

S&DS 265 / 565

Introductory Machine Learning

Stochastic Gradient Descent and Bias-Variance Tradeoffs

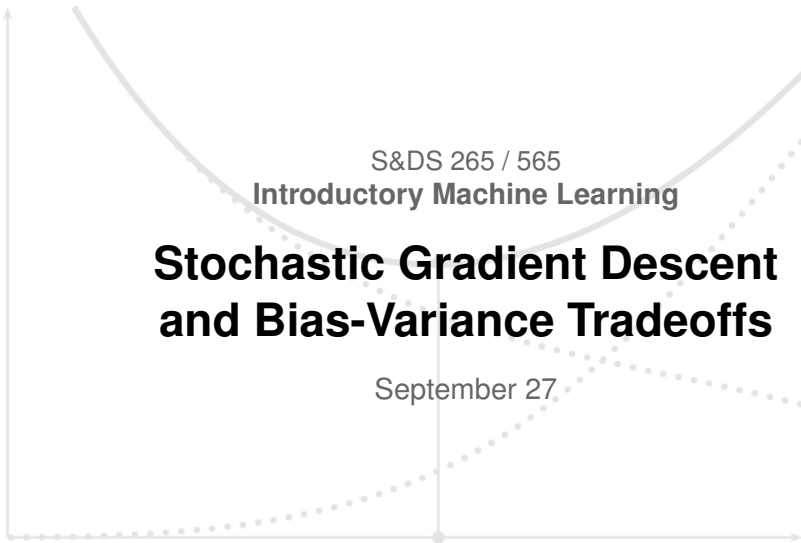
September 27

Risk

Bias squared

Variance

Yale



Goings on

- Assignment 1 due Thursday
- Assignment 2 posted Thursday
- Quiz 2 scores out: Average 84%, great!
- Quiz 3 posted next week
- Midterm: October 18 (in class)

Outline for today

- Stochastic gradient descent (redux)
- Regularization
- Jupyter notebook example
- Bias-variance tradeoffs

SGD idea

- For each parameter β_j , see what happens to the loss if that parameter is increased a little bit.
- If the loss goes down (up), then increase (decrease) β_j proportionately
- Do this simultaneously for all of the parameters
- Rinse and repeat

Stochastic gradient descent

Initialize all parameters to zero: $\beta_j = 0, j = 1, \dots, p$.

Read through the data one record at a time, and update the model.

- 1 Read data item x
- 2 Make a prediction $\hat{y}(x)$
- 3 Observe the true response/label y
- 4 Update the parameters β so \hat{y} is closer to y

Stochastic gradient descent

Suppose we are doing *linear regression*. We initialize all parameters to zero: $\beta_j = 0, j = 1, \dots, p$.

We read through the data one record at a time, and update the model.

- 1 Read data item x
- 2 Make a prediction $\hat{y}(x) = \sum_{j=1}^p \beta_j x_j$
- 3 Observe the true response/label y
- 4 Update the parameters β so \hat{y} is closer to y

SGD idea

Change β_j by a little bit:

$$\beta_j \rightarrow \beta_j + \varepsilon$$

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What happens to the squared error?

$$\begin{aligned}(y - \hat{y})^2 &\rightarrow (y - \hat{y} - \varepsilon x_j)^2 \\ &\approx (y - \hat{y})^2 + \underbrace{-2(y - \hat{y})x_j}_{\text{derivative of loss}} \varepsilon\end{aligned}$$

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Use adjustment

$$\begin{aligned}\beta_j &\rightarrow \beta_j + \overbrace{-\eta \cdot \text{derivative of loss}}^{\varepsilon} \\ &= \beta_j + \eta \cdot 2(y - \hat{y})x_j\end{aligned}$$

SGD idea

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Use adjustment

$$\begin{aligned}\beta_j &\rightarrow \beta_j + \overbrace{-\eta \cdot \text{derivative of loss}}^{\varepsilon} \\ &= \beta_j + \eta \cdot 2(y - \hat{y})x_j\end{aligned}$$

Squared error then decreases:

$$(y - \hat{y})^2 \approx (y - \hat{y})^2 - \eta \cdot \text{derivative of loss} \underline{\text{squared}}$$

SGD for general loss

Suppose $L(y, \beta^T x)$ is the loss for an input (x, y) , e.g., $(y - \beta^T x)^2$

SGD update:

$$\beta_j \leftarrow \beta_j - \eta \frac{\partial L(y, \beta^T x)}{\partial \beta_j}$$

$$\beta \leftarrow \beta - \eta \nabla_{\beta} L(y, \beta^T x) \quad (\text{vector notation})$$

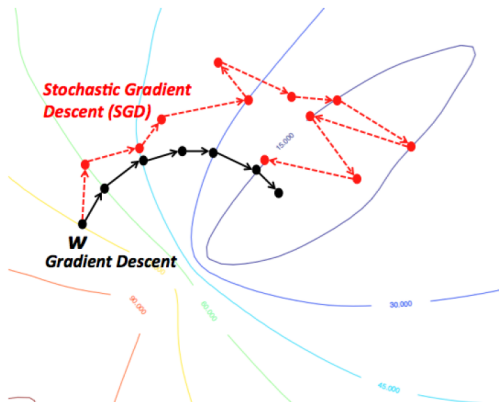
“Batch” gradient descent uses the entire training set in each step of gradient descent.

Stochastic gradient descent computes a quick approximation to this gradient, using only a single or a small “mini-batch” of data points

Batch vs. stochastic gradient descent

- The average derivative over a mini-batch can be thought of as a noisy version of the average derivative over the entire data set
- (Which can in turn be thought of as a sample estimate of a population)
- The stochastic gradient is computed more cheaply, and updating the parameters makes progress more quickly

Batch vs. stochastic gradient descent



<https://wikidocs.net/3413>

SGD for logistic regression

SGD Update:

$$\beta_j \longleftarrow \beta_j + \eta(y - p(x))x_j$$

$$\beta_j x_j \longleftarrow \beta_j x_j + \eta(y - p(x))x_j^2$$

$$p(x) = \frac{1}{1 + \exp(-\beta^T x)}$$

Case checking:

- Suppose $y = 1$ and probability $p(x)$ is high?

SGD for logistic regression

SGD Update:

$$\beta_j \longleftarrow \beta_j + \eta(y - p(x))x_j$$

$$\beta_j x_j \longleftarrow \beta_j x_j + \eta(y - p(x))x_j^2$$

$$p(x) = \frac{1}{1 + \exp(-\beta^T x)}$$

Case checking:

- Suppose $y = 1$ and probability $p(x)$ is high? *small change*
- Suppose $y = 1$ and probability $p(x)$ is small?

SGD for logistic regression

SGD Update:

$$\beta_j \longleftarrow \beta_j + \eta(y - p(x))x_j$$

$$\beta_j x_j \longleftarrow \beta_j x_j + \eta(y - p(x))x_j^2$$

$$p(x) = \frac{1}{1 + \exp(-\beta^T x)}$$

Case checking:

- Suppose $y = 1$ and probability $p(x)$ is high? *small change*
- Suppose $y = 1$ and probability $p(x)$ is small? *big change* \uparrow
- Suppose $y = 0$ and probability $p(x)$ is small?

SGD for logistic regression

SGD Update:

$$\beta_j \longleftarrow \beta_j + \eta(y - p(x))x_j$$

$$\beta_j x_j \longleftarrow \beta_j x_j + \eta(y - p(x))x_j^2$$

$$p(x) = \frac{1}{1 + \exp(-\beta^T x)}$$

Case checking:

- Suppose $y = 1$ and probability $p(x)$ is high? *small change*
- Suppose $y = 1$ and probability $p(x)$ is small? *big change* \uparrow
- Suppose $y = 0$ and probability $p(x)$ is small? *small change*
- Suppose $y = 0$ and probability $p(x)$ is big?

SGD for logistic regression

SGD Update:

$$\beta_j \longleftarrow \beta_j + \eta(y - p(x))x_j$$

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$$p(x) = \frac{1}{1 + \exp(-\beta^T x)}$$

Case checking:

- Suppose $y = 1$ and probability $p(x)$ is high? *small change*
- Suppose $y = 1$ and probability $p(x)$ is small? *big change* \uparrow
- Suppose $y = 0$ and probability $p(x)$ is small? *small change*
- Suppose $y = 0$ and probability $p(x)$ is big? *big change* \downarrow

SGD: choice of learning rate

A conservative choice of learning rate is

$$\eta_t = \frac{1}{t}$$

A more aggressive choice is

$$\eta_t = \frac{1}{\sqrt{t}}$$

In practice: Try learning rates C/\sqrt{t} for different choices of C , and monitor the error

$$\frac{1}{T} \sum_{t=1}^T (Y_t - \hat{Y}_t)^2$$

SGD: Regularization

A “ridge” penalty $\frac{1}{2}\lambda \sum_{j=1}^d \beta_j^2$ is easily handled.

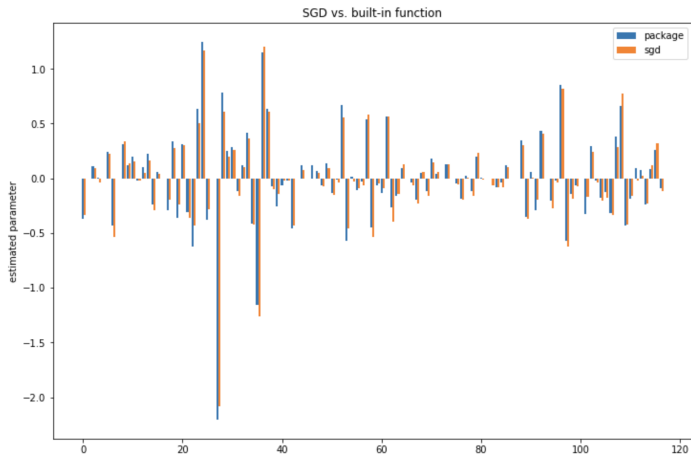
Gradient changes by an additive term $2\lambda\beta_j$. Update becomes

$$\begin{aligned}\beta_j &\longleftarrow \beta_j + \eta\{(y - p(x))x_j - \lambda\beta_j\} \\ &= (1 - \eta\lambda)\beta_j + \eta(y - p(x))x_j\end{aligned}$$

Check that this “does the right thing” whether β_j wants to be large positive or negative.

- *The penalty shrinks β_j toward zero*

Recall from demo



Bias and variance

Bias: How much are we off—on average?

Variance: How variable are we—on average?

Bias and variance

$$\text{Bias: } \theta - \mathbb{E}\hat{\theta}$$

$$\text{Variance: } \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2$$

Bias and variance

Examples of $\theta, \hat{\theta}$:

Estimating height, population, election outcome, ad click rate...

Bias and variance

$$\text{Bias: } \theta - \mathbb{E}\hat{\theta}$$

$$\text{Variance: } \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2$$

Bias and variance

$$\text{Bias: } \theta - \mathbb{E}\hat{\theta}$$

$$\text{Variance: } \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2$$

- $\hat{\theta}$ is an estimate from a sample
- \mathbb{E} is the expectation (average) with respect to the sample
- So $\mathbb{E}\hat{\theta}$ is the average estimate
- We can only directly compute $\hat{\theta}$ for the sample we have
- *We don't know θ*

Bias and variance

Bias and variance are two sides of the same coin: As squared bias goes up, variance goes down

Bias-variance tradeoff

$$\text{Risk} = \text{Bias}^2 + \text{Variance}$$

Bias-variance tradeoff

$$\mathbb{E}(\theta - \hat{\theta})^2 = \text{Bias}(\hat{\theta})^2 + \text{Variance}(\theta)$$

Bias-variance tradeoff

$$\mathbb{E}(\theta - \hat{\theta})^2 = (\theta - \mathbb{E}\hat{\theta})^2 + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2$$

Bias-variance tradeoff

Proof:

$$\mathbb{E}(\theta - \hat{\theta})^2 = \mathbb{E}(\theta - \mathbb{E}\hat{\theta} + \mathbb{E}\hat{\theta} - \hat{\theta})^2$$

Bias-variance tradeoff

Proof:

$$\begin{aligned}\mathbb{E}(\theta - \hat{\theta})^2 &= \mathbb{E}(\theta - \mathbb{E}\hat{\theta} + \mathbb{E}\hat{\theta} - \hat{\theta})^2 \\ &= \mathbb{E}(\theta - \mathbb{E}\hat{\theta})^2 - 2\mathbb{E}\left\{(\theta - \mathbb{E}\hat{\theta})(\hat{\theta} - \mathbb{E}\hat{\theta})\right\} + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2\end{aligned}$$

Bias-variance tradeoff

Proof:

$$\begin{aligned}\mathbb{E}(\theta - \hat{\theta})^2 &= \mathbb{E}(\theta - \mathbb{E}\hat{\theta} + \mathbb{E}\hat{\theta} - \hat{\theta})^2 \\ &= \mathbb{E}(\theta - \mathbb{E}\hat{\theta})^2 - 2\mathbb{E}\left\{(\theta - \mathbb{E}\hat{\theta})(\hat{\theta} - \mathbb{E}\hat{\theta})\right\} + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2 \\ &= \mathbb{E}(\theta - \mathbb{E}\hat{\theta})^2 - 2(\theta - \mathbb{E}\hat{\theta})\mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta}) + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2\end{aligned}$$

Bias-variance tradeoff

Proof:

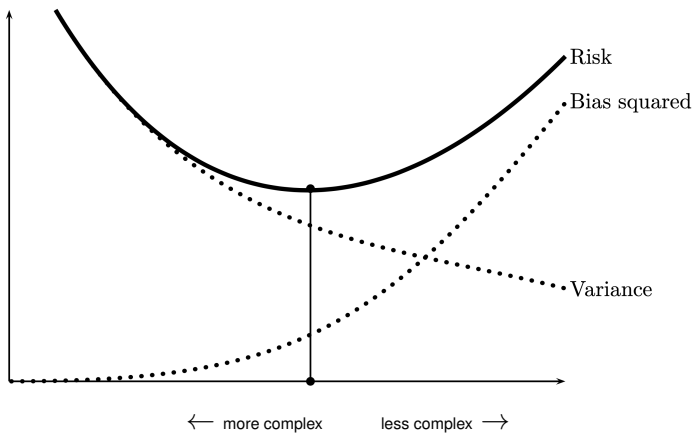
$$\begin{aligned}\mathbb{E}(\theta - \hat{\theta})^2 &= \mathbb{E}(\theta - \mathbb{E}\hat{\theta} + \mathbb{E}\hat{\theta} - \hat{\theta})^2 \\&= \mathbb{E}(\theta - \mathbb{E}\hat{\theta})^2 - 2\mathbb{E}\left\{(\theta - \mathbb{E}\hat{\theta})(\hat{\theta} - \mathbb{E}\hat{\theta})\right\} + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2 \\&= \mathbb{E}(\theta - \mathbb{E}\hat{\theta})^2 - 2(\theta - \mathbb{E}\hat{\theta})\mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta}) + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2 \\&= \mathbb{E}(\theta - \mathbb{E}\hat{\theta})^2 + \mathbb{E}(\hat{\theta} - \mathbb{E}\hat{\theta})^2\end{aligned}$$

Bias-variance tradeoff

Proof:

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Bias-variance tradeoff



Example: Regularization

Suppose that $\mathbb{E}(Y) = \theta^*$ and we estimate

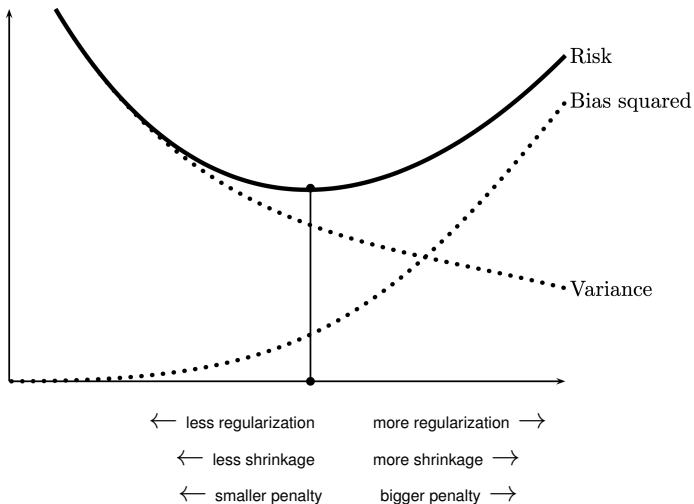
$$\hat{\theta} = \arg \min_{\theta} (Y - \theta)^2 + \lambda \theta^2$$

Then $\hat{\theta} = \frac{Y}{1+\lambda}$. What are the squared bias and variance?

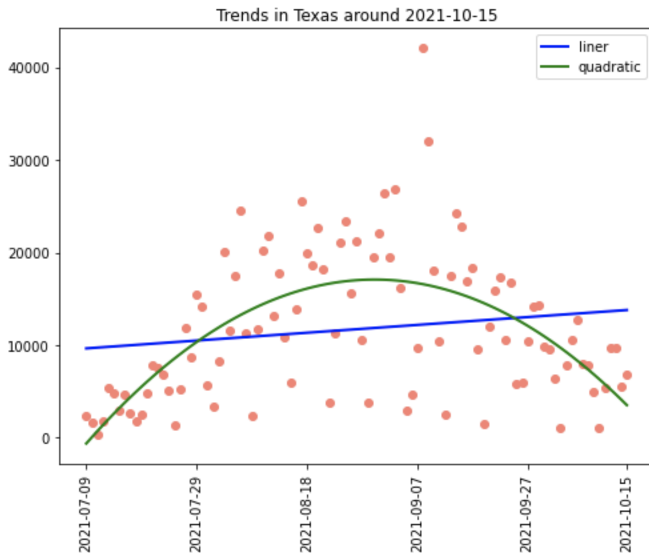
$$\text{Bias}^2 = \theta^{*2} \left(\frac{\lambda}{1+\lambda} \right)^2$$

$$\text{Variance} = \left(\frac{1}{1+\lambda} \right)^2 \text{Variance}(Y)$$

Bias-variance tradeoff



Let's go to the notebook



What did we learn today?

- In SGD, a parameter is updated according to how much the loss changes when that parameter is changed by a little bit
- Mean squared error splits into squared bias plus variance
- As model complexity increases, squared bias decreases while variance increases