Step Response

"Step response" is defined as the output of a system when the input is a step signal.

$$u[n],\ u(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow s[n],\ s(t)$$

In the case that $\mathcal{H}$ is LTI and the impulse response is $h[n]$, $h(t)$

DT $\quad s[n] = h[n] * u[n]$

CT $\quad s(t) = h(t) * u(t)$

DT

$$s[n] = \sum_{k=-\infty}^{+\infty} h[k]\, u[n-k]$$

$u[n-k] = 0 \quad k > n$
$u[n-k] = 1 \quad k \leq n$

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k]} \quad * \text{ DT Step response}$$

CT

$$s(t) = \int_{-\infty}^{+\infty} h(\tau)\, u(t-\tau)\, d\tau$$

Since $u(t-\tau) = 0 \quad \tau > t$
$\qquad u(t-\tau) = 1 \quad \tau < t$

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau} \quad \text{CT step response}$$

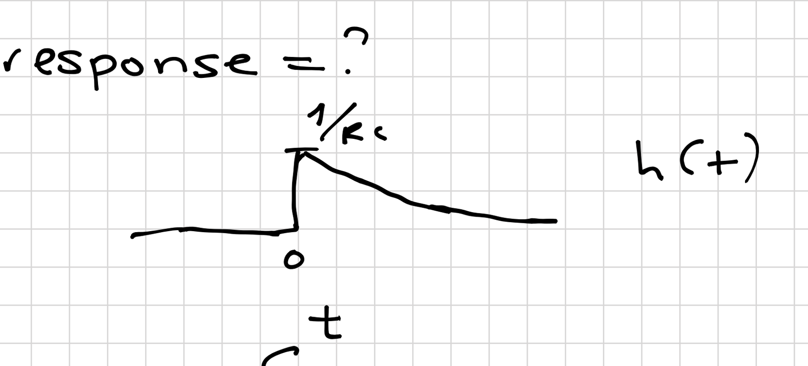
To find the impulse response from the step response

DT $\quad h[n] = s[n] - s[n-1]$

CT $\quad h(t) = \frac{d}{dt} s(t)$

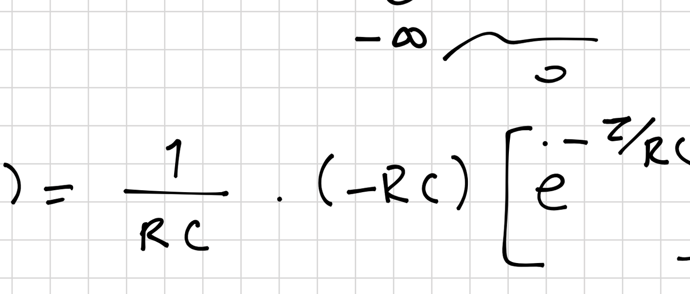
/*
$$s[n] = \underbrace{\cdots + h[n-2] + h[n-1]}_{s[n-1]} + h[n] + 0 + \cdots$$
*/

Ex



If $x(t) = \delta(t)$
then
$$y(t) = h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Step response = ?

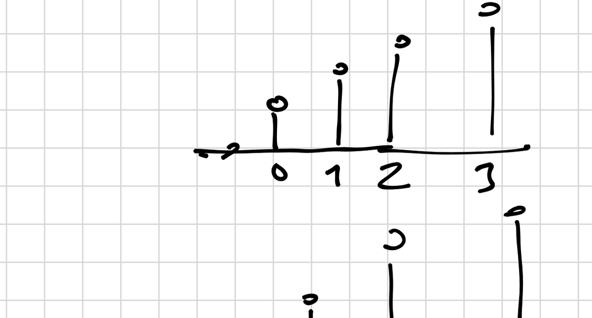


$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

* $t < 0 \Rightarrow s(t) = 0$
* $t > 0 \quad s(t) = \int_{-\infty}^{0} 0\, d\tau + \int_{0}^{t} \frac{1}{RC} e^{-\tau/RC}\, d\tau$

$$s(t) = \frac{1}{RC} \cdot (-RC) \left[e^{-\tau/RC} \right]_0^t$$
$$= 1 - e^{-t/RC} \quad t > 0$$

$$\boxed{s(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t/RC}, & t > 0 \end{cases}}$$

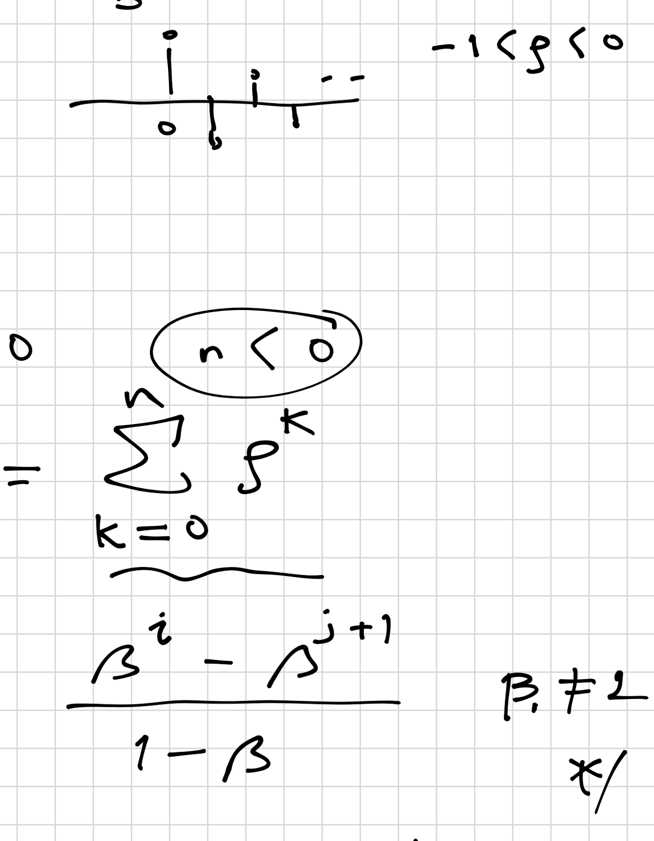


Ex

$\mathcal{H}$ is an DT LTI system,
$$h[n] = \rho^n u[n]$$
Find the step response

$|\rho| > 1$

$|\rho| < 1$



* $|\rho| > 1 \Rightarrow$

$s[n] = 0 \quad n < 0$

$n \geq 0 \quad s[n] = \sum_{k=0}^{n} \rho^k$

/* $\displaystyle\sum_{k=i}^{j} \beta^k = \frac{\beta^i - \beta^{j+1}}{1-\beta} \quad \beta \neq 1$ */

Then
$$s[n] = \frac{\rho^0 - \rho^{n+1}}{1-\rho}$$
$$= \frac{1-\rho^{n+1}}{1-\rho} \quad \text{for } n \geq 0$$

$$\therefore \quad s[n] = \frac{1-\rho^{n+1}}{1-\rho} u[n]$$

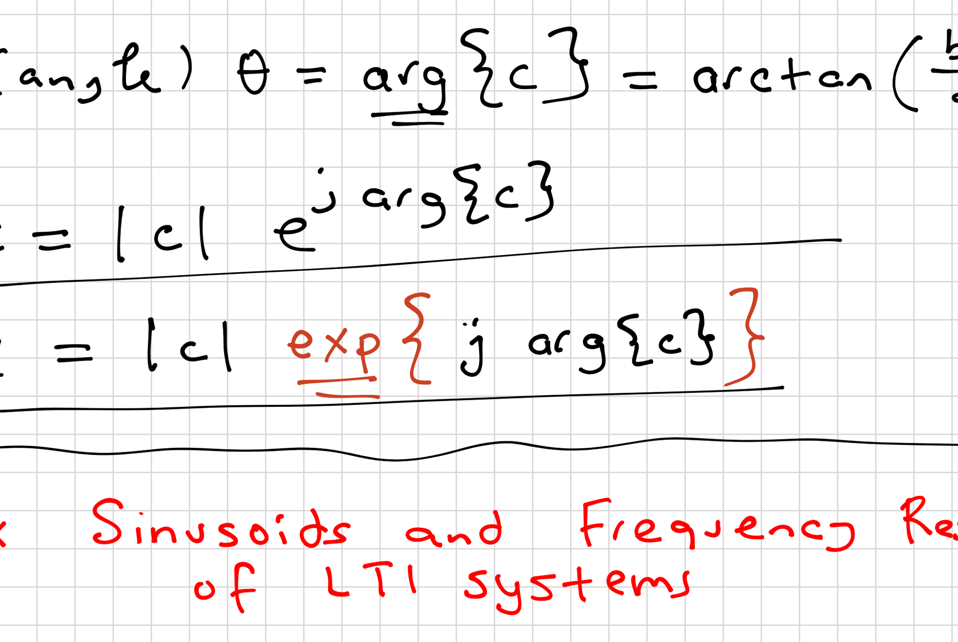
FOURIER REPRESENTATION OF LTI SYSTEMS

$$\rightarrow x[n] = \sum_{k=-\infty}^{+\infty} x[k]\, \delta[n-k]$$

$$\rightarrow x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

In this chapter:
Representation of signals as weighted superposition of complex sinusoids.

Euler's Formula } $e^{j\theta} = \cos\theta + j \sin\theta$



$c = a + jb$

$$|c| = \sqrt{a^2 + b^2}$$

(angle) $\theta = \arg\{c\} = \arctan\left(\frac{b}{a}\right)$

$$c = |c|\, e^{j \arg\{c\}}$$

$$c = |c| \exp\{ j \arg\{c\} \}$$

Complex Sinusoids and Frequency Response of LTI systems

DT

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$$

$$y[n] = \mathcal{H}\{x[n]\}$$

Impulse Response } $h[n] = \mathcal{H}\{\delta[n]\}$

$$y[n] = \mathcal{H}\{x[n]\}$$
$$= x[n] * h[n]$$
$$= h[n] * x[n]$$
$$= \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

Let $x[n] = e^{j\Omega n} \qquad \Omega$: frequency

$$= \cos(\Omega n) + j \sin(\Omega n)$$

$$y[n] = \mathcal{H}\{e^{j\Omega n}\}$$
$$= \sum_{k=-\infty}^{+\infty} h[k]\, e^{j\Omega(n-k)}$$
$$= e^{j\Omega n} \sum_{k=-\infty}^{+\infty} h[k] e^{-j\Omega k}$$

Let's define
$$\underbrace{H(e^{j\Omega})}_{\text{Frequency Response}} \triangleq \sum_{k=-\infty}^{+\infty} h[k]\, e^{-j\Omega k}$$

— Not a function of time but frequency!

$$y[n] = \boxed{H(e^{j\Omega})} \cdot e^{j\Omega n} = \mathcal{H}\{e^{j\Omega n}\}$$

$$e^{j\Omega n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow \boxed{H(e^{j\Omega})} \cdot e^{j\Omega n}$$

CT

$$x(t) \rightarrow \boxed{\mathcal{H}} \rightarrow y(t) = \mathcal{H}\{x(t)\}$$

Let $x(t) = e^{jwt}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{jw(t-\tau)}\, d\tau$$

$$= e^{jwt} \int_{-\infty}^{+\infty} h(\tau)\, e^{-jw\tau}\, d\tau$$

Frequency Response:

$$H(jw) = \underbrace{\int_{-\infty}^{+\infty} h(\tau) \exp(-jw\tau)\, d\tau}_{\text{not dependent on time !}}$$

So,

$$y(t) = \mathcal{H}\{e^{jwt}\} = e^{jwt} \cdot H(jw)$$

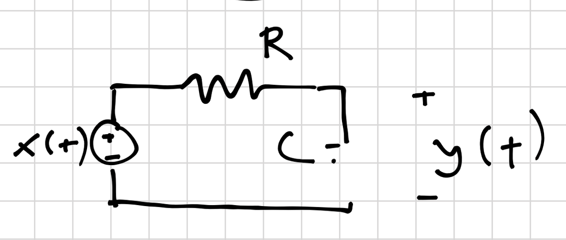
$$e^{jwt} \rightarrow \boxed{\mathcal{H}} \rightarrow e^{jwt}\, H(jw)$$

Polar form

$$\mathcal{H}\{e^{jwt}\} = |H(jw)| \cdot \exp\left\{ j(wt + \arg\{H(jw)\}) \right\}$$

$$H(jw) = \underbrace{|H(jw)|}_{\text{magnitude respond}}\, e^{-j \underbrace{\arg\{H(jw)\}}_{\text{phase response}}}$$

Ex



$x(t)$: input voltage
$y(t)$: output voltage
Let $\alpha \triangleq RC$

$$h(t) = \frac{1}{\alpha} \exp\left\{-\frac{t}{\alpha}\right\} u(t)$$

Frequency Response

$$H(jw) = \int_{-\infty}^{+\infty} h(\tau) \exp\{-jw\tau\}\, d\tau$$

$$= \int_{0}^{+\infty} \frac{1}{\alpha} e^{-\tau/\alpha} \exp\{-jw\tau\}\, d\tau$$

$$= \frac{1}{\alpha} \int_{0}^{\infty} \exp\left\{-\left(jw + \frac{1}{\alpha}\right)\tau\right\} d\tau$$

$$= \frac{1}{\alpha} \cdot \frac{-1}{jw + 1/\alpha} \cdot \Big[\underbrace{\exp\{\cdot\}}_{-1}\Big]_0^{\infty}$$

$$H(jw) = \frac{1/\alpha}{jw + 1/\alpha}$$ Frequency response

* Magnitude response

/* $|a + jb| = \sqrt{a^2 + b^2}$ */

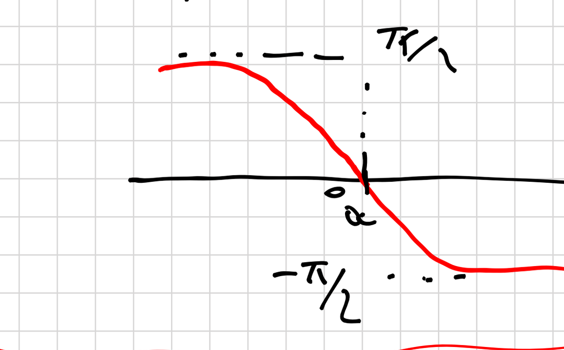
$$H(jw) = \frac{1/\alpha}{1/\alpha + jw} \cdot \frac{1/\alpha - jw}{1/\alpha - jw}$$

$$= \frac{1/\alpha\, (1/\alpha - jw)}{1/\alpha^2 - (jw)^2}$$

$$= \frac{1/\alpha\, (1/\alpha - jw)}{1/\alpha^2 + w^2}$$

$$|H(jw)| = \frac{1/\alpha}{1/\alpha^2 + w^2} \sqrt{1/\alpha^2 + (jw)^2}$$

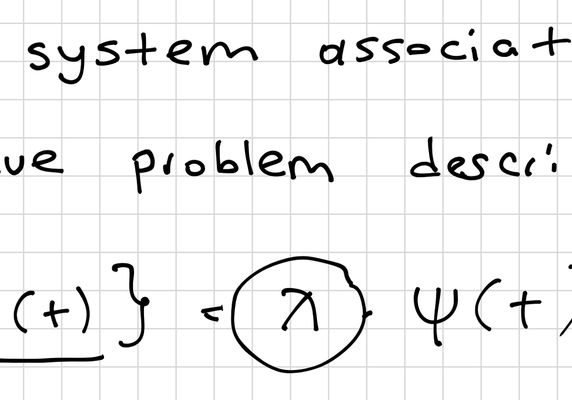
$$= \frac{1/\alpha}{\sqrt{1/\alpha^2 + w^2}}$$



$|H(jw)|$ Magnitude Response

Phase Response

$$\arg\{H(jw)\} = -\arctan\{\alpha w\}$$



Eigenfunction

Linear Algebra
Eigenvalue
vector

$\psi(t) = e^{jwt}$ is an eigenfunction of the LTI system associated with the eigenvalue problem described by

$$H\{\psi(t)\} = \lambda\, \psi(t)$$

$$\psi(t) \rightarrow \boxed{H} \rightarrow H(jw)\, \psi(t)$$

By representing arbitrary signals as weighted superposition of eigenfunction we can transform the convolution to multiplication!

Let $x(t)$ be a weighted sum of M complex sinusoids such that:

$$x(t) = \sum_{k=1}^{M} a_k \cdot e^{jw_k t}$$

If $e^{jw_k t}$ is an eigenfunction of the system with an eigenvalue $H(jw)$, then each term in $x(t)$ produces an output $a_k \cdot H(jw_k) \cdot e^{jw_k t}$

$$y(t) = \sum_{k=1}^{M} a_k\, H(jw_k)\, e^{jw_k t}$$

Ex Let $y(t) = x(t-3)$

If the input is $x(t) = e^{j2t}$

then $y(t) = e^{j2(t-3)} = e^{j2t}\, e^{-j6}$

e^{j2t} is the eigenfunction associated with the eigenvalue $H(j6) = e^{-j6}$

$$e^{j2t} \rightarrow \boxed{\mathcal{H}} \rightarrow H(j6)\, e^{j2t}$$

Impulse response is
$h(t) = \delta(t-3)$

$$H(jw) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-jw\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty} \delta(\tau - 3)\, e^{-jw\tau} = e^{-3jw}$$

$$H(j2) = e^{-3j \cdot 2} = e^{-6j}$$