Number Theory and Cryptography

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CHAPTER 1

FOUNDATIONS OF ALGEBRAIC GEOMETRY & VARIETIES

1.1 Introduction & Motivation

Finite fields and algebraic closures

We begin with finite fields. The finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, can be extended to larger fields such as $\mathbb{F}_{p^2}$, and so on, up to the algebraic closure $\overline{\mathbb{F}_p}$. Thus, we have the inclusion:

$$\mathbb{F}_p \subset \mathbb{F}_{p^2} \subset \overline{\mathbb{F}_p}.$$

We are going to study polynomials over these fields, specifically $\mathbb{F}_p[x_1, \ldots, x_n]$. Through these notes, we will primarily focus on the case where $n = 2$, i.e., polynomials in two variables.[**? ? ? ? ?**]

Arithmetic geometry

Algebraic Geometry studies polynomial equations such as:

$$x^2 - 2 = 0, \quad y^2 = x^3 + x, \quad y^2 = x^n + x,$$

from a geometric point of view. This means we are interested in properties that can be understood or even *visualized*, sometimes even over finite fields like $\mathbb{F}_p$.

Arithmetic Geometry

Arithmetic Geometry, on the other hand, studies *counting questions*. For example, we may ask: How many points (or solutions, roots, or zeros) exist on the curve

$$y^2 = x^3 \pmod{p}\,?$$

Solution. We want to find the number of solutions

$$(x, y) \in \mathbb{F}_p \times \mathbb{F}_p \text{ to } y^2 \equiv x^3 \pmod{p}.$$

For each fixed $x \in \mathbb{F}_p$, the number of corresponding y-values is

$$1 + \chi(x^3),$$

where χ denotes the Legendre symbol modulo p, with $\chi(0) = 0$.
Hence, the total number of affine solutions is

$$N = \sum_{x \in \mathbb{F}_p} \Big(1 + \chi(x^3)\Big) = p + \sum_{x \in \mathbb{F}_p} \chi(x^3).$$

For $x \neq 0$, we have

$$\chi(x^3) = \chi(x)^3 = \chi(x),$$

since $\chi(x) = \pm 1$ for nonzero x. Therefore,

$$\sum_{x \in \mathbb{F}_p} \chi(x^3) = \sum_{x \in \mathbb{F}_p^\times} \chi(x) = 0,$$

because the Legendre symbol sums to zero over all nonzero elements of $\mathbb{F}_p$.
Thus, the total number of affine solutions is

$$N = p.$$

If we also include the single point at infinity in the projective curve, the total number of points is

$$N = p + 1.$$

(Later on we will see that this is a special case of a more general result.)

Algorithmic complexity and NP-completeness

In these notes, we are also interested in algorithmic problems. Given polynomials $f_i \in \mathbb{F}_p[x_1, \ldots, x_n]$, we ask:

$$\text{Does there exist } P \in \mathbb{F}_p^n \text{ such that } f_i(P) = 0 \text{ for all } i\,?$$

A natural question is: How difficult is this problem?

Complexity and Simplifications

It turns out that this problem is **NP-complete**, even when the degree of each f_i is only 2 and $p = 2$. Therefore, efficient or practical algorithms are unlikely to exist in general.

However, in these notes, we will focus on the simpler case where $n = 2$, that is, on *curves* over $\mathbb{F}_q$. This case is expected to be much easier to handle. In fact, much of modern

mathematics has grown from studying such objects.

Course Overview

The first part of these notes will cover the fundamentals of Algebraic Geometry that are needed for counting zeros of polynomials. The second part will apply these ideas to Computer Science problems.

We will study curves using the modern language of Algebraic Geometry—through concepts such as *varieties*, *morphisms*, and *function fields*. The guiding idea is that the geometric properties of a curve are reflected in the algebraic properties of the functions defined on it.

Studying Curves Algebraically

To study the roots of an equation like

$$y^2 = x^3 + x,$$

we look at the quotient ring:

$$\mathbb{F}_p[x, y]/\langle y^2 - x^3 - x \rangle.$$

One important result we will encounter is the **Riemann–Roch theorem**, which defines the *genus* of a curve in a purely algebraic way.
We will study this in detail later on in Chapter **??**, where we will develop the necessary tools to state and prove the Riemann–Roch theorem.
We will also prove an estimate for the number of points on a curve over a finite field—this is known as the **Riemann Hypothesis for Curves**.

Applications

There are many applications of these ideas, including:

- Algorithms for counting points on curves (an open problem: finding a fast algorithm for curves),
- Integer factorization algorithms (open problem),
- Computing $\sqrt{a}$ or k-th roots of a in $\mathbb{F}_p$,
- Cryptographic systems,
- Coding theory.

1.2 Affine Varieties

Affine n-space and coordinate rings

Let k be a field (e.g., $\mathbb{F}_p$, $\mathbb{C}$).

Definition 1.2.1: Affine n-space

Affine n-space over k is the set

$$\mathbb{A}_k^n := k^n.$$

If $P \in \mathbb{A}_k^n$ and $P = (a_1, \ldots, a_n)$, then $a_i \in k$ is the i-th coordinate of P.

The polynomial ring

$$A := k[x_1, \ldots, x_n]$$

is the *coordinate ring* of $\mathbb{A}_k^n$.

Functions on affine space

For $f \in A$, define a function $f : \mathbb{A}_k^n \to k$ by

$$P \mapsto f(P).$$

The *zeros* of f are

$$Z(f) = \{P \in \mathbb{A}_k^n \mid f(P) = 0\} \subseteq \mathbb{A}_k^n.$$

For a subset $T \subseteq A$, define

$$Z(T) := \{P \in \mathbb{A}_k^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

Example 1.2.1:

$$Z(x_1^2, x_1 + x_2) = \{(0,0)\}, \quad Z(x_1^2) = \{(0,t) \mid t \in k\}.$$

Definition 1.2.2: Algebraic Set

A subset $Y \subseteq \mathbb{A}_k^n$ is *algebraic* or *closed* if $Y = Z(T)$ for some $T \subseteq A$.

Example 1.2.0.1. *$\mathbb{C} = \mathbb{A}_{\mathbb{C}}^1$ is closed (as $\mathbb{C} = Z(0)$). $\mathbb{C} \setminus \{0\}$ is not closed.*

Remark. The complement of a closed set is *open*.

Definition 1.2.3: Ideal of a Set

For $Y \subseteq \mathbb{A}_k^n$, define

$$I(Y) := \{f \in A \mid f(P) = 0 \text{ for all } P \in Y\}.$$

An ideal $I \subseteq A$ satisfies $I \triangleleft A$ if I is a subring and $a \cdot I \subseteq I$ for all $a \in A$. We have the associations:

$$\begin{array}{ccc} \text{closed subsets of } \mathbb{A}^n_k & \xrightarrow{I(\cdot)} & \text{ideals of } A \\ Y & \mapsto & I(Y) \\ Z(T) & \leftarrow & T \\ & \xleftarrow{Z(\cdot)} & \end{array}$$

Remark. If I is the ideal generated by the set T, denoted $I = \langle T \rangle$, then $Z(T) = Z(I)$. Therefore, every algebraic set can be written as $Z(I)$ for some ideal $I \triangleleft A$. By the Hilbert Basis Theorem, every ideal in A is finitely generated, so we can always take T to be a finite set of polynomials.

The Zariski Topology

Proposition 1.2.0.2 (Properties of Open Sets in the Zariski Topology)**.** (a) The empty set ϕ and the entire space $\mathbb{A}^n$ are open.

(b) The union of an arbitrary collection of open sets, $\{Y_i\}_{i \in I}$, is open.

(c) The intersection of a finite collection of open sets, $\{Y_i\}_{i=1}^m$, is open.

Proof. **(a) Proof that ϕ and $\mathbb{A}^n$ are open:**
A set is open if its complement is a closed (algebraic) set.

- The complement of $\mathbb{A}^n$ is the empty set, ϕ. The empty set can be written as the zero set of the constant polynomial 1, i.e., $\phi = Z(1)$. Since ϕ is a closed set, its complement, $\mathbb{A}^n$, is **open**.
- The complement of ϕ is the entire space $\mathbb{A}^n$. The entire space can be written as the zero set of the zero polynomial, i.e., $\mathbb{A}^n = Z(0)$. Since $\mathbb{A}^n$ is a closed set, its complement, ϕ, is **open**.

(b) Proof that an arbitrary union of open sets is open:
Let $\{Y_i\}_{i \in I}$ be an arbitrary collection of open sets.

- By definition, each Y_i is the complement of a closed set $Z(T_i)$ for some set of polynomials T_i. So, $Y_i = \mathbb{A}^n \setminus Z(T_i)$.
- We examine the union:
$$\bigcup_{i \in I} Y_i = \bigcup_{i \in I} (\mathbb{A}^n \setminus Z(T_i))$$
- Using De Morgan's laws, the union of complements is the complement of the intersection:
$$\bigcup_{i \in I} Y_i = \mathbb{A}^n \setminus \left(\bigcap_{i \in I} Z(T_i) \right)$$

- The arbitrary intersection of closed sets is a closed set. Specifically, $\bigcap_{i\in I} Z(T_i) = Z(\bigcup_{i\in I} T_i)$.
- Since $\bigcap_{i\in I} Z(T_i)$ is a closed set, its complement, $\bigcup_{i\in I} Y_i$, is by definition **open**.

(c) Proof that a finite intersection of open sets is open:
Let $\{Y_i\}_{i=1}^m$ be a finite collection of open sets.

- By definition, each $Y_i = \mathbb{A}^n \setminus Z(T_i)$ for some set of ideals (or sets of polynomials) T_i.
- We examine the intersection:

$$\bigcap_{i=1}^{m} Y_i = \bigcap_{i=1}^{m} (\mathbb{A}^n \setminus Z(T_i))$$

- Using De Morgan's laws, the intersection of complements is the complement of the union:

$$\bigcap_{i=1}^{m} Y_i = \mathbb{A}^n \setminus \left(\bigcup_{i=1}^{m} Z(T_i) \right)$$

- The union of a finite number of closed sets is a closed set. We know that $Z(T_1) \cup Z(T_2) = Z(T_1T_2)$, where T_1T_2 is the set of all products of polynomials from T_1 and T_2. By induction, the finite union $\bigcup_{i=1}^m Z(T_i)$ is equal to $Z(T_1T_2\cdots T_m)$, which is a closed set.
- Since $\bigcup_{i=1}^m Z(T_i)$ is a closed set, its complement, $\bigcap_{i=1}^m Y_i$, is by definition **open**.

□

E.g., $\bigcup_{\alpha\in\mathbb{C}\setminus\{0\}}\{\alpha\} = \mathbb{A}^1_{\mathbb{C}} \setminus S \neq Z(T)$ (Not closed).

Definition 1.2.4: Zariski Topology

The **Zariski topology** on $\mathbb{A}^n_k$ is defined by taking the open subsets to be the complements of the algebraic sets.

To verify this is a valid topology, we must show that the empty set and the whole space are closed, and that finite unions and arbitrary intersections of closed sets are also closed.

1.3 Quasi-Affine Varieties

Varieties and Irreducibility

Definition 1.3.1: Reducible and Irreducible Sets

A non-empty topological space Y is **reducible** if it can be written as the union of two proper closed subsets, $Y = Y_1 \cup Y_2$, where $Y_1 \subsetneq Y$ and $Y_2 \subsetneq Y$. A non-empty topological space is **irreducible** if it is not reducible. By convention, the empty set is considered reducible.

If a space Y is reducible, $Y = Y_1 \cup Y_2$, we can study its properties by studying its simpler components Y_1 and Y_2. If these are also reducible, we can decompose them further. E.g., $\mathbb{A}^1_{\mathbb{C}}$ is irreducible.

$$\Big[\text{Say, } \mathbb{A}^1_{\mathbb{C}} = Y_1 \cup Y_2 := Z(I_1) \cup Z(I_2) = Z(I_1 I_2) \implies I_1 I_2 = \langle 0 \rangle \implies I_1 = \langle 0 \rangle \text{ or } I_2 = \langle 0 \rangle.\Big]$$

Affine and Quasi-affine Varieties

Definition 1.3.2: Affine Variety

An **affine variety** is an irreducible closed subset of $\mathbb{A}^n$.

Definition 1.3.3: Quasi-affine Variety

A **quasi-affine variety** is an open subset of an affine variety.

Example 1.3.1:

1. $Y = Z(x_1 x_2) \subset \mathbb{A}^2$. This is the union of the two coordinate axes: $Y = Z(x_1) \cup Z(x_2)$. Both $Z(x_1)$ (the x_2-axis) and $Z(x_2)$ (the x_1-axis) are proper closed subsets of Y. Therefore, Y is reducible and is not an affine variety.
2. $Y = Z(x_1^2) \subset \mathbb{A}^2$. Any point (a_1, a_2) where $a_1^2 = 0$ must have $a_1 = 0$. Thus, $Z(x_1^2) = Z(x_1)$, which is the x_2-axis. This is an irreducible set (isomorphic to $\mathbb{A}^1$), so it is an affine variety.
3. $Z(x_1^2) \setminus Z(x_1^2, x_2)$ is quasi-affine (The reason for it to be quasi-affine is that it is an open subset of the affine variety $Z(x_1^2)$).

The Nullstellensatz and the Algebra-Geometry Dictionary

The main goal is to prove one of the most fundamental theorems in algebraic geometry, Hilbert's Nullstellensatz. For an ideal J in a polynomial ring $A = k[x_1, \ldots, x_n]$ (where k is an algebraically closed field like $\mathbb{C}$), the theorem states that the ideal of polynomials vanishing on the zero set of J is precisely the radical of J.

In mathematical notation:

$$I(Z(J)) = \sqrt{J}$$

Where:

- $Z(J)$ **(The Zero Set):** The set of all points $(a_1, \ldots, a_n) \in k^n$ where every polynomial in the ideal J evaluates to zero. This is the **geometric** object.
- $I(V)$ **(The Ideal of a Set):** The set of all polynomials that evaluate to zero for every point in the set V.
- $\sqrt{J}$ **(The Radical Ideal):** The set of all polynomials g such that some power of g is in J. That is, $g \in \sqrt{J}$ if $g^e \in J$ for some positive integer e. This is the **algebraic** object.

The proof relies on a simpler version of the theorem, known as the Weak Nullstellensatz.

The Foundation: Weak Nullstellensatz

Theorem (Weak Nullstellensatz)

Let k be an algebraically closed field and J be an ideal in $A = k[x_1, \ldots, x_n]$. The zero set $Z(J)$ is empty if and only if J is the entire ring A (i.e., $1 \in J$).

Proof Sketch (for $Z(J) = \emptyset \implies J = A$):

1. **Assumption:** Assume $Z(J) = \emptyset$, but for the sake of contradiction, assume J is a proper ideal ($J \neq A$).
2. **Find a Maximal Ideal:** Since J is a proper ideal, it must be contained within some maximal ideal $\mathfrak{m}$, so $J \subseteq \mathfrak{m}$.
3. **From Ideals to Zero Sets:** This inclusion implies $Z(\mathfrak{m}) \subseteq Z(J)$. If we can show that $Z(\mathfrak{m}) \neq \emptyset$, we will have a contradiction.
4. **The Key Insight (Zariski's Lemma):**
   - Since $\mathfrak{m}$ is a maximal ideal, the quotient ring $A/\mathfrak{m}$ is a field.
   - This field is a finitely generated k-algebra, and it contains k.
   - By Zariski's Lemma, this implies $A/\mathfrak{m}$ is a finite algebraic extension of k.
   - **Because k is algebraically closed**, the only such extension is k itself.
   - Therefore, we must have an isomorphism $A/\mathfrak{m} \cong k$.
5. **Finding a Point:** This isomorphism gives a surjective homomorphism $\phi : A \to k$ with $\ker(\phi) = \mathfrak{m}$. Let $a_i = \phi(x_i) \in k$. This defines a point $P = (a_1, \ldots, a_n) \in k^n$. For any polynomial $f \in \mathfrak{m}$, we have $\phi(f) = f(a_1, \ldots, a_n) = 0$. This means the point P is in the zero set of $\mathfrak{m}$. Thus, $Z(\mathfrak{m}) \neq \emptyset$.

6. **Contradiction:** We have found a point in $Z(\mathfrak{m}) \subseteq Z(J)$, which contradicts our initial assumption that $Z(J)$ is empty. Therefore, the assumption that J was a proper ideal must be false, so $J = A$.

Proof of the Full Nullstellensatz: $I(Z(J)) = \sqrt{J}$

Part A: The "Easy" Direction ($\sqrt{J} \subseteq I(Z(J))$)

- Let $g \in \sqrt{J}$. By definition, this means $g^e \in J$ for some integer $e \geq 1$.
- Let $P \in Z(J)$. By definition, $f(P) = 0$ for all $f \in J$.
- Since $g^e \in J$, we must have $g^e(P) = (g(P))^e = 0$.
- Since $g(P)$ is an element of the field k, this implies $g(P) = 0$.
- Because this holds for every point $P \in Z(J)$, g vanishes on $Z(J)$. By definition, this means $g \in I(Z(J))$.

Part B: The "Hard" Direction ($I(Z(J)) \subseteq \sqrt{J}$)

This proof uses the famous **Rabinowitsch trick**.

1. **Setup:** Let $g \in I(Z(J))$. This means g vanishes everywhere on $Z(J)$. We want to show that $g \in \sqrt{J}$.
2. **The Trick:** Introduce a new variable, y, and consider the polynomial ring $A' = A[y] = k[x_1, \ldots, x_n, y]$. In this new ring, define a new ideal:

$$J' = JA' + \langle 1 - gy \rangle$$

3. **Zero Set of J':** Consider the zero set $Z(J') \subseteq k^{n+1}$. A point $(P, b) \in k^{n+1}$ is in $Z(J')$ if $f(P) = 0$ for all $f \in J$ and $1 - g(P)b = 0$.
4. The first condition means $P \in Z(J)$. But by our initial assumption, $g \in I(Z(J))$, so $g(P) = 0$. Substituting this into the second condition gives $1 - (0) \cdot b = 0$, which simplifies to the contradiction $1 = 0$.
5. **Apply Weak Nullstellensatz:** Since there are no points in $Z(J')$, we have $Z(J') = \emptyset$. By the Weak Nullstellensatz, the ideal J' must be the entire ring A'. This means $1 \in J'$.

6. **Unpack the Algebra:** Since $1 \in J'$, we can write 1 as a linear combination of its generators. Let $J = \langle f_1, \ldots, f_m \rangle$.

$$1 = \sum_{i=1}^{m} a_i(x, y) f_i(x) + a_0(x, y)(1 - gy)$$

for some polynomials $a_0, a_i \in A'$.

7. **Final Step:** In the field of fractions of A', we can make the formal substitution $y = 1/g$:

$$1 = \sum_{i=1}^{m} a_i(x, 1/g) f_i(x)$$

The terms $a_i(x, 1/g)$ are rational functions in g. To clear the denominators, we multiply the entire equation by a sufficiently large power of g, say g^e:

$$g^e = \sum_{i=1}^{m} \underbrace{(g^e \cdot a_i(x, 1/g))}_{\text{This is a polynomial in A}} \cdot f_i(x)$$

This shows that g^e is a linear combination of the generators of J, which means $g^e \in J$.

8. **Conclusion:** By the definition of the radical ideal, if $g^e \in J$, then $g \in \sqrt{J}$.

Summary: The Algebra-Geometry Dictionary

The Nullstellensatz establishes a profound duality between geometry and algebra.

- There is a **one-to-one, order-reversing correspondence** between **algebraic sets** in k^n (the geometric side) and **radical ideals** in $k[x_1, \ldots, x_n]$ (the algebraic side).
- The maps that form this correspondence are $V \mapsto I(V)$ and $J \mapsto Z(J)$.
- This allows geometric properties of sets to be studied using the algebraic properties of their corresponding ideals, and vice versa. For example, an irreducible algebraic set corresponds to a prime ideal.

1.4 The Correspondence

The Nullstellensatz establishes a fundamental, order-reversing correspondence between geometry (algebraic sets) and algebra (ideals).

Corollary 1.4.1: Ideal-Variety Correspondence There is a one-to-one, inclusion-reversing correspondence between:

$$\{\text{algebraic sets in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } A\}$$

Given by the maps $\mathcal{I}$ and $\mathcal{Z}$.

Under this correspondence, we can ask what properties of an algebraic set correspond to what properties of its ideal.

Proposition 1.4.1: Affine Varieties and Prime Ideals

An algebraic set Y is an affine variety (i.e., is irreducible) if and only if its ideal $\mathcal{I}(Y)$ is a prime ideal.

Proof. ($\Rightarrow$) Let Y be an affine variety, so it is irreducible. We know $I = \mathcal{I}(Y)$ is a radical ideal. Suppose $fg \in I$ for some $f, g \in A$. Then $Y \subseteq \mathcal{Z}(fg) = \mathcal{Z}(f) \cup \mathcal{Z}(g)$. Let $Y_1 = Y \cap \mathcal{Z}(f)$ and $Y_2 = Y \cap \mathcal{Z}(g)$. These are closed sets in Y. We have $Y = Y_1 \cup Y_2$. Since Y is irreducible, either $Y = Y_1$ or $Y = Y_2$. If $Y = Y_1$, then $Y \subseteq \mathcal{Z}(f)$, which means f vanishes on all of Y. Thus, $f \in \mathcal{I}(Y) = I$. If $Y = Y_2$, then $Y \subseteq \mathcal{Z}(g)$, which means $g \in \mathcal{I}(Y) = I$. In either case, we have shown that if $fg \in I$, then $f \in I$ or $g \in I$. This is the definition of a prime ideal.

($\Leftarrow$) Let I be a prime ideal. A prime ideal is always radical, so $Y = \mathcal{Z}(I)$ is an algebraic set with $\mathcal{I}(Y) = I$. Suppose $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are closed subsets of Y. Then $I = \mathcal{I}(Y) = \mathcal{I}(Y_1 \cup Y_2) = \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2)$. Assume for contradiction that $Y_1 \neq Y$ and $Y_2 \neq Y$. Then $\mathcal{I}(Y_1) \supsetneq I$ and $\mathcal{I}(Y_2) \supsetneq I$. So we can pick $f \in \mathcal{I}(Y_1) \setminus I$ and $g \in \mathcal{I}(Y_2) \setminus I$. The product fg vanishes on Y_1 (since f does) and on Y_2 (since g does). So fg vanishes on $Y_1 \cup Y_2 = Y$. This means $fg \in \mathcal{I}(Y) = I$. But since I is prime, this implies $f \in I$ or $g \in I$, which is a contradiction. Therefore, our assumption was false, and either $Y_1 = Y$ or $Y_2 = Y$. This shows Y is irreducible. □

This correspondence can be refined further. The smallest non-empty algebraic sets are single points.

Proposition 1.4.0.1. The points of $\mathbb{A}^n$ are in one-to-one correspondence with the maximal ideals of A.

Proof. Let $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$. Consider the ideal $m_P = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. The quotient ring A/m_P is isomorphic to $\Bbbk$ via the evaluation map $f \mapsto f(P)$. Since $\Bbbk$ is a field, m_P is a maximal ideal. Conversely, let M be a maximal ideal of A. By the proof of the Weak Nullstellensatz, we know that $\mathcal{Z}(M)$ is a single point $P = (a_1, \ldots, a_n)$ and that M is the ideal of all functions vanishing at that point, which is precisely m_P. So $\mathcal{I}(P) = m_P$. □

Summary of the Correspondence

Geometry (Subsets of $\mathbb{A}^n$)	**Algebra (Ideals of $A = \Bbbk[x_1, \ldots, x_n]$)**
Affine space $\mathbb{A}^n$	Zero ideal $\langle 0 \rangle$
Empty set $\emptyset$	The whole ring A
Algebraic Set	Radical Ideal
Affine Variety (irreducible)	Prime Ideal
Point $P \in \mathbb{A}^n$	Maximal Ideal

1.5 Functions on Varieties

Definition 1.5.1: Coordinate Ring

Let $Y \subseteq \mathbb{A}^n$ be a closed set (an algebraic set). The **coordinate ring** of Y, denoted $A(Y)$, is the quotient ring

$$A(Y) := A/\mathcal{I}(Y) = \Bbbk[x_1, \ldots, x_n]/\mathcal{I}(Y)$$

The elements of $A(Y)$ can be thought of as equivalence classes of polynomials, where two polynomials are equivalent if they agree on all points of Y. Thus, $A(Y)$ is the ring of polynomial functions from Y to $\Bbbk$.

Remark 1.5.1: Important Special Cases

- $A(\mathbb{A}^n) = A/\mathcal{I}(\mathbb{A}^n) = A/\langle 0 \rangle \cong A = \Bbbk[x_1, \ldots, x_n]$.
- If Y is an affine variety, then $\mathcal{I}(Y)$ is a prime ideal. Therefore, the coordinate ring $A(Y) = A/\mathcal{I}(Y)$ is an integral domain.

Example 1.5.1: Let $Y = \mathcal{Z}(f)$ where f is an irreducible polynomial in A. Since A is a Unique Factorization Domain, the ideal $\langle f \rangle$ is prime. Thus Y is an affine variety. Its coordinate ring is $A(Y) = A/\mathcal{I}(Y) = A/\sqrt{\langle f \rangle} = A/\langle f \rangle$.

Definition 1.5.2: Field of Fractions and Function Field

Let B be an integral domain. The **field of fractions** of B, denoted $\mathrm{Frac}(B)$, is the set of formal fractions $\{a/b \mid a, b \in B, b \neq 0\}$, with the usual rules for addition and multiplication. If Y is an affine variety, its coordinate ring $A(Y)$ is an integral domain. The **function field** of Y, denoted $\Bbbk(Y)$, is the field of fractions of $A(Y)$.

$$\Bbbk(Y) := \mathrm{Frac}(A(Y))$$

Its elements are the rational functions on Y.

Exercise Every closed set $Y \subseteq \mathbb{A}^n$ can be expressed as a finite union $Y = \bigcup_{i=1}^m Y_i$ of affine varieties Y_i. If we require that $Y_i \not\subseteq Y_j$ for $i \neq j$, this decomposition is unique up to reordering. The Y_i are called the irreducible components of Y.

Proof. This is a standard result for Noetherian topological spaces. The ring $A = \Bbbk[x_1, \ldots, x_n]$ is a Noetherian ring (by Hilbert's Basis Theorem). This implies that any descending chain of closed sets $Y_1 \supseteq Y_2 \supseteq \ldots$ in $\mathbb{A}^n$ must eventually stabilize. If Y is reducible, we can write $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper closed subsets. If either is reducible, we decompose it further. The Noetherian property guarantees this process must terminate in a finite number of steps, leaving a decomposition into irreducible closed sets (affine varieties). Uniqueness is proven by showing that any irreducible component in one decomposition must be equal to an irreducible component in another. □

Exercise In $A = \Bbbk[x_1, \ldots, x_n]$, every ideal $I \triangleleft A$ is finitely generated.

Proof. This is a direct statement of the **Hilbert Basis Theorem**. The proof is a classic result in commutative algebra. It proceeds by induction on the number of variables, showing that if a ring R is Noetherian, then the polynomial ring $R[x]$ is also Noetherian. Since a field $\Bbbk$ is trivially Noetherian, it follows that $\Bbbk[x_1]$ is Noetherian, and by induction $\Bbbk[x_1, \ldots, x_n]$ is Noetherian. Being a Noetherian ring means every ideal is finitely generated. □

Exercise Any radical ideal I can be expressed as a finite intersubsection $I = \bigcap_{i=1}^m P_i$ where the P_i are prime ideals.

Proof. This is the algebraic counterpart to the first exercise. Let $Y = \mathcal{Z}(I)$. From Exercise 1, we can write $Y = \bigcup_{i=1}^m Y_i$ where Y_i are affine varieties. Taking the ideal of both sides, we get:

$$I = \mathcal{I}(Y) = \mathcal{I}(\bigcup_{i=1}^{m} Y_i) = \bigcap_{i=1}^{m} \mathcal{I}(Y_i)$$

Let $P_i = \mathcal{I}(Y_i)$. Since each Y_i is an affine variety, each P_i is a prime ideal. This gives the desired decomposition . □

1.6 Dimension of Affine Varieties

Intuitive and Formal Definitions

Intuitively, we are familiar with the concept of dimension from linear algebra. For example, we know that:

- $\dim(\{\overline{0}\}) = 0$

We wish to establish a rigorous definition for the dimension of algebraic sets.

Definition 1.6.1: Dimension of an Affine Variety

The **dimension** of an affine variety (AV) Y, denoted $\dim(Y)$, is the maximum integer $n \in \mathbb{N}$ such that there exists a chain of distinct irreducible algebraic subsets (subvarieties) of the form:

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n = Y$$

where Z_0 is a point $\{P\}$.

Example 1.6.1: Consider the affine line $\mathbb{A}^1_k$. Let $A = k[x_1]$. Any prime ideal in A is of the form $\langle 0 \rangle$ or $\langle x_1 - \alpha \rangle$ for some $\alpha \in k$. This gives rise to the maximal chain of varieties:

$$\mathcal{Z}(\langle x_1 - \alpha \rangle) \subsetneq \mathcal{Z}(\langle 0 \rangle)$$

which corresponds to:

$$\{\alpha\} \subsetneq \mathbb{A}^1_k$$

This is a chain of length 1. Thus, $\dim(\mathbb{A}^1_k) = 1$.

Example 1.6.2: Consider the affine plane $\mathbb{A}^2_k$. We can construct the chain:

$$\mathcal{Z}(\langle x_1, x_2 \rangle) \subsetneq \mathcal{Z}(\langle x_1 \rangle) \subsetneq \mathcal{Z}(\langle 0 \rangle)$$

which corresponds to:

$$\{(0,0)\} \subsetneq \mathbb{A}^1_k \times \{0\} \subsetneq \mathbb{A}^2_k$$

This is a chain of length 2, which implies that $\dim(\mathbb{A}^2_k) \geq 2$. In fact, it can be shown that $\dim(\mathbb{A}^2_k) = 2$.

Definition 1.6.2: Affine Curve

An affine variety of dimension 1 is called an **affine curve**.

Theorem 1.6.1: Hypersurfaces

An affine variety Y in $\mathbb{A}^n_k$ has dimension $n-1$ if and only if $Y = \mathcal{Z}(f)$ for some non-constant, irreducible polynomial $f \in k[x_1, \ldots, x_n]$. Such a variety is called a hypersurface.

Proof. Let $Y = \mathcal{Z}(f_1, \ldots, f_m)$. The core idea is to show that adding a polynomial to the ideal reduces the dimension. Specifically, one can show that if $Y_1 = \mathcal{Z}(f_1)$ is a variety, then $\dim(\mathcal{Z}(f_1, f_2)) < \dim(\mathcal{Z}(f_1))$, provided $\mathcal{Z}(f_2)$ does not contain Y_1. Since the dimension cannot be larger than n, a single irreducible polynomial defines a variety of dimension $n-1$. □

Example 1.6.3: A curve Y in $\mathbb{A}_k^2$ is the zero set $Y = \mathcal{Z}(f(x_1, x_2))$ for an irreducible polynomial $f \in k[x_1, x_2]$.

Remark 1.6.1: The set $Y = \mathcal{Z}(x_1, x_2)$ in $\mathbb{A}_k^2$ is the point $\{(0, 0)\}$, which has dimension 0, not 1. It is not a curve because it cannot be defined by a single polynomial.

Dimension and Transcendence Degree

There is a more algebraic and often more computable way to define dimension using the concept of transcendence degree.

Theorem 1.6.2: Dimension and Transcendence Degree

The dimension of an affine variety Y is equal to the transcendence degree of its coordinate ring $A(Y)$ over the field k.

$$\dim(Y) = \operatorname{trdeg}_k A(Y)$$

Definition 1.6.3: Transcendence Degree

Let B be a domain that is a k-algebra. The **transcendence degree** of the field of fractions $k(B)$ over k, denoted $\operatorname{trdeg}_k(B)$, is the maximum number of algebraically independent elements $y_1, \ldots, y_t \in B$ over k. A set of elements $\{y_1, \ldots, y_t\}$ is **algebraically independent** over k if there is no non-zero polynomial $P \in k[Y_1, \ldots, Y_t]$ such that $P(y_1, \ldots, y_t) = 0$. The elements $y_1, \ldots, y_t$ form a **transcendence basis** if $k(B)$ is a finite (algebraic) extension of the field $k(y_1, \ldots, y_t)$.

Example 1.6.4:

1. $\operatorname{trdeg}_k k = 0$.
2. $\operatorname{trdeg}_k k[x] = 1$. The element x is a transcendence basis.
3. $\operatorname{trdeg}_k k[x, y] = 2$. The elements x, y are a transcendence basis.
4. Let $B = k[x, y]/\langle y^2 - x^3 \rangle$. In this ring, x and y are algebraically dependent. The element $\bar{x}$ (the image of x in B) is a transcendence basis. Thus, $\operatorname{trdeg}_k(B) = 1$.

Example 1.6.5: The field extension $\mathbb{Q} \subset \mathbb{Q}(\mathbb{P})$ is a transcendental extension. The field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$ is an algebraic extension.

Proof. Let $d = \dim(Y)$. Then there exists a maximal chain of affine varieties in Y:

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d = Y$$

where $Z_i = \mathcal{Z}(\mathfrak{p}_i)$ for prime ideals $\mathfrak{p}_i$. This corresponds to a reversed chain of prime ideals in the coordinate ring $A(Y)$:

$$\mathfrak{p}_d \subsetneq \mathfrak{p}_{d-1} \subsetneq \cdots \subsetneq \mathfrak{p}_0$$

where $\mathfrak{p}_d = \langle 0 \rangle$ and $\mathfrak{p}_0$ is a maximal ideal. This in turn gives us a chain of field extensions of the fields of fractions:

$$k(A(Y)/\mathfrak{p}_0) \subsetneq k(A(Y)/\mathfrak{p}_1) \subsetneq \cdots \subsetneq k(A(Y)/\mathfrak{p}_d) = k(Y)$$

This is a maximal chain of field extensions within $k(Y)$. It is a result from commutative algebra that the length of such a chain is equal to the transcendence degree. Each field extension in this chain can be shown to have a transcendence degree of 1 relative to the previous one. Therefore, $\text{trdeg}_k k(A(Y)) = d$. □

Fundamental Computational Problems

Given a set of polynomials $f_1, \ldots, f_m$ in $A = k[x_1, \ldots, x_n]$, let $\mathfrak{a} = \langle f_1, \ldots, f_m \rangle$. We can ask several fundamental questions:

1. **Emptiness Test**: Is the algebraic set $\mathcal{Z}(\mathfrak{a})$ empty? This is equivalent to asking if $1 \in \mathfrak{a}$ (Hilbert's Nullstellensatz). Are there fast algorithms (polynomial time in $n, m, \deg(f_i)$, etc.) for this?

2. **Dimension Computation**: What is $\dim(\mathcal{Z}(\mathfrak{a}))$? The dimension can range from -1 (for the empty set) to n.

3. **Finding Points**: Find a root of the system $f_1 = \cdots = f_m = 0$.

4. **Radical Computation**: Compute the radical of the ideal, $\sqrt{\mathfrak{a}}$.

5. **Irreducibility Test**: Is $\mathcal{Z}(\mathfrak{a})$ an affine variety (i.e., is $\sqrt{\mathfrak{a}}$ a prime ideal)? If not, find its decomposition into irreducible components, i.e., find prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ such that $\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\mathfrak{p}_1) \cup \mathcal{Z}(\mathfrak{p}_2)$.

1.7 Projective Varieties

Motivation

The affine space $\mathbb{A}^n$ is not "complete" in a certain sense (e.g., parallel lines never meet). Projective space is a natural compactification of affine space. The key idea is to work with objects that are invariant under scaling. If $(a_1, \ldots, a_n)$ is a root of a system of equations, we want to consider all points on the line through the origin and $(a_1, \ldots, a_n)$, i.e., all points $(ca_1, \ldots, ca_n)$ for $c \in k^*$, as equivalent. This leads to the study of homogeneous polynomials.

A polynomial f is **homogeneous** if all its monomials have the same total degree.

- $f = x^2 - xy + y^2$ is homogeneous of degree 2.
- $g = x^2 + y^3$ is not homogeneous.

For a homogeneous polynomial f of degree d, we have the property:

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$$

This means that if $f(a_1, \ldots, a_n) = 0$, then $f(\lambda a_1, \ldots, \lambda a_n) = 0$ for any $\lambda \in k$. The zero sets of homogeneous polynomials are unions of lines through the origin.

Projective Space

Definition 1.7.1: Projective n-space

Consider the set $(\mathbb{A}^{n+1})^* = \mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$. We define an equivalence relation $\sim$ on this set: For $\overline{a} = (a_0, \ldots, a_n)$ and $\overline{b} = (b_0, \ldots, b_n)$ in $(\mathbb{A}^{n+1})^*$, we say $\overline{a} \sim \overline{b}$ if there exists a scalar $\lambda \in k^*$ such that $\overline{a} = \lambda \cdot \overline{b}$.
Projective n-space over k, denoted $\mathbb{P}^n_k$, is the set of equivalence classes:

$$\mathbb{P}^n_k := (\mathbb{A}^{n+1}_k)^* / \sim$$

The equivalence class of a point $(a_0, \ldots, a_n)$ is denoted by $(a_0 : a_1 : \cdots : a_n)$ and these are called **homogeneous coordinates**.

Geometrically, $\mathbb{P}^n_k$ is the space of all lines passing through the origin in $\mathbb{A}^{n+1}_k$.

Example 1.7.1: If $k = \mathbb{F}_p$ (the finite field with p elements), then $|\mathbb{A}^{n+1}_k| = p^{n+1}$. Each equivalence class in $(\mathbb{A}^{n+1}_k)^*$ has $p-1$ elements. Therefore, the number of points in projective space is:

$$|\mathbb{P}^n_k| = \frac{p^{n+1} - 1}{p - 1}$$

Example 1.7.2: In $\mathbb{P}^1_{\mathbb{C}}$, the points $(2, 1-i)$ and $(1+i, 1)$ are the same, because $(1+i)(1-i) = 2$. So $(2, 1-i) = (1-i)(1+i, 1)$. Thus $(2 : 1-i) = (1+i : 1)$.

Homogeneous Ideals and Projective Varieties

What is the algebraic analogue of $\mathbb{P}^n_k$? We work with the polynomial ring in $n+1$ variables, $S = k[x_0, \ldots, x_n]$. This ring has a natural **grading**:

$$S = \bigoplus_{d \geq 0} S_d$$

where S_d is the k-vector space of all homogeneous polynomials of degree d. This grading respects multiplication: for any $d, e \in \mathbb{N}$, if $f \in S_d$ and $g \in S_e$, then $f \cdot g \in S_{d+e}$.

Definition 1.7.2: Homogeneous Ideal

An ideal $I \subseteq S$ is called **homogeneous** if for every polynomial $f \in I$, all of its homogeneous components are also in I.

Proposition 1.7.1: Characterizations of Homogeneous Ideals

The following are equivalent for an ideal $I \subseteq S$:

1. I is homogeneous.
2. $I = \bigoplus_{d \geq 0} (I \cap S_d)$.
3. I can be generated by a set of homogeneous polynomials.

Example 1.7.3: Counter-example The ideal $I = \langle x_1^2 + x_2 \rangle$ in $k[x_1, x_2]$ is not homogeneous. Its generator is not homogeneous. We have $I \cap S_1 = \{0\}$ and $I \cap S_2$ does not contain x_1^2 (only multiples of $x_1^2 + x_2$), so $I \neq (I \cap S_1) \oplus (I \cap S_2) \oplus \dots$.

Proposition 1.7.2: Operations on Homogeneous Ideals

The sum, product, intersubsection, and radical of homogeneous ideals are homogeneous.

Definition 1.7.3: Projective Variety

Let T be a set of homogeneous polynomials in $S = k[x_0, \dots, x_n]$. The **projective zero set** of T is

$$\mathcal{Z}_p(T) := \{P \in \mathbb{P}_k^n \mid \forall f \in T, f(P) = 0\}$$

Note that the condition $f(P) = 0$ is well-defined for homogeneous polynomials because if $f(a_0, \dots, a_n) = 0$, then $f(\lambda a_0, \dots, \lambda a_n) = \lambda^{\deg(f)} f(a_0, \dots, a_n) = 0$. A subset of $\mathbb{P}_k^n$ is a **projective algebraic set** if it is of the form $\mathcal{Z}_p(I)$ for some homogeneous ideal $I \subseteq S$.

Example 1.7.4: In $\mathbb{P}_{\mathbb{C}}^1$, with coordinates $(x_0 : x_1)$:

- $\mathcal{Z}_p(x_0^2 - x_1^2) = \mathcal{Z}_p((x_0 - x_1)(x_0 + x_1)) = \{(1:1), (1:-1)\}$.
- $\mathcal{Z}_p(x_0^2) = \{(0:1)\}$.

Projective Varieties and the Zariski Topology

Let k be an algebraically closed field ($k = \overline{k}$). Let $S = k[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables. We refer to the elements of S as polynomials, and if all monomials

in a polynomial have the same total degree, we call it a homogeneous polynomial.

Definition 1.7.4: Projective Algebraic Set

A subset $Y \subseteq \mathbb{P}^n$ is called a **closed set** (or an **algebraic set**) if there exists a set of homogeneous polynomials $T \subseteq S$ such that Y is the zero locus of T. We denote this as:

$$Y = \mathcal{Z}(T) = \{P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in T\}$$

An **open set** in $\mathbb{P}^n$ is the complement of a closed set, i.e., a set of the form $\mathbb{P}^n \setminus Y$ where Y is closed.

Remark 1.7.1: Zariski Topology The family of open sets defined above forms a topology on $\mathbb{P}^n$, which is known as the **Zariski topology**.

Definition 1.7.5: Projective Variety

A **projective variety** is an irreducible closed subset of $\mathbb{P}^n$. A topological space is irreducible if it cannot be written as the union of two proper closed subsets.

Definition 1.7.6: Quasi-Projective Variety

A **quasi-projective variety** is an open subset of a projective variety.

Remark 1.7.2: Dimension of Projective and Quasi-Projective Varieties The dimension of a projective or quasi-projective variety is defined, similarly to the affine case, as the supremum of lengths of chains of irreducible closed subsets.

$\dim(Y) = \sup\{m \mid \emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m \subseteq Y, \text{ where each } Z_i \text{ is irreducible and closed in } Y\}$

Example 1.7.5:

1. In $\mathbb{P}^1_{\mathbb{C}}$, consider the ideal $T = \langle x_0^2 \rangle$. The zero set $\mathcal{Z}(x_0^2) = \mathcal{Z}(x_0)$ is the point $[0:1]$. A single point is an irreducible closed set, so it is a projective variety of dimension 0.

2. In $\mathbb{P}^1_{\mathbb{C}}$, consider $T = \langle x_0^2 - x_1^2 \rangle$. Since $x_0^2 - x_1^2 = (x_0 - x_1)(x_0 + x_1)$, the zero set is $\mathcal{Z}(x_0 - x_1) \cup \mathcal{Z}(x_0 + x_1)$, which corresponds to the two distinct points $\{[1:1], [1:-1]\}$. This is a closed set, but it is not irreducible. Therefore, it is a projective algebraic set, but not a projective variety.

3. In $\mathbb{P}^2_{\mathbb{C}}$, consider $T = \langle x_0^2, x_1 \rangle$. The zero set $\mathcal{Z}(x_0^2, x_1)$ consists of points $[a_0 : a_1 : a_2]$ where $a_0^2 = 0$ and $a_1 = 0$. This implies $a_0 = 0$ and $a_1 = 0$, so the only point is $[0:0:1]$. This is a projective variety. *[Note: The original document calls this a "curve!", which is incorrect; it is a single point and thus 0-dimensional.]*

Ideals and Coordinate Rings

Just as there is a correspondence between algebraic sets in affine space and ideals, a similar relationship exists in projective space, with the condition that all polynomials and ideals must be homogeneous.

Definition 1.7.7: Homogeneous Ideal

For a closed set $Y \subseteq \mathbb{P}^n$, the **homogeneous ideal of Y**, denoted $\mathcal{I}(Y)$, is the ideal generated by all homogeneous polynomials in S that vanish on every point in Y.

$$\mathcal{I}(Y) = \langle f \in S \mid f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y \rangle$$

Definition 1.7.8: Homogeneous Coordinate Ring

The **homogeneous coordinate ring** of a closed set $Y \subseteq \mathbb{P}^n$ is the quotient ring:

$$S(Y) := S/\mathcal{I}(Y)$$

Remark 1.7.3: Projective Nullstellensatz The diagram on page 3 of the notes illustrates the fundamental correspondence of the Projective Nullstellensatz for an algebraically closed field k. Let $S_+ = \langle x_0, \ldots, x_n \rangle$ be the irrelevant ideal.

Geometry in $\mathbb{P}^n$		**Algebra in $S = k[x_0, \ldots, x_n]$**
Closed subsets of $\mathbb{P}^n$	$\longleftrightarrow$	Radical homogeneous ideals $\neq S_+$
$\cup$		$\cap$
$\cap$		$+$
Irreducible closed subsets (Projective Varieties)	$\longleftrightarrow$	Homogeneous prime ideals $\neq S_+$
Points	$\longleftrightarrow$	Homogeneous prime ideals maximal among proper homogeneous ideals other than S_+

The maps are given by $Y \mapsto \mathcal{I}(Y)$ and $I \mapsto \mathcal{Z}(I)$.

Affine Covering of Projective Space

A powerful technique for studying projective space is to cover it with open sets that are themselves affine spaces.

Proposition 1.7.3: Affine Covering of Projective Space

Projective n-space, $\mathbb{P}^n$, has an open covering by $n+1$ sets, each of which is homeomorphic to affine n-space, $\mathbb{A}^n$.

Proof. For each $i \in \{0, 1, \ldots, n\}$, define the open set U_i as:

$$U_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n \mid a_i \neq 0\}$$

Since a point in $\mathbb{P}^n$ must have at least one non-zero coordinate, it is clear that $\mathbb{P}^n = \bigcup_{i=0}^n U_i$.

We define a map $\varphi_i : U_i \to \mathbb{A}^n$ by dehomogenizing with respect to the i-th coordinate:

$$\varphi_i([a_0 : \cdots : a_n]) = \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

This map is well-defined because the ratios are independent of the choice of homogeneous coordinates. One can show that φ_i is a bijection and a homeomorphism, meaning it establishes a one-to-one correspondence between the open sets of U_i (in the subspace topology) and the open sets of $\mathbb{A}^n$ (in the Zariski topology). □

This property is inherited by any projective or quasi-projective variety $Y \subseteq \mathbb{P}^n$. We can study Y by examining its intersubsection with each affine chart:

$$Y = \bigcup_{i=0}^{n} (Y \cap U_i)$$

Each piece $Y_i = Y \cap U_i$ is an open subset of Y, and $\varphi_i(Y_i)$ is a quasi-affine variety in $\mathbb{A}^n$.

1.8 Regular Functions and Morphisms

To study maps between varieties, we need a notion that preserves their algebraic structure. This leads to the concept of morphisms, which are built upon regular functions.

Continuity

In topology, a map $\phi : X \to Y$ is continuous if for every open set $V \subseteq Y$, its preimage $\phi^{-1}(V)$ is open in X. While maps like $\phi(x) = x^2$ on $\mathbb{R}$ are continuous in the standard topology, the Zariski topology is much coarser, so continuity is a weaker condition. We need an additional algebraic condition.

Regular Functions

Definition 1.8.1: Regular Function

Let Y be a quasi-projective variety.

1. A function $f : Y \to k$ is **regular at a point** $P \in Y$ if there exists an open neighborhood U with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S$ of the *same degree* such that $h(Q) \neq 0$ for all $Q \in U$ and $f(Q) = \frac{g(Q)}{h(Q)}$ for all $Q \in U$.

2. A function f is **regular on** Y if it is regular at every point $P \in Y$.

For a quasi-affine variety $Y \subseteq \mathbb{A}^n$, the definition is analogous, but g and h are simply polynomials in the affine coordinate ring $A(Y) = k[x_1, \ldots, x_n]/\mathcal{I}(Y)$, and the condition of being homogeneous of the same degree is not required.

Example 1.8.1:

1. Consider $f(x) = \frac{1}{x^2+1}$ on $\mathbb{A}^1$.
   - On $\mathbb{A}^1_{\mathbb{R}}$, the denominator is never zero, so f is regular everywhere.
   - On $\mathbb{A}^1_{\mathbb{C}}$, the denominator is zero at $x = \pm i$. Thus, f is not regular on all of $\mathbb{A}^1_{\mathbb{C}}$. It is only regular on the open set $\mathbb{A}^1_{\mathbb{C}} \setminus \{i, -i\}$.
2. Consider the function $f([x_0 : x_1]) = \frac{x_0}{x_1}$ on $\mathbb{P}^1_{\mathbb{C}}$. Here $g = x_0$ and $h = x_1$ are homogeneous of degree 1. The function is defined where $x_1 \neq 0$, which is the open set U_1. Thus, f is regular on U_1. It is regular at the point $[0 : 1]$ but not at the point $[1 : 0]$ where the denominator vanishes. *[Note: The original document states the opposite, which appears to be a mistake.]*

Remark 1.8.1: A key theorem states that the only functions that are regular on all of an affine variety Y are the elements of its coordinate ring, $A(Y)$. For $Y = \mathbb{A}^n_k$ with k algebraically closed, the only globally regular functions are the polynomials in $k[x_1, \ldots, x_n]$.

Morphisms

A "good " map between varieties should be continuous and should preserve the structure of regular functions.

Definition 1.8.2: Morphism

Let X and Y be varieties. A **morphism** $\phi : X \to Y$ is a continuous map (in the Zariski topology) such that for every open set $V \subseteq Y$ and every regular function $g : V \to k$, the composition

$$g \circ \phi : \phi^{-1}(V) \to k$$

is a regular function on the open set $\phi^{-1}(V) \subseteq X$. This is often summarized as "a morphism is a continuous map that pulls back regular functions to regular functions."

Rational Functions and Function Fields

In the study of algebraic varieties, we initially define regular functions as those that can be represented by polynomials on affine varieties, or locally as quotients of homogeneous polynomials of the same degree on projective varieties. The ring of globally regular functions on a variety Y is denoted by $\mathcal{O}(Y)$.

However, there are many other functions that are well-behaved (i.e., regular) on some open subset $U \subseteq Y$, but not necessarily on all of Y. This leads to a more general notion. We will use the pair (U, f) to denote a function f that is regular on an open set $U \subseteq Y$.

These pairs allow us to define an equivalence relation which identifies functions that agree on the intersubsection of their domains.

Definition 1.8.3: Equivalence of Local Regular Functions

Let (U, f) and (V, g) be two pairs where f is regular on an open set $U \subseteq Y$ and g is regular on an open set $V \subseteq Y$. We say that they are equivalent, denoted $(U, f) \sim (V, g)$, if the functions agree on the intersubsection of their domains, i.e., $f|_{U \cap V} = g|_{U \cap V}$.
This equivalence relation allows us to define the concept of a rational function. A rational function is an equivalence class of such pairs $[(U, f)]$.

Definition 1.8.4: Function Field

The set of all equivalence classes of pairs (U, f), where U is a non-empty open subset of Y and f is a regular function on U, is called the **function field** of Y. It is denoted by $K(Y)$.

$$K(Y) := \{[(U, f)] \mid U \subseteq Y \text{ is open and non-empty, } f \text{ is regular on } U\}$$

The elements $\alpha \in K(Y)$ are called **rational functions** on Y. If Y is an irreducible variety, $K(Y)$ is a field.

Example 1.8.2:

1. Let $Y = \mathbb{A}^1_k$ be the affine line. The ring of regular functions (polynomials) is $\mathcal{O}(Y) = k[x]$. The function field is the field of rational functions in one variable, $K(Y) = k(x)$. An element of $K(Y)$ is of the form $f(x)/g(x)$ where f, g are polynomials and $g \not\equiv 0$.

2. Let $Y = \mathbb{P}^1_k$ be the projective line. The globally regular functions are just the constants, so $\mathcal{O}(Y) = k$. The function field, however, is much larger. If the homogeneous coordinates on $\mathbb{P}^1_k$ are $[x_0 : x_1]$, then $K(Y)$ is the field of rational functions $f(x_0, x_1)/g(x_0, x_1)$ where f and g are homogeneous polynomials of

the same degree and $g \not\equiv 0$. This field is isomorphic to the field of rational functions in one variable, $k(t)$, where $t = x_1/x_0$.

$$K(\mathbb{P}^1_k) \cong K(\mathbb{A}^1_k) = k(x)$$

The Stalk of Germs at a Point

For a specific point $P \in Y$, it is useful to study the behavior of functions in arbitrarily small neighborhoods of P. This leads to the idea of "germs" of functions.

Definition 1.8.5: Germs of Functions

The set of **germs** of regular functions on Y near a point $P \in Y$ is the set of equivalence classes of pairs (U, f) where $P \in U \subseteq Y$ is an open neighborhood of P and f is regular on U. The equivalence relation is the same as before: $(U, f) \sim (V, g)$ if $f = g$ on $U \cap V$. This set is denoted by $\mathcal{O}_P(Y)$ or $\mathcal{O}_{Y,P}$.

$$\mathcal{O}_P(Y) := \{[(U, f)] \mid P \in U \subseteq Y \text{ is open, } f \text{ is regular on } U\}/\sim$$

This set $\mathcal{O}_P(Y)$ forms a ring, often called the **local ring of Y at P** or the **stalk** of the sheaf of regular functions at P.

Definition 1.8.6: Ideal of Germs Vanishing at a Point

Within the ring $\mathcal{O}_P(Y)$, the set of germs of functions that vanish at P forms a special subset.

$$\mathfrak{m}_P := \{[(U, f)] \in \mathcal{O}_P(Y) \mid f(P) = 0\}$$

This set $\mathfrak{m}_P$ is an ideal of the ring $\mathcal{O}_P(Y)$.

Example 1.8.3: Example: The Affine Line Let $Y = \mathbb{A}^1_k$ with coordinate ring $\mathcal{O}(Y) = k[x]$, and let P be the origin, corresponding to the point $x = 0$. A rational function $g(x)/h(x)$ is regular in a neighborhood of P if and only if $h(P) \neq 0$. Therefore, the local ring at the origin is:

$$\mathcal{O}_P(Y) = \left\{ \frac{g(x)}{h(x)} \mid g, h \in k[x], h(0) \neq 0 \right\}$$

For instance:

- $1/x \notin \mathcal{O}_P(Y)$ because the denominator is zero at $P = (0)$.
- $1/(1 - x) \in \mathcal{O}_P(Y)$ because the denominator is non-zero at $P = (0)$.
- $1/(x^2 + x + 1) \in \mathcal{O}_P(Y)$ for the same reason.

The ideal of germs vanishing at P is:

$$\mathfrak{m}_P = \left\{ \frac{g(x)}{h(x)} \in \mathcal{O}_P(Y) \mid g(0) = 0 \right\}$$

This ideal is generated by the function x. Examples of elements in $\mathfrak{m}_P$ include x, $x/(1-x)$, etc.

Properties of the Local Ring $\mathcal{O}_P(Y)$

The ring of germs $\mathcal{O}_P(Y)$ has a very special algebraic structure.

Proposition 1.8.1: Properties of the Local Ring at a Point

Let P be a point on a variety Y over a field k.

1. $\mathcal{O}_P(Y)$ is a k-algebra, and $\mathfrak{m}_P$ is an ideal of $\mathcal{O}_P(Y)$.
2. The quotient ring $\mathcal{O}_P(Y)/\mathfrak{m}_P$ is a field extension of k. This implies that $\mathfrak{m}_P$ is a maximal ideal.
3. $\mathfrak{m}_P$ is the *unique* maximal ideal of $\mathcal{O}_P(Y)$.

Rings that have a unique maximal ideal are called **local rings**.

Proof Sketch

1. **(k-algebra with ideal $\mathfrak{m}_P$):** Addition and multiplication of germs are defined by performing the operations on representative functions on the intersubsection of their domains. This makes $\mathcal{O}_P(Y)$ a ring. It is a k-algebra because we can embed k into $\mathcal{O}_P(Y)$ via constant functions. It is straightforward to check that $\mathfrak{m}_P$ is an ideal.
2. **($\mathfrak{m}_P$ is maximal):** We can define an evaluation map $ev_P : \mathcal{O}_P(Y) \to k$ by $ev_P([(U, f)]) = f(P)$. This map is a surjective ring homomorphism whose kernel is precisely $\mathfrak{m}_P$. By the First Isomorphism Theorem, $\mathcal{O}_P(Y)/\mathfrak{m}_P \cong k$. Since k is a field, the ideal $\mathfrak{m}_P$ must be maximal.
3. **($\mathfrak{m}_P$ is the unique maximal ideal):** To show $\mathcal{O}_P(Y)$ is a local ring, we need to show that any element not in $\mathfrak{m}_P$ is a unit (i.e., has a multiplicative inverse). Let $[(U, f)] \in \mathcal{O}_P(Y) \setminus \mathfrak{m}_P$. By definition of $\mathfrak{m}_P$, this means $f(P) \neq 0$. Since f is continuous, there exists a smaller open neighborhood $V \subseteq U$ of P where f is non-zero everywhere. On this neighborhood V, the function $1/f$ is well-defined and regular. Therefore, the germ $[(V, 1/f)]$ is an element of $\mathcal{O}_P(Y)$ and serves as the inverse to $[(U, f)]$. Thus, every element outside $\mathfrak{m}_P$ is a unit.

Now, if I is any proper ideal of $\mathcal{O}_P(Y)$, it cannot contain any units. Therefore, I must be a subset of the set of non-units, which we have just shown is exactly $\mathfrak{m}_P$. This implies that every proper ideal is contained in $\mathfrak{m}_P$, making $\mathfrak{m}_P$ the unique maximal ideal.

1.9 Localization of Rings

The algebraic structure of the local ring $\mathcal{O}_P(Y)$ can be described elegantly using the concept of localization from commutative algebra.

Definition 1.9.1: Localization of a Ring

Let A be a commutative ring and let $T \subseteq A$ be a multiplicatively closed subset (i.e., $1 \in T$ and for all $s, t \in T$, we have $st \in T$). The **localization of A with respect to T** is the ring denoted $T^{-1}A$, defined as the set of equivalence classes of pairs (a, t) with $a \in A, t \in T$:

$$T^{-1}A := \left\{ \frac{a}{t} \mid a \in A, t \in T \right\}$$

where two fractions $\frac{a}{t}$ and $\frac{a'}{t'}$ are equivalent if there exists some $s \in T$ such that $s(at' - a't) = 0$. If A is an integral domain, this simplifies to $at' = a't$. The set $T^{-1}A$ forms a ring, and there is a natural ring homomorphism $A \to T^{-1}A$ given by $a \mapsto a/1$.

Example 1.9.1: Examples of Localization

1. **Field of Fractions:** If A is an integral domain, the set $T = A \setminus \{0\}$ is multiplicatively closed. The localization $(A \setminus \{0\})^{-1}A$ is the **field of fractions** of A, denoted $\mathrm{Frac}(A)$. For example, $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$, and if $A(Y)$ is the coordinate ring of an irreducible affine variety, $\mathrm{Frac}(A(Y)) = K(Y)$, the function field.

2. **Localization at a Prime Ideal:** If $\mathfrak{p} \subseteq A$ is a prime ideal, then the set $T = A \setminus \mathfrak{p}$ is multiplicatively closed. The localization $(A \setminus \mathfrak{p})^{-1}A$ is called the **localization of A at $\mathfrak{p}$** and is denoted by $A_{\mathfrak{p}}$.

   For example, let $A = \mathbb{Z}$ and $\mathfrak{p} = \langle 2 \rangle$. Then $T = \mathbb{Z} \setminus \langle 2 \rangle$ is the set of all odd integers. The localization is:

   $$\mathbb{Z}_{\langle 2 \rangle} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \text{ is odd} \right\}$$

   This ring $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. The quotient by this maximal ideal gives the residue field, $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p})$. For our example, $\mathbb{Z}_{\langle 2 \rangle}/\langle 2 \rangle\mathbb{Z}_{\langle 2 \rangle} \cong \mathbb{Z}/\langle 2 \rangle = \mathbb{F}_2$.

Localization and Varieties

For an irreducible affine variety Y, its coordinate ring $A(Y)$ is an integral domain. For a point $P \in Y$, there is a corresponding maximal ideal $\mathfrak{m}_P = \{f \in A(Y) \mid f(P) = 0\}$. The local ring of germs at P is precisely the localization of the coordinate ring at this maximal ideal.

Theorem 1.9.1: Local Ring as Localization

For an affine variety Y and a point $P \in Y$, there is a natural isomorphism:

$$\mathcal{O}_P(Y) \cong A(Y)_{\mathfrak{m}_P}$$

Under this isomorphism, both are subrings of the function field $K(Y)$. The function field itself can be seen as a localization at the zero ideal: $K(Y) \cong A(Y)_{\langle 0 \rangle}$.
The dimension of the variety can also be defined algebraically in terms of these rings:

$$\dim Y = \text{tr.deg}_k A(Y) = \text{tr.deg}_k K(Y)$$

Example 1.9.2: Example: A Cuspidal Cubic Curve Consider the curve $Y = Z(x_2^2 - x_1^3) \subset \mathbb{A}_k^2$. Its coordinate ring is $A(Y) = k[x_1, x_2]/\langle x_2^2 - x_1^3 \rangle$. The function field is $K(Y) = \text{Frac}(A(Y))$. We can describe this as $k(x_1)[x_2]/\langle x_2^2 - x_1^3 \rangle$, which is isomorphic to $k(x_1)(\sqrt{x_1^3})$. A simpler description comes from the parametrization $x_1 = t^2, x_2 = t^3$, which shows $K(Y) \cong k(t)$.
Let's look at the origin $P = (0, 0) \in Y$. The corresponding maximal ideal in $A(Y)$ is $\mathfrak{m}_P = \langle x_1, x_2 \rangle$. The local ring at the origin is:

$$\mathcal{O}_P(Y) = A(Y)_{\mathfrak{m}_P} = \left(\frac{k[x_1, x_2]}{\langle x_2^2 - x_1^3 \rangle} \right)_{\langle x_1, x_2 \rangle}$$

This ring contains elements like $1/(1 + x_1 + x_2)$, since the denominator is not in $\mathfrak{m}_P$. It does not contain elements like $1/(x_1 - x_2)$ since $x_1 - x_2 \in \mathfrak{m}_P$. The rational function x_1/x_2 is in $K(Y)$ but not in $\mathcal{O}_P(Y)$. Using the parametrization, $x_1/x_2 = t^2/t^3 = 1/t$, which is not defined at the origin ($t = 0$).

1.10 The Sheaf of Regular Functions and Functoriality

The concepts discussed can be formalized using the language of sheaves. For a variety X, we can define a sheaf $\mathcal{O}_X$, called the **sheaf of regular functions**.

- To each open set $U \subseteq X$, it assigns the ring of regular functions on that set, $\mathcal{O}_X(U)$.

For the whole variety, $\mathcal{O}_X(X) = \mathcal{O}(X)$.

- The **stalk** of this sheaf at a point $P \in X$, denoted $\mathcal{O}_{X,P}$, is precisely the local ring of germs $\mathcal{O}_P(X)$.

This assignment $X \mapsto \mathcal{O}_X(X)$ is a bridge from geometry to algebra. It is a **contravariant functor**. This means that a morphism of varieties $\phi : X \to Y$ induces a ring homomorphism in the opposite direction, called the pullback map:

$$\phi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) \quad \text{defined by} \quad \phi^*(f) = f \circ \phi$$

Essentially, arrows are reversed when moving from the geometric category of varieties to the algebraic category of rings. Deeper connections exist, for instance, "covering maps" in geometry often correspond to ring extensions in algebra.

Distinguished Open Sets

A special and important class of open sets on an affine variety X are the distinguished open sets.

Definition (Distinguished Open Set) For any regular function $f \in A(X)$ on an affine variety X, the **distinguished open set** defined by f is the set of points where f does not vanish:

$$X_f := \{P \in X \mid f(P) \neq 0\}$$

This is an open set because it is the complement of the closed set $Z(f) = \{P \in X \mid f(P) = 0\}$. These sets form a basis for the Zariski topology on X.

1.11 The Sheaf of Regular Functions and Distinguished Open Sets

The structure sheaf $\mathcal{O}_X(\cdot)$ is a fundamental contravariant functor in algebraic geometry, which translates geometric structures into algebraic ones. Contravariance means that it reverses the direction of morphisms. Given a morphism of varieties $\phi : X \to Y$, it induces a homomorphism of rings (a "pullback") in the opposite direction:

$$\phi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$$

This correspondence is central; for instance, inclusions of affine varieties $X \hookrightarrow Y$ correspond to surjective ring extensions $A(Y) \to A(X)$.

Let's consider a special and important class of open subsets of an affine variety X.

Definition 1.11.1: Distinguished Open Set

For an affine variety X with coordinate ring $A(X)$, and an element $f \in A(X)$, the **distinguished open set** associated with f is defined as:

$$X_f := \{P \in X \mid f(P) \neq 0\} = X \setminus \mathcal{Z}(f)$$

where $\mathcal{Z}(f)$ is the zero set of f.

Warning 1.11.1: Any open set $U \subseteq X$ can be expressed as a union of distinguished open sets. If $U = X \setminus \mathcal{Z}(J)$ for some ideal $J = \langle f_1, \ldots, f_m \rangle$, then:

$$U = X \setminus \bigcap_{i=1}^{m} \mathcal{Z}(f_i) = \bigcup_{i=1}^{m} (X \setminus \mathcal{Z}(f_i)) = \bigcup_{i=1}^{m} X_{f_i}$$

This means the set $\{X_f \mid f \in A(X)\}$ forms a basis for the Zariski topology on X.

A key result connects the ring of regular functions on a distinguished open set to an algebraic construction: localization.

Proposition 1.11.1: Regular Functions on Distinguished Open Sets

The ring of regular functions on the distinguished open set X_f is isomorphic to the localization of the coordinate ring $A(X)$ at the multiplicative set generated by f.

$$\mathcal{O}_X(X_f) \cong A(X)_f := S^{-1}A(X), \quad \text{where } S = \{f^n \mid n \geq 0\}$$

Proof. First, we show that $A(X)_f \subseteq \mathcal{O}_X(X_f)$. An element of $A(X)_f$ has the form g/f^n for some $g \in A(X)$ and $n \geq 0$. This defines a function on X_f because for any point $P \in X_f$, the denominator $f^n(P) = (f(P))^n \neq 0$. This is a regular function by definition.

Conversely, let $\phi \in \mathcal{O}_X(X_f)$ be a regular function. By definition, for every point $P \in X_f$, there exists an open neighborhood $U_P \subseteq X_f$ and polynomials $g_P, h_P \in A(X)$ such that $h_P(Q) \neq 0$ for all $Q \in U_P$ and $\phi(Q) = g_P(Q)/h_P(Q)$ for all $Q \in U_P$.

We can cover X_f with a finite number of such neighborhoods. A more direct approach is as follows: Define an ideal $J \subseteq A(X)$ by:

$$J := \{h \in A(X) \mid h \cdot \phi \text{ extends to a regular function on all of } X, \text{i.e., } h\phi \in A(X)\}$$

This is indeed an ideal of $A(X)$. For any point $P \in X_f$, there is some h_P in our local definition of ϕ such that $h_P(P) \neq 0$ and $h_P \in J$. This means that no point in X_f is in the zero set of J.

$$X_f \cap \mathcal{Z}(J) = \emptyset$$

This implies that the zero set of J must be contained within the complement of X_f, which

is $\mathcal{Z}(f)$.

$$\mathcal{Z}(J) \subseteq \mathcal{Z}(f)$$

By Hilbert's Nullstellensatz, this inclusion of zero sets implies a reverse inclusion for their corresponding radical ideals. Taking the ideal of functions vanishing on these sets, we get $\mathcal{I}(\mathcal{Z}(J)) \supseteq \mathcal{I}(\mathcal{Z}(f))$. This means $\sqrt{J} \supseteq \sqrt{\langle f \rangle}$, which implies that some power of f must lie in J.

$$f^r \in J \quad \text{for some integer } r > 0.$$

By the definition of J, this means $f^r \cdot \phi = g$ for some $g \in A(X)$. Therefore, on X_f, we can write $\phi = g/f^r$, which is an element of $A(X)_f$. □

Corollary 1.11.1: Stalk of the Structure Sheaf at a Point The stalk of the structure sheaf at a point $P \in X$ (which is an affine variety) is the localization of the coordinate ring at the maximal ideal $m_P = \mathcal{I}(P)$ corresponding to that point.

$$\mathcal{O}_{X,P} = A(X)_{m_P}$$

The Projective Case

The situation for a projective variety $X \subseteq \mathbb{P}^n$ is similar but involves graded rings. Let $S(X)$ be the homogeneous coordinate ring of X.

- For a homogeneous polynomial $f \in S(X)$, the ring of regular functions on the distinguished open set $X_f = X \setminus \mathcal{Z}(f)$ is the degree-zero part of the localization of $S(X)$ at f:

$$\mathcal{O}_X(X_f) = (S(X)_f)_0 := \left\{ \frac{g}{f^k} \mid g, f \in S(X) \text{ are homogeneous}, k \in \mathbb{N}, \deg(g) = \deg(f^k) \right\}$$

- A key difference is that for any projective variety X, the only globally regular functions are constants: $\mathcal{O}_X(X) = k$.
- The stalk at a point $P \in X$ is the degree-zero part of the localization of $S(X)$ at the homogeneous prime ideal m_P corresponding to P: $\mathcal{O}_{X,P} = (S(X)_{(m_P)})_0$.

1.12 Rational Maps and Birational Equivalence

We now define a more general type of map between varieties that is not necessarily defined everywhere.

Definition 1.12.1: Rational Map

For varieties X and Y, a **rational map** $\phi : X \dashrightarrow Y$ is an equivalence class of pairs (U, ϕ_U), where $U \subseteq X$ is a non-empty open set and $\phi_U : U \to Y$ is a morphism. Two pairs (U, ϕ_U) and (V, ϕ_V) are considered equivalent if ϕ_U and ϕ_V agree on the intersubsection $U \cap V$. The largest possible open set on which ϕ is defined as a morphism is called the domain of definition of ϕ.

Remark 1.12.1: Rational Functions and Function Fields If we take $Y = \mathbb{A}^1_k$, this definition recovers the concept of a **rational function** on X. The set of all rational functions on X forms a field, the **function field** of X, denoted $K(X)$.

Definition 1.12.2: Dominant Map

A rational map $\phi : X \dashrightarrow Y$ is **dominant** if for some (and therefore any) representative morphism $\phi_U : U \to Y$, the image $\phi_U(U)$ is a dense subset of Y.

Warning 1.12.1: In the Zariski topology, any non-empty open subset of an irreducible variety Y is dense. Therefore, ϕ is dominant if its image contains an open subset of Y.

A dominant rational map $\phi : X \dashrightarrow Y$ induces a field homomorphism of the corresponding function fields, $\phi^* : K(Y) \to K(X)$.

Definition 1.12.3: Birational Equivalence

Varieties X and Y are **birationally equivalent** (or **birational**), written $X \cong_{bir} Y$, if there exists a rational map $\phi : X \dashrightarrow Y$ which has a rational inverse $\psi : Y \dashrightarrow X$. This means that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$ as rational maps (i.e., they are the identity on their domains of definition).

Theorem 1.12.1: Birational Equivalence and Function Fields

Two varieties X and Y are birationally equivalent if and only if their function fields are isomorphic as k-algebras.

$$X \cong_{bir} Y \iff K(X) \cong K(Y)$$

Proof. ($\Rightarrow$) Let $\phi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ be inverse rational maps. Since ϕ must be dominant, it induces an injective field homomorphism $\phi^* : K(Y) \to K(X)$. Similarly, the dominant map ψ induces $\psi^* : K(X) \to K(Y)$. The compositions correspond to $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. Since the compositions are identity maps, the field homomorphisms must be inverses of each other. Thus, $K(X) \cong K(Y)$.

($\Leftarrow$) Conversely, suppose there is a k-algebra isomorphism $\Psi : K(Y) \to K(X)$. Since

function fields of affine varieties are the fraction fields of their coordinate rings, we have $K(Y) = \mathrm{Frac}(A(Y))$ and $K(X) = \mathrm{Frac}(A(X))$. We can find an affine open set $U \subseteq X$ and an affine open set $V \subseteq Y$ such that Ψ restricts to an isomorphism of their coordinate rings, $\Psi : A(V) \to A(U)$. An isomorphism of coordinate rings corresponds to an isomorphism of affine varieties, $\phi : U \to V$. This morphism can be extended to a rational map $\phi : X \dashrightarrow Y$. The inverse isomorphism gives an inverse rational map $\psi : Y \dashrightarrow X$. Thus, X and Y are birationally equivalent. □

Example: A Cusp and the Affine Line

Consider the affine line $X = \mathbb{A}^1_k$ and the cuspidal cubic curve $Y = \mathcal{Z}(y^2 - x^3) \subseteq \mathbb{A}^2_k$.

- The coordinate rings are $A(X) = k[t]$ and $A(Y) = k[x, y]/\langle y^2 - x^3 \rangle$. These rings are not isomorphic (e.g., the local ring at the origin of Y is not a regular local ring, while all local rings of X are). Thus, X and Y are not isomorphic.
- Consider the morphism $\phi : X \to Y$ given by $t \mapsto (t^2, t^3)$. This is a morphism.
- Consider the rational map $\psi : Y \dashrightarrow X$ given by $(x, y) \mapsto y/x$. This is defined on the open set $Y \setminus \{(0, 0)\}$.
- The maps are inverses: $\psi(\phi(t)) = t^3/t^2 = t$, and $\phi(\psi(x, y)) = ((y/x)^2, (y/x)^3) = (y^2/x^2, y^3/x^3)$. Since on the curve $y^2 = x^3$, this is $(x^3/x^2, y^3/y^2) = (x, y)$.
- Their function fields are $K(X) = k(t)$ and $K(Y) = k(x, y)$ where $y^2 = x^3$. We can express $x = (y/x)^2$ and $y = (y/x)^3$ in terms of the rational function $t = y/x$. So $K(Y) = k(y/x) \cong k(t)$.

Since their function fields are isomorphic, $\mathbb{A}^1_k$ and the cusp Y are birationally equivalent.

1.13 Birationality and Hypersurfaces

An important result states that from a birational point of view, all varieties are as simple as hypersurfaces.

Proposition 1.13.1: Birational Equivalence to Hypersurfaces

Any irreducible algebraic variety of dimension r is birationally equivalent to a hypersurface in $\mathbb{A}^{r+1}$.

Proof. Let X be a variety of dimension r. The dimension is equal to the transcendence degree of its function field over the base field k.

$$r = \dim(X) = \mathrm{tr.deg}_k(K(X))$$

This means there exist r algebraically independent elements $x_1, \ldots, x_r \in K(X)$ over k. Let $K' = k(x_1, \ldots, x_r)$ be the purely transcendental extension of k. The extension K'/k is the function field of $\mathbb{A}^r$.

The field extension $K(X)/K'$ is a finite algebraic extension. By the Primitive Element Theorem, if the extension is separable (which is always true for fields of characteristic zero), we can find a single element $x_{r+1} \in K(X)$ that generates the extension.

$$K(X) = K'(x_{r+1}) = k(x_1, \ldots, x_r, x_{r+1})$$

Since x_{r+1} is algebraic over K', it satisfies a minimal polynomial relation:

$$F(x_{r+1}) = 0, \quad \text{where } F \text{ is an irreducible polynomial with coefficients in } K'.$$

By clearing denominators, we can choose F to be an irreducible polynomial in $k[X_1, \ldots, X_r, X_{r+1}]$. The zero set of this polynomial, $H = \mathcal{Z}(F)$, is a hypersurface in $\mathbb{A}^{r+1}$. Its function field is $K(H) = \mathrm{Frac}(k[X_1, \ldots, X_{r+1}]/\langle F \rangle) \cong k(x_1, \ldots, x_{r+1}) = K(X)$.

Since $K(X) \cong K(H)$, the variety X is birationally equivalent to the hypersurface H. □

The Primitive Element Theorem

This subsection briefly touches upon the Primitive Element Theorem, which is a fundamental result in field theory stating that every finite separable field extension is a simple extension. This concept is useful in algebraic geometry for simplifying the structure of function fields of varieties.

Exercise Given a function field $K(X)$ over a base field k, can we write $K(X)$ as a simple extension $k(\alpha)$ for some element $\alpha \in K(X)$?
The Primitive Element Theorem provides a positive answer in the case of separable extensions. If $K(X)/k$ is a finite separable extension, then such an α, called a primitive element, exists.

The minimal polynomial of this primitive element, let's call it $f(y) = \mathrm{minpoly}_{\alpha,k'}(y)$, where k' is a subfield, satisfies the isomorphism:

$$k'(\alpha) := k'[y]/\langle f(y) \rangle \cong K(x)$$

This implies that the variety X is birationally equivalent to a hypersurface $H = \mathcal{Z}(f)$. This birational map is a key tool for studying varieties.

Idea of the Proof of the Primitive Element Theorem (P.E.T.): Consider a field extension generated by two elements, $K = k(\alpha_1, \alpha_2)$, where the extension K/k is separable.

The theorem asserts that we can find a single element α such that $K = k(\alpha)$.

Example 1.13.1: Example: Combining Square RootsA classic example is the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. This can be expressed as a simple extension $\mathbb{Q}(\sqrt{2} + \sqrt{3})$.

The constructive proof for finding a primitive element often involves taking a "random" linear combination of the generators. For a field k of characteristic 0 (or a sufficiently large finite field), we can almost always write:

$$k(\alpha_1, \alpha_2) = k(c_1\alpha_1 + c_2\alpha_2) \quad \text{for some } c_1, c_2 \in k.$$

Typically, we can set $\alpha = \alpha_1 + c\alpha_2$ for some $c \in k$.

Exercise Prove the existence of suitable constants c_1, c_2 by considering a linear system.

1.14 Non-singular (or Smooth) Varieties

We now shift our focus to the local geometry of an algebraic variety. The key idea is to approximate the variety near a point with a linear space, known as the tangent space.

Let $X \subseteq \mathbb{A}_k^n$ be an affine variety of dimension r. Let its defining ideal be $\mathcal{I}(X) = \langle f_1, \ldots, f_t \rangle \lhd k[x_1, \ldots, x_n]$. To study the local structure of X at a point $P \in X$, we can, without loss of generality, assume $P = \bar{0} = (0, \ldots, 0)$ by applying a coordinate transformation. Since $P \in X$, we have $f_i(P) = 0$ for all $i = 1, \ldots, t$.

To reduce the non-linear structure of X at P to a linear one (a k-vector space), we linearize the defining polynomials f_i. The Taylor expansion of f_i around $P = \bar{0}$ is:

$$f_i(x_1, \ldots, x_n) = \underbrace{f_i(\bar{0})}_{=0} + \underbrace{\sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_P \cdot x_j}_{\text{linear part } f_i^{(1)}} + (\text{higher order terms})$$

Definition 1.14.1: Zariski Tangent Space

Let $X \subseteq \mathbb{A}_k^n$ be an affine variety and $P \in X$. Assume $P = \bar{0}$ and $\mathcal{I}(X) = \langle f_1, \ldots, f_t \rangle$. The **tangent space** to X at P, denoted $T_{X,P}$, is the affine linear subspace of $\mathbb{A}_k^n$ defined by the vanishing of the linear parts of the polynomials in $\mathcal{I}(X)$.

$$T_{X,P} := Z(f_1^{(1)}, \ldots, f_t^{(1)}) = Z\left(\left\{\sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_P \cdot x_j \quad \forall i \in [t]\right\}\right)$$

The ideal of these linear forms is denoted $\mathcal{I}(X)^{(1)} = \langle f_1^{(1)}, \ldots, f_t^{(1)} \rangle$.

The linear functions on the tangent space form its dual vector space.

Definition 1.14.2: Cotangent Space

The dual space to the tangent space, known as the **cotangent space**, is given by

$$T_{X,P}^{\vee} := A^{(1)}/\mathcal{I}(X)^{(1)}$$

where $A^{(1)}$ is the space of all linear forms in $k[x_1, \ldots, x_n]$.

The dimension of the tangent space is determined by the rank of the Jacobian matrix.

Proposition 1.14.1: Dimension of the Tangent Space

The dimension of the tangent space $T_{X,P}$ is given by:

$$\dim_k T_{X,P} = n - \operatorname{rank}(J_P)$$

where $J_P = \left(\frac{\partial f_i}{\partial x_j}\Big|_P\right)_{i\in[t],j\in[n]}$ is the Jacobian matrix of the generators of $\mathcal{I}(X)$ evaluated at P.

Proof. The tangent space $T_{X,P}$ is the solution space to the system of linear equations:

$$\forall i \in [t], \quad f_i^{(1)} = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_P \cdot x_j = 0$$

This is a homogeneous linear system in the variables $x_1, \ldots, x_n$. The coefficient matrix of this system is precisely the Jacobian matrix J_P. By the rank-nullity theorem, the dimension of the solution space is $n - \operatorname{rank}(J_P)$. □

Relation to the Local Ring

The tangent space, which is defined extrinsically using embeddings in $\mathbb{A}^n$, has an intrinsic algebraic characterization in terms of the local ring of the variety at the point P.

Let $\mathcal{O}_{X,P}$ be the local ring of X at P, and let M_P be its unique maximal ideal, consisting of rational functions on X that vanish at P. The quotient M_P/M_P^2 is a k-vector space.

Proposition 1.14.2: Cotangent Space and Local Ring

The cotangent space $T_{X,P}^{\vee}$ is canonically isomorphic to M_P/M_P^2.

$$T_{X,P}^{\vee} \cong M_P/M_P^2$$

Proof. Consider the natural map $\varphi : T_{X,P}^{\vee} \to M_P/M_P^2$. Since $P = \bar{0}$, a linear form $f \in T_{X,P}^{\vee}$ vanishes at P, so its germ is in M_P. The map is defined by sending the class of a linear form f in $T_{X,P}^{\vee}$ to its class in M_P/M_P^2. One can show this map is a well-defined k-vector space isomorphism.

- **Injectivity:** If $\varphi(f) = 0$, then $f \in M_P^2$. A linear form that lies in the square of the maximal ideal must be the zero form. Thus, $f = 0$ in $T_{X,P}^\vee$.
- **Surjectivity:** Let $\tau = f/g \in M_P$ be an arbitrary element (a germ of a function vanishing at P). We can assume, without loss of generality, that $g(P) = 1$. The linear approximation of τ at P is the linear form $\tau' := \sum_{j=1}^n (\partial_j \tau)_P \cdot x_j$. One can show that $\tau - \tau' \in M_P^2$ using a Taylor expansion argument. This means $\tau \equiv \tau' \pmod{M_P^2}$, so the class of τ is the image of the linear form τ' under φ.

This proves that φ is an isomorphism. □ □

Remark 1.14.1: Geometric Interpretation The tangent space $T_{X,P}$ and its dual $T_{X,P}^\vee$ provide a first-order linear approximation of the variety X (or more precisely, the neighborhood of P in X) and the germs of functions at P, respectively.

1.14.1 Singular and Non-Singular Points

A fundamental result in algebraic geometry relates the dimension of the tangent space to the local dimension of the variety at that point.

Theorem 1.14.1: Dimension Inequality for Tangent Spaces

For any point P on an affine variety X, we have the inequality:

$$\dim_k T_{X,P} \geq \dim X$$

(More precisely, $\dim_k T_{X,P} \geq \dim \mathcal{O}_{X,P}$, the local dimension of X at P.)

Exercise Prove this result.

Proof. The tangent space is constructed from the linear parts of the defining equations, "forgetting" the higher-order constraints. If the dimension of this linear approximation is strictly larger than the dimension of the variety itself, it means the linear approximation is a poor fit and has "lost" local information about the variety at that point. This happens at points we call "singular". □

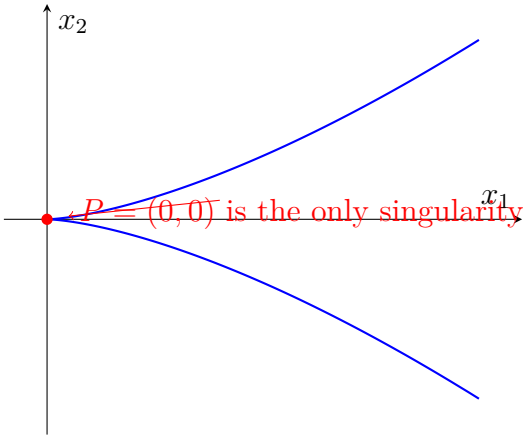
Definition 1.14.3: Singularity

A point P on a variety X is **non-singular** (or **smooth**, or a **simple point**) if $\dim_k T_{X,P} = \dim X$.
If $\dim_k T_{X,P} > \dim X$, then P is a **singular point** of X.
The variety X is called non-singular if every point $P \in X$ is non-singular.

Example 1.14.1: Let $X = \mathcal{Z}(x_2 - x_1^3) \subseteq \mathbb{A}^2$, and let $P = (0,0)$. The dimension of the curve is $\dim X = 1$. The Jacobian matrix is $J = [\partial_{x_1} f, \partial_{x_2} f] = [-3x_1^2, 1]$. At $P = (0,0)$, the Jacobian is $J_P = [-3(0)^2, 1] = [0,1]$. The rank is $\operatorname{rank}(J_P) = 1$. The dimension of the tangent space is $\dim_k T_{X,P} = n - \operatorname{rank}(J_P) = 2 - 1 = 1$. Since $\dim_k T_{X,P} = \dim X$, the point $P = (0,0)$ is a smooth point.

Example 1.14.2: A Singular Curve: The Cusp Let $X = \mathcal{Z}(x_2^2 - x_1^3) \subseteq \mathbb{A}^2$, and let $P = (0,0)$. This is the cuspidal cubic. The dimension of the curve is $\dim X = 1$. The Jacobian matrix is $J = [\partial_{x_1} f, \partial_{x_2} f] = [-3x_1^2, 2x_2]$. At $P = (0,0)$, the Jacobian is $J_P = [-3(0)^2, 2(0)] = [0,0]$. The rank is $\operatorname{rank}(J_P) = 0$. The dimension of the tangent space is $\dim_k T_{X,P} = n - \operatorname{rank}(J_P) = 2 - 0 = 2$. Here, $\dim_k T_{X,P} = 2 > \dim X = 1$. Therefore, the point $P = (0,0)$ is a singularity of X. (In fact, P is the only singularity on this curve).

The Cuspidal Cubic $x_2^2 = x_1^3$



At the singular point $P = \bar{0}$, the tangent space is $T_{X,P} = \mathbb{A}_k^2$, the entire ambient plane. For any other point $P \neq \bar{0}$ on the curve, the tangent space is a line, $T_{X,P} \cong \mathbb{A}_k^1$. Even though X is singular, it is birational to the affine line $\mathbb{A}_k^1$.

CHAPTER 2

RESOLUTION OF SINGULARITIES & VALUATIONS

Exercise Can we find a non-singular curve that is birationally equivalent to a given singular curve X?

The answer is yes. A major theorem in algebraic geometry states that every curve is birational to a non-singular projective curve. This process of finding such a smooth model is called "resolution of singularities."

Let's see how to resolve the singularity at $P = \bar{0}$ for the cuspidal cubic $X = \mathcal{Z}(x_2^2 - x_1^3)$.

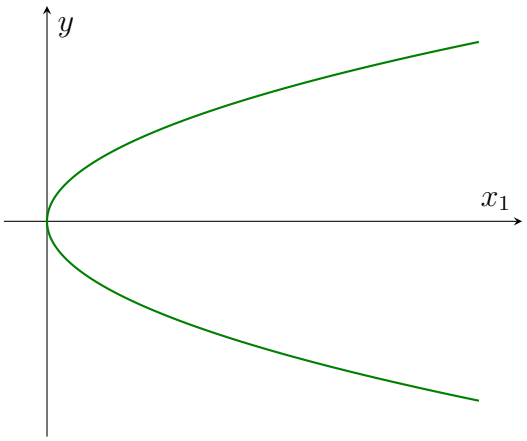
The issue at the origin is that the curve "crosses itself" in a way that is not smooth. The rational function $y = x_2/x_1$ is not regular at P (it is in the form "0/0"), but it encodes the direction of approach to the origin. Let's use this.

Consider a new curve $\tilde{X}$ in the (x_1, y)-plane defined by the equation we get from substituting $x_2 = yx_1$ into the original equation: $(yx_1)^2 = x_1^3 \implies y^2x_1^2 = x_1^3 \implies x_1^2(y^2 - x_1) = 0$. The interesting component is $\tilde{X} = \mathcal{Z}(y^2 - x_1)$. This is a non-singular parabola.

We have a birational map $\varphi : X \dashrightarrow \tilde{X}$ given by $(x_1, x_2) \mapsto (x_1, y = x_2/x_1)$. Its inverse is a regular map $\psi : \tilde{X} \to X$ given by $(x_1, y) \mapsto (x_1, x_1y) = (y^2, y^3)$.

The curve $\tilde{X} = \mathcal{Z}(y^2 - x_1)$ is a non-singular curve and is birational to our original singular curve X. We have "resolved" the singularity.

The Smooth Parabola $\tilde{X} = \mathcal{Z}(y^2 - x_1)$



The general goal is to develop systematic procedures (like blowing up points) to achieve this resolution for any curve, even those with multiple or more complex singularities.

2.1 Resolving Singularities: Blowing-Up

The process of resolving a singularity is also known as "blowing-up". The name comes from the fact that a singular point is mapped to a higher-dimensional space. For instance, a singular point (x_1, x_2) can be mapped to a point (x_1, yx_1, y). A common example is blowing up the origin $(0, 0)$ in the plane, where the point $(0, 0)$ might be mapped to a point like $(0, 0, 1)$ in a new space.

The goal of this process is to replace a singular point with a non-singular object (like a line). This transformation should satisfy certain desirable properties.

Property I: Resolving the Local Ring Structure

The first property concerns the algebraic structure of the local ring at the point. We want the new ring to be "simpler" in a specific way.

Definition 2.1.1: Blowing-Up Property I

[Property I:] The transformed ring, which we can denote as $A(\tilde{X})_{\langle x_1, y\rangle}$, is a local domain whose unique maximal ideal is now a principal ideal.

Example 2.1.1: Consider the cusp curve defined by $x_2^2 - x_1^3 = 0$. The local ring at the origin, $(k[x_1, x_2]/\langle x_2^2 - x_1^3\rangle)_{\langle x_1, x_2\rangle}$, has a maximal ideal $\langle x_1, x_2\rangle$ which is not principal.
The blowing-up process involves the substitution $x_2 = yx_1$. The equation becomes $(yx_1)^2 - x_1^3 = 0$, which simplifies to $x_1^2(y^2 - x_1) = 0$. On the new chart defined by $y^2 - x_1 = 0$, the maximal ideal corresponding to the original singularity becomes $\langle x_1, y\rangle$. In the new coordinate ring $A(\tilde{X}) = k[x_1, y]/\langle y^2 - x_1\rangle$, we can see that $x_1 = y^2$. Therefore, the ideal simplifies:

$$\langle x_1, y\rangle = \langle y^2, y\rangle = \langle y\rangle$$

This new ideal is principal, generated by y. The original non-principal ideal has been resolved into a principal one.

2.2 Discrete Valuation Rings (DVRs)

Definition and examples

The property of having a principal maximal ideal in a local domain is central to the theory of resolving singularities. This leads to the definition of a Discrete Valuation Ring (DVR).

Definition 2.2.1: Discrete Valuation Ring (DVR)

A ring R is called a **Discrete Valuation Ring (dvr)** if it is a local domain with a principal maximal ideal.

Example 2.2.1:

- The ring $(k[x_1, y]/\langle y^2 - x_1\rangle)_{\langle x_1, y\rangle}$ from the previous example is a dvr.
- The ring $(k[x_1, x_2]/\langle x_2^2 - x_1^3\rangle)_{\langle x_1, x_2\rangle}$ is **not** a dvr because its maximal ideal $\langle x_1, x_2\rangle$ is not principal.
- The ring of integers $\mathbb{Z}$ is not a dvr because it is not a local ring, although every maximal ideal (generated by a prime p) is principal. However, its localizations $\mathbb{Z}_{(p)}$ are dvrs.

2.3 Valuation Theory

The second property relates dvrs to the concept of a valuation on a field.

Definition 2.3.1: Blowing-Up Property II

A dvr R with maximal ideal m and field of fractions $K = k(R)$ gives rise to a discrete valuation on K. The field $k \cong R/m$ is the residue field.

Definition 2.3.2: Discrete Valuation

A **discrete valuation** of a field K is a surjective map $v : K^* \to \mathbb{Z}$ (where $K^* = K \setminus \{0\}$) such that for all $\alpha, \beta \in K^*$:

1. $v(\alpha\beta) = v(\alpha) + v(\beta)$ (i.e., v is a group homomorphism).
2. $v(\alpha + \beta) \geq \min(v(\alpha), v(\beta))$ (the non-Archimedean or triangle inequality).

By convention, we set $v(0) = \infty$.

Remark 2.3.1: A useful property that follows from the axioms is that if $v(\alpha) \neq v(\beta)$, then $v(\alpha + \beta) = \min(v(\alpha), v(\beta))$.

Constructing a Valuation from a DVR

If R is a dvr, its maximal ideal m is principal, so $m = \langle u \rangle$ for some element $u \in R$. This generator u is called a **uniformizer** of R.

Any non-zero element $\alpha \in K = k(R)$ can be uniquely written in the form $\alpha = u^e \cdot \alpha'$, where $e \in \mathbb{Z}$ and $\alpha' \in R \setminus m$ (i.e., α' is a unit in R).

We can then define a map $v : K^* \to \mathbb{Z}$ by:

$$v(\alpha) := e$$

This map v is a discrete valuation.

Example 2.3.1: Computing Valuations in a DVR Let's return to $R = (k[x_1, y]/\langle y^2 - x_1 \rangle)_{\langle x_1, y \rangle}$, which is a dvr with uniformizer y. Any element α in its fraction field K can be written as $\alpha = y^e \cdot \alpha'$ for a unit α'. The valuation $v(\alpha)$ is this exponent e.

- $v(x_1) = v(y^2) = 2$.
- In the original singular curve setting, $x_2 = yx_1 = y(y^2) = y^3$. Thus, $v(x_2) = v(y^3) = 3$.
- $v(1 + x_1)$: Since $1 + x_1$ is not in the maximal ideal $\langle y \rangle$, it is a unit in R. Thus, $v(1 + x_1) = 0$.
- $v(1/x_1) = v((y^2)^{-1}) = v(y^{-2}) = -2$.

An arbitrary element in K can be written as $\alpha = \frac{a(x_1)+y \cdot b(x_1)}{c(x_1)}$ for polynomials a, b, c.

Key Properties of DVRs

There is a deep connection between a ring being a dvr, the existence of a valuation, and the property of being integrally closed.

Characterizing a DVR via its Valuation

Proposition 2.3.1: Fractional Characterization

If R is a dvr with fraction field K and corresponding valuation v, then the ring R, its maximal ideal m, and its group of units R^* can be described entirely in terms of v:

- $R = \{\alpha \in K \mid v(\alpha) \geq 0\} \cup \{0\}$
- $m = \{\alpha \in K \mid v(\alpha) > 0\} \cup \{0\}$

- $R^* = \{\alpha \in K \mid v(\alpha) = 0\}$ (the units in R)

Conversely, if a field K has a valuation $v : K^* \to \mathbb{Z}$, then the set $R = \{\alpha \in K \mid v(\alpha) \geq 0\} \cup \{0\}$ is a dvr with K as its field of fractions.

Property III: Integral Closure

The third key property of a dvr is that it is integrally closed.

Definition 2.3.3: Blowing-Up Property III

A dvr R is integrally closed in its field of fractions K.

Definition 2.3.4: Integrally Closed Domain

A domain R is **integrally closed** in its field of fractions K if every element $\alpha \in K$ that is a root of a monic polynomial with coefficients in R is already in R. That is, if $\alpha \in K$ satisfies an equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where all $a_i \in R$, then it must be that $\alpha \in R$.

Example 2.3.2: The ring of integers $\mathbb{Z}$ is integrally closed in the field of rational numbers $\mathbb{Q}$. For instance, if $\alpha = a/b \in \mathbb{Q}$ is a root of $\alpha^2+2\alpha+3 = 0$, then substituting and clearing denominators gives $a^2 + 2ab + 3b^2 = 0$. From this, one can deduce that b must divide a, which means $\alpha = a/b$ is an integer.

The idea for resolving singularities is to replace the local ring of the singular point, $\mathcal{O}_{X,P}$, with its integral closure in the function field $K(X)$.

The Equivalence Theorem

For a large class of rings relevant to geometry, these three properties are equivalent.

Proposition 2.3.2: Equivalence Theorem

For a local domain R whose fraction field K has transcendence degree 1 over a base field k, the following are equivalent (TFAE):

1. R is a dvr.
2. R is the valuation ring for some discrete valuation on K.
3. R is integrally closed in K.

Proof. **(1) ⇔ (2):** As shown previously, a dvr defines a valuation via its uniformizer , and a valuation defines a dvr as the set of elements with non-negative valuation. The maximal ideal $m = \{\alpha \in K \mid v(\alpha) > 0\}$ is generated by any element $u \in m$ with the least positive valuation, making R a dvr.

(2) ⇒ (3): Let R be a valuation ring for v, and let $\alpha \in K$ be integral over R. Then α satisfies $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$ for $a_i \in R$. Suppose, for contradiction, that $\alpha \notin R$. This means $v(\alpha) < 0$. We have $\alpha^n = -(a_{n-1}\alpha^{n-1} + \cdots + a_0)$. Applying the valuation v:

$$v(\alpha^n) = v(a_{n-1}\alpha^{n-1} + \cdots + a_0) \geq \min_{0 \leq i \leq n-1}\{v(a_i\alpha^i)\}$$

This gives $n \cdot v(\alpha) \geq \min_{0 \leq i \leq n-1}\{v(a_i) + i \cdot v(\alpha)\}$. Since $a_i \in R$, we have $v(a_i) \geq 0$. As $v(\alpha) < 0$, the term $i \cdot v(\alpha)$ is strictly increasing with i. Therefore, $v(a_i) + i \cdot v(\alpha) > v(a_j) + j \cdot v(\alpha)$ if $i < j$. The minimum value on the right is achieved when i is largest, i.e., $i = n - 1$. So, we have:

$$n \cdot v(\alpha) \geq v(a_{n-1}) + (n-1)v(\alpha)$$

Since $v(a_{n-1}) \geq 0$, this implies $n \cdot v(\alpha) \geq (n-1)v(\alpha)$, which means $v(\alpha) \geq 0$. This contradicts our assumption that $v(\alpha) < 0$. Thus, α must be in R.

(3) ⇒ (1): (This direction is more involved). Let R be integrally closed. For any two elements $\alpha, \beta \in m$, the transcendence degree condition ensures they are algebraically dependent over k, so there is a polynomial relationship $F(\alpha, \beta) = 0$. One can use this relationship to show that the element α/β is integral over $R[\beta]$. Since R is integrally closed, this implies that for any $\alpha, \beta \in m$, either $\alpha/\beta \in R$ or $\beta/\alpha \in R$. This property implies that the ideal m must be principal, hence R is a dvr. □

Classification of Valuations on $k(x)$

We now seek to find all the valuations (and thus all the dvrs) of the rational function field $K = k(x)$, the function field of the affine line $\mathbb{A}^1$.

This problem is analogous to finding all valuations on $\mathbb{Q}$. For $\mathbb{Q}$, the discrete valuations correspond to the prime numbers p. For each prime p, the valuation ring is the localization of $\mathbb{Z}$ at the prime ideal $\langle p \rangle$, denoted $\mathbb{Z}_{(p)}$, which is a dvr. The valuation $v_p(a/b)$ is defined as the exponent of p in the prime factorization of a minus the exponent of p in the factorization of b. It is a known result that these, along with the trivial valuation, are all the discrete valuations on $\mathbb{Q}$.

For the field $K = k(x)$, the role of prime numbers is played by irreducible polynomials.

Definition 2.3.5: Localization at an Irreducible Polynomial

For any irreducible polynomial $f \in k[x]$, we define a subring of $K = k(x)$ as:

$$R_f := \left\{ \frac{g}{h} \in K \mid g, h \in k[x], f \nmid h \right\}$$

This is the localization of $k[x]$ at the prime ideal $\langle f \rangle$.

Theorem 2.3.1: Distinct Discrete Valuations on $k(x)$

The distinct dvrs (and thus the distinct non-trivial discrete valuations) of the field $K = k(x)$ are precisely the following:

1. The rings R_f for every monic irreducible polynomial $f \in k[x]$.
2. A single additional ring, $R_{x^{-1}}$, which is defined by viewing x^{-1} as a variable and localizing the ring $k[x^{-1}]$ at the ideal $\langle x^{-1} \rangle$. This is the "valuation at the point at infinity".

Proof that these are the only ones. Let R be a dvr whose field of fractions is $K = k(x)$. Let m be its maximal ideal. We consider two cases for the element $x \in K$.

Case I: $x \in R$. In this case, the polynomial ring $k[x]$ is a subring of R, i.e., $k[x] \subseteq R$. Consider the ideal $P = m \cap k[x]$. This is a prime ideal of $k[x]$.

- P cannot be the zero ideal $\{0\}$. If it were, every non-zero polynomial in $k[x]$ would not be in m, and thus would be a unit in the local ring R. This would imply that the entire field $k(x)$ is contained in R, so $R = K$, which is not a dvr (a dvr must be a proper subring of its fraction field).
- Therefore, P is a non-zero prime ideal in $k[x]$. Since $k[x]$ is a Principal Ideal Domain (PID), every non-zero prime ideal is maximal and generated by a unique monic irreducible polynomial $f(x)$. So, $m \cap k[x] = \langle f \rangle$ for some irreducible f.

Any element $g/h \in K$ where $f \nmid h$ has a denominator $h \notin \langle f \rangle = m \cap k[x]$. So $h \notin m$, which means h is a unit in R. Thus $g/h = g \cdot h^{-1} \in R$. This shows $R_f \subseteq R$. Since R_f is a dvr (and thus a maximal subring), it must be that $R = R_f$. The valuation v is v_f.

Case II: $x \notin R$. If x is not in R, then its valuation must be negative, $v(x) < 0$. For a valuation ring, this implies that the element x^{-1} must have positive valuation, $v(x^{-1}) = -v(x) > 0$. Therefore, x^{-1} must be in the maximal ideal m. This implies that the polynomial ring in the variable x^{-1}, denoted $k[x^{-1}]$, is a subring of R. We can now apply the exact same logic as in Case I, but with the ring $k[x^{-1}]$ and variable $y = x^{-1}$. The ideal $m \cap k[x^{-1}]$ is a non-zero prime ideal in $k[x^{-1}]$, so it must be generated by an irreducible polynomial $f(x^{-1})$. Since $x^{-1} \in m$, the generator must be x^{-1} itself (up to a constant). Thus $m \cap k[x^{-1}] = \langle x^{-1} \rangle_{k[x^{-1}]}$. This implies that $R = R_{x^{-1}}$, the ring of

rational functions $g(x)/h(x)$ such that when written in terms of x^{-1}, the denominator is not divisible by x^{-1}. This is equivalent to saying $\deg(h) \geq \deg(g)$. This completes the classification. □

Extension of Valuation Rings

Curves and Function Fields

Any field K with transcendence degree $\text{tr deg}_k(K) = 1$ can be written as a finite algebraic extension of a purely transcendental extension, i.e., $k(x) \subseteq K$ is a finite extension.

Theorem 2.3.2: Local Domain Extension to DVR

Let R be a local domain in a field K, and let m_R be its unique maximal ideal. Then, there exists a discrete valuation ring (DVR) B in K, with unique maximal ideal m_B, such that $R \subseteq B$ and $m_R \subseteq m_B$.

Proof. If R is integrally closed in K, then R is already a DVR, and we are done.

Suppose R is not integrally closed. Consider the set $\mathcal{F}$ of all local domains R' that dominate R:

$$\mathcal{F} = \{R' \text{ local domain} \mid R \subseteq R' \text{ and } m_R \subseteq m_{R'}\}$$

Let R^* be a maximal element in $\mathcal{F}$ (such an element exists by Zorn's Lemma). This maximal element R^* must be integrally closed. If it were not, its integral closure would be a larger local domain dominating R, contradicting maximality.

Thus, R^* is an integrally closed local domain. A noetherian local domain of dimension 1 is a DVR if and only if it is integrally closed. Therefore, R^* is a DVR in K that extends (dominates) R. □

Example 2.3.3: Resolving a Singularity Consider the curve defined by $X = Z(x_2^2 - x_1^3)$ and the local ring at the origin, $R = A(X)_{\langle x_1, x_2 \rangle}$. This ring R is not a DVR in its field of fractions $k(X)$.
The element $y := x_2/x_1 \in K(X)$ is integral over R (since $y^2 = x_1$), but $y \notin R$. We can extend the ring R by adjoining this element to get $B = R[y]$. This new ring is a DVR in $K(X)$ that extends R, effectively "resolving" the singularity at the origin.

A key question arises: How do we repeat this process to resolve multiple singular points? The answer lies in the fact that an algebraic curve has only a finite number of singular points (the singular locus is a closed set).

Abstract Curves

Consider any field K with $\text{tr deg}_k(K) = 1$. We can view this field as an "abstract curve" through the lens of its valuations.

Definition 2.3.6: The Set of Points

Let C_K be the set of all discrete valuations v on K. Each valuation v corresponds to a DVR R_v with a unique maximal ideal m_v. We can think of the elements of C_K as the "points" of our curve.

We can define a topology on C_K:

- **Closed sets** are the finite subsets of C_K, along with C_K itself.
- **Open sets** are the complements of the closed sets.

Definition 2.3.7: Regular Functions

For any open set $U \subseteq C_K$, we define the ring of regular functions on U as:

$$\mathcal{O}(U) := \bigcap_{v \in U} R_v$$

Each function $f \in \mathcal{O}(U)$ defines a map $v \mapsto f(v)$, where $f(v)$ is the image of f in the residue field $R_v/m_v \cong k$. Two functions $f, g \in \mathcal{O}(U)$ are the same if and only if $f \equiv g \pmod{m_v}$ for all $v \in U$. This means $f - g \in \bigcap_{v \in U} m_v$. If U is infinite, this intersubsection is $\{0\}$, so $f = g$ in K.

Lemma 2.3.1:

For any $f \in K$, there exists an open set $U \subseteq C_K$ such that $f \in \mathcal{O}(U)$.

Proof. Let $f = g/h$ for some g, h in a polynomial ring. The function f is not defined at a valuation (point) $v \in C_K$ if $v(h) > 0$, which is equivalent to $h \in m_v$. The set of such "bad points" where h vanishes is finite. The complement of this finite set is an open set U on which f is regular. □

Definition 2.3.8: Abstract Curve

We call the set C_K, together with the sheaf of regular functions $\mathcal{O}(\cdot)$ and the function field K, an **abstract curve**.

Definition 2.3.9: Morphism of Curves

A **morphism** $\phi : X \to Y$ between two abstract curves is a continuous map such that for every open set $V \subseteq Y$ and every regular function $f \in \mathcal{O}_Y(V)$, the pullback $f \circ \phi$ is a regular function on the open set $\phi^{-1}(V) \subseteq X$, i.e., $f \circ \phi \in \mathcal{O}_X(\phi^{-1}(V))$.

Discrete valuations on K^*

Examples and computations

2.4 Properties of DVRs

Valuation characterisation

Local domains of rank 1

2.5 Integral Closure

Definition

Examples (Gauss lemma)

2.6 Characterisation Theorem

Equivalence of DVR, valuation ring, integrally closed domain

2.7 Valuations of $K = k(x)$

Classification of valuations

Case analysis

2.8 Extension of Valuation Rings

To algebraic extensions / curves

2.9 Resolving Multiple Singular Points

Finite singular set

Strategy of resolution

2.10 Abstract Curves

Valuations as points

Regular functions

Morphisms of curves

CHAPTER 3

DIVISORS, FUNCTION FIELDS & APPROXIMATION

Existence of Non-Singular Models

There is a deep connection between abstract curves and geometric curves.

Lemma 3.0.1:
Every non-singular quasi-projective curve is isomorphic to an abstract curve.

Proof Sketch. Let X be a non-singular quasi-projective curve. We can define a map $\phi : X \to C_{K(X)}$ by sending a point $P \in X$ to the valuation v_P corresponding to its local ring $\mathcal{O}_{X,P}$. Since X is non-singular, each local ring $\mathcal{O}_{X,P}$ is a DVR. This map can be shown to be an isomorphism. □

Theorem 3.0.1: Non-Singular Projective Model Existence Theorem
Let K be a field extension with $\text{tr deg}_k(K) = 1$. Then the abstract curve C_K is isomorphic to a non-singular projective curve.

Proof Idea. The idea is to "glue" together finitely many affine non-singular models, one for each "problematic" region of the curve, to form a complete non-singular projective curve.

1. For each singular point of a given projective model, we find a valuation $v \in C_K$ that is not represented by a non-singular point on that model.

2. The DVR R_v can be realized as the local ring $\mathcal{O}_{V,P}$ of a point P on some non-singular quasi-affine curve V. This V gives us an affine "chart" that resolves the singularity.

3. An algebraic curve has finitely many singular points, so we only need a finite number of such charts, $\{V_i\}$. Each chart V_i corresponds to an open set $U_i \subseteq C_K$. We get isomorphisms $\phi_i : U_i \to V_i$.

4. We can show that each ϕ_i extends to a morphism $\bar{\phi}_i : C_K \to Y_i$, where Y_i is the projective closure of V_i. This is done by clearing denominators in the coordinate representation of the map. If $\phi_i(p) = (f_0(p) : \cdots : f_n(p))$ and all $f_j(p)$ are zero at a point $q \notin U_i$, we can rescale by dividing by the f_s with the minimum valuation at q.

5. We combine these maps into a single "union" morphism:

$$\Phi : C_K \to Y_1 \times Y_2 \times \cdots \times Y_n$$

$$v \mapsto (\bar{\phi_1}(v), \bar{\phi_2}(v), \ldots, \bar{\phi_n}(v))$$

6. Using the Segre embedding, which embeds a product of projective spaces into a larger projective space (e.g., $SE : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$), we embed the target space into a single $\mathbb{P}^N$.

7. The image $Y = \mathrm{cl}(\Phi(C_K))$ is the desired non-singular projective curve isomorphic to C_K.

□

This result shows that any curve (defined by its function field) is birational to a unique non-singular projective curve.

Summary of Correspondences

- Non-singular point $P \in X \longleftrightarrow$ The local ring $\mathcal{O}_{X,P}$ is a DVR in $k(X)$.
- Tangent space rank $\mathrm{rk}(T_{X,P}) = 1 \longleftrightarrow T^*_{X,P} \cong m_P/m_P^2$.
- Non-singular projective curve $X \longleftrightarrow$ Abstract curve $C_{k(X)}$.

Points on a Smooth Curve

From now on, we study only non-singular (smooth) projective curves. If $k = \bar{k}$ (algebraically closed), any smooth projective curve C admits a finite morphism to the projective line $\mathbb{P}^1_k$.

$$\pi : C \to \mathbb{P}^1_k$$

A point $P \in C$ corresponds to a DVR $\mathcal{O}_{C,P}$ in the function field $k(C)$. The inclusion of function fields $k(x) \hookrightarrow k(C)$ (where $k(x)$ is the function field of $\mathbb{P}^1_k$) induces the map π. A point $Q \in \mathbb{P}^1_k$ corresponds to a valuation on $k(x)$, and the points in the fiber $\pi^{-1}(Q)$ correspond to the extensions of this valuation to $k(C)$. The number of points in the fiber is related to ramification.

Curves, Points, and Valuations

We study non-singular (simple/smooth) projective curves.

Points on a Smooth Curve

A smooth projective curve C over an algebraically closed field $k = \overline{k}$ can be seen as a cover of the projective line $\mathbb{P}^1_k$.

$$\begin{array}{c} C \\ \downarrow \\ \mathbb{P}^1_k \end{array}$$

A point $P \in C$ corresponds to a discrete valuation ring (DVR), specifically the local ring at P, denoted $\mathcal{O}_{C,P}$. This gives a map from points on the curve to valuations on its function field.

The setup involves a tower of field extensions: $k \subset k(x) \subset K := k(C)$.

The correspondence works in both directions:

- A point $P \in C$ gives a DVR $\mathcal{O}_{C,P} \subset K$. This in turn gives a point $Q \in \mathbb{P}^1_k$ via the uniformizer. This mapping from C to $\mathbb{P}^1_k$ can have ramification, where multiple points on C map to a single point on $\mathbb{P}^1_k$.
- A point $Q \in \mathbb{P}^1_k$ gives a DVR $R \subset k(x)$. We can extend this valuation to a valuation (and a corresponding DVR $\tilde{R}$) on the larger field $K = k(C)$. This gives a point $P \in C$ via the uniformizer of $\tilde{R}$. Note that the extension of the DVR from $k(x)$ to $K(C)$ is not necessarily unique, and neither is the resulting point P.

Exercise For a finite field $\mathbb{F}_q$, what is the value of $|C(\mathbb{F}_q)| - |\mathbb{P}^1(\mathbb{F}_q)| = |C(\mathbb{F}_q)| - (q+1)$?

A point corresponds to a DVR in K; it may be a cluster of points over the algebraic closure.

Definition 3.0.1: Point on a Curve

Let K be the function field of a curve C over $k = \mathbb{F}_q$. A point P on the curve defines a valuation v_P on K, given by the DVR $\mathcal{O}_{C,P}$.

Degree of a Point

Definition 3.0.2: Residue Field and Degree of a Point

The **residue field** at a point P is $k_P := \mathcal{O}_{C,P}/m_P$, where m_P is the maximal ideal of the local ring $\mathcal{O}_{C,P}$. The residue field k_P is a finite extension of the base field k.

Definition 3.0.3: Degree of a Point

The **degree** of a point P is defined as $d(P) := [k_P : k]$.

Proposition 3.0.1: Properties of Degree and Valuation at a Point

Let P be a point on a smooth curve C with function field K. Let m_P be the maximal ideal of $\mathcal{O}_{C,P}$ and let v_P be the corresponding valuation.

(i) The degree $d(P)$ is finite.

(ii) $\bigcap_{i>0} m_P^i = \{0\}$.

(iii) If $m_P = \langle u \rangle$ (where u is a uniformizer), then for any $\alpha \in K$, $v_P(\alpha)$ is the largest integer i such that $\alpha \in u^i \mathcal{O}_{C,P}$.

Proof. (i) Let $m_P = \langle u \rangle \mathcal{O}_{C,P}$. The degree can be interpreted geometrically as the number of points in the intersubsection of the zero locus of the uniformizer with the curve over the algebraic closure, $d(P) = |Z(u) \cap C(\bar{k})|$, which is finite. Algebraically, this corresponds to the number of conjugates of the point under the Galois action of k_P/k.

(ii) Let $y \in \bigcap_{i>0} u^i$. By Krull's Intersubsection Theorem for the local ring $\mathcal{O}_{C,P}$, this implies $y = 0$.

(iii) This is clear from the definition of the valuation associated with a DVR. □

Any function $f \in K(C)$ gives rise to a sequence of integers $\{v_P(f)\}_{P \in C}$. A natural question arises: given such a sequence, does a corresponding function f exist? Generally, the answer is no. However, we can show that a function f can be constructed to match any finite part of such a sequence. This is the essence of the Approximation Theorem.

The Approximation Theorem

Theorem 3.0.2: Weak Approximation Theorem

Let K be the function field of a curve C. Let $P_1, \ldots, P_h \in C$ be distinct points with corresponding valuations $v_1, \ldots, v_h$. Let $u_1, \ldots, u_h \in K$ and $m_1, \ldots, m_h \in \mathbb{Z}$. Then, there exists an element $u \in K$ such that for all $i \in \{1, \ldots, h\}$, we have $v_i(u - u_i) \geq m_i$.

Proof. The proof relies on several claims.

Lemma 3.0.2: Claim 1

Given distinct valuations $v_1, \ldots, v_{h-1}$ and integers $e_1, \ldots, e_{h-1} \in \mathbb{Z}$, there exists an element $u \in K$ such that $v_i(u) = e_i$ for all $i \in \{1, \ldots, h-1\}$.

Proof of Claim 1. For each $i \in \{1, \ldots, h-1\}$, we can find $w_i \in K$ such that $v_i(w_i) = e_i$. Now consider the system of congruences:

$$v_i(u - w_i) \geq e_i + 1 \quad \forall i \in \{1, \ldots, h-1\}$$

By the induction hypothesis of the main theorem (assuming it holds for $h-1$ points), such a $u \in K$ exists. We can then check the valuation of u:

$$v_i(u) = v_i(w_i + (u - w_i))$$

Since $v_i(u - w_i) \geq e_i + 1 > e_i = v_i(w_i)$, the properties of non-archimedean valuations imply that $v_i(u) = v_i(w_i) = e_i$. □

Lemma 3.0.3: Claim 2

The valuations $v_1, \ldots, v_h$ corresponding to distinct points $P_1, \ldots, P_h$ are $\mathbb{Q}$-linearly independent.

Proof of Claim 2. Suppose not. Then there exists a non-trivial relation $\sum_{i=1}^{h} \lambda_i v_i(u) = 0$ for all $u \in K$, with $\lambda_i \in \mathbb{Q}$. Without loss of generality, let $\lambda_h = -1$, so $v_h(u) = \sum_{i=1}^{h-1} \lambda_i v_i(u)$.

Consider the base case $h = 2$, with $v_2(u) = \lambda_1 v_1(u)$. If we assume $\lambda_1 > 0$, this implies that for any $z \in K$, $v_1(z) \geq 0 \iff v_2(z) \geq 0$. This means their corresponding valuation rings are identical: $R_{v_1} = R_{v_2}$. For valuations arising from points on a curve, this implies that the points themselves are identical, $P_1 = P_2$, which contradicts our assumption that the points are distinct. A similar argument holds for $\lambda_1 < 0$. The general case for $h > 2$ can be reduced to this case. □

Lemma 3.0.4: Claim 3

There exist elements $z_1, \ldots, z_h \in K^*$ such that the matrix $U = (v_i(z_j))_{h \times h}$ is invertible (i.e., its determinant is non-zero).

Proof of Claim 3. This is a direct consequence of the $\mathbb{Q}$-linear independence of the valuations $v_1, \ldots, v_h$. We can inductively find elements $z_1, \ldots, z_h$ such that the vectors $(v_1(z_j), \ldots, v_h(z_j))$ for $j = 1, \ldots, h$ are linearly independent over $\mathbb{Q}$. □

Construction of the element u

Now, we construct the element u for the theorem. We first construct auxiliary elements y_m with specific valuation properties. Let $U = (v_i(z_j))$ be the invertible matrix from Claim 3. We can solve the matrix equation $U \cdot C = A$ for a matrix $C = (c_{jm})$, where A is the $h \times h$ matrix with -1 on the diagonal and 1 everywhere else.

$$\sum_{j=1}^{h} c_{jm} v_i(z_j) = \begin{cases} -1 & \text{if } i = m \\ 1 & \text{if } i \neq m \end{cases}$$

Let d be an integer that clears the denominators of the rational entries c_{jm}, so $d \cdot c_{jm} \in \mathbb{Z}$

for all j, m. Define y_m for $m \in \{1, \ldots, h\}$ as:

$$y_m := \prod_{j=1}^{h} z_j^{d \cdot c_{jm}}$$

The valuation of y_m at P_i is then:

$$v_i(y_m) = \sum_{j=1}^{h} (d \cdot c_{jm}) v_i(z_j) = d \sum_{j=1}^{h} c_{jm} v_i(z_j) = \begin{cases} -d & \text{if } i = m \\ d & \text{if } i \neq m \end{cases}$$

Now define $x_m \in K^*$ for $m \in \{1, \ldots, h\}$ as:

$$x_m := (1 + y_m^{-1})^{-1} = \frac{y_m}{y_m + 1}$$

Let's choose $d > 0$. The valuations of x_m and $x_m - 1$ are as follows:

- For $i \neq m$: $v_i(y_m) = d > 0$, so $v_i(y_m^{-1}) = -d < 0$. Thus, $v_i(1 + y_m^{-1}) = v_i(y_m^{-1}) = -d$. This gives $v_i(x_m) = -v_i(1 + y_m^{-1}) = d$.
- For $i = m$: $v_m(y_m) = -d < 0$, so $v_m(y_m^{-1}) = d > 0$. Thus, $v_m(1 + y_m^{-1}) = v_m(1) = 0$. This gives $v_m(x_m) = 0$.
- For $i = m$, we also compute $v_m(x_m - 1) = v_m(\frac{-1}{y_m + 1})$. Since $v_m(y_m) = -d < 0$, $v_m(y_m + 1) = v_m(y_m) = -d$. Therefore, $v_m(x_m - 1) = -v_m(y_m + 1) = d$.

Finally, we define our desired element u as a linear combination of the target elements u_i:

$$u := \sum_{m=1}^{h} x_m u_m$$

Let's check the condition $v_i(u - u_i) \geq m_i$. We can write $u - u_i$ as:

$$u - u_i = (x_i - 1)u_i + \sum_{m \neq i} x_m u_m$$

Now we evaluate v_i of each term:

- $v_i((x_i - 1)u_i) = v_i(x_i - 1) + v_i(u_i) = d + v_i(u_i)$
- $v_i(x_m u_m) = v_i(x_m) + v_i(u_m) = d + v_i(u_m)$ for $m \neq i$

Using the property $v(A + B) \geq \min(v(A), v(B))$, we get:

$$v_i(u - u_i) \geq \min\left(d + v_i(u_i), \min_{m \neq i}\{d + v_i(u_m)\}\right) = d + \min_{m \in \{1, \ldots, h\}}\{v_i(u_m)\}$$

We can choose the integer d to be large enough such that for all $i \in \{1, \ldots, h\}$:

$$d + \min_{m \in \{1,\ldots,h\}} \{v_i(u_m)\} \geq m_i$$

Such a d exists because the set of points and target values is finite. This completes the proof. □

> **Corollary 3.0.1: Corollary: Existence of Functions with Prescribed Zeros and Poles**Let $S \subset C$ be a finite set of points. Let $\{m_P | P \in S\} \subset \mathbb{Z}$ be a set of prescribed integer orders. Then there exists a function $f \in K(C)^*$ such that $v_P(f) = m_P$ for all $P \in S$.

Proof. This is a direct application of the construction used to prove Claim 1, which itself relies on the main theorem. □

> **Warning 3.0.1:** The function f is only guaranteed to have the specified orders at the points in S. The full set of its zeros and poles, $Z(f) \cup Z(1/f)$, may be strictly larger than S.

Future Direction: Divisors

Exercise Take $C = \mathbb{P}^1$ and find functions f for a given finite set $S \subset \mathbb{P}^1$ and prescribed orders $\{m_P\}_{P \in S}$.

Divisors of a Curve (or K)

Group of Divisors The free abelian group $\mathrm{Div}(C) = \bigoplus_{P \in C} \mathbb{Z}P$ is the group of divisors of a smooth projective curve. ($P \in C$ iff $\mathcal{O}_{C,P}$ is a DVR in $k(C)$).

Note: $d(P)$ may not be equal to 1.

Elements of $\mathrm{Div}(C)$ have finite support by definition. Therefore, $\sum_{i=1}^{\infty} P_i$ is not in $\mathrm{Div}(C)$.

Maps on the Group of Divisors

Any element $D \in \mathrm{Div}(C)$ is a divisor.

- **Define the order map**, $\mathrm{ord}_P : \mathrm{Div}(C) \to \mathbb{Z}$ such that for $D = \sum_{P \in C} a_P P$, the map is $D \mapsto a_P$.

  The map $_P$ is a group homomorphism.

- **Define the degree map**, $d : \mathrm{Div}(C) \to \mathbb{Z}$ such that for $D = \sum_{P \in C} a_P P$, the map is $D \mapsto \sum_{P \in C} a_P d(P)$.

The map $d(\cdot)$ is a group homomorphism.

- The kernel of the degree map, $\ker(d)$, is the group of degree-0 divisors, denoted by $\mathrm{Div}_0(C)$.

Properties of Divisors

- The **support of D** is defined as $\mathrm{supp}(D) := \{P \in C \mid_P (D) \neq 0\}$.
- A divisor D is called **integral/positive/effective** if $\forall P \in C, _P (D) \geq 0$. We write this as $D \geq 0$.
- We say D_1 **divides** D_2 if $D_2 - D_1 \geq 0$. We write this as $D_2 \geq D_1$ or $D_1 \leq D_2$.

These divisors (written additively) are new objects.

Principal Divisors

Divisors are interesting because each rational function $x \in K^*$ has an associated **principal divisor**, defined as:

$$(x) := \sum_{P \in C} (x) \cdot P \in \mathrm{Div}(C)$$

This is well-defined because $(x) \neq 0$ for only finitely many $P \in C$.

Example Calculation

Let $C = Z(x_2^2 x_0 - x_1^3 + x_1 x_0^2) \subset \mathbb{P}_k^2$. For $f := x_1/x_2 \in K(C)$, what is (f)?

- $P_1 = \langle x_1, x_0 \rangle = \langle x_1 \rangle_{R_{P_1}}$
  $v_{P_1}(f) = v_{P_1}(x_1) - v_{P_1}(x_2) = 1 - 0 = 1.$
- $P_2 = \langle x_1, x_2 \rangle = \langle x_2 \rangle_{R_{P_2}}$
  $v_{P_2}(f) = v_{P_2}(x_1) - v_{P_2}(x_2) = 2 - 1 = 1.$
- $P_3 = \langle x_2, x_1 + x_0 \rangle = \langle x_2 \rangle_{R_{P_3}}$ (up to units in R_{P_3})
  $v_{P_3}(f) = v_{P_3}(x_1) - v_{P_3}(x_2) = 0 - 1 = -1.$
- $P_4 = \langle x_2, x_1 - x_0 \rangle = \langle x_2 \rangle_{R_{P_4}}$ (up to units in R_{P_4})
  $v_{P_4}(f) = v_{P_4}(x_1) - v_{P_4}(x_2) = 0 - 1 = -1.$

So, the principal divisor is:

$$(x_1/x_2) = P_1 + P_2 - P_3 - P_4 \in \mathrm{Div}(C)$$

Note that $d((f)) = 0$, which implies $(f) \in \mathrm{Div}_0(C)$. We will later prove this for all principal divisors.

Question: Is the converse true? i.e., $\forall D \in \mathrm{Div}_0(C)$, does there exist an $f \in K$ such that $D = (f)$?

430.20575ptProposition 3.0.2:

Proposition 3.0.2: The set $\mathrm{Div}_a(C) := \{(x) \mid x \in K^*\}$ is a subgroup of $\mathrm{Div}(C)$.

Proof. $(x) + (y) = (xy), \forall x, y \in K^*$. This follows from the valuation axiom $\nu_p(xy) = \nu_p(x) + \nu_p(y)$. □

We have the following subgroup relationships: $\mathrm{Div}_a(C) \triangleleft \mathrm{Div}_0(C) \triangleleft \mathrm{Div}(C)$. Our long-term goal is to compare these subgroups quantitatively.

3.1 Divisors on a Curve

Definition of $\mathrm{Div}(C)$

Order map and degree map

3.2 Effective and Integral Divisors

Support of a divisor

Divisibility of divisors

3.3 Principal Divisors

Definition of (x)

Examples

3.4 Subgroups of Divisors

Principal subgroup

Degree-zero divisors

3.5 Equivalence Modulo a Divisor

Definition of $x \equiv y \mod D$

Equivalence relation

Definition and intuition

Restricted $L_S(D)$

Vector space structure

3.6 Dimension Formula

$\dim L(D')/L(D) = d(D') - d(D)$

3.7 Bases via Approximation

Constructing a basis using residue fields

3.8 Corollaries

Finiteness of $L(D)$

Behaviour for large divisors

3.9 Degree of Principal Divisors

Degree of zeros and poles

Key theorem on degrees

CHAPTER 4

ZETA FUNCTION OF A CURVE OVER FINITE FIELDS

4.0.1 Introduction: Counting Points and the Zeta Function

The Setup

Let C be a smooth projective curve defined over a finite field $k = \mathbb{F}_q$, where $q = p^e$ for some prime p. The function field of C is denoted by K.

Our goal is to count the number of points on the curve C over finite extensions of k. For any positive integer $n \in_{>0}$, we define N_n as the number of points on C defined over the extension field $\mathbb{F}_{q^n}$:

$$N_n := |C(\mathbb{F}_{q^n)|}$$

A closed point $P \in C$ corresponds to a Galois orbit of points over the algebraic closure $\bar{k}$. The residue field at P, denoted $k(P)$, is a finite extension of k. The degree of the point P, denoted $d(P)$, is the degree of this field extension: $d(P) = [k(P) : k]$. A point is in $C(\mathbb{F}_{q^n)}$ if and only if its residue field $k(P)$ is a subfield of $\mathbb{F}_{q^n}$. This occurs precisely when the degree $d(P)$ divides n. Therefore, we can write N_n as a sum over the closed points of C:

$$N_n = \sum_{P \in C, d(P)|n} d(P)$$

The notes seem to use a slightly different definition $N_n = \#\{P \in C \mid d(P) \text{ divides } n\}$ based on the proof sketch provided. Let's proceed with the standard definition of N_n as the number of $\mathbb{F}_{q^n}$-rational points. The connection between the counts N_n and the degrees of closed points is fundamental.

Definition of the Zeta Function

To study the sequence of numbers $\{N_n\}_{n \geq 1}$, it is useful to encode them in a generating function:

$$G(t) := \sum_{n \geq 1} N_n t^n \in \mathcal{Z}[[t]]$$

However, a more structured object, called the Zeta function of C over k, proves to be better behaved.

Definition 4.0.1.1 (Zeta Function)**.** *The Zeta function of the curve C over k is the formal power series $Z(t) \in \mathcal{Z}[[t]]$ defined by the Euler product:*

$$Z(t) := \prod_{P \in C} (1 - t^{d(P)})^{-1}$$

where the product is taken over all closed points P of the curve C.

By expanding each term in the product as a geometric series, $(1-t^{d(P)})^{-1} = \sum_{i=0}^{\infty} t^{i \cdot d(P)}$, we can see the connection to effective divisors. An effective divisor is a formal sum $D = \sum_P n_P P$ where $n_P \geq 0$ are integers and almost all are zero. The degree of such a divisor is $d(D) = \sum_P n_P d(P)$. The product expansion gives:

$$Z(t) = \prod_{P \in C} \left(\sum_{i=0}^{\infty} t^{i \cdot d(P)} \right) = \sum_{D \geq 0} t^{d(D)}$$

where the final sum is taken over all effective divisors D on C.

4.0.2 Rewriting Z(t) using Divisor Classes

We can group the terms in the sum $\sum_{D \geq 0} t^{d(D)}$ by the degree of the divisor. The degree map $d : \mathrm{Div}(C) \to \mathcal{Z}$ is a group homomorphism. Its image is a subgroup of $\mathcal{Z}$, and hence must be of the form $\delta\mathcal{Z}$ for some integer $\delta \geq 1$.

We can stratify the sum by first summing over degrees $d \in \delta\mathcal{Z}_{\geq 0}$, then over divisor classes $\mathcal{D}$ of degree d, and finally over effective divisors D within that class:

$$Z(t) = \sum_{d \in \delta\mathcal{Z}_{\geq 0}} \sum_{\mathcal{D} \in {}_d(C)} \sum_{D \in \mathcal{D}, D \geq 0} t^d$$

where ${}_d(C)$ is the set of divisor classes of degree d.

To evaluate this, we need to count the number of effective divisors in a given class.

Lemma 4.0.2.1. *Let $\mathcal{D}$ be a divisor class. The number of effective divisors in $\mathcal{D}$ is given by*

$$|\{D \in \mathcal{D} \mid D \geq 0\}| = \frac{q^{l(D_0)} - 1}{q - 1}$$

where D_0 is any divisor in $\mathcal{D}$ and $l(D_0) = \dim_k H^0(C, \mathcal{O}(D_0))$ is the dimension of the Riemann-Roch space $L(D_0)$.

Proof. Fix a divisor $D_0 \in \mathcal{D}$. Any other divisor $D' \in \mathcal{D}$ is linearly equivalent to D_0, meaning $D' - D_0 = (f)$ for some function $f \in K^\times$, where K is the function field of C. The condition $D' \geq 0$ is equivalent to $(f) + D_0 \geq 0$, which means $f \in L(D_0) \setminus \{0\}$.

Two functions $f_1, f_2 \in L(D_0) \setminus \{0\}$ define the same divisor if and only if $(f_1) = (f_2)$, which means $f_1 = c \cdot f_2$ for some constant $c \in k^\times = \mathbb{F}_{q^\times}$. Therefore, the number of distinct effective divisors in $\mathcal{D}$ is the number of 1-dimensional subspaces in the k-vector space $L(D_0)$. This is the number of points in the projective space $\mathbb{P}(L(D_0))$, which is

$$\frac{|L(D_0)| - 1}{|k^\times|} = \frac{q^{l(D_0)} - 1}{q - 1}$$

□

Next, we relate the number of divisor classes of different degrees.

Lemma 4.0.2.2. *For any degree $d \in \delta\mathcal{Z}$, the set of divisor classes ${}_d(C)$ has the same cardinality as the set of degree-0 divisor classes, ${}_0(C)$:*

$$|{}_d(C)| = |{}_0(C)|$$

Proof. Fix a divisor D_0 of degree d. The map $\mathcal{D} \mapsto \mathcal{D} - [D_0]$ is a bijection from ${}_d(C)$ to ${}_0(C)$. □

An important fact is that ${}_0(C)$ is a finite group.

Definition 4.0.2.3. *The **class number** of C over k, denoted $h(C)$, is the size of the degree-0 divisor class group:*

$$h(C) := |{}_0(C)|$$

Using these results, we can rewrite the formula for $Z(t)$. For any given degree d, the dimension $l(D)$ is the same for all divisors D belonging to classes of degree d if d is large enough, but not necessarily for small d. The notes simplify this by writing $l(D)$ as if it only depends on d. Let's denote the sum over classes as an average. Summing over all effective divisors of degree d:

$$\sum_{\mathcal{D} \in {}_d(C)} \frac{q^{l(D_\mathcal{D})} - 1}{q - 1} = \frac{h(C)}{q - 1} \left(\sum_{\mathcal{D} \in {}_d(C)} q^{l(D_\mathcal{D})} \right) - \frac{h(C)}{q - 1}$$

where $D_\mathcal{D}$ is any divisor in class $\mathcal{D}$. The expression for $Z(t)$ becomes:

$$Z(t) = \sum_{d \in \delta\mathcal{Z}_{\geq 0}} \left(\sum_{\mathcal{D} \in {}_d(C)} \frac{q^{l(D_\mathcal{D})} - 1}{q - 1} \right) t^d$$

Proposition 4.0.2.4 (Convergence)**.** The power series $Z(t)$ converges for all $t \in$ with $|t| < q^{-1}$. Let $t = q^{-s}$. Then the function $\zeta(s, C) := Z(q^{-s})$ is defined by the series

$$\zeta(s, C) = \sum_{D \geq 0} N(D)^{-s} = \prod_{P \in C} (1 - N(P)^{-s})^{-1}$$

where $N(P) := q^{d(P)}$ is the norm of the prime divisor P. This series converges for $\Re(s) > 1$. This function is analogous to the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$.

4.0.3 Rationality and the Functional Equation

One of the most important properties of the zeta function is that it is a rational function of t.

Theorem 4.0.3.1 (Functional Equation)**.** *Let g be the genus of the curve C.*

(i) The zeta function $Z(t)$ is a rational function of t, i.e., $Z(t) \in \mathbb{Q}(t)$.

(ii) $Z(t)$ satisfies the following functional equation:

$$Z(t) = (qt^2)^{g-1} Z\left(\frac{1}{qt}\right)$$

Proof Sketch. From the previous subsection, we have $(q-1)Z(t) = \sum_{D \geq 0} (q^{l(D)} - 1) t^{d(D)}$. We can write this as:

$$(q-1)Z(t) = \sum_{d,\mathcal{D}} h(C) q^{l(D)} t^d - \sum_{d,\mathcal{D}} h(C) t^d$$

The second sum is a geometric series: $\sum_{d \in \delta \mathcal{Z}_{\geq 0}} h(C) t^d = \frac{h(C)}{1-t^\delta}$.

For the first sum, $\sum h(C) q^{l(D)} t^d$, we use the Riemann-Roch theorem. The canonical divisor class W has degree $d(W) = 2g-2$. We split the sum over degrees d into two parts: $0 \leq d \leq 2g-2$ and $d > 2g-2$. Let's assume for simplicity that $\delta = 1$. Let $S_1 = \sum_{d=0}^{2g-2} \sum_{\mathcal{D} \in_d(C)} h(C) q^{l(D)} t^d$ and $S_2 = \sum_{d=2g-1}^{\infty} \sum_{\mathcal{D} \in_d(C)} h(C) q^{l(D)} t^d$. S_1 is a finite sum, hence a polynomial in t.

For $d > 2g-2$, the Riemann-Roch theorem states $l(D) = d + 1 - g$. Thus, $l(D)$ only depends on the degree d.

$$S_2 = \sum_{d=2g-1}^{\infty} h(C) q^{d+1-g} t^d = h(C) q^{1-g} \sum_{d=2g-1}^{\infty} (qt)^d$$

This is a geometric series which sums to a rational function:

$$S_2 = h(C) q^{1-g} \frac{(qt)^{2g-1}}{1-qt}$$

Since $(q-1)Z(t) = S_1 + S_2 - \frac{h(C)}{1-t}$, and all terms on the right are rational functions, $Z(t)$ must be a rational function.

To prove the functional equation, one shows that the expression for $Z(t)$ is invariant under the transformation $t \mapsto \frac{1}{qt}$ up to the factor $(qt^2)^{g-1}$. This is done by applying the

Riemann-Roch theorem in the form $l(D) = d(D) + 1 - g + l(W - D)$ to the sum S_1. This relates the sum over degrees $d \in [0, 2g-2]$ to itself in a symmetric way. The notes show this for functions $F(t)$ and $G(t)$ which correspond to the sums, eventually yielding:

$$Z\left(\frac{1}{qt}\right) = (qt^2)^{1-g} Z(t)$$

which is equivalent to the stated functional equation. □

Corollary 4.0.3.2. *The function $\zeta(s, C)$ has a meromorphic continuation to all $s \in$ and satisfies the functional equation:*

$$\zeta(1-s, C) = (q^{2g-2})^{s-1/2} \zeta(s, C)$$

This reveals a symmetry about the critical line $\Re(s) = 1/2$.

Corollary 4.0.1: Degree GCD is 1 The zeta function can be written in the form

$$Z(t, C) = \frac{L(t, C)}{(1-t)(1-qt)}$$

where $L(t, C) \in \mathcal{Z}[t]$ is a polynomial of degree $2g$. The poles of $Z(t)$ are simple poles at $t = 1$ and $t = 1/q$. This implies that $\delta = 1$.

4.0.4 Behavior under Base Change

[cite$_s$*tart*]$We now consider how the zeta function changes when we extend the base field from$k=F$_q$ to $k_n = \mathbb{F}_{q^n}$[cite: 135]. [cite$_s$*tart*]LetC$_n$ be the curve C considered over the field k_n[cite: 136].

Theorem 4.0.4.1 (Base Change)**.** *[cite$_s$tart]The zeta function of C_n over k_n is related to the zeta function of C over k by the following identity[cite: 137, 169]:*

$$Z(t^n, C_n) = \prod_{\eta^n = 1} Z(\eta t, C)$$

where the product is over all n-th roots of unity η.

Proof Sketch. We compare the Euler products on both sides. Let P be a closed point of C over k with degree $d(P)$. [cite$_s$*tart*]$Over the extension field$k$_n$, the point P splits into e closed points $Q_1, \ldots, Q_e$ of C_n[cite: 145, 146, 153]. [cite$_s$*tart*]$The number of points is$e = $\gcd(n, d(P))$[cite: 151]. The degree of each new point Q_i over k_n is $d'(Q_i) = d(P)/e$.

The contribution from the points $\{Q_i\}$ to the LHS, $Z(t^n, C_n)$, is:

$$\prod_{i=1}^{e} \left(1 - (t^n)^{d'(Q_i)}\right)^{-1} = \left(1 - t^{n \cdot d(P)/e}\right)^{-e}$$

[cite$_s tart$][$cite$: 155].$The contribution from the point \mathrm{P to the RHS}, \prod_{\eta^n=1} Z(\eta t, C)$, is:

$$\prod_{\eta^n=1} \left(1 - (\eta t)^{d(P)}\right)^{-1}$$

[cite$_s tart$][$cite$: 168].[$cite_s tart$]$The theorem follows from the algebraic identity$[$cite$: 158] :$(1$-$X^{n/e})^e = \prod_{\eta^n=1}(1 - (\eta Y)^{d/e})$ where X=t$^{d(P)}$, $Y = t$, $d = d(P)$. Let $u = t^{d(P)}$ and $m = d(P)$. We need to prove that $(1 - u^{n/e})^e = \prod_{\eta^n=1}(1 - \eta^m u)$. A standard identity is $\prod_{\zeta^k=1}(1 - \zeta x) = 1 - x^k$. Let $d = d(P)$. [cite$_s tart$]As η runs through the n-th roots of unity, η^d runs through the (n/e)-th roots of unity, with each root appearing e times[cite: 160, 161]. Therefore,

$$\prod_{\eta^n=1} (1 - (\eta t)^d) = \prod_{\eta^n=1} (1 - \eta^d t^d) = \left(\prod_{\zeta^{n/e}=1} (1 - \zeta t^d) \right)^e = (1 - (t^d)^{n/e})^e = (1 - t^{nd/e})^e$$

Taking the reciprocal of both sides proves that the factors are equal, and thus the identity holds. □

Corollary 4.0.4.2 (cite$_s tart$). *The integer $\delta = \gcd_{P \in C}\{d(P)\}$ must be 1[cite: 170].*

Proof Sketch. [cite$_s tart$]$The poles of$ Z(t,C)$ are related to the roots of$ (1-t$^{\delta}$) and $(1-(qt)^{\delta})$[cite: 134, 173]. If $\delta > 1$, the poles would be at the δ-th roots of unity (and $1/q$ times these roots). Consider the base change to $k_\delta = \mathbb{F}_{q^\delta}$. From the theorem:

$$Z(t^\delta, C_\delta) = \prod_{\eta^\delta=1} Z(\eta t, C) = \prod_{\eta^\delta=1} \frac{L(\eta t, C)}{(1 - \eta t)(1 - q\eta t)}$$

The product in the denominator is $(1 - t^\delta)(1 - (qt)^\delta)$. This implies that $Z(T, C_\delta)$, with $T = t^\delta$, has poles at $T = 1$ and $T = 1/q$. This means the gcd of degrees of points on C_δ is 1. By construction, for any point P on C of degree $d(P)$, its split components on C_δ have degree $d(P)/\gcd(\delta, d(P))$. Since δ divides all $d(P)$, this degree is $d(P)/\delta$. The gcd of these new degrees is $\frac{1}{\delta}\gcd(d(P)) = \frac{\delta}{\delta} = 1$. This is consistent. The Weil bounds imply the existence of a rational point over some finite extension, which can be used to show $\delta = 1$ if the base field is chosen appropriately (e.g., large enough). The notes suggest this is a direct corollary of the base change formula itself, likely by analyzing and comparing the pole structures. □

4.1 Point Counting

Definition of N_n

Counting points over extensions

4.2 Generating Functions

Definition of $G(t)$

Definition of $Z(t)$

Properties of the power series

4.3 Expansion Using Divisor Classes

Class groups

Counting techniques

4.4 Class Number

Definition of $h(C)$

4.5 Convergence

Convergence of $Z(t)$ and $\zeta(s)$

4.6 Functional Equation

Rationality of $Z(t)$

Symmetry around $s = 1/2$

4.7 Base Change of Fields

Behaviour under $k \to \mathbb{F}_{q^n}$

Contribution of valuations

4.8 Final Formula

Expression of $Z(t)$ via L-polynomial

CHAPTER 5

WEIL BOUNDS, COHOMOLOGY & L-FUNCTIONS

Page 1

On the other hand,

$$Z(t, c_\delta) := \frac{L'(t)}{(1-t^8)\cdot(1-qt^9)}$$

$$\Rightarrow Z(t^8, c_\delta) = \frac{L'(t^8)}{(1-t^{88})\cdot(1-q^8\cdot t^{88})}$$

which has only simple poles.
$\Rightarrow$ The two expressions imply $\delta = 1$.

- Note that $L(0) = Z(0) = 1$.
- Also, $(q-1)\cdot Z(t)\cdot(t-1)|_{t=1} = h(c)$ [as ??=1].
  $(-1)\cdot\frac{L(t)}{qt-1}|_{t=1} = h(c) \Rightarrow L(1) = h(c)$.

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Theorem 5.0.0.1 (L-function)**.** *(i) We've an exact sequence*

$$^0(C) \to \mathbb{Z} \to 0.$$

(ii) $Z(t, C) = L(t)/((1-t)(1-qt))$
where $L(t) \in \mathcal{Z}[t]$ *has* $= 2g$ *&*

$$L(t) = (qt^2)^g \cdot L(1/qt)$$

(iii) $L(0) = 1$ *&* $L(1) = h(C)$.

Qn. Can we compute $D \in^{-1}(1)$?
Qn. Can we compute L(t) efficiently, given C/k?

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Consequences to counting pts.

- Euler product: $Z = Z(t, C) = \prod_{P \in C}(1 - t^{(P)})^{-1}$.
- Idea: Use dlog (log-derivative operator). [dlog $f = f'/f$]

$$\begin{aligned} d\log Z &= \sum_{P \in C} d\log(1 - t^{(P)})^{-1} \\ &= \sum_{P \in C} \frac{(P) \cdot t^{(P)-1}}{(1 - t^{(P)})} \\ &= t^{-1} \cdot \sum_{P \in C} (P) \cdot \sum_{n \geq 1} t^{n \cdot (P)} \\ &= t^{-1} \cdot \sum_{m \geq 1} t^m \left(\sum_{P \in C : (P) | m} (P) \right) = t^{-1} \cdot \sum_{m \geq 1} N_m \cdot t^m \end{aligned}$$

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- Definite integration $\int_0^t$ gives:

Proposition 5.0.0.2. $\log Z(t) = \sum_{m \geq 1} N_m \frac{t^m}{m}$.

- $Z(t) = \exp\left(\sum_{m \geq 1} N_m \cdot \frac{t^m}{m}\right)$.
- Now, use

$$Z(t) = \frac{L(t)}{(1-t)(1-qt)} = \frac{\prod_{i=1}^{2g}(1 - \alpha_i t)}{(1-t)(1-qt)}$$

  ($L(0) = 1$, reciprocal roots of L are α_i)
- Plugging in the first formula above, we get:

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$$\log Z(t) = \sum_{i=1}^{2g} \log(1 - \alpha_i t) - \log(1 - t) - \log(1 - qt)$$

Comparing the series expansion of this with $\log Z(t) = \sum_{m \geq 1} N_m \cdot \frac{t^m}{m}$ yields:

$$N_m = q^m + 1 - \sum_{i=1}^{2g} \alpha_i^m$$

The term $\sum_{i=1}^{2g} \alpha_i^m$ is viewed as the error-term.

Qn. How big can the error be?

- By functional equation symmetry we get: We can label α_i's s.t. $\alpha_i \cdot \alpha_{i+g} = q$, $\forall i \in \{1, \ldots, g\}$.

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The Riemann Hypothesis (RH)

- RH conjectures a strong symmetry, namely

$$|\alpha_i| = \sqrt{q} = q^{1/2}$$

- In terms of $Z(s, C) = Z(q^{-s}, C)$, it means that the zeros of Z are on the line $Re(s) = \frac{1}{2}$ [just like the original unproved RH!]
- It means: $|N_m - (q^m + 1)| = |\sum_{i=1}^{2g} \alpha_i^m| \leq \sum_{i=1}^{2g} |\alpha_i|^m \leq 2g \cdot q^{m/2}$.

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RH $\Rightarrow$ #pts on a curve (smooth projective) are in the range $q + 1 \pm 2g\sqrt{q}$.

- We'll prove RH by doing a sequence of reductions, and finally using the l(D)-sheaf.

Proposition 5.0.0.3 (Base-change)**.** RH is true for $Z(t, C)$ iff RH is true for $Z(t, C_n)$, for $n > 1$.

Proof. We know Z(t, C) ...
We've: $(1 - \beta t)$ is a factor of $Z(t, C_n) \iff (1 - \beta^{1/n} t)$ is a factor of $Z(t^n, C_n)$. □

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- Thus, RH for $C_n \Rightarrow |\beta| = q^{n/2}$
- $\Rightarrow |\beta^{1/n}| = (q^{n/2})^{1/n} = q^{1/2}$
- $\Rightarrow$ All roots of $Z(t, C)$ satisfy $|\alpha| = \sqrt{q}$
  $\Rightarrow$ RH for C.
- Conversely, assume RH for $Z(t, C)$.
- As, $(1 - \alpha t)$ is a factor of $Z(t, C)$
- We deduce, $\prod_{\eta^n=1}(1 - \alpha\eta t)$ are the factors of $Z(t^n, C_n)$.
- $(1 - \alpha^n t^n)$ is a factor of $Z(t^n, C_n) \Rightarrow (1 - \alpha^n t)$ is a factor of $Z(t, C_n)$.
- $|\alpha^n| = |\alpha|^n = (q^{1/2})^n = q^{n/2} \Rightarrow$ RH for C_n.

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Proposition 5.0.0.4. The following are equivalent (TFAE):

(i) RH for $Z(t, C)$. [i.e., $|\alpha_i| = q^{1/2}, \forall i$]

(ii) $|N_d - (q^d + 1)| \leq A + B \cdot q^{d/2}$ for some constants $A, B, N \in \mathcal{Z}$ & for all multiples d of N. (independent of d)

Proof. (i) $\Rightarrow$ (ii): We've proved this already ($\forall d$).
(ii) $\Rightarrow$ (i): Replace the base field $\mathbb{F}_q$ by $\mathbb{F}_{q^N}$. As we vary d (over multiples of N), we can rename $\mathbb{F}_{q^N}$ to be the new base field $\mathbb{F}_q$. The hypothesis becomes $|N_d - (q^d+1)| \leq A + B \cdot q^{d/2}$ for all $d \geq 1$.
We've $\prod_{i=1}^{2g} \alpha_i = \prod_{i=1}^{g} (\alpha_i \cdot \alpha_{i+g}) = q^g$ (by symmetry). □

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It suffices to show: $\forall i, |\alpha_i| \leq \sqrt{q}$. (The original text had "Kils" here).
Recall $L(t) = \prod_{i=1}^{2g}(1 - \alpha_i t)$.

$$\Rightarrow \log \frac{1}{L(t)} = \sum_{d \geq 1} \left(\sum_{i=1}^{2g} \alpha_i^d \right) \cdot \frac{t^d}{d}$$

Let's consider the curve C_N over $\mathbb{F}_{q^N}$. Let $q' = q^N$ and its roots be $\alpha_i' = \alpha_i^N$.

$$\left| \log \frac{1}{L(t)} \right| \leq \sum_{d \geq 1} (A + B \cdot (q')^{d/2}) \cdot \frac{|t|^d}{d}$$

$$= A \cdot \log \frac{1}{1 - |t|} + B \cdot \log \frac{1}{1 - |t|\sqrt{q'}}$$

The LHS converges for $t \in \mathbb{C}$ if $|t| < 1/\sqrt{q'}$.
$\Rightarrow$ Zeros of $L(t)$ (which are $1/\alpha_i'$) must have norm $\geq 1/\sqrt{q'}$.

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$\Rightarrow |1/\alpha_i'| \geq 1/\sqrt{q'}$, which implies $|\alpha_i'| \leq \sqrt{q'}$.
Combining with the functional equation (which implies $|\alpha_i'| \geq \sqrt{q'}$), we get

$$\forall i \in [2g], \quad |\alpha_i'| = \sqrt{q'}.$$

RH for $Z(t, C_N)$ holds. By the base-change proposition, RH holds for $Z(t, C)$.

Next reduction that we need is to make the Cover Galois;

$$C \to^1; \quad k(x_1)[x_2]/\langle F \rangle \supseteq k(x_1)$$

, we want $k(C)$ to have all roots of F wrt x_2 (think of x_1 fixed in k).

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Move to the Galois Cover of C. Let C (in affine patch $x_0 = 1$) be

$$x_2^3 - x_2 - x_1^2 = 0 \quad \text{over } k = \mathbb{F}_q.$$

$\Rightarrow k(C) =: K = k(x_1)[x_2]/\langle x_2^3 - x_2 - x_1^2 \rangle$ is a $= 3$ extension of $k(x_1) = k(^1)$ which may not be a splitting field.

- We correct this by moving to the splitting field K' of $x_2^3 - x_2 - x_1^2$ over $k(x_1)$.

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$[K' : k(x_1)]$ is typically $3! = 6$. For this specific polynomial, the Galois group is S_3.

$$K' \supset K \supset k(x_1) \supset k = \mathbb{F}_q$$

K'/K & $K'/k(x_1)$ are Galois extensions (i.e., they're separable & normal.)
Let C' be the curve defined by K' (as it's a function field in 1 variable over k).

$$C' \to C \to^1$$

is called the Galois Cover of C.

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Let $f(y) = y^3 - y - x_1^2$. If x_2 is a root, we can divide $f(y)$ by $(y - x_2)$. The fiber over a point $x_1 \in k(x_1)$ corresponds to the roots of $y^3 - y - x_1^2 = 0$.

$$\begin{array}{ccc} P_0, P_1, P_2 & \in & k(x_1)[x_2, y_2]/\langle \dots \rangle \cong k(C') \\ \downarrow & & \\ P \in C & \in & k(x_1)[x_2]/\langle \dots \rangle \cong k(C) \\ \downarrow & & \\ x_1 & \in & k(x_1) \end{array}$$

$P_0 := (x_1, x_2)$, $P_1 := (x_1, y_2)$& $P_2 := (x_1, -x_2 - y_2)$ are the three roots, given x_1.
Let F be the Frobenius & σ be a Galois automorphism of $k(C')$ over $k(C)$. For instance, an element $\sigma \in \mathrm{Gal}(K'/K)$ would fix P_0 and permute P_1, P_2.

$$P_0^\sigma = P_0, \quad P_1^\sigma = P_2, \quad P_2^\sigma = P_1.$$

E.g., for $(x_1, x_2) \in \mathbb{F}_{q \times \mathbb{F}_q}$ s.t. $x_2^3 - x_2 - x_1^2$ has only one root in $\mathbb{F}_q$.

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The other two roots are in $\mathbb{F}_{q^2 \setminus \mathbb{F}_q}$. The Frobenius F acts as $F(P_0) = P_0$, $F(P_1) = P_2$, $F(P_2) = P_1$.
P_1, P_2 are conjugates over $\mathbb{F}_{q(C)}$, via F.

- $P_1 = P_2$ for some $x_1 \in \mathbb{F}_q$ implies x_1 is a zero of the discriminant of the polynomial, or $\text{Res}_{x_2}(f(x_2, x_1), \frac{\partial f}{\partial x_2}(x_2, x_1))$.
- Thus, $P_1 = P_2$ happens for $\leq (\deg f)^2$ many x_1's.

Definition 5.0.0.5. *Let $G = Gal(k(C')/k(C))$ be the group of $k(C)$-automorphisms of $K' := k(C')$.*

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Proposition 5.0.0.6. Let C be a smooth projective curve over k & let C' be a Galois cover.

$$|\text{Gal}(k(C')/k(C))| = [k(C') : k(C)]$$

Proof. $k(C')/k(C)$ is a finite Galois extension. By the primitive element theorem, $k(C') = k(C)(\alpha)$. $(\text{minpoly}_\alpha) = [k(C') : k(C)]$.
& $k(C')$ is the splitting-field of minpoly_α. & so it has all its conjugates.

$$|\text{Gal}(k(C')/k(C))| = \#\text{ conjugates} = [k(C') : k(C)].$$

□

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$\forall \sigma \in \text{Gal}(k(C')/k(C))$, can be seen to act on pts. $P' \in C'$ via its action on $k(C')$. Say, pt. $P' = (x_1, x_2)$.

- $\sigma(x_1)\&\sigma(x_2)$ can be seen as functions in $k(C')$.
- Use them to define $\sigma(P')$.
- Qn: What about poles of $\sigma(x_2)$?

Frobenius F: $k(C) \to k(C)$; $f(x_1, x_2) \mapsto f(x_1^q, x_2^q)$. This induces a map on points (the geometric Frobenius): $(\alpha, \beta) \mapsto (\alpha^{q^{-1}}, \beta^{q^{-1}})$.
F is injective. (k-Monomorphism).

5.0.1 Introduction and Setup

This document outlines a proof of the Hasse-Weil bound (also known as the Riemann Hypothesis for curves over finite fields). The method presented is based on the work of Bombieri and Stepanov. The core idea is to bound the number of rational points on a curve by constructing a function with specific properties related to its zeros and poles.

Let C_0 be a curve defined over a finite field . Let $k_0 = (C_0)$ be its function field. We consider the curve C over the algebraic closure $k = \Bbbk$, with function field $K = k_0 \cdot k$. The points of C rational over , denoted $C()$, are the points fixed by the geometric Frobenius endomorphism F.

- The Frobenius map F acts on the coordinates of points. For a point $P = (x_1, \ldots, x_n)$, $F(P) = (x_1^q, \ldots, x_n^q)$. [cite$_s tart$]

- Consequently, the set of -rational points is precisely the set of fixed points of Frobenius[cite: 5]:
$$C() = \{P \in C \mid F(P) = P\}$$
The number of these points is denoted $N_1(C) = C()$.

- The action of Frobenius on functions in the function field $K(C)$ is given by raising the coefficients to the q-th power. [cite$_s tart$]$For example, the action on a function$(\mathrm{x}_1 - \alpha)$ is $F^*(x_1 - \alpha) = x_1^q - \alpha^q$[cite: 1].

- This action relates the order of vanishing of a function at a point P to the order of vanishing of its image at the point $F(P)$. [cite$_s tart$]$Specifically, _P(f \circ F) = q \cdot_{F(P)} (f)$[cite: 2, 69].

Galois Covers and Averaging

The proof strategy involves relating the point count on an arbitrary curve C to the point count on a related curve C' which is a Galois cover.

[cite$_s tart$]$Let \phi : C' \to C$ be a Galois cover, with $G = (k(C')/k(C))$ being the Galois group of the corresponding function field extension[cite: 8]. For any $\sigma \in G$, we define a set of "twisted" fixed points on C': [cite$_s tart$]$\bar{N}_1(C', \sigma) := \{P \in C' \mid \sigma^{-1} \circ F(P) = P\}$ [cite: 8]$The number of such points is denoted \mathrm{N}_1(C', \sigma)$. [cite$_s tart$]$The standard set of rational points sc$ is the identity element, $N_1(C') = N_1(C', 1)$[cite: 8].

A key tool is the following proposition, which connects the number of rational points on C to the average of these twisted point counts on C'.

Proposition 5.0.1.1 (Averaging over G)**.** The number of -rational points on C is given by: [cite$_s tart$]$N_1(C) = \frac{1}{G} \sum_{\sigma \in G} N_1(C', \sigma) + O_\delta(1)$ [cite: 11][$cite_s tart$]$where the error term \mathrm{O}_\delta(1)$

depends on the degree δ of the extension defining C, specifically on the number of ramified points[cite: 11, 14].

Proof Sketch.

- For any point $P \in C$, consider its fiber under the cover, $\phi^{-1}(P) = \{Q_1, \ldots, Q_r\}$[cite: 12]. [cite$_s$*tart*]$The size of the fiber$,r, $is equal to$|G|ifP$is unramified, and$r $<$ |G|ifP$is ramified$[*cite* : 13, 18].[*cite*$_s$*tart*]$The number of ramified points is bounded, e.g., byδ^2, contributing the $O_\delta(1)$ term[cite: 14]. [cite$_s$*tart*]

- If $P \in C()$ is an unramified point, then for any $Q \in \phi^{-1}(P)$, the point $F(Q)$ is also in the fiber $\phi^{-1}(P)$ because $F(P) = P$[cite: 12]. The group G acts transitively on the fiber. For any Q_i, Q_j in the fiber, there is a unique $\sigma \in G$ such that $Q_j = \sigma(Q_i)$.

- Let's count the size of the sum $\sum_{\sigma \in G} N_1(C', \sigma)$. [cite$_s$*tart*]$A point$Q' $\in C'$ contributes to this sum if there exists some $\sigma \in G$ such that $\sigma^{-1}(F(Q')) = Q'$[cite: 19].

- If Q' contributes to the sum, let $P' = \phi(Q')$. Then $\phi(\sigma^{-1}(F(Q'))) = \phi(Q')$, which implies $\phi(F(Q')) = \phi(Q')$. [cite$_s$*tart*]$This simplifies to$F$(\phi(Q')) = \phi(Q')$, meaning $P' = \phi(Q')$ is an -rational point on C[cite: 21]. [cite$_s$*tart*]

- So, the sum only counts points Q' that lie above the -rational points of C[cite: 22, 23].

- For each unramified point $P \in C()$, and for each $Q \in \phi^{-1}(P)$, the set $\{\sigma^{-1}(F(Q)) \mid \sigma \in G\}$ is precisely the fiber $\phi^{-1}(P)$. Therefore, for each Q in the fiber, there is exactly one σ that fixes it in this twisted sense. This means each unramified rational point $P \in C()$ contributes exactly $|G|$ to the total sum $\sum_{\sigma \in G} N_1(C', \sigma)$.

- Summing over all unramified points, we get $\sum_{\sigma \in G} N_1(C', \sigma) \approx |G| \cdot N_1(C)$. [cite$_s$*tart*]$Including the error fr$ 29]. □

5.0.2 The Main Theorem and its Implication

The proof hinges on the following theorem, which provides a bound for the number of twisted fixed points on a Galois cover of the projective line .

Theorem 5.0.2.1 (Weil, Bombieri-Stepanov)**.** *[cite$_s$tart]Let $C \to$ be a Galois cover over [cite: 39]. Let g be the genus of C. [cite$_s$tart]Assume $q=p^\alpha$ with α even, and that $q > (g+1)^4$[cite: 39]. Then, for any $\sigma \in Aut(C/)$, we have the bound: [cite$_s$tart]$N_1(C, \sigma) \leq q + 1 + (2g+1)\sqrt{q}$ [cite: 40]*

This theorem for a special case (Galois covers of) can be leveraged to prove the Riemann Hypothesis for any curve C_0.

Implication for General Curves Let C_0 be an arbitrary curve.

1. Construct the smallest Galois extension K' of the function field $K = k(C_0)$ and let C' be the corresponding curve. [cite$_s tart$]$ThisgivesaGaloiscover$C' $\to C_0$[cite: 47, 48]. [cite$_s tart$]LetH = (K'/K)[$cite$: 50].[cite$_s tart$]

1. By the Averaging Proposition: $N_1(C_0) = |H|^{-1} \sum_{h \in H} N_1(C', h) + O_\delta(1)$[cite: 53].

2. Now view C' as a cover of the projective line . [cite$_s tart$]LetG = (K'/(x))$bethefullGaloisgroup, whe$
$\geq H$[cite: 54].

3. We know $N_1() = q + 1$. Applying the Averaging Proposition to the cover $C' \to$: [cite$_s tart$]$q + 1 = N_1() = |G|^{-1} \sum_{\sigma \in G} N_1(C', \sigma) + O_\delta(1)$ [cite: 57]

3. The theorem gives an upper bound on each term in the sum: $N_1(C', \sigma) \leq q + 1 + O(g\sqrt{q})$. Combining this with the equality from the previous step implies that the terms $N_1(C', \sigma)$ must be close to $q + 1$ on average. This forces a lower bound as well, leading to the two-sided estimate: [cite$_s tart$]$N_1(C', \sigma) = q + 1 + O(g\sqrt{q} + \delta^2)$ [cite: 57]

3. Substituting this estimate back into the formula for $N_1(C_0)$ from step 2 (noting that H is a subset of G), we get: [cite$_s tart$]$N_1(C_0) = \frac{1}{|H|} \sum_{h \in H} \left(q + 1 + O(g\sqrt{q} + \delta^2)\right) =$
q+1+$O(g\sqrt{q}$+$\delta^2)$ [cite: 57][cite$_s tart$]$Thisisthecelebrated Hasse-Weilbound, whichisthestatem$
58].

5.0.3 Proof of the Main Theorem

[cite$_s tart$]$Wenowfocusonprovingthetheoremfor aGaloiscover$C $\to$[cite: 59]. Let ϕ be the composite map $\sigma^{-1} \circ F$. We want to bound the number of points $P \in C$ such that $\phi(P) = P$.

Construction of the Auxiliary Function

If $N_1(C, \sigma)$ is empty, the theorem is trivial. [cite$_s tart$]$Otherwise, pickapoint$P $\in C$ such that $\phi(P) = P$[cite: 65]. [cite$_s tart$]$Foranintegera > 2$g-2$, considertheRiemann - Rochspace$L$_a = L(aP)$, which is the vector space of functions $f \in k(C)$ whose only poles are at P with order at most a[cite: 66]. [cite$_s tart$]$BytheRiemann-Rochtheorem, itsdimensionis$l$_a =$ $\dim(L_a) = a + 1 - g$[cite: 67].

We will construct a non-zero function that vanishes at many of the points we want to count. The argument relies on two actions on these function spaces:

1. The map $\phi = \sigma^{-1} \circ F$. If $f \in L_a$, then $f \circ \phi$ has a pole of order at most qa at P (since $\phi(P) = P$). [cite$_s tart$]$Thisdefinesalinearmap$L$_a \to L_{aq}$[cite: 74, 76].

2. The absolute Frobenius map $F_{abs} : x \mapsto x^p$. Let $q = p^\alpha$. For an integer μ, we consider the map $f \mapsto f \circ F_{abs}^\mu = f^{p^\mu}$. If $f \in L_b$, then f^{p^μ} has a pole of order at most bp^μ at P. [cite$_s tart$]$This gives a linear map \mathrm{L}_b \to L_{bp^\mu}$[cite: 88, 91].

Now, consider the multiplication map on the tensor product of these transformed spaces:

$$\Psi : L_b^{p^\mu} \otimes L_a^q \to L_{bp^\mu + aq}$$

[cite$_s tart$]$(s \circ F_{abs}^\mu) \otimes (f \circ \phi) \mapsto (s \circ F_{abs}^\mu) \cdot (f \circ \phi)$ [cite: 95]

[cite$_s tart$]$The multiplication map \Psi$ is injective provided that $bp^\mu < q$[cite: 95].

Proof of Claim 1. [cite$_s tart$]$Let \{\mathrm{f}_i\}_{i=1}^{l_a}$ be a basis of L_a such that the pole orders at P are strictly increasing: $v_P(f_i) < v_P(f_{i+1})$, where $v_P = -_P$[cite: 104]. [cite$_s tart$]$An arbitrary element in the ten$
$= \sum_i (s_i \circ F_{abs}^\mu) \otimes (f_i \circ \phi)$, where $s_i \in L_b$[cite: 110]. Suppose its image under Ψ is zero: [cite$_s tart$]$\sum_i (s_i \circ F_{abs}^\mu) \cdot (f_i \circ \phi) = 0$ [cite: 112][$cite_s tart$]$Let h be the smallest index for which s_h \neq$ 0[cite: 113]. We can write: [cite$_s tart$]$(s_h \circ F_{abs}^\mu) \cdot (f_h \circ \phi) = -\sum_{i>h} (s_i \circ F_{abs}^\mu) \cdot (f_i \circ \phi)$ [cite: 115]$Now we compare the pole order sat \mathrm{P} for both sides. The valuation \mathrm{v}_P$ transforms as $v_P(g \circ \phi) = q \cdot v_P(g)$ and $v_P(g \circ F_{abs}^\mu) = p^\mu \cdot v_P(g)$.

$$v_P(\text{LHS}) = p^\mu v_P(s_h) + q v_P(f_h)$$
$$v_P(\text{RHS}) \geq \min_{i>h} \{p^\mu v_P(s_i) + q v_P(f_i)\}$$

Since $v_P(f_i) > v_P(f_h)$ for $i > h$, and $v_P(s_i) \leq b$ for all i, we get:

$$p^\mu v_P(s_h) + q v_P(f_h) \geq \min_{i>h} \{p^\mu v_P(s_i) + q v_P(f_i)\} > p^\mu v_P(\text{some } s_i) + q v_P(f_h)$$

$p^\mu (v_P(s_h) - v_P(s_i)) > q(v_P(f_h) - v_P(f_i))$ (This step in the notes seems slightly off, let's follow the note

[cite$_s tart$]$The logic in the notes [cite : 118] leads to \mathrm{p}^\mu v_P(s_h) \geq -bp^\mu + q(v_P(f_i) - v_P(f_h))$. Since $v_P(f_i) - v_P(f_h) > 0$, if we have $q - bp^\mu > 0$, a contradiction arises, implying that our assumption (that some $s_h \neq 0$) must be false. [cite$_s tart$]$Thus, all \mathrm{s}_i = 0$ and the map is injective[cite: 119, 121, 122]. □

The Counting Argument

To create our desired function, we need a non-zero element in the kernel of some map. The previous map is injective under the condition $bp^\mu < q$. So instead, we consider a different map and find an element in its kernel. Let's consider the space $L_b \otimes L_a$. We require its dimension, $l_b \cdot l_a$, to be greater than the dimension of the target space $L_{bp^\mu + aq}$. This will guarantee the existence of a non-zero function in the kernel of a related map.

Let $G_0 = \sum_i s_i \otimes f_i$ be a non-zero element in $L_b \otimes L_a$. We construct the function

$$G = \sum_i (s_i \circ F_{abs}^{\mu}) \cdot (f_i \circ \phi)$$

[cite$_s$*tart*]*We can choose* G_0 such that a related function $\tau(G_0) = \sum_i s_i^{p^\mu} f_i$ is identically zero[cite: 140]. The function G is not identically zero (by a modification of Claim 1's argument). [cite$_s$*tart*]*By construction,* G *lies in the space* $L_{bp^\mu+aq}$, so the degree of its pole divisor is at most $bp^\mu + aq$[cite: 144].

Now let's find the zeros of G. Let Q be any point in our target set, i.e., $Q \in N_1(C, \sigma)$ and $Q \neq P$. By definition, $\phi(Q) = Q$. At such a point:

$$G(Q) = \sum_i s_i(F_{abs}^{\mu}(Q)) \cdot f_i(\phi(Q)) = \sum_i s_i(Q)^{p^\mu} \cdot f_i(Q) = \tau(G_0)(Q) = 0$$

[cite$_s$*tart*]*So,* G *vanishes at every point in* $N_1(C, \sigma) \setminus \{P\}$[cite: 141, 142].

[cite$_s$*tart*]*The argument in the notes* [*cite* : 141] *is that the related function* $\tau(G_0)$ is a p^μ-th power, and thus vanishes with high multiplicity. This high order of vanishing at the points Q is crucial. The degree of the zero divisor of G must be at least $p^\mu \cdot (|N_1(C, \sigma)| - 1)$. Since the degree of the zero divisor equals the degree of the pole divisor: [cite$_s$*tart*]$p^\mu(N_1(C, \sigma) - 1) \leq \deg((G)_\infty) \leq bp^\mu + aq$ [cite: 143, 144] *Rearranging this inequality gives our initial bound* : [*cite*$_s$*tart*]$N_1(C, \sigma) \leq 1 + b + \frac{aq}{p^\mu}$ [cite: 145]

Parameter Optimization

[cite$_s$*tart*]*The final step is to choose the integer parameters* a, b, μ to make this bound as strong as possible, while satisfying the necessary conditions for the proof to work[cite: 147].

Let $q = p^\alpha$ with α even. [cite$_s$*tart*]*Choose the parameters as follows* [*cite* : 148] :

$\mu = \alpha/2$, which implies $p^\mu = \sqrt{q}$[cite: 149]. [cite$_s$*tart*]

$a = \lfloor \sqrt{q} \rfloor + 2g$[cite: 149]. [cite$_s$*tart*]

$b = \left\lfloor \frac{g\sqrt{q}}{g+1} \right\rfloor + g + 1$[cite: 150].

Then for $q > (g+1)^4$, the following conditions hold:

(i) $bp^\mu < q$[cite: 151]. [cite$_s$*tart*]

(ii) $l_b \cdot l_a > l_{bp^\mu+a}$ (a slightly different target space from the notes to ensure a non-zero element)[cite: 152]. [cite$_s$*tart*]

(iii) The bound becomes $1 + b + a\sqrt{q} \leq q + 1 + (2g+1)\sqrt{q}$[cite: 154].

Proof of Claim 2.

- **For (i):** $b = \left\lfloor \frac{g\sqrt{q}}{g+1} \right\rfloor + g + 1 \leq \frac{g\sqrt{q}}{g+1} + g + 1 = \frac{g\sqrt{q}+(g+1)^2}{g+1}$. Since we assume $q > (g+1)^4$, we have $\sqrt{q} > (g+1)^2$. $b < \frac{g\sqrt{q}+\sqrt{q}}{g+1} = \frac{(g+1)\sqrt{q}}{g+1} = \sqrt{q}$. Therefore, $b \cdot p^\mu = b\sqrt{q} < \sqrt{q} \cdot \sqrt{q} = q$. [cite$_s$*tart*]*The condition holds* [*cite* : 157].

- **For (ii):** We need to show $\dim(L_b \otimes L_a) > \dim(L_{bp^\mu+a})$. Using Riemann-Roch for large degrees, this is approximately $(b-g+1)(a-g+1) > bp^\mu+a-g+1$. [cite$_s tart$]$The notes show that this is equival$ g-p$^\mu$) $> g(a+1-g)$[cite: 164]. Substituting $a \approx \sqrt{q}$ and $p^\mu = \sqrt{q}$, this is roughly $b(\sqrt{q} - \sqrt{q}) > g\sqrt{q}$, which doesn't seem right. [cite$_s tart$]$Let's follow the derivation in the notes [cite : 165, 166] : the condition becomes$b $> g + g$\sqrt{q}\frac{}{g+1}$. Our choice for b is $b = \lfloor \frac{g\sqrt{q}}{g+1} \rfloor + g + 1$, which is indeed greater than $g + \frac{g\sqrt{q}}{g+1}$. The condition holds.
- **For (iii):** We substitute our choices into the bound $1 + b + aq/p^{-\mu} = 1 + b + a\sqrt{q}$.

$$
\begin{aligned}
N_1(C, \sigma) &\leq 1 + \left(\left\lfloor \frac{g\sqrt{q}}{g+1} \right\rfloor + g + 1 \right) + (\lfloor \sqrt{q} \rfloor + 2g)\sqrt{q} \\
[cite_s tart] &\leq 1 + \frac{g\sqrt{q}}{g+1} + g + 1 + (\sqrt{q} + 2g)\sqrt{q} \quad \text{[cite: 167]} \\
&= 1 + \frac{g\sqrt{q}}{g+1} + g + 1 + q + 2g\sqrt{q} \\
&= q + g + 2 + \sqrt{q}\left(2g + \frac{g}{g+1} \right) \\
&= q + g + 2 + \sqrt{q}\left(\frac{2g(g+1) + g}{g+1} \right)
\end{aligned}
$$

This simplifies to $q + 1 + \sqrt{q}(2g + \frac{g}{g+1}) + g + 1$. [cite$_s tart$]$For large $q, this is bounded by $q+1 + (2g+1)\sqrt{q}$[cite: 167].

[cite$_s tart$]$This completes the proof of the theorem$[cite : 168]. $\square$

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Thus, we've shown for any Smooth proj. curve C over $\mathbb{F}_q$, genus g, the roots of $L(t)$ satisfy $|\alpha| = \sqrt{q}$.

$$-2g\sqrt{q} \leq N_1(C) - (q+1) \leq 2g\sqrt{q}$$

For large q, $N_1(C) \approx q + 1 = N_1(\mathbb{P}^1)$.

RM has tons of implications. For ex. in CS we use

Corollary (Weil estimate for χ-sums): Let $\chi = \chi_2 : \mathbb{F}_q \to \{-1, 0, +1\}$ be the character.

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$a^{(q-1)/2} \equiv 1$ iff a is square in $\mathbb{F}_q^*$.

Let $f(x)$ be deg-d polynomial. Then,

$$\left| \sum_{x \in \mathbb{F}_q} \chi(f(x)) \right| \leq (d-1)\sqrt{q}$$

Pf: Consider the curve C for $K := \mathbb{F}_q(x)[y]/\langle y^2 - f(x)\rangle$.

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}_q} \chi(f(\alpha)) &= \sum_{\alpha;\chi(f(\alpha))=1} 1 - \sum_{\alpha;\chi(f(\alpha))=-1} 1 \\ &= (\#\text{ of points with 2 preimages}) - (\#\text{ of points with 0 preimages}) \\ &\approx N_1(C) - (q+1) \quad \text{(ignoring points at infinity, roots of } f) \\ &= N_1(C) - q + O_d(1).\end{aligned}$$

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Exercise: Do this for other exponential sums.

Q's: (i) Given $C/\mathbb{F}_q$, how do we compute $N_r(C)$ in poly$(\log q, \dots)$-time?
(ii) Is there another interpretation of $L(t)$ that can help in computing?

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Cohomological interpretation of $L(t)$

See Frob (q-th) as an isogeny on Jacobian.

Defn: Isogeny $\alpha : J_C(k) \to J_C(k)$ is a surjective morphism with finite $\ker(\alpha)$. (respecting the group & the variety)

$\deg(\alpha) := |\ker(\alpha)|$.

- $\pi : J \to J$ is an isogeny with $\deg(\pi) = q^g$.
- For $n \in \mathbb{Z}$, $[n] : J \to J; D \mapsto nD$ is an isogeny.

What's $\deg([n])$ =?

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Defn: For prime $l \in \mathbb{N}$ we call $\ker([l])$ the l-torsion of J, denoted $J[l]$.

$T_lJ := \varprojlim_i J[l^i]$ is the l-adic Tate module of J. (T_lJ is built from $\bigcup_{i\geq 1} J[l^i]$ which is the l-adic torsion of J.)

Any isogeny $\alpha : J \to J$ gives an isogeny $T_l\alpha : T_lJ \to T_lJ; D \mapsto \alpha(D)$.

Now, we can study $T_l\alpha$ for α coming from π & $[n]$.

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Theorem (Weil):

(a) Linear map $T_l(\pi)$ on T_lJ has Charpoly $t^{2g}L(1/t)$. (Assume $l \neq p = \text{char}(\mathbb{F}_q)$.)

(b) $T_lJ \cong (\mathbb{Z}_l)^{2g}$, i.e. T_lJ is a finite rank $\mathbb{Z}_l$-module.

(c) For any $n \in \mathbb{N}$, $\deg([n]) = n^{2g}$ & $J[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Pf: Recall, $\deg(\pi - 1) = |J(\mathbb{F}_q)| = |\mathrm{Cl}_0(C)| = h(C) = L(1) = \prod(\alpha_i - 1)$.

Similarly, $\forall m \in \mathbb{N}$, $\deg(\pi^m - 1) = |J(\mathbb{F}_{q^m})| = \prod(\alpha_i^m - 1)$ [Base-change of $L(t)$].

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Going to $\mathbb{Z}_l$, we can view π in $M_{2g}(\mathbb{Z}_l)$.

Let π act on $T_l J$ with eigenvalues $\beta_i, i \in \{1, \ldots, 2g\}$.

$\Rightarrow \deg(\pi^m - 1) = |T_l J/(\pi^m - 1)T_l J| = \det(T_l(\pi^m - 1)) = \prod_i(\beta_i^m - 1)$. ($|\ker| = |\mathrm{coker}| = \det = \prod$ eigenval)

Since this holds $\forall m$, we deduce that $\{\alpha_i\} = \{\beta_i\}$ as multisets. $\Rightarrow$ $\mathrm{Charpoly}(\pi|_{T_l J}) = \det(tI - T_l\pi) = \prod(t - \beta_i) = \prod(t - \alpha_i) = t^{2g}L(1/t)$ over $\mathbb{Z}_l$.

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Also, $\mathrm{Charpoly}(\pi^{-1}|_{T_l J}) = q^{-g}L(t)$, over $\mathbb{Q}_l$.

(b) & (c) are implied by $\deg(L) = 2g$ & l-torsion resp. n-torsion of J.

$\rightarrow$ These properties are computationally useful, as one can work with the Frobenius action on torsion points.

Current time-complexity for $L(t)$ is the min of:

- $\mathrm{poly}(p, g, \log q)$ [Kedlaya, 2001] (by computing $L(t)$ p-adically.)
- $\mathrm{poly}(\log q, g^{O(g)})$ [Pila, 1990]

5.1 Weil Bound

Eigenvalues satisfy $|\alpha_i| = \sqrt{q}$

Estimate for N_1

5.2 Character Sums

Quadratic character example

Bound on exponential sums

5.3 Cohomological Interpretation of $L(t)$

Jacobian varieties

Isogenies

Torsion points

5.4 Weil's Theorem

Characteristic polynomial of Frobenius

Structure of $T_\ell J$

Degree formulas

5.5 Computational Implications

Computing $L(t) \bmod \ell$

Complexity: Kedlaya, Pila