YData: An Introduction to Data Science

Lecture 26: The Normal Curve

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Credit: data8.org

Announcements

Questions for This Week

- How can we quantify natural concepts like "center" and "variability"?
- Why do many of the empirical distributions that we generate come out bell shaped?
- How is sample size related to the accuracy of an estimate?

Standard Deviation (Review)

How Far from the Average?

- Standard deviation (SD) measures roughly how far the data are from their average
- SD $=$ $\underset{5}{\text{root}}\quad \underset{4}{\text{mean}}\quad \underset{3}{\text{square}}\quad \underset{2}{\text{of deviations}}\quad \underset{1}{\text{from average}}$
- SD has the same units as the data

Why Use the SD?

There are two main reasons.

- **The first reason**:
  No matter what the shape of the distribution,
  the bulk of the data are in the range "average $\pm$ a few SDs"

- **The second reason**:
  Coming up later this lecture.

How Big are Most of the Values?

No matter what the shape of the distribution, the bulk of the data are in the range "average $\pm$ a few SDs"

Chebyshev's Inequality
No matter what the shape of the distribution, the proportion of values in the range "average $\pm$ z SDs" is

$$\text{at least } 1 - 1/z^2$$

Chebyshev's Bounds

Range	Proportion
average ± 2 SDs	at least 1 - 1/4 (75%)
average ± 3 SDs	at least 1 - 1/9 (88.888...%)
average ± 4 SDs	at least 1 - 1/16 (93.75%)
average ± 5 SDs	at least 1 - 1/25 (96%)

No matter what the distribution looks like

(DEMO)

Standard Units

Standard Units

- How many SDs above average?

- z = (value - average)/SD

Negative z:	value below average
Positive z:	value above average
z = 0:	value equal to average

- When values are in standard units: average = 0, SD = 1

- Chebyshev: At least 96% of the values of z are between -5 and 5

(DEMO)

Discussion Question

Find whole numbers that are close to:
(a) the average age (b) the SD of the ages

Age in Years	Age in Standard Units
27	-0.0392546
33	0.992496
28	0.132704
23	-0.727088
25	-0.383171
33	0.992496
23	-0.727088
25	-0.383171
30	0.476621
27	-0.0392546

... (1164 rows omitted)

(DEMO)

The SD and the Histogram

- Usually, it's not easy to estimate the SD by looking at a histogram.

- But if the histogram has a bell shape, then you can.

The SD and Bell-Shaped Curves

If a histogram is bell-shaped, then

- the average is at the center
- the SD is the distance between the average and the points of inflection on either side

(DEMO)

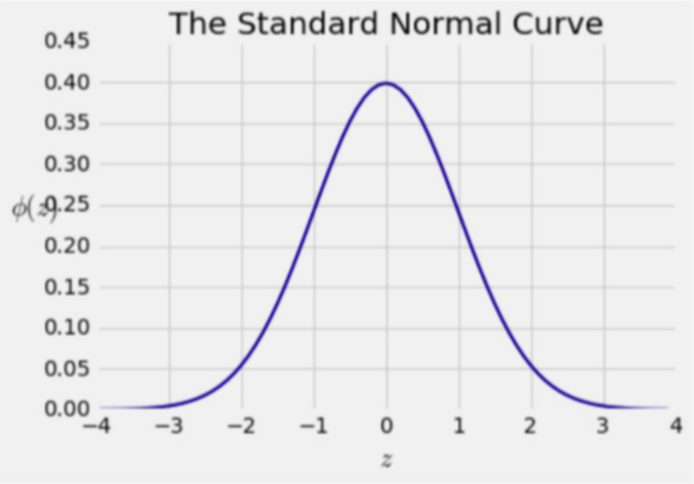
The Normal Distribution

The Standard Normal Curve

A beautiful formula that we won't use at all:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad -\infty < z < \infty$$

Bell Curve


The Standard Normal Curve
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$\phi(z)$
−4
−3
−2
−1
0
1
2
3
4
z

Normal Proportions

How Big are Most of the Values?

No matter what the shape of the distribution,

- the bulk of the data are in the range
  "average $\pm$ a few SDs"

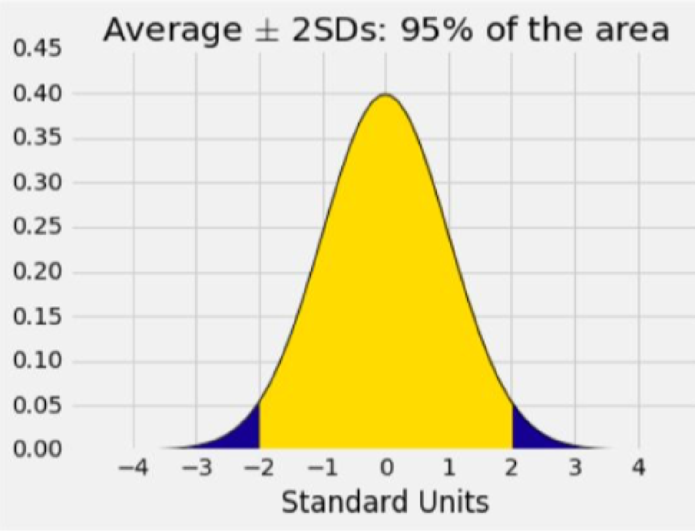
If a histogram is bell-shaped, then

- Almost all of the data are in the range
  "average $\pm$ 3 SDs"

Bounds and Normal Approximations

Percent in Range	All Distributions	Normal Distribution
average $\pm$ 1 SD	at least 0%	about 68%
average $\pm$ 2 SDs	at least 75%	about 95%
average $\pm$ 3 SDs	at least 88.888...%	about 99.73%

A "Central" Area



Central Limit Theorem

Second Reason for Using the SD

If the sample is

- large, and
- drawn at random with replacement,

Then, *regardless of the distribution of the population*,

the probability distribution of the sample sum (or of the sample average) is roughly normal

(DEMO)