Number Theory and Cryptography

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Introduction

Number theory forms the mathematical foundation of modern cryptography. Concepts like modular arithmetic, prime factorization, and elliptic curves secure data in the digital age.

Preliminaries of Number Theory

- **Greatest Common Divisor :** For integers $a, b \neq 0$, $\gcd(a,b) = \max\{d > 0 : \ d \mid a, \ d \mid b\}$.
- **Euclidean Algorithm :** The Euclidean algorithm computes $\gcd(a,b)$; the extended version finds $x, y \in \mathbb{Z}$ with $ax + by = \gcd(a,b)$, which yields modular inverses when $\gcd(a,b) = 1$.
- **Fundamental Theorem of Arithmetic :** Every integer $n > 1$ factors uniquely as $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \quad p_i$ primes.
- **Totient Function :** $\varphi(n) = \#\{1 \leq k \leq n : \gcd(k,n) = 1\}$. It counts the units of $(\mathbb{Z}/n\mathbb{Z})^\times$ and is multiplicative on coprime inputs.

Theorems of Euler and Fermat

- **Euler's Theorem :** If $\gcd(a,n) = 1$ then $a^{\varphi(n)} \equiv 1 \pmod{n}$.
  - It ensures that encrypting and then decrypting a message returns the original via $(m^e)^d \equiv m \pmod{n}, \quad ed \equiv 1 \pmod{\varphi(n)}$, with Euler's theorem providing the mathematical reversibility that enables RSA to work.
- **Fermat's Little Theorem :** For prime $p \nmid a$, $a^{p-1} \equiv 1 \pmod{p}$.
  - It provides modular exponentiation and inversion—essential for encryption, decryption, and key exchange. It ensures predictable modular behavior for secure, reversible transformations in public-key systems like RSA and Diffie–Hellman.

Chinese Remainder Theorem

If $n_1, n_2, \ldots, n_k$ are pairwise coprime, then the system of congruences

$$x \equiv a_1 \pmod{n_1}, \quad x \equiv a_2 \pmod{n_2}, \quad \ldots, \quad x \equiv a_k \pmod{n_k}$$

has a unique solution modulo $N = n_1 n_2 \cdots n_k$. Explicitly,

$$x \equiv \textstyle\sum_{i=1}^{k} a_i N_i M_i \pmod{N},$$

where $N_i = N n_i^{-1}$ and $M_i \equiv N_i^{-1} \pmod{n_i}$.

- **Applications:** The CRT is crucial in RSA decryption, enabling computations modulo the prime factors p and q of $n = pq$ separately and recombining results via CRT, reducing exponentiation time by a factor of 4.

RSA Cryptography

A public-key scheme on the group $(\mathbb{Z}/n\mathbb{Z})^\times$, with $n = pq$ (large primes p, q).

Public key: (n, e). **Private key:** d with $ed \equiv 1 \pmod{\varphi(n)}$.

Encryption: $c \equiv m^e \pmod{n}$. **Decryption:** $m \equiv c^d \pmod{n}$.

- **Security basis:** Computational difficulty of factoring n and inverting modular exponentiation without d.
- **Mathematical core:** Relies on Euler's theorem $m^{\varphi(n)} \equiv 1 \pmod{n}$ for correct decryption.
- **Essence:** Modular exponentiation is a one-way trapdoor function—easy to compute but hard to invert without the secret key.

Diffie-Hellman Key Exchange

A public key exchange over insecure channels. Let $G = \langle g \rangle$ be cyclic of order q.

Alice $\to g^a$, Bob $\to g^b$; both derive g^{ab}.

Security: Rooted in the **Discrete Logarithm Problem (DLP)** — recovering a from g^a is computationally hard.

RSA vs ECC

- **ECC advantages:** Higher security per bit; compact keys and efficient scalar multiplication on $E(\mathbb{F}_q)$ minimize computation, memory, and energy — ideal for constrained systems.
- **RSA advantages:** Based on modular exponentiation in $(\mathbb{Z}/n\mathbb{Z})^\times$; simple arithmetic, long-term standardization, and broad interoperability.
- **Practical considerations:** ECC needs safe curve choice and constant-time arithmetic to avoid side-channel attacks; RSA requires large moduli ($n = pq$) for comparable security, raising computation cost.

Feature	RSA	ECC
Algebraic Base	$(\mathbb{Z}/n\mathbb{Z})^\times$	$E(\mathbb{F}_q)$, elliptic curve group
Security Basis	Integer Factorization	Elliptic Curve Discrete Logarithm (ECDLP)
Key Size	Large (2048–4096 bits)	Compact ($\approx$ 256 bits for equivalent strength)
Efficiency	Classical, slower	Modern, faster, energy-efficient

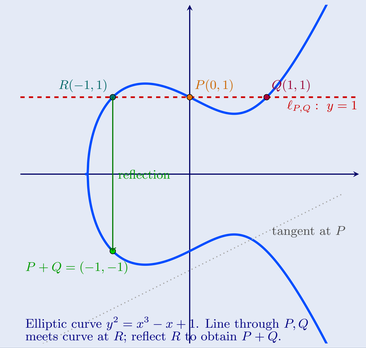
Interpretation: Both use one-way functions, but ECC gains extra security from algebraic geometry.

What Next?

- **Quantum Threat:** Shor's algorithm breaks RSA & ECC.
- **Post-Quantum Shift:** Move to quantum-resistant schemes — Lattice (NTRU, Kyber), Code (McEliece), Hash & Multivariate.
- **Core Idea:** From number/elliptic arithmetic $\to$ lattice geometry & combinatorial hardness.

Introduction to Elliptic Curves

An elliptic curve over a field K is a smooth projective curve of genus one possessing a rational base point O. In affine form (for char $K \neq 2, 3$): $E : y^2 = x^3 + ax + b, \quad \Delta = -16(4a^3 + 27b^2) \neq 0$



Geometric group law: for $P, Q \in E$ the line through P and Q meets E at R', and reflecting R' across the x-axis gives $R = P + Q$. The point at infinity $\mathcal{O}$ is the identity, and the inverse of (x, y) is $(x, -y)$.

Elliptic curves power ECC, pairings, and signatures — secure, and efficient.

Group Law on Elliptic Curves

The set $E(K)$ of K-rational points is an abelian group: for $P, Q \in E(K)$ the line through P, Q meets E at R, and

$$P + Q = -R$$

(the reflection of R across the x-axis). This defines a complete algebraic group law, so E is a group variety; the tangent case $P = Q$ yields the duplication formula.

Mordell's Theorem: $E(K) \cong E(K)_{\text{tors}} \oplus \mathbb{Z}^r$,

Elliptic Curve Cryptography

ECC is built upon the arithmetic of elliptic curves over finite fields $\mathbb{F}_q$. Its **security** relies on the **Elliptic Curve Discrete Logarithm Problem (ECDLP)** — given points $P, Q \in E(\mathbb{F}_q)$, find k such that $Q = kP$.
The difficulty of solving ECDLP underpins asymmetric cryptographic schemes like **ECDH** (key exchange) and **ECDSA** (digital signatures). ECC achieves **RSA-equivalent security** with **exponentially smaller key sizes**, due to the rich structure of $E(\mathbb{F}_q)$.

References

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- J. H. Silverman & J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.