CSE 450 Assignment 4

13^{rd} November, 2022

Submission Instructions: Deadline is **11:59pm on 11/21/2022**. Late submissions will be penalized, therefore please ensure that you submit (file upload is completed) before the deadline. Additionally, you can download the submitted file to verify if the file was uploaded correctly. **Please TYPE UP YOUR SOLUTIONS and submit a PDF** electronically, via *Canvas*.
Furthermore, please note that the graders will grade 2 out of the 4 questions randomly. Therefore, if the grader decides to check questions 1 and 4,and you haven't answered question 4, you'll lose points for question 4. Hence, please answer all the questions.

1. Suppose that we have a set of activities to schedule among a large number of lecture halls, where any activity can take place in any lecture hall. We wish to schedule all the activities using as few lecture halls as possible. Give an efficient greedy algorithm to determine which activity should use which lecture hall. [**25 points**]

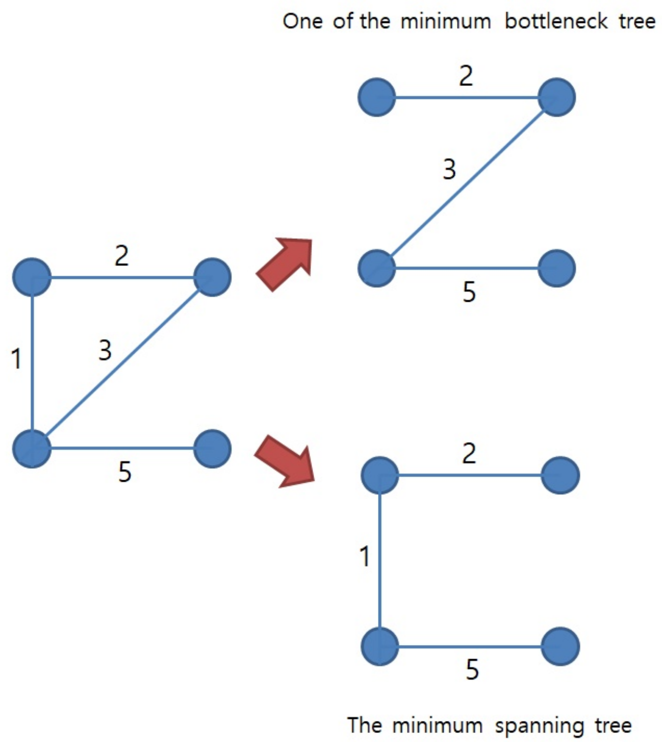
   Solution: Maintain a set of free (but already used) lecture halls F and currently busy lecture halls B. Sort the classes by start time. For each new start time which you encounter, remove a lecture hall from F, schedule the class in that room, and add the lecture hall to B. If F is empty, add a new, unused lecture hall to F. When a class finishes, remove its lecture hall from B and add it to F. This is optimal for following reason, suppose we have just started using the mth lecture hall for the first time. This only happens when ever classroom ever used before is in B. But this means that there are m classes occurring simultaneously, so it is necessary to have distinct lecture halls in use

2. One of the basic motivations behind the Minimum Spanning Tree Problem is the goal of designing a spanning network for a set of nodes with minimum total cost. Here we explore another type of objective: designing a spanning network for which the most expensive edge is as cheap as possible. Specifically, let $G = (V, E)$ be a connected graph with n vertices, m edges, and positive edge costs that you may assume are all distinct. Let $T = (V, E')$ be a spanning tree of G; we define the bottleneck edge of T to be the edge of T with the greatest cost. A spanning tree T of G is

a minimum-bottleneck spanning tree if there is no spanning tree T' of G with a cheaper bottleneck edge. **[25 points]**

(a) Is every minimum-bottleneck tree of G a minimum spanning tree of G? Prove or give a counterexample.

Solution: False. We can see a counterexample as follows.



(b) Is every minimum spanning tree of G a minimum-bottleneck tree of G? Prove or give a counterexample.

Solution: True. Suppose minimum spanning tree T is not minimum bottleneck tree T_b of G. The bottleneck of T is edge (a, b), which will be larger than any edge of T_b. There exits a path from a to b in T_b. Suppose it is (a, p_1, ... p_k, b). Thus, (a, b) $>$ (p_k, b). In the minimum spanning tree T, there exit a path from a to p_k. Now, in the tree T, put edge (p_k, b) in T and take the edge (a, b) out of T. In this way we get a new spanning tree T' which has smaller summation of cost. However, T is the minimum spanning tree. So we get contradiction. Thus, the minimum spanning tree should be minimum bottleneck tree.

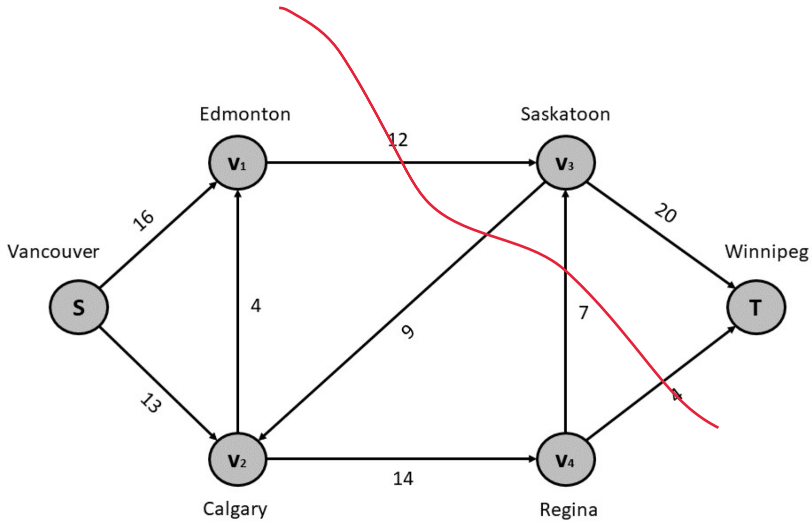


Figure 1: Network for Q3

3. For the network shown in Figure 1, compute the maximum flow from Vancouver to Winnipeg. **Show all your work**. [**25 points**]

   Solution: The maximum flow in this network is 23. The minimum cut $C(S:\bar{S})$ is across $S = \{S, v_1, v_2, v_4\}$ and $\bar{S} = \{v_3, T\}$.

4. Will the maximum flow from the source to the destination node in the Ford-Fulkerson Algorithm will be equal to the capacity of the minimum cut, if the capacity of a cut is defined in the following manner? For each definition, if you agree, then please provide arguments as to why. Else, provide a counter example to show that the definition does not hold: [**25 points**]

   (a) $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) + \sum_{e\in(\overline{S}:S)} C(e)$
   Solution: Consider the following network with three nodes S, A, T and directed edges SA with capacity 10, ST with a capacity 7, AT with capacity 5 and TS with capacity 4. Maximum flow in this network is 12. We can have two cuts here:

   - $S = \{S, A\}, \bar{S} = \{T\}$, Here, $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) + \sum_{e\in(\overline{S}:S)} C(e) = 12 + 4 = 16 \neq 12$, **False**.
   - $S = \{S\}, \bar{S} = \{A, T\}$, Here, $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) + \sum_{e\in(\overline{S}:S)} C(e) = 17 + 4 = 21 \neq 12$, **False**.

   Therefore, maximum flow is not equal to the capacity of the minimum cut for this definition.

(b) $C(S:\overline{S}) = \sum_{e \in (S:\overline{S})} C(e)$
Solution: True. This has been shown to be true in class. Refer class slides.