

S&DS 365 / 665  
Intermediate Machine Learning

# Lasso, Smoothing and Kernels

Monday, January 31

# Reminders

- OH posted to Canvas / EdD
- Reminder: Slides updated often; please refresh
- Assignment 1 next week
- Any questions?

# Topics for today

- Recap of lasso
- A simple algorithm for the lasso
- Nonparametric regression
- Smoothing methods
- Bias, variance, and curse of dimensionality

# Recall from last time

- For low dimensional (linear) prediction, we can use least squares.
- For high dimensional linear regression, we face a bias-variance tradeoff: omitting too many variables causes bias while including too many variables causes high variance.
- The key is to select a good subset of variables.
- The *lasso* ( $\ell_1$ -regularized least squares) is a fast way to select variables.
- If there are good, sparse linear predictors, lasso will work well.

# Regression

Given the training data  $\mathcal{D} = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$  we want to construct  $\hat{m}$  to make

$$\text{prediction risk} = R(\hat{m}) = \mathbb{E}(Y - \hat{m}(X))^2$$

small. Here,  $(X, Y)$  are a new pair.

**Key fact:** Bias-variance decomposition:

$$R(\hat{m}) = \int \text{bias}^2(x)p(x)dx + \int \text{var}(x)p(x) + \sigma^2$$

where

$$\text{bias}(x) = \mathbb{E}(\hat{m}(x)) - m(x)$$

$$\text{var}(x) = \text{Variance}(\hat{m}(x))$$

$$\sigma^2 = \mathbb{E}(Y - m(X))^2$$

# Bias-Variance Tradeoff

More generally, we need to tradeoff **approximation error** against **estimation error**:

$$R(\hat{f}) - R^* = \underbrace{R(\hat{f}) - \inf_{f \in \mathcal{F}} R(f)}_{\text{estimation error}} + \underbrace{\inf_{f \in \mathcal{F}} R(f) - R^*}_{\text{approximation error}}$$

where  $R^*$  is the smallest possible risk and  $\inf_{f \in \mathcal{F}} R(f)$  is smallest possible risk using class of estimators  $\mathcal{F}$ .

- Approximation error is a generalization of squared bias
- Estimation error is a generalization of variance
- Decomposition holds more generally, even for classification

# Sparse Linear Regression

Ridge regression does not take advantage of **sparsity**.

Maybe only a small number of covariates are important predictors.  
How do we find them?

We could fit many submodels (with a small number of covariates) and choose the best one. This is called *model selection*.

The inaccuracy is

$$\text{prediction error} = \text{bias}^2 + \text{variance} + \sigma^2$$

Now the **bias** is the errors due to omitting important variables, and the **variance** is the error due to having to estimate many parameters.

# Sparsity Meets Convexity

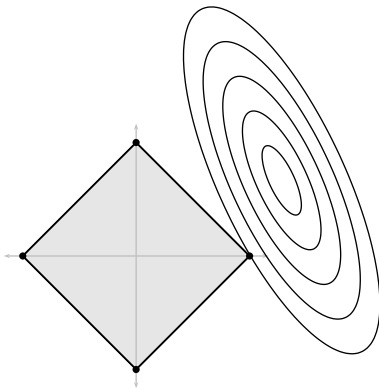
## Lasso regression

$$\hat{\beta} = \arg \min_{\beta} \frac{1}{2n} \sum_{i=1}^n (Y_i - \beta^T X_i)^2 + \lambda \|\beta\|_1$$

where  $\|\beta\|_1 = \sum_j |\beta_j|$ .

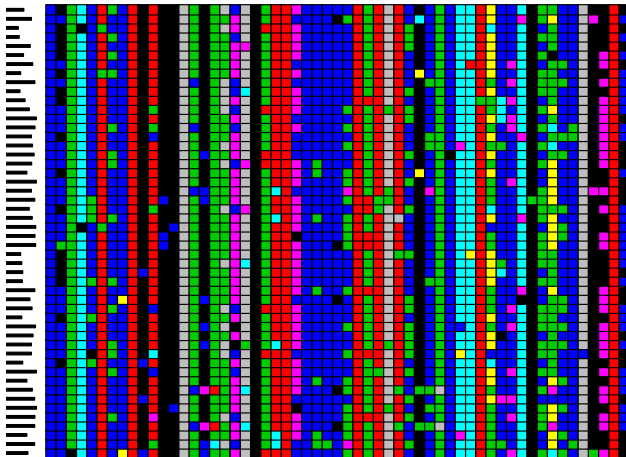


# Sparsity: How corners create sparse estimators

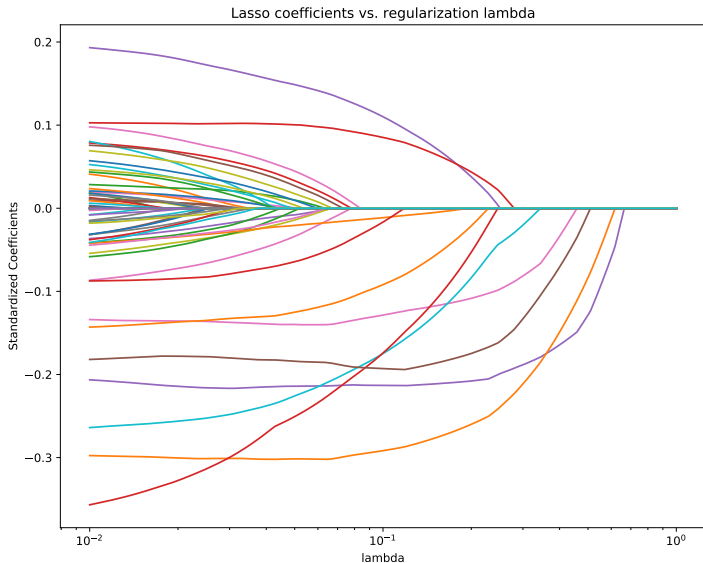


# The lasso: HIV example

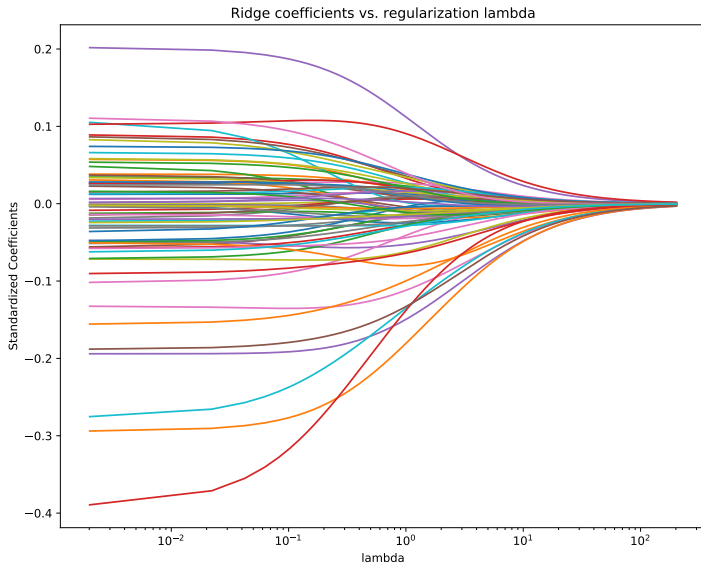
- $Y$  is resistance to HIV drug.
- $X_j$  = amino acid in position  $j$  of the virus.
- $p = 99$ ,  $n \approx 100$ .



# The lasso: HIV example



# Contrast with ridge regression



# The lasso

- $\hat{\beta}(\lambda)$  is called the **lasso** estimator. Selected set of variables is

$$\hat{S}(\lambda) = \left\{ j : \hat{\beta}_j(\lambda) \neq 0 \right\}.$$

- After you find  $\hat{S}(\lambda)$ , you should re-fit the model by doing least squares on the sub-model  $\hat{S}(\lambda)$ .

# Selecting $\lambda$

To choose  $\lambda$  by risk estimation:

Re-fit the model with the non-zero coefficients. Then apply leave-one-out cross-validation:

$$\hat{R}(\lambda) = \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{Y}_{(i)})^2 = \frac{1}{n} \sum_{i=1}^n \frac{(Y_i - \hat{Y}_i)^2}{(1 - H_{ii})^2} \approx \frac{1}{n} \frac{RSS}{(1 - \frac{s}{n})^2}$$

where  $RSS$  is residual sum of squares and  $H$  is the hat matrix and  $s = \|\hat{\beta}\|_0 = \#\{j : \hat{\beta}_j \neq 0\}$ .

Choose  $\hat{\lambda}$  to minimize  $\hat{R}(\lambda)$ .

# The lasso

The complete steps are:

- 1 Find  $\hat{\beta}(\lambda)$  and  $\hat{S}(\lambda)$  for each  $\lambda$ .
- 2 Choose  $\hat{\lambda}$  to minimize estimated risk.
- 3 Let  $\hat{S}$  be the selected variables.
- 4 Let  $\hat{\beta}$  be the least squares estimator using only  $\hat{S}$ .
- 5 Prediction:  $\hat{Y} = X^T \hat{\beta}$ .

# An algorithm for the lasso: Derived in steps

We'll derive a simple algorithm for computing the lasso solution in steps.

I'll do the first step in detail. The next steps only require simple calculations that I'll leave to you.



# An algorithm for the lasso: Step 1

First consider minimizing

$$\frac{1}{2}(y - \beta)^2 + \lambda|\beta|$$

where  $y$  is a single number.

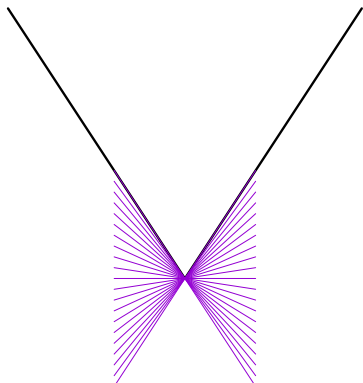
Taking the derivative and setting to zero, we get

$$\beta - y + \lambda v = 0$$

where

$$v \begin{cases} = \text{sign}(\beta) & \text{if } |\beta| > 0 \\ \in [-1, 1] & \text{if } \beta = 0. \end{cases}$$

## Subdifferential for $|\cdot|$



The set of vectors  $v$  pass through the tip at 0 and have slope between -1 and 1.

# An algorithm for the lasso: Step 1

Solution can be written as

$$\hat{\beta} = \begin{cases} y - \lambda & \text{if } \beta > 0 \\ y + \lambda & \text{if } \beta < 0 \\ y - \lambda \left(\frac{y}{\lambda}\right) & \text{if } \beta = 0. \end{cases}$$

Equivalently:

$$\hat{\beta} = \begin{cases} y - \lambda & \text{if } y > \lambda \\ y + \lambda & \text{if } y < -\lambda \\ 0 & \text{if } |y| \leq \lambda. \end{cases}$$

# An algorithm for the lasso: Step 1

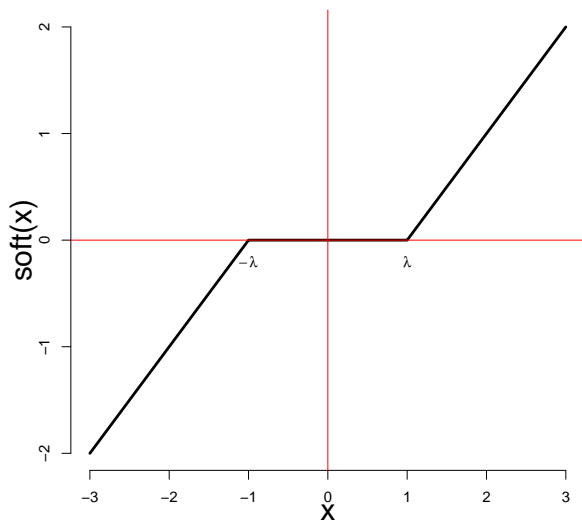
Equivalently:

$$\hat{\beta} = \begin{cases} y - \lambda & \text{if } y > \lambda \\ y + \lambda & \text{if } y < -\lambda \\ 0 & \text{if } |y| \leq \lambda. \end{cases}$$

## Soft thresholding

$$\begin{aligned} \hat{\beta} &= \text{Soft}_{\lambda}(y) \\ &\equiv \text{sign}(y) (|y| - \lambda)_+ = \left(1 - \frac{\lambda}{|y|}\right)_+ y \end{aligned}$$

# Soft thresholding



## An algorithm for the lasso: Step 2

Next consider minimizing

$$\frac{1}{2}(y - x\beta)^2 + \lambda|\beta|$$

where  $y$  and  $x$  are a single numbers.

Exercise: Show that

$$\hat{\beta} = \text{Soft}_{\frac{\lambda}{x^2}} \left( \frac{xy}{x^2} \right)$$

## An algorithm for the lasso: Step 3

Now consider minimizing

$$\frac{1}{2n} \sum_{i=1}^n (y_i - x_i \beta)^2 + \lambda |\beta|$$

for data  $(x_1, y_1), \dots, (x_n, y_n)$  with  $p = 1$ .

Exercise: Show that

$$\hat{\beta} = \text{Soft}_{\lambda_x} \left( \hat{\beta}_{OLS} \right)$$

where  $\hat{\beta}_{OLS}$  is the least squares estimator and  $\lambda_x = \frac{\lambda}{\frac{1}{n} \sum_{i=1}^n x_i^2}$ .

## An algorithm for the lasso: Step 4

Finally, consider minimizing

$$\frac{1}{2n} \sum_{i=1}^n \left( y_i - \sum_{j=1}^p x_{ij} \beta_j \right)^2 + \lambda |\beta|$$

for data  $(x_1, y_1), \dots, (x_n, y_n)$  with  $p \geq 1$ . Apply previous algorithm to estimate  $\beta_1$ , holding other coefficients fixed.

Exercise: Show that

$$\hat{\beta}_1 = \text{Soft}_{\lambda_1} \left( \hat{\beta}_{OLS} \right)$$

where  $\beta_{OLS}$  is the least squares estimator for data  $(x_{i1}, r_i)$  with  $r_i = y_i - \sum_{j \neq 1} x_{ij} \beta_j$ . and  $\lambda_1 = \frac{\lambda}{\frac{1}{n} \sum_{i=1}^n x_{i1}^2}$ .



# The lasso: Computing $\hat{\beta}$

To minimize  $\frac{1}{2n} \sum_i (y_i - \beta^T x_i)^2 + \lambda \|\beta\|_1$ :

## Lasso by coordinate descent

- Set  $\hat{\beta} = (0, \dots, 0)$ , then iterate until convergence:
- for  $j = 1, \dots, p$ :
  - ▶ set  $r_i = y_i - \sum_{s \neq j} \hat{\beta}_s x_{si}$
  - ▶ Set  $\hat{\beta}_j$  to be least squares fit of  $r_i$ 's on  $x_j$ .
  - ▶  $\hat{\beta}_j \leftarrow \text{Soft}_{\lambda_j}(\hat{\beta}_j)$  where  $\lambda_j = \frac{\lambda}{\frac{1}{n} \sum_i x_{ij}^2}$ .
- Then use least squares  $\hat{\beta}$  on selected subset  $\hat{S}(\lambda)$ .

**Next up**

Nonparameteric regression by smoothing

# Nonparametric Regression

Given  $(X_1, Y_1), \dots, (X_n, Y_n)$  predict  $Y$  from  $X$ .

Assume only that  $Y_i = m(X_i) + \epsilon_i$  where  $m(x)$  is a smooth function of  $x$ .

The most popular methods are *kernel methods*. However, there are two types of kernels:

- 1 Smoothing kernels
- 2 Mercer kernels

Smoothing kernels involve local averaging.  
Mercer kernels involve regularization.

## Smoothing kernel estimator

$$\hat{m}_h(x) = \frac{\sum_{i=1}^n Y_i K_h(X_i, x)}{\sum_{i=1}^n K_h(X_i, x)}$$

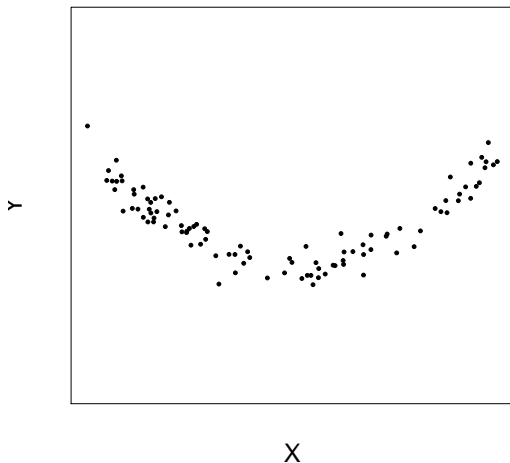
where  $K_h(x, z)$  is a *kernel* such as

$$K_h(x, z) = \exp\left(-\frac{\|x - z\|^2}{2h^2}\right)$$

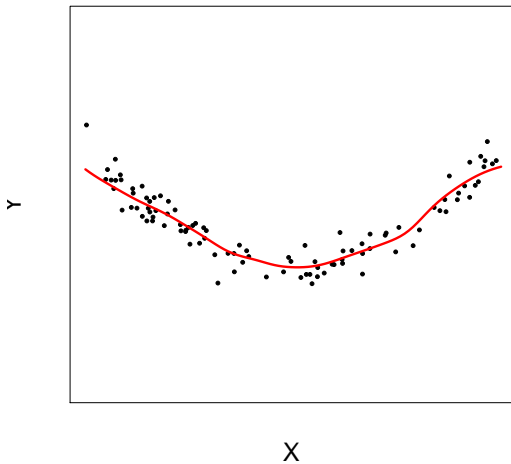
and  $h > 0$  is called the *bandwidth*

- $\hat{m}_h(x)$  is just a local average of the  $Y_i$ 's near  $x$ .
- The bandwidth  $h$  controls the bias-variance tradeoff:  
*Small  $h$  = large variance* while *large  $h$  = large bias*.

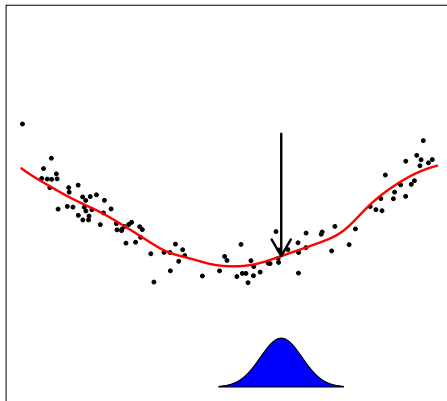
## Example: Some Data – Plot of $Y_i$ versus $X_i$



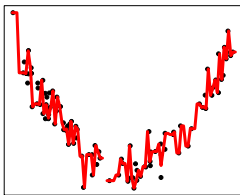
**Example:**  $\hat{m}(x)$



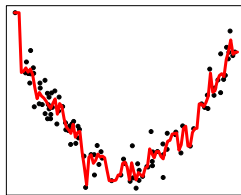
$\hat{m}(x)$  is a local average



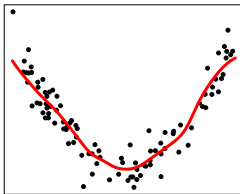
# Effect of the bandwidth $h$



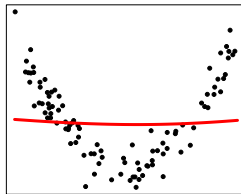
very small bandwidth



small bandwidth



medium bandwidth



large bandwidth



Let's go to the notebook

# Smoothing Kernels

$$\text{Risk} = \mathbb{E}(Y - \hat{m}_h(X))^2 = \text{bias}^2 + \text{variance} + \sigma^2.$$

Under mild assumptions on the distribution of the data:

$$\begin{aligned}\text{bias}^2 &\approx h^4 \\ \text{variance} &\approx \frac{1}{nh^p}\end{aligned}$$

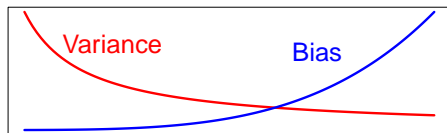
where  $p$  = dimension of  $X$ .

$\sigma^2 = \mathbb{E}(Y - m(X))^2$  is the unavoidable prediction error.

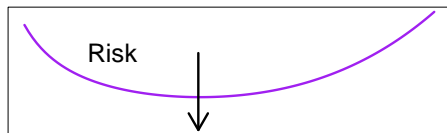
*small  $h$* : low bias, high variance (undersmoothing)

*large  $h$* : high bias, low variance (oversmoothing)

# Risk Versus Bandwidth



$h$



optimal  $h$

# Estimating the Risk: Cross-Validation

To choose  $h$  we need to estimate the risk  $R(h)$ . We can estimate the risk by using *cross-validation*.

- 1 Omit  $(X_i, Y_i)$  to get  $\hat{m}_{h,(i)}$ , then predict:  $\hat{Y}_{(i)} = \hat{m}_{h,(i)}(X_i)$ .
- 2 Repeat this for all observations.
- 3 The cross-validation estimate of risk is:

$$\hat{R}(h) = \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{Y}_{(i)})^2.$$

*Shortcut formula:*

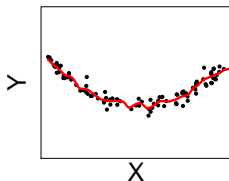
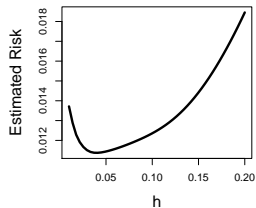
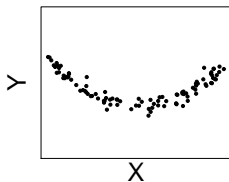
$$\hat{R}(h) = \frac{1}{n} \sum_{i=1}^n \left( \frac{Y_i - \hat{Y}_i}{1 - L_{ii}} \right)^2$$

where  $L_{ij} = K_h(X_i, X_j) / \sum_t K_h(X_i, X_t)$ .

# Summary so far

- 1 Compute  $\hat{m}_h$  for each  $h$ .
- 2 Estimate the risk  $\hat{R}(h)$ .
- 3 Choose bandwidth  $\hat{h}$  to minimize  $\hat{R}(h)$ .
- 4 Let  $\hat{m}(x) = \hat{m}_{\hat{h}}(x)$ .

# Example



# The curse of dimensionality

The method is easily applied in high dimensions — but it doesn't work well.

- The squared bias scales as  $h^4$  and the variance scales as  $\frac{1}{nh^p}$
- As a result, the risk goes down no faster than  $n^{-4/(4+p)}$  (Exercise)
- Suppose we want to make this small, of size  $\epsilon$ —how many data points do we need?

$$n \geq \left(\frac{1}{\epsilon}\right)^{1+p/4}$$

- Grows exponentially with dimension—*the curse of dimensionality*

# Additive models

A compromise is to use *additive models* of the form

$$\hat{m}(x) = \hat{m}_1(x_1) + \hat{m}_2(x_2) + \cdots \hat{m}_p(x_p)$$

- Each function  $\hat{m}_j(x_j)$  is estimated by smoothing, holding the other functions fixed
- Soft thresholding can be used in high dimensions, leading to a generalization of the lasso



# Summary for today

- The lasso can be computed by iterative soft thresholding
- Smoothing methods compute local averages, weighting points by a kernel
- The curse of dimensionality limits use of kernel methods to low dimensions