A Solution to Question 3 of Exercise 6

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Proposition 1. *Let μ be a spectral radius of G, then*

$$\delta(G) \leq \mu \leq \Delta(G).$$

Proof. First we show that $\mu \leq \Delta(G)$. Let A be the adjacent matrix of G, since μ is an eigenvector, thus

$$\exists\, v\, (Av = \mu v). \tag{1}$$

(1).

$$e^T Av = e^T \mu v = \mu e^T v. \tag{2}$$

Write v as $[v_1, v_2, \ldots, v_n]$, then the right-hand side can be written as $\mu \sum_{i=1}^n v_i$. For the left-hand side, we have

$$e^T Av = (e^T A^T)v = (Ae)^T v = \begin{bmatrix}\deg(v_1) & \ldots & \deg(v_n)\end{bmatrix} v \leq \Delta(G) \sum_{i=1}^n v_i. \tag{3}$$

Thus by Equation (2) and Equation (3), we have $\mu \leq \Delta(G)$.

Now we try to show that $\mu \geq \delta(G)$. Let $e = \begin{bmatrix}1 & 1 & \ldots & 1\end{bmatrix}^T$, note that Ae can be regarded as a linear combination of columns C_i of A and the sum of elements of the vector of C_i is $\deg(v_i)$. In this case, we try to study the sum of vector elements in Equation. Note that in the above proof, we have found that $Ae = \begin{bmatrix}\deg(v_1) & \ldots & \deg(v_n)\end{bmatrix}^T$ and thus the sum of its elements is

$$e^T Ae = \sum_{i=1}^n \deg(v_i) = 2\,|E(G)|. \tag{4}$$

Note that e can also be written as a linear combination of eigenbasis. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthonormal eigenbasis of A, then

$$\exists r_1, r_2, \ldots, r_n\, (e = \sum_{i=1}^n r_i \alpha_i).$$

Then

$$\begin{aligned} e^T Ae &= (\sum_{i=1}^n r_i \alpha_i^T) \left(A(\sum_{i=1}^n r_i \alpha_i) \right) = (\sum_{i=1}^n r_i \alpha_i^T)(\sum_{i=1}^n \lambda_i r_i \alpha_i) = \sum_{i=1}^n \lambda_i r_i^2 \\ &\leq \mu \sum_{i=1}^n r_i^2 = \mu e^T e = \mu n. \end{aligned} \tag{5}$$

By Equation (4) and Equation (5), one may find that

$$\mu n \geq 2|E(G)| \geq n\delta(G).$$

Thus $\mu \geq \delta(G)$. ∎