Vector Field Mid Review

Yifan Li

November 2023

1 Coordinate

There are three commonly used coordinates: rectangular (Cartesian), cylindrical, and spherical coordinates. You need to know their conversions.

Proposition 1.1. *Conversion of coordinates*

Conversion	Formulae		
$(r,\theta,z) \to (x,y,z)$	$x = r\ \cos\theta$	$y = r\ \sin theta$	$z = z$
$(x,y,z) \to (r,\theta,z)$	$r = \sqrt{x^2+y^2}$	$\tan\theta = y/x$	$z = z$
$(\rho,\phi,\theta) \to (r,\theta,z)$	$r = \rho\sin\theta$	$\theta = \theta$	$z = \rho\cos\phi$
$(r,\theta,z) \to (\rho,\phi,\theta)$	$\rho = \sqrt{r^2+z^2}$	$\tan\phi = r/z$	$\theta = \theta$
$(\rho,\phi,\theta) \to (x,y,z)$	$x = \rho\sin\phi\cos\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$(x,y,z) \to (\rho,\phi,\theta)$	$\rho = \sqrt{x^2+y^2+z^2}$	$\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan\theta = \frac{y}{x}$

There is another thing that is important for the conversion of coordinates. When we integrate along a curve, we may need to know the "element" of arc length, ds. In addition, we also need to consider the "element" of the area of a surface or that of volume. Thanks to Renjing and Prof. Andrew, they brought me a fabulous way to think about the element, div and curl.

Consider the conversion of coordiante $(x^1, \ldots, x^n) \to (u^1, \ldots, u^n)$. Then the position can be written as

$$\vec{r}(u^1, \ldots, u^n) = x^1\hat{e}_{x^1} + \ldots + x^n\hat{e}_{x^n}$$

Take the differentiation, we get

$$d\vec{r}(u^1, \ldots, u^n) = \sum_{i=1}^{n} \frac{\partial\vec{r}}{\partial u^i}\, du^i.$$

We hope that $\{\frac{\partial\vec{r}}{\partial u^i}\}$ forms a basis. To check whether the new coordiante system $(u^1, \ldots, u^n)$ is acceptable (the vectors above is a basis), we need to check the determinant

$$\begin{vmatrix} \frac{\partial\vec{r}}{\partial u^1} & \ldots & \frac{\partial\vec{r}}{\partial u^n} \end{vmatrix}$$

If the determinant is not 0, then the conversion is bijective. The determinant above is called Jacobian, denoted by

$$J(u^1,\,\ldots,\,u^n) = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^n} \end{vmatrix}$$

However, one may find that $\{\frac{\partial \vec{r}}{\partial u^1},\ldots,\frac{\partial \vec{r}}{\partial u^n}\}$ may not be unit vectors, thus we need to normalize them.

$$\vec{e_1} = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u^1},\ldots,\vec{e_n} = \frac{1}{h_n}\frac{\partial \vec{r}}{\partial u^n}$$
$$h_1 = \|\frac{\partial \vec{r}}{\partial u^1}\|,\ldots,h_n = \|\frac{\partial \vec{r}}{\partial u^n}\|$$

For an orthogonal coordiante system, $d\vec{r}$ can be rewritten as

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \ldots + \frac{\partial \vec{r}}{\partial u_n}du_n = h_1\,du_1\vec{e_1} + \ldots + h_n\,du_n\vec{e_n}. \tag{1}$$

Then the element of square of arc length can be written as

$$ds^2 = d\vec{r}\cdot d\vec{r} = h_1^2 du_1^2 + \ldots + h_n^2 du_n^2 \tag{2}$$

and

$$dV = dx_1\,dx_2\,dx_3 = \frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}du_1\,du_2\,du_3.$$

The statements above might be abstract, so an example is given below.

Example 1.1. *Consider the cylindrical coordinate, we know that a point in the space can be written as $\vec{r} = (r\cos\theta, r\sin\theta, z)$. Then we have*

$$h_1 = h_r = \|\frac{\partial \vec{r}}{\partial r}\| = \|(\cos\theta,\sin\theta,0)\| = 1$$
$$h_2 = h_\theta = \|\frac{\partial \vec{r}}{\partial \theta}\| = \|(-r\sin\theta,r\cos\theta,0)\| = r$$
$$h_3 = h_z = \|\frac{\partial \vec{r}}{\partial z}\| = \|(0,0,1)\| = 1$$

thus we have

$$\vec{e_r} = (\cos\theta,\sin\theta,0)$$
$$\vec{e_\theta} = (-\sin\theta,\cos\theta,0)$$
$$\vec{e_z} = (0,0,1)$$

One can check that they are orthogonal and satisfy the right-hand rule. It also explains that it is $r\,d\theta$ in $d\vec{r} = dr\,\vec{e_r} + rd\theta\,\vec{e_\theta} + dz\,\vec{e_z}$ instead of $d\theta$ itself.

Remark: You can also use dimension (unit) analysis to check that there should be an r in the coefficient of e_θ.

2 Gradient, Divergence and Curl

2.1 Directional Derivative

Before we introduce the gradient, we need to know what is the directional derivative.

Definition 2.1. *Given a scalar field f (in fact it is just a scalar function) and a vector v, the directional derivative of f in the direction $\vec{v} = v_1, \ldots, v_n$ at point p is defined to be*

$$\frac{\partial f}{\partial v}\Big|_p = \lim_{t \to 0} \frac{f(p + tv)}{t}.$$

One may check that $\frac{\partial f}{\partial v} = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \cdot \vec{v}$

If we restrict $\|\vec{v}\| = 1$, we may find that

$$|\frac{\partial f}{\partial v}| \leq \|(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})\|, \tag{3}$$

the equal holds when $\vec{v}$ is parallel to $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

Note that here scalar field and vector field can be regarded as functions $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{A} : \mathbb{R}^n \to \mathbb{R}^n$, respectively.

2.2 Gradient

From the inequality(3), it is quite nature to define the vector $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

Definition 2.2. *Gradient*

Given a scalar field f, the gradient of f is defined to be

$$\triangledown f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}). \tag{4}$$

<u>The gradient indicates the direction the the scalar field increases the most rapidly. In addition, its norm is the rate of the maximum increase</u>.

2.3 Divergence

Definition 2.3. *Divergence*

The divergence of a vector field $\vec{A} = (f_1, f_2, \ldots, f_n)$ is defined to be

$$div\,\vec{A} = \triangledown \cdot \vec{A} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \cdot \vec{A} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \tag{5}$$

We can not get anything intuitively from the definition. To have a better understanding, we may start with the Gauss theorem

$$\iiint_V div\,\vec{A}\,dV = \iint_S \vec{A} \cdot \vec{n}\,ds \tag{6}$$

divide each side by $\|V\|$ and let $\|V\| \to 0$, we get

$$div\,\vec{A} = \lim_{\|V\|\to 0} \frac{1}{\|V\|} \iint_S \vec{A}\cdot\vec{n}\,ds \tag{7}$$

One may find that the divergence is the flux of the vector field on an element of volume.

2.4 Curl

Definition 2.4. *For a vector field $\vec{A} = (f, g, h)$, we may define its curl as*

$$curl\,\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{8}$$

It is also hard to understand what is curl from the definition. Hence one can try to use Stokes' theorem which contains curl for help.

$$\oint_C \vec{A}\cdot d\vec{r} = \iint_S curl\,\vec{A}\cdot\vec{n}\,dS.$$

Divide each side by $\|S\|$ (the area of surface) and let $\|S\| \to 0$, then we get

$$curl\,\vec{A}\cdot\vec{n} = \lim_{\|S\|\to 0} \frac{1}{\|S\|} \oint_C \vec{A}\cdot d\vec{r} \tag{9}$$

Since $\|curl\,\vec{A} \geq |curl\,\vec{A}\cdot\vec{n}|$, one may find that the curl of a vector space can be regarded as the maximum of circulating flow on an element of area.

3 Some Basic Method for Integration

3.1 Line Integral

Integrate on Scalar Field Let f be a scalar function and C a curve. Then we may have a parametric representation $s : I \to \mathbb{R}^n,\ s = s(t)$ for the curve C.

The integral of f along C can be written as

$$\int_C f(s)\,ds = \lim_{\delta_{s_i}\to 0} \sum_{i=1}^{N} f(s_i)\delta_{s_i}$$

By equation(2), we know that the integral can be written as

$$\int_I f(s(t))\sqrt{h_1^2(\frac{ds_1}{dt})^2 + \ldots + h_n^2(\frac{ds_n}{dt})^2}\,dt \tag{10}$$

For rectangular coordinates, equation(10) is

$$\int_I f(s(t))\sqrt{(\frac{dx_1}{dt})^2+\ldots+(\frac{dx_n}{dt})^2}\,dt$$

For cylindrical coordinates, equation(10) is

$$\int_I f(s(t))\sqrt{(\frac{dr}{dt})^2+r^2(\frac{d\theta}{dt})^2+(\frac{z}{dt})^2}\,dt$$

In this case, the line integral is transformed into proper integral.

Integral on Vector Field Let $\vec{F}(x,y,z)=(f(x,y,z),g(x,y,z),h(x,y,z))$ a vector field, C a curve in $\mathbb{R}^3$ with representation $\vec{r}(t)$. Then the line integral of $\vec{F}$ along C is

$$\int_C \vec{F}(x,y,z)\cdot d\vec{r}=\int_C f(x,y,z)\,dx+g(x,y,z)\,dy+h(x,y,z)\,dz$$

To calculate the integral, one could use $d\vec{r}=\frac{d\vec{r}(t)}{dt}\,dt$ and compute the inner product $\vec{F}\cdot\frac{d\vec{r}(t)}{dt}$ first and integral along I. It would not be so hard.

3.2 Fundamental Theorem of Line Integral

Theorem 3.1. *If $\vec{F}=\nabla\phi$ for some scalar field ϕ, then*

$$\int_C \vec{F}\cdot d\vec{r}=\phi(p_1)-\phi(p_0)$$

where p_1 and p_0 are the end point and starting point, correspondingly.

Proof.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r}&=\int_C \frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz\\&=\int_a^b(\frac{\partial\phi}{\partial x}\frac{dx}{dt}+\frac{\partial\phi}{\partial y}\frac{dy}{dt}+\frac{\partial\phi}{\partial z}\frac{dz}{dt})\,dt\\&=\int_a^b\frac{d}{dt}\phi(x(t),y(t),z(t))\,dt\\&=\phi(x(t),y(t),z(t))_{t=a}^{t=b}=\phi(p_1)-\phi(p_0)\end{aligned}$$

□

Theorem 3.2. *Let $\vec{F}(x,y)=(f(x,y),g(x,y))$ where f and g have continuous first partial derivatives in an open simple connected region, then*

$$\exists\phi(\nabla\phi=\vec{F})\Leftrightarrow(\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x})$$

at each point of the region.

3.3 Green's Theorem

Theorem 3.3. *Let R be a simply connected plane region whose boundary is a piecewise smooth closed curve C oriented counterclockwise. If f and g have continuous first derivatives on some open set containing the region R, then*

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \tag{11}$$

Proof. The proof is on pages 47-50 of the note for Chapter 3. □

In fact, one can regard it as the $2-dim$ version of Stokes' theorem.

4 Surface Integration

Before we step into the surface integration, we may introduce the orientation first. For a curve, the orientation is the direction of the derivative of the representation $\vec{r}(t)$. However, the surface would have two normal vectors, so we need to define its orientation.

Definition 4.1. *Oriented Surface*

Let $S(u,v) : D \to \mathbb{R}^3$ be a parametric representation of the surface S. Define $\vec{n} : S(D) \to S^3$ mapping each point on the surface to the unit sphere. We say that S is oriented by $\vec{n}$ if for any curve $\alpha(t)$ on the surface, $\vec{n} \circ \alpha$ is continuous.

Intuitively, one could think that the normal vector changes continuously on the surface.

In addition, if there is $\vec{n}$ such that the surface S is oriented, we say S is orientable, and $\vec{n}$ an orientation of S. Otherwise, S is not orientable.

Notice: The Möbius strip is an example of non-orientable surfaces.

Definition 4.2. *Surface Integrals*

Let S be surface, g a scalar field. Then the integral of g on S is defined to be

$$\iint_S g\,dS = \lim \sum_{i=1}^{m} g_i\,\delta S_i \tag{12}$$

where m is the (supremum) area of δS_i.

The definition is not so convenient to compute the integral, so we need a further theorem.

Theorem 4.1. *If we can project the surface injectively onto the subset R of the $x-y$ plane, we have*

$$\iint_S g\,dS = \iint_R g\,\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

where $\vec{n}$ is the unit orientation of surface and $\vec{k} = (0,0,1)$.

Remark: the inner product of $\vec{n}$ and $\vec{k}$ should be always > 0 or < 0 since it is also continuous and the denominator should not be 0.