Numerical Methods for Nonlinear Equation and Programming RobustPCA

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Introduction

PCA (Principle Component Analysis)

PCA

$$L = \arg\min_{L} \|M - L\|$$

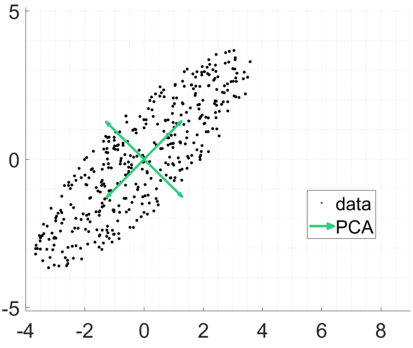
- L : low rank matrix. (i.e rank$(L) \leq k$)

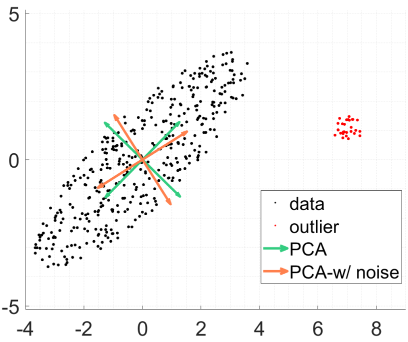
Solution by truncated singular value decomposition.

$$M = \mathrm{U}\Sigma V^* = \sum_i \sigma_i u_i v_i^* \Rightarrow L = \sum_{i \leq k} \sigma_i u_i v_i^*$$

PCA

It's too senitive to outlier points.





Method

Robust PCA[1]

Robust PCA

$$\min_{L,S} \operatorname{rank}(L) + \lambda \|S\|_0 \quad \longrightarrow \quad \min_{L,S} \|L\|_* + \lambda \|S\|_1$$
$$s.t. \quad L + S = M \qquad\qquad s.t. \quad L + S = M$$

- $\|L\|_* = \sum_i \sigma_i(L)$
- $\|S\|_1 = \sum_{i,j} |S_{i,j}|$
- $\lambda > 0$: regularization parameter that balances two terms.

The Augmented Lagrangian

$$\mathcal{L}(L,S,Y) = \underbrace{\|L\|_* + \lambda\|S\|_1 + \langle Y, M-L-S\rangle_F}_{\text{Lagrangian}} + \frac{\mu}{2}\|M-L-S\|_F^2$$

- Y : Lagrange multiplier

Sub derivative

Definition

The definition of $f : \mathbb{R}^n \to \mathbb{R}$ sub derivative

$$\partial f(x_0) = \{g \in \mathbb{R}^n | f(x) \geq f(x_0) + g^\top (x - x_0), \quad \forall x \in \mathbb{R}^n\}$$

eg.

Let $f(x) = | x |$, where $f : \mathbb{R}^1 \to \mathbb{R}^1$

$\partial f(0) = \{g \in \mathbb{R}^1 \big| |x| \geq g \cdot x, \forall x \in \mathbb{R}^1\} = [-1, 1]$

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eg.

Let $f(x) = \mid x \mid$, where $f : \mathbb{R}^1 \to \mathbb{R}^1$
$\partial f(0) = \{g \in \mathbb{R}^1 || x| \geq g \cdot x, \forall x \in \mathbb{R}^1\} = [-1, 1]$

Property

- $\partial(af_1 + f_2) = a\partial(f_1) + \partial(f_2)$
- $f(x^*) = \min_x f(x) \iff 0 \in \partial f(x^*)$

$$\mathcal{L}(L, S, Y) = \|L\|_* + \lambda \|S\|_1 + \langle Y, M - L - S \rangle_F + \frac{\mu}{2} \|M - L - S\|_F^2.$$

$$\mathcal{L}(L, S^*, Y) = \min_S \mathcal{L}(L, S, Y) \iff 0 \in \partial \mathcal{L}(L, S^*, Y).$$

$\min_{S} \mathcal{L}(L, S, Y)$

$$\partial \mathcal{L}(S^*) = \lambda \partial \|S\|_1 - Y - \mu(M - L - S) = 0$$

$$\Rightarrow \frac{\lambda}{\mu} \partial \|S\|_1 + S = M - L + \frac{1}{\mu} Y \equiv C$$

$$\Rightarrow \forall S_{i,j} \in S, \quad S_{i,j} = sign(c_{i,j}) \max(|c_{i,j}| - \tau, 0), \text{ where } C = \left[c_{i,j}\right]$$

$\min_{S} \mathcal{L}(L, S, Y)$

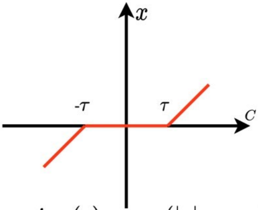
$$\partial \mathcal{L}(S^*) = \lambda \partial \|S\|_1 - Y - \mu(M - L - S) = 0$$

$$\Rightarrow \frac{\lambda}{\mu} \partial \|S\|_1 + S = M - L + \frac{1}{\mu} Y \equiv C$$

$$\Rightarrow \forall S_{i,j} \in S, \quad S_{i,j} = sign(c_{i,j}) \max(|c_{i,j}| - \tau, 0), \text{ where } C = [c_{i,j}]$$

- $\tau \partial \|x\|_1 + x = c$

x	0	$(0, \infty)$	$(-\infty, 0)$
$\partial \|x\|_1$	$[-1, 1]$	1	-1
c	$[-\tau, \tau]$	(τ, ∞)	$(-\infty, -\tau)$



$$x = sign(c) \max(|c| - \tau, 0)$$

Algorithm

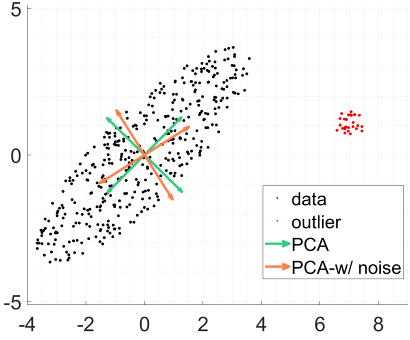
Algorithm RobustPCA

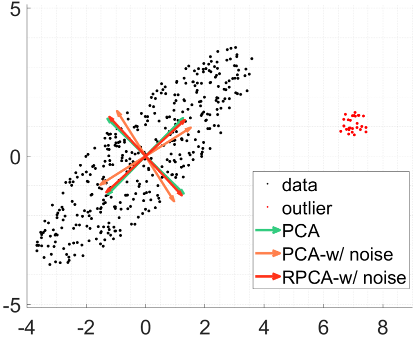
1: **Input:** M, λ, μ
2: **Initial:** $L_0 = S_0 = Y_0 = O_{m \times n}$
3: **while** not converged **do**
4: $\quad L_{k+1} = D_{1/\mu}(M - S_k + \mu^{-1} Y_k)$
5: $\quad S_{k+1} = S_{\lambda/\mu}(M - L_{k+1} + \mu^{-1} Y_k)$
6: $\quad Y_{k+1} = Y_k + \mu(M - L_{k+1} - S_{k+1})$
7: **end while**
8: **return** L, S

- $D_\tau(X) = \mathrm{U} S_\tau(\Sigma) V^*$, where $X = \mathrm{U}\Sigma V^*$.(SVD)
- $S_\tau(x) = sign(x) \max(|x| - \tau, 0)$

Comaprison (PCA v.s RPCA)

Robust PCA can adjust the vector with a given penalty term λ.





Application

Human face reconstruction

Human face reconstruction

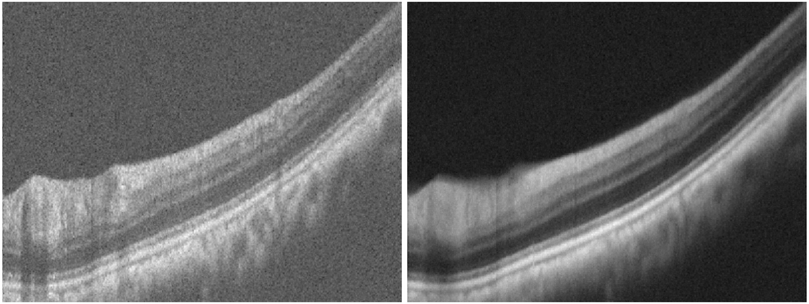
- **Result**

M L S

where $M = L + S$.

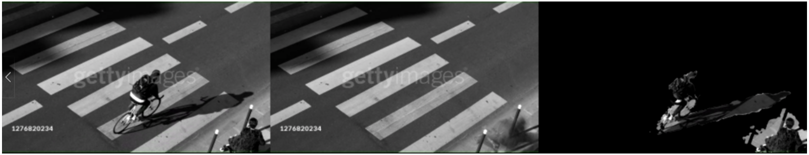
Artifact remove in the OCT

Experiment on the OCT image to remove the artifacts and maintain the anatomical features.



OCT Movie Link.

Foreground extraction



Road record Movie Link.

Reference I

[1] Emmanuel J. Candès et al. "Robust Principal Component Analysis?" In: *J. ACM* 58.3 (June 2011). ISSN: 0004-5411. DOI: 10.1145/1970392.1970395. URL: https://doi.org/10.1145/1970392.1970395.

Q&A SESSION

Thank you for your attention.