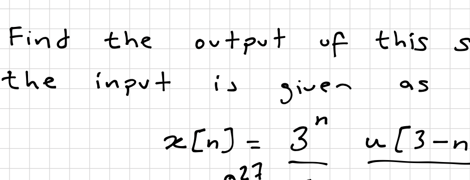
Ex

$\mathcal{H}$ is a discrete LTI system, the impulse response of $\mathcal{H}$ is given as:

$$h[n] = u[n-2]$$

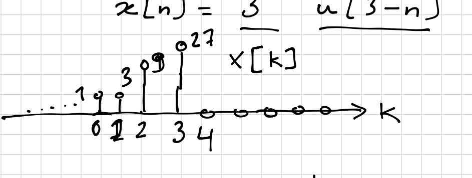
Find the output of this system when the input is given as

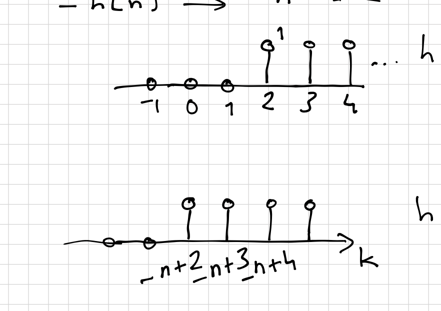
$$x[n] = 3^n \, u[3-n]$$

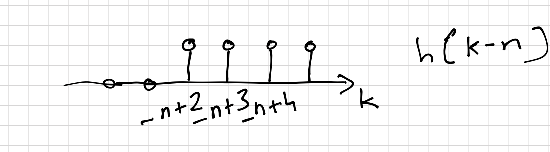


$h[n-k] = u[n-k-2] = u[(n-2)-k]$

$-h[n] \rightarrow n \rightarrow k$







$$w_n[k] = h[n-k] \cdot x[k] \qquad y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$$

1) $n-2 \leq 3$ $\qquad w_n[k] = \begin{cases} 3^k \cdot 1, & k \leq n-2 \\ 0, & k > n-2 \end{cases}$

$-\infty < n-2 \leq 3$

2) $n-2 > 3$ $\qquad w_n[k] = \begin{cases} 3^k, & k \leq 3 \\ 0, & k > 3 \end{cases}$

1) $y[n] = \sum_{k=-\infty}^{n-2} 3^k = \dots = 3^n/6$ for $n \leq 5$

2) $n > 5 \Rightarrow y[n] = \sum_{k=-\infty}^{3} 3^k = \frac{81}{2}$

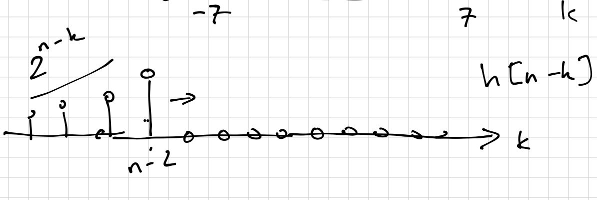
$$\boxed{y[n] = \begin{cases} 3^n/6, & n \leq 5 \\ \frac{81}{2}, & n > 5 \end{cases}}$$

Ex

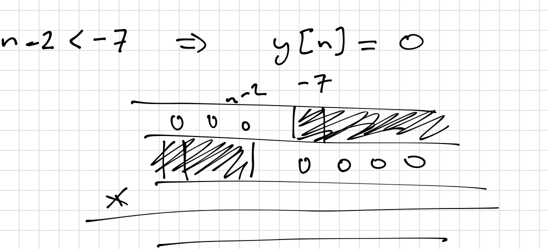
$\mathcal{H}$ is a discrete LTI system

$h[n] = 2^n u[n-2]$

$\rightarrow x[n] = \left(\frac{1}{2}\right)^n \{u[n+7] - u[n-8]\}$



1) $n-2 < -7 \Rightarrow y[n] = 0$



$2^{-k} \cdot 2^{-k} = 4^{-k}$

2) $-7 \leq n-2 \leq 7$

$$y[n] = \sum_{k=-7}^{n-2} 2^{n-k} \cdot \left(\frac{1}{2}\right)^k = 2^n \sum_{k=-7}^{n-2} 4^{-k} = \frac{4}{3} 2^n (4^7 - 4^{-(n+1)})$$

3) $n-2 > 7$, $n > 9$

$$y[n] = \sum_{k=-7}^{7} 2^{n-k} \left(\frac{1}{2}\right)^k = 21845{,}3 \cdot 2^n$$

$$\boxed{y[n] = \begin{cases} 0, & n < -5 \\ \frac{4}{3} 2^n (4^7 - 4^{-(n+1)}), & -5 \leq n \leq 9 \\ 21845{,}3 \cdot 2^n, & n > 9 \end{cases}}$$

EXAMPLES

Ex Consider the system $\mathcal{H}$, defined by the following input-output relationship:

$$y[n] = x[n] \cdot \sum_{k=-\infty}^{+\infty} \delta[n-2k]$$

$$\delta[n-2k] = \begin{cases} 1, & n = 2k \\ 0, & n \neq 2k \end{cases}$$

$$\therefore \sum_{k=-\infty}^{+\infty} \delta[n-2k] = \begin{cases} 1, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

$$\Rightarrow y[n] = \begin{cases} x[n], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

memoryless? $y[n]$ depends on only the current values of $x[n]$ $\therefore$ Memoryless

Causal? All memoryless systems are causal.

Linearity

1) Homogeneity

$$H\{\alpha x[n]\} = \begin{cases} \alpha \cdot x[n], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} = \alpha y[n]$$

$\therefore$ Homogeneity is satisfied.

2) Superposition.

Let $x_1[n] \xrightarrow{\mathcal{H}} y_1[n]$

$x_2[n] \xrightarrow{\mathcal{H}} y_2[n]$

$x[n] = x_1[n] + x_2[n]$

$$\mathcal{H}\{x[n]\} = \mathcal{H}\{x_1[n] + x_2[n]\} = \begin{cases} x_1[n] + x_2[n], & n \text{ is even} \\ 0, & \text{otherwise} \end{cases} = y_1[n] + y_2[n]$$

$\therefore$ Superposition is satisfied

$\Rightarrow \mathcal{H}$ is Linear!

Time Invariance

Let's say $y_1[n] = \mathcal{H}\{x[n-n_0]\} = \begin{cases} x[n-n_0], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$

$y_2[n] = y(n-n_0) = \begin{cases} x[n-n_0], & n-n_0 \text{ is even} \\ 0, & n-n_0 \text{ is odd} \end{cases}$

$y_1[n] \neq y_2[n]$

$\therefore \mathcal{H}$ is not Time invariant!

Stability

Assume $|x[n]| \leq M_x < \infty$ for $\forall n$

Then $|y[n]| = |x[n]| \, \Big|\sum_{k=-\infty}^{+\infty} \delta[n-2k]\Big|$, where $|x[n]| \leq M_x$ and $\Big|\sum \delta[n-2k]\Big| \leq 1$

$|y[n]| \leq M_x$

$\therefore \mathcal{H}$ is stable.

Ex Consider a DT system which has the following impulse response:

$$h[n] = \begin{cases} 2, & n = -1, +1 \\ 0, & \text{otherwise} \end{cases}$$

Find the output when the input is:

$$x[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ -1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k]$$

$$= x[0] \cdot h[n] + x[1] h[n-1] + x[3] h[n-3]$$

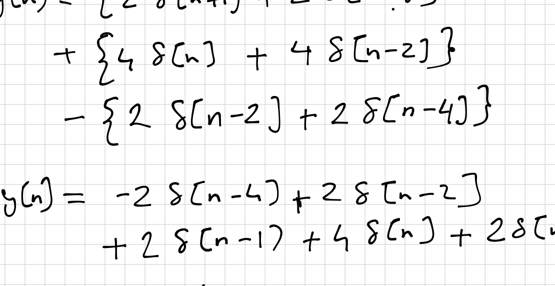
$$= h[n] + 2h[n-1] - h[n-3]$$

-3	-2	-1	0	1	2	3	4	n
		2	0	2				$h[n]$
			4	0	4			$2h[n-1]$
					-2	0	-2	$-h[n-3]$
		2	4	2	2	0	-2	$y[n]$

$h[n] = 2\delta[n+1] + 2\delta[n-1]$

$y[n] = \{2\delta[n+1] + 2\delta[n-1]\} + \{4\delta[n] + 4\delta[n-2]\} - \{2\delta[n-2] + 2\delta[n-4]\}$

$y[n] = -2\delta[n-4] + 2\delta[n-2] + 2\delta[n-1] + 4\delta[n] + 2\delta[n+1]$



Ex CT system

$$y(t) = \mathcal{H}\{x(t)\} = e^{-2t} x(t) u(t)$$

Stable?

Assume $|x(t)| \leq M_x < \infty$

$\therefore |y(t)| \leq |e^{-2t} M_x u(t)|$

$|e^{-2t} u(t)| \leq 1 \quad \forall t$

$|y(t)| \leq M_x \cdot 1$

$\therefore \mathcal{H}$ is BIBO-STABLE

Ex $y[n] = \mathcal{H}\{x[n]\} = 2x[n-1](u[n] - u[n-4])$

Memory $\rightarrow$ Current values of output depends on the past values of the input. $\rightarrow$ not-memoryless

causality $\rightarrow$ Yes, causal.

Stability $\rightarrow$ Assume $|x[n]| \leq M_x < \infty$

$|y[n]| = |2x[n-1](u[n] - u[n-4])| \leq 2 M_x 1$

Finite $\therefore \mathcal{H}$ is stable

Linearity:

Homogeneity

$H\{\alpha x[n]\} = 2 \cdot \alpha x[n-1] (u[n] - u[n-4]) = \alpha y[n]$

Superposition

Let $\mathcal{H}\{x_1[n]\} = y_1[n]$

$\mathcal{H}\{x_2[n]\} = y_2[n]$

$\mathcal{H}\{x_1[n] + x_2[n]\} = 2 \cdot \{x_1[n] + x_2[n]\}(u[n] - u[n-4]) = y_1[n] + y_2[n]$

$\mathcal{H}$ is Linear

Time Invariance $\quad =? \begin{cases} y_1[n] = y[n-n_0] \\ y_2[n] = \mathcal{H}\{x[n-n_0]\} \end{cases}$

$y_2[n] = 2x[n-n_0-1](u[n] - u[n-4])$

$y_1[n] = 2x[n-n_0-1] \cdot (u[n-n_0] - u[n-n_0-4])$

not equal

$\therefore$ not TI