BIMU3009
Signal Processing
Final Exam Solutions

Istanbul University - Cerrahpaşa

Computer Engineering Department - Fall 2021

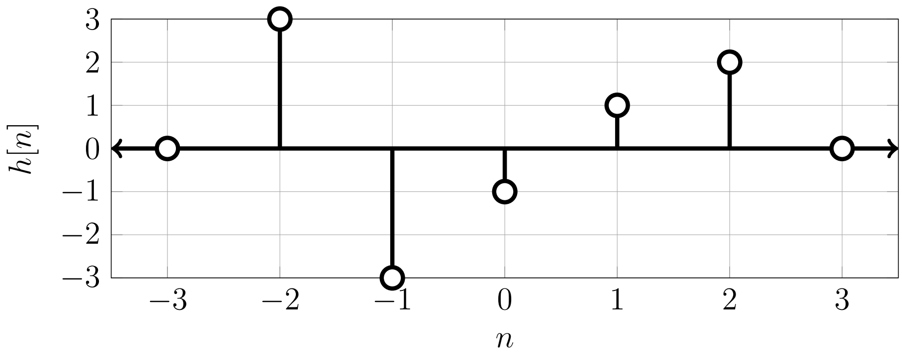
January 7^{th}, 2022 14:00-15:10

`Son güncelleme: 2022-01-15 17:36`

Some useful equations

$$\cos^2(\theta) = \frac{1}{2}\Big[1 + cos(2\,\theta)\Big] \qquad \int \cos(\alpha x)\ \mathrm{d}x = \frac{1}{\alpha}\sin(\alpha x) + c$$

Q1: The impulse response of a DISCRETE TIME Linear-Time Invariant system $\mathcal{H}$ is given below.



(a) `(10 pts)` Is $\mathcal{H}$ stable? Explain.

Solution 1a:

For an DT LTI system to be BIBO-stable the following must be true for the impulse response:

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

Then:

$$\sum_{k=-\infty}^{\infty} |h[n]| = |3| + |-3| + |-1| + |1| + |2|$$
$$= 10 < \infty$$

Therefore $\mathcal{H}$ is BIBO-stable.

(b) `(10 pts)` Is $\mathcal{H}$ causal and/or memoryless? Explain.

Solution 1b:

For an DT LTI system to be memoryless the following must be true for the impulse response:

$$h[n] = 0 \quad \text{for} \quad n \neq 0$$

Therefore $\mathcal{H}$ is NOT memoryless.

For an DT LTI system to be causal the following must be true for the impulse response:

$$h[n] = 0 \quad \text{for} \quad n < 0$$

Therefore $\mathcal{H}$ is NOT Causal.

(c) `(10 pts)` Calculate and sketch the step response of this system.

Solution 1c:

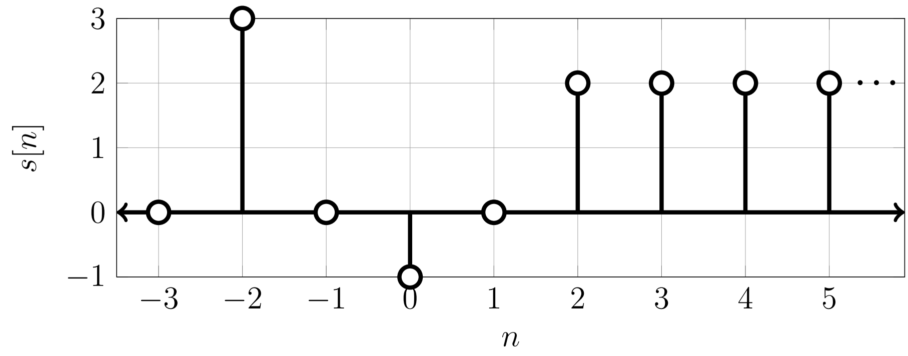
Step function of a DT LTI system:

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

So, we can calculate the step function point by point:

$$\begin{aligned}
s[n] &= 0 \qquad \text{for} \qquad n < -2 \\
s[-2] &= 3 \\
s[-1] &= 3 - 3 = 0 \\
s[0] &= 0 - 1 = -1 \\
s[1] &= -1 + 1 = 0 \\
s[2] &= 0 + 2 = 2 \\
s[n] &= 2 \qquad \text{for} \qquad n > 2
\end{aligned}$$

If we sketch this:



(d) `(10 pts)` The following input is applied to this system. Calculate and sketch the output.

$$x[n] = 2\delta[n+2] + \delta[n] - 3\delta[n+2]$$

Solution 1d:

We know that:

$$a[n] * \delta[n-k] = a[n-k]$$

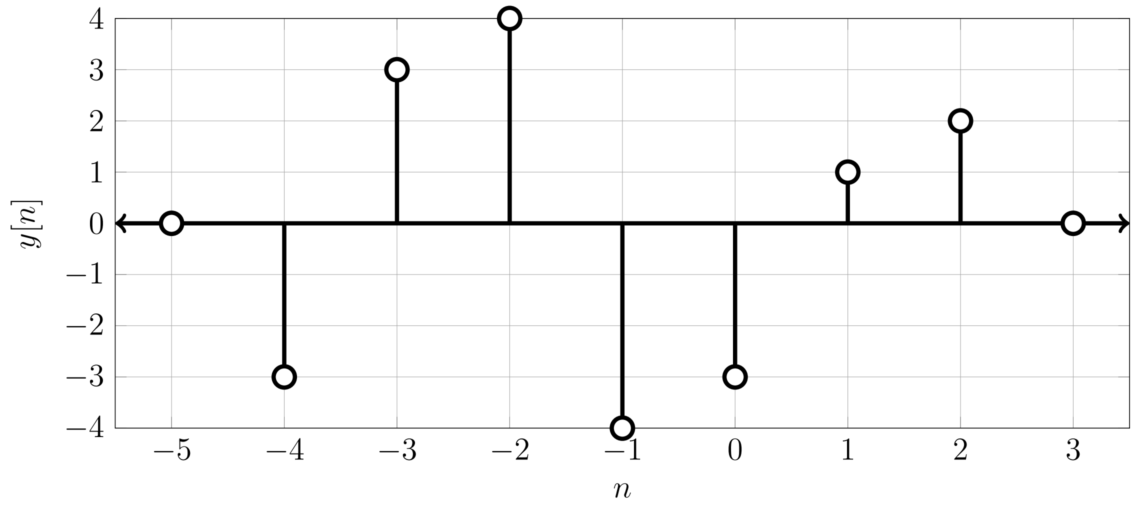
So,

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * \left\{ 2\delta[n+2] + \delta[n] - 3\delta[n+2] \right\} \\ &= h[n] * \left\{ \delta[n] - \delta[n+2] \right\} \\ &= h[n] - h[n+2] \end{aligned}$$

Putting these signals on a table for easy addition:

n	-5	-4	-3	-2	-1	0	1	2	3	4
$-h[n+2]$	0	-3	3	1	-1	-2	0	0	0	0
$h[n]$	0	0	0	3	-3	-1	1	2	0	0
$y[n]$	0	-3	3	4	-4	-3	1	2	0	0

If we sketch this:



Q2: Evaluate the following Continuous-Time Convolutions.

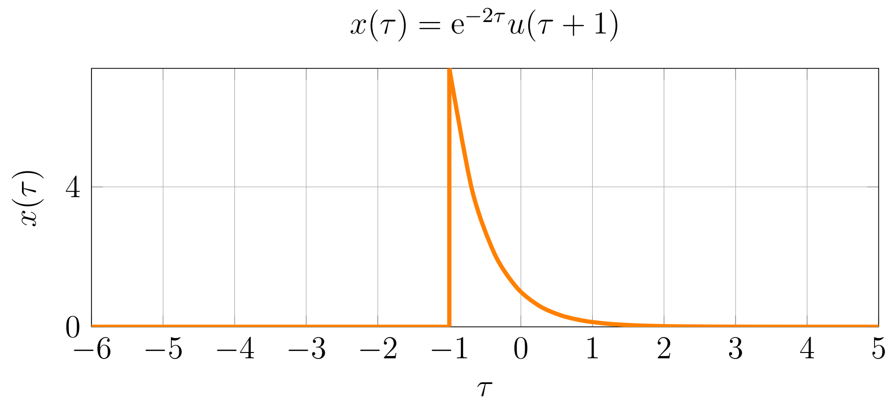
(a) `(10 pts)` $e^{t}\ u(-t) * e^{-2t}\ u(t+1)$

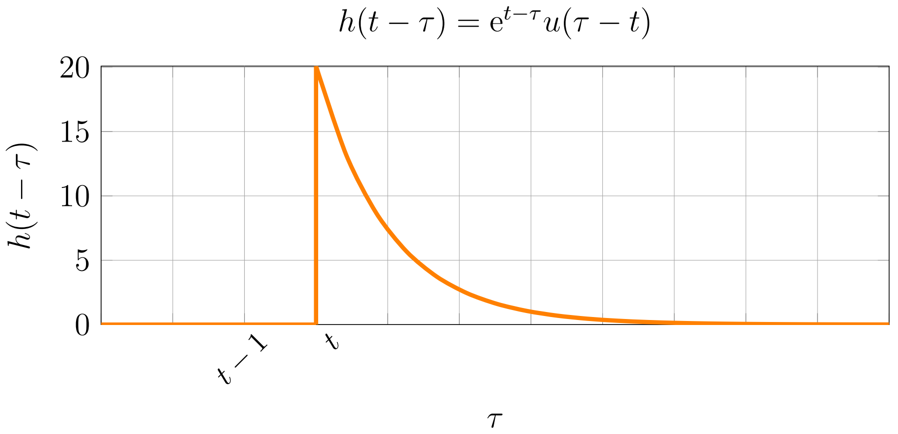
Solution 2a:

Let:

$$
\begin{aligned}
x(t) &= e^{-2t}\ u(t+1) \\
h(t) &= e^{t}\ u(-t) \\
y(t) &= x(t) * h(t) \\
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau
\end{aligned}
$$

Let's plot the intermediate signals, $x(\tau)$ and $h(t-\tau)$ with respect to τ.





So, for $t < -1$:

$$
\begin{aligned}
y(t) &= \int_{-1}^{\infty} \mathrm{e}^{-2\tau}\ \mathrm{e}^{t-\tau}\ \mathrm{d}\tau \\
&= \mathrm{e}^{t} \int_{-1}^{\infty} \mathrm{e}^{-3\tau}\ \mathrm{d}\tau \\
&= \mathrm{e}^{t} \frac{1}{-3} \left[\mathrm{e}^{-3\tau}\right]_{-1}^{\infty} \\
&= \frac{1}{3}\ \mathrm{e}^{t+3}
\end{aligned}
$$

So, for $t \geq -1$:

$$
\begin{aligned}
y(t) &= \int_{t}^{\infty} \mathrm{e}^{-2\tau}\ \mathrm{e}^{t-\tau}\ \mathrm{d}\tau \\
&= \mathrm{e}^{t} \int_{t}^{\infty} \mathrm{e}^{-3\tau}\ \mathrm{d}\tau \\
&= \mathrm{e}^{t} \frac{1}{-3} \left[\mathrm{e}^{-3\tau}\right]_{t}^{\infty} \\
&= \frac{1}{3}\ \mathrm{e}^{-2t}
\end{aligned}
$$

(b) `(10 pts)` $[u(t+3) - 2u(t) + u(t-3)] * u(t-2)$

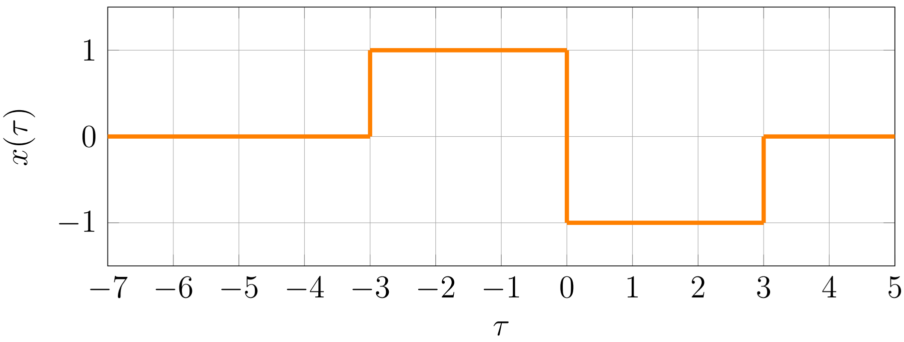
Solution 2a:

Let:

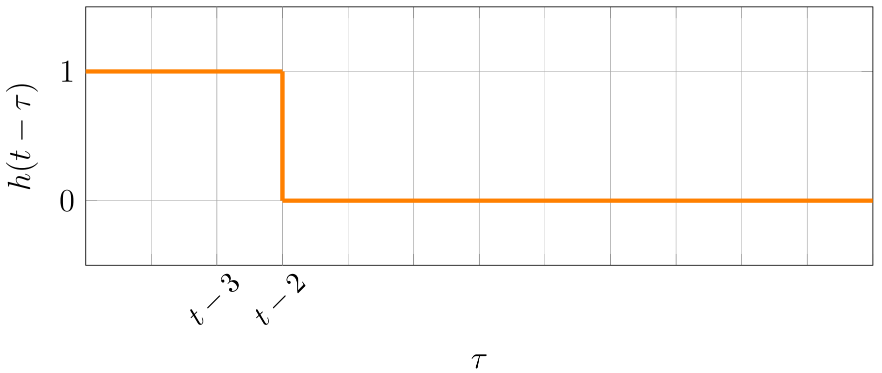
$$
\begin{aligned}
x(t) &= u(t+3) - 2u(t) + u(t-3) \\
h(t) &= u(t-2) \\
y(t) &= x(t) * h(t) \\
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ \mathrm{d}\tau
\end{aligned}
$$

Let's plot the intermediate signals, $x(\tau)$ and $h(t-\tau)$ with respect to τ.

$x(\tau) = u(t+3) - 2u(t) + u(t-3)$



$h(t-\tau) = u(t-\tau-2)$



So, for $t-2 < -3$ which is $t < -1$: $y(t) = 0$.

For $-3 \leq t-2 < 0$ which is $-1 \leq t < 2$:

$$
\begin{aligned}
y(t) &= \int_{-3}^{t-2} 1\ \mathrm{d}t \\
&= t - 2 + 3 = t + 1
\end{aligned}
$$

.

For $0 \leq t-2 < 3$ which is $2 \leq t < 5$:

$$\begin{aligned} y(t) &= \int_{-3}^{0} 1 \text{ d}t + \int_{0}^{t-2} -1 \text{ d}t \\ &= (0+3) - [(t-2) - 0] = 5 - t \end{aligned}$$

.

For $t - 2 \geq 3$ which is $t \geq 5$:

$$\begin{aligned} y(t) &= \int_{-3}^{0} 1 \text{ d}t + \int_{0}^{3} -1 \text{ d}t \\ &= (0+3) - (3-0) = 0 \end{aligned}$$

.

Thus:

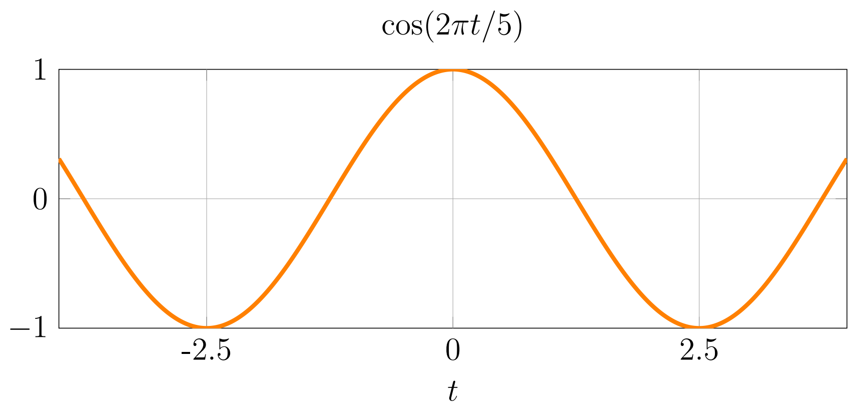
$$y(t) = \begin{cases} 0 & , \quad t < -1 \\ t+1 & , \quad -1 \leq t < 2 \\ 5-t & , \quad 2 \leq t < 5 \\ 0 & , \quad 5 \leq t \end{cases}$$

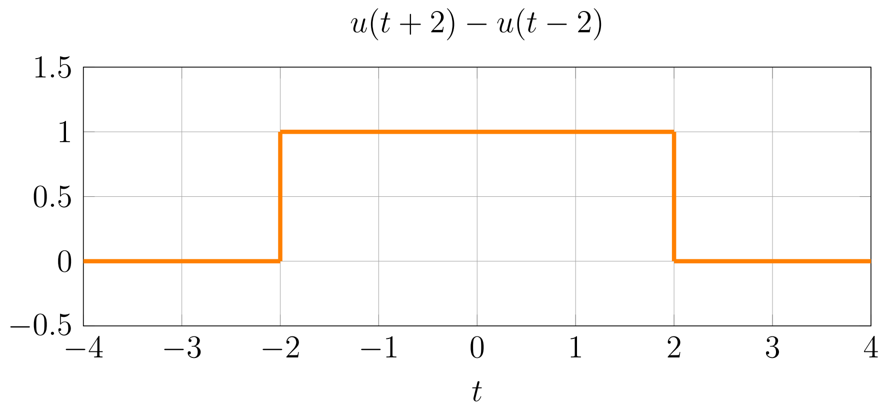
Q3: Consider the following CT signal. Answer the following questions.

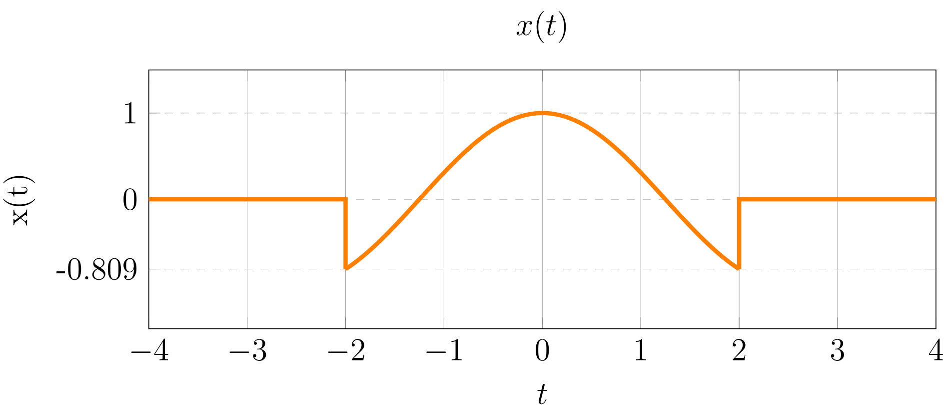
$$x(t) = \cos\left(\frac{2\pi t}{5}\right) [u(t+2) - u(t-2)]$$

(a) `(10 pts)` Carefully sketch $x(t)$. Show your work.

Solution 3a:







(b) `(10 pts)` Determine the energy and average power of $x(t)$. Is it an energy signal, power signal or neither?

Solution 3b:

$$
\begin{aligned}
E &= \int_{-2}^{2} [\cos(2\pi t/5)]^2 \ \mathrm{d}t \\
&= \int_{-2}^{2} \frac{1}{2}(1+\cos(4\pi t/5)) \ \mathrm{d}t \\
&= \frac{1}{2}(4 + \frac{5}{4\pi}[\sin(4\pi t/5)]_{-2}^{2}) \\
&= 1.6216
\end{aligned}
$$

It is an energy signal, so its average power is zero.

Q4: Consider the following DT signal. Answer the following questions.

$$
x[n] = \cos\left(\frac{2\pi n}{5}\right)
$$

(a) `(10 pts)` Is $x[n]$ periodic? If so, find the period and the frequency.

Solution 4a:

$$
\begin{aligned}
\Omega_0 &= \frac{2\pi}{5} \quad \text{radians} \\
N &= \frac{2\pi}{\Omega_0} \\
&= \frac{2\pi}{\frac{2\pi}{5}} \\
N &= 5 \quad \text{cyles}
\end{aligned}
$$

Since N is an integer, $x[n]$ is periodic.

(b) `(10 pts)` If you find that the answer to Q4a is "periodic", then determine the DTFS coefficients of $x[n]$.

Solution 4a:

$$
x[n] = \sum_{k=<N>} X[k] \ \mathrm{e}^{jk\Omega_0 n}
$$

If we use Euler's formula:

$$
x[n] = \frac{1}{2}\left[\mathrm{e}^{j\frac{2\pi n}{5}} + \mathrm{e}^{-j\frac{2\pi n}{5}}\right]
$$

So, over a single period between $[-2, 2]$

$$X[k] = \begin{cases} 1/2 & , \quad k = 1, -1 \\ 0 & , \quad k = -2, 0, 2 \end{cases}$$