MATH 165: Sp 15 **TEST 2** **SOLUTIONS**

1. (20 pts) Let $f(x) = 3x^5 - 20x^3$ be defined on the whole real line $(-\infty, \infty)$.

i. Find the critical point(s) of f.

Show work: $f'(x) = 15x^4 - 60x^2 = 15x^2(x^2 - 4) = 15x^2(x+2)(x-2)$.

Answer: critical points are -2, 0, 2.

ii. Find the interval(s) where the function increases/decreases. (Use interval notation.)

Show work: draw a number line and indicate where $f' > 0$ and where $f' < 0$.

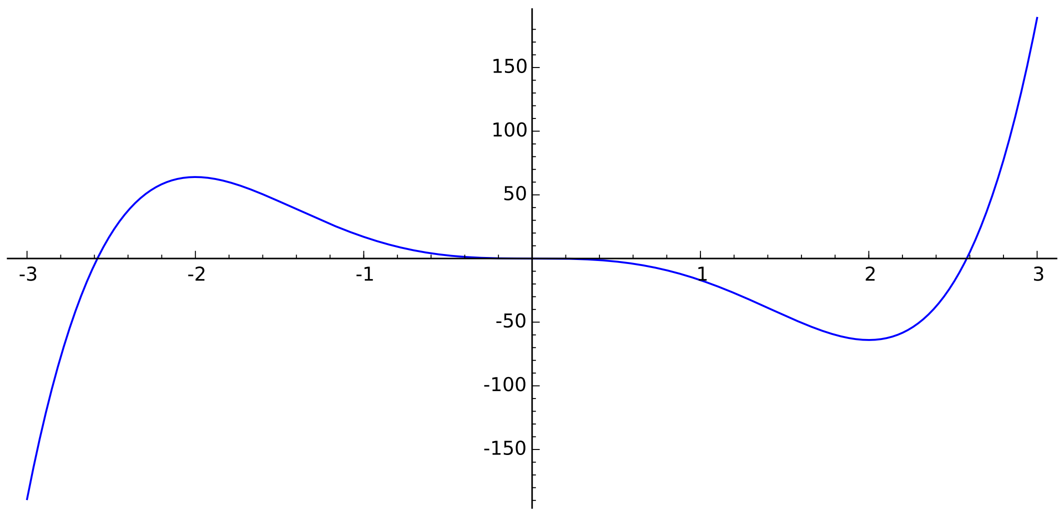
Answer: f increases on $(-\infty, -2) \cup (2, \infty)$
f decreases on $(-2, 0) \cup (0, 2)$.

iii. Locate the inflection point(s) of f. (Justify your answer.)

Show work: $f''(x) = 60x^3 - 120x = 60x(x^2 - 2) = 60x(x + \sqrt{2})(x - \sqrt{2})$, so potential inflection points are at $-\sqrt{2}, 0, \sqrt{2}$; draw a numberline and plot potential inflection points; mark intervals where $f'' > 0$ and where $f'' < 0$ to see that concavity changes at each of the points.

Answer: inflection points are at $-\sqrt{2}, 0, \sqrt{2}$.

iv. Sketch the graph of f.



2. (20 pts) Evaluate the limits. Show your work and please **CIRCLE** your final answer.
(Hint: L'Hopital's Rule)

i.

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2}}{1} = \frac{1}{2}.$$

ii.

$$\lim_{x \to 0^-} \frac{\sin x}{x^2} \overset{\text{H}}{=} \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty, \quad \text{so the limit } \textit{does not exist.}$$

iii.

$$\lim_{t \to 0} \frac{t - \sin t}{t^3} \overset{\text{H}}{=} \lim_{t \to 0} \frac{1 - \cos t}{3t^2} \overset{\text{H}}{=} \lim_{t \to 0} \frac{\sin t}{6t} = \frac{1}{6}.$$

iv.

$$\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{1/x}{(-1/2)x^{-3/2}} = \lim_{x \to 0^+} \frac{-2x^{3/2}}{x} = \lim_{x \to 0^+} -2\sqrt{x} = 0.$$

3. (10 pts) Let $f(x) = x\sqrt{\ln x}$ and $g(x) = e^{x/10}$. As $x \to \infty$, does $f(x)$ grow faster than $g(x)$, does $g(x)$ grow faster than $f(x)$, or do the two functions grow at the same rate? Be sure to justify your answer using limits. *No credit given for mere answers.*

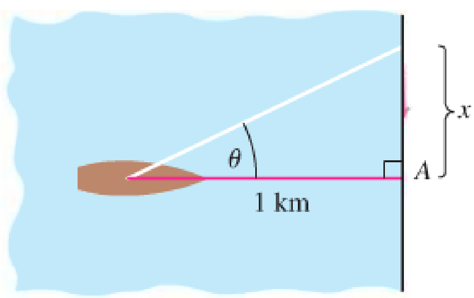
Show work:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x\sqrt{\ln x}}{e^{x/10}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sqrt{\ln x} + x(\frac{1}{2}\ln x)(1/x)}{\frac{1}{10}e^{x/10}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{2}(\ln x)^{-1/2}(1/x) + \frac{1}{2x}}{\frac{1}{100}e^{x/10}} = 0.$$

Answer

☐ $f(x)$ grows faster ☒ $g(x)$ grows faster ☐ they grow at the same rate.

4. (18 pts) A boat 1 km offshore is sweeping the beach with a searchlight. The light turns at a rate $\theta'(t) = -2.4$ rad/s. How fast is the light moving along the shore when it reaches point A? Make sure to include correct units with your answer.



Answer

$$\tan(\theta(t)) = \frac{x(t)}{1} \quad \Rightarrow \quad (\sec^2\theta)\theta'(t) = x'(t).$$

At the point A, $\theta = 0$, so $\sec^2\theta = 1/\cos^2\theta = 1$.

Therefore, when $x(t) = A$, the light is moving at $x'(t) = -2.4$ kilometers per second.

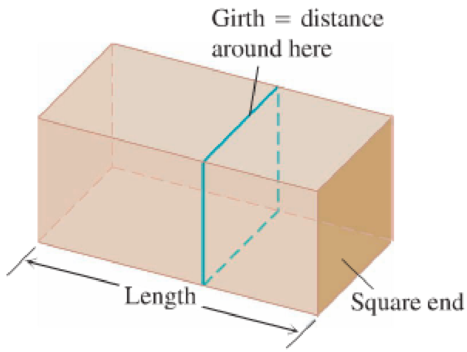


Figure 1: Girth + Length = 228

5. (18 pts) The Post Office will accept a box only if the sum of its length and girth (distance around the box) does not exceed 228 in. (See Figure 1.)

What dimensions will give a box with a square end the largest possible volume?

Use Calculus to both find the maximal dimensions, and to justify they give the maximum (rather than a minimum). Your answer should give a Length and a Width in inches. (Here, "Width" refers to the length of one edge of the square end.)

Solution:

Volume $= V(\ell, w) = \ell w^2$.

$228 = \text{Girth} + \text{length} = 4w + \ell \quad \Rightarrow \quad \ell = 228 - 4w.$

Therefore, the volume,

$$V(w) = (228 - 4w)w^2 = 228w^2 - 4w^3,$$

is a function of one variable, the width, and in this problem we may take the domain of $V(w)$ to be the interval $[0, 57]$ (since w is at least 0 and $4w$ is at most 228).

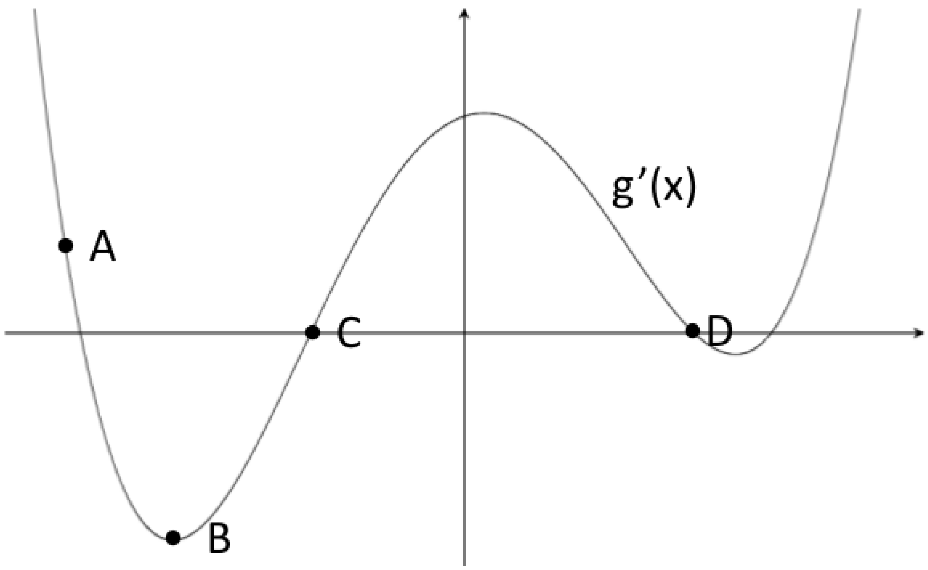
To maximize volume, take its derivative and set it equal to 0 to find critical points:

$$V'(w) = 456w - 12w^2 = 12w(38 - w),$$

so critical points are 0 and 38. The critical point $w = 0$ (which is also an end point) gives a volume of 0. So we need only consider $w = 38$ and the other end point, $w = 57$. The latter gives 0 length and 0 volume, so $w = 38$ is the only reasonable candidate. Indeed, $V''(w) = 456 - 24w$ implies $V''(38) < 0$, so V has a local max at $w = 38$, and from the foregoing, it is the absolute max on the interval $[0, 57]$

Answer: length = 76 width = 38

6. (14 pts) The graph below is the graph of *the derivative of* a function $g(x)$. (Note: this is *not* the graph of $g(x)$; it is the graph of $g'(x)$.)



For each of the points above, circle all the statements that apply at each of the points A, B, C, and D.

	Concerning g"(x)			Concerning g(x)		
Point A	g" > 0	g" = 0	g" < 0	local min	local max	inflection pt
Point B	g" > 0	g" = 0	g" < 0	local min	local max	inflection pt
Point C	g" > 0	g" = 0	g" < 0	local min	local max	inflection pt
Point D	g" > 0	g" = 0	g" < 0	local min	local max	inflection pt