ECE584 Homework2

Yao Yijiang

September 2025

1 Problem 1

1.1 Problem 1.1

The prediction score for label i is always the top-1 score for all $x \in S$, which can be expressed as

$$\forall x \in S, \quad y_i > y_j \quad \text{for all } j \neq i \tag{1}$$

Convert it into a Satisfiability Problem Formulation:

$$\exists x, \quad x \in S \wedge (y = f(x)) \wedge \left(\bigvee_{i \neq j} (y_i \leq y_j) \right) \tag{2}$$

If this formula is unsatisfiable, then our requirement is verified.
If this formula is satisfiable, then there exists a counterexample where label i is not the top-1 score.
Optimization-based Formulation

$$\min_{x \in S} g(x) := \min_{j \neq i} (y_i - y_j) \tag{3}$$

If $\min g(x) > 0$, then the requirement is verified.
If $\min g(x) \leq 0$, then the requirement is not verified.

1.2 Problem 1.2

The prediction score for label j can never be the top-1 score for all $x \in S$, which can be expressed as

$$\forall x \in S, \quad \exists k \neq i, \quad y_k > y_i \tag{4}$$

Convert it into a Satisfiability Problem Formulation:

$$\exists x, \quad x \in S \wedge \left(\bigwedge_{k \neq j} (y_i \geq y_j) \right) \tag{5}$$

If this formula is unsatisfiable, then our requirement is verified
If this formula is satisfiable, then there exists a counterexample where label i is the top-1 score.
Optimization-based Formulation

$$\max_{x \in S} g(x) := \min_{j \neq k}(y_k - y_j) \tag{6}$$

If $\min g(x) > 0$, then the requirement is verified.
If $\min g(x) \leq 0$, then the requirement is not verified.

2 Problem 2

$$y = W^{(3)}(\hat{z^{(2)}} + z^{(1)}) \tag{7}$$
$$\hat{z^{(2)}} = ReLU(z^{(2)}) \tag{8}$$
$$z^{(2)} = W^{(2)}\hat{z^{(1)}} + b^{(2)} \tag{9}$$
$$\hat{z^{(1)}} = ReLU(z^{(1)}) \tag{10}$$
$$z^{(1)} = W^{(1)}x + b^{(1)} \tag{11}$$

Input: $x \in \mathbf{R}^2$ and $y \in \mathbf{R}$. The input set $S = \{x| -1 \leq x_1 \leq 1, \quad -1 \leq x_2 \leq 1\}$
The network weights:

$$W^{(1)} = W^{(2)} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, W^{(3)} = \begin{bmatrix} -1 & 1 \end{bmatrix}, b^{(1)} = \begin{bmatrix} 1 & 1 \end{bmatrix}, b^{(2)} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

2.1 Problem 2.1

Use interval bound propagation(IBP) to calculate the lower bound of y

$$z^{(1)} = W^{(1)}x + b^{(1)} \tag{12}$$
$$z^{(1)} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{13}$$
$$z^{(1)} = \begin{bmatrix} x_1 - x_2 + 1 \\ 2x_1 - 2x_2 + 1 \end{bmatrix} \tag{14}$$

Since $-1 \leq x_1 \leq 1, \quad -1 \leq x_2 \leq 1$

$$z_1^{(1)} \in [-1, 3], \quad z_2^{(2)} \in [-3, 5] \tag{15}$$
$$ReLU(z_1^{(1)}) \in [0, 3], \quad ReLU(z_1^{(1)}) \in [0, 5] \tag{16}$$
$$\tag{17}$$

Second layer:

$$z^{(2)} = W^{(2)} \hat{z^{(1)}} + b^{(2)} \tag{18}$$

$$z^{(2)} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} z_1^{(1)} \\ z_2^{(1)} \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{19}$$

$$z^{(2)} = \begin{bmatrix} z_1^{(1)} - z_2^{(1)} + 2 \\ 2z_1^{(1)} - 2z_2^{(1)} + 2 \end{bmatrix} \tag{20}$$

Similarly,

$$z_1^{(2)} \in [-3, 5], \quad z_2^{(2)} \in [-8, 8] \tag{21}$$

$$ReLU(z_1^{(1)}) \in [0, 5], \quad ReLU(z_1^{(1)}) \in [0, 8] \tag{22}$$

Therefore,

$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} (\begin{bmatrix} \hat{z_1^{(2)}} \\ \hat{z_2^{(2)}} \end{bmatrix} + \begin{bmatrix} z_1^{(1)} \\ z_2^{(1)} \end{bmatrix}) \tag{23}$$

$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} (\begin{bmatrix} [-1, 3] \\ [-3, 5] \end{bmatrix} + \begin{bmatrix} [0, 5] \\ [0, 8] \end{bmatrix}) \tag{24}$$

$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} [-1, 8] \\ [-3, 13] \end{bmatrix}) \tag{25}$$

$$y = [-8, 1] + [-3, 13] = [-11, 14] \tag{26}$$

IBP Lower Bound: $y \geq -11$

2.2 Problem 2.2

Use CROWN to calculate the intermediate layer lower and upper bounds of $z^{(2)}$.
$z_1^{(1)} \in [-1, 3], z_2^{(1)} \in [-3, 5]$, since $|u| \geq |l|$, $\alpha_1^{(1)} = \alpha_2^{(1)} = 1$
Upper bound:

$$\hat{z_1^{(1)}} \leq \frac{3}{4} z_1^{(1)} + \frac{3}{4}, \hat{z_2^{(1)}} \leq \frac{5}{8} z_1^{(1)} + \frac{15}{8} \tag{27}$$

Lower bound:

$$\hat{z_1^{(1)}} \geq z_1^{(1)}, \hat{z_2^{(1)}} \geq z_2^{(1)} \tag{28}$$

Since,

$$z^{(2)} = \begin{bmatrix} z_1^{(1)} - z_2^{(1)} + 2 \\ 2z_1^{(1)} - 2z_2^{(1)} + 2 \end{bmatrix} \tag{29}$$

Upper bound:

$$z_1^{\hat{(2)}} \leq (\frac{3}{4}z_1^{(1)} + \frac{3}{4}) - (z_2^{(1)}) + 2 = -\frac{5}{4}x_1 + \frac{5}{4}x_2 + \frac{5}{2} \leq 5 \tag{30}$$

$$z_2^{\hat{(2)}} \leq 2(\frac{3}{4}z_1^{(1)} + \frac{3}{4}) - 2z_2^{(1)} + 2 = -\frac{5}{2}x_1 + \frac{5}{2}x_2 + 3 \leq 8 \tag{31}$$

Lower bound:

$$z_1^{\hat{(2)}} \geq z_1^{(1)} - (\frac{5}{8}z_2^{(1)} + \frac{15}{8}) + 2 = -\frac{1}{4}x_1 + \frac{1}{4} + \frac{1}{2} \geq 0 \tag{32}$$

$$z_2^{\hat{(2)}} \geq 2z_1^{(1)} - 2(\frac{5}{8}z_2^{(1)} + \frac{5}{8}) + 2 = -\frac{1}{2}x_1 + \frac{1}{2}x_2 - 1 \geq -2 \tag{33}$$

Therefore,

$$z_1^{\hat{(2)}} \in [0, 5], z_2^{\hat{(2)}} \in [-2, 8] \tag{34}$$

2.3 Problem 2.3

Using the intermediate layer bounds in step 2, calculate CROWN lower bound on the output y

$$y = W^{(3)}(z^{\hat{(2)}} + z^{(1)}) = -z_1^{\hat{(2)}} + z_2^{\hat{(2)}} - z_1^{(1)} + z_2^{(1)} \tag{35}$$

Since $z_1^{\hat{(2)}}$ is always nonneigative,

$$z_1^{\hat{(2)}} = z_1^{(2)} \tag{36}$$

Since $z_2^{\hat{(2)}} \in [-2, 8]$, The upper bound:

$$z_2^{\hat{(2)}} \leq \frac{4}{5}z_2^{(2)} + \frac{8}{5} \tag{37}$$

The lower bound:

$$z_2^{\hat{(2)}} \geq z_2^{(2)} \tag{38}$$

Therefore, the lower bound for y is:

$$y \geq -z_1^{(2)} + z_2^{(2)} - z_1^{(1)} + z_2^{(1)} \tag{39}$$

Now we apply,

$$\hat{z}_2^{(1)} \leq \tfrac{5}{8}z_2^{(1)} + \tfrac{15}{8} \tag{40}$$

Thus,

$$\begin{aligned} y &\geq z_1^{(1)} - \left(\tfrac{5}{8}z_2^{(1)} + \tfrac{15}{8}\right) - z_1^{(1)} + z_2^{(1)} = \left(1 - \tfrac{5}{8}\right)z_2^{(1)} - \tfrac{15}{8} &(41)\\ &= \tfrac{3}{8}z_2^{(1)} - \tfrac{15}{8}. &(42) \end{aligned}$$

Substitute $z_2^{(1)} = 2x_1 - 2x_2 + 1$:

$$y \geq \frac{3}{8}(2x_1 - 2x_2 + 1) - \frac{15}{8} = \frac{3}{4}x_1 - \frac{3}{4}x_2 - \frac{3}{2}. \tag{43}$$

Since inputs are in $S = \{x| -1 \leq x_1 \leq 1, \quad -1 \leq x_2 \leq 1\}$ this linear function attains its minimum at $(x_1, x_2) = (-1, 1)$, giving

$$y \geq -3 \tag{44}$$

2.4 Problem 2.4

Maximum number of α variables in α-CROWN is 4. The network contains two ReLU layers with 2 neurons each, so there are $2 \times 2 = 4$ ReLU units in total. In the worst case all are unstable.