

4. GREEDY ALGORITHMS II

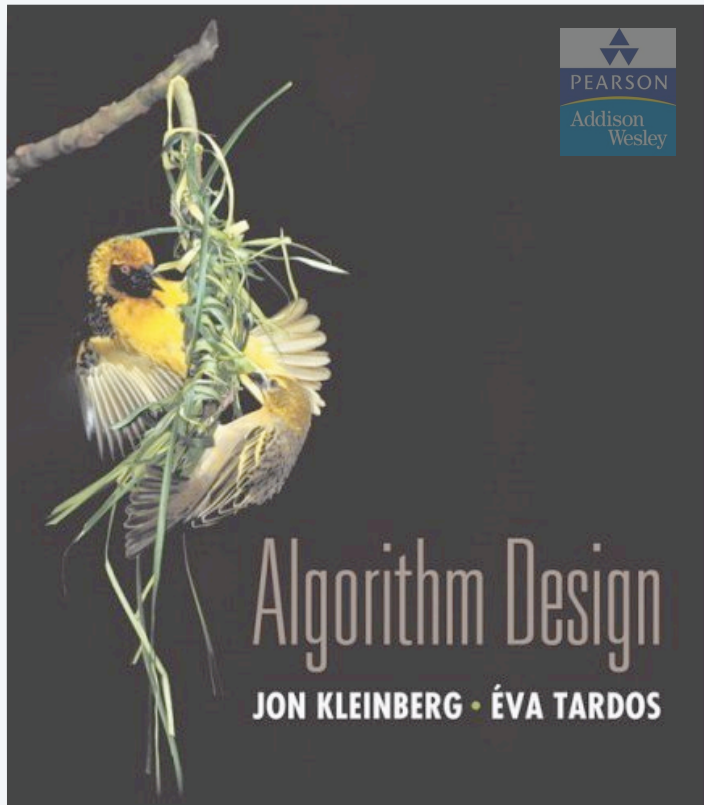
- ▶ *Dijkstra's algorithm*
- ▶ *minimum spanning trees*
- ▶ *Prim, Kruskal, Boruvka*
- ▶ *single-link clustering*
- ▶ *min-cost arborescences*

Lecture slides by Kevin Wayne

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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>



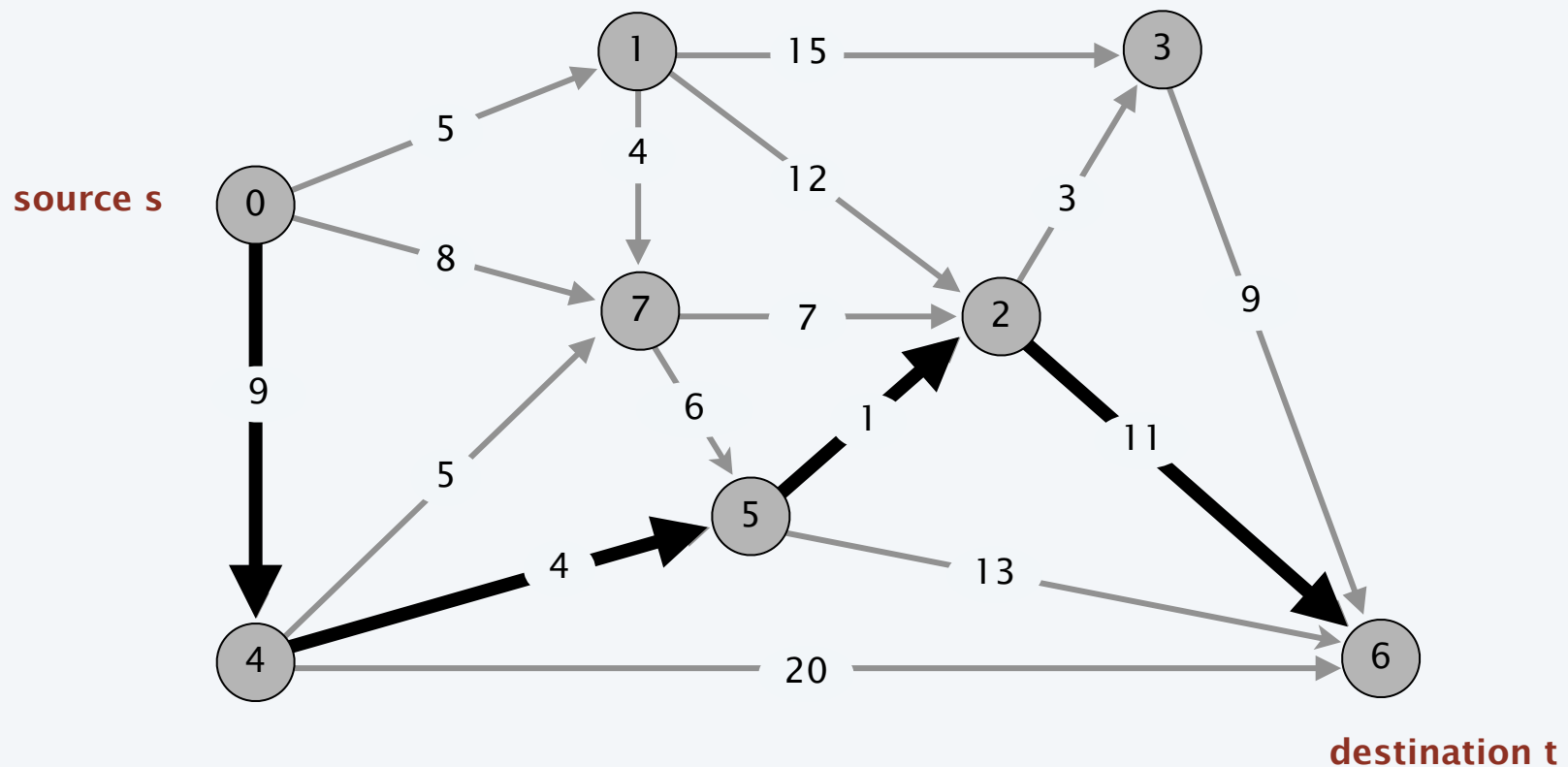
SECTION 4.4

4. GREEDY ALGORITHMS II

- ▶ *Dijkstra's algorithm* 
- ▶ *minimum spanning trees*
- ▶ *Prim, Kruskal, Boruvka*
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Shortest-paths problem

Problem. Given a digraph $G = (V, E)$, edge lengths $\ell_e \geq 0$, source $s \in V$, and destination $t \in V$, find the shortest directed path from s to t .



$$\text{length of path} = 9 + 4 + 1 + 11 = 25$$

Car navigation



Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in LaTeX.
- Urban traffic planning.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

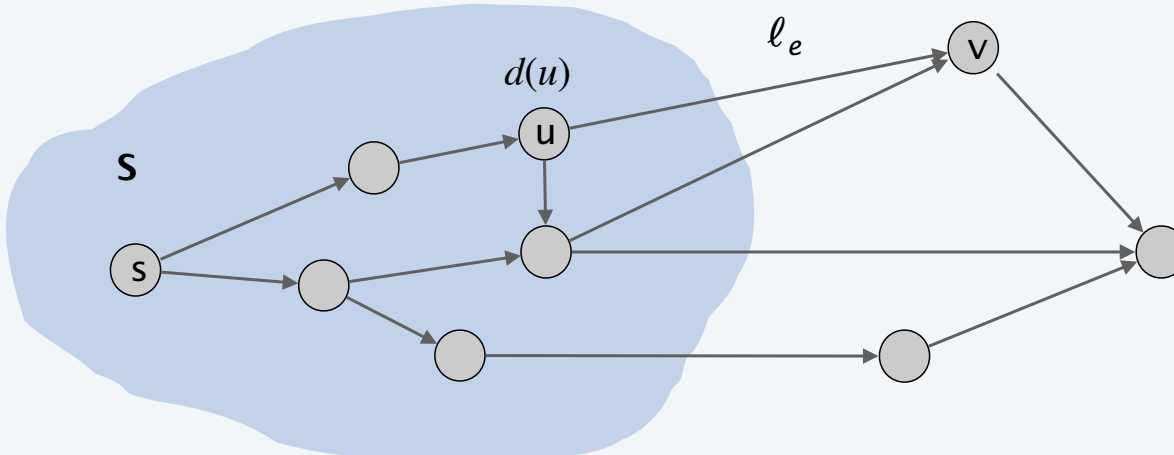
Dijkstra's algorithm

Greedy approach. Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u .

- Initialize $S = \{s\}$, $d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e,$$

shortest path to some node u in explored part,
followed by a single edge (u, v)



Dijkstra's algorithm


Greedy approach. Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u .

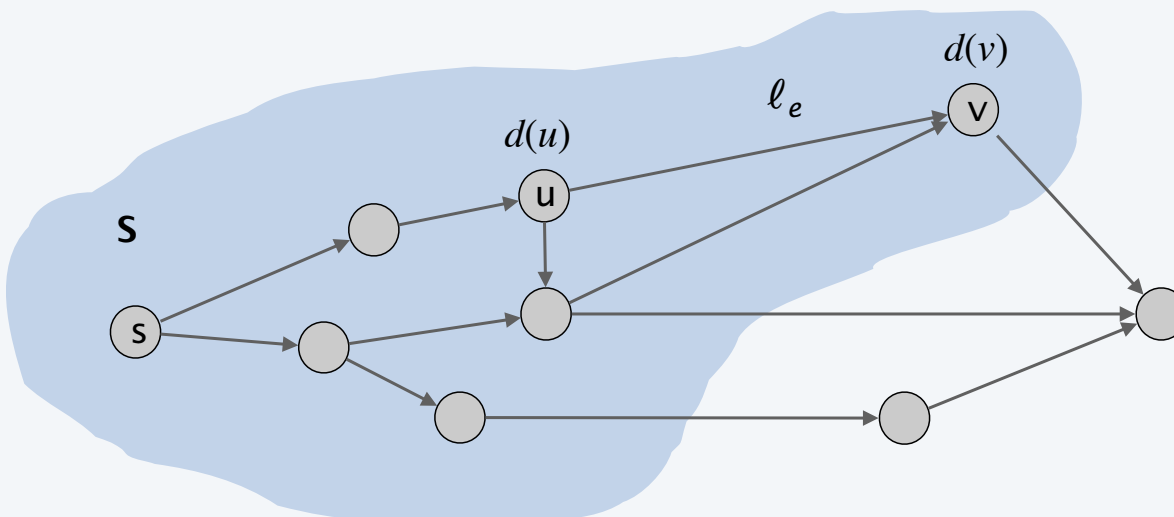


- Initialize $S = \{s\}$, $d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e,$$

add v to S , and set $d(v) = \pi(v)$. 

 shortest path to some node u in explored part, followed by a single edge (u, v)



Dijkstra's algorithm: proof of correctness

Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest $s \rightarrow u$ path.

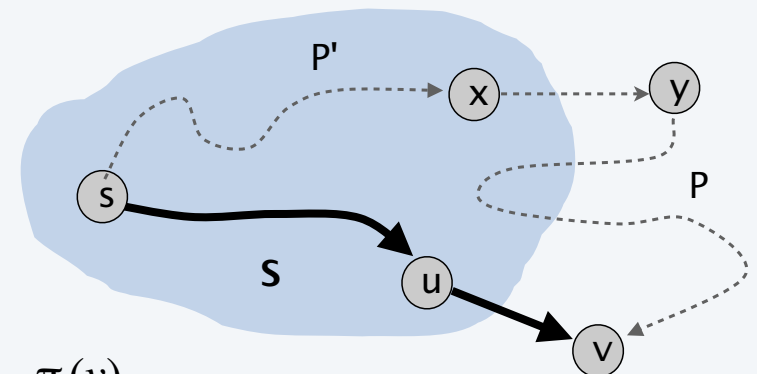
Pf. [by induction on $|S|$]

Base case: $|S| = 1$ is easy since $S = \{ s \}$ and $d(s) = 0$.



Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S , and let (u, v) be the final edge.
- The shortest $s \rightarrow u$ path plus (u, v) is an $s \rightarrow v$ path of length $\pi(v)$.
- Consider any $s \rightarrow v$ path P . We show that it is no shorter than $\pi(v)$.
- Let (x, y) be the first edge in P that leaves S , and let P' be the subpath to x .
- P is already too long as soon as it reaches y .



$$\ell(P) \geq \ell(P') + \ell(x, y) \geq d(x) + \ell(x, y) \geq \pi(y) \geq \pi(v) \quad \blacksquare$$

↑
nonnegative
lengths

↑
inductive
hypothesis

↑
definition
of $\pi(y)$

↑
Dijkstra chose v
instead of y

Dijkstra's algorithm: efficient implementation

Critical optimization 1. For each unexplored node v , explicitly maintain $\pi(v)$ instead of computing directly from formula:



$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e .$$

- For each $v \notin S$, $\pi(v)$ can only decrease (because S only increases).
- More specifically, suppose u is added to S and there is an edge (u, v) leaving u . Then, it suffices to update:

$$\pi(v) = \min \{ \pi(v), d(u) + \ell(u, v) \}$$

Critical optimization 2. Use a **priority queue** to choose the unexplored node that minimizes $\pi(v)$.

Dijkstra's algorithm: efficient implementation

Implementation.

- Algorithm stores $d(v)$ for each explored node v .
- Priority queue stores $\pi(v)$ for each unexplored node v .
- Recall: $d(u) = \pi(u)$ when u is deleted from priority queue.

DIJKSTRA(V, E, s)

Create an empty priority queue.

FOR EACH $v \neq s$: $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.

FOR EACH $v \in V$: *insert* v with key $d(v)$ into priority queue.

WHILE (the priority queue *is not empty*)

$u \leftarrow$ *delete-min* from priority queue.

 FOR EACH edge $(u, v) \in E$ leaving u :

 IF $d(v) > d(u) + \ell(u, v)$

decrease-key of v to $d(u) + \ell(u, v)$ in priority queue.

$d(v) \leftarrow d(u) + \ell(u, v)$.



Dijkstra's algorithm: which priority queue?

Performance. Depends on PQ: n insert, n delete-min, m decrease-key.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci/Brodal best in theory, but not worth implementing.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
d-way heap (Johnson 1975)	$O(d \log_d n)$	$O(d \log_d n)$	$O(\log_d n)$	$O(m \log_{m/n} n)$
Fibonacci heap (Fredman-Tarjan 1984)	$O(1)$	$O(\log n)^\dagger$	$O(1)^\dagger$	$O(m + n \log n)$
Brodal queue (Brodal 1996)	$O(1)$	$O(\log n)$	$O(1)$	$O(m + n \log n)$

† amortized

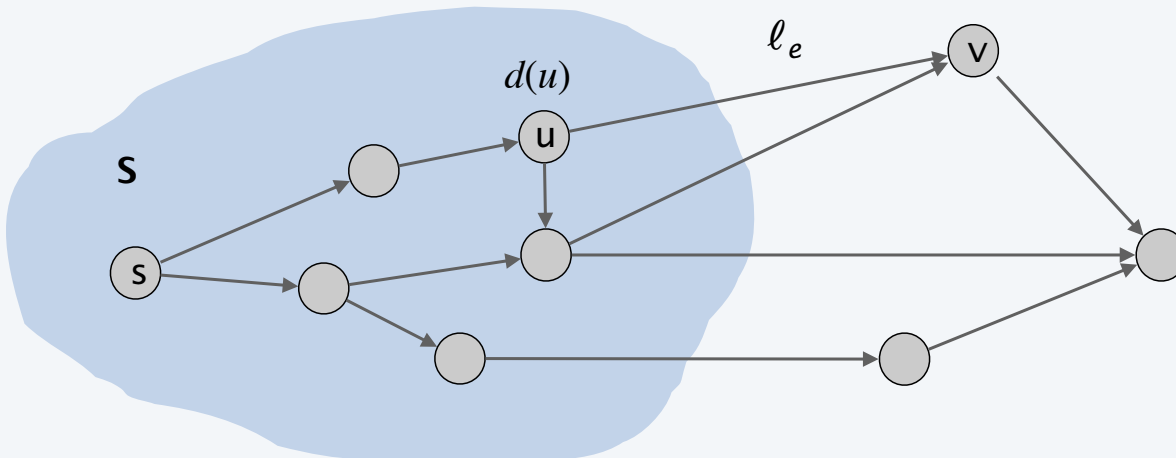
Extensions of Dijkstra's algorithm

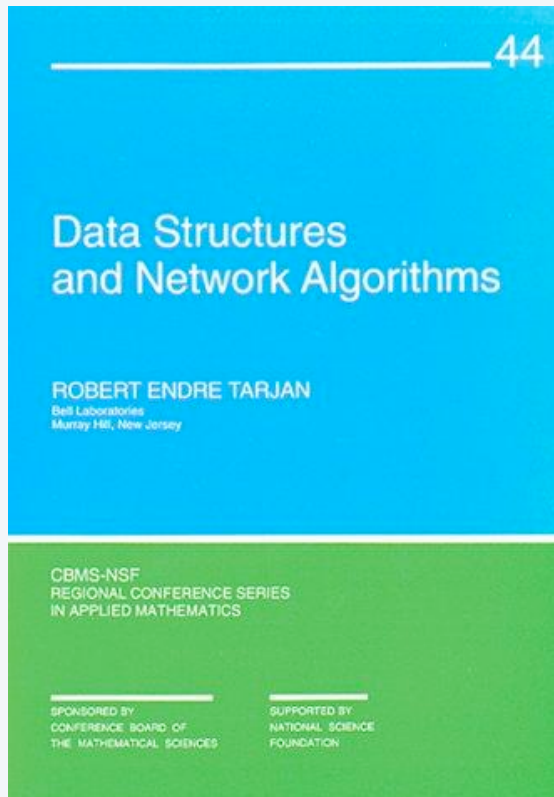
Dijkstra's algorithm and proof extend to several related problems:



- Shortest paths in undirected graphs: $d(v) \leq d(u) + \ell(u, v)$.
- Maximum capacity paths: $d(v) \geq \min \{ \pi(u), c(u, v) \}$.
- Maximum reliability paths: $d(v) \geq d(u) \times \gamma(u, v)$.
- ...


Key algebraic structure. Closed semiring (tropical, bottleneck, Viterbi).





SECTION 6.1

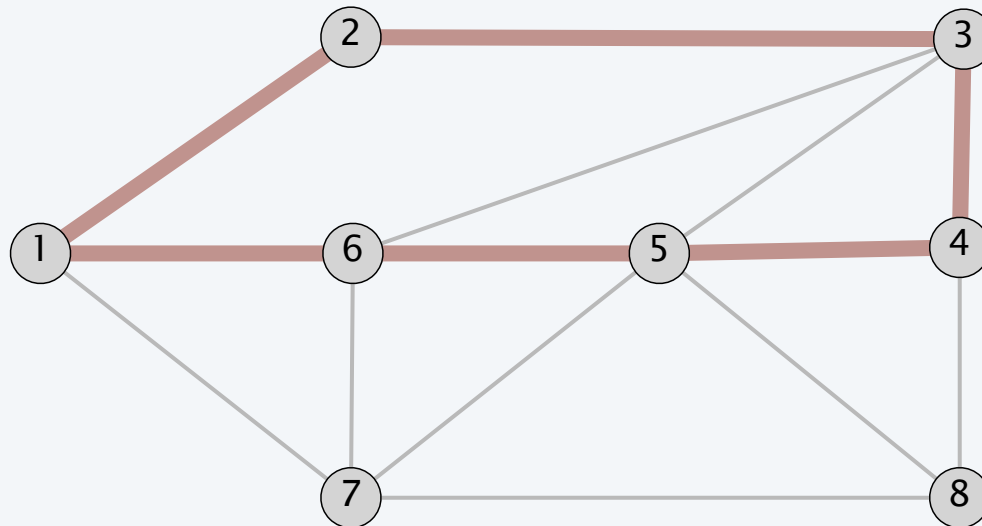
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Cycles and cuts

Def. A **path** is a sequence of edges which connects a sequence of nodes.

Def. A **cycle** is a path with no repeated nodes or edges other than the starting and ending nodes.

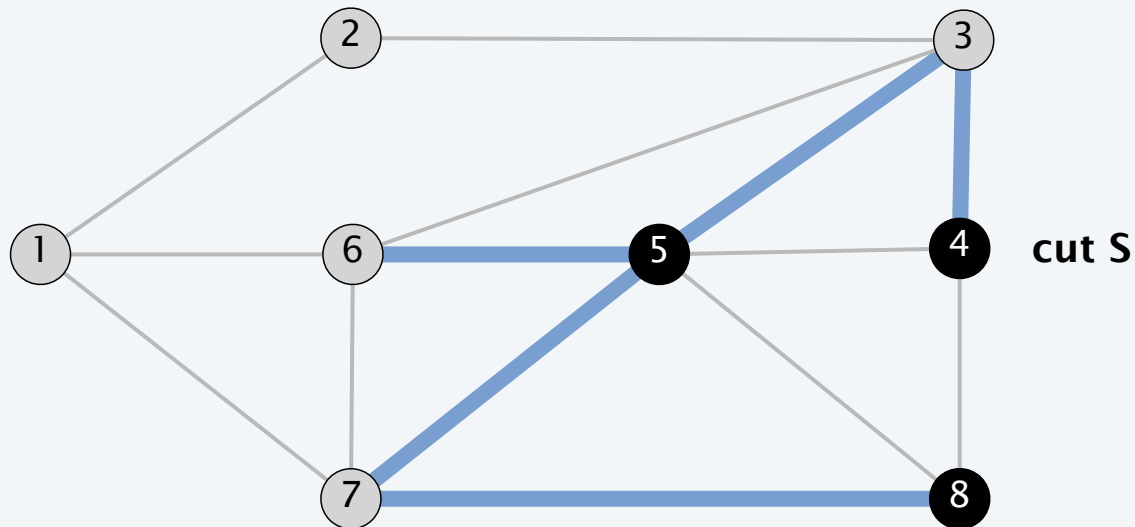


cycle $C = \{ (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) \}$

Cycles and cuts

Def. A **cut** is a partition of the nodes into two nonempty subsets S and $V - S$.

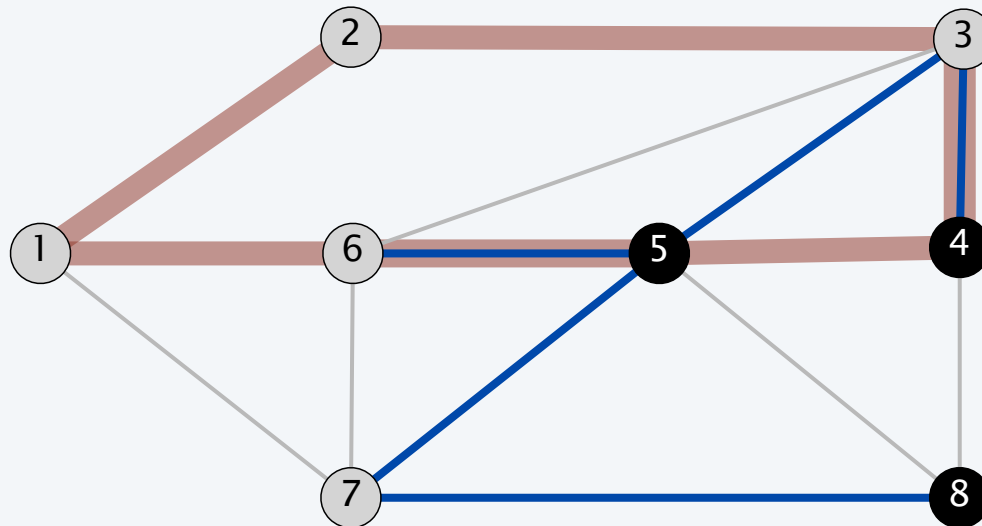
Def. The **cutset** of a cut S is the set of edges with exactly one endpoint in S .



cutset $D = \{ (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) \}$

Cycle-cut intersection

Proposition. A cycle and a cutset intersect in an **even** number of edges. 



cutset $D = \{ (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) \}$

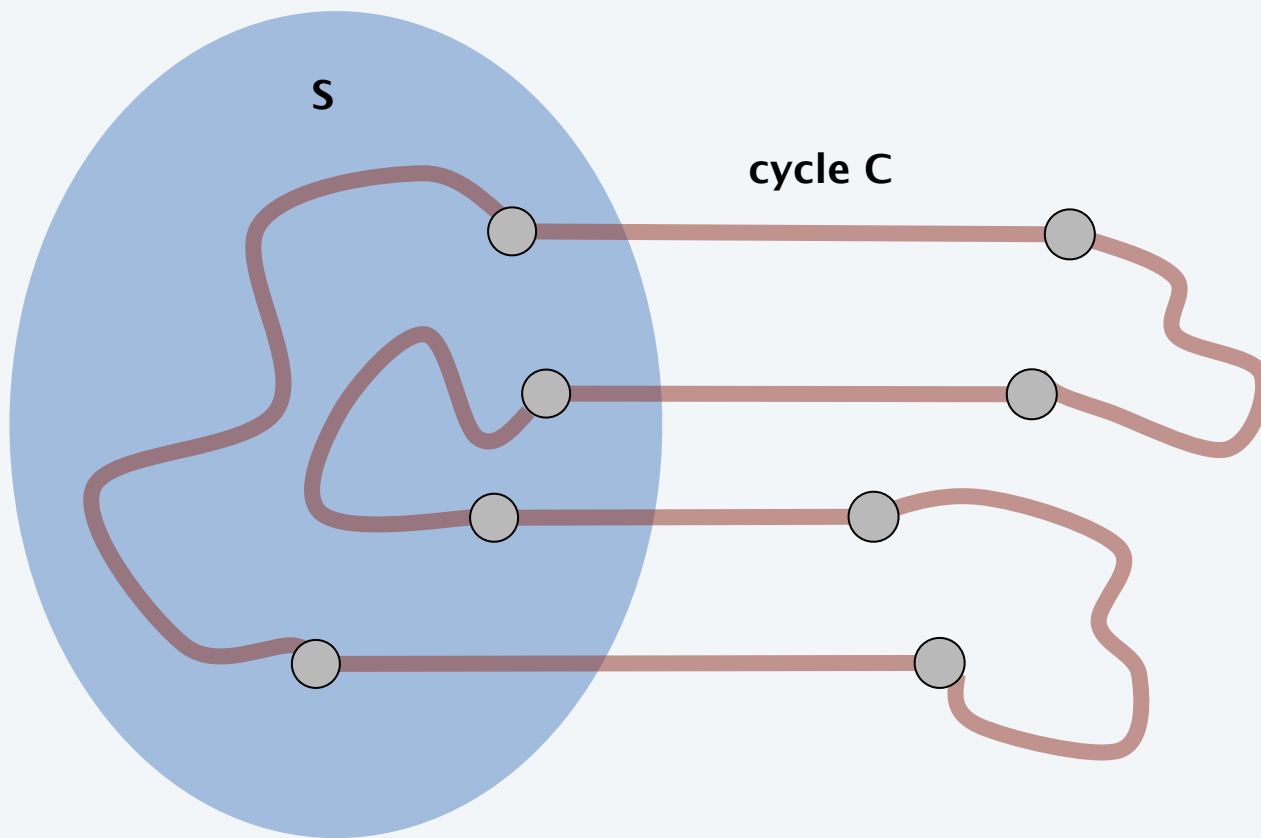
cycle $C = \{ (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) \}$

intersection $C \cap D = \{ (3, 4), (5, 6) \}$

Cycle-cut intersection

Proposition. A cycle and a cutset intersect in an **even** number of edges.

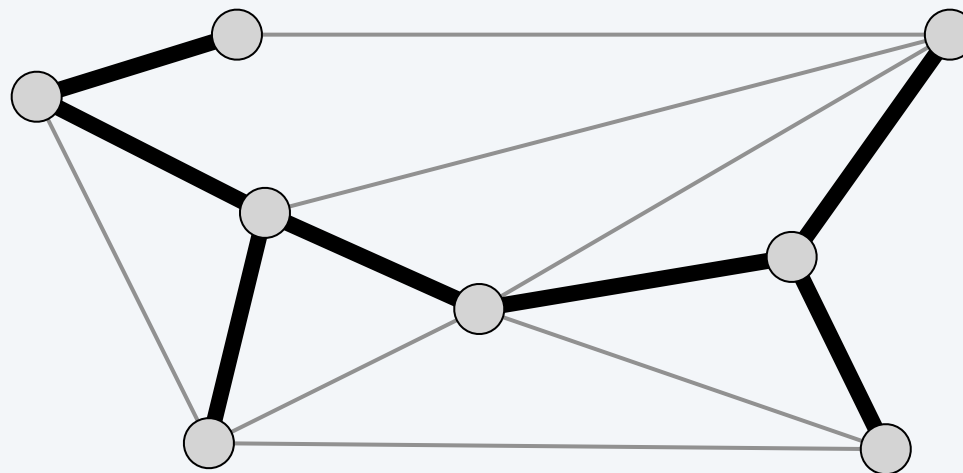
Pf. [by picture]



Spanning tree properties

Proposition. Let $T = (V, F)$ be a subgraph of $G = (V, E)$. TFAE: 

- T is a spanning tree of G .
- T is acyclic and connected.
- T is connected and has $n - 1$ edges.
- T is acyclic and has $n - 1$ edges.
- T is minimally connected: removal of any edge disconnects it.
- T is maximally acyclic: addition of any edge creates a cycle.
- T has a unique simple path between every pair of nodes.

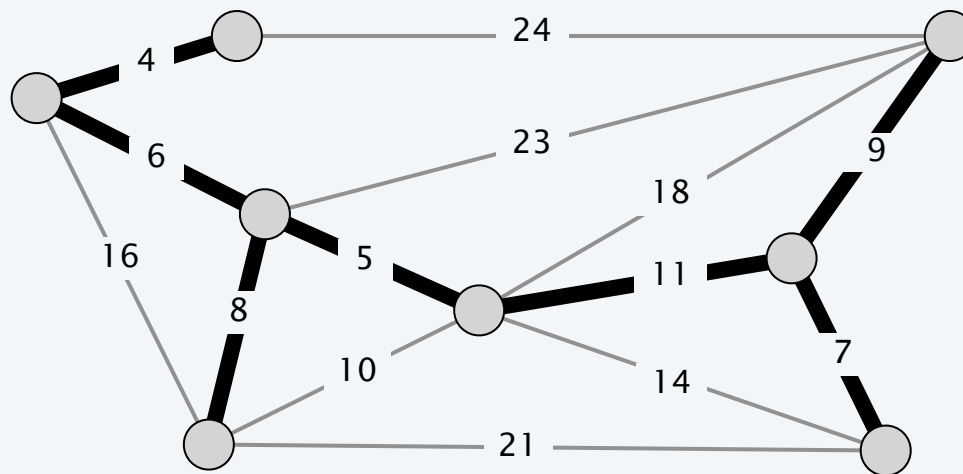


$T = (V, F)$

Minimum spanning tree



Given a connected graph $G = (V, E)$ with edge costs c_e , an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge costs is minimized.



$$\text{MST cost} = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7$$

Cayley's theorem. There are n^{n-2} spanning trees of K_n . ← can't solve by brute force



Applications

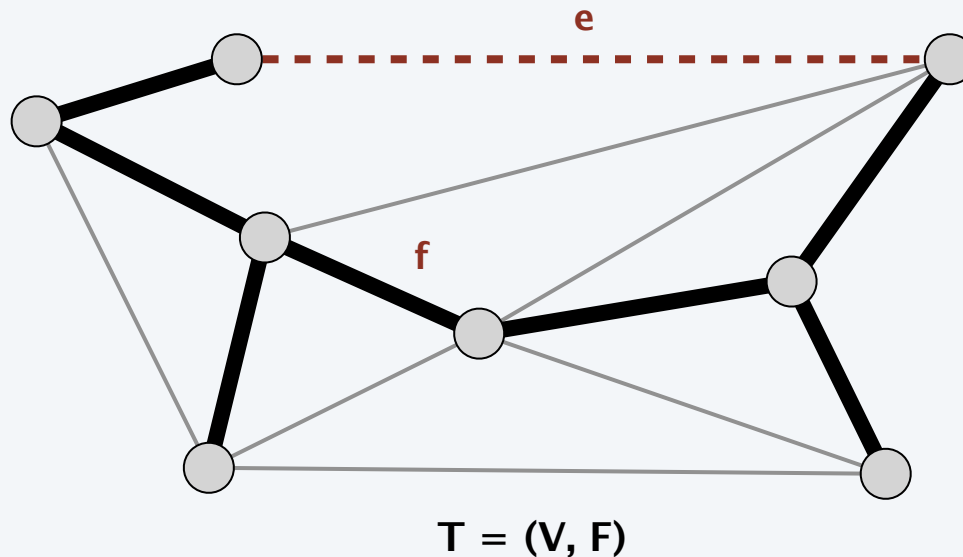
MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, computer, road).

Fundamental cycle

Fundamental cycle.

- Adding any non-tree edge e to a spanning tree T forms unique cycle C .
- Deleting any edge $f \in C$ from $T \cup \{e\}$ results in new spanning tree.



Observation. If $c_e < c_f$, then T is not an MST.

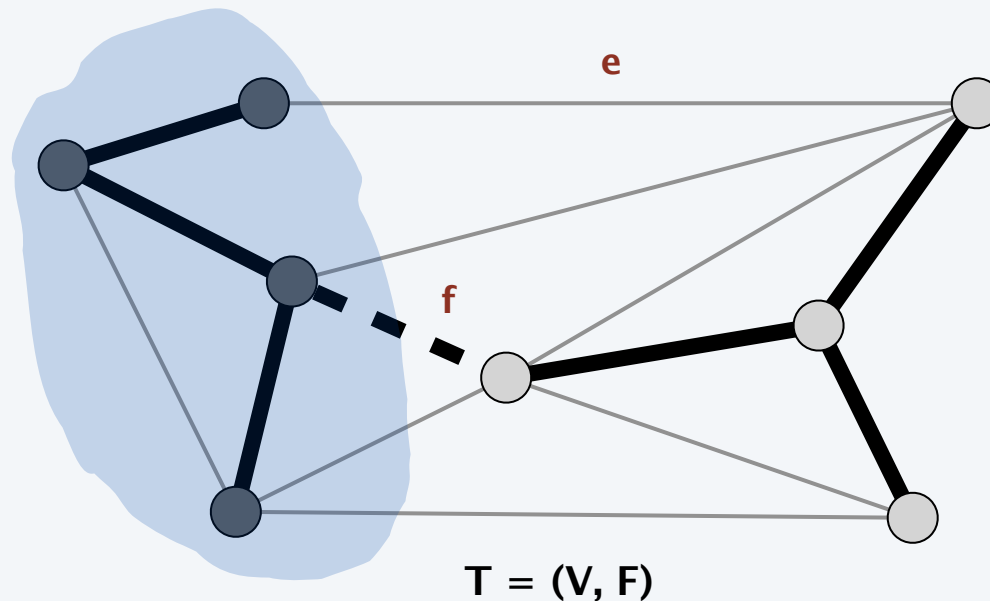


Fundamental cutset



Fundamental cutset.

- Deleting any tree edge f from a spanning tree T divide nodes into two connected components. Let D be cutset.
- Adding any edge $e \in D$ to $T - \{f\}$ results in new spanning tree.



Observation. If $c_e < c_f$, then T is not an MST.

The greedy algorithm



Red rule.

- Let C be a cycle with no red edges.
- Select an uncolored edge of C of max weight and color it red.



Blue rule.

- Let D be a cutset with no blue edges.
- Select an uncolored edge in D of min weight and color it blue.

Greedy algorithm.

- Apply the red and blue rules (non-deterministically!) until all edges are colored. The blue edges form an MST.
- Note: can stop once $n - 1$ edges colored blue.



Greedy algorithm: proof of correctness

Color invariant. There exists an MST T^* containing all of the blue edges and none of the red edges.

Pf. [by induction on number of iterations]



Base case. No edges colored \Rightarrow every MST satisfies invariant.

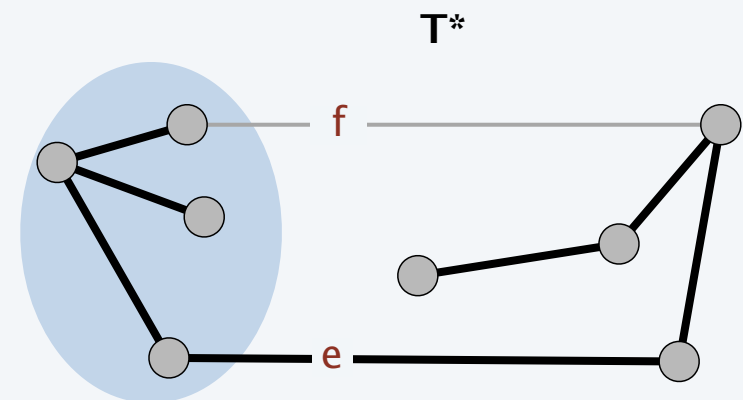
Greedy algorithm: proof of correctness

Color invariant. There exists an MST T^* containing all of the blue edges and none of the red edges.

Pf. [by induction on number of iterations]

Induction step (blue rule). Suppose color invariant true before **blue** rule. 

- let D be chosen cutset, and let f be edge colored blue.
- if $f \in T^*$, T^* still satisfies invariant.
- Otherwise, consider fundamental cycle C by adding f to T^* .
- let $e \in C$ be another edge in D .
- e is uncolored and $c_e \geq c_f$ since
 - $e \in T^* \Rightarrow e$ not red
 - blue rule $\Rightarrow e$ not blue and $c_e \geq c_f$
- Thus, $T^* \cup \{f\} - \{e\}$ satisfies invariant.



Greedy algorithm: proof of correctness

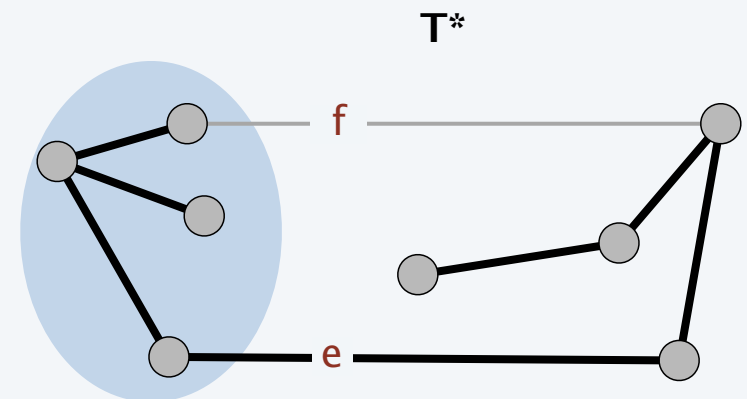
Color invariant. There exists an MST T^* containing all of the blue edges and none of the red edges.

Pf. [by induction on number of iterations]

Induction step (red rule). Suppose color invariant true before **red** rule.



- let C be chosen cycle, and let e be edge colored red.
- if $e \notin T^*$, T^* still satisfies invariant.
- Otherwise, consider fundamental cutset D by deleting e from T^* .
- let $f \in D$ be another edge in C .
- f is uncolored and $c_e \geq c_f$ since
 - $f \notin T^* \Rightarrow f$ not blue
 - red rule $\Rightarrow f$ not red and $c_e \geq c_f$
- Thus, $T^* \cup \{f\} - \{e\}$ satisfies invariant. ■

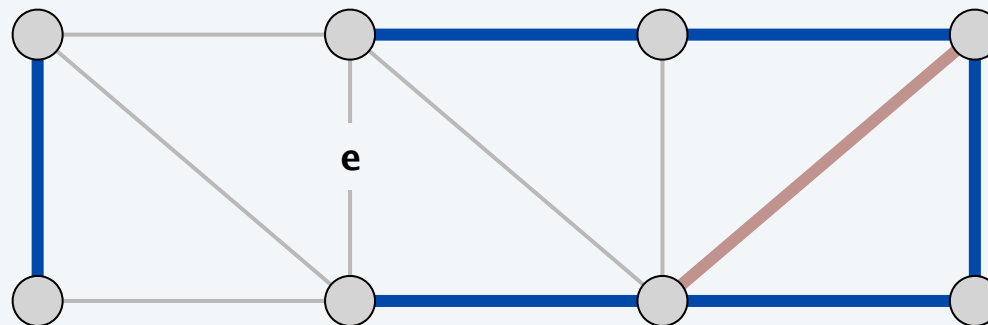


Greedy algorithm: proof of correctness

Theorem. The greedy algorithm terminates. Blue edges form an MST.

Pf. We need to show that either the red or blue rule (or both) applies.

- Suppose edge e is left uncolored.
- Blue edges form a forest.
- Case 1: both endpoints of e are in same blue tree.
⇒ apply red rule to cycle formed by adding e to blue forest.



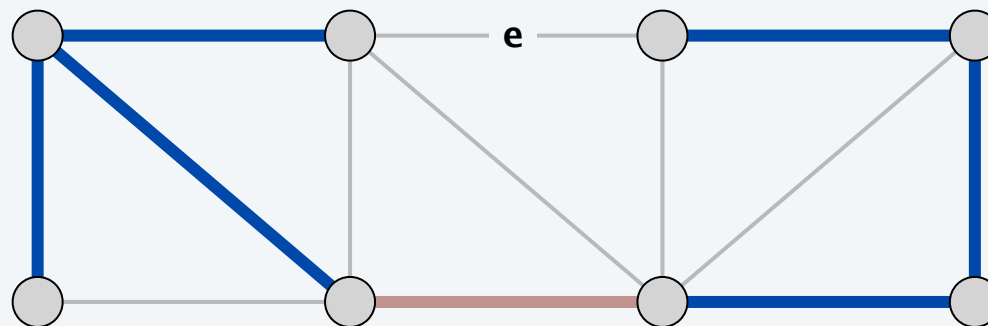
Case 1

Greedy algorithm: proof of correctness

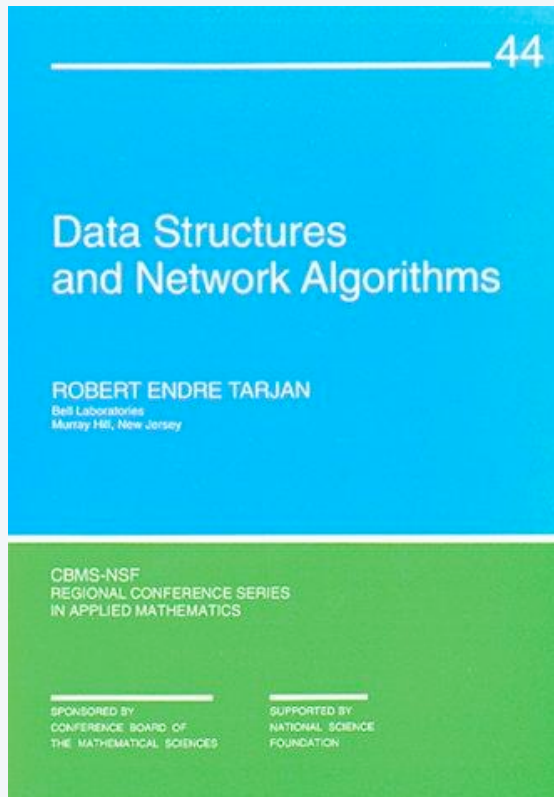
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- Suppose edge e is left uncolored.
- Blue edges form a forest.
- Case 1: both endpoints of e are in same blue tree.
⇒ apply red rule to cycle formed by adding e to blue forest.
- Case 2: both endpoints of e are in different blue trees.
⇒ apply blue rule to cutset induced by either of two blue trees. ■



Case 2



SECTION 6.2

4. GREEDY ALGORITHMS II

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Prim's algorithm



Initialize $S =$ any node.

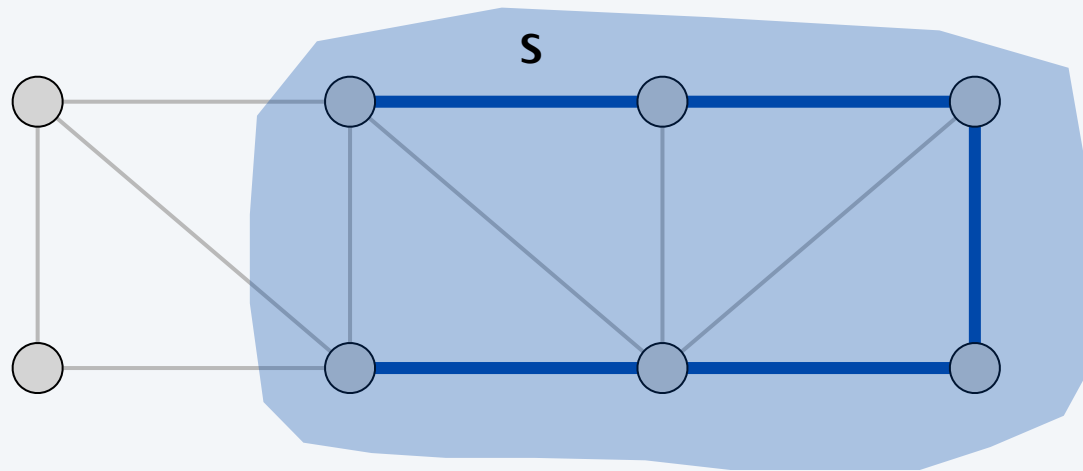
Repeat $n - 1$ times:

- Add to tree the min weight edge with one endpoint in S .
- Add new node to S .



Theorem. Prim's algorithm computes the MST.

Pf. Special case of greedy algorithm (blue rule repeatedly applied to S). ■




Prim's algorithm: implementation

Theorem. Prim's algorithm can be implemented in $O(m \log n)$ time.

Pf. Implementation almost identical to Dijkstra's algorithm.

[$d(v)$ = weight of cheapest known edge between v and S]

PRIM (V, E, c) 

Create an empty priority queue.

$s \leftarrow$ any node in V .

FOR EACH $v \neq s$: $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.

FOR EACH v : *insert* v with key $d(v)$ into priority queue.

WHILE (the priority queue *is not empty*)

$u \leftarrow$ *delete-min* from priority queue.

 FOR EACH edge $(u, v) \in E$ incident to u :

 IF $d(v) > c(u, v)$

decrease-key of v to $c(u, v)$ in priority queue.

$d(v) \leftarrow c(u, v)$.

Kruskal's algorithm

Consider edges in ascending order of weight:

- Add to tree unless it would create a cycle.



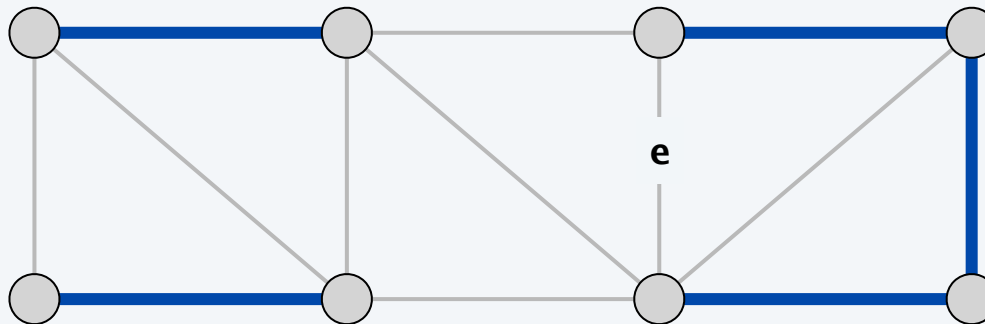
Theorem. Kruskal's algorithm computes the MST.

Pf. Special case of greedy algorithm.

- Case 1: both endpoints of e in same blue tree.
⇒ color red by applying red rule to unique cycle.
- Case 2. If both endpoints of e are in different blue trees.
⇒ color blue by applying blue rule to cutset defined by either tree. ■

all other edges in cycle are blue

no edge in cutset has smaller weight
(since Kruskal chose it first)



Kruskal's algorithm: implementation

Theorem. Kruskal's algorithm can be implemented in $O(m \log m)$ time.

- Sort edges by weight.
- Use **union-find** data structure to dynamically maintain connected components.

KRUSKAL (V, E, c)

SORT m edges by weight so that $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$

$S \leftarrow \phi$

FOREACH $v \in V$: **MAKESET**(v).

FOR $i = 1$ **TO** m

$(u, v) \leftarrow e_i$

IF **FINDSET**(u) \neq **FINDSET**(v) \leftarrow are u and v in same component?

$S \leftarrow S \cup \{e_i\}$

UNION(u, v). \leftarrow make u and v in same component

RETURN S

Reverse-delete algorithm

Consider edges in descending order of weight:



- Remove edge unless it would disconnect the graph.

Theorem. The reverse-delete algorithm computes the MST.

Pf. Special case of greedy algorithm.

- Case 1: removing edge e does not disconnect graph.
⇒ apply red rule to cycle C formed by adding e to existing path between its two endpoints
- Case 2: removing edge e disconnects graph.
⇒ apply blue rule to cutset D induced by either component. ■

any edge in C with larger weight would have been deleted when considered

e is the only edge in the cutset
(any other edges must have been colored red / deleted)

Fact. [Thorup 2000] Can be implemented in $O(m \log n (\log \log n)^3)$ time.



Review: the greedy MST algorithm

Red rule.

- Let C be a cycle with no red edges.
- Select an uncolored edge of C of max weight and color it red.

Blue rule.

- Let D be a cutset with no blue edges.
- Select an uncolored edge in D of min weight and color it blue.

Greedy algorithm.

- Apply the red and blue rules (non-deterministically!) until all edges are colored. The blue edges form an MST.
- Note: can stop once $n - 1$ edges colored blue.

Theorem. The greedy algorithm is correct.


Special cases. Prim, Kruskal, reverse-delete, ...

Borůvka's algorithm

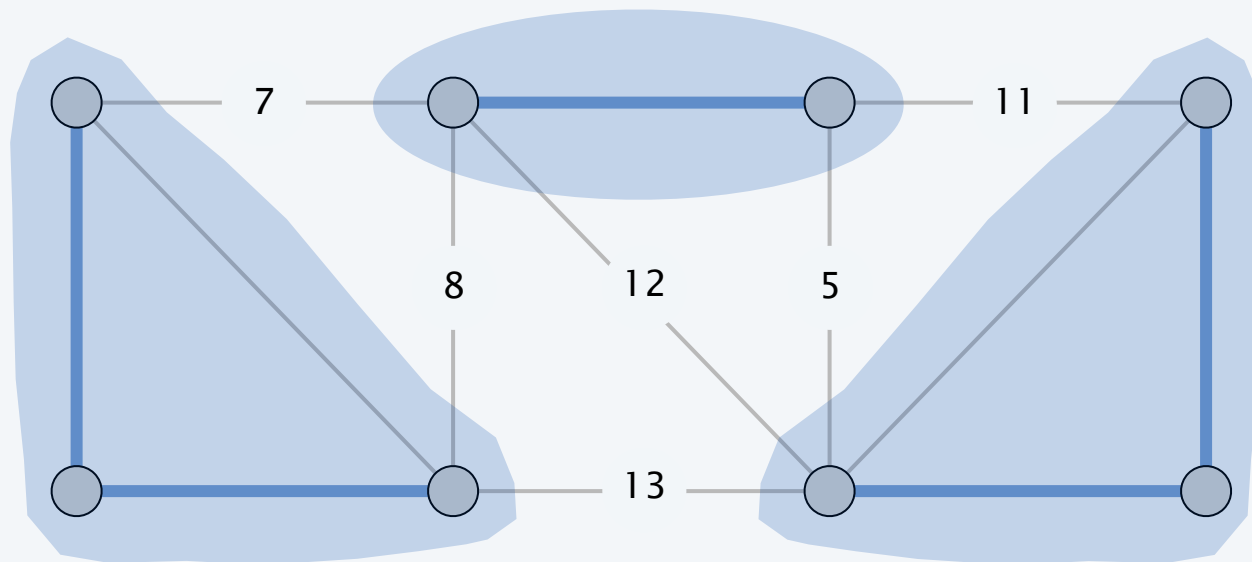
Repeat until only one tree.

- Apply blue rule to cutset corresponding to **each** blue tree.
- Color all selected edges blue.



Theorem. Borůvka's algorithm computes the MST.  assume edge costs are distinct

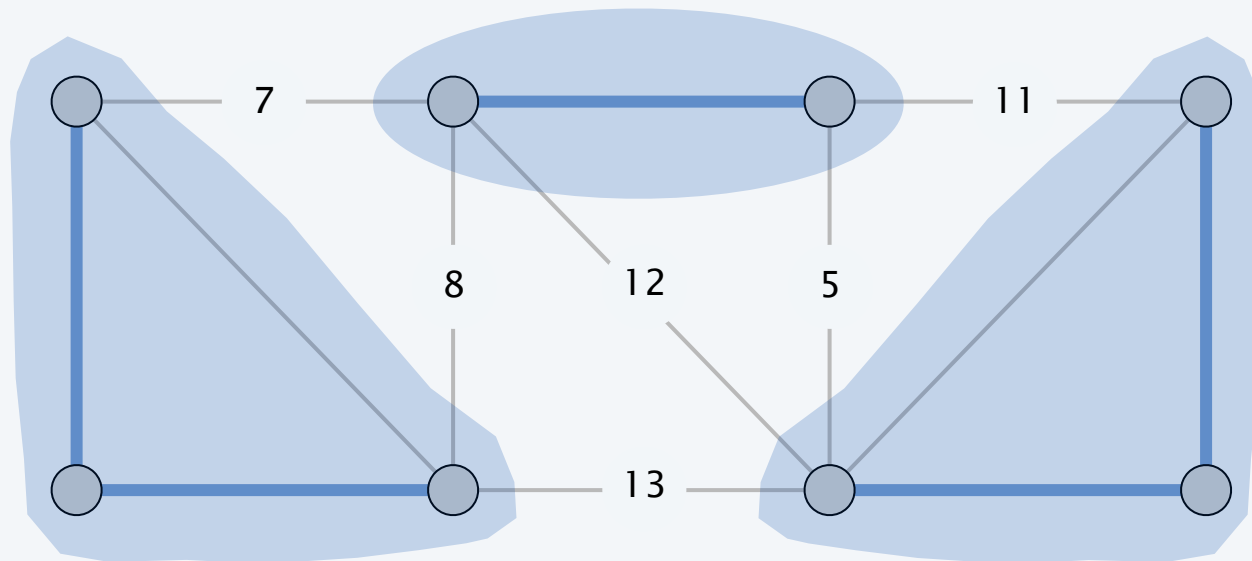
Pf. Special case of greedy algorithm (repeatedly apply blue rule). ▀



Borůvka's algorithm: implementation

Theorem. Borůvka's algorithm can be implemented in $O(m \log n)$ time.
Pf.

- To implement a phase in $O(m)$ time:
 - compute connected components of blue edges
 - for each edge $(u, v) \in E$, check if u and v are in different components; if so, update each component's best edge in cutset
- At most $\log_2 n$ phases since each phase (at least) halves total # trees. ■

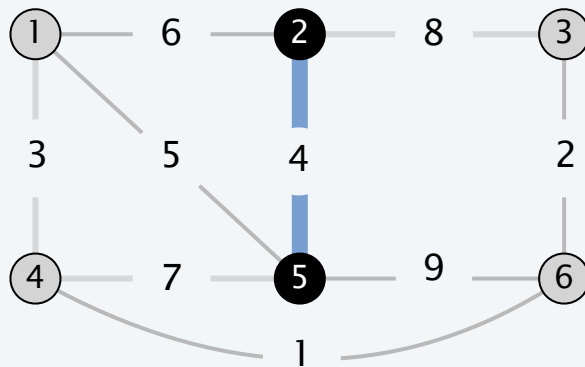


Borůvka's algorithm: implementation

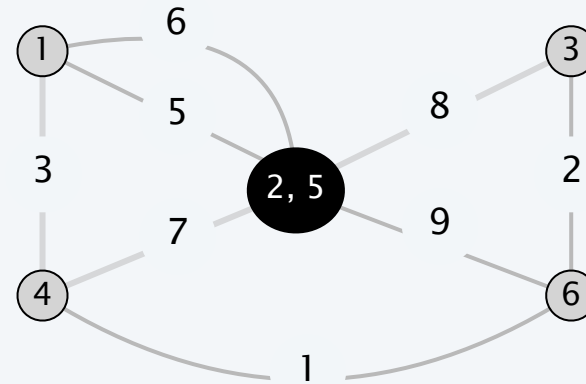
Node contraction version.

- After each phase, **contract** each blue tree to a single supernode.
- Delete parallel edges (keeping only cheapest one) and self loops.
- Borůvka phase becomes: take cheapest edge incident to each node.

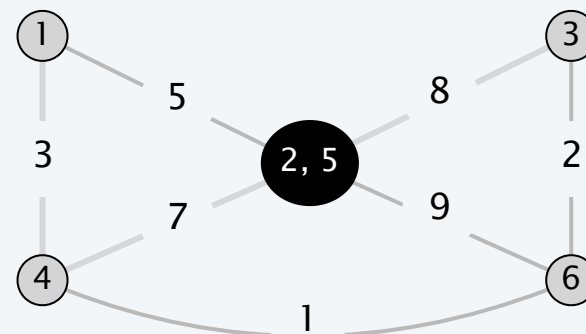
graph G



contract nodes 2 and 5



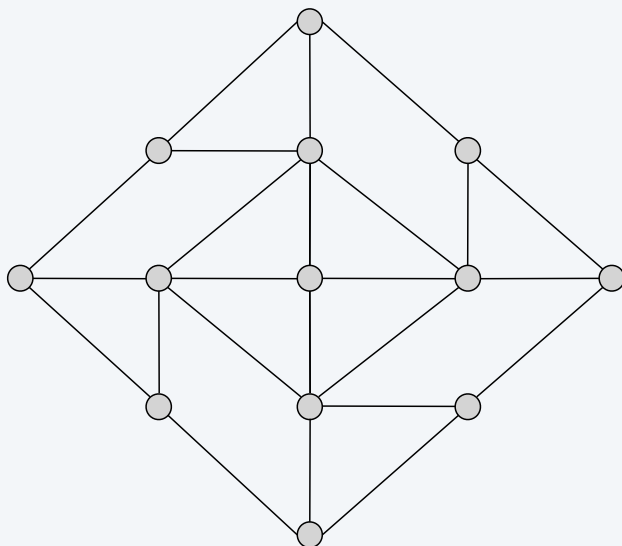
delete parallel edges and self loops



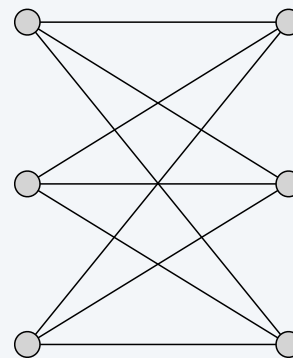
Borůvka's algorithm on planar graphs

Theorem. Borůvka's algorithm runs in $O(n)$ time on planar graphs.
Pf.

- To implement a Borůvka phase in $O(n)$ time:
 - use contraction version of algorithm
 - in planar graphs, $m \leq 3n - 6$.
 - graph stays planar when we contract a blue tree
- Number of nodes (at least) halves.
- At most $\log_2 n$ phases: $cn + cn/2 + cn/4 + cn/8 + \dots = O(n)$. ▀



planar



not planar

Borůvka-Prim algorithm


Borůvka-Prim algorithm.

- Run Borůvka (contraction version) for $\log_2 \log_2 n$ phases.
- Run Prim on resulting, contracted graph.

Theorem. The Borůvka-Prim algorithm computes an MST and can be implemented in $O(m \log \log n)$ time.


Pf.

- Correctness: special case of the greedy algorithm.
- The $\log_2 \log_2 n$ phases of Borůvka's algorithm take $O(m \log \log n)$ time; resulting graph has at most $n / \log_2 n$ nodes and m edges.
- Prim's algorithm (using Fibonacci heaps) takes $O(m + n)$ time on a graph with $n / \log_2 n$ nodes and m edges. ■


$$O\left(m + \frac{n}{\log n} \log\left(\frac{n}{\log n}\right)\right)$$

Does a linear-time MST algorithm exist?

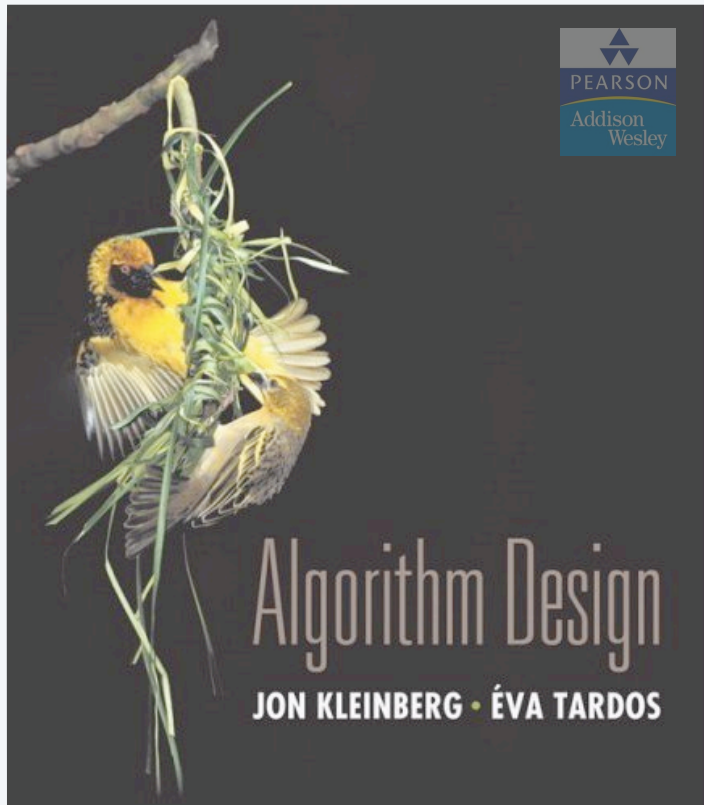
deterministic compare-based MST algorithms

year	worst case	discovered by
1975	$O(m \log \log n)$	Yao
1976	$O(m \log \log n)$	Cheriton-Tarjan
1984	$O(m \log^* n)$ $O(m + n \log n)$	Fredman-Tarjan 
1986	$O(m \log (\log^* n))$	Gabow-Galil-Spencer-Tarjan
1997	$O(m \alpha(n) \log \alpha(n))$	Chazelle
2000	$O(m \alpha(n))$	Chazelle
2002	<i>optimal</i>	Pettie-Ramachandran
20xx	$O(m)$???



Remark 1. $O(m)$ randomized MST algorithm. [Karger-Klein-Tarjan 1995]

Remark 2. $O(m)$ MST verification algorithm. [Dixon-Rauch-Tarjan 1992]



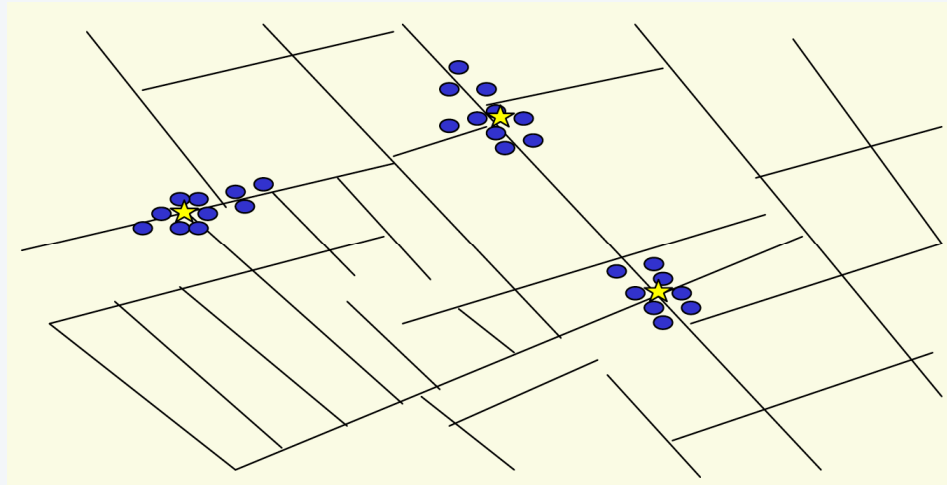
SECTION 4.7

4. GREEDY ALGORITHMS II

- ▶ *Dijkstra's algorithm*
- ▶ *minimum spanning trees*
- ▶ *Prim, Kruskal, Boruvka*
- ▶ *single-link clustering*
- ▶ *min-cost arborescences*

Clustering

Goal. Given a set U of n objects labeled p_1, \dots, p_n , partition into clusters so that objects in different clusters are far apart.



outbreak of cholera deaths in London in 1850s (Nina Mishra)

Applications.

- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.
- ...

Clustering of maximum spacing

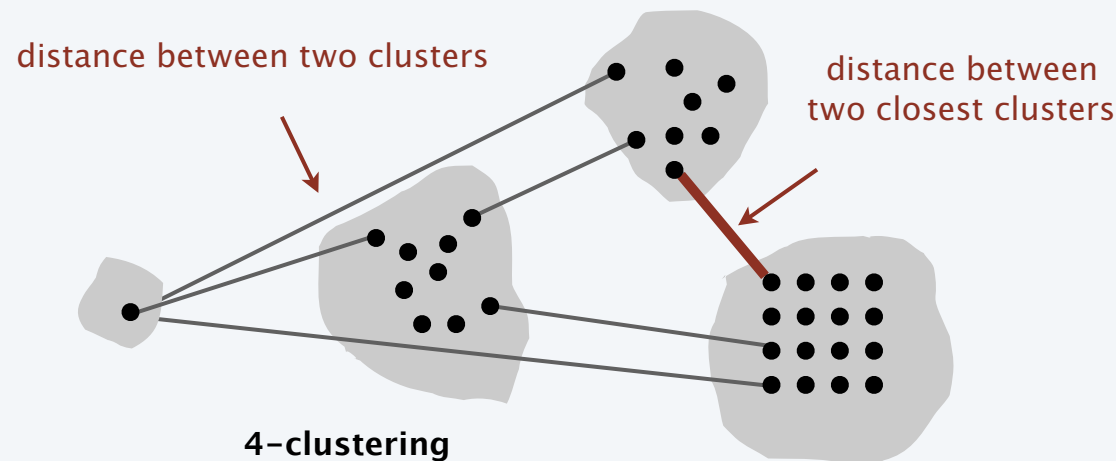
k-clustering. Divide objects into k non-empty groups.

Distance function. Numeric value specifying "closeness" of two objects. 

- $d(p_i, p_j) = 0$ iff $p_i = p_j$ [identity of indiscernibles]
- $d(p_i, p_j) \geq 0$ [nonnegativity]
- $d(p_i, p_j) = d(p_j, p_i)$ [symmetry]

Spacing. Min distance between any pair of points in different clusters. 

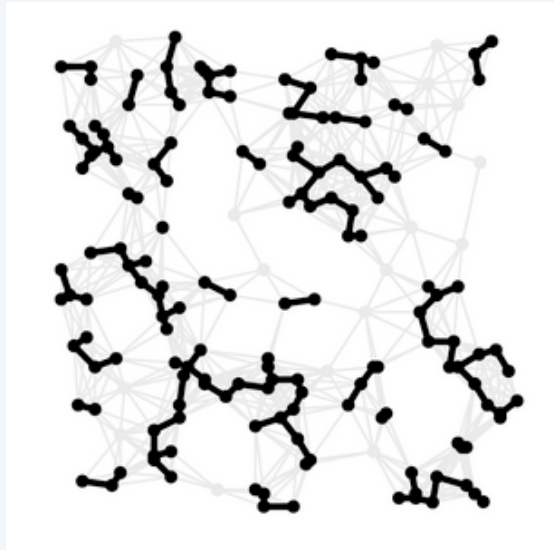
Goal. Given an integer k , find a k -clustering of maximum spacing. 




Greedy clustering algorithm

“Well-known” algorithm in science literature for single-linkage k -clustering:

- Form a graph on the node set U , corresponding to n clusters.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat $n - k$ times until there are exactly k clusters.



Key observation. This procedure is precisely Kruskal's algorithm (except we stop when there are k connected components).

Alternative. Find an MST and delete the $k - 1$ longest edges. 

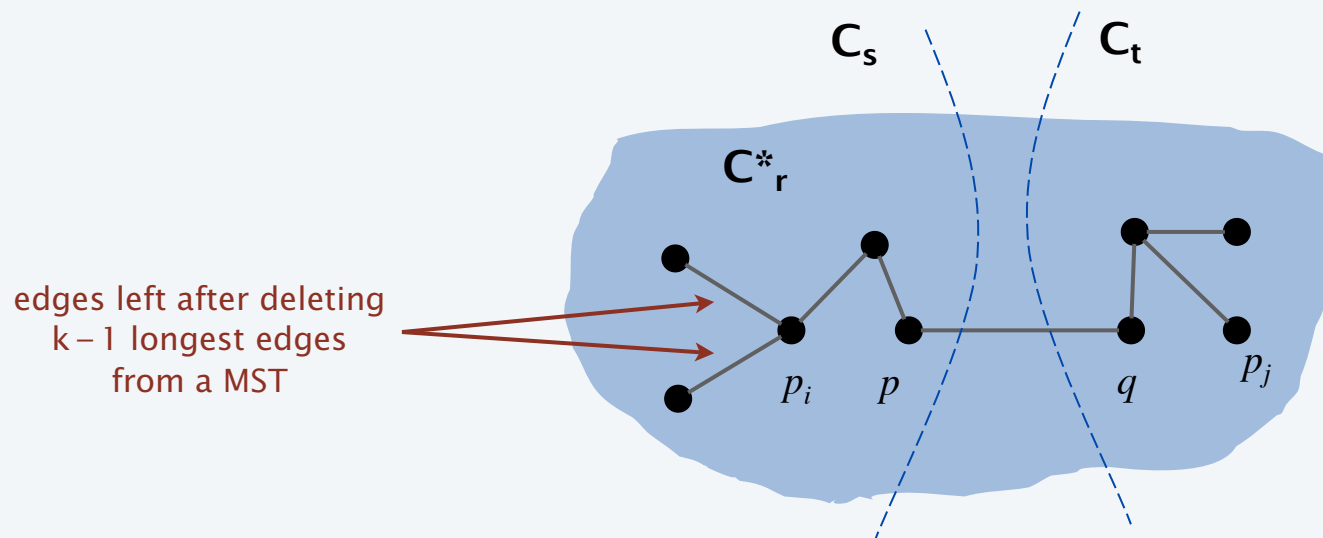
Greedy clustering algorithm: analysis

Theorem. Let C^* denote the clustering C^*_1, \dots, C^*_k formed by deleting the $k - 1$ longest edges of an MST. Then, C^* is a k -clustering of max spacing.

Pf. Let C denote some other clustering C_1, \dots, C_k .

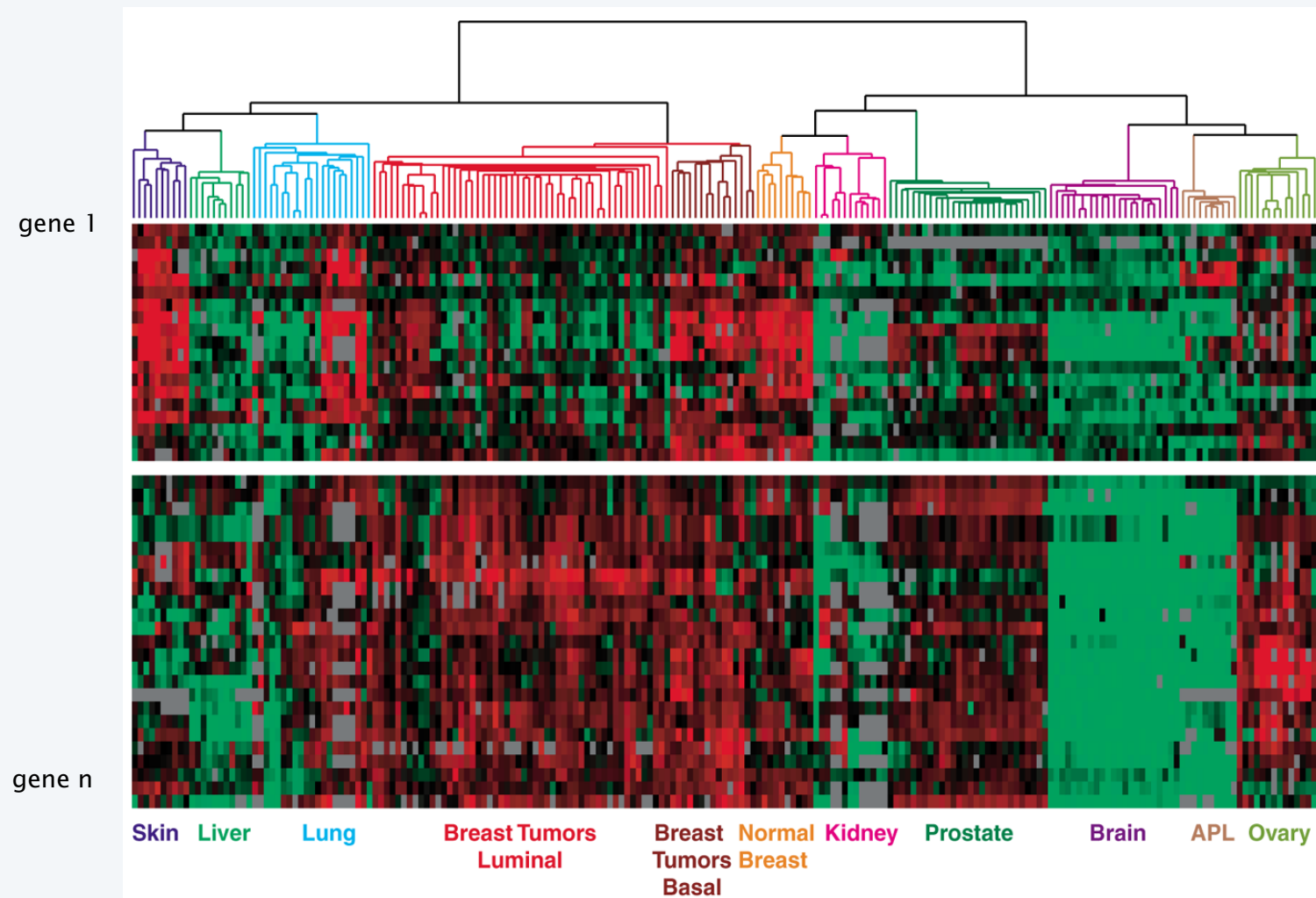


- The spacing of C^* is the length d^* of the $(k - 1)^{\text{st}}$ longest edge in MST.
- Let p_i and p_j be in the same cluster in C^* , say C^*_r , but different clusters in C , say C_s and C_t .
- Some edge (p, q) on $p_i - p_j$ path in C^*_r spans two different clusters in C .
- Edge (p, q) has length $\leq d^*$ since it wasn't deleted.
- Spacing of C is $\leq d^*$ since p and q are in different clusters. ■



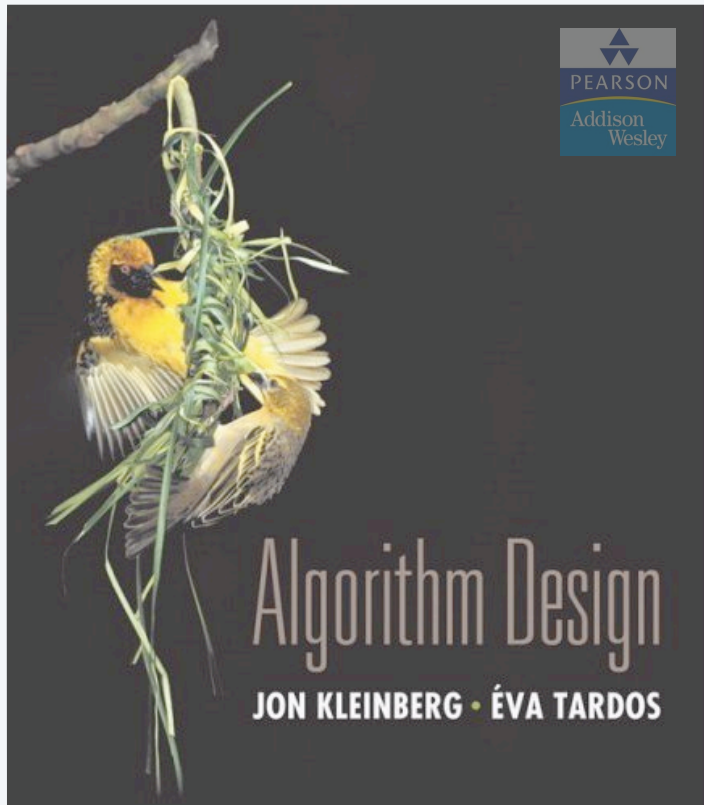
Dendrogram of cancers in human

Tumors in similar tissues cluster together.




Reference: Botstein & Brown group

■ gene expressed
■ gene not expressed



SECTION 4.9

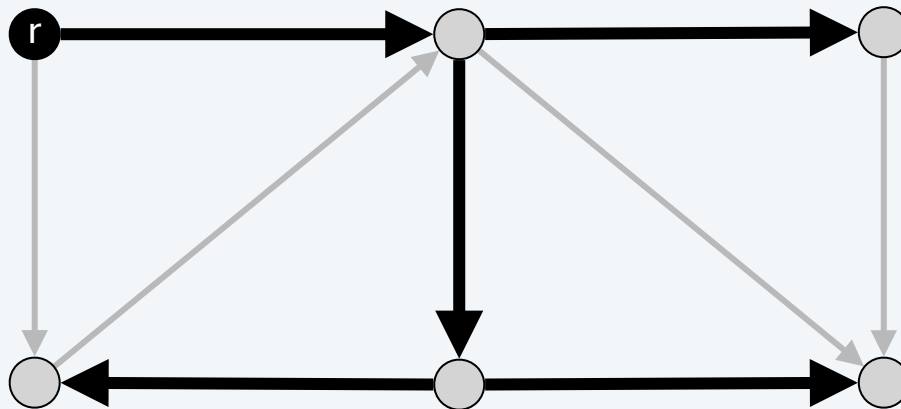
4. GREEDY ALGORITHMS II

- ▶ *Dijkstra's algorithm*
- ▶ *minimum spanning trees*
- ▶ *Prim, Kruskal, Boruvka*
- ▶ *single-link clustering*
- ▶ ***min-cost arborescences*** 

Arborescences

Def. Given a digraph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that

- T is a spanning tree of G if we ignore the direction of edges.
- There is a directed path in T from r to each other node $v \in V$.



Warmup. Given a digraph G , find an arborescence rooted at r (if one exists).

Algorithm. BFS or DFS from r is an arborescence (iff all nodes reachable).

Arborescences

Def. Given a digraph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that

- T is a spanning tree of G if we ignore the direction of edges.
- There is a directed path in T from r to each other node $v \in V$.

Proposition. A subgraph $T = (V, F)$ of G is an arborescence rooted at r iff T has no directed cycles and each node $v \neq r$ has exactly one entering edge.

Pf.

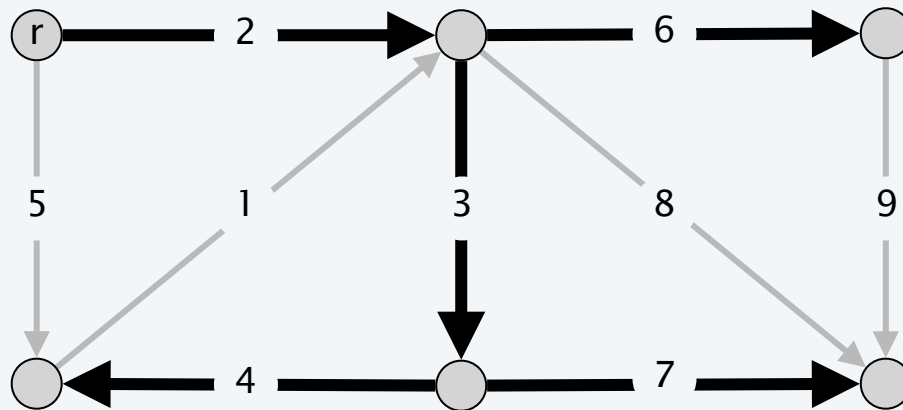
\Rightarrow If T is an arborescence, then no (directed) cycles and every node $v \neq r$ has exactly one entering edge—the last edge on the unique $r \rightarrow v$ path.

\Leftarrow Suppose T has no cycles and each node $v \neq r$ has one entering edge.

- To construct an $r \rightarrow v$ path, start at v and repeatedly follow edges in the backward direction.
- Since T has no directed cycles, the process must terminate.
- It must terminate at r since r is the only node with no entering edge. ■

Min-cost arborescence problem

Problem. Given a digraph G with a root node r and with a nonnegative cost $c_e \geq 0$ on each edge e , compute an arborescence rooted at r of minimum cost.



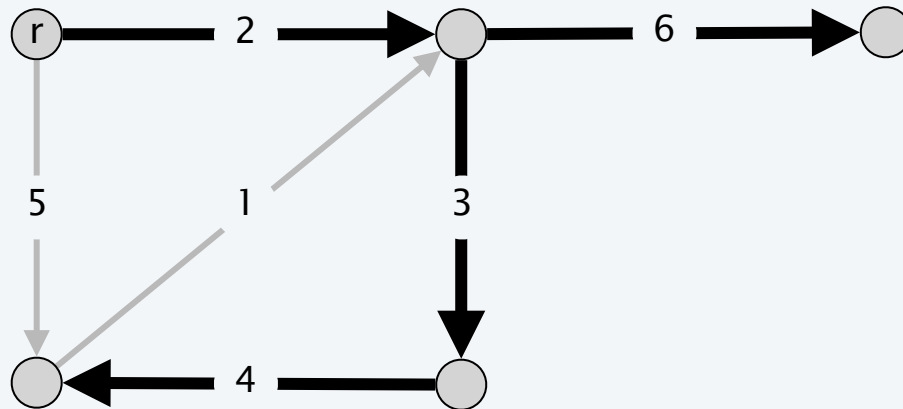
Assumption 1. G has an arborescence rooted at r .

Assumption 2. No edge enters r (safe to delete since they won't help).

Simple greedy approaches do not work

Observations. A min-cost arborescence need not:

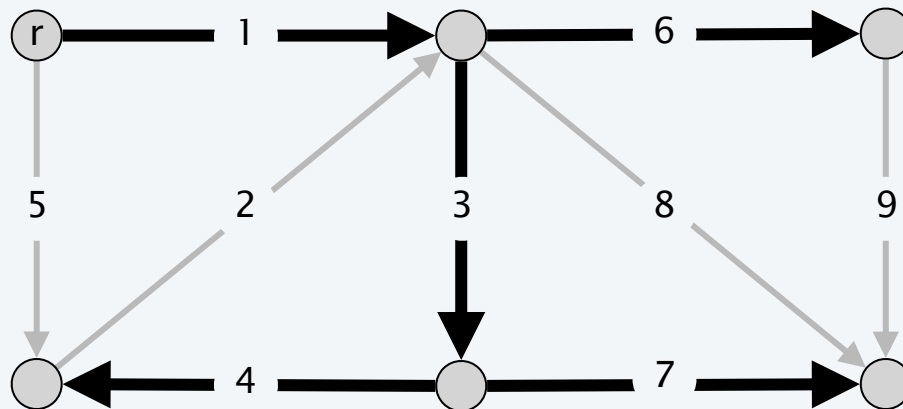
- Be a shortest-paths tree.
- Include the cheapest edge (in some cut).
- Exclude the most expensive edge (in some cycle).



A sufficient optimality condition

Property. For each node $v \neq r$, choose one cheapest edge entering v and let F^* denote this set of $n - 1$ edges. If (V, F^*) is an arborescence, then it is a min-cost arborescence.

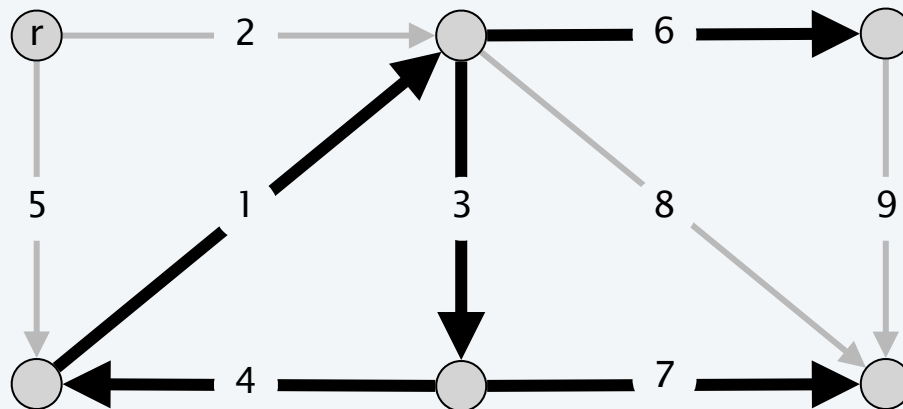
Pf. An arborescence needs exactly one edge entering each node $v \neq r$ and (V, F^*) is the cheapest way to make these choices. ■



A sufficient optimality condition

Property. For each node $v \neq r$, choose one cheapest edge entering v and let F^* denote this set of $n - 1$ edges. If (V, F^*) is an arborescence, then it is a min-cost arborescence.

Note. F^* may not be an arborescence (since it may have directed cycles).



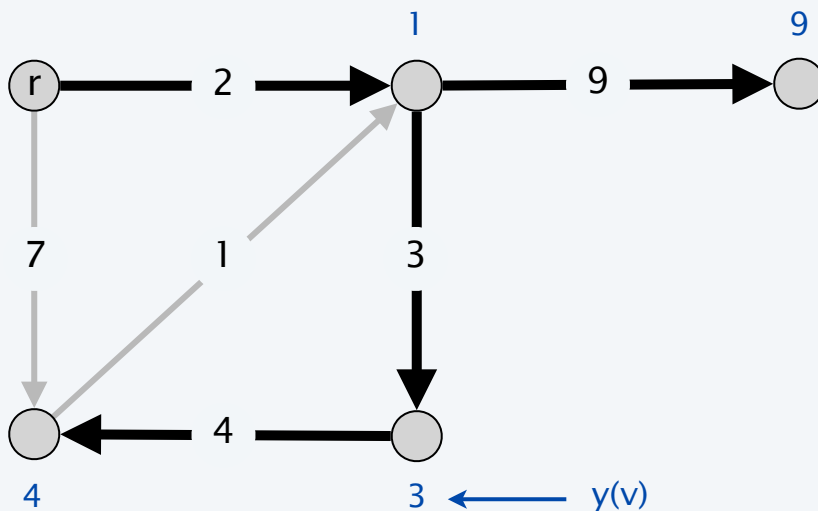
Reduced costs

Def. For each $v \neq r$, let $y(v)$ denote the min cost of any edge entering v .
The **reduced cost** of an edge (u, v) is $c'(u, v) = c(u, v) - y(v) \geq 0$.

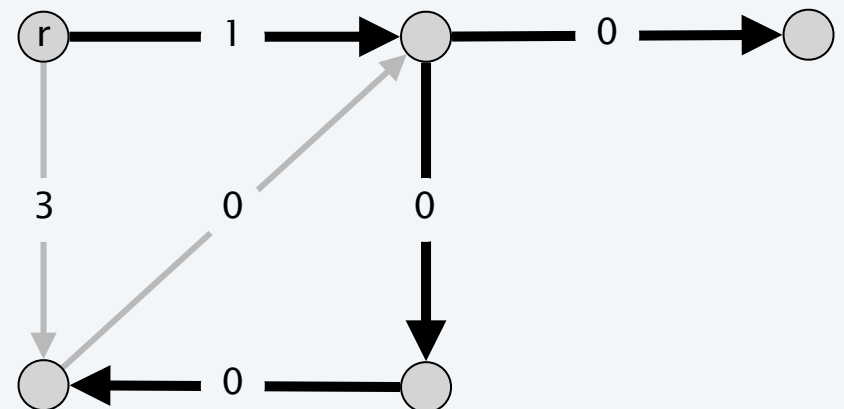
Observation. T is a min-cost arborescence in G using costs c iff T is a min-cost arborescence in G using reduced costs c' .

Pf. Each arborescence has exactly one edge entering v .

costs c



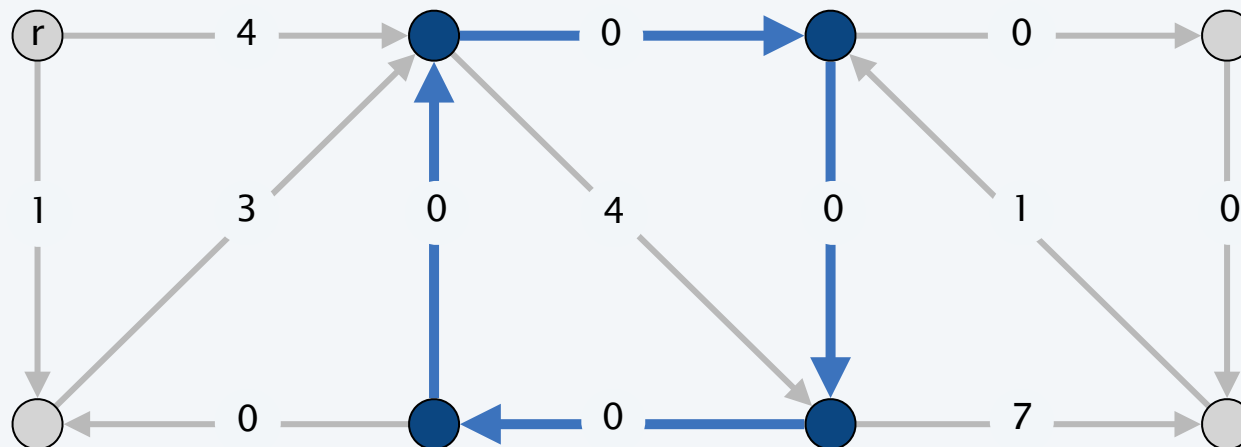
reduced costs c'



Edmonds branching algorithm: intuition

Intuition. Recall F^* = set of cheapest edges entering v for each $v \neq r$.

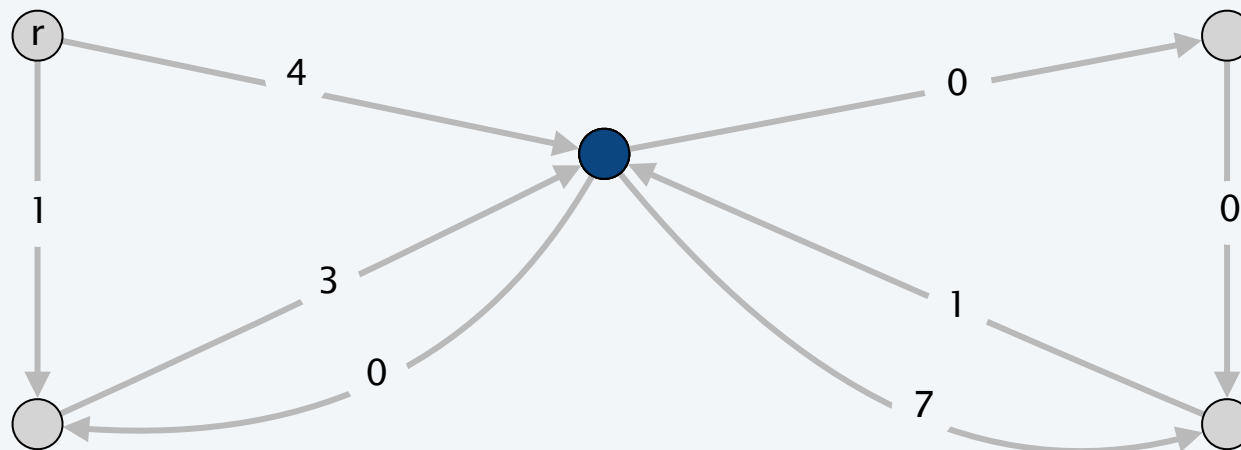
- Now, all edges in F^* have 0 cost with respect to costs $c'(u, v)$.
- If F^* does not contain a cycle, then it is a min-cost arborescence.
- If F^* contains a cycle C , can afford to use as many edges in C as desired.
- **Contract nodes** in C to a supernode.
- Recursively solve problem in contracted network G' with costs $c'(u, v)$.



Edmonds branching algorithm: intuition

Intuition. Recall F^* = set of cheapest edges entering v for each $v \neq r$.

- Now, all edges in F^* have 0 cost with respect to costs $c'(u, v)$.
- If F^* does not contain a cycle, then it is a min-cost arborescence.
- If F^* contains a cycle C , can afford to use as many edges in C as desired.
- **Contract nodes** in C to a supernode (removing any self-loops).
- Recursively solve problem in contracted network G' with costs $c'(u, v)$.



Edmonds branching algorithm



EDMONDSBRANCHING(G, r, c)

FOREACH $v \neq r$

$y(v) \leftarrow$ min cost of an edge entering v .

$c'(u, v) \leftarrow c(u, v) - y(v)$ for each edge (u, v) entering v .

FOREACH $v \neq r$: choose one 0-cost edge entering v and let F^* be the resulting set of edges.

IF F^* forms an arborescence, RETURN $T = (V, F^*)$.

ELSE

$C \leftarrow$ directed cycle in F^* .

Contract C to a single supernode, yielding $G' = (V', E')$.

$T' \leftarrow$ EDMONDSBRANCHING(G', r, c')

Extend T' to an arborescence T in G by adding all but one edge of C .

RETURN T .

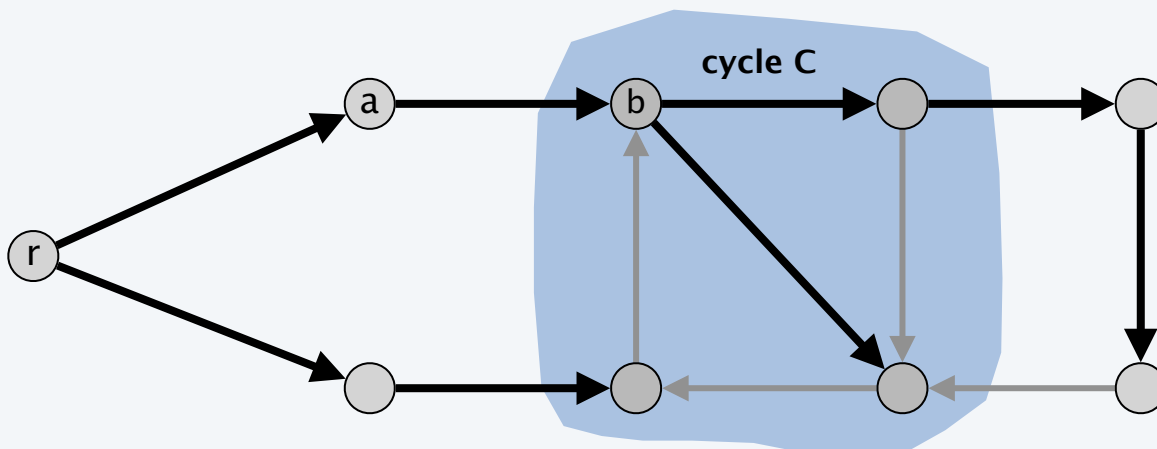
Edmonds branching algorithm

Q. What could go wrong?

A.

- Min-cost arborescence in G' has exactly one edge entering a node in C (since C is contracted to a single node)
- But min-cost arborescence in G might have more edges entering C .

min-cost arborescence in G



Edmonds branching algorithm: key lemma

Lemma. Let C be a cycle in G consisting of 0-cost edges. There exists a min-cost arborescence rooted at r that has exactly one edge entering C .

Pf. Let T be a min-cost arborescence rooted at r .

Case 0. T has no edges entering C .

Since T is an arborescence, there is an $r \rightarrow v$ path for each node $v \Rightarrow$ at least one edge enters C .

Case 1. T has exactly one edge entering C .

T satisfies the lemma.

Case 2. T has more than one edge that enters C .

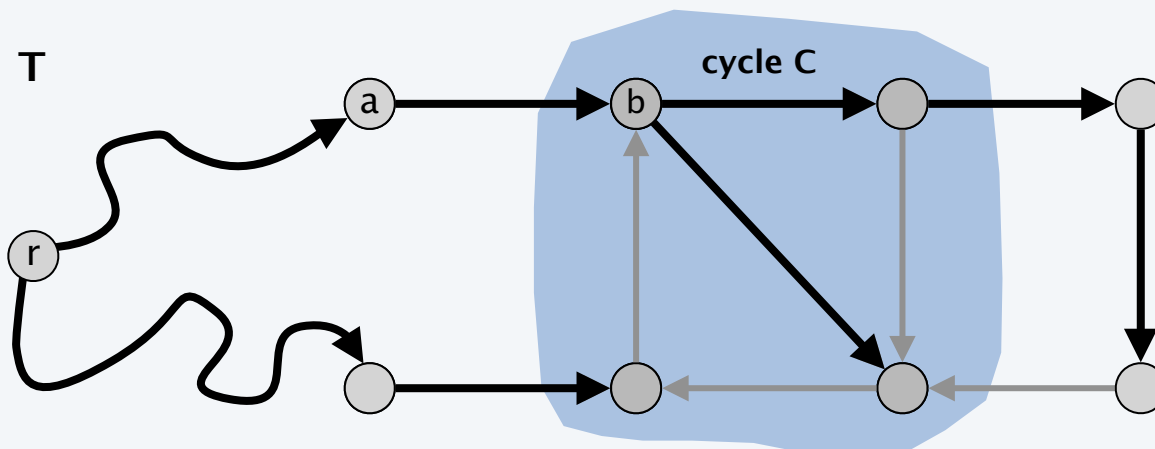
We construct another min-cost arborescence T' that has exactly one edge entering C .

Edmonds branching algorithm: key lemma

Case 2 construction of T' .

- Let (a, b) be an edge in T entering C that lies on a shortest path from r .
- We delete all edges of T that enter a node in C except (a, b) .
- We add in all edges of C except the one that enters b .

path from r to C uses
only one node in C



Edmonds branching algorithm: key lemma

Case 2 construction of T' .

- Let (a, b) be an edge in T entering C that lies on a shortest path from r .
- We delete all edges of T that enter a node in C except (a, b) .
- We add in all edges of C except the one that enters b .

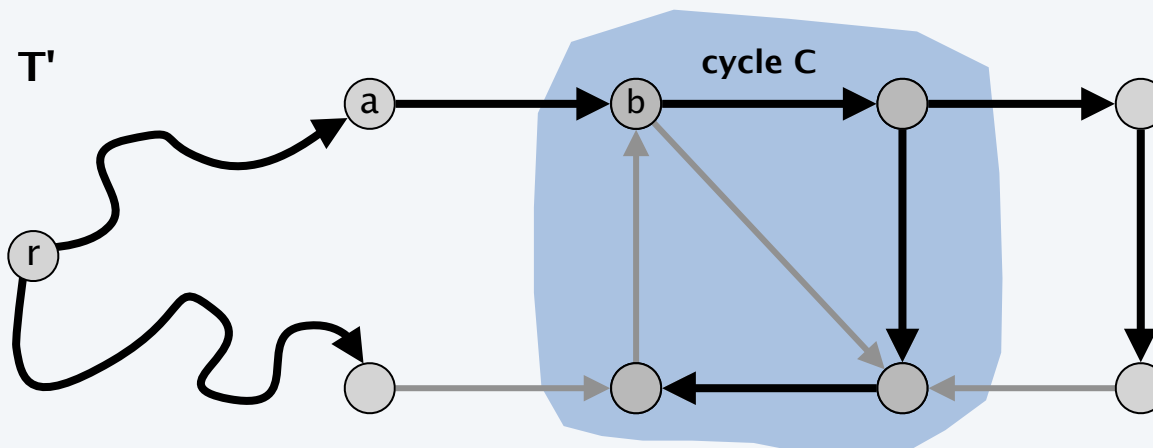
path from r to C uses only one node in C

Claim. T' is a min-cost arborescence.

- The cost of T' is at most that of T since we add only 0-cost edges.
- T' has exactly one edge entering each node $v \neq r$.
- T' has no directed cycles.

T is an arborescence rooted at r

(T had no cycles before; no cycles within C ; now only (a, b) enters C)



and the only path in T' to a is the path from r to a (since any path must follow unique entering edge back to r)

Edmonds branching algorithm: analysis

Theorem. [Chu-Liu 1965, Edmonds 1967] The greedy algorithm finds a min-cost arborescence.

Pf. [by induction on number of nodes in G]

- If the edges of F^* form an arborescence, then min-cost arborescence.
- Otherwise, we use reduced costs, which is equivalent.
- After contracting a 0-cost cycle C to obtain a smaller graph G' , the algorithm finds a min-cost arborescence T' in G' (by induction).
- Key lemma: there exists a min-cost arborescence T in G that corresponds to T' . ■

Theorem. The greedy algorithm can be implemented in $O(mn)$ time.

Pf.

- At most n contractions (since each reduces the number of nodes).
- Finding and contracting the cycle C takes $O(m)$ time.
- Transforming T' into T takes $O(m)$ time. ■

Min-cost arborescence

Theorem. [Gabow-Galil-Spencer-Tarjan 1985] There exists an $O(m + n \log n)$ time algorithm to compute a min-cost arborescence.

COMBINATORICA 6 (2) (1986) 109—122

EFFICIENT ALGORITHMS FOR FINDING MINIMUM SPANNING TREES IN UNDIRECTED AND DIRECTED GRAPHS

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Received 23 January 1985

Revised 1 December 1985

Recently, Fredman and Tarjan invented a new, especially efficient form of heap (priority queue). Their data structure, the *Fibonacci heap* (or F-heap) supports arbitrary deletion in $O(\log n)$ amortized time and other heap operations in $O(1)$ amortized time. In this paper we use F-heaps to obtain fast algorithms for finding minimum spanning trees in undirected and directed graphs. For an undirected graph containing n vertices and m edges, our minimum spanning tree algorithm runs in $O(m \log \beta(m, n))$ time, improved from $O(m\beta(m, n))$ time, where $\beta(m, n) = \min \{i | \log^{(i)} n \leq m/n\}$. Our minimum spanning tree algorithm for directed graphs runs in $O(n \log n + m)$ time, improved from $O(n \log n + m \log \log \log_{(m/n+2)} n)$. Both algorithms can be extended to allow a degree constraint at one vertex.