CT Convolution (Continued from last week)

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t) \quad \Big\}\ \text{C.T. Convolution}$$

Impulse response of an LTI system: $h(t) = \mathcal{H}\{\delta(t)\}$

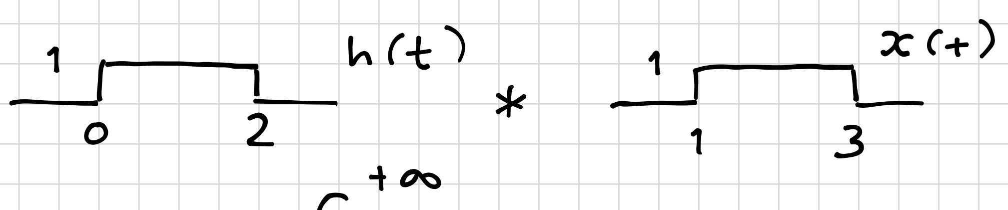
CT Integral Evaluation

Ex $\mathcal{H}$ is an LTI system.

$h(t) = u(t) - u(t-2)$

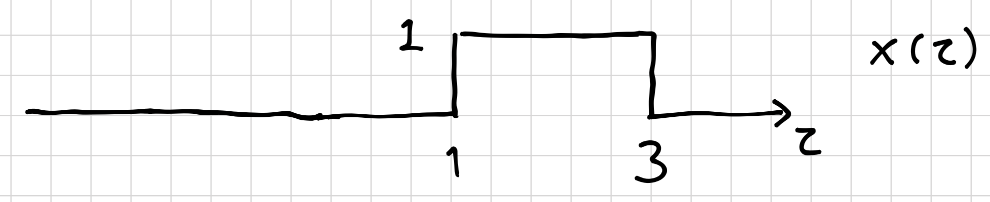
Find the output of $\mathcal{H}$ when the input is

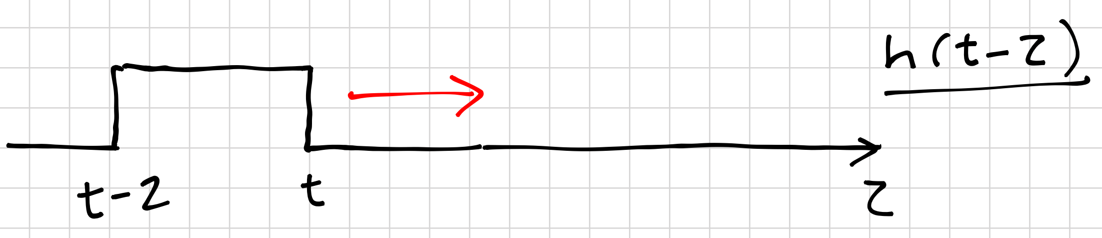
$x(t) = u(t-1) - u(t-3)$



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

① Graph $x(\tau)$ and $h(t-\tau)$ with respect to τ





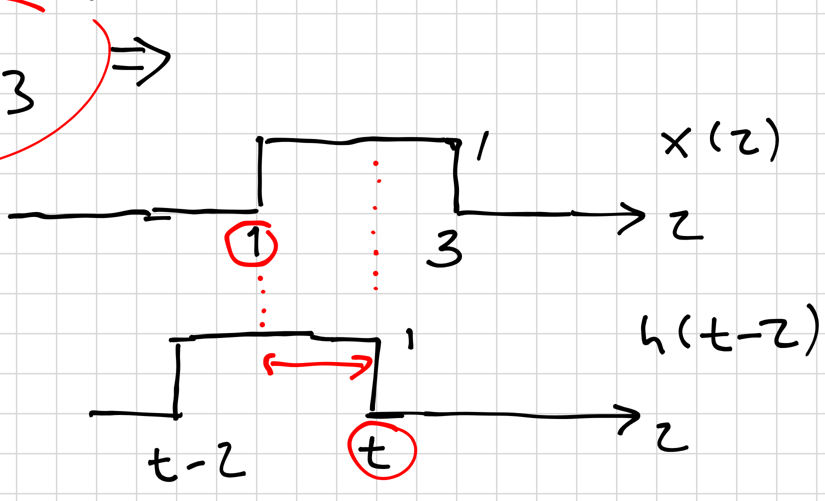
② Define intermediate signal

$$w_t(\tau) = x(\tau) \cdot h(t-\tau)$$

*) When $\boxed{t < 1}$ $\quad w_t(\tau) = 0$

$y(t) = \int_{-\infty}^{+\infty} w_t(\tau)\, d\tau = \boxed{0}$

⊛ when $t > 1$, $t < 3$ $\Rightarrow$



$$w_t(\tau) = \begin{cases} 1, & 1 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

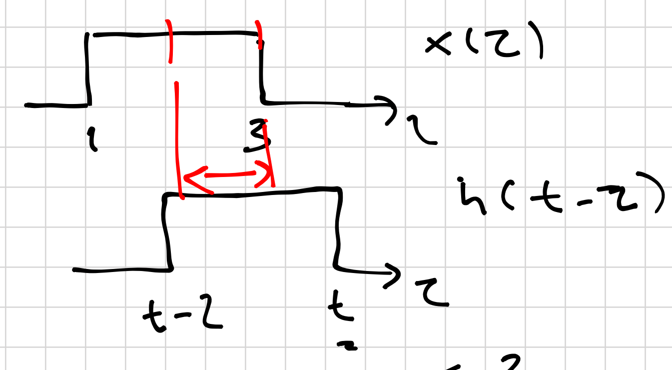
$$y(t) = \int_1^t 1\, dt = t - 1$$

⊛

$t > 3$

$t - 2 < 3$

$3 < t < 5$



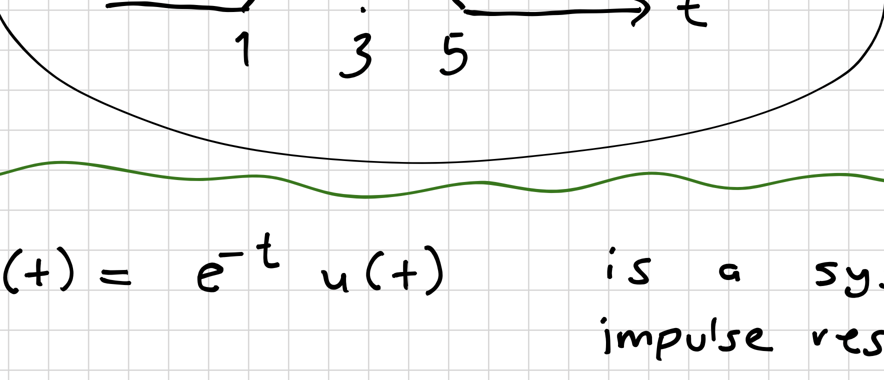
$$w_t(\tau) = \begin{cases} 1, & t-2 < \tau < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \int_{t-2}^{3} 1.\ dt = 3 - (t-2) = 5 - t$$

⊛ $t > 5$ $\quad w_t(\tau) = 0$

$y(t) = 0$

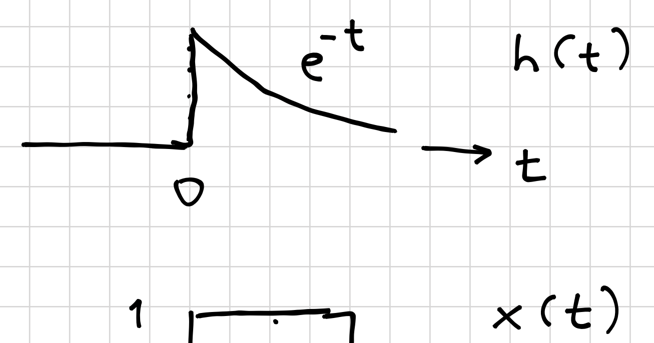
$$y(t) = \begin{cases} t-1, & 1 < t \leq 3 \\ 5-t, & 3 < t \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

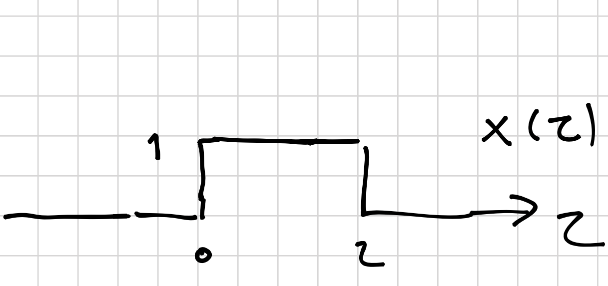


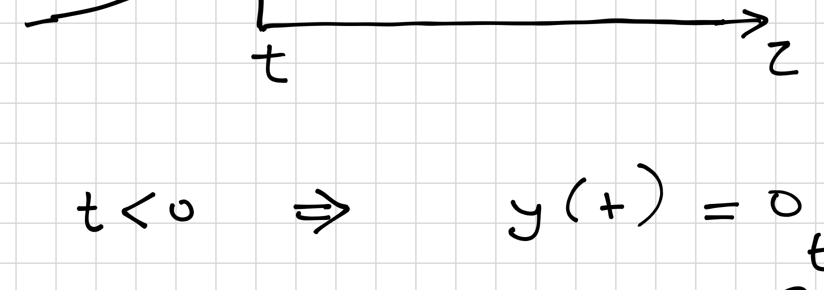
Ex $h(t) = e^{-t} u(t)$ is a system's impulse response

Find the output when the input is

$x(t) = u(t) - u(t-2)$







$$h(t-\tau) = \begin{cases} e^{\tau - t}, & \tau < t \\ 0, & 0. \end{cases}$$

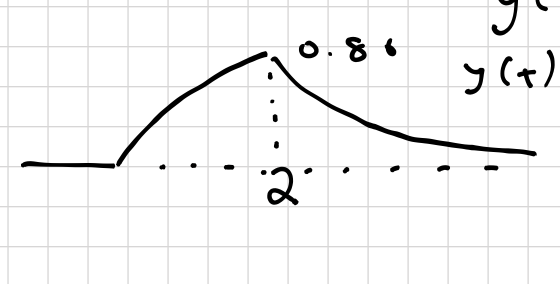
① $t < 0 \Rightarrow y(t) = 0$

② $0 \leq t < 2 \Rightarrow y(t) = \int_0^t 1.\, e^{\tau - t}\, d\tau$

$= \boxed{1 - e^{-t}}$

③ $t \geq 2$ $\quad y(t) = \int_0^2 1.\, e^{\tau - t}\, d\tau$

$= e^{-t}(e^2 - 1)$



Properties of LTI Systems and Convolution
(Kitapta 2.6)

① The Commutative Property

$$x(t) * h(t) = h(t) * x(t)$$

Same for DT.

② The distributive property

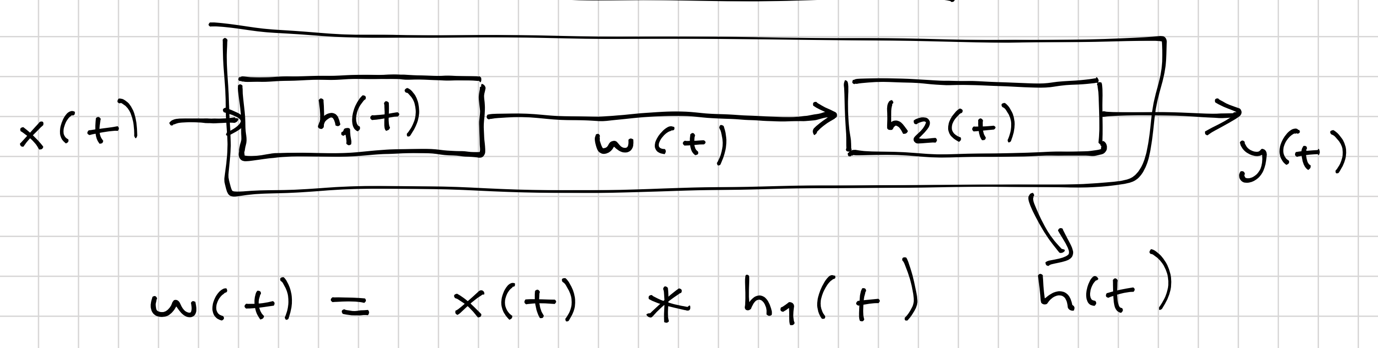
$$y(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

$$= [x_1(t) + x_2(t)] * h(t)$$

$$= h(t) * [x_1(t) + x_2(t)]$$

(Same applies to DT)

③ The Associative property



$$w(t) = x(t) * h_1(t)$$

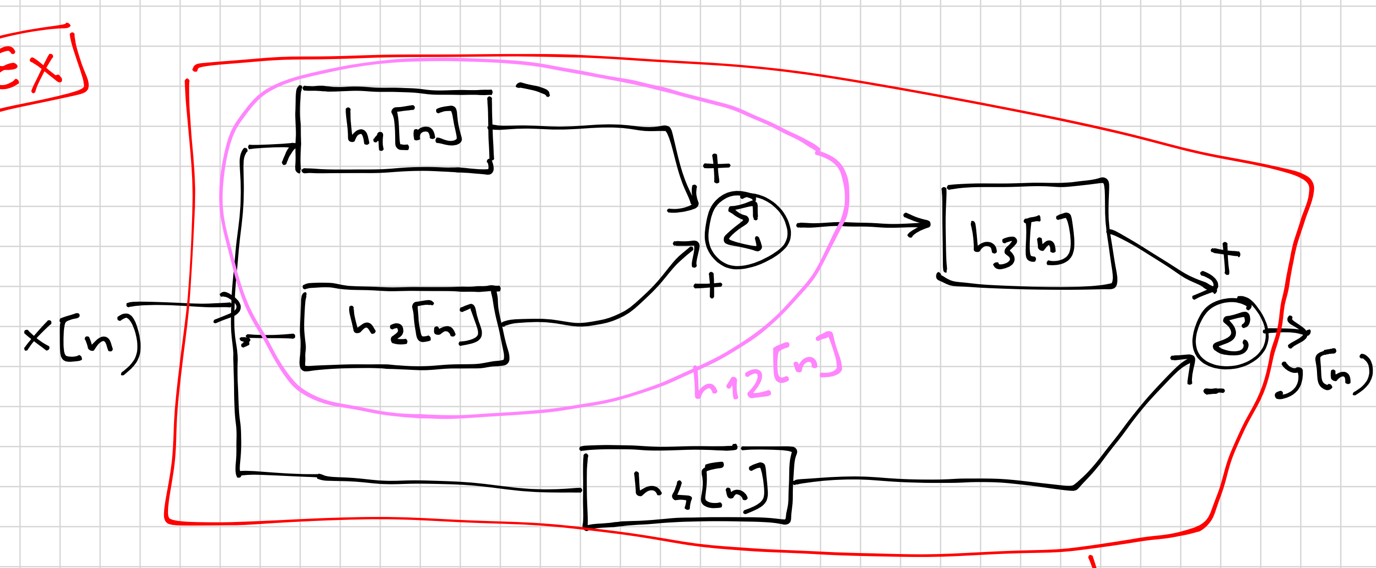
$$y(t) = w(t) * h_2(t)$$

$$= \{x(t) * h_1(t)\} * h_2(t)$$

$$= x(t) * \underbrace{\{h_1(t) * h_2(t)\}}_{h(t)}$$

$$= x(t) * h(t)$$

Ex

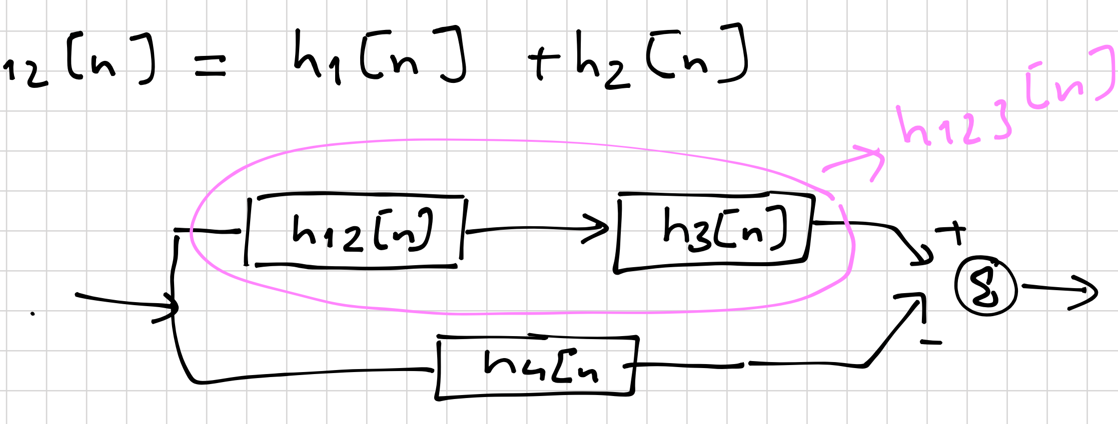


$h[n] = ?$

$$h_1[n] = u[n]$$
$$h_2[n] = u[n+2] - u[n]$$
$$h_3[n] = \delta[n-2]$$
$$h_4[n] = \alpha^n u[n]$$

Find the overall impulse response

$$h_{12}[n] = h_1[n] + h_2[n]$$



$$h_{123}[n] = h_{12}[n] * h_3[n]$$

$$h[n] = h_{123}[n] - h_4[n]$$

$x[n] * \delta[n-n_0] = x[n-n_0]$

$$h_{12}[n] = u[n] + u[n+2] - u[n]$$
$$= u[n+2]$$

$$h_{123}[n] = u[n+2] * \delta[n-2]$$
$$= u[n+2-2] = u[n]$$

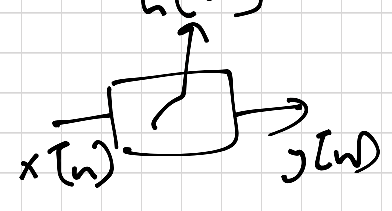
$$h[n] = u[n] - \alpha^n u[n]$$

$$\boxed{h[n] = (1-\alpha^n)\, u[n]}$$

Relationship Between LTI System Properties and the Impulse Response

① Memoryless LTI Systems.

$h[n]$: Impulse Response.



$y[n]$ must depend only on the current values of $x[n]$

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] \cdot x[n-k]$$

$$= \dots + h[-2]\, x[n+2] = 0$$
$$+ h[-1]\, x[n+1] = 0$$
$$+ h[0]\, x[n]$$
$$+ h[1]\, x[n-1] = 0$$
$$+ h[2]\, x[n-2] = 0$$
$$+ \dots$$

In order to have a memoryless system

$$h[k] = 0 \quad \text{for } k \neq 0$$

or

$$h[n] = c\,\delta[n] \quad DT$$
$$h(t) = c\,\delta(t) \quad CT$$

② CAUSAL SYSTEMS.

For a causal LTI system:

$$h[n] = 0 \quad \text{for } n < 0 \ (DT)$$
$$h(t) = 0 \quad \text{for } t < 0 \ (CT)$$

③ STABLE SYSTEMS

If a system is BIBO-stable:

Assuming $|x[n]| < M_x < \infty$

$|y[n]| \leq M_y < \infty$ must be true for all n

$$y[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] \right|$$

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]|\, |x[n-k]|$$

$$\leq \sum_{k=-\infty}^{+\infty} |h[k]|\, M_x$$

$$\leq M_x \sum_{k=-\infty}^{+\infty} |h[k]| \leq M_y$$

must be finite

⊙ In order to have a BIBO-stable DT LTI system

$h[n]$ must be "absolutely summable"

that is

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty \quad DT$$

For a BIBO stable DT LTI system, $h(t)$ must be "absolutely integrable":

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$$

Invertible Systems and Deconvolution

A system is invertible if and only if the input of the system can be recovered from the output.

$$x(t) \to \boxed{\mathcal{H}} \xrightarrow{y(t)} \boxed{\mathcal{H}^{inv}} \to x(t)$$

If $\mathcal{H}^{inv}$ exists then $\mathcal{H}$ is invertible

→ If an LTI system is invertible the the inverse system is also LTI

$$x(t) \to \underset{h(t)}{\boxed{\mathcal{H}}} \xrightarrow{y(t)} \underset{h^{inv}(t)}{\boxed{\mathcal{H}^{inv}}} \to x(t)$$

$$\{x(t) * h(t)\} * h^{inv}(t) = x(t)$$

$$x(t) * \underbrace{\{h(t) * h^{inv}(t)\}}_{\delta(t)} = x(t)$$

CT $\quad h(t) * h^{inv}(t) = \delta(t)$

DT $\quad h[n] * h^{inv}[n] = \delta[n]$

Ex $\quad y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha x[n-1]$

Find a $\boxed{\text{causal}}$ inverse system of $\mathcal{H}$

Sol

Let's find the impulse response.

$$h[n] = \mathcal{H}\{\delta[n]\} = \delta[n] + \alpha \delta[n-1]$$

$$h[n] * h^{inv}[n] = \delta[n]$$

$$\{\delta[n] + \alpha \delta[n-1]\} * h^{inv}[n] = \delta[n]$$

$$h^{inv}[n] + \alpha h^{inv}[n-1] = \delta[n]$$

1. Since $h^{inv}[n]$ is causal

$$h^{inv}[n] = 0 \qquad n < 0$$

2. $n = 0$, $\delta[0] = 1$ $\qquad h^{inv}[0] + \alpha \underbrace{h^{inv}[-1]}_{} = 1$

$$h^{inv}[0] = 1$$

3. $n > 0 \qquad \delta[n] = 0$

$$h^{inv}[n] + \alpha h^{inv}[n-1] = 0$$

$$h^{inv}[n] = -\alpha h^{inv}[n-1]$$

$$h^{inv}[1] = -\alpha$$

$$h^{inv}[2] = -\alpha(-\alpha) = (-\alpha)^2$$

$$h^{inv}[3] = (-\alpha)^3$$

$$\vdots$$

$$h^{inv}[n] = (-\alpha)^n \qquad n \geq 0$$

$$h^{inv}[n] = (-\alpha)^n u[n]$$

Is $h^{inv}[n]$ stable?

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

$$\sum_{n=0}^{\infty} |(-\alpha)^n| \quad \begin{cases} |\alpha| < 1 \Rightarrow \mathcal{H}^{inv} \text{ is stable} \\ |\alpha| > 1 \Rightarrow \mathcal{H}^{inv} \text{ is unstable} \end{cases}$$