Hyperbolic Geometry

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These is my personal viewpoint to hyperbolic geometry. They are only one of the ways to look at this topic, and in particular, all errors are almost surely mine.

1 Fundamental Groups and Covering Spaces

1.1 Fundamental Groups

Definition (Homotopy)**.** Two functions $f, g : X \to Y$ are **homotopic** if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(s,0) = f(s)$ and $F(s,1) = g(s)$.

Definition (Path)**.** A **path** in a space X is a continuous map $\gamma : [0,1] \to X$.

Definition (Homotopy of Paths)**.** Two paths $\gamma_0, \gamma_1 : [0,1] \to X$ are **homotopic** if there is a homotopy $F : [0,1] \times [0,1] \to X$ such that $F(s,0) = \gamma_0(s)$ and $F(s,1) = \gamma_1(s)$. In addition, we require that $F(0,t) = \gamma_0(0)$ and $F(1,t) = \gamma_0(1)$ for all $t \in [0,1]$.

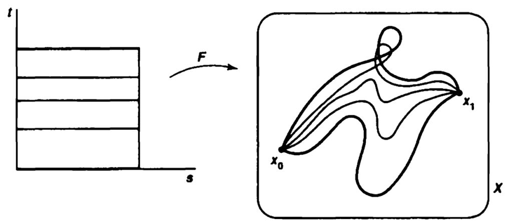

Figure 1: Homotopy of paths

Remark. Normally, we use straight line homotopy to prove the homotopy of paths. Given two paths $\gamma_0, \gamma_1 : [0,1] \to X$, we define $F(s,t) = \gamma_0(s)(1-t) + \gamma_1(s)t$ for all $t \in [0,1]$. But one should be careful that generally F is not a homotopy.

Proposition. Homotopy is an equivalence relation.

Proof. First, given a path γ, let $F(s,t) = \gamma(s)$ for all $t \in [0,1]$. Then F is a homotopy of γ to itself. Second, if $F(s,t)$ is a homotopy of γ_0 to γ_1, then $G(s, 1-t)$ is a homotopy of γ_1 to γ_0. Finally, if F is a homotopy of γ_0 to γ_1 and F' is a homotopy of γ_1 to γ_2, then

$$G(s,t) = \begin{cases} F(s,2t) & 0 \le t \le 1/2 \\ F'(s,2t-1) & 1/2 \le t \le 1 \end{cases}$$

is a homotopy of γ_0 to γ_2. So homotopy is an equivalence relation. $\square$

Definition (Loop)**.** A **loop** in a space X is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1)$.

Observation. For a path-connected space X, the initial point of a path is not important, it would not imfluence the homotopy class of the path.

Definition (Concatenation of Paths)**.** Given two paths $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$ satisfying $\gamma_1(1) = \gamma_2(0)$. The **concatenation** of is the path $\gamma_1 * \gamma_2 : [0,1] \to X$ defined by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Proposition. The concatenation has the following properties:

1. The concatenation of paths is associative.
2. The concatenation of paths is homotopic to the concatenation of their homotopic paths.
3. The concatenation of path with its reverse is homotopic to the constant path at the initial point.
4. Given $x \in X$, the constant path at x is the identity element of the concatenation of loops based at x.

Proof. 1. Let $\gamma_1, \gamma_2, \gamma_3 : [0,1] \to X$ be paths. Then

$$(\gamma_1 * (\gamma_2 * \gamma_3))(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(4t-2) & 1/2 \le t \le 3/4 \\ \gamma_3(4t-3) & 3/4 \le t \le 1 \end{cases} = (\gamma_1 * \gamma_2) * \gamma_3(t).$$

2. Let F_1 be a homotopy of γ_1 to γ_1' and F_2 be a homotopy of γ_2 to γ_2'. Then

$$F(x,t) = \begin{cases} F_1(x,2t) & 0 \le t \le 1/2 \\ F_2(x,2t-1) & 1/2 \le t \le 1 \end{cases}$$

is a homotopy of $\gamma_1 * \gamma_2$ to $\gamma_1' * \gamma_2'$.

3. Let $\gamma : [0,1] \to X$ be a path. Then $F(x,t) = t(\gamma * \gamma^{-1})(x) + (1-t)\gamma(0)$ is a homotopy of $\gamma * \gamma^{-1}$ to the constant path at $\gamma(0)$.

□

Definition (Fundamental Group)**.** The **fundamental group** $\pi_1(X, x_0)$ of a space X at a point $x_0 \in X$ is the set of homotopy classes of loops based at x_0 with the operation induced by concatenation: $[\gamma_1][\gamma_2] = [\gamma_1 * \gamma_2]$.

One may easily check that the operation is well-defined and the fundamental group is indeed a group with identity element being the homotopy class of the constant path at x_0. In addition, by the observation above, the fundamental group of path-connected space is independent of the choice of base point.

Definition (Simply Connected)**.** A path-connected space X is **simply connected** if its fundamental group at any point is trivial.

Proposition. Continuous maps induce homomorphisms between fundamental groups. Let $f : X \to Y$ be a continuous map between path-connected spaces. Then we have a homomorphism

$$\begin{aligned} f_* : \pi_1(X, x_0) &\to \pi_1(Y, f(x_0)) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned},$$

and

$$f_*([\gamma_1][\gamma_2]) = f_*([\gamma_1 * \gamma_2]) = [f \circ (\gamma_1 * \gamma_2)] = [(f \circ \gamma_1) * (f \circ \gamma_2)] = f_*([\gamma_1])f_*([\gamma_2]).$$

1.2 Covering Spaces

Definition (Evenly Covered)**.** Given a continuous and surjective map $p : E \to B$, an open set U is **evenly covered** by p if $p^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$ in E, and each of which is mapped homeomorphically onto U by p.

Definition (Covering Space)**.** A **covering space** of a space B is a space E with a continuous and surjective map $p : E \to B$ such that each point in B has a neighborhood that is evenly covered. In addition, the mapping p is called the **covering map**.

Example. The map $p : \mathbb{R} \to S^1$ defined by $p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering map.

Definition (Universal Cover)**.** A covering space $p : E \to B$ is a **universal cover** if E is simply connected.

Proposition. The universal cover of a path-connected and locally path-connected space is unique up to homeomorphism.

Definition (Deck Transformation)**.** A **deck transformation** of a covering space $p : E \to B$ is a homeomorphism $\sigma : E \to E$ such that $p \circ \sigma = p$.

$$\begin{tikzcd}
E \arrow[rr, "d"'] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

Remark. The set of deck transformations of a covering space $p : E \to B$ is a group under composition. The group is called the **group of deck transformations** and denoted by $\mathcal{C}(E, p, B)$.

Proposition. Let $p : E \to B$ be a covering map. If E is simply connected, then $\mathcal{C}(E, p, B)$ is isomorphic to the fundamental group of B.

$$\mathcal{C}(E, p, B) \cong \pi_1(B, b_0).$$

Example. The universal cover of S^1 is $\mathbb{R}$ and the group of deck transformations $\mathcal{C}(\mathbb{R}, p, S^1)$ is isomorphic to $\mathbb{Z}$. Then the fundamental group of S^1 is $\mathbb{Z}$.

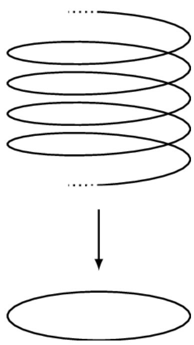

Figure 2: The universal cover of S^1

1.3 Group Actions

Definition (Orbit)**.** Let G be a group acting on a space X.

- The **orbit** of a point $x \in X$ under $G \circlearrowright X$ is the set $\{g \cdot x \mid g \in G\}$.

- The **orbit space** of $G \circlearrowright X$ is the set of orbits of X under G.

$$X/G = \{G \cdot x \mid x \in X\}.$$

In a sense, the orbit space describes the original object "up to symmetry"

Definition (Stabilizer). The **stabilizer** of a point $x \in X$ under a group action $G \circlearrowright X$ is the set $\{g \in G \mid g \cdot x = x\}$, denoted by G_x. It is the subgroup that fixes x.

Definition (Fixed Set). The **fixed set** of a subgroup $H \leq G$ acting on X is the set of points fixed by all elements of H.

Definition (Free Action). A group action $G \circlearrowright X$ is **free** if the stabilizer of each point is trivial. (The only element fixing a point is the identity element.)

Definition (Transitive Action). A group action $G \circlearrowright X$ is **transitive** if there is only one orbit. (Any point can be moved to any other point by some group element.)

Definition (Torsion-Free Group). A group is **torsion-free** if it has no nontrivial elements of finite order.

Definition (Properly Discontinuous Action). A group action $G \circlearrowright X$ is **properly discontinuous** if for each $x \in X$, there is a neighborhood U of x such that $g \cdot U \cap U \neq \varnothing$ implies $g = e$. (It moves in a discontinuous way.)

Theorem. Let X be a path-connected and locally path-connected space; let G be a group acting properly discontinuously on X. Then the orbit space X/G is a topological space and the projection map $p : X \to X/G$ is a covering map. In addition, G is isomorphic to the group of deck transformations of X/G.

2 Hyperbolic Geometry

Definition (Hyperbolic Space). In every dimension $n \geq 2$, there is a unique simply connected complete Riemannian manifold of constant sectional curvature -1. This manifold is called the **hyperbolic space** and denoted by $\mathbb{H}^n$.

2.1 Basic Definitions

Definition (Metrc Tensor). The **metric tensor** of a Riemannian manifold is a symmetric positive definite tensor field (or simply, represented by a matrix) that defines the inner product of tangent vectors at each point.

Example (Euclidean Space). The metric tensor of the Euclidean space $\mathbb{R}^n$ is the identity matrix, or precisely, the metric tensor is given by

$$g(x, y) = x^T I_n y = \sum_{i=1}^{n} x_i y_i$$

at every tangent space $T_p\mathbb{R}^n$.

The metric tensor would induce a norm on the tangent space, and the norm would induce the length of curves, given by

$$L(\gamma) = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

In this way, we would also define the distance between two points as the infimum of the length of curves connecting them.

$$d(p, q) = \inf\{L(\gamma) \mid \gamma : [0, 1] \to M \text{ is a curve connecting } p \text{ and } q\}.$$

Definition (Geodesic)**.** A **geodesic** is a curve that locally minimizes the length between two points. More precisely, a geodesic is a curve $\gamma : I \to M$ with constant speed k such that for any $t \in I$, there is a neighborhood $U = [t_0, t_1]$ of t such that

$$d(\gamma(t_0), \gamma(t_1)) = L(\gamma|_U) = k(t_1 - t_0).$$

2.2 Hyperboloid Model

Definition (Lorentzian scalar product)**.** The **Lorentzian scalar product** of two vectors $x, y \in \mathbb{R}^{n+1}$ is defined by

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

Definition (Hyperboloid Model)**.** The **hyperboloid model** of the hyperbolic space is the set

$$I^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1, x_0 > 0\}$$

with the metric tensor induced by the Lorentzian scalar product.

Note that the level set $H = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1\}$ has two connected components, but we only need one of them. There is a equivalent way to define I^n by quotienting H by the relation $x \sim y$ if $x = -y$, i.e., $I^n = H/\{\pm 1\}$.

Isometry of the Hyperboloid Model We have known that the metric tensor would induce a norm on the tangent space, and thus the length of curves and the distance between points. The isometry of the hyperboloid model is defined as the map that preserves the distance between points. Thus if the inner product of two vectors is preserved, the map is an isometry. The converse is also true, i.e., if a map is an isometry, then it preserves the inner product.

Proposition. The isometries of the hyperboloid model are precisely the linear transformations of $\mathbb{R}^{n+1}$ that preserve the Lorentzian scalar product.

$$Isom(I^n) = PO(n, 1) = O(n, 1)/\{\pm I\} = \{g \in \mathrm{GL}(n + 1, \mathbb{R}) \mid \langle gx, gy \rangle = \langle x, y \rangle\}/\{\pm I\}.$$

In the viewpoint of matrices, the orthogonal group can be written as

$$O(n, 1) = \{A \in \mathrm{GL}(n + 1, \mathbb{R}) \mid A^T J A = J\},$$

where

$$J = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1. \end{pmatrix}$$

The reason why we quotient by $\{\pm I\}$ is that we only conseder one sheet of the hyperboloid.

Definition (Subspace of Hyperboloid Model)**.** The **k-dimensional subspace** of the hyperboloid model is the intersection of the model with a $(k + 1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Proposition. Geodesics in the hyperboloid model are also 1-dimensional subspaces of the model, which can be parametrized by

$$\ell_\theta = p \cosh \theta + v \sinh \theta,$$

where p is a point in the hyperboloid model, v is a unit vector in the tangent space at p, and $\theta \in \mathbb{R}$.

2.3 Poincaré Disc Model

Definition (Poincaré Disc Model)**.** The **Poincaré ball model** of the hyperbolic space is the set

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

with the metric tensor given by

$$g(x, y) = \frac{4}{(1 - \|x\|^2)(1 - \|y\|^2)} \sum_{i=1}^{n} x_i y_i.$$

Remark. The Poincaré ball model is obtained by the projection towards (-1,0,...,0) from the hyperboloid model, and the projection is given by

$$\pi(x) = \frac{x}{1 + x_0}.$$

One may check that the metric tensor of the Poincaré ball model is indeed what we have given above.

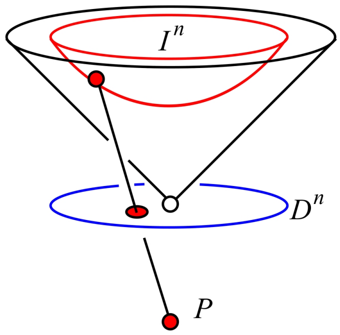

Figure 3: The Poincaré ball model

Unfortunately, the isometries of the Poincaré ball model are not easy to describe and the geodesics are not always straight lines in the Euclidean space. Then we need to find a better model to describe the hyperbolic space.

2.4 Klein Model

Definition (Klein Model)**.** The **Klein model** of the hyperbolic space is the set

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

It is like the Poincaré disc model, but with a different metric tensor. This model is obtained by embedding the hyperbolic space into the projective space $\mathbb{RP}^n$ and then reading in chart $x_0 = 1$. In this way, the geodesics in the Klein model are straight lines in the Euclidean space. In addition, the distance between two points in the Klein model is given by the cross ratio

$$d(p, q) = \frac{1}{2} \log((p, q; r, s)),$$

where the cross ratio is defined by

$$(p, q; r, s) = \frac{\|p - r\| \|q - s\|}{\|p - s\| \|q - r\|}.$$

2.5 Hyperbolic Mainfold

Since that the hyperbolic space is defined as a complete simply connected Riemannian manifold and the hyperbolic manifold is good enough: it always has a universal cover since it is connected and locally simply connected. Thus we can apply the results in the previous section to the hyperbolic manifold.

Proposition. The fundamental group of the hyperbolic manifold M is isomorphic to the group of deck transformations Γ of the universal cover, which is discrete and torsion-free. In addition, $\mathbb{H}^n/\Gamma \cong M$.