1. (a) $T(n) = 3T(\frac{n}{2}) + n^2$

by Master Theorem

$c = \log_b a$ 其中 $a=3$, $b=2$, 故 $c = \log_2 3$

又 $f(n) = n^2$ 且 $f(n) \in \Omega(n^{\log_2 3 + \varepsilon})$, $\varepsilon > 0$

$3f(\frac{n}{2}) \le df(n)$ 其中 $d \in R^+$ 且 $d<1$

$T(n) = \Theta(n^2)$

(b) $T(n) = 4T(\frac{n}{2}) + n^2$

by Master Theorem, $c = \log_b a$

其中 $c = \log_2 4 = 2$

$f(n) = n^2$, $f(n) \in \Theta(n^{\log_2 4})$

故 $T(n) \in \Theta(n^2 \log n)$

(c) $T(n) = 16T(\frac{n}{4}) + n$

by Master Theorem, $c = \log_4 16 = 2$

$f(n) = n$, $f(n) \in \Theta(n^{\log_4 16 - \varepsilon})$, $\varepsilon > 0$

故 $T(n) = \Theta(n^2)$

(d) $T(n) = T(\frac{n}{2}) + 2^n$

by Master Theorem $c = \log_2 1 = 0$

$f(n) = 2^n$, $f(n) \in \Omega(n^{\log_2 1 + \varepsilon})$, $\varepsilon = n\log\frac{2}{n}$

$\because n>0$ ε 必大於 0

$\Rightarrow T(n) = \Theta(2^n)$

2、(a) $n^2+3n+1 \in O(n^2)$

若存在 $c, n_0 > 0$，且在 $n \geq n_0$ 時 $T(n) \leq c\phi(n)$

$\Rightarrow n^2+3n+1 \leq cn^2$

$\frac{n^2+3n+1}{n^2} \leq c \Rightarrow 1+\frac{3}{n}+\frac{1}{n^2} \leq c$

$c=2$ 時 $1+\frac{3}{n}+\frac{1}{n^2} \leq 2 \Rightarrow \frac{3}{n}+\frac{1}{n^2} \leq 1$

$\forall n \geq 4$ 成立

(b) $\log_b n \in O(n^k) \quad \forall b>1, k>0$

if $\exists\ c>0, n_0>0$ when $n \geq n_0$, $T(n) \leq c\phi(n)$

$\Rightarrow \log_b n \leq cn^k \Rightarrow \frac{\log_b n}{n^k} \leq c$

$\lim_{n\to\infty} \frac{\log_b n}{n^k} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n\log b}}{kn^{k-1}} = 0$ 故 $\log_b n \in O(n^k)$

羅必達法則

(c) $n^k \in O(b^n)$

If $\exists\ c>0, n_0>0$ when $n \geq n_0$, $T(n) \leq c\phi(n)$

$\Rightarrow n^k \leq cb^n \quad \frac{n^k}{cb^n} \leq 0$

$\lim_{n\to\infty} \frac{n^k}{cb^n} \Rightarrow \lim_{n\to\infty} \frac{kn^{k-1}}{cb^n \ln b} = 0$ 故 $n^k \in O(b^n)$

exp.

3、$\log(n!) \in \Theta(n\log n)$

$\log 1 + \cdots \log n \leq (\log n + \cdots \log n) \in O(n\log n)$

$\log 1 + \cdots \log n \geq \log\frac{n}{k} + \cdots \log n$ 剩一半

$\log\frac{n}{2} \cdots \log n \geq \log\frac{n}{k} + \cdots \log\frac{n}{k}$ 把 $\log n$ 減 $\log k$

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而 $\underbrace{\log\frac{n}{k} + \cdots + \log\frac{n}{k}}_{\frac{n}{k}} \Rightarrow \frac{n}{k}\log\frac{n}{k} = \frac{n}{k}(\log n - \log k)$

$\in \Omega(n\log n)$

其中 $k>1$ 之 constant

故 $\log n! \in \Theta(n\log n)$