Exercise1_5

yifan.li21

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Question: Prove that the number of subsets of $\{1, 2, \ldots, n\}$ whose size is divisible by 3 is either $\lfloor \frac{2^n}{3} \rfloor$ or $\lceil \frac{2^n}{3} \rceil$.

Proof:

Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ whose size is $3i$ where $i = 1, 2, 3, \ldots$, similarly, let b_n be the number of subsets whose size is $3i - 1$ and c_n be the number of subsets whose size is $3i - 2$. Similar to the problem of odd and even case, we try to use induction.

To show that $a_n = \lfloor \frac{2^n}{3} \rfloor$ or $a_n = \lceil \frac{2^n}{3} \rceil$, we need to show $|a_n - \frac{2^n}{3}| < 1$. Let $A_n = a_n - \frac{2^n}{3}$, similarly, let $B_n = b_n - \frac{2^n}{3}$ and $C_n = c_n - \frac{2^n}{3}$. Then what we need is $|A_n| < 1$ for $n \in \mathbb{N}$

Since the total number of subsets of $\{1, 2, \ldots, n\}$ is 2^n, thus

$$A_n + B_n + C_n = 0 \tag{1}$$

For $n = 1$, it is easy to check that $|A_n|, |B_n|, |C_n|$ are all less than 1. Suppose for $n = k$, they are also less than 1, then we need to show that $|A_{k+1} + B_{k+1}| < 1$, which is equivalent to $|C_{k+1}| < 1$ by equation(1).

Then we try to find some relations between $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ to help us. For A_{k+1}, we know that $A_{k+1} = A_k + B_k = -C_k$ since $a_{k+1} = a_k + b_k$ and $A_{k+1} = a_{k+1} - \frac{2^{k+1}}{3}$. Similarly, $C_{k+1} = C_k + A_k = -B_k$ and $B_{k+1} = B_k + C_k = -A_k$. Thus we have $|A_{k+1}|$, $|B_{k+1}|$, $|C_{k+1}|$ are all less than 1. Then by induction, we have $|a_n - \frac{2^n}{3}| < 1$ for any $n \in \mathbb{N}$.