Basic Concepts in MTH207

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1 Scalar Field and Vector Field

Definition 1 (**Scalar Field**).

Definition 2 (**Vector Field**).

Definition 3 (**Consevation Field (Irrotational)**). *A vector field* $\vec{A}$ *is consevation if* $\operatorname{curl}\vec{A} = 0$.

Definition 4 (**Solenoidal Field**). *A vector field* $\vec{A}$ *is solenoidal if* $\operatorname{div}\vec{A} = 0$.

Definition 5 (**Convergent and Divergent Vector Field**). *A vector field is called divergent if* $\operatorname{div}\vec{A} > 0$. *Otherwise, it is convergent.*

Theorem 1 (**Differential Operator Formulae**).

$$
\begin{aligned}
&\triangledown\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\times\left(\triangledown\times\vec{B}\right) + \vec{B}\times\left(\triangledown\times\vec{A}\right) + \left(\vec{A}\cdot\triangledown\right)\vec{B} + \left(\vec{B}\cdot\triangledown\right)\vec{A},\\
&\operatorname{div}\left(\vec{A}\times\vec{B}\right) = \left(\triangledown\times\vec{A}\right)\cdot\vec{B} - \left(\triangledown\times\vec{B}\right)\cdot\vec{A},\\
&\triangledown\times\left(\vec{A}\times\vec{B}\right) = \left(\triangledown\cdot\vec{B}\right)\vec{A} - \left(\triangledown\cdot\vec{A}\right)\vec{B} + \left(\vec{B}\cdot\triangledown\right)\vec{A} - \left(\vec{A}\cdot\triangledown\right)\vec{B}
\end{aligned}
$$

Definition 6 (**Laplace Equation and Poisson Equation**). *Laplace equation is*

$$\triangledown^2 G = 0.$$

Poisson equation is

$$\triangledown^2 G = f(x,\, y,\, z).$$

2 Line integral

Definition 7 (**Simple Curve and Simply Connected Region**). *Simple curve is defined to be a curve with no self-intersection. Simply connected region is a region whose boundary is a simple curve.*

Theorem 2 (**Independence of Parameterization**).

Theorem 3 (**Reversal of Orientation**). *(get the negative value)*

Definition 8 (**Conservation Field (Vector field)**). *The line integral of the field does not depends on path.*

Theorem 4 (**Equivalent Statements**). $\vec{F}$ *conservative* $\Leftrightarrow \exists\Psi\left(\vec{F} = \nabla\Psi\right) \Leftrightarrow \operatorname{curl}\vec{F} = 0$.

Theorem 5 (**Fundamental Theorem of Line Integral**)**.** *If $\vec{F}$ is conservative, then* $\int_{\vec{x}_0}^{\vec{x}_1} \vec{F} \cdot d\vec{r} = \Psi(\vec{x}_1) - \Psi(\vec{x}_0)$ *along and curve from* $\vec{x}_1$ *to* $\vec{x}_0$*.*

Theorem 6 (**Green's Theorem**)**.**

$$\int_{\partial S} f\,dx + g\,dy = \iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

3 Surface Integral

Theorem 7 (**Projection of the surface (onto x-y plane)**)**.**

$$\iint_S G\,dS = \iint_R G\,\frac{dx\,dy}{|\hat{n} \cdot \hat{k}|},$$

where R is the projection of S on x-y plane. Note that the projection should be injective. (The note says that the surface sould not intersect x-y plane, but I do not think it is reasonable. What we only need to ensure is that the transformation is injective).

Definition 9 (**Orientation, Oriented Surface, Orientable Surface**)**.** *Define a map from the points on the surface to a unit vector perpendicular to the surface at that point. If along any curve on the surface, the map is continuous, then we say that it is an orientation of the surface.*

An oriented surface is a surface together with its orientation. An orientable surface is a surface such that there exists an orientation.

Definition 10 (**Orientation of the Boundary of Surfaces**)**.** *Consider a gear rotating acrossing to the right hand rule on a surface's normal surface. Orientation on the boundary is the direction in which the dot moves as they are pushing by the moving gear.*

Proposition 1 (**A quick way to compute the surface integral of vector field**)**.**

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS = \iint_R \vec{F} \cdot \frac{\nabla G}{|\nabla G|} \frac{dA}{|\hat{n} \cdot \hat{k}|} = \iint_R \vec{F} \cdot \nabla G\,dA.$$

Note that here we need $\frac{\partial G}{\partial z} = 1$*.*

Theorem 8 (**Stokes's Theorem**)**.**

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \mathrm{curl}\vec{F} \cdot d\vec{S}.$$

Theorem 9 (**Divergence Theorem**)**.**

$$\iint_{\partial \Omega} \vec{F} \cdot \hat{n} dS = \iiint_{\Omega} \mathrm{div}\vec{F} dV.$$

Note that divergence theorem can also applied to line integral (just let third element being zero).

Theorem 10 (**Chain Complex**)**.**

$$\Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\mathrm{curl}} \Omega^2 \xrightarrow{\mathrm{div}} \Omega^3 \longrightarrow 0$$

4 Curvilinear Coordinates

Definition 11 (Curvilinear Coordinates). *The curvilinear coordinate* (u_1, u_2, u_3) *is defined by the transformation equation*

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3).$$

Definition 12 (Basis Vectors).

$$\hat{e}_i = \frac{\partial \vec{r}}{\partial u_i} \Big/ \left| \frac{\partial \vec{r}}{\partial u_i} \right|.$$

Definition 13 (Orthogonal System). *The system* (u_1, u_2, u_3) *is orthogonal if* $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ *is orthogonal.*

Definition 14 (Scalar Factor).

$$h_i = \left| \frac{\partial \vec{r}}{\partial u_i} \right|.$$

Definition 15 (Differential arc length).

$$h_i \, du_i.$$

5 Fluid Mechanics

Definition 16 (Lagrangian and Eulerian Discription). *Lagrangian discription is that we follow the fluid particle and Eulerian discription is that we observe the character in laboratory coordinate over the whole region.*

Definition 17 (Streamline). *Curves that are tangential to the velocity at each point.*

Definition 18 (Stationary Point). *Point where the velocity vanish.*

Definition 19 (Pathline). *Trajectory followed by any given fluid particle for a given time interval.*

Definition 20 (Steady). *The velocity does not depend on time* t. *That is,* $\dfrac{\partial \vec{V}}{\partial t} = 0$

Definition 21 (Uniform). *The velocity does not depend on space coordinate.*

Proposition 2 (Material Derivative). *It gives the Lagrangian time rate of change in terms of Eulerian measurements.*

Theorem 11 (Continuous equation).

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{V}) = 0,$$

or

$$\frac{D\vec{V}}{Dt} + \rho \mathrm{div} \vec{V}.$$

Definition 22 (**Incompressible**).

$$\frac{D\vec{V}}{Dt} = 0,$$

or

$$\mathrm{div}\vec{V} = 0.$$

(Density of fluid particle does not change).

Definition 23 (**Streamline for Incompressible 2-dimentional flow**).

$$\mathrm{div}\vec{V} = 0 \Rightarrow \exists \vec{A} = (0,\, 0,\, \psi)\ \left(\mathrm{curl}\vec{A} = \vec{V}\right).$$

We call such $\vec{A}$ to be ***vector potential*** *and such ψ to be* ***stream function****.*

Note that in polar coordinate, we have

$$\vec{V} = \frac{\partial}{r\partial\theta}\psi\,\hat{e}_r - \frac{\partial}{\partial r}\psi\,\hat{e}_\theta$$

Example 1 (**Simple Source**).

$$\vec{V} = V_r\hat{e}_r \Rightarrow \left(\frac{\partial}{r\partial\theta}\psi = f(r) \wedge \frac{\partial}{\partial r}\psi = 0\right) \Rightarrow \left(\psi = A\theta \wedge \vec{V} = \frac{A}{r}\,\hat{e}_r\right)$$

Definition 24 (**True Dipole**). *A source and a equal sink. The distance between them tends to* 0*, the strength times distance remains as a constant. The* ***strength*** *mentioned above is the total outflow, i.e. the flow through the cycle containing the source.*

$$\begin{aligned}
&\psi(P) = -A\theta + A\theta_1 = Ah\left(\frac{\theta_1 - \theta}{h}\right), \\
&\lim_{h\to 0}\frac{\theta_1 - \theta}{h} = \frac{\partial\theta}{\partial x}, \\
&\theta = \arctan\frac{y}{x}.
\end{aligned}$$

Thus one may get that

$$\psi = Ah\frac{\sin\theta}{r}$$

Definition 25 (**Vorticity**).

$$\vec{\omega} = \mathrm{curl}\vec{V}.$$

Definition 26 (**Velocity Potential for Incompressible Flow**). *ϕ such that $\nabla\phi = \vec{V}$.*

Proposition 3. *Incompressible and irrotational implies that $\triangle\phi = \nabla^2\phi = 0$ (so does streamfunction).*

Definition 27 (**Equipotential Line**). *Level curve of ϕ.*

Definition 28 (**Flow Net**). *On streamlines, points that have the same value of ϕ (same velocity potential).*

Definition 29 (**Body Force**). *External forces act at each point within a fluid (in this module, it is usually force per unit mass, but I think using force per unit volumn is better, the difference is just a density).*

Definition 30 (**Surface Force**). *Forces act across a surface, including normal stress and shear stress. One may use tensor to represent it*

$$\mathcal{T} = -pI + D.$$

and the first column means the force act on surface parallel to y-z plane. Also, $\mathcal{T}$ is symmetric since the small cube would not rotate.

Definition 31 (**viscous**). *Fluid that exhibit internal friction. Then $D \neq 0$.*

Definition 32 (**inviscid**). *Fluid that offer no resistance, $D = 0$.*

Definition 33 (**Newtonian Fluid**).

$$D = \mu \left(-\frac{2}{3} \mathrm{div} \vec{V} I + \epsilon \right),$$

where μ is the ***viscocity*** *and ϵ is the* ***rate of strain*** *(which is also symmetric).*

Theorem 12 (**Navier-Stokes Equation**).

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{F},$$

where $\vec{F}$ is the external force per unit mass, $\mu \nabla^2 \vec{V}$ the viscosity and $-\nabla p$ the pressure gradient. One may regard it as the application of Newton's second law.

Definition 34 (**Rigid Boundary**). *$\vec{v} \cdot \hat{n} = 0$ for inviscid flow on the boundary, and $\vec{V} = 0$ for viscous flow on the boundary.*

Theorem 13 (**Bernoulli Equation**). *For inviscid flow that has steady motion, conservative external force and constant density, we have the equation*

$$\frac{\vec{V}^2}{2} - \Psi + \frac{p}{\rho} = constant,$$

along any fixed streamline.

Happy New Year and Good Luck!
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