The Foundations for Probability Limiting Theory by Zhengyan Lin & Chuanrong Lu & Zhonggen Su

Lecture Notes for Probability Limiting Theory Seminar

(2020 Spring, SUSTech)

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Lastest Updated: January 30, 2020, 15:08

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Lecture 1

Preliminary

1.1 Basis for Martingale

Let $(\Omega, \mathscr{A}, P)$ is a probability space and X is an integrable random variable, $\mathscr{G}$ G is a subalgebra of $\mathscr{A}$.

Definition 1.1.1. For an integrable random variable X, if there exist a random varibale Y such that

- Y is $\mathscr{G}-$ measurable;
- for any $A \in \mathscr{G}$ we have

$$\int_A Y dP = \int_A X dP,$$

then we say Y is the conditional expectation of X given $\sigma-$algebra $\mathscr{G}$. We can write it as $\mathsf{E}[X|\mathscr{G}]$.

Remark 1.1.2. The conditional expectation is unique in the sense of *a.s.*, see Jacod and Protter [2012].

Remark 1.1.3.

- For any real number c_1 and c_2, we have

$$\mathsf{E}\,(c_1 X_1 + c_2 X_2|\mathscr{G}) = c_1 \mathsf{E}\,(X_1|\mathscr{G}) + c_2 \mathsf{E}\,(X_2|\mathscr{G}) \quad a.s.$$

- Let Y is $\mathscr{G}-$measurable, $\mathsf{E}|XY| < \infty, \mathsf{E}|X| < \infty$, then we have

$$E(XY|\mathscr{G}) = YE(X|\mathscr{F}) \quad a.s.$$

If $Y = 1_B(\omega)$, $\forall B$ is $\mathscr{G}$-measurable.
$\forall A$ is $\mathscr{G}$-measurable, we have
$\int_A E(X 1_B|\mathscr{G})dp = \int_A X 1_B \, dp$.
$= \int_{A\cap B} X \, dp$.
$= \int_{A\cap B} E(X|\mathscr{G})dp$.
$= \int_A 1_B \cdot E(X|\mathscr{G}) dp$.

$\forall A$ is $\mathscr{G}_1$- measurable

- Let $\mathscr{G}_1 \subset \mathscr{G}_2 \subset \mathscr{A}$, then we have

$\int_A E[E(X|\mathscr{G}_2)|\mathscr{G}_1]dp = \int_A E(X|\mathscr{G}_2)\, dp$
$= \int_A X dp = \int_A E(X|\mathscr{G}_1)dp$.

$$E\left[E\left(X|\mathscr{G}_2\right)|\mathscr{G}_1\right] = E\left(X|\mathscr{G}_1\right) = E\left[E\left(X|\mathscr{G}_1\right)|\mathscr{G}_2\right] \quad a.s.$$

Note that when $\mathscr{G}_1 = \emptyset$ the first equality will turn to the normal Adam's law.

因为 $\mathscr{G}_1 \subset \mathscr{G}_2$，故 A是 $\mathscr{G}_2$-measurable.

Let $\{X_n\}$ is a sequence of independent random variables with zero mean and $S_n = \sum_{j=1}^{n} X_j$. Then we have

$$\begin{aligned} E\left(S_{n+1}|X_1, \cdots, X_n\right) &= E\left(S_n + X_{n+1}|X_1, \cdots, X_n\right) \\ &= S_n + EX_{n+1} = S_n \quad a.s., \end{aligned}$$

which indicates that the conditional expectation of S_{n+1} only relates to the previous n random variables.

discrete-time

$\mathscr{A}_n \subset \mathscr{A}_{n+1} \cdots \subset \mathscr{A}$

Definition 1.1.4. Let $\{\mathscr{A}_n\}$ is an increasing sequence of sub $\sigma-$algebra of $\mathscr{A}$. We say a seqence of random variables $\{S_n\}$ is a **martingale** if

(i) S_n is $\mathscr{A}_n-$measurable;

(ii) $\mathsf{E}[S_n] < \infty$;

(iii) for $m < n$, $\mathsf{E}(S_n|\mathscr{A}_m) = S_m$ a.s.,

if the equality in (iii) turns out to be $\leq$ $(\geq)$ then we say $\{S_n\}$ is a supermartingale (submartingale).

Example 1.1.1. See pp.24.

Example 1.1.2. See pp.24.

Lemma 1.1.5. (i) $\{S_n\}$ is a submartingale (supermartingale), Φ is a non-decreasing convex (concave) real function. if $\mathsf{E}|\Phi(S_n)| < \infty$, then $\{\Phi(S_n)\}$ is also a submartingale (supermartingale).

(ii) $\{S_n\}$ is a martingale, Φ is a real convex function. if $\mathsf{E}|\Phi(S_n)| < \infty$, then $\{\Phi(S_n)\}$ is a submartingale.

why convex function is measurable: $\forall a \in \mathbb{R}$, we consider $A = \{x : \Phi(x) < a\}$ Let $x_1 = \sup\{x : \Phi(x) < a\}$, $x_2 = \inf\{x : \Phi(x) < a\}$. $\forall \varepsilon > 0$, $\theta \in [0,1]$ we have $\Phi(\theta(x_1-\varepsilon) + (1-\theta)(x_2+\varepsilon)) \leq \theta\Phi(x_1-\varepsilon) + (1-\theta)\Phi(x_2+\varepsilon) \leq a$. Thus, $\forall \varepsilon > 0$ $[x_2+\varepsilon, x_1-\varepsilon] \subset A$. $\Downarrow$ $(x_2, x_1) \subset A$. By the definition of x_1 and x_2, we have the otherside naturally.

Proof. We only prove the submartingale case in (i).

- (Measurability.) Since Φ is a convex function on real line, we can see that $\{x : f(x) < a\}$ will be either an empty set or an inteval for any $a \in \mathbb{R}$. And the mixture of measurable function will be mearable function. Thus $\{\Phi(S_n)\}$ is $\mathscr{A}_n-$measurable.
- (Inequality.) By Jensen's inequality, we have

$$\mathsf{E}[\Phi(S_n)|\mathscr{A}_{n-1}] \geqslant \Phi[\mathsf{E}(S_n|\mathscr{A}_{n-1})] \geqslant \Phi(S_{n-1}) \quad a.s..$$

□

Example 1.1.3. $\Phi(x) = x^+$ and $\Phi(x) = |x|^p$.

Let $\{\mathscr{A}_n\}$ is an increasing sequence of sub $\sigma-$algebra of $\mathscr{A}$ and $N_\infty = \{1, 2, \ldots, \infty\}$.

Definition 1.1.6. Let α is a random variable:

$$\alpha : \Omega \to N_\infty.$$

If α satisfies

$$\{\alpha = n\} \in \mathscr{A}_n, \quad n \geqslant 1$$

or equvilently

$$\{\alpha \leqslant n\} \in \mathscr{A}_n, \quad n \geqslant 1 \tag{1.1.1}$$

then we say α is a **stopping time**.

Remark 1.1.7. It means that the "decision" of whether to stop at time n must be based only on the information present at time n, not on any future info.

Definition 1.1.8. Let α is a stopping time and $\mathscr{A}_\infty = \sigma\left(\{\mathscr{A}_n, n \geqslant 1\}\right)$. We define

$$\mathscr{A}_\alpha = \{E : E \in \mathscr{A}_\infty,\ E \cap \{\alpha = n\} \in \mathscr{A}_n, n \geqslant 1\}$$

as the stopping time $\sigma-$algebra $\mathscr{A}_\alpha$.

Note: $E^c \cap \{\alpha = n\} = \{\alpha = n\} \setminus (E \cap \{\alpha = n\})$.

By checking $\mathscr{A}_\alpha$ is closed under complement of sets and union of countable sets we can see that this definition is reasonable.

Lemma 1.1.9 (Doob's). Let $\{S_n\}$ is a martingale (submartingale) and α, β are two bounded stopping time. If $\alpha \leqslant \beta$, then we have

$$E\left(S_\beta | \mathscr{A}_\alpha\right) = S_\alpha \left(\geqslant S_\alpha\right) \quad a.s.$$

Proof. We only prove the case of submartingale. To compelte our proof, we need to verify

- integrability of S_β;
- S_α is $\mathscr{A}_\alpha-$measurable; → 等式要有意义的.
- Inequality.

Assume α and β can be bounded by m. We have $S_\beta = \sum_{n=1}^m S_n \mathbb{1}\{\beta = n\}$ and $\{S_n\}$ is a martingale. It follows that S_β is integrable. Let B is a Borel set on real line. Since

$$\{S_\alpha \in B\} = \bigcup_{k=1}^{m} \cancel{{}_{k \in N_\infty}} \left(\overset{\in \mathscr{A}_k}{\{S_k \in B\}} \cap \overset{\in \mathscr{A}_k}{\{\alpha = k\}}\right) \in \mathscr{A}_\infty$$

$$\{S_\alpha \in B\} \cap \{\alpha = n\} = \overset{\in \mathscr{A}_n}{\{S_n \in B\}} \cap \overset{\in \mathscr{A}_n.}{\{\alpha = n\}} \in \mathscr{A}_n.$$

Thus, S_α is $\mathscr{A}_\alpha-$measurable. To prove the inequality we only need to prove that for any $\Lambda \in \mathscr{A}_\alpha$

$$\int_\Lambda S_\alpha dP \leqslant \int_\Lambda \mathsf{E}\left(S_\beta | \mathscr{A}_\alpha\right) dP = \int_\Lambda S_\beta dP. \tag{1.1.2}$$

What we want to do is to fix the index in (1.1.2) and change problem into fixed α and fixed β. Thus we define $\Lambda_n = \Lambda \cap \{\alpha = n\}$ to fix α and $\Lambda_n \cap \{\beta > k\}$ to fixed β, here $k \geqslant j$. The details of proof can be found in pp.25. □

Definition 1.1.10. (**Continuous case.**) See pp.26.

Lecture 2

Infinite Divisible Distribution Function

2.1 Definition of Infinite Divisible Distribution Function

Definition 2.1.1. (**From the perspective of c.f.**) We say a characteristic function $f(t)$ is **infinite divisible** (i.d.) iff. $\forall n \in \mathbb{Z}^*$, there exist some characteristic function $f_n(t)$ such that

$$f(t) = [f_n(t)]^n.$$

And the corresponding distribution function F is called infinite divisible distribution function.

Definition 2.1.2. (**From the perspective of r.v.**) Let X is a random variable, we say **the distribution of X is infinite divisible** iff. $\forall n \in \mathbb{Z}$, there exist an i.i.d random variable sequence $\{X_k^{(n)}\}_{n \geqslant k \geqslant 1}$ such that

$$X = X_1^{(n)} + X_2^{(n)} + \dots X_n^{(n)}.$$

Definition 2.1.3. (**From the perspective of d.f.**) We say a distribution function F is a **infinite divisible distribution function** iff. $\forall n \in \mathbb{Z}$, there exist another distribution function F_n such that

$$F = F_n^{*n}.$$

Remark 2.1.4. From Definition (2.1.1) we have

$$[f(t)]^{\frac{1}{n}} = f_n(t).$$

Thus, for i.d.c.f. $f(t)$, $[f(t)]^{\frac{1}{n}}$ will always be a c.f. (对于i.d.的c.f.，其开$1/n$次方后还是c.f.)

Example 2.1.1. See pp.30.

2.2 Properties of i.d.c.f.

Theorem 2.2.1. (i) I.d.c.f. are closed under finite multiplications; (ii) if f is a i.d.c.f., so does $|f|$.

Theorem 2.2.2. If $f(t)$ is a i.d.c.f., then $f(t) \neq 0$.

Proof. It's hard to consider the zeros of a complex function, thus we consider $g = |f|^2$. We assme $f(t) = [f_n(t)]^n$, then $g = |f|^2 = |f_n|^{2n} \triangleq g_n^n$, where $g_n = |f_n|^2$ is also a c.f. Here we transform a problem of complex function into a problem of real function. We define

$|f_n(t)|^2 = f_n(t) \cdot f_n(-t)$.

$$h(t) \triangleq \lim_{n\to\infty} g_n = \begin{cases} 1, & if\ g(t) > 0, \\ 0, & if\ g(t) = 0. \end{cases}$$

we only need to show $h \equiv 1$. Note that $g(0) = 1$ which indicates that $h(0) = 1$. Since g_ns are continous, thus $h(t)$ will also be continous. Thus, h is nowhere 0.

因为 c.f. is uniform conti. □

Theorem 2.2.3. Let $\{f^{(m)}(t)\}$ is a sequence of i.d.c.f. If there exist a c.f. $f(t)$ such that

$$f^{(m)}(t) \to f(t),$$

then $f(t)$ is a i.d.c.f.

condition: ① $f^{(m)}(t) = (f_n^{(m)}(t))^n$, where $f_n^{(m)}(t) = (f^{(m)}(t))^{\frac{1}{n}}$. And $f_n^{(m)}(t) \to (f(t))^{\frac{1}{n}}$ $(m \to +\infty)$. ② $f(t)$ is conti at 0 & $f_n^{(m)}(t)$ are c.f. $\Rightarrow (f(t))^{\frac{1}{n}}$ is c.f. $\Downarrow$ $f(t)$ is i.d.c.f.

$(f^{(m)}(t))^{\frac{1}{n}} = \exp\{\frac{1}{n}\log f^{(m)}(t)\}$

Proof. We only need to note that $f^{\frac{1}{n}}(t)$ is a c.f. and Levy's Continuity theorem. □

Remark 2.2.4. Note that this theorem does not tell us that i.d.c.f. are closed in the sense of limitation because we need the limiting function is a c.f.

Definition 2.2.5. If c.f. $f(t)$ satisfies

$$f(t) = \exp\{i\alpha t + \lambda(e^{i\beta t} - 1)\},$$

where $\lambda \geq 0$ and α, β are real number, then we say $f(t)$ is a **poisson type c.f.**

Theorem 2.2.6. $f(t)$ is an i.d.c.f. iff. it can be ~~write~~ written as the limitation of the product of finite poisson type c.f.

任意极限均可.

Proof. For the sufficient part we need to prove $f(t)$ is a c.f. and it is i.d. We assume

$$f(t) = \left[\lim \prod_{k=1}^{n} \exp\{i\frac{\alpha_k}{m}t + \frac{\lambda_k}{m}(e^{i\beta_k t} - 1)\}\right]^m \qquad f(t) \underset{\triangle}{=} \lim \prod_{k=1}^{n} \exp\{i\alpha_k t + \lambda_k(e^{i\beta_k t} - 1)\}.$$

从表达式可看出

Note that the finite multiplication of c.f. is also a c.f. and $f(t)$ is continous at 0, we conlude that $f(t)$ is a c.f. And it's easy to see $f(t)$ is i.d. The proof of necessary condition can be found at pp.31. □

Remark 2.2.7. This theorem implies that the limitation of a sequenc of i.d.c.f. must be a c.f. Keep this result in mind, and combine it with Theorem 2.2.3 indicate that the i.d.c.f. class is closed under limitation.

[illegible]一个都可以写成[illegible]极限的形式.

Example 2.2.1. See pp.32.

2.3 Levy-Khinchine Representation

Let γ is a real constant and $G(x)$ is a non-decreasing bounded and left continous function. We define

$$\psi(t) = i\gamma t + \int_{-\infty}^{\infty} g(t, x) \mathrm{d}G(x),$$

where

$$g(t,x) = \begin{cases} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}, & if\ x \neq 0, \\ -\frac{t^2}{2}, & if\ x = 0. \end{cases} \to -\frac{t^2}{2}\ (x \to 0).$$

Thus it's easy to see $g(t,x)$ is continous on $\mathbb{R}^2$.(Check the limit of $g(t,x)$ at $x = 0$.)

Theorem 2.3.1. $e^{\psi(t)}$ is a i.d.c.f. (e的积分式，用L-S积分的定义证).

Proof. Note that $g(t,x)$ has different form at 0 and other points, we seperate the integral into three differenc parts. For any $1 > \epsilon > 0$, let

$$\epsilon = x_0 < x_1 < \cdots < x_n = 1/\epsilon,\ x_k \le \xi_k < x_{k+1}\ (k = 0,1,\ldots,n-1).$$

We consider

$$\begin{aligned}
&\int_{\varepsilon}^{1/\epsilon} g(t,x)\mathrm{d}G(x) \quad \text{(g(t,x)在x=0和x≠0的表达式不同，故分段写).} \\
&= \int_{\varepsilon}^{1/\epsilon} \left(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right)\frac{1+x^2}{x^2} dG(x) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\mathrm{e}^{\mathrm{i}t\xi_k} - 1 - \frac{\mathrm{i}t\xi_k}{1+\xi_k^2}\right)\frac{1+\xi_k^2}{\xi_k^2}\left[G\left(x_{k+1}\right) - G\left(x_k\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}(\mathrm{i}\frac{\xi_k(1+\xi_k^2)[G(x_{k+1}) - G(x_k)]}{(1+\xi_k^2)\xi_k^2}t + \frac{1+\xi_k^2}{\xi_k^2}\left[G\left(x_{k+1}\right) - G\left(x_k\right)\right](\mathrm{e}^{\mathrm{i}\xi_k t} - 1)).
\end{aligned}$$

Thus, by Theorem 2.2.6 we see that

$$f_1^{(\epsilon)}(t) = \exp\{\int_{\varepsilon}^{1/\epsilon} g(t,x)\mathrm{d}G(x)\}$$

is an i.d.c.f. By Lebesgue's dominated convergence theorem (LDCT) we got

$$f_1^{(\epsilon)}(t) \to I_1(t) = \exp\{\int_{x>0} g(t,x)\mathrm{d}G(x)\}, \quad (\varepsilon \to +\infty).$$

again, by Theorem 2.2.3 or 2.2.6 we can see that $I_1(t)$ is also a i.d.c.f. Similarly, if we consider the integral of $g(t,x)$ on $(-\epsilon, -\frac{1}{\epsilon})$ we can prove that

$$I_2(t) = \exp\{\int_{x<0} g(t,x)\mathrm{d}G(x)\}$$

is also an i.d.c.f. Now we write

$$e^{\psi(t)} = I_1 \times I_2 \times e^{\mathrm{i}\gamma t} \times (-\frac{t^2}{2})[G(+0) - G(-0)].$$

Then we can see that $e^{\psi(t)}$ is a i.d.c.f. $\square$

Our goal is to prove the sufficiency of Theorem 2.3.1. Before that we define

$$\Lambda(x) = \int_{-\infty}^{x} A(y)\mathrm{d}G(y),$$

where

$$A(y) = \begin{cases} \left(1 - \frac{\sin y}{y}\right) \frac{1+y^2}{y^2}, & if\ y \neq 0; \\ \frac{1}{3!}, & if\ y = 0. \end{cases}$$

and

$$\lambda(t) = \psi(t) - \int_0^1 \frac{\psi(t+h) - \psi(t-h)}{2} \mathrm{d}h.$$

Then we have following observation:

- $A(y)$ is continous and bounded. (Because it is bounded at $[\epsilon, \infty]$ and $[0, \epsilon]$.)
- $\Lambda(x)/\Lambda(+\infty)$ is a d.f.

Bibliography

Jean Jacod and Philip Protter. ***Probability essentials***. Springer Science & Business Media, 2012.