⑤ Time-Invariance

Given $y(t) = \mathcal{H}\{x(t)\}$

If $\mathcal{H}$ is Time-Invariant, then

$$y(t-t_0) = \mathcal{H}\{x(t-t_0)\} \quad \forall t_0, t \in \mathbb{R}$$

otherwise, $\mathcal{H}$ is time-variant.

Ex

$$y(t) = \mathcal{H}\{x(t)\} = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Is $\mathcal{H}$ Time-Invariant?

- First shift the input and apply the system

① $y_2(t) = \mathcal{H}\{x(t-t_0)\}$

$$= \int_{-\infty}^{t} x(\tau - t_0)\, d\tau \quad =?$$

② $y_1(t) = y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$

$\tau' \triangleq \tau - t_0 \qquad d\tau' = d\tau$

$$y_2(t) = \int_{-\infty}^{t-t_0} x(\tau')\, d\tau'$$

equal

so $\mathcal{H}$ is T.I

Ex

$$y[n] = \mathcal{H}\{x[n]\} = r^n x[n]$$

* $y_1[n] = \mathcal{H}\{x[n-n_0]\}$

$$= r^n x[n-n_0]$$

* $y_2[n] = y[n-n_0] = r^{n-n_0} x[n-n_0]$

$\therefore y_1[n] \neq y_2[n]$ TIME-VARIANT SYSTEM.

⑥ Linearity.

A system, $\mathcal{H}$, is Linear if it satisfies the following two properties:

① Superposition

For any signal $x_1(t)$ and $x_2(t)$ let:

$$y_1(t) = \mathcal{H}\{x_1(t)\}$$
$$y_2(t) = \mathcal{H}\{x_2(t)\}$$
$$y(t) = y_1(t) + y_2(t)$$
$$x(t) = x_1(t) + x_2(t)$$

IF

$y(t) = \mathcal{H}\{x(t)\}$ then the system $\mathcal{H}$ satisfies the principle of superposition.

② Homogeneity

Let $y(t) = \mathcal{H}\{x(t)\}$

If $\alpha\, y(t) = \mathcal{H}\{\alpha x(t)\}$ for $\forall \alpha \in \mathbb{R}$

the $\mathcal{H}$ satisfies the principle of homogeneity.

Generally:

Let $x(t) = \sum_{i=1}^{N} \alpha_i \cdot x_i(t)$

$$y(t) = \mathcal{H}\{x(t)\}$$

If $y(t) = \sum_{i=1}^{N} \alpha_i\, \mathcal{H}\{x_i(t)\}$

then $\mathcal{H}$ is LINEAR

Let $y_i(t) = \mathcal{H}\{x_i(t)\} \quad i = 1, \dots, N$

$$x_i(t) \rightarrow \boxed{\mathcal{H}} \rightarrow y_i(t)$$

$$[\alpha_1 x_1(t) + \alpha_2 x_2(t) + \dots + \alpha_N x_N(t)] \rightarrow \boxed{\mathcal{H}} \rightarrow \alpha_1 y_1(t) + \dots + \alpha_N y_N(t)$$

Ex

$$y(t) = (t+1)^2 x(t) \qquad \text{Linear?}$$

Homogeneity

$$\mathcal{H}\{\alpha x(t)\} = (t+1)^2 \alpha x(t)$$
$$= \alpha y(t)$$

Homogeneity is satisfied.

Superposition Test

Let $\mathcal{H}\{x_1(t)\} = y_1(t)$

$\mathcal{H}\{x_2(t)\} = y_2(t)$

$$\mathcal{H}\{x_1(t) + x_2(t)\} = (t+1)^2 [x_1(t) + x_2(t)]$$
$$= (t+1)^2 x_1(t) + (t+1)^2 x_2(t)$$
$$= y_1(t) + y_2(t)$$

✓ Superposition satisfied.

$\therefore$ $\mathcal{H}$ is LINEAR

Ex

$$y(t) = x(t)\, x(t-1) \qquad \text{Linear?}$$

Homogeneity

$$\mathcal{H}\{\alpha x(t)\} = (\alpha x(t)) \cdot (\alpha x(t-1))$$
$$= \alpha^2 x(t) \cdot x(t-1)$$
$$\alpha y(t) = \alpha x(t) \cdot x(t-1)$$

$\neq$ not equal

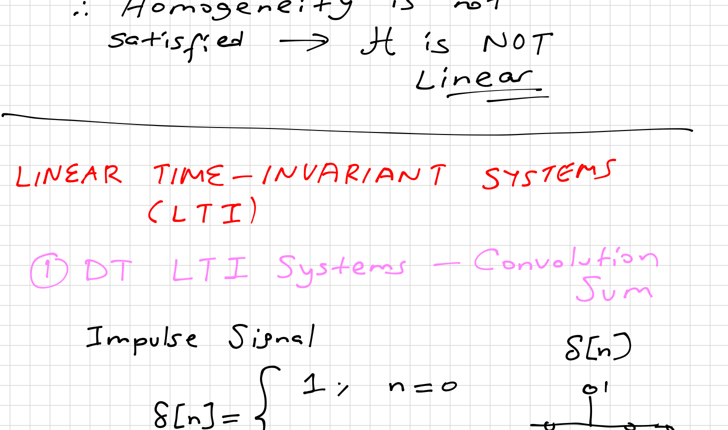
$\therefore$ Homogeneity is not satisfied $\rightarrow$ $\mathcal{H}$ is NOT Linear

LINEAR TIME-INVARIANT SYSTEMS (LTI)

① DT LTI Systems — Convolution Sum

Impulse Signal

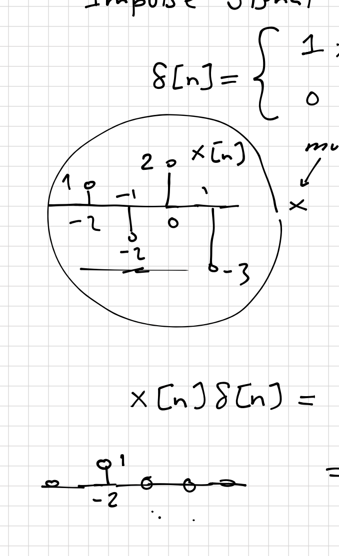
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



multiplication

$$x[n] \times \delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n]\delta[n] = x[0]\,\delta[n]$$



$$= x[-2]\,\delta[n+2]$$
$$= x[-1]\,\delta[n+1]$$
$$= x[0]\,\delta[n]$$
$$= x[1]\,\delta[n-1]$$

$$x[n] = x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

Let the system $\boxed{\mathcal{H}}$ be an LTI system

$$x[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n]$$

$$y[n] = \mathcal{H}\{x[n]\}$$
$$= \mathcal{H}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]\right\}$$

Superposition $\rightarrow$ $= \sum_{k=-\infty}^{+\infty} \mathcal{H}\{x[k]\,\delta[n-k]\}$

Homogeneity $\rightarrow$ $= \sum_{k=-\infty}^{+\infty} x[k]\,\boxed{\mathcal{H}\{\delta[n-k]\}}$

$$h[n-k] \triangleq \mathcal{H}\{\delta[n-k]\}$$

Because of Time-Invariance

$h[k] = \mathcal{H}\{\delta[n]\}$ $\rightarrow$ IMPULSE Response

$$\delta[n] \rightarrow \boxed{\mathcal{H}} \rightarrow h[n]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k] \quad \Big\} \text{ DT Convolution}$$

$$= x[n] * h[n]$$

Convolution operator!

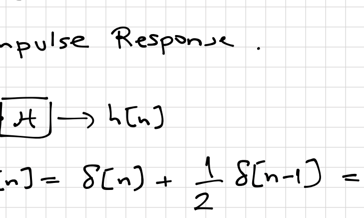
Ex

$$y[n] = \mathcal{H}\{x[n]\} = x[n] + \tfrac{1}{2} x[n-1]$$

a) Impulse Response.

$$\delta[n] \rightarrow \boxed{\mathcal{H}} \rightarrow h[n]$$

$$h[n] = \delta[n] + \frac{1}{2}\delta[n-1] = \begin{cases} 1, & n = 0 \\ 1/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$



b)
$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the output when the input is $x[n]$.

$$x[n] = 2\,\delta[n] + 4\,\delta[n-1] - 2\,\delta[n-2]$$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$
$$= x[0] \cdot h[n] + x[1]\,h[n-1] + x[2] \cdot h[n-2]$$
$$= 2h[n] + 4h[n-1] - 2h[n-2]$$

	0	1	2	3
$2h[n]$	2	1		
$4h[n-1]$		4	2	
$-2h[n-2]$			-2	-1
+	2	5	0	-1

$y[\,]$

$$y[n] = \begin{cases} 2, & n = 0 \\ 5, & n = 1 \\ -1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

DT Convolution Evaluation Procedure

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{+\infty} \underbrace{x[k]\,h[n-k]}_{w_n[k]}$$

$$w_n[k] \triangleq x[k] \cdot h[n-k]$$

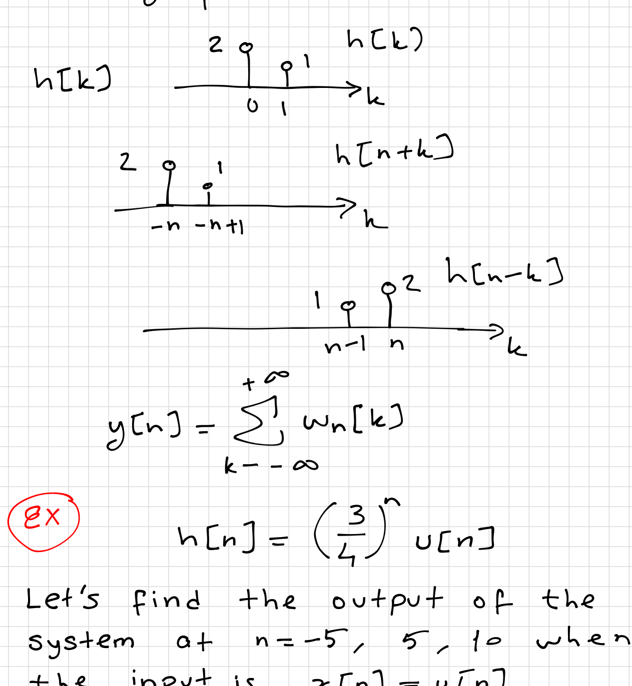
k is the independent variable!

$h[n-k]$?

$h[n] \rightarrow h[k]$

$h[k+n]$ shift left by n

$h[-k+n]$ Reflect around origin



$$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$$

Ex

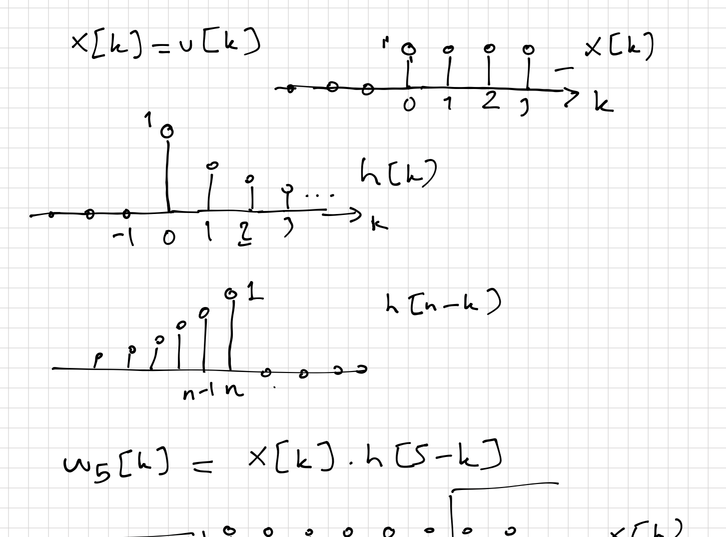
$$h[n] = \left(\frac{3}{4}\right)^n u[n]$$

Let's find the output of the system at $n = -5, 5, 10$ when the input is $x[n] = u[n]$

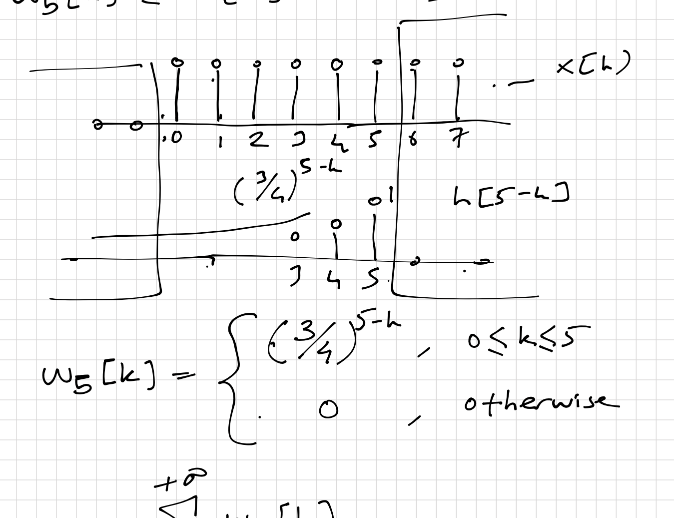
$$y[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]\,h[n-k]}_{w_n[k]}$$

$$h[n-k] = \left(\frac{3}{4}\right)^{n-k} u[n-k]$$
$$= \begin{cases} (3/4)^{n-k}, & k \leq n \\ 0, & k > n \end{cases}$$

$x[k] = u[k]$



$$w_5[k] = x[k] \cdot h[5-k]$$



$$w_5[k] = \begin{cases} (3/4)^{5-k}, & 0 \leq k \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

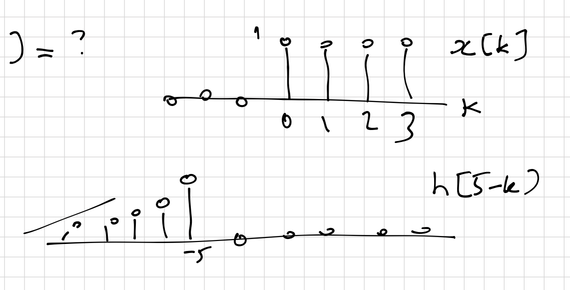
$$y[5] = \sum_{k=-\infty}^{+\infty} w_5[k]$$
$$= \sum_{k=0}^{5} \left(\frac{3}{4}\right)^{5-k}$$
$$= \left(\frac{3}{4}\right)^5 \sum_{k=0}^{5} \left(\frac{4}{3}\right)^k$$
$$= \left(\frac{3}{4}\right)^5 \frac{1 - (4/3)^6}{1 - 4/3}$$
$$= 3.288$$

$y[10] = ?$

$$w_{10}[k] = \begin{cases} (3/4)^{10-k}, & 0 \leq k \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

$$y[10] = \sum_{k=0}^{10} \left(\frac{3}{4}\right)^{10-k} = 3.831$$

$y[-5] = ?$



$$w_{-5}[k] = 0$$

$$y[-5] = 0$$