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1.4.2 B-spline curve

A B-spline curve is defined as a linear combination of control points $\mathbf{p}_i$ and B-spline basis functions $N_{i,k}(t)$ given by

$$\mathbf{r}(t) = \sum_{i=0}^{n} \mathbf{p}_i N_{i,k}(t), \qquad n \geq k-1, \qquad t \in [t_{k-1}, t_{n+1}] \ . \qquad (1.62)$$

In this context the control points are called *de Boor points*. The basis function $N_{i,k}(t)$ is defined on a *knot vector*

$$\mathbf{T} = (t_0, t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+k}) \ , \qquad (1.63)$$

where there are $n+k+1$ elements, i.e. the number of control points $n+1$ plus the order of the curve k . Each knot *span* $t_i \leq t \leq t_{i+1}$ is mapped onto a polynomial curve between two successive joints $\mathbf{r}(t_i)$ and $\mathbf{r}(t_{i+1})$. Normalization of the knot vector, so it covers the interval [0,1], is helpful in improving numerical accuracy in floating point arithmetic computation due to the higher density of floating point numbers in this interval [133,300].
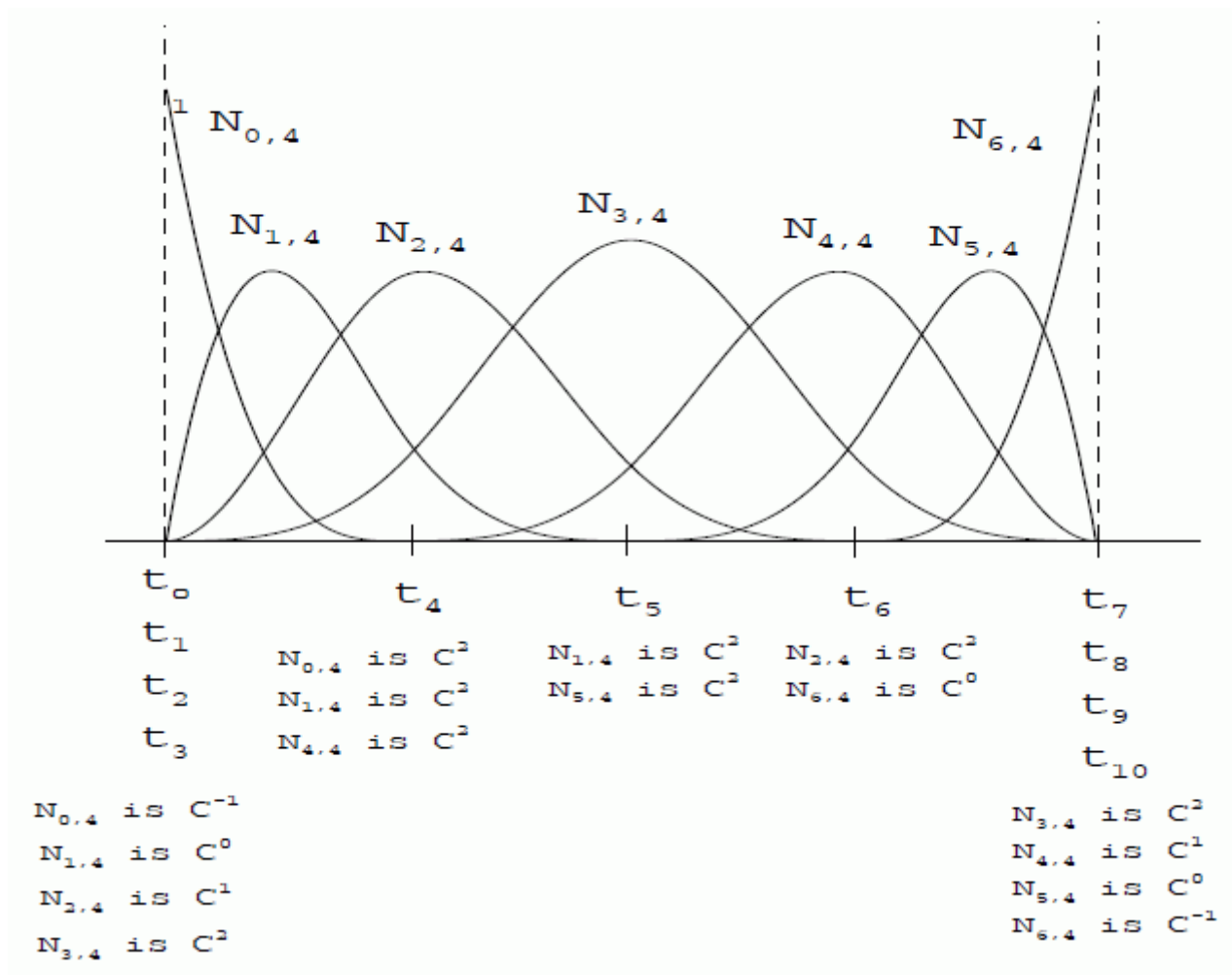
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Next: 1.4.3 Algorithms for B-spline **Up:** 1.4 B-spline curves and **Previous:** 1.4.1 B-splines **Contents** **Index**

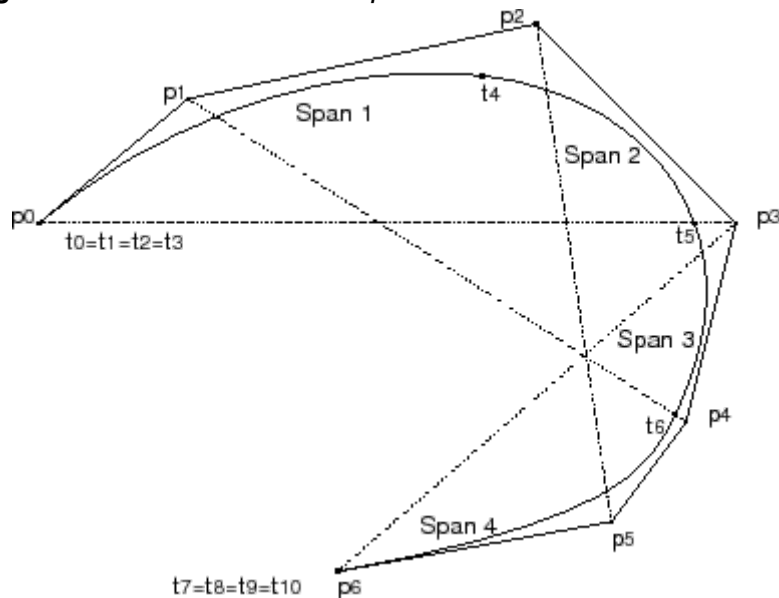**Figure 1.10:** *An order four B-spline basis functions with uniform knot vector*

Figure 1.11: *A clamped cubic B-spline curve*

A B-spline curve has the following properties:

- *Geometry invariance property*: Partition of unity property of the B-spline assures the invariance of the shape of the B-spline curve under translation and rotation.
- *End points geometric property*:
  - Unlike Bézier curves, B-spline curves do not in general pass through the two end control points. Increasing the multiplicity of a knot reduces the continuity of the curve at that knot. Specifically, the curve is $(k - p - 1)$

    times continuously differentiable at a knot with multiplicity p $(\leq k)$, and

thus has $C^{(k-p-1)}$ continuity. Therefore, the control polygon will coincide with the curve at a knot of multiplicity $k-1$, and a knot with multiplicity

k indicates C^{-1} continuity, or a discontinuous curve. Repeating the knots at the end k times will force the endpoints to coincide with the control polygon. Thus the first and the last control points of a curve with a knot vector described by

$$\mathbf{T} = (\underbrace{t_0, t_1, \ldots, t_{k-1}}_{k \text{ equal knots}}, \underbrace{t_k, t_{k+1}, \ldots, t_{n-1}, t_n}_{n\text{-}k+1 \text{ internal knots}}, \underbrace{t_{n+1}, \ldots, t_{n+k}}_{k \text{ equal knots}}) , \quad (1.64)$$

coincide with the endpoints of the curve. Such knot vectors and curves are known as *clamped* [314]. In other words, *clamped/unclamped* refers to whether both ends of the knot vector have multiplicity equal to k or not. Figure 1.10 shows cubic B-spline basis functions defined on a knot vector $\mathbf{T} = (t_0 = t_1 = t_2 = t_3, \ t_4, \ t_5, \ t_6, \ t_7 = t_8 = t_9 = t_{10})$. A

clamped cubic B-spline curve based on this knot vector is illustrated in Fig. 1.11 with its control polygon.
  - B-spline curves with a knot vector (1.64) are tangent to the control polygon at their endpoints. This is derived from the fact that the first derivative of a B-spline curve is given by [175]

$$\dot{\mathbf{r}}(t) = \sum_{i=0}^{n-1}(k-1)\left(\frac{\mathbf{P}_{i+1} - \mathbf{P}_i}{t_{i+k} - t_{i+1}}\right) N_{i,k-1}(t) , \quad (1.65)$$

where the knot vector is obtained by dropping the first and last knots from (1.64), i.e.

$$\mathbf{T}' = (\underbrace{t_1, \ldots, t_{k-1}}_{k\text{-}1 \text{ equal knots}}, \underbrace{t_k, t_{k+1}, \ldots, t_{n-1}, t_n}_{n\text{-}k+1 \text{ internal knots}}, \underbrace{t_{n+1}, \ldots, t_{n+k-1}}_{k\text{-}1 \text{ equal knots}}) , \quad (1.66)$$

and

$$\dot{\mathbf{r}}(0) = \frac{k-1}{t_k - t_1}(\mathbf{P}_1 - \mathbf{P}_0) , \quad (1.67)$$

$$\dot{\mathbf{r}}(1) = \frac{k-1}{t_{n+k-1} - t_n}(\mathbf{P}_n - \mathbf{P}_{n-1}) . \quad (1.68)$$

- *Convex hull property*: The convex hull property for B-splines applies locally, so that a span lies within the convex hull of the control points that affect it. This provides a tighter convex hull property than that of a Bézier curve, as can be seen in Fig. 1.11. The i -th span of the cubic B-spline curve in Fig. 1.11 lies within the convex hull formed by control points $\mathbf{P}_{i-1}$, $\mathbf{P}_i$, $\mathbf{P}_{i+1}$, $\mathbf{P}_{i+2}$. In

other words, a B-spline curve must lie within the union of all such convex hulls formed by k successive control points [130].

- *Local support property*: A single span of a B-spline curve is controlled only by k control points, and any control point affects k spans. Specifically, changing $\mathbf{P}_i$

affects the curve in the parameter range $t_i < t < t_{i+k}$ and the curve at a point

t where $t_r < t < t_{r+1}$ is determined completely by the control points

$\mathbf{P}_{r-(k-1)}, \ldots, \mathbf{P}_r$ as shown in Fig. 1.11.

- *Variation diminishing property*:
  - 2-D: The number of intersections of a straight line with a planar B-spline curve is no greater than the number of intersections of the line with the control polygon. A line intersecting the convex hull of a planar B-spline curve may intersect the curve transversally, be tangent to the curve, or not intersect the curve at all. It may not, however, intersect the curve more times than it intersects the control polygon.
  - 3-D: The same relation holds true for a plane with a 3-D space B-spline curve.
- *B-spline to Bézier property*: From the discussion of end points geometric property, it can be seen that a Bézier curve of order k (degree $k-1$) is a B-spline curve with no internal knots and the end knots repeated k times. The knot vector is thus

$$\mathbf{T} = (\underbrace{t_0, t_1, \ldots, t_{k-1}}_{k \text{ equal knots}}, \underbrace{t_{n+1}, \ldots, t_{n+k}}_{k \text{ equal knots}}) \,, \qquad (1.69)$$
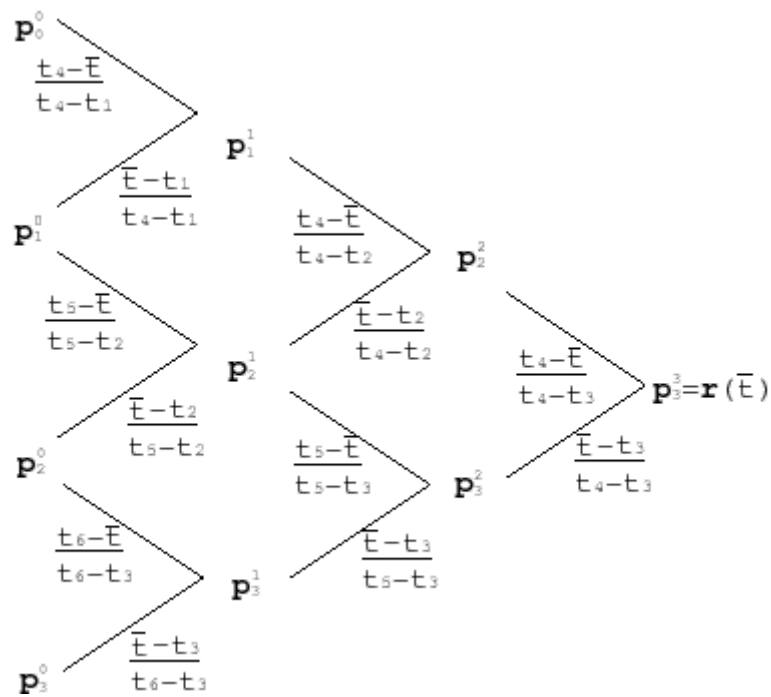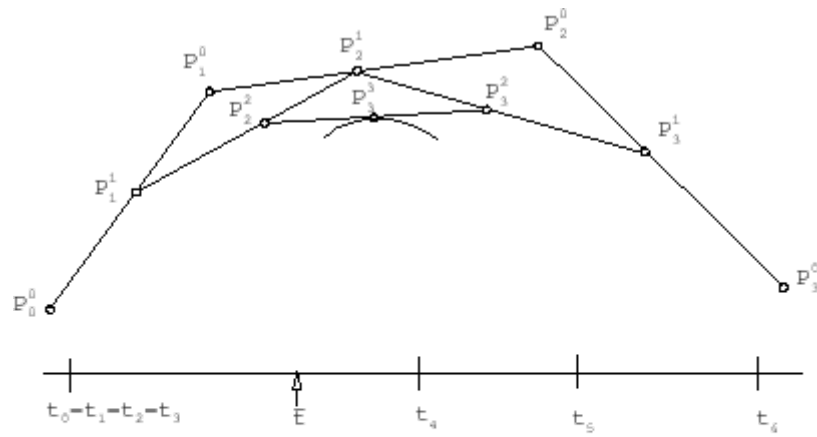
where $n + k + 1 = 2k$ or $n = k - 1$.

Figure 1.12: *The de Boor algorithm*

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