}>0$.
Then for any $|y|>b$, $A(y)=(1-\frac{\sin y}{y})(1+\frac{1}{y^2})\geq 1-\frac{1}{b}>0 \Rightarrow \exists C_1>0$, s.t. $A(y)>C_1$.

② If $|y|\geq b$, $|A(y)|=\left|1-\sin y/y\right|\left|\frac{1+y^2}{y^2}\right|\leq\left|1+\frac{1}{b}\right|\cdot\left|\frac{2|y|}{y^2}\right|=\left|1+\frac{1}{b}\right|\cdot 2\frac{1}{|y|}\leq\frac{2}{b}|1+b|$.
If $|y|\leq b$, $|A(y)|=\left|1-(y-\frac{y^3}{3!}+o(y^4))/y\right|\left|\frac{1+y^2}{y^2}\right|=\left|(1+y^2)/3!+o(y)(1+y^2)\right|\leq\left|(1+b^2)/3!+b(1+b^2)\right|$
$\Rightarrow \exists C_2>0$, s.t. $A(y)\leq C_2$.

Then we can see that $e^{\psi(t)}$ is a i.d.c.f. □

Our goal is to prove the sufficiency of Theorem 2.1.13. Before that we assume $G(-\infty)=0$ and define

$$\Lambda(x)=\int_{-\infty}^{x}A(y)\mathrm{d}G(y), \tag{2.1.1}$$

where

$$A(y)=\begin{cases}\left(1-\frac{\sin y}{y}\right)\frac{1+y^2}{y^2}, & if\ y\neq 0;\\ \frac{1}{3!}, & if\ y=0.\end{cases}$$

and

$$\lambda(t)=\psi(t)-\int_0^1\frac{\psi(t+h)-\psi(t-h)}{2}\mathrm{d}h. \tag{2.1.2}$$

Then we have following observations:

Lemma 2.1.14. $\lambda(x)/\Lambda(+\infty)$ is the characteristic function of $\Lambda(x)/\Lambda(+\infty)$.

Proof. **Step 1. $\Lambda(x)/\Lambda(+\infty)$ is a distribution function.** By the definition of $A(y)$, we can easily see that $A(y)$ is a continuous function, and there exists $0<C_1<C_2$ such that

$$C_1\leq A(y)\leq C_2.$$

Thus, $\Lambda(x)$ is well-defined and non-decreasing. By the boundeness and left-continuity of G, we can see that $\Lambda(x)$ is also a bounded and left continuous function. Hence, $\Lambda(x)/\Lambda(+\infty)$ is a distribution function.

Step 2. Construction of $\mu_\Lambda \Rightarrow \mu_\Lambda\ll\mu_G\Rightarrow\frac{\mathrm{d}\mu_\Lambda}{\mathrm{d}\mu_G}=(1-\frac{\sin x}{x})\frac{1+x^2}{x^2}$.

(1) Let $\mathcal{S}=\{(a,b]:a,\in\mathbb{R}\}$ and define set function $\mu_\Lambda((a,b])=\Lambda(b)-\Lambda(a)$. It is easy to verify that $\mathcal{S}$ is a semiring (closed under finite intersection + difference) and μ_Λ is a premeasure (finite additivity + countably monotone, see pp.437, Royden and Fitzpatrick). Then by Caratheodory-Hahn Theorme, we can extend μ_Λ onto $\sigma(\mathcal{S})$, and the extension of μ_Λ, which is still denoted by μ_Λ, is the corresponding measure on $\sigma(\mathcal{S})$. Moremover, by the definition of $\mathcal{S}$, we have $\sigma(\mathcal{S})=\mathcal{B}$, where $\mathcal{B}$ denotes the Borel $\sigma-$algebra.

(2) We only need to prove: for any $B\in\mathcal{B}$

$$\mu_\Lambda(B)=\int_B A(y)\mathrm{d}\mu_G(y). \tag{2.1.3}$$

By transfinite induction we can prove that to prove certain facts about Borel sets, it is sufficient to prove it for open sets. (The deail of the proof can be found here.) Thus, by the definition of $\mu_\Lambda((a,b])$, we obtain (2.1.3).

(3) By the conclusion in (2), we have $\frac{\mathrm{d}\mu_\Lambda}{\mathrm{d}\mu_G}=(1-\frac{\sin x}{x})\frac{1+x^2}{x^2}$. (the uniqueness of R-N derivative)

Step 3. We take $\psi(t)$ into (2.1.2), then

$$\begin{aligned}\lambda(t)&=\psi(t)-\int_0^1\frac{\psi(t+h)-\psi(t-h)}{2}\mathrm{d}h\\&=\int_0^1\int_{-\infty}^{+\infty}e^{itx}(1-cos(hx))\frac{1+x^2}{x^2}\mathrm{d}G(x)\mathrm{d}h\end{aligned}$$

$$= \int_{-\infty}^{+\infty} \int_{0}^{1} e^{itx}(1 - cos(hx)) \frac{1+x^2}{x^2} \mathrm{d}h \mathrm{d}G(x) \text{ (Boundedness (Measurability) + Fubini's Theorem)}$$
$$= \int_{-\infty}^{+\infty} e^{itx}(1 - \frac{\sin(x)}{x}) \frac{1+x^2}{x^2} \mathrm{d}G(x)$$
$$= \int_{-\infty}^{+\infty} e^{itx} \mathrm{d}\Lambda(x).$$

□

To proof the other side of Theorem 2.1.13 we need the following two lemmas.

Lemma 2.1.15. There is a one-to-one corresponding between $\psi(t)$ and (γ, G).

Proof. It is obvious that each pair of (γ, G) will determine a $\psi(t)$. We will consider the other side. The following chain holds:

$$\psi(t) \Rightarrow \lambda(t) \Rightarrow \Lambda(x),$$

then we only need to prove that $\Lambda(x)$ can determine a unique $G(x)$

Step 1. $\mu_G \ll \mu_\Lambda$. By (2.1.3) and the fact that $A(y)$ is a positive function, we can see that for any measurable set A such that $\mu_\Lambda(A) = 0$, it must follow that $\mu_G(A) = 0$.

Step 2. $\frac{\mathrm{d}\mu_G}{\mathrm{d}\mu_\Lambda} = (1 - \frac{\sin x}{x})^{-1} \frac{x^2}{1+x^2}$ By the chain rule, the conclusion can be obtained immediately.

Step 3. By Step 2. we have

$$\left(1 - \frac{\sin x}{x}\right)^{-1} \frac{x^2}{1+x^2} \mathrm{d}\mu_\Lambda = \mathrm{d}\mu_G,$$

then the μ_G can be deduced as

$$\mu_G(A) = \int_A \left(1 - \frac{\sin x}{x}\right)^{-1} \frac{x^2}{1+x^2} \mathrm{d}\mu_\Lambda, \text{ for any } A \in \mathcal{B}.$$

It can be verified that

$$G(x) = \mu_G((-\infty, x)) = \int_{-\infty}^{x} \left(1 - \frac{\sin x}{x}\right)^{-1} \frac{x^2}{1+x^2} \mathrm{d}\mu_\Lambda, \tag{2.1.4}$$

is a bounded, non-decreasing, left continuous function with zero value at $-\infty$. □

Remark 2.1.16. From now on, we use $\psi = (\gamma, G)$ to denote the $\psi(t)$ that determined by (γ, G) or the (γ, G) that determined by $\psi(t)$.

Lemma 2.1.17. Let

$$\psi_n(t) = \mathrm{i}\gamma_n t + \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{1+x^2}{x^2} \mathrm{d}G_n(x),$$

where $\gamma_n > 0$, $G_n(x)$ is a non-decreasing bonded and left continous function, $n \in \mathbb{N}^*$. Then we have

1. if $\gamma_n \to \gamma$ and $G_n \xrightarrow{\mathscr{D}} G$, then we have

$$\psi_n(t) \to \psi(t) = (\gamma, G).$$

2. If $\psi_n(t) \to \psi(t)$ and $\psi(t)$ is continous at $t = 0$. Then there exist a constant γ and a a non-decreasing bonded left continous function G, such that $\gamma_n \to \gamma$, $G_n \xrightarrow{\mathscr{D}} G$ and $\psi = (\gamma, G)$.

Proof. We want to show: the convergence of $\psi_n(t)$ implies the convergence of γ_n and G_n. To do this we only neet to follow the following chain:

$$\boxed{\begin{array}{ccccc} \psi_n(t) & \lambda_n(t) & \Lambda_n(t) & G_n(t) & \gamma_n \\ \downarrow\overset{①}{\Longrightarrow} & \downarrow\overset{②}{\Longrightarrow} & \overset{d}{\downarrow}\overset{③}{\Longrightarrow} & \downarrow\overset{④}{\Longrightarrow} & \downarrow \\ \psi(t) & \lambda(t) & \Lambda(t) & G(t) & \gamma \end{array}} \tag{2.1.5}$$

By the unifom convergence of $\psi_n(t)$ arrow ① can be easily seen and $\lambda(t) = \psi(t) - \int_0^1 \frac{\psi(t+h)-\psi(t-h)}{2}\mathrm{d}h$. Note the corresponding $\Lambda_n(x)$ and $\Lambda(x)$ w.r.t $\lambda_n(t)$ and $\lambda(t)$, to prove arrow ②, we need to prove $\lambda_n(t)/\Lambda_n(+\infty) \to \lambda(t)/\Lambda(+\infty)$. Since $\lambda_n(-\infty) = \lambda(-\infty) = 0$, $\lambda_n(0) \to \lambda(0)$ and

$$\lambda_n(0) = \int_{-\infty}^{\infty} \mathrm{d}\Lambda_n(x), \quad \lambda(0) = \int_{-\infty}^{\infty} \mathrm{d}\Lambda(x),$$

we have $\Lambda_n(\infty) \to \Lambda(\infty)$. It follows that $\Lambda_n \xrightarrow{d} \Lambda$. Combine this fact with (2.1.4) and by the definition of weak convergence, we can obtain ④. Thus we have

$$\mathrm{i}\gamma_n t \to \psi(t) - \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) \frac{1+x^2}{x^2} \mathrm{d}G(x).$$

Finally, the conclusion in 1. tells us $\psi = (\gamma, G)$. □

Theorem 2.1.18. $f(t)$ is a i.d.c.f. if and only if there exist a constant γ and a non-decreasing bounded left-continous function $G(x)$ such that

$$f(t) = \exp\left\{i\gamma t + \int_{-\infty}^{\infty} g(t,x)\mathrm{d}G(x)\right\}.$$

Proof. By the proof of Theorem 2.1.11 we have

$$\mathrm{Log} f(t) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} n(e^{itx} - 1)\mathrm{d}F_n(x), \tag{2.1.6}$$

where $F_n(x)$ is the corresponding distribution function of $f(t)^{\frac{1}{n}}$. Then

$$\begin{aligned} (2.1.6) &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} n(e^{\mathrm{i}tx} - 1 + \frac{itx}{1+x^2} - \frac{itx}{1+x^2})\mathrm{d}F_n(x) \\ &= \lim_{n\to\infty} \left[\mathrm{i}\int_{-\infty}^{+\infty} \frac{nx}{1+x^2}\mathrm{d}F_n(x)t + \int_{-\infty}^{+\infty} n(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2})\mathrm{d}F_n(x)\right] \\ &= \lim_{n\to\infty} \left[\mathrm{i}\gamma_n t + \int_{-\infty}^{+\infty} (e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2})\frac{1+x^2}{x^2}\mathrm{d}G_n(x)\right], \end{aligned}$$

where $\gamma_n = \int_{-\infty}^{+\infty} \frac{nx}{1+x^2} \mathrm{d}F_n(x)$ and

$$\boxed{G_n(x) = \int_{-\infty}^{x} \frac{ny^2}{1+y^2} \mathrm{d}F_n(y)}. \tag{2.1.7}$$

Thus by the second result in Lemma 2.1.17 we proved

$$\operatorname{Log} f(t) = \lim_{n\to\infty} \psi_n(t) = \psi(t) = (\gamma, G),$$

where $\gamma = \lim_{n\to\infty} \gamma_n$ and $G_n \xrightarrow{\mathscr{D}} G$. Thus $f(t) = e^{\psi(t)}$. □

2.2 Asymptotic Distribution for the Sum of Independent Random Variables.

2.2.1 Goal.

We consider a broad generalization of the central limit theorem. Consider the following sequence of random variables:

$$\begin{array}{cccc} X_{11}, & X_{12}, & \ldots, & X_{1K_1}, \\ X_{21}, & X_{22}, & \ldots, & X_{2K_2}, \\ & \cdots & \cdots & \\ X_{n1}, & X_{n2}, & \ldots, & X_{nK_n}, \\ & \cdots & \cdots & \end{array} \tag{2.2.1}$$

where the elements within each rows are independent. Let $S_n = \sum_{k=1}^{k_n} X_{nk}$, then we want to consider the limiting distribution of S_n as n goes to infinity. Note that the question reduce to normal central limiting theorem as there is only one row.

2.2.2 Uniformly Asymptotically Negligible Condition.

Note that if the problem is assumption-free, the limiting distribution of S_n could be any well-defined distribution function. In fact, for any $n > 0$, if let X_{n1} follows some distribution $F(x)$ whlie other r.v. equal to zero, then S_n will asymptotically distributed as $F(x)$. This is a trivil situation, thus we need some reasonable condition on the random variables such that the limiting process of S_n will not be impacted significantly by few random variables.

Definition 2.2.1. For random varible sequences (2.2.1), if for any $\epsilon > 0$

$$\lim_{n\to\infty} \max_{1\le k\le n} P\left(|X_{nk}| > \varepsilon\right) = 0,$$

then we say $\{X_{nk}\}$ follows the **uniformly asymptotically negligible (u.a.n.) condition**.

Proposition 2.2.2. Following statements are equvilent:

- $\{X_{nk}\}$ follows the u.a.n. condition;

- $\max_k \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mathrm{d}F_{nk}(x) \to 0$.
- let $b > 0$, then for any $|t| \le b$

$$\max_k |f_{nk}(t) - 1| \Rightarrow 0.$$

Proof. (ii) $\Rightarrow$ (i). Note that

$$\max_k \mathrm{P}(|X_{nk}| \geq \epsilon) = \max_k \int_{|x|\geq\epsilon} \mathrm{d}F_{nk}(x) \leq \max_k \frac{1+\epsilon^2}{\epsilon^2} \int_{|x|\geq\epsilon} \frac{x^2}{1+x^2} \mathrm{d}F_{nk}(x).$$

(i) $\Rightarrow$ (iii). Note that for any $\epsilon > 0$

$$\begin{aligned}
\max_k |f_{nk}(t) - 1| &\leq \max_k \int |e^{itx} - 1| \mathrm{d}F_{nk}(x) \\
&\leq \int_{|x|\leq\epsilon} |itx + o(x)| \mathrm{d}F_{nk}(x) + 2\max_k \mathrm{P}(|X_{nk}| \geq \epsilon) \\
&\leq b\epsilon + 2\max_k \mathrm{P}(|X_{nk}| \geq \epsilon).
\end{aligned}$$

(iii) $\Rightarrow$ (ii). Note that for any d.f. $F(x)$ and corresponding c.f. $f(x)$ we have

$$\boxed{\int_0^{\infty} e^{-t}(1 - \Re(f(t)))\mathrm{d}t = \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mathrm{d}F_{nk}(x)}. \tag{2.2.2}$$

Then we have for any $T > 0$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mathrm{d}F_{nk}(x) &\leq \int_0^{\infty} |e^{-t}(1 - \Re(f_{nk}(t)))| \mathrm{d}t \\
&\leq \int_0^{T} \max_k |1 - f_{nk}(t)| \mathrm{d}t + 2e^{-T}.
\end{aligned}$$

□

Proposition 2.2.3. If $\{X_{nk}\}$ follows the u.a.n. condition, then for any $\tau > 0$, $r > 0$ we have $\max_k |mX_{nk}| \to 0$ and $\max_k \int_{|X|<\tau} |x|^r \mathrm{d}F_{nk}(x)$.

Proof. Note for any $\epsilon > 0$

$$1 = 1 - \lim_{n\to\infty} \max_{1\leq k\leq n} P(|X_{nk}| \geq \varepsilon) = \lim_{n\to\infty} \min_{1\leq k\leq n} P(|X_{nk}| < \varepsilon).$$

Thus for n large sufficently, it follows

$$\min_{1\leq k\leq n} P(|X_{nk}| < \varepsilon) > 1/2.$$

By the definition of medium number we have $\max_k |mX_{nk}| < \epsilon$. Also, for $0 < \epsilon < \tau$

$$\max_k \int_{|x|<\tau} |x|^r \mathrm{d}F_{nk}(x) \leq \max_k \int_{|x|<\epsilon} |x|^r \mathrm{d}F_{nk}(x) + \max_k \int_{\epsilon\leq|x|<\tau} |x|^r \mathrm{d}F_{nk}(x)$$

$$\leq \epsilon^r + \tau^r \max_k P\left(|X_{nk}| \geq \varepsilon\right).$$

□

2.2.3 Fundamental Theorem.

Theorem 2.2.4. Suppose $\{X_{nk}\}$ follows u.a.n. condition, then there is an one-to-one corresponding betwee the limiting distribution of S_n and i.d. family.

Proof. **Part 1. First we prove that for any i.d.c.f. $F(x)$ is a limiting distribution function of some S_n.**

Suppose that the c.f. of $F(x)$ is $f(t)$ and by the definition of i.d.d.f. we know that for any k_n, there exist a c.f. $f_{k_n}(t)$ such that $f(t) = [f_{k_n}(t)]^{k_n}$. We draw random variables X_{nk} such that the c.f. of X_{nk} is $f_{k_n}(t)$, $k = 1, 2, \ldots, k_n$. Then the c.f. of S_n is $\prod_{k=1}^{k_n} f_{k_n}(t) = (f_{k_n}(t))^{k_n} = f(t)$. If we take limit on both sides, we can see that the c.f. of S_n will tend to $f(t)$ which is a i.d.c.f. Now we only need to verify if the $\{X_{nk}\}$ we defined above will follow the u.a.n. condition. To do this, we need we only

Part2.

First we introduce some notations: for any $0 < \tau < \infty$,

$$a_{nk} = \int_{|x|<\tau} x \mathrm{d}F_{nk}(x), \quad \overline{F}_{nk}(x) = F_{nk}\left(x + a_{nk}\right), \quad \overline{f}_{nk}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}\overline{F}_{nk}(x).$$

Lemma 2.2.5. Suppose that $\{X_{nk}\}$ follows the u.a.n. condition, then $\{\overline{X}_{nk}\}$, where $\overline{X}_{nk} = X_{nk} - a_{nk}$ will also follow the u.a.n. condition.

Proof. From Proposition 2.2.3

$$\max_k |a_{nk}| \leq \max_k \int_{|x|<\tau} |x| \mathrm{d}F_{nk}(x) \to 0.$$

For any $\epsilon > 0$ and n large enough, we have

$$\max_k P\left(\left|\overline{X}_{nk}\right| \geqslant \varepsilon\right) \leqslant \max_k P\left(|X_{nk}| + |a_{nk}| \geqslant \varepsilon\right) \leqslant \max_k P\left(|X_{nk}| \geqslant \frac{\varepsilon}{2}\right).$$

□

□

Bibliography

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