② Memory –

A system is said to be "memoryless" if its output signal depends <u>only</u> on the current values of the input signal

Ex $y[n] = (2x[n] - x^2[n])^2$ memoryless

It is said to "possess memory" if it depends on the future or the past values of the input.

Ex $y(t) = x(t-1)$ not memoryless.

$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$? not memoryless.

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③ CAUSALITY

A system is "causal" if the current output of the system depends <u>only</u> on the past and/or present values of the input.

Ex $y[n] = x^2[n-1] + x[n]$ Causal.

Ex $y(t) = x(t+1)$ Not-causal

Ex $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ CAUSAL

④ INVERTIBILITY

A system is <u>invertible</u> if <u>distinct</u> inputs lead to <u>distinct</u> outputs, that is, if the input of the system can be recovered from the output, that system is <u>invertible</u>.

$$x(t) \rightarrow \boxed{\mathcal{H}} \xrightarrow{y(t)} \boxed{\mathcal{H}^{inv}} \rightarrow x(t)$$

If $\mathcal{H}^{inv}$ exists then $\mathcal{H}$ is invertible!

$x(t) =$

$$\mathcal{H}^{inv}\{y(t)\} = \mathcal{H}^{inv}\{\mathcal{H}\{x(t)\}\}$$

$$x(t) = \boxed{\mathcal{H}^{inv}\mathcal{H}}\,\{x(t)\}$$

$I = \mathcal{H}^{inv}.\mathcal{H}$ identity system.

Ex $y(t) = 2x(t) = \mathcal{H}\{x(t)\}$

$$x(t) \rightarrow \boxed{\times 2} \rightarrow y(t)$$

$$x(t) = \frac{1}{2} y(t) = \mathcal{H}^{inv}\{y(t)\}$$

$\therefore \mathcal{H}$ <u>is</u> invertible

Ex $y(t) = x^2(t) = \mathcal{H}\{x(t)\}$

$$\sqrt{x^2(t)} = \begin{cases} -x(t) \\ x(t) \end{cases}$$

Since $x(t)$ and $-x(t)$ produce the same output $\mathcal{H}$ is <u>not</u> invertible.

⑤ TIME INVARIANCE

A system is said to be "time invariant" if a time delay or a time advance of the input signal leads to an identical time shift in the output signal.

$$y(t) = \mathcal{H}\{x(t)\}$$

If for any $t_0 \in \mathbb{R}$

$$y(t-t_0) = \mathcal{H}\{x(t-t_0)\}, \quad \forall t \qquad (TI)$$

then $\mathcal{H}$ is Time-Invariant. Otherwise $\mathcal{H}$ is time-variant.

Ex $y(t) = \mathcal{H}\{x(t)\} = \int_{-\infty}^{t} x(\tau)\, d\tau$

Is $\mathcal{H}$ T.I?

First shift the input

① $y_2(t) = \mathcal{H}\{x(t-t_0)\}$

$$= \int_{-\infty}^{t} x(\tau - t_0)\, d\tau$$

② $y(t-t_0) = y_1(t) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$

$\tau' = \tau - t_0 \qquad d\tau' = d\tau$

$$y_2(t) = \int_{-\infty}^{t-t_0} x(\tau')\, d\tau'$$

$y_1(t) = y_2(t) \quad \therefore \mathcal{H}$ is T.I !

Ex $y[n] = \mathcal{H}\{x[n]\} = r^n x[n]$ T.I?

$$y_1[n] = \mathcal{H}\{x(n-n_0)\} = r^n x[n-n_0]$$

$$y_2[n] = y[n-n_0] = r^{n-n_0} x(n-n_0)$$

$y_1[n] \neq y_2[n] \quad \therefore$ Time Variant System

⑥ LINEARITY

A system, $\mathcal{H}$, is Linear if it satisfies the following two properties.

Superposition

Let for any signal $x_1(t)$ and $x_2(t)$

$$y_1(t) = \mathcal{H}\{x_1(t)\}$$
$$y_2(t) = \mathcal{H}\{x_2(t)\}$$

Let $y(t) = y_1(t) + y_2(t)$

$x(t) = x_1(t) + x_2(t)$

<u>If</u> $y(t) = \mathcal{H}\{x(t)\}$ then the system $\mathcal{H}$ satisfies the principle of superposition.

Homogeneity

Let $y(t) = \mathcal{H}\{x(t)\}$

<u>If</u> $\alpha y(t) = \mathcal{H}\{\alpha x(t)\}$ for $\forall \alpha \in \mathbb{R}$

<u>then</u> $\mathcal{H}$ satisfies the principle of homogeneity!

Generally

Let $x(t) = \sum_{i=1}^{N} \alpha_i \cdot x_i(t)$

$$y(t) = \mathcal{H}\{x_i(t)\}$$
$$= \mathcal{H}\{\sum_{i=1}^{N} \alpha_i x_i(t)\}$$

If $y(t) = \sum_{i=1}^{N} \alpha_i \underbrace{\mathcal{H}\{x_i(t)\}}_{y_i(t)}$ $\sum_{i=1}^{N} \alpha_i y_i(t)$ then $\mathcal{H}$ is linear!

Let $y_i = \mathcal{H}\{x_i\}$

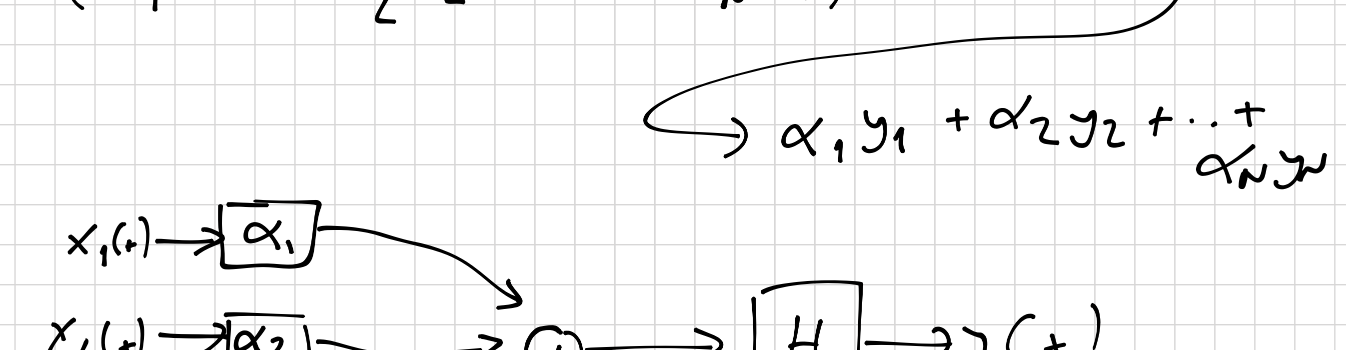
$x_1 \to \boxed{\mathcal{H}} \to y_1$

$x_2 \to \boxed{\mathcal{H}} \to y_2$

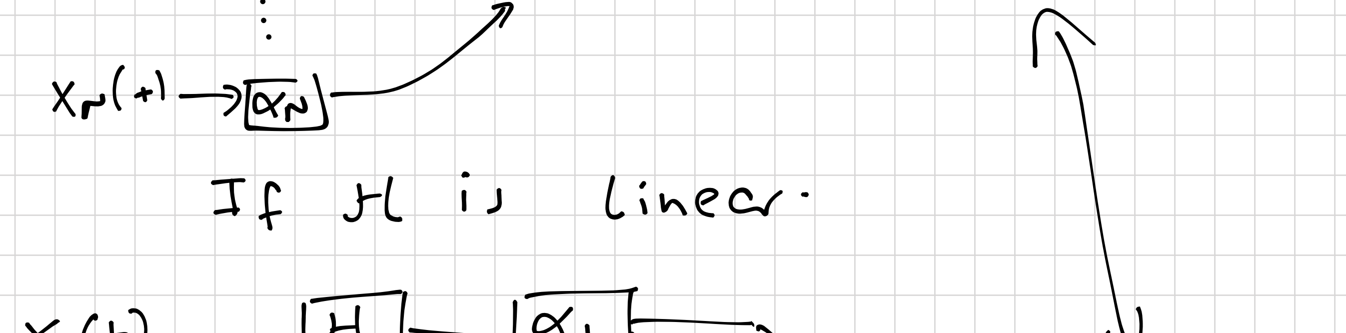
$\vdots$

$x_N \to \boxed{\mathcal{H}} \to y_N$

$(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_N x_N) \to \boxed{\mathcal{H}} \to \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_N y_N$



If $\mathcal{H}$ is linear.



Ex $\quad y(t) = (t+1)^2\, x(t)$ $\qquad$ Is this system Linear ?

Homogeneity test

$$\mathcal{H}\{\alpha\, x_1(t)\} = (t+1)^2\, \alpha\, x(t)$$
$$= \alpha\left[(t+1)^2 x(t)\right]$$
$$= \alpha \cdot y(t)$$

✓ Homogeneity is satisfied.

Superposition Test

Let $\mathcal{H}\{x_1(t)\} = y_1(t)$

$\mathcal{H}\{x_2(t)\} = y_2(t)$

$$\mathcal{H}\{x_1(t) + x_2(t)\}$$
$$= (t+1)^2\left[x_1(t) + x_2(t)\right]$$
$$= (t+1)^2 x_1(t) + (t+1)^2 x_2(t)$$
$$= y_1(t) + y_2(t)$$

✓ Superposition is satisfied.

$\therefore$ The system is LINEAR

Ex

$$y(t) = x(t) \cdot x(t-1)$$

Homogeneity test

$$\mathcal{H}\{\alpha\, x(t)\} = (\alpha\, x(t)) \cdot (\alpha\, x(t-1))$$
$$= (\alpha^2)\, x(t)\, x(t-1)$$

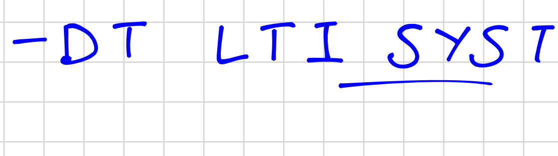
$$\alpha\, y(t) = \alpha \cdot x(t) \cdot x(t-1)$$

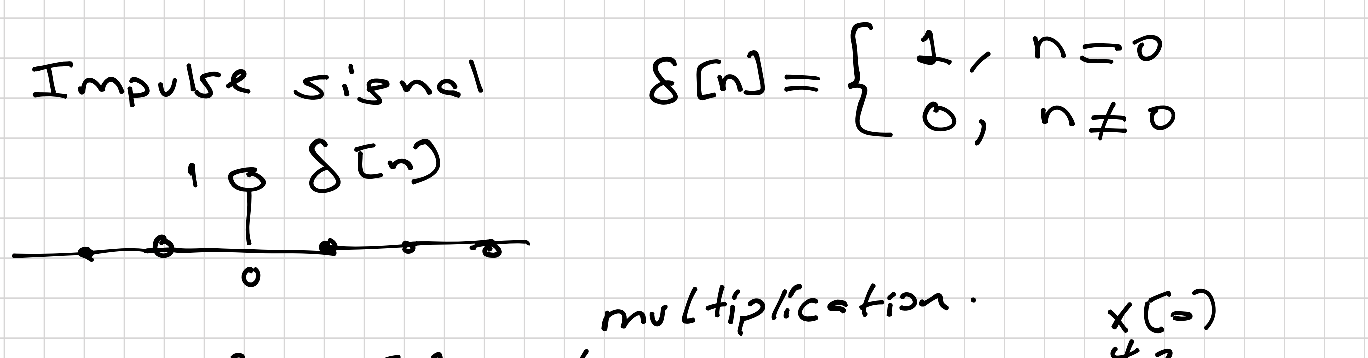
not equal $\therefore$ The system is NOT Linear !

LINEAR – TIME INVARIANT SYSTEMS

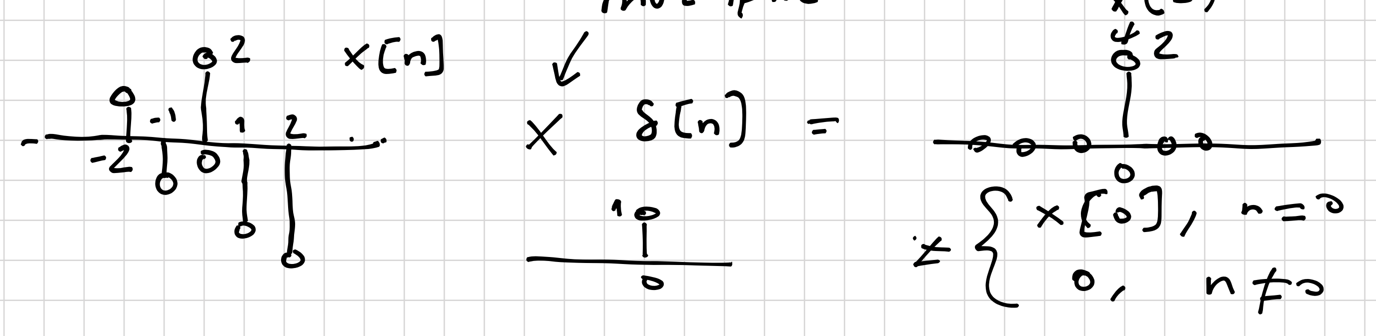
– DT LTI SYSTEMS : Convolution Sum

Impulse signal $\qquad \delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$





$$x[n]\,\delta[n] = x[0]\,\delta[n]$$



$$x[n]\,\delta[n-1] = x[1]\,\delta[n-1]$$

$$x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$$

$$= x[-2]\cdot\delta[n+2] + x[-1]\cdot\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

Let $\mathcal{H}$ be LTI

$x[n] \to \boxed{\mathcal{H}} \to y[n]$

$$y[n] = \mathcal{H}\{x[n]\} = \mathcal{H}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]\right\}$$

$$y[n] = \sum_{k=-\infty}^{+\infty} \mathcal{H}\{x[k]\,\delta[n-k]\} \qquad (\mathcal{H} \text{ is linear}) \text{ - Superposition}$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\,\mathcal{H}\{\delta[n-k]\} \qquad \text{Homogeneity}$$

$$h[n-k] \triangleq \mathcal{H}\{\delta[n-k]\}$$

$\mathcal{H}$ is also T.I.

$\therefore\ h[n] = \mathcal{H}\{\delta[n]\}$ $\to$ IMPULSE RESPONSE

$\delta[n] \to \boxed{\mathcal{H}} \to h[n]$

So

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k] \quad \Big\}\ \text{DT convolution!}$$

$$y[n] = x[n] * h[n]$$

$*$ ← convolution operator.

Ex

$$y[n] = \mathcal{H}\{x[n]\}$$
$$= x[n] + \frac{1}{2}\,x[n-1]$$

a) Impulse Response

$\delta[n] \to \boxed{\mathcal{H}} \to h[n]$

$$h[n] = \delta[n] + \frac{1}{2}\,\delta[n-1] = \begin{cases} 1, & n=0 \\ 1/2 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

b) $\quad x[n] = \begin{cases} 2, & n=0 \\ 4, & n=1 \\ -2, & n=2 \\ 0, & \text{otherwise.} \end{cases}$

$$x[n] = 2\,\delta[n] + 4\,\delta[n-1] - 2\,\delta[n-2]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$
$$= x[0]\,h[n] + x[1]\,h[n-1] + x[2]\,h[n-2]$$
$$= 2\,h[n] + 4\,h[n-1] - 2\,h[n-2]$$

0	1	2	3	
2	1			$2h[n]$
	4	2		$4h[n-1]$
		-2	-1	$-2h[n-2]$
2	5	0	-1	$y[n]$

$$y[n] = \begin{cases} 2, & n=0 \\ 5, & n=1 \\ -1, & n=3 \\ 0, & \text{otherwise} \end{cases}$$