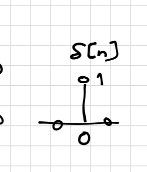
LTI Systems Cont'd

– DT

Impulse Signal $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$



$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$

Impulse Response
$h[n] \triangleq \mathcal{H}\{\delta[n]\}$

$y[n] = \mathcal{H}\{x[n]\}$

$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k]$

$y[n] = x[n] * \underline{h[n]}$ convolution operation

$\boxed{y[n]} = \underline{x[n]} * \underline{h[n]}$

DT Convolution Evaluation Procedure

$y[n] = x[n] * h[n]$

$y[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k] \cdot h[n-k]}_{w_n[k]}$

$w_n[k] \triangleq x[k] \cdot h[n-k]$

independent variable

$x[n] \longrightarrow x[k]$

$h[n] \longrightarrow h[n-k]$

↓

Find $h[k]$

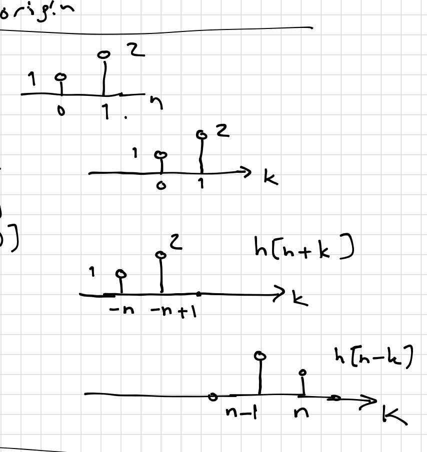
↓

$h[k+n]$ shift left by n

↓

vertically reflect around origin → $h[-k+n]$

Ex



$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$

Ex $h[n] = \left(\frac{3}{4}\right)^n u[n]$

Let's find the output of the system at $n = \underline{-5}, \underline{5}, \underline{10}$ when the input is $x[n] = u[n]$

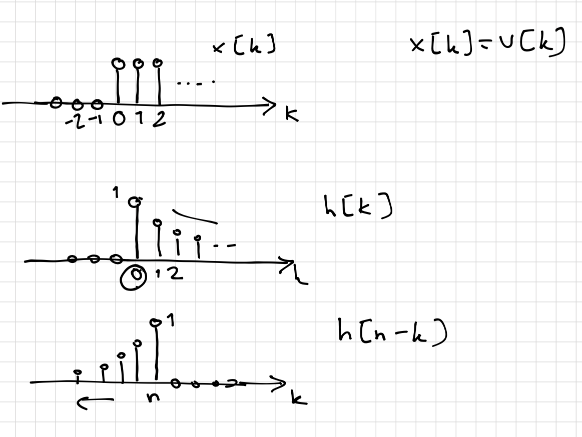
$y[-5]$, $y[5]$, $y[10]$

$y[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k] \cdot h[n-k]}_{w_n[k]}$

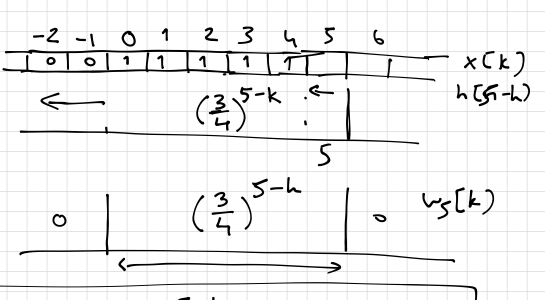
$h[n-k] = \left(\frac{3}{4}\right)^{n-k} u[\underline{n-k}]$

$h[n-k] = \begin{cases} (3/4)^{n-k}, & k \leq n \\ 0, & \text{otherwise} \end{cases}$

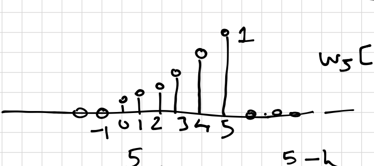
$x[k] = u[k]$



$w_5[k] = x[k] \cdot h[5-k]$



$\boxed{w_5[k] = \begin{cases} \left(\frac{3}{4}\right)^{5-k}, & 0 \leq k \leq 5 \\ 0, & 0 \end{cases}}$



$y[5] = \sum_{k=-\infty}^{+\infty} w_5[k] = \sum_{k=0}^{5} \left(\frac{3}{4}\right)^{5-k}$

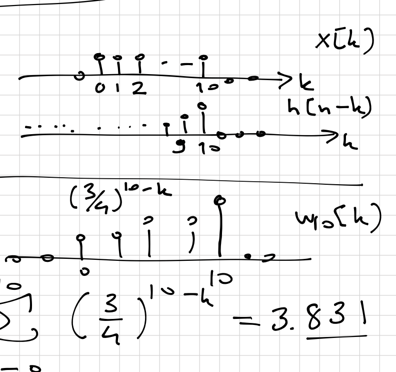
$= \left(\frac{3}{4}\right)^5 \sum_{k=0}^{5} \left(\frac{4}{3}\right)^k$

$= \left(\frac{3}{4}\right) \cdot \frac{1-(4/3)^6}{1-4/3}$

$= 3.288$

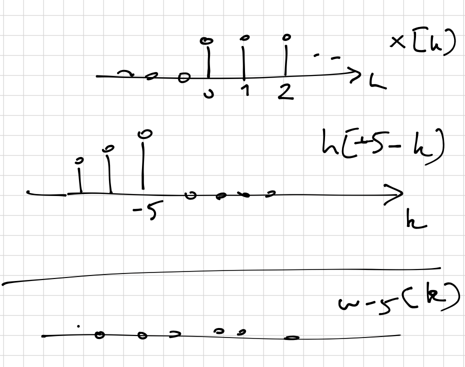
$\Rightarrow \boxed{y[5] = 3.288}$

$y[10] = ?$



$y[10] = \sum_{k=0}^{10} \left(\frac{3}{4}\right)^{10-k} = \underline{3.831}$

$y[-5]$



$\therefore$ $y[-5] = 0$

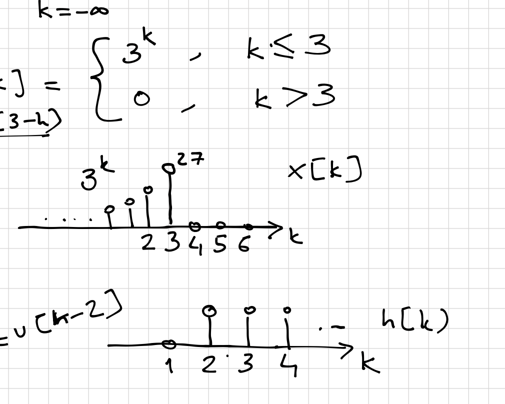
Ex $\mathcal{H}$ is an LTI system, the impulse response of $\mathcal{H}$ is given as $h[n] = u[n-2]$

Find the output of this system when the input is given as

$x[n] = 3^n \cdot \underline{u[3-n]}$

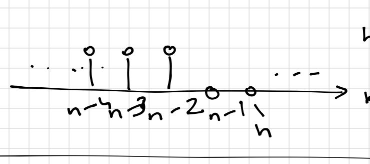
$y[n] = x[n] * h[n]$
$= \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k]$

⊙ $x[k] = \begin{cases} 3^k, & k \leq 3 \\ 0, & k > 3 \end{cases}$
$= 3^k u[3-k]$



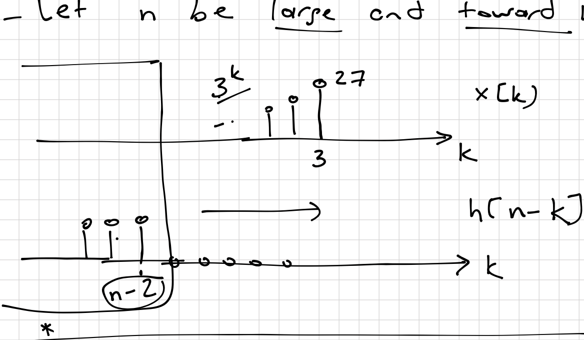
• $h[k] = u[k-2]$

– $h[n-k] = u[n-k-2]$
$= u[(n-2)-k]$
$= \begin{cases} 1 & k \leq n-2 \\ 0, & k > n-2 \end{cases}$



$w_n[k] = x[k] \cdot h[n-k]$

– Let n be $\underline{\text{large}}$ and $\underline{\text{toward}}$ left



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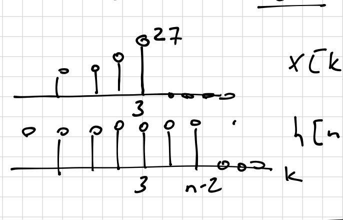
$w_n[k] = \begin{cases} 1 \cdot 3^k, & k \leq n-2 \\ 0, & \text{otherwise} \end{cases}$

Is the same formula for $-\infty < n-2 \leq 3$

• Shift $h[n-k]$ towards $\underline{\text{right}}$



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$w_n[k] = \begin{cases} 3^k, & \underline{k \leq 3} \\ 0, & \text{otherwise} \end{cases}$

for $n-2 > 3$

① For $n-2 \leq 3 \rightarrow n \leq 5$

$y[n] = \sum_{k=-\infty}^{n-2} 3^k = \dots = \frac{1}{6} 3^n$

② $n-2 > 3 \Rightarrow n > 5$

$y[n] = \sum_{k=-\infty}^{3} 3^k = \dots = \frac{81}{2}$

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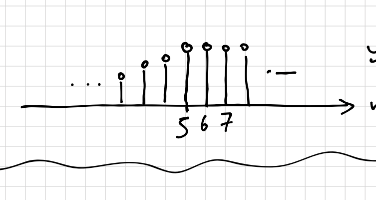
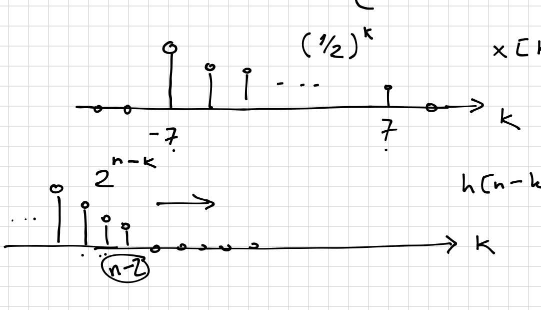
$y[n] = \begin{cases} 3^n/6, & n \leq 5 \\ 81/2, & n > 5 \end{cases}$



Ex $\mathcal{H} \rightarrow$ LTI

$h[n] = 2^n \, u[n-2]$

$x[n] = \left(\frac{1}{2}\right)^n \{u[n+7] - u[n-8]\}$



$n-2 < -7$ $\quad y[n] = 0$

$\underline{-7 \leq n-2 \leq 7}$ $\quad y[n] = \sum_{k=-7}^{n-2} \left(\frac{1}{2}\right)^k (2)^{n-k}$

$= 2^n \sum_{k=-7}^{n-2} 4^{-k}$

$= \frac{4}{3} 2^n \left(4^7 - 4^{-(n+1)}\right)$

$n-2 > 7$ $\quad y[n] = \sum_{k=-7}^{7} \left(\frac{1}{2}\right)^k 2^{n-k}$

$= \dots$

$= 2^n \times 21845{,}33$

$\boxed{y[n] = \begin{cases} 0, & n < -5 \\ \frac{4}{3} 2^n (4^7 - 4^{-(n+1)}), & -5 \leq n \leq 9 \\ 21845.3 \times 2^n, & n > 9 \end{cases}}$

CT Convolution

$\mathcal{H}$: an $\overset{CT}{\text{LTI}}$ system

$$y(t) = \mathcal{H}\{x(t)\}$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\delta(t) = 0 \quad t = 0$$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

$$y(t) = \mathcal{H}\left\{ \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \right\}$$

Because $\mathcal{H}$ is an LTI system

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$

where $h(t) \triangleq \mathcal{H}\{\delta(t)\}$

Impulse Response

$$y(t) = x(t) * h(t)$$

convolution operator