# Dragon Problem

Yifan Li

February 1, 2024

The dragon of luck time-travels bringing happiness and prosperity to the year he visits. To jump between the years, he only uses two bi-directional transits $N \leftrightarrow 3N + 1$ or $N \leftrightarrow 2N$. Prove that there is a sequence of transits that the dragon can use to come to visit us in 2024, no matter which year he is currently in. (Here we only consider the positive integer)

**Definition 1.**

$$T_1 : \ \mathbb{Z}^+ \to \mathbb{Z}^+, \ T_1(n) = 2n,$$
$$T_2 : \ \mathbb{Z}^+ \to \mathbb{Z}^+, \ T_2(n) = 3n + 1,$$
$$\mathcal{T} = \{T_1, \ T_2\},$$
$$T_3 : \ \mathbb{Z}^+ \to \mathbb{Z}^+, \ T_3(n) = 2n + 1,$$

To prove that the dragon can jump to 2024 from any year, we prove the following proposition.

**Proposition 1.** *For $\forall m, \ n \in \mathbb{Z}^+$, there exists a combination $T$ of $T_1$, $T_1^{-1}$, $T_2$ and $T_2^{-1}$ such that $T(m) = n$.*

*Proof.* To show that an arbitrary positive integer can be transfered to any positive integer, we can show that any positive integer can be transfered to 1 since the operation we use is in the free group $F_{\mathcal{T}}$ and the inverse of operation is still in $F_{\mathcal{T}}$.

For an even number, we may apply $T_1^{-1}$ to transit it into an odd number. Then we only need to consider the situation of odd number. For an odd number $a$, it has the form $a = 2b + 1$ for some $b \in \mathbb{N}$. In this case, we have $a = T_3(b)$, if $b$ is still an odd number, we can apply $T_3^{-1}$ again and finally get an even natural number.

We separate the situation into two cases: 0 and positive even integer $2k$. For the situation of 0, we may write $a$ as $T_3 \circ \ldots \circ T_3(0) = 2(2(\ldots (2(0) + 1) \ldots) + 1) + 1 = 1 + 2 + 4 + \ldots + 2^n$ for some $n \in \mathbb{N}$. For the situation of positive even integer $2k$, we may write $a$ as $T_3 \circ \ldots \circ T_3(2k) = 2(2(\ldots (2(2k) + 1) \ldots) + 1) + 1 = 2^{n+2}k + 1 + 2 + 4 + \ldots + 2^n$ for some $n \in \mathbb{Z}^+$.

- $T_3 \circ \ldots \circ T_3(0)$. For $a = 1+2+4+\ldots+2^n = 2^{n+1}-1$. Applying $T_1^{-1} \circ T_2$ for $n+1$ times, we can get $\frac{3^{n+1}-1}{2} = 1+3+9+\ldots+3^n$. Notice that $T_2^{-1}(\frac{3^{n+1}-1}{2}) = T_2^{-1}(1+3+\ldots+3^n) = 1+3+\ldots+3^{n-1}$. Thus applying $T_2^{-1}$ for $n$ times, we would get 1.

- $T_3 \circ \ldots \circ T_3(2k)$. For $a = 2^{n+2}k + 1 + 2 + 4 + \ldots + 2^n = 2^{n+2}k + 2^{n+1} - 1$. Applying $T_1^{-1} \circ T_2$ for $n + 1$ times, we can get $3^{n+1}(2k + 1) - 1$. Applying $T_1^{-1}$, we have $3^{n+1}k + \frac{3^{n+1}-1}{2} = 3^{n+1}k + 1 + 3 + \ldots + 3^n$. Notice that $T_2^{-1}(3^{n+1}k + 1 + 3 + \ldots + 3^n) = 3^n k + 1 + 3 + \ldots + 3^{n-1}$. Thus applying $T_2^{-1}$ for $n$ times, we would get $k + 1$, which is smaller than $a$. Repeat the previous steps and we can finally reduce to 1.

$\square$