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One-factor Hull-White Model Calibration and Analytical Pricing of Interest Rate Caps and Floors

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Abstract

In this thesis, I explore the calibration and application of the one-factor Hull-White model for pricing interest rate derivatives, specifically interest rate caps and floors. The Hull-White model, a widely used framework in financial mathematics, allows for the analytical pricing of these derivatives by modeling the evolution of interest rates over time. I discuss the theoretical foundations of the model, its calibration to market data, and the practical implications of using this model in real-world financial scenarios with Python programming. I also highlight the limitations of the Hull-White model, particularly in its ability to capture the complex dynamics of interest rates and suggest potential areas for further research and improvement.

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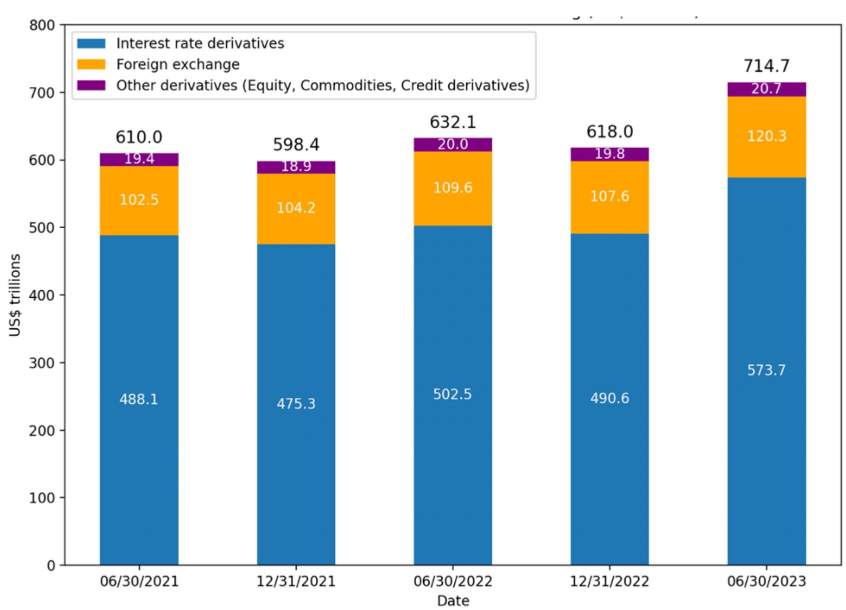
Chapter 1. Introduction

The interest rate derivative market is a critical component of the financial system, providing mechanisms for managing interest rate risk. Instruments such as interest rate swaps, caps, floors, and swaptions are actively traded to hedge against fluctuations in interest rates or to speculate on future movements. These derivatives are utilized by a wide range of participants, including financial institutions, corporations, and governments, to stabilize cash flows and manage the cost of funding. Global interest rate derivatives are predominantly traded over the counter (OTC), meaning the process of trading financial instruments is organized directly between two parties without going through a formal exchange.

The global interest rate derivative OTC market has been experiencing significant growth, driven in part by the recent trend of rising interest rates. This trend has increased the demand for interest rate derivatives as financial institutions and corporations seek to hedge against interest rate risk and capitalize on changing market conditions. Recent data from Bank for International Settlements (BIS) highlights an increase in the notional outstanding amounts, gross market value, and gross credit exposure of OTC derivatives in the first half of 2023, compared to the same period in 2022 (Bank for International Settlements, 2023). This growth has been primarily driven by the expansion in interest rate and foreign exchange (FX) derivatives, coinciding with rising interest rates across major currencies.

Figure 1. Global OTC Derivatives Notional Outstanding (US$ trillions)



Source: Bank for International Settlements (BIS)

Modeling interest rates and accurately pricing interest rate derivatives require sophisticated mathematical frameworks. One such framework is the one-factor Hull-White model, which is an extension of the Vasicek model. It allows for a flexible fit to the initial term structure and incorporates mean-reversion property in interest rates. The Hull-White model's ability to capture the dynamics of interest rates over time, combined with its simplicity, makes it a popular choice for financial analysts and practitioners.

This thesis delves into the mechanics and practical applications of the one-factor Hull-White model in this evolving market landscape. By focusing on the model's ability to price interest rate caps and floors, I aim to provide a comprehensive analysis of its theoretical underpinnings, calibration techniques, and practical implementations. I explore the model's formulation, parameter estimation, and how it can be calibrated to market data. Additionally, I investigate the effectiveness of the Hull-White model in capturing market realities and identify its limitations.

It is important to note that this thesis does not aim for a rigorous mathematical derivation of the model. Instead, the focus is on following the general theoretical flow around the Hull-White model and demonstrating its practical applications using Python programming. This thesis illustrates the application of the Hull-White model in pricing interest rate caps and floors, and compares model output to market data, discussing the implications of any discrepancies observed.

Chapter 2. Interest Rates and Interest Rate Derivatives

2.1. Zero rates and forward rates

Zero rates, also known as spot rates, are the annualized interest rates for zero-coupon bonds, which are bonds that pay no coupons but pay a face value at maturity. Zero rates can be thought of yield-to-maturity on a zero-coupon bond, from which the name "zero" is originated. To understand zero rates, consider a zero-coupon bond that matures at time T. Let $P(0,T)$ denote the price of this bond today ($t = 0$) and the face value of the bond is 1 ($P(T,T) = 1$), then the zero rate $R(0,T)$ for this bond with continously compounding is the rate that satisfies the following condition:

$$P(0,T) = e^{-R(0,T)\cdot\tau(0,T)} \qquad 2.1$$

$\tau(0,T)$ is the time difference in years with a day-count convention between time 0 and T. Solving for $R(0,T)$ gives us:

$$R(0,T) = -\frac{ln\, P(0,T)}{\tau(0,T)} \qquad 2.2$$

Forward rates are annualized interest rates agreed upon today for loans or investments that will occur in the future. The forward rate from time T to time $T + \delta$ at time 0 is denoted by $F(0,T,T+\delta)$.

The above zero rates and forward rates are defined at present ($t = 0$). However, once we move to the perspective of a future time t, we would get $R(t,T)$ for the zero rate with maturity $\tau(0,T)$ and $F(t,T,T+\delta)$ for the forward rate.

There is a close relationship between zero and forward rates, such that if we know one, the other one can be derived. This direct relationship comes from the fact that forward rates are defined as expected future interest rate at any time $t < T$ that makes investors indifferent between buying a zero-coupon bond maturing at time $T + \delta$ and investing with forward rate from T to $T + \delta$ following a zero-coupon bond maturing at time T. This can be written down as:

$$e^{R(t,T+\delta)\cdot\tau(t,T+\delta)} = e^{R(t,T)\cdot\tau(t,T)} \cdot e^{F(t,T,T+\delta)\cdot\tau(T,T+\delta)} \qquad 2.3$$

which simplifies to:

$$R(t,T+\delta)\cdot\tau(t,T+\delta) = R(t,T)\cdot\tau(t,T) + F(t,T,T+\delta)\cdot\tau(T,T+\delta) \quad 2.4$$

and delivers the formula for the forward rate:

$$F(t,T,T+\delta) = \frac{R(t,T+\delta)\cdot\tau(t,T+\delta) - R(t,T)\cdot\tau(t,T)}{\tau(T,T+\delta)} \quad 2.5$$

2.2. Instantaneous short rates and forward rates

In a one-factor short rate model, the target variable that a stochastic process aims to replicate is the instantaneous short rate. The instantaneous short rate, $r(t)$, is the spot rate at which borrowing occurs over an infinitesimally small period from time t. The instantaneous short rate is defined in a period between initiation time t and maturity time $t+\delta$ of the spot rate $R(t,t+\delta)$ when δ goes to zero in the limit:

$$r(t) = \lim_{\delta\to 0} R(t,t+\delta) \quad 2.6$$

Likewise, the instantaneous forward rate is defined in the limit. Given a forward rate from T to $T+\delta$ at time t, $F(t,T,T+\delta)$, the instantaneous forward rate $f(t,T)$ is formulated in the limit where the period between T and $T+\delta$ goes to zero:

$$f(t,T) = \lim_{\delta\to 0} F(t,T,T+\delta) \quad 2.7$$

Intuitively, the instantaneous forward rate can be understood as the instantaneous short rate $r(T)$ from the viewpoint of a future time t. Likewise in zero rates and forward rates, the instantaneous short rates and forward rates are also defined in annualized term.

2.3. Money market accounts, discount factors, and zero-coupon bond (ZCB) prices

Let $B(t)$ be the value of a money market account at time $t \geq 0$ and $B(0) = 1$. The dynamics of this money market account evolves with instantaneous short rate $r(t)$ so that its value at time t is:

$$B(t) = e^{\int_0^t r(s)ds}, \qquad B(0) = 1 \tag{2.8}$$

Next, let us define the discount factor between time t and T as the ratio between the money market account values:

$$D(t,T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r(s)ds} \tag{2.9}$$

This discount factor can be understood as the equivalent amount at time t of one unit of money at time T. When the short rate $r(t)$ is deterministic, the price at time t of a zero-coupon bond (ZCB) that pays a unit of currency at maturity T under no-arbitrage assumption is the same as the discount factor between t and T:

$$P(t,T) = e^{-\int_t^T r(s)ds} = D(t,T) \tag{2.10}$$

However, when the short rate is stochastic, the discount factor also becomes stochastic so the price of a ZCB must be evaluated under a probability measure, since the ZCB prices are theoretically observed values. In this case, the price at time t of a ZCB is expressed as the expectation under the risk-neutral measure Q:

$$P(t,T) = E^Q\left[e^{-\int_t^T r(s)ds} | \mathcal{F}_t\right] = E^Q[D(t,T)| \mathcal{F}_t] \tag{2.11}$$

where $\mathcal{F}_t$ is a filtration that contains all information up to time t.

This formulation accounts for the uncertainty associated with the stochastic evolution of the short rate. By pricing under the risk-neutral measure, we incorporate market expectations into zero-coupon bond (ZCB) pricing by calibrating the above theoretical prices to observed market prices, ensuring the absence of arbitrage opportunities. In this sense, the risk-neutral measure represents the implied probability measure derived from market prices. This simplifies the pricing of derivatives by eliminating risk preferences of individual investors from valuation process and establishing prices that only offer the risk-free rate for future payoffs. Equation 2.11 implies that once the stochastic process of the short rate is known, the price of a ZCB at time t can be calculated accordingly. This, in turn, determines the corresponding zero curves.

The instantaneous forward rate is also related to the price of a ZCB:

$$f(t,T) = -\lim_{\tau(T,T+\delta)\to 0} \frac{\ln P(t,T+\delta) - \ln P(t,T)}{\tau(T,T+\delta)} = -\frac{\partial \ln P(t,T)}{\partial \delta} \qquad 2.12$$

Lastly, notice that even though t or T are assumed to be continuous in the above formulas, in practice we can only observe the data in a finite number of moments. For this reason, interpolation methods are utilized to construct curves of interest rates introduced above, and accordingly ZCB prices at any given t or T. Interpolation methods will be discussed in Chapter 2.7.

2.4. Change of measure

A numeraire is a unit of measurement against which the value of other financial assets or contracts are expressed. Switching from one numeraire to another, thereby adjusting the measure to a new measure associated with the new numeraire, is termed as a change-of-measure. Under risk-neutral measure, we use money market account $B(t)$ as a numeraire.

The future payoff of a derivative on the underlying interest rate at maturity date T is given by $V(T)$, then the price of this derivative at time t under risk-neutral measure Q is:

$$V(t) = E^Q\left[e^{-\int_t^T r(s)ds} \cdot V(T) \mid \mathcal{F}_t\right] = E^Q\left[\frac{B(t)}{B(T)} \cdot V(T) \mid \mathcal{F}_t\right] \qquad 2.13$$

When the underlying interest rates are modeled as stochastic processes, both the discount factor $e^{-\int_t^T r(s)ds}$ and the payoff $V(T)$ are random variables. Therefore, calculating the expectation involves considering the joint distribution of the discount factor and the future payoff under the risk-neutral measure, which adds complexity to the pricing process.

But using the price of a ZCB, $P(t,T)$, as a numeraire, derivative pricing process gets more tractable and simpler, which even leads to analytical formulas for the price of interest rate caps and floors under a few stochastic interest rate models, including the Hull-White model. The corresponding measure to this T maturity date ZCB numeraire is called a T-forward measure, or simply a forward measure.

Forward measure Q^T is an equivalent martingale measure on which the payoff of a derivative $V(t)$ in the numeraire $P(t,T)$ is a martingale:

$$\frac{V(t)}{P(t,T)} = E^{Q^T}\left[\frac{V(T)}{P(T,T)} \mid \mathcal{F}_t\right] = E^{Q^T}[V(T) \mid \mathcal{F}_t] \qquad 2.14$$

The second equality holds from the fact that the ZCB pays one unit currency at its maturity date T, so $P(T,T) = 1$. From the second equality we can derive the tractable relationship between $V(t)$ and $V(T)$:

$$V(t) = P(t,T) \cdot E^{Q^T}[V(T) \mid \mathcal{F}_t] = E^{Q}\left[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\right] \cdot E^{Q^T}[V(T) \mid \mathcal{F}_t] \qquad 2.15$$

The pricing formula becomes much simpler so that we only need to compute the expectation of the future payoff under the forward measure and then multiply it by the price of a ZCB at time t calculated in risk-neutral measure. As we will see in a subsequent chapter, the Hull-White model provides analytical formulas for both ZCB prices and the expectation of interest rate caps/floors' future payoff under the forward measure.

Given that the value of $P(t,T)$ is known at time t and $P(T,T) = 1$, combining this information with Equation 2.13 gives us:

$$V(t) = E^{Q^T}\left[\frac{P(t,T)}{P(T,T)} \cdot V(T) \mid \mathcal{F}_t\right] = E^{Q}\left[\frac{B(t)}{B(T)} \cdot V(T) \mid \mathcal{F}_t\right] \qquad 2.16$$

In other words, we can achieve the same target result, $V(t)$, under different measures. When working with financial models, it is often advantageous to switch to a measure where the resulting calculations become simpler. The equation above serves as a perfect example of this principle.

With slight modification of the notation from Brigo (2001), the forward measure is defined by the following Radon-Nikodym derivative:

$$\frac{dQ^T}{dQ} = \frac{\frac{B(t)}{B(T)}}{\frac{P(t,T)}{P(T,T)}} = \frac{P(T,T) \cdot B(t)}{P(t,T) \cdot B(T)} = \frac{e^{-\int_t^T r(s)ds}}{P(t,T)} \tag{2.17}$$

It is straightforward from Equation 2.10 that both measures coincide when the short rates are deterministic.

2.5. European call and put options on a ZCB

A European call option is a financial contract which gives the buyer of the option the right, but not the obligation, to buy the underlying asset from the seller of the option at a set price (strike price) at a predetermined date. Once the price of the underlying is realized above the strike price at the predetermined date (in-the-money), the buyer will exercise the option to take the profit. Otherwise, the option will not be exercised. Likewise, a European put option is a financial contract in which the purchaser of the option has the right, but not the obligation, to sell the underlying asset to the seller of the option at a strike price at a set date. The put option will be exercised when the price of the underlying ends below the strike price at the set date (in-the-money), otherwise will remain unexercised. Since the purchaser of an option only has right, which automatically gives the obligation to the seller, an upfront payment, so-called option premium, is paid from the buyer to the seller.

Using the notation from Brigo (2001), the price at time t of a European call and put option with maturity date T and strike price of X written on a ZCB that pays a unit currency at time $S > T$ under risk-neutral measure Q are:

$$ZCBC(t,T,S,X) = E^Q\left[e^{-\int_t^T r(s)ds} \cdot max(P(T,S) - X, 0)|\, \mathcal{F}_t\right] \tag{2.18}$$

$$ZCBP(t,T,S,X) = E^Q\left[e^{-\int_t^T r(s)ds} \cdot max(X - P(T,S),0) \mid \mathcal{F}_t\right]$$

Using Equation 2.15, the call and put price equations above are transformed into:

$$ZCBC(t,T,S,X) = P(t,T) \cdot E^{Q^T}[max(P(T,S) - X,0)| \mathcal{F}_t] \qquad 2.19$$
$$ZCBP(t,T,S,X) = P(t,T) \cdot E^{Q^T}[max(X - P(T,S),0) \mid \mathcal{F}_t]$$

2.6. Interest rate caps (IRC) and floors (IRF)

An interest rate cap is a financial agreement where the purchaser receives payments at the end of each period whenever the reference interest rate surpasses the agreed strike rate. Likewise, an interest rate floor is a financial agreement where the purchaser receives payments at the end of each period whenever the interest rate falls beneath the predetermined strike rate. A cap allows borrowers to protect against increasing interest rates and a floor allows lenders to protect against decreasing interest rates.

Using the notation in Brigo (2001), let's say when $T_\alpha > t \geq 0$, a cap resets at $T_\alpha,\ T_{\alpha+1}, \dots,\ T_{\alpha+A-1}$ and settles at $T_{\alpha+1},\ T_{\alpha+2}, \dots,\ T_{\alpha+A}$ with day count fraction τ_i between T_{i-1} and T_i for $i = \alpha + 1, \dots, \alpha + A - 1, \alpha + A$. Since $T_{\alpha+A}$ is the last settlement date, this also could be understood as the maturity of the cap. The price of this cap at time t struck at cap rate K with notional value N under risk-neutral measure Q is given by:

$$Cap(t,\mathcal{T},N,K) = N \cdot \sum_{i=\alpha+1}^{A} E^Q[D(t,T_i) \cdot \tau_i \cdot max\ (R(T_{i-1},T_i) - K,0)| \mathcal{F}_t] \qquad 2.20$$

$D(t,T_i)$ is a discount factor between valuation time t and each settlement time T_i and $R(T_{i-1},T_i)$ is a future reference spot rate that resets at time T_{i-1} with maturity τ_i. Similarly, the price of a floor is defined just by flipping the sign for the first element in the max function above. Clearly, the price of a cap or a floor is the sum of the discounted future payoffs at each settlement time $T_{\alpha+1}$, $T_{\alpha+2}, \dots,\ T_{\alpha+A}$. Such payoff for each settlement can also be understood as a single caplet or floorlet payoff as described in the following form:

$$Cpl(t, T_{i-1}, T_i, \tau_i, N, K) = N \cdot E^Q[D(t, T_i) \cdot \tau_i \cdot max\,(R(T_{i-1}, T_i) - K, 0) | \mathcal{F}_t] \qquad 2.21$$

With decent modification of Equation 2.21 from Brigo (2001), the price of a caplet is expressed by a European put option on a ZCB:

$$\begin{aligned} &Cpl(t, T_{i-1}, T_i, \tau_i, N, K) \qquad 2.22 \\ &= N \cdot (1 + K \cdot \tau_i) \cdot ZCBP\left(t, t_{i-1}, t_i, \frac{1}{1 + K\tau_i}\right) \\ &= N \cdot (1 + K \cdot \tau_i) \cdot P(t, t_{i-1}) \cdot E^{Q^T}\left[max\left(\frac{1}{1 + K\tau_i} - P(t_{i-1}, t_i), 0\right) | \mathcal{F}_t\right] \end{aligned}$$

In other words, an interest rate cap can be understood as the sum of multiple European put options on corresponding ZCBs. The second equality in Equation 2.22 comes from the put option pricing formula under forward measure in Equation 2.19.

Similarly, an interest rate floor can be defined by the sum of multiple European call options on corresponding ZCBs and the price of a floorlet is expressed as:

$$\begin{aligned} &Fll(t, T_{i-1}, T_i, \tau_i, N, K) \qquad 2.23 \\ &= N \cdot (1 + K \cdot \tau_i) \cdot ZCBC\left(t, t_{i-1}, t_i, \frac{1}{1 + K\tau_i}\right) \\ &= N \cdot (1 + K \cdot \tau_i) \cdot P(t, t_{i-1}) \cdot E^{Q^T}\left[max\left(P(t_{i-1}, t_i) - \frac{1}{1 + K\tau_i}, 0\right) | \mathcal{F}_t\right] \end{aligned}$$

2.7. Compounding and interpolation methods

In financial modeling, the assumption about compounding and interpolation methods is crucial for accurate valuation of financial instruments. In this thesis, continuous compounding with the 30/360-day count convention was adopted and a few interpolation methods were considered.

Continuous compounding assumes that interest accrues infinitely many times per year, which is mathematically represented by the exponential function. This method simplifies the integration and differentiation processes involved in advanced financial models. Let $R(t, T)$ denote the annualized continuously compounded spot rate from time t to time T then the relationship with the price of a zero-coupon bonds $P(t,T)$ is:

$$e^{R(t,T)\tau(t,T)} \cdot P(t,T) = 1 \qquad 2.24$$

$\tau(t,T)$ represents time difference in years between time t and T with a day-count convention, which, in this thesis, is the 30/360-day count convention. Solving Equation 2.24 for $R(t,T)$ gives us:

$$R(t,T) = \frac{-lnP(t,T)}{\tau(t,T)} \qquad 2.25$$

The above equation is used very often for convenient conversion between zero rates and discount factors in financial modeling.

Unlike compounding method, due to the instability of differentiating the market initial yield curve, several interpolation methods were considered. Differentiating the yield curve is required when determining the market instantaneous forward rate, $f^M(0,t)$ as seen in Equation 2.12, which is needed for model calibration and simulation. Since we only have a few observable data points of market initial term structure, interpolation becomes essential and in fact we end up differentiating the interpolation function. Consequently, the choice of interpolation method exerts a significant impact on model calibration and simulation results. The interpolation methods considered in this thesis include linear, log-linear, and cubic spline. All these interpolation methods are applied to discount factors rather than to zero rates. Ultimately, the cubic spline method was chosen as the benchmark one, despite the remaining issues like the zigzag patterns that can occur at early maturities. The selection of the final interpolation method is discussed in Chapter 4.1.2, where the market instantaneous forward rate is determined and plotted with each interpolation method.

The linear interpolation method is the simplest interpolation method and is widely used in many applications. This method estimates the values between two known data points by assuming that the change between these points is linear. This straightforward approach is useful for cases where the data is assumed to vary at a constant rate between points. Where $t_i < t < t_{i+1}$ for $i = 1, \dots n -$

1, the unknown discount factor $D(0,t)$ between the two known values $D(0,t_i)$ and $D(0,t_{i+1})$ calculated using the linear interpolation method is:

$$D(0,t) = D(0,t_i) + \frac{\tau(t,t_{i+1})}{\tau(t_i,t_{i+1})} \cdot \left(D(0,t_{i+1}) - D(0,t_i)\right) \tag{2.26}$$

Here $\tau(t_i,t_{i+1})$ represents time difference in years between time t_i and t_{i+1} with the 30/360-day count convention.

The log-linear interpolation method constructs a piecewise linear function of the logarithm of the discount factors between known data points, ensuring a smooth and realistic yield curve. This technique is preferred for its ability to capture the exponential nature of interest rates, reflecting market behavior more accurately than simple linear interpolation. The unknown discount factor $D(0,t)$ between the two known values $D(0,t_i)$ and $D(0,t_{i+1})$ using log-linear interpolation method is:

$$D(0,t) = e^{\ln D(0,t_i) + \frac{\tau(t,t_{i+1})}{\tau(t_i,t_{i+1})} \cdot (\ln D(0,t_{i+1}) - \ln D(0,t_i))} \tag{2.27}$$

The cubic spline interpolation method assumes a set of piecewise cubic polynomial functions between known data points. The unknown discount factor $D(0,t)$ between the two known values $D(0,t_i)$ and $D(0,t_{i+1})$ using cubic spline interpolation method is:

$$D(0,t) = S_i(t) = a_i t^3 + b_i t^2 + c_i t + d_i \tag{2.28}$$

Here, $S_i(t)$ is a cubic interpolating function between t_i and t_{i+1} for $i = 1, \dots n-1$. The four constants at each ith interval are estimated by assuming the following boundary conditions which are set to ensure continuity:

$$\begin{aligned} S_i(t_i) &= D(0,t_i) \\ S_i(t_{i+1}) &= D(0,t_{i+1}) \\ S_i'(t_{i+1}) &= S_{i+1}'(t_{i+1}) \\ S_i''(t_{i+1}) &= S_{i+1}''(t_{i+1}) \end{aligned} \tag{2.29}$$

S' and S'' are first and second derivatives of a cubic spline interpolating function S, resepectively. We have $4 * (n - 1)$ unknowns for n known data points and the above boundary conditions give us $2 * (n - 1) + 2 * (n - 2)$ equations. The commonly used additional boundary conditions for the missing 2 equations are to assume the second derivatives of first and last data point are zero (Kong, Siauw, and Bayen, 2020):

$$\begin{aligned} S_1''(t_1) &= 0 \\ S_{n-1}''(t_n) &= 0 \end{aligned} \qquad 2.30$$

Chapter 3. Short Rate Model

3.1. Endogenous and exogenous one-factor short rate models

A one-factor short-rate model is a type of interest rate models used in finance to describe the movement of the instantaneous short-term interest rate over time. It assumes that a single source of risk factor influences the entire yield curve. Once the dynamics of the instantaneous short rate is described by an appropriate model, interest rates with longer maturities are also determined.

One of the most distinguishing characteristics that categorizes a short rate model is whether it produce the initial yield curve as output or take it as input. The former model is called an endogenous interest rate model, and the latter is called an exogenous interest model. Exogenous interest rate models have fewer parameters since they don't need to be calibrated to fit the observed initial term structure, which leads to their relative simplicity. In other words, regardless of how well these models are estimated, they cannot accurately replicate the initial yield curve. This drawback has been criticized by researchers and market experts, which led to the birth of endogenous interest rate models. Endogenous interest rate models provide higher complexity by calibrating the model output to the initial yield curve observed in the market. To achieve this result, endogenous models incorporate time-dependent parameters. Vasicek (1977) model and Cox, Ingersoll and Ross (1985) model are exemplary representatives of an exogenous short-rate model, while Ho-Lee (1986) model and Hull-White (1990) model are notable examples of an endogenous short-rate model.

3.2. Affine term structure model

An affine term structure model is a mathematical framework used to describe how short rates behave over time with regards to ZCB prices. The term "affine" refers to a particular mathematical property that these models show that the relationship between short rates and ZCB prices follows a deterministic pattern. Essentially, an affine term structure model allows us to model the dynamics of interest rates in a way that is both flexible and mathematically tractable. This relationship is described in such a form as:

$$P(t,T) = A(t,T) \cdot e^{-B(t,T) \cdot r(t)} \qquad 3.1$$

$A(t,T)$ and $B(t,T)$ are deterministic functions of time. Vasicek model, Cox, Ingersoll, and Ross model, and Hull-White model are considered as such affine term structure models.

3.3. Hull-White model

Hull-White model, also called Hull-White extended Vasicek model, was proposed by Hull and White (1990) to address the problem of inaccurate fitting to the initial yield curve by Vasicek model. Hull-White model resolves the issue by introducing time-dependency to the constant mean level parameter of Vasicek model.

The Vasicek (1977) model is described in a stochastic differential equation:

$$dr(t) = \big(\theta - \alpha r(t)\big) \cdot dt + \sigma dW(t) \qquad 3.2$$

where α is a reversion speed, θ/α is a long-term equilibrium level, σ is an instantaneous volatility, and $W(t)$ is a Wiener process under risk neutral measure. α and σ are positive constants. The reversion speed, represented by α, governs how rapidly the variable being modeled returns to its long-term mean level, θ/α. This long-term mean level serves as an equilibrium level towards which the process reverts over time. The instantaneous volatility, denoted by σ, captures the degree of random fluctuation or uncertainty inherent in the process. While σ determines the general degree of stochasticity on the instantaneous short rate process, α also serves as a relative volatility of long and short rates. For instance, higher α forces short rate to converge to long-term mean level quicker, relatively limiting long-term volatility. The differential of Wiener process, $dW(t)$, signifies the continuous-time stochastic component with normally distributed increments, reflecting random fluctuations in the short-rate process over an infinitesimally small-time interval dt.

By introducing time-dependency to θ, Hull-White model is formulated as:

$$dr(t) = \big(\theta(t) - \alpha r(t)\big) \cdot dt + \sigma dW(t) \qquad 3.3$$

where $\theta(t)/\alpha$ is a time-varying equillibrium level and $\theta(t)$ is a deterministic function of time t, which is calibrated to exactly match the initial yield curve. Allowing time-dependency of the long-term mean level in this continuous time process mathematically implies infinite number of parameters to be calibrated. The reason is that the initial term structure is constructed for infinite number of maturities, meaning that the exact matching requires infinite parameters $\theta(t)$.

The biggest advantage of the Hull White model comes from the fact that even with the time-dependent mean level parameter it is still mathematically tractable due to the Gaussianity of the process. The assumption about the Gaussian distribution of the short rates changes at time t in Vasicek and Hull-White models allows for negative rates. This feature, which was considered as a disadvantage in the past, nowadays is viewed as a benefit since there has been a rise in the number of countries experiencing negative interest rates after the great financial crisis of 2007-2008.

The analytical formula for $\theta(t)$ can be derived by solving Equation 3.3 for $r(t)$, constructing the price of a ZCB using Equation 2.11, and finally matching the price of a ZCB in $r(t)$ with the price of a ZCB in $f(t,T)$ in Equation 2.12. By setting $t = 0$, the relationship is described with market observed instantaneous forward rate $f^M(0,T)$ derived from the market initial term structure and Hull-White model parameters α and σ. This way, $\theta(t)$ is calibrated to initial market term structure.

$$\theta(t) = \alpha f^M(0,t) + \frac{\partial f^M(0,t)}{\partial t} + \frac{\sigma^2}{2\alpha} \cdot (1 - e^{-2\alpha t}) \tag{3.4}$$

Using the notation from Brigo (2001), integrated with Equation 3.3 gives us the conditional distribution of $r(t)$ on $\mathcal{F}_s$ for $s < t$, which follows normal distribution with the following mean and variance:

$$\begin{aligned} E(r(t)|\,\mathcal{F}_s) &= r(s) \cdot e^{-\alpha(t-s)} + g(t) - g(s) \cdot e^{-\alpha(t-s)} \\ Var(r(t)|\,\mathcal{F}_s) &= \frac{\sigma^2}{2\alpha} \cdot \left(1 - e^{-2\alpha(t-s)}\right) \\ g(t) &= f^M(0,t) + \frac{\sigma^2}{2\alpha} \cdot (1 - e^{-\alpha t})^2 \end{aligned} \tag{3.5}$$

The conditional normal distribution with the above mean and variance equations is utilized for the simulation of short rate paths in Chapter 4.3.

3.4. Analytical formula for the price of a ZCB

Assuming stochasticity of the instantaneous short rate, the price of a ZCB is calculated by taking the expectation of the integral of the instantaneous short rate under risk neutral measure as indicated in Equation 2.11. Hull-White model is an affine term structure model denoted by Equation 3.1 and with the use of the notation from Brigo (2001), solving Equation 2.11 yields:

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \cdot e^{B(t,T)\cdot f^M(0,t) - \frac{\sigma^2}{4\alpha}\cdot\left(1-e^{-2\alpha t}\right)\cdot B(t,T)^2}, \qquad 3.6$$

$$B(t,T) = \frac{1}{\alpha} \cdot [1 - e^{-\alpha(T-t)}]$$

$P^M(0,t)$ is the market observed price of a ZCB at time 0 for maturity date t, which corresponds to a discount factor with the same maturity.

3.5. Analytical formula for the price of interest rate caps (IRC) and floors (IRF)

We saw interest rate caps and floors can be considered as the sum of multiple European put and call options, respectively, written on the corresponding ZCBs from Equation 2.22 and 2.23. Since the latter expectations in the both equations are defined under forward measure, Hull-White model short rate dynamics under forward measure must be found. The Hull-White model short rate $r(t)$ conditional on $\mathcal{F}_s$ $(s < t)$ under the forward measure Q^T follows Gaussian distribution. From the notation in Brigo (2001), the European call and put option analytical pricing formulas in Hull-White model are given:

$$ZCBC^{HW}(t,T,S,X) = P(t,S) \cdot \Phi(h) - X \cdot P(t,T) \cdot \Phi\left(h - \sigma_p\right), \qquad 3.7$$

$$ZCBP^{HW}(t,T,S,X) = X \cdot P(t,T) \cdot \Phi\left(-h + \sigma_p\right) - P(t,S) \cdot \Phi(-h),$$

$$\sigma_p = \sigma \cdot \sqrt{\frac{1 - e^{-2\alpha(T-t)}}{2\alpha}} \cdot B(T,S),$$

$$h = \frac{1}{\sigma_p} \cdot \ln \frac{P(t,S)}{P(t,T) \cdot X} + \frac{\sigma_p}{2}$$

Then the price of a cap or a floor under Hull-White model can be analytically calculated by plugging the findings above in Equation 2.22 and 2.23 as:

$$Cap(t,\mathcal{T},N,K) \\ = N\sum_{i=\alpha+1}^{\beta} P(t,t_{i-1})\Phi\left(-h_i+\sigma_p^i\right)-(1+K\tau_i)P(t,t_i\,)\Phi(-h_i), \tag{3.8}$$

$$Flr(t,\mathcal{T},N,K) \\ = N\sum_{i=\alpha+1}^{\beta}(1+K\tau_i)P(t,t_i\,)\Phi(h_i)-P(t,t_{i-1})\Phi\left(h_i-\sigma_p^i\right),$$

$$\sigma_p^i=\sigma\sqrt{\frac{1-e^{-2\alpha(t_{i-1}-t)}}{2\alpha}}B(t_{i-1},t_i),$$

$$h_i=\frac{1}{\sigma_p^i}\ln\frac{P(t,t_i)(1+K\tau_i)}{P(t,t_{i-1}\,)}+\frac{\sigma_p^i}{2}$$

Chapter 4. Calibration of Model Parameters and Simulation

The implementation is conducted in Python programming and is uploaded on my GitHub[1].

4.1. Data

I use two types of data to calibrate the Hull-White model, Euro short-term rate (ESTR) discount factors (initial term structure) and at-the-money interest cap and floor prices. These vital datasets are sourced from Eikon, a leading financial data platform renowned for its comprehensive and reliable data offerings.

4.1.1. Euro Short-Term Rate (ESTR) discount factors

Euro short-term rate (ESTR) is “a rate which reflects the wholesale Euro unsecured overnight borrowing costs of Euro area banks.” (European Central Bank, 2021). This is an alternative risk-free rate to current benchmarks in Euro area, such as the Euro interbank offered rate (EURIBOR) and the Euro overnight index average (EONIA). This global transition to corresponding risk-free rates (RFRs) kicked off in the aftermath of the LIBOR manipulation scandal by major banks in 2012 and ESTR is selected as a RFR in Euro area.

Since ESTR is an in-arrears overnight risk-free rate, constructing a yield curve given maturities involves daily compounding publicly available historical ESTR rates (European Central Bank, 2021). In this thesis, ESTR is assumed to be continuously compounded and have 30/360 day-count convention without considering holidays. The provided data in Table 1 presents ESTR discount factors and zero rates corresponding to various tenors. The tenors range from overnight (ON) to 30 years (30Y) and the data was procured on 1st April 2024. Zero rates are derived with continuous compounding from the given discount factors and year fraction with 30/360-day count convention.

Table 1. ESTR OIS Discount Factors and Zero Rates on 1st April 2024

Tenor	Date	Year Fraction	Discount	Zero Rate
0D	*Apr 1, 2024*	*0.00000*	*1.000000*	*0.03885102*
ON	*Apr 2, 2024*	*0.00278*	*0.999892*	*0.03885102*
SW	*Apr 10, 2024*	*0.02500*	*0.999026*	*0.03897899*
2W	*Apr 17, 2024*	*0.04444*	*0.998266*	*0.03905277*
3W	*Apr 24, 2024*	*0.06389*	*0.997514*	*0.03895907*
1M	*May 3, 2024*	*0.08889*	*0.996546*	*0.03892428*

[1] https://github.com/YANJINI/One-Factor-Hull-White-Model-Calibration-with-CAF

2M	*Jun 3, 2024*	*0.17222*	*0.993222*	*0.03949062*
3M	*Jul 3, 2024*	*0.25556*	*0.990140*	*0.03877341*
4M	*Aug 5, 2024*	*0.34444*	*0.986880*	*0.0383429*
5M	*Sep 3, 2024*	*0.42222*	*0.984079*	*0.03801123*
6M	*Oct 3, 2024*	*0.50556*	*0.981287*	*0.03736511*
7M	*Nov 4, 2024*	*0.59167*	*0.978453*	*0.03681533*
8M	*Dec 3, 2024*	*0.67222*	*0.975944*	*0.03622337*
9M	*Jan 3, 2025*	*0.75556*	*0.973358*	*0.03573949*
10M	*Feb 3, 2025*	*0.83889*	*0.970875*	*0.03523412*
11M	*Mar 3, 2025*	*0.92222*	*0.968705*	*0.03447675*
12M	*Apr 3, 2025*	*1.00556*	*0.966373*	*0.03401626*
15M	*Jul 3, 2025*	*1.25556*	*0.959921*	*0.03257852*
18M	*Oct 3, 2025*	*1.50556*	*0.954107*	*0.03120397*
21M	*Jan 5, 2026*	*1.76111*	*0.948336*	*0.03012101*
2Y	*Apr 7, 2026*	*2.01667*	*0.942805*	*0.02920448*
3Y	*Apr 5, 2027*	*3.01111*	*0.922607*	*0.02675157*
4Y	*Apr 3, 2028*	*4.00556*	*0.903406*	*0.02536055*
5Y	*Apr 3, 2029*	*5.00556*	*0.884216*	*0.02458344*
6Y	*Apr 3, 2030*	*6.00556*	*0.864765*	*0.02419383*
7Y	*Apr 3, 2031*	*7.00556*	*0.845061*	*0.02403041*
8Y	*Apr 5, 2032*	*8.01111*	*0.824882*	*0.02403099*
9Y	*Apr 4, 2033*	*9.00833*	*0.804566*	*0.02413902*
10Y	*Apr 3, 2034*	*10.00556*	*0.783991*	*0.02432225*
11Y	*Apr 3, 2035*	*11.00556*	*0.763235*	*0.02455025*
12Y	*Apr 3, 2036*	*12.00556*	*0.742533*	*0.02479584*
15Y	*Apr 4, 2039*	*15.00833*	*0.683701*	*0.0253349*
20Y	*Apr 4, 2044*	*20.00833*	*0.605786*	*0.02505099*
25Y	*Apr 5, 2049*	*25.01111*	*0.548030*	*0.02404632*
30Y	*Apr 3, 2054*	*30.00556*	*0.500307*	*0.02308017*

The column 'Year Fraction' is calculated based on the column 'Date' using 30/360-day count convention without considering holidays.

Figure 2. ESTR Yield Curve on 1st April 2024

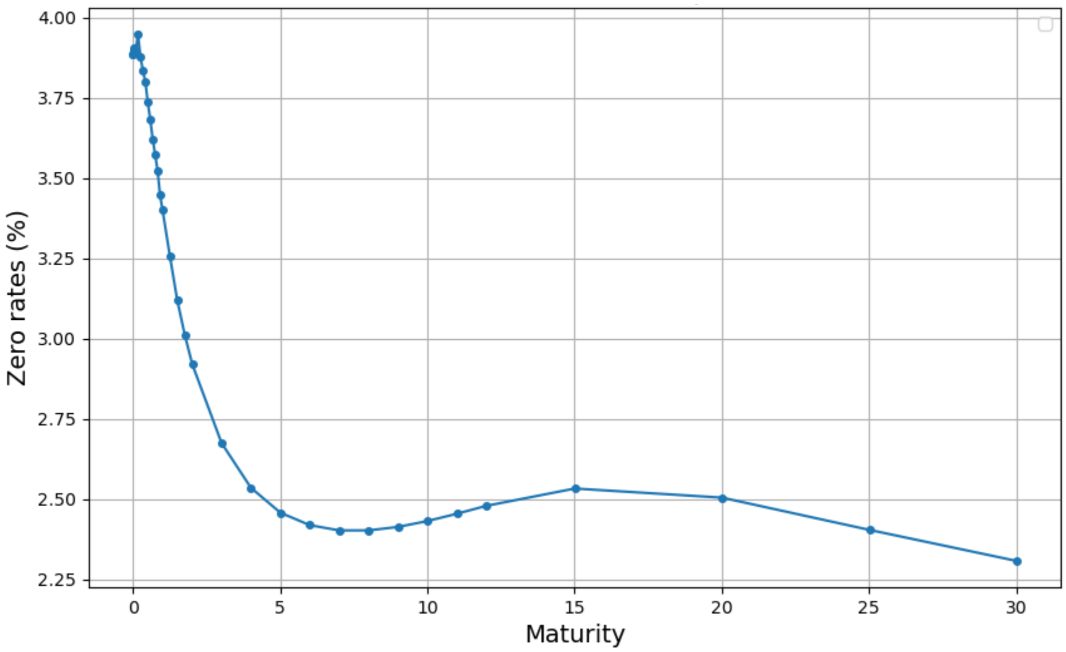


Figure 3. ESTR Discount Factor on 1st April 2024

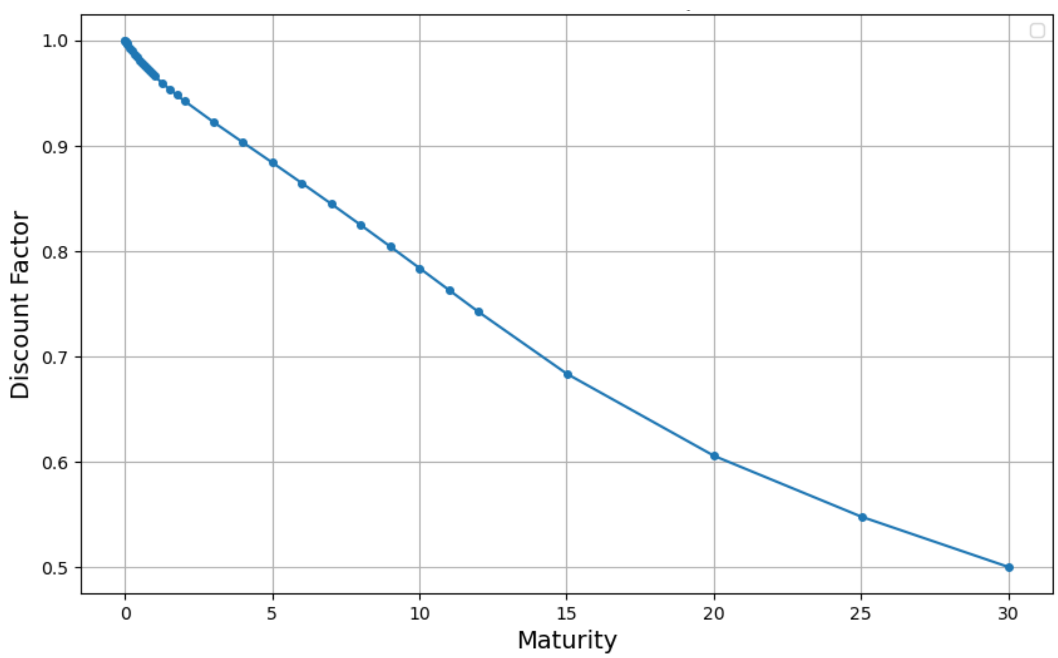


Figure 2 and Figure 3**Error! Reference source not found.** show the ESTR yield curve and discount factor observed on 1st April 2024. The vertical axis represents the zero rate in percentage or discount factor, while the horizontal axis represents maturity expressed in years. The key observations from the zero rate figure are (i) an initial decline in rates from approximately 4.00% to 2.50% for maturities below 5 years, (ii) a flattening out of the curve between 5 and 10 years with rates hovering around 2.50%, (iii) a slight increase in rates between 10 and 20 years, and (iv) a gradual decline in rates from 20 years to 30 years, ending at approximately 2.25%. The bumps in the yield curve refers to the non-monotonic behavior observed in the very beginning and around the 10-20 year tenor. This deviation can be attributed to various factors such as changes in market expectations, central bank policies, or macroeconomic conditions.

The ability of the Hull-White model to replicate these bumps in the yield curve depends on its parameters. The single-factor version might struggle to capture these bumps precisely due to its inherent limitations in accommodating complex term structure movements driven by multiple sources of risk. Adjusting the mean-reversion rate and volatility can improve the fit, but capturing the specific nuances of the bumps requires more complex models.

4.1.2. Market instantaneous forward rate

Throughout the model calibration and simulation, the market instantaneous forward rate, $f^M(0,t)$ must be determined and this involves differentiating interpolation function of market initial term structure. For the rest of this sub-chapter, plots of interpolated initial term structure and market instantaneous forward rate with different interpolation methods are displayed to justify the choice of the cubic spline method.

By discretizing the Equation 2.12 with small finite time difference (e.g. dt=1e-6), we can write the following formula of the market instantaneous forward rate in discrete world:

$$f^M_{discrete}(0,t) = -\frac{ln\frac{P(0,t+dt)}{P(0,t)}}{dt} \qquad \textit{4.1}$$

The attached plots below show the market instantaneous forward rates calculated using different interpolation methods. These rates were plotted for monthly intervals in years (1/12) on x-axis, starting from time zero (present) and extending up to the longest maturity of 30 years. This means the forward rates were calculated and plotted for each month over a 30-year period, resulting in a comprehensive view of how the forward rates change over time with different interpolation approaches.

Figure 4. Market Instantaneous Forward Rate with Linear Interpolation

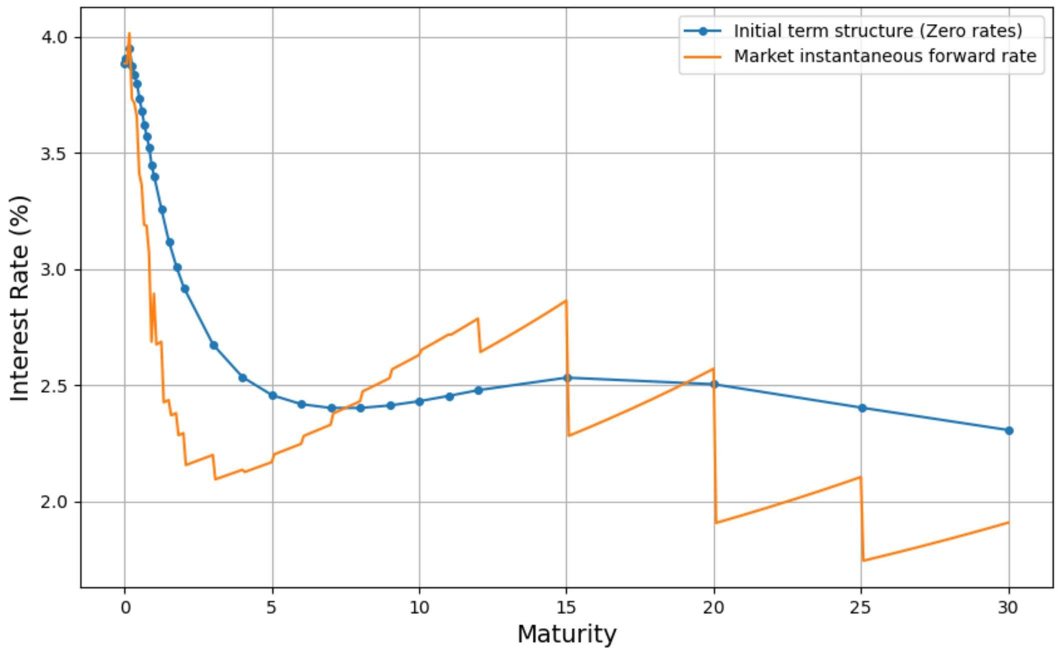


Figure 5. Market Instantaneous Forward Rate with Log-Linear Interpolation

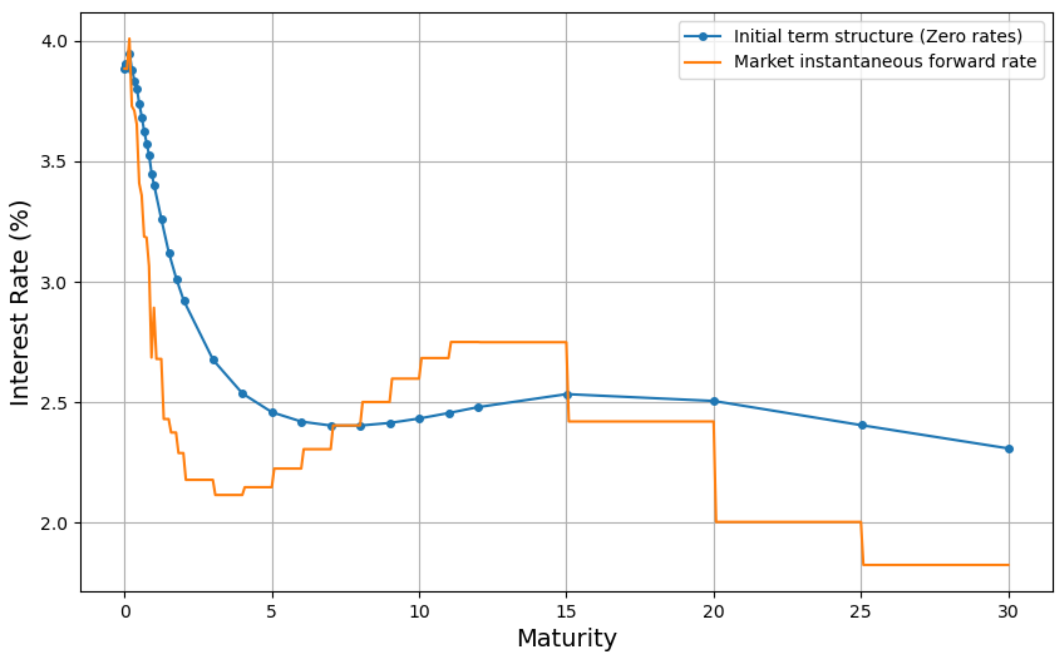
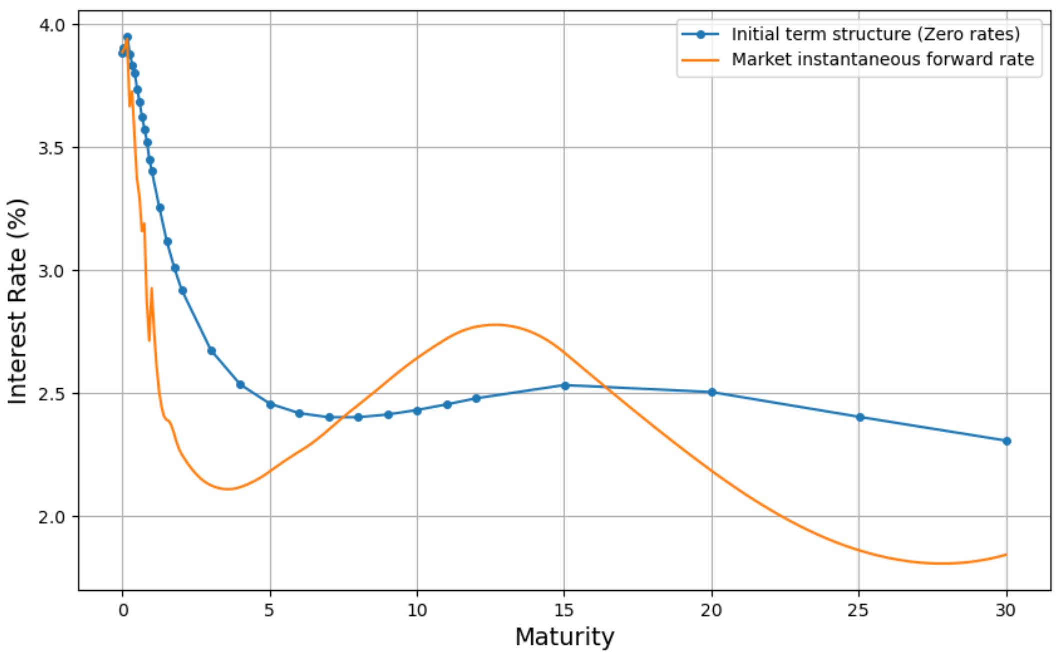


Figure 4 and Figure 5 depict the market instantaneous forward rates using linear and log-linear interpolation methods, respectively. In both figures the orange lines fluctuate drastically. It starts at a high point, dips steeply, and then exhibits sharp oscillations particularly noticeable at mid to long-term tenors. The market instantaneous forward rates are derived by differentiating the interpolation functions of the market initial term structure. Since the first derivative of a linear function such as linear and log-linear interpolation functions is simply a constant, we observe these abrupt, constant-looking changes at points where the interpolation segments meet. This issue expresses a need for a "smoother" interpolating function.

Figure 6. Market Instantaneous Forward Rate with Cubic-Spline Interpolation



The cubic-spline interpolation method effectively addresses the fluctuation issues seen in the previous two methods, generally producing a smooth and continuous curve across all maturities. However, the interpolated market instantaneous forward rate curve still exhibits minor oscillations in the earlier maturities. These oscillations can be attributed to several factors related to the underlying data and the nature of cubic-spline interpolation. Specifically, unevenly spaced data points and abrupt changes at the beginning can exaggerate fluctuations in the instantaneous forward rates. This explains why this issue appears consistently across all figures plotted using the three different interpolation methods. Additionally, inappropriate boundary conditions for cubic-spline fitting can lead to unrealistic oscillations at the edges of the data set. Despite these zigzag issues that cannot be fully resolved with the cubic spline method, this method remains effective and is therefore chosen in this thesis.

4.1.3. ESTR caps and floors

At-the-money ESTR cap prices are used for the calibration of the Hull-White model and ESTR floor prices are used for test of the fit. The dataset includes cap and floor identifiers, frequencies, maturities in years, market prices, normal volatilities (bp), strikes (atm), and notional.

Table 2. ESTR Cap and Floor Data on 1st April 2024

Capflr Id	Freq	Maturity	Market Price	Norm Vol (bp)	Strike	Notional
cap1	*0.25*	*1*	*2496.69235*	*65.4364287*	*3.37477673*	*1000000*
cap2	*0.5*	*2*	*8335.82411*	*82.7259843*	*2.91421598*	*1000000*
cap3	*0.5*	*3*	*15168.4449*	*90.1694824*	*2.66924638*	*1000000*
cap4	*0.5*	*4*	*22684.6755*	*93.2744268*	*2.52859833*	*1000000*
cap5	*0.5*	*5*	*30230.9885*	*93.3312736*	*2.45222801*	*1000000*
cap7	*0.5*	*7*	*45834.6795*	*91.7825445*	*2.39653664*	*1000000*
cap9	*0.5*	*9*	*62268.6946*	*90.1374826*	*2.40308508*	*1000000*
cap11	*0.5*	*11*	*79124.5513*	*88.5541697*	*2.43851021*	*1000000*
cap13	*0.5*	*13*	*96227.4514*	*87.2127778*	*2.47851675*	*1000000*
cap15	*0.5*	*15*	*112929.949*	*85.7063372*	*2.50552797*	*1000000*
cap20	*0.5*	*20*	*152250.457*	*81.6270884*	*2.48483404*	*1000000*
cap25	*0.5*	*25*	*192418.938*	*79.3543214*	*2.40776767*	*1000000*
cap30	*0.5*	*30*	*228054.521*	*76.2490177*	*2.33360786*	*1000000*
flr1	*0.5*	*1*	*2575.0261*	*65.7535937*	*3.38850248*	*1000000*
flr2	*0.25*	*2*	*8349.68199*	*86.0578616*	*2.90397982*	*1000000*
flr3	*0.25*	*3*	*14980.7246*	*91.359377*	*2.66058151*	*1000000*
flr4	*0.25*	*4*	*22447.8436*	*94.3168995*	*2.52075162*	*1000000*
flr5	*0.25*	*5*	*29777.8334*	*93.5477494*	*2.44481004*	*1000000*
flr6	*0.5*	*6*	*37865.0667*	*92.5185463*	*2.41359945*	*1000000*
flr7	*0.25*	*7*	*45253.3236*	*91.8553763*	*2.38941025*	*1000000*
flr8	*0.5*	*8*	*54003.9761*	*90.9799892*	*2.39434949*	*1000000*
flr9	*0.25*	*9*	*61552.4326*	*90.0790792*	*2.39588785*	*1000000*
flr10	*0.5*	*10*	*70670.0481*	*89.2938084*	*2.41888332*	*1000000*
flr11	*0.25*	*11*	*78243.0598*	*88.3580276*	*2.43107905*	*1000000*
flr12	*0.5*	*12*	*87709.9311*	*87.9120493*	*2.45913588*	*1000000*
flr13	*0.25*	*13*	*95229.453*	*86.9629369*	*2.47083033*	*1000000*
flr14	*0.5*	*14*	*104689.015*	*86.5138925*	*2.49458038*	*1000000*

flr15	*0.25*	*15*	*111943.321*	*85.511685*	*2.49767049*	*1000000*
flr16	*0.5*	*16*	*121079.641*	*84.8653807*	*2.51030898*	*1000000*
flr17	*0.5*	*17*	*129173.331*	*84.1607939*	*2.50964682*	*1000000*
flr18	*0.5*	*18*	*137038.258*	*83.3450021*	*2.50465128*	*1000000*
flr19	*0.5*	*19*	*144812.924*	*82.5517356*	*2.49615633*	*1000000*
flr20	*0.25*	*20*	*150913.465*	*81.2855373*	*2.47710245*	*1000000*
flr21	*0.5*	*21*	*159774.798*	*80.8167664*	*2.47129177*	*1000000*
flr22	*0.5*	*22*	*168112.538*	*80.5122004*	*2.45633169*	*1000000*
flr23	*0.5*	*23*	*176199.401*	*80.1036072*	*2.44044008*	*1000000*
flr24	*0.5*	*24*	*184274.726*	*79.6945442*	*2.42405051*	*1000000*
flr25	*0.25*	*25*	*191385.161*	*79.217038*	*2.40050478*	*1000000*
flr26	*0.5*	*26*	*200574.635*	*79.0440352*	*2.39191577*	*1000000*
flr27	*0.5*	*27*	*208775.033*	*78.746749*	*2.37644778*	*1000000*
flr28	*0.5*	*28*	*213717.654*	*77.3023268*	*2.36169545*	*1000000*
flr29	*0.5*	*29*	*220953.634*	*76.7778344*	*2.34738388*	*1000000*
flr30	*0.25*	*30*	*226869.073*	*76.0733981*	*2.32678142*	*1000000*

Figure 7. Implied Normal Volatility of At-The-Money ESTR Interest Rate Cap and Floor

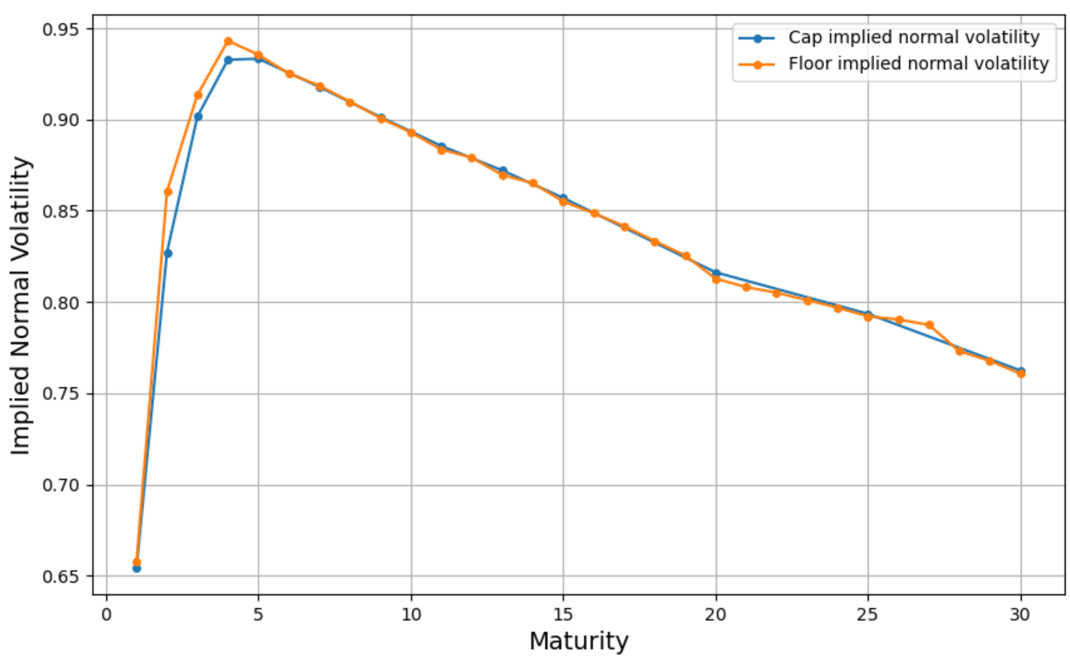


Figure 7 shows an initial increase in implied normal volatility, which can be explained by the shape of the initial term structure observed in Figure 2. The initial term structure exhibits a steep decline in the short end of the yield curve, which indicates significant uncertainty and higher volatility expectations for short-term interest rates. As the maturity increases, the term structure flattens, reflecting more stable long-term rate expectations and reduced uncertainty. The initial spike in the implied normal volatility thus seems to come from the steep initial term structure in earlier maturities, while the subsequent decline corresponds to the more stable long-term rate expectations, which are related to the mean-reversion feature of the short-term rates.

The overlapping of the cap and floor implied normal volatility curves indicates that the market perceives similar levels of volatility for both instruments, especially since they are at-the-money (ATM) strikes. ATM options are particularly sensitive to volatility because the current interest rate is close to the strike price, making the likelihood of the option being exercised (either as a cap or a floor) higher. This symmetry in the volatility levels reflects the balanced expectation for both upward and downward movements in interest rates, consistent with the market's neutral stance on future rate directions.

4.2. Calibration

The prices of interest rate derivatives such as caps, floors, and zero-coupon bonds depend on the underlying interest rate process. We aim to describe this underlying process using stochastic interest rate model. The stochastic interest rate models consist of several parameters, and our goal is to estimate these parameter values as closely as possible to replicate observed zero rates or interest rate derivative prices in the market. This process is called calibration. In the Hull-White model, we have one time-varying parameter (mean level) and two constant parameters (mean reversion rate and volatility) to be estimated. The mean level is simply calibrated to initial market term structure and other two constant parameters can be estimated using various optimization methods, which leads to a minimization problem in a two-dimensional space. Since we have the analytical formulas for prices of interest rate cap and floor under Hull-White model, this optimization can be done by choosing the optimal values of the two constant parameters that minimize the difference between the analytically calculated model-implied prices and observed market prices.

For calibrating the mean reversion rate and volatility of the Hull-White model to market data for interest rate caps and floors, the Nelder-Mead optimization method is employed. This method is chosen for its simplicity and effectiveness in handling non-linear optimization problems, making it suitable for calibrating complex financial models, such as the Hull-White model. The primary objective of the calibration process is to minimize the difference between the market prices of caps and floors and the model prices generated by the Hull-White model. This difference is measured using multiple error functions including Mean Error (ME), Mean Absolute Error (MAE), and Root Mean Squared Error (RMSE), for both the levels and log-levels of the prices.

Using levels, the error functions directly compare the absolute differences between the market prices and the model prices. This method provides a straightforward way of evaluating the performance of the model but tends to give more weight to instruments with longer maturities, where the absolute prices are higher. Since higher absolute values result in larger differences, this approach is prone to over-penalizing caps and floors with longer maturities

On the other hand, using log-levels helps capture the relative differences between market and model prices. This approach is particularly beneficial when pricing instruments with a wide range of maturities, as it penalizes smaller percentage errors for shorter maturities more heavily than using levels. By considering both levels and log-levels in the calibration process, the Hull-White model can be adjusted to fit market prices more accurately across different maturities.

The choice of error function also plays a critical role: while ME captures the overall bias, MAE emphasizes the average magnitude of deviations, and RMSE gives more weight to larger errors, making it sensitive to outliers. The combination of these error metrics in both levels and log-levels provides a comprehensive framework for assessing model performance and ensuring a robust calibration across the entire spectrum of cap and floor prices.

In this thesis, RMSE using log-levels of prices is selected for the calibration process. Additionally, a table is provided that summarizes the six combinations of error metrics and scales, offering a comprehensive view of the model's performance across different evaluation criteria.

The calibration begins with an initial guess for the parameters of the Hull-White model. The Nelder-Mead algorithm iteratively adjusts these parameters to minimize the objective function. At each iteration, the Hull-White model is used to price the caps and floors with the current set of parameters, and these model prices are then compared to the market prices to calculate the error. The optimization process continues until the changes in the objective function fall below a specified tolerance level, indicating that the parameters have converged to their optimal values. The implementation of this calibration method is facilitated by the Scipy library in Python, which provides robust tools for performing numerical optimization. Using Scipy's ‘minimize’ function with the Nelder-Mead method, we can effectively calibrate the Hull-White model parameters to market data for ESTR caps and floors.

For calibration of Hull-White model, only cap data is used, and floor data is held out for test. After such calibration in Python, we end up with mean reversion rate of 0.17964 and volatility of 0.017.

Table 3. Different Error Metrics with Different Scales

Scale	Metric	Train (Cap)	Test (Floor)
Log Level	*ME*	*0.00192631*	*0.02194369*
	MAE	*0.06276621*	*0.08328333*
	RMSE	*0.08046592*	*0.12879554*
Level	*ME*	*-700.11758*	*439.015089*
	MAE	*3664.27832*	*5627.53915*
	RMSE	*5533.79323*	*7010.30915*

Table 3 summarizes the error metrics using different scales, providing insights into the calibrated model performance.

Figure 8. Comparison between Model Prices and Market Prices of Cap (Train)

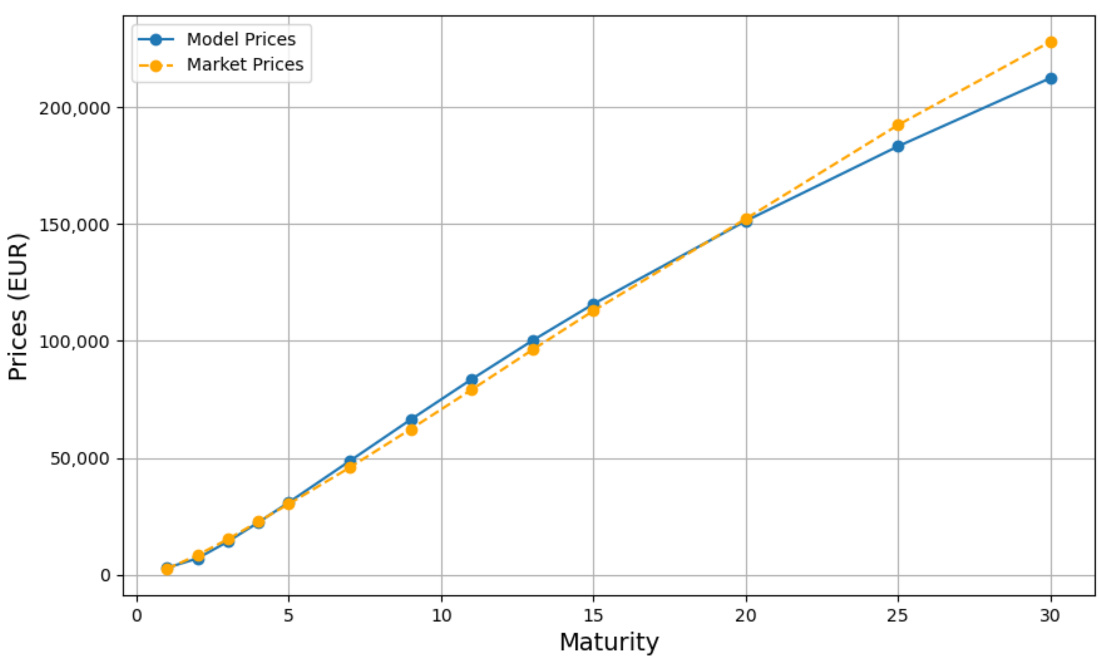


Figure 9. Comparison between Model Prices and Market Prices of Floor (Test)

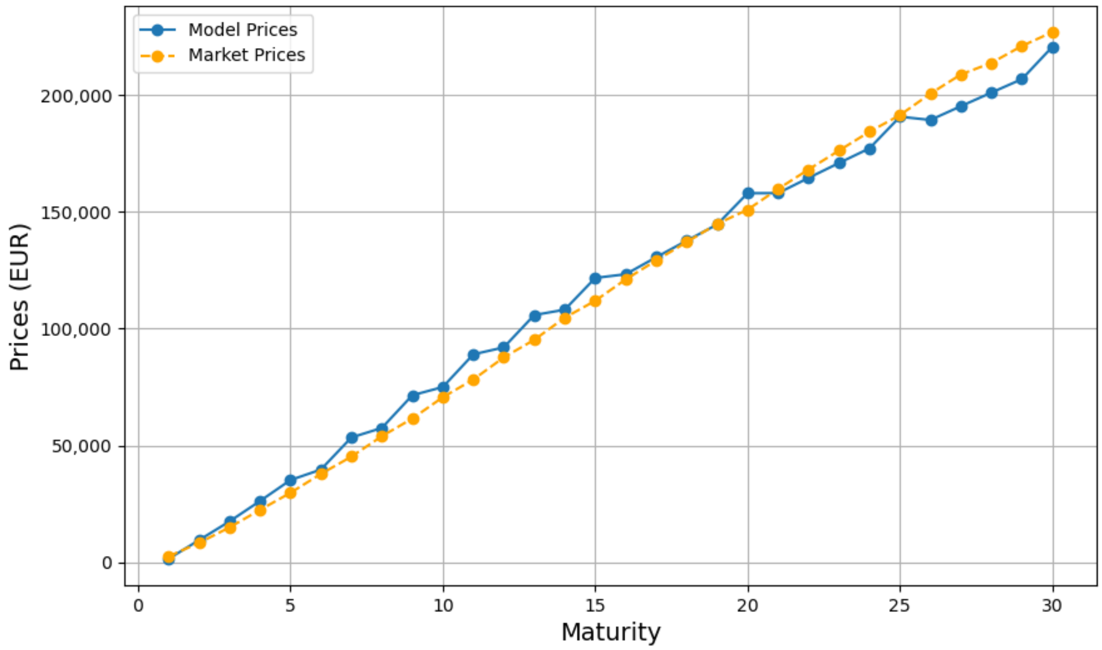


Figure 8 and Figure 9 compare model prices and market prices of caps and floors, respectively, showcasing the accurate calibration of the Hull-White model. The x-axis represents the maturity of the cap and floor in years, while the y-axis represents the prices in EUR. Overall, the two lines are very close to each other, indicating that the model prices closely match the market prices regardless of caps or floors. The relatively bigger deviations in longer maturities of the both figures are because optimization is conducted with log-levels of prices. **Error! Reference source not found.** and **Error! Reference source not found.** below show comparison of the model prices and market prices in log scale. This close alignment suggests that the model is well-calibrated to the market data and accurately reflects the market's pricing of caps and floors.

Figure 10. Comparison of Logged Prices for Train (Cap) Data

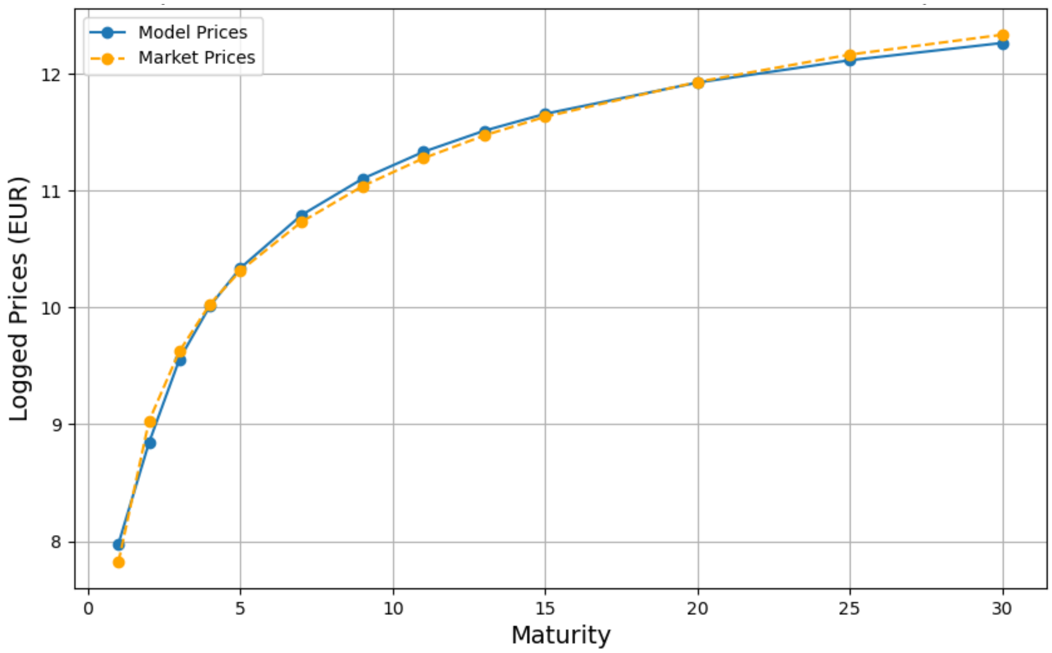
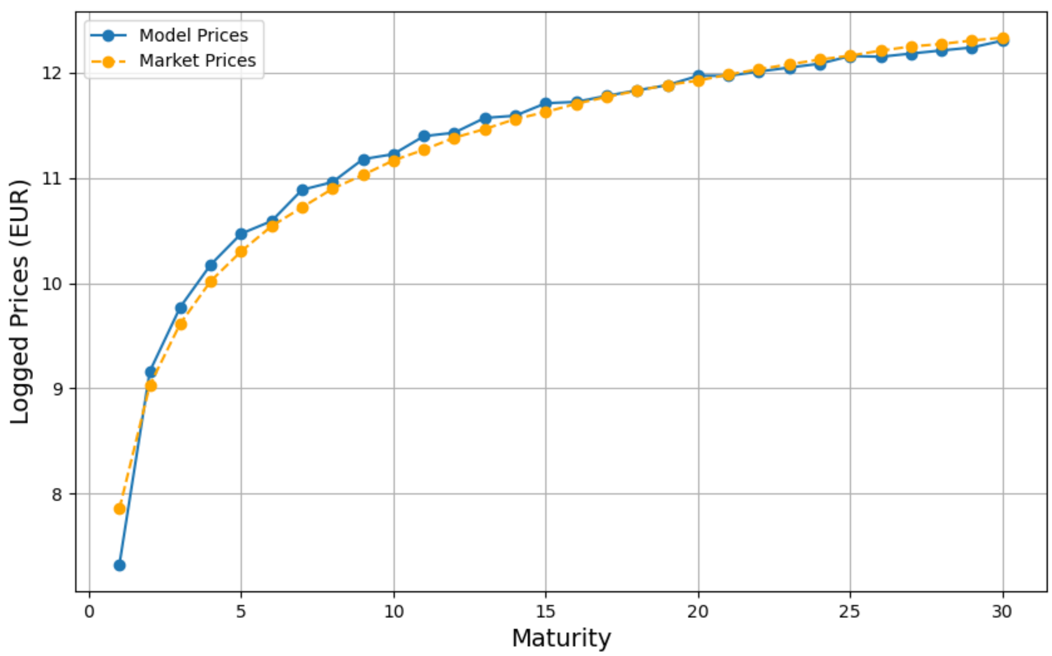


Figure 11. Comparison of Logged Prices for Test (Floor) Data



4.3. Simulation with Euler-Maruyama discretization

So far, we have compared the analytical model prices of interest rate caps and floors to the actual market prices. However, if we want to compare the model's instantaneous short rate, which is the target variable of the Hull-White model, or the zero rates derived from this short rate to market initial term structure, we need to conduct a simulation. For the simulation of stochastic differential equations (SDEs) such as the Hull-White model, discretization is required. This is because the model deals with continuous variables, and we need to approximate these continuous processes with discrete steps. The Euler-Maruyama discretization scheme is a numerical method used to approximate solutions to SDEs. In the context of the Hull-White model, the scheme involves the following steps for small step, $dt = t_{i+1} - t_i$ for $i = 1, \dots, n-1$:

$$r(t_{i+1}) = r(t_i) + E\left(r(t_{i+1}) \mid \mathcal{F}_{t_i}\right) + \sqrt{Var\left(r(t_{i+1}) \mid \mathcal{F}_{t_i}\right)} \cdot Z \tag{4.2}$$
$$Z \sim N(0,1)$$

The Equation 4.2 comes from the conditional normal distribution of $r(t)$ given $\mathcal{F}_s$ in Equation 3.5. During the simulation, short rate paths are generated and all of them start from the initial short rate, $r(0)$, which in this thesis is assumed to be the overnight ESTR zero rate, 0.03885102. Then, the paths evolve according to the Euler-Maruyama discretization in Equation 4.2. In this thesis 10000 paths are generated with max maurity of 30 years and 1/12 as each time interval, which leads to a two-dimensional array with the shape of $(10000, 361 = 30/(1/12) + 1)$. After simulation, we take the average of the 10000 short rate paths to get the simulated mean short rate curve. Once short rates are simulated, we can easily derive zero rates by taking the cumulative product of the zero-coupon bond prices $P(t_i, t_{i+1})$ calculated using the simulated short rates at each ith time interval. The plots of the simulated short rates and zero rates with comparison to market instantaneous forward rates are presented below.

Figure 12. Simulated Short Rates and Zero Rates

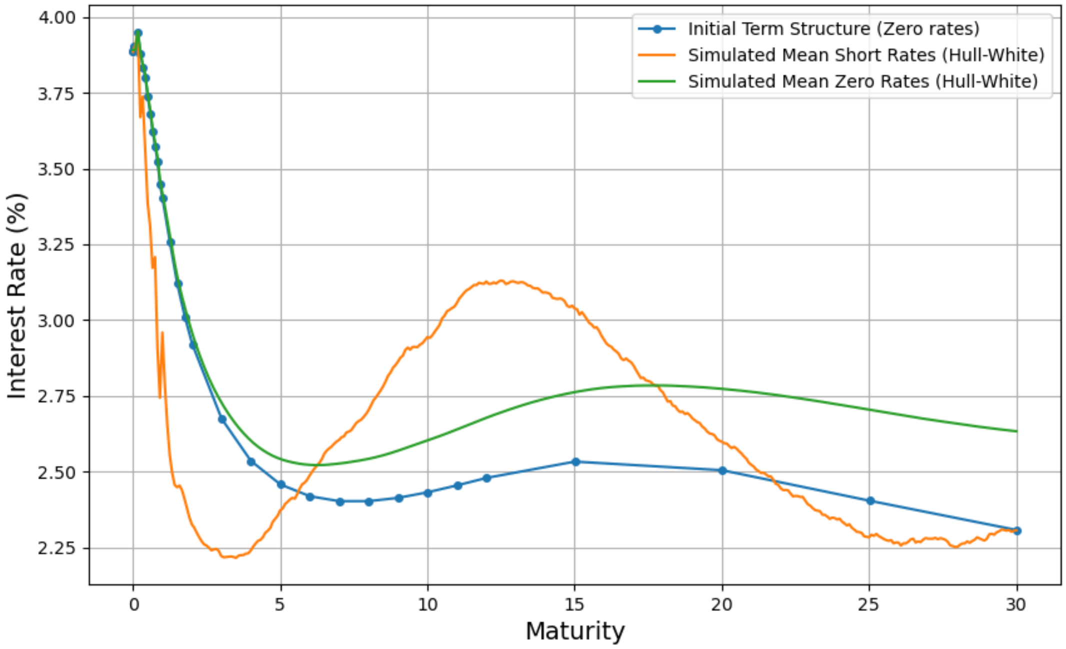


Figure 13. Simulated Short Rates and Market Instantaneous Forward Rates

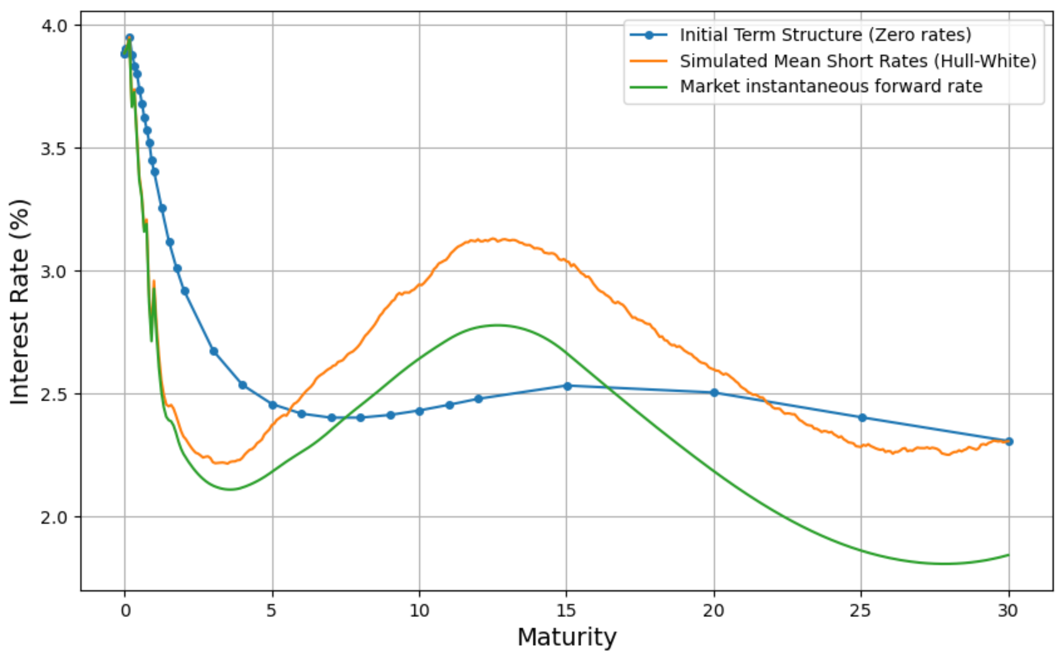


Figure 12 compares the initial term structure of zero rates, simulated mean short rates, and simulated mean zero rates, using the calibrated Hull-White model. The blue line represents the initial term structure of zero rates, which shows the interest rates at different maturities. The orange line indicates the mean of the simulated short rates using the Hull-White model. Short rates, being the rates defined in a short period, incorporatees a volatility term in Hull-White model that makes them have the fluctuations over time. The green line represents the simulated mean zero rates. These reflect the cumulative sum of discount factors (or zero-coupon bond prices) derived from the simulated short rates. Due to that property, despite being derived from the volatile short rates, the simulated zero rates are smoother. However, there is a notable misshoot of the simulated zero rates in the second bump that appears in initial zero rates, which can be seen as a limitation of the simple Hull-White model. This misshoot indicates that it may not fully align with the actual observed zero rates over longer tenors.

Figure 13 compares the initial term structure of zero rates, simulated mean short rates, and the market instantaneous forward rate. This time, the green line represents the market instantaneous forward rate, which reflects the market's expectation of future short-term interest rates. The market instantaneous forward rate is the main component in the calculation of short rate simulation path, which makes it show similar pattern to the simulated short rate.

Chapter 5. Conclusion

The one-factor Hull-White model presents a robust framework for modeling interest rates and pricing interest rate derivatives such as caps and floors. Through detailed calibration and analysis, this thesis demonstrates the model's utility in capturing the dynamics of interest rates and providing analytical pricing solutions. However, the study also identifies certain limitations of the Hull-White model, particularly its oversimplification of interest rate movements and the resultant discrepancies in pricing accuracy.

While the model is effective in generating zero rates and forward rates, it tends to overshoot the short rate fluctuations and misshoot certain market features, such as the second bump in yield curves. These drawbacks highlight the need for more sophisticated models or enhancements to the Hull-White framework to better capture the complexities of real-world interest rate behaviors.

In conclusion, the Hull-White model remains a valuable tool in the arsenal of financial analysts and practitioners, but its application should be complemented with an awareness of its limitations and potential improvements. Future research could explore multi-factor models or integrate stochastic volatility to address these challenges, thereby enhancing the precision and reliability of interest rate derivative pricing in the ever-evolving financial markets.

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