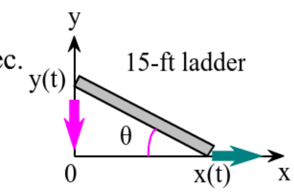
A 15-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving away at the rate of 18 ft/sec.
a. At what rate is the top of the ladder sliding down the wall then?
b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?



c. At what rate is the angle between the ladder and the ground changing then?

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a. Let L be the length of the ladder. The given values are shown below.

$$L = 15 \text{ ft}, x = 12 \text{ ft}, \frac{dx}{dt} = 18 \text{ ft/sec}$$

The lengths x and y are related by $x^2 + y^2 = L^2$.

Differentiate $x^2 + y^2 = L^2$ with respect to t using the chain rule. Notice that L is not a function of t.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Notice that the rate equation also involves y as an unknown. Use $x^2 + y^2 = L^2$ to solve for y.

$$\begin{aligned} (12)^2 + y^2 &= (15)^2 \\ y &= 9 \text{ ft} \end{aligned}$$

Substitute the known values and solve for the rate of change of the height of the top of the ladder.

$$\begin{aligned} 2(12)(18) + 2(9)\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -24 \text{ ft/sec} \end{aligned}$$

Therefore, the rate of change of the height of the top of the ladder is -24ft/sec.

b. The area of the triangle is given by the equation $A = \frac{1}{2}xy$.

Use the chain rule and the product rule to differentiate this equation with respect to t. In this case, A, x, and y are all functions of t.

$$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}y\frac{dx}{dt}$$

Then substitute the known values to find the rate of change of the area of the triangle.

$$\frac{dA}{dt} = \frac{1}{2}(12)(-24) + \frac{1}{2}(9)(18)$$

$$\frac{dA}{dt} = -63\ \text{ft}^2/\text{sec}$$

c. There are several trigonometric definitions that can be used to relate $\boldsymbol{\theta}$ to x, y, and L. Choose $\mathbf{cos}\ \theta = \frac{x}{L}$.

Differentiate this equation with respect to t using the chain rule on the left side. Here, both $\boldsymbol{\theta}$ and x are functions of t.

$$-\mathbf{sin}\ \theta \frac{d\theta}{dt} = \frac{1}{L}\frac{dx}{dt}$$

Use the triangle to find $\mathbf{sin}\ \theta = \frac{\text{opposite}}{\text{hypotenuse}}$.

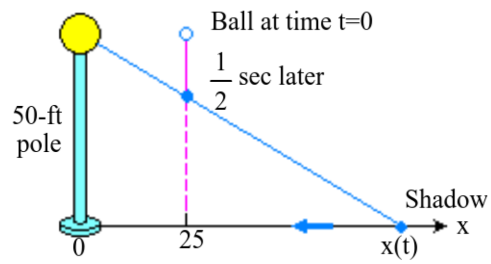
$$\mathbf{sin}\ \theta = \frac{9}{15}$$

Then substitute the known values and solve for the rate of change of the angle.

$$-\frac{9}{15}\frac{d\theta}{dt} = \frac{1}{15}(18)$$

$$\frac{d\theta}{dt} = -2\ \text{rad/sec}$$

A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 25 ft away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ sec later? (Assume the ball falls a distance $s = 16t^2$ in t sec.)



To find the velocity of the shadow, $\frac{dx}{dt}$, write an equation that relates the position of the shadow, x, to the distance the ball has fallen, s.

Notice that there are similar triangles in the figure. The triangle formed by the light, the base of the pole, and the shadow is similar to the triangle formed by the position of the ball, the impact point of the ball, and the shadow.

Write an equation relating x to s using these similar triangles.

$$\frac{50 - s}{50} = \frac{x - 25}{x}$$

Now simplify the equation using the cross product and solve for x.

$$\begin{aligned} (50 - s)x &= 50(x - 25) \\ 50x - sx &= 50x - 1250 \\ x &= \frac{1250}{s} \end{aligned}$$

Differentiate the equation with respect to t.

$$\frac{dx}{dt} = -\frac{1250}{s^2}\frac{ds}{dt}$$

The expression for $\frac{dx}{dt}$ contains s. Therefore, calculate s, the distance the ball has fallen for $t = \frac{1}{2}$ sec.

$$\begin{aligned} s &= 16t^2 \\ s &= 4 \text{ ft} \end{aligned}$$

The expression also contains $\frac{ds}{dt}$. Differentiate $s = 16t^2$ to find $\frac{ds}{dt}$.

$$\frac{ds}{dt} = 32t$$

Then evaluate $\frac{ds}{dt}$ for $t = \frac{1}{2}$.

$$\frac{ds}{dt} = 32\left(\frac{1}{2}\right) = 16 \text{ ft/sec}$$

Substitute the values of s and $\frac{ds}{dt}$ into the expression for $\frac{dx}{dt}$.

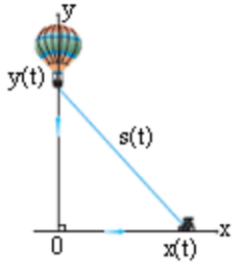
$$\frac{dx}{dt} = -\frac{1250}{s^2}\frac{ds}{dt} = -\frac{1250}{(4)^2}(16)$$

Simplify to find the velocity of the shadow.

$$\frac{dx}{dt} = -\frac{1250}{(4)^2}(16) = -1250 \text{ ft/sec}$$

Therefore, the shadow of the ball is moving along the ground at -1250 ft/sec.

A balloon is rising vertically above a level, straight road at a constant rate of 2 ft / sec. Just when the balloon is 30 ft above the ground, a bicycle moving at a constant rate of 9 ft / sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 seconds later?



Refer to the picture of the problem situation on the left.

x(t) represents the distance that bicycle has traveled t seconds after passing under the balloon.

y(t) represents the height of the balloon above the ground t seconds after the bicycle passes below it.

s(t) represents the distance between the bicycle and the balloon t seconds after the bicycle passes under the balloon.

x(t), y(t), and s(t) are all differentiable functions of time, t. The rates of change of x and y, or $\frac{dx}{dt}$ and $\frac{dy}{dt}$, are given in the problem statement.

$$\frac{dx}{dt} = 9 \text{ ft / sec}$$
$$\frac{dy}{dt} = 2 \text{ ft / sec}$$

Determine the rate of increase of $s(t)$, or $\frac{ds}{dt}$, at $t = 3$ seconds.

Start by finding the relationship between the variables x, y, and s. Note that the triangle formed by x, y, and s is a right triangle. Use the Pythagorean theorem to write s^2 in terms of x and y.

$$s^2 = x^2 + y^2$$

Now, use the chain rule to differentiate s with respect to t.

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} + \frac{ds}{dy} \cdot \frac{dy}{dt}$$

$$2s\frac{ds}{dt} = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$

Solve this equation for $\frac{ds}{dt}$ and write s in terms of x and y.

$$\frac{ds}{dt} = \frac{x\left(\frac{dx}{dt}\right) + y\left(\frac{dy}{dt}\right)}{\sqrt{x^2 + y^2}}$$

Substitute the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into $\frac{ds}{dt}$.

$$\frac{ds}{dt} = \frac{9x + 2y}{\sqrt{x^2 + y^2}}$$

To evaluate $\frac{ds}{dt}$ at $t = 3$ seconds, find x and y, or the positions of the bicycle and the balloon, 3 seconds after the bicycle passes under the balloon.

Calculate x, or the distance the bicycle has traveled in these 3 seconds.

$$\begin{aligned} x &= (9 \text{ ft / sec})(3 \text{ sec}) \\ &= 27 \text{ feet} \end{aligned}$$

Now find y, or the height of the balloon above the ground, 3 seconds after the bicycle passes below it.

$$\begin{aligned} y &= 30 + (2 \text{ ft / sec}) \cdot (3 \text{ sec}) \\ &= 36 \text{ feet} \end{aligned}$$

Substitute these values into the expression for $\frac{ds}{dt}$ and evaluate.

$$\frac{ds}{dt} = \frac{9(27) + 2(36)}{\sqrt{27^2 + 36^2}}$$

$$\frac{ds}{dt} = 7 \text{ ft / sec}$$

Therefore, 3 seconds after the bicycle passes under the balloon, the distance between the bicycle and the balloon is increasing by 7 ft / sec.