From Mean-Value Property to Maximum Principle

Yifan Li

June 16, 2024

Proof. As U is path-connected, we can connect x_0 to x by a continuous curve $\gamma : [0,1] \to U$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. Then $\gamma([0,1])$ is compact as γ is continuous and $[0,1]$ is compact. Similarly as Method 1, set

$$d = \inf_{y \in \gamma([0,1])} d(y, \partial U),$$

and clearly $d > 0$. Observe that the following collection

$$\mathcal{U} = \{B(y, d/3) : y \in \gamma([0,1])\}$$

is an open cover of $\gamma([0,1])$. As $\gamma([0,1])$ is compact, there exists a finite subcover

$$\mathcal{U}' = \{B(y_i, d/3) : i = 1, \ldots, n\}.$$

Note that if u attains the maximum at the center y_i of ball $B(y_i, d/3)$, then u is constant on $B(y_i, d/3)$ by the mean-value property. Also note that u attains the maximum at $x_0 \in \gamma([0,1])$, thus $x_0 \in B(y_i, d/3)$ for some i. Therefore, u is constant on $B(y_i, d/3)$. Now we want to deduce that u is constant in all balls of the open cover, and thus u is constant on $\gamma([0,1])$.

Observing that if two balls $B(y_i, d/3)$ and $B(y_j, d/3)$ intersect, then $d(y_i, y_j) < 2d/3 < d$, thus if we double the radius of the balls, then the centers of the balls would lie in the other ball.

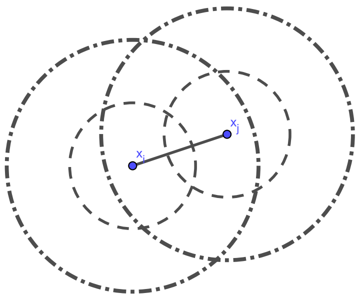

Figure 1: The balls $B(y_i, d/3)$ and $B(y_j, d/3)$ intersect.

Then u attaining the maximum at ball $B(y_i, 2d/3)$ implies that u is constant on both $B(y_i, 2d/3)$ and $B(y_j, 2d/3)$. Inspired by the above argument, we can try to show that u is constant on all balls of the open cover $\mathcal{U}$. To have a better intuition, we can form a graph G. Let the center of open balls of the finite subcover $\mathcal{U}'$ be the vertices of the graph, and two vertices are connected by an edge if the corresponding balls intersect. Then the graph is connected as the balls cover $\gamma([0,1])$ and $\gamma([0,1])$ is connected. By the previous observation, we can double the radius of the balls. Since u attains the maximum in $B(y_i, 2d/3)$, u is constant on $B(y_i, 2d/3)$ and all balls intersecting with $B(y_i, 2d/3)$, i.e., u is constant on all balls centered at the neighbors $N(y_i)$ of y_i in the graph G. Since G is connected, u is constant on all balls of the open cover $\mathcal{U}$ by induction.

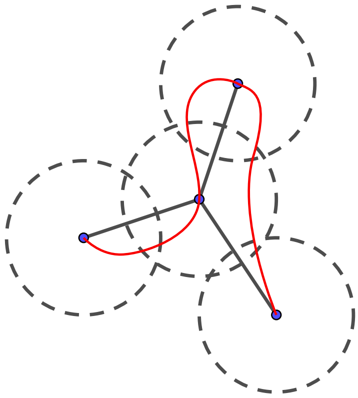

Figure 2: The graph G formed by the centers of the balls.

Finally, as u is constant on $\gamma([0,1])$, $u(x) = u(x_0)$. $\square$

Remark. It is not so trivial that the graph G is connected. The connectedness of G is equivalent to the connectedness of the union of the balls in the subcover $\mathcal{U}'$. The union covers $\gamma([0,1])$, and any two balls are joint by γ. Thus there is a corresponding path in G connecting the two vertices.