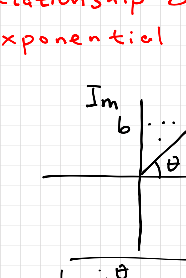
Relationship Between Sinusoidal and Complex Exponential Signal



$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$ Euler's Identity

CT

$x(t) = A \cdot e^{j\omega t}$
$= A[\cos(\omega t) + j\sin(\omega t)]$

$Re\{x(t)\} = A\cos(\omega t)$
$Im\{x(t)\} = A\sin(\omega t)$

DT

$x[n] = A \cdot e^{j\Omega n} = A[\cos(\Omega n) + j\sin(\Omega n)]$
$Re\{x[n]\} = A\cos(\Omega n)$
$Im\{x[n]\} = A\sin(\Omega n)$

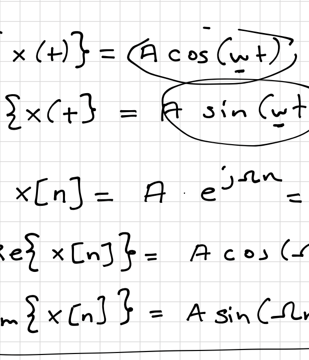
---

⑥ Exponentially Damped Sinusoidal Signals

CT $\quad x(t) = A\, e^{-\alpha t}\sin(\omega t + \theta)$

$\alpha > 0$

$\alpha < 0$



Ex

Given two periodic CT signals, $x_1(t)$ and $x_2(t)$, of which the fundamental periods are T_1 and T_2 respectively,

$x(t) = x_1(t) + x_2(t)$

Is $x(t)$ periodic? If what is the period?

(*) $x_1(t) = x_1(t + T_1) = x_1(t + mT_1)$ $\quad m \in \mathbb{Z}^+$

(*) $x_2(t) = x_2(t + T_2) = x_2(t + kT_2)$ $\quad k \in \mathbb{Z}^+$

$x(t) = x_1(t + mT_1) + x_2(t + kT_2)$
If $x(t)$ is periodic with a period T

$x(t) = x(t + T)$
$= x_1(t + T) + x_2(t + T)$

So:

$$\boxed{T = mT_1 = kT_2}$$

$\frac{T_1}{T_2} = \frac{k}{m}$ → must be a rational number.

– If we cannot find an $(m,k) \in \mathbb{Z}^+$ pair, $x(t)$ is not periodic.
– Find the smallest (m,k) pairs to determine the period.

Ex Same question for DT.

$x_1[n]$ is periodic, N_1
$x_2[n]$ is " , N_2
$x[n] = x_1[n] + x_2[n]$ periodic? $N = ?$

$x_1[n] = x_1[n + mN_1]$ $\quad m \in \mathbb{Z}^+$
$x_2[n] = x_2[n + kN_2]$ $\quad k \in \mathbb{Z}^+$
Assume period is N for $x[n]$
$x[n] = x_1[n + N] + x_2[n + N]$
⇒
$N = mN_1 = kN_2$

$\frac{m}{k} = \frac{N_2}{N_1}$ → Always rational!
Always periodic!

Find the smallest (m,k) pair

— SYSTEMS —

$x(t) \to \boxed{\text{CT System}} \to y(t)$

$x[n] \to \boxed{\text{DT System}} \to y[n]$

$\mathcal{H}$ — System

$y(t) = \mathcal{H}\{x(t)\}$
$y[n] = \mathcal{H}\{x[n]\}$

Ex

ρ: frictional force
v: velocity.
m: mass

$f(t) \longrightarrow v(t)$

$\frac{d}{dt}v(t) = \frac{1}{m}[f(t) - \rho v(t)]$

$$\boxed{v(t) = \mathcal{H}\{f(t)\}}$$

Properties of Systems

① Stability

A system is bounded-input bounded-output (BIBO) stable if and only if every bounded input results in a bounded output.

$y(t) = \mathcal{H}\{x(t)\}$
$\mathcal{H}$ is BIBO-stable if

$|y(t)| \leq M_y < \infty$, for all t
M_y is a finite positive number

when $|x(t)| \leq M_x < \infty$ for all t

/* Same applies to DT systems */

Ex Let's say that DT system has the following input–output relationship:

$y[n] = \mathcal{H}\{x[n]\}$
$= \frac{1}{3}(x[n-1] + x[n] + x[n+1])$

is $\mathcal{H}$ stable?

Assume $|x[n]| \leq M_x < \infty$, $\forall n$ (for all)

$|y[n]| = \left|\frac{1}{3}(x[n-1] + x[n] + x[n+1])\right|$
$= \frac{1}{3}|\quad\quad|$ $\quad$ $c = a+b$, $c \leq |a| + |b|$
$\leq \frac{1}{3}(\underbrace{|x[n-1]|}_{\leq M_x} + \underbrace{|x[n]|}_{\leq M_x} + \underbrace{|x[n+1]|}_{\leq M_x})$
$\leq \frac{1}{3}(M_x + M_x + M_x)$
$|y[n]| \leq M_x$ → finite!
∴ $\mathcal{H}$ is stable. $\quad$ $y = ab$, $|y| = |a||b|$

Ex

$y[n] = \mathcal{H}\{x[n]\} = r^n x[n]$
$r > 1$ ⇒ is $\mathcal{H}$ stable?

Assume $|x[n]| \leq M_x < \infty$, $\forall n$

$|y[n]| = |r^n| \cdot |x[n]| \leq |r^n| \cdot M_x$

If $r > 1$ ⇒ $|r^n|$ will diverge as n increases. Thus, we cannot say $y[n]$ is bounded when $x[n]$ is bounded!

∴ For $r > 1$, $\mathcal{H}$ is not stable.

---

② Memory

A system is said to be "memoryless" if its output depends only on the current values of the input signal.

Ex $y[n] = (2x[n] + x^2[n])^2$ $\quad$ Memoryless

A system is said to "possess memory" if it depends on the future or the past values of the input.

Ex $y(t) = x(t-1)$ $\quad$ not memoryless

Ex $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ $\quad$ not memoryless.

---

③ Causality $\quad$ Causal × Casual

A system is "causal" if the current output of the system depends only on the past and/or the present values of the input.

Ex $y[n] = x^2[n-1] + x[n]$: CAUSAL

Ex $y(t) = x(t+1)$: non-causal

Ex $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$: causal

---

④ Invertibility

A system is invertible if distinct inputs lead to distinct outputs, that is, if the input of the system can be recovered from the output, then the system is invertible.

$x(t) \to \boxed{\mathcal{H}} \xrightarrow{y(t)} \boxed{\mathcal{H}^{inv}} \to x(t)$

If $\mathcal{H}^{inv}$ exists then $\mathcal{H}$ is invertible

$\mathcal{H}^{inv}\{y(t)\} = \mathcal{H}^{inv}\{\mathcal{H}\{x(t)\}\}$
$x(t) = (\mathcal{H}^{inv}\mathcal{H})\,x(t)$

$I = \mathcal{H}^{inv} \cdot \mathcal{H}$. $\quad$ identity system!

Ex

$y(t) = \mathcal{H}\{x(t)\} = 2 \cdot x(t)$
$x(t) = \mathcal{H}^{inv}\{y(t)\} = \frac{1}{2}y(t)$

Since $\mathcal{H}^{inv}$ exists
$\mathcal{H}$ is invertible

Ex

$y(t) = x^2(t) = \mathcal{H}\{x(t)\}$

$\sqrt{y(t)} \to x(t)$
$\sqrt{y(t)} \to -x(t)$

Since $x(t)$ and $-x(t)$ produce the same output $\mathcal{H}$ is not invertible.

⑤ TIME-INVARIANCE

A system is said to be "time-invariant" if a time delay or a time-advance of the input signal leads to an identical time shif in the output signal.