An Examples in Complex Analysis

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Example 1. *Show that*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2},$$

where $u \notin \mathbb{Z}$ being a constant.

Solution 1. *Let $f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$ and consider the inetgral alone a cycle centered as 0 with radius $R = N + \frac{1}{2}$ where $N > |u|$. Now we study the residue of f.*

First we find the singularities of f, the singularities of $\cot z$ are $\mathbb{N}$ of order 1 and the zero of $(u+z)^2$ is $-u$ with order 2. Then we calculate the residue of each singularity.

For $z = -u$, we have

$$\operatorname{Res}(f,\, -u) = \lim_{z \to -u} \left((z+u)^2 f(z)\right)' = -\lim_{x \to -u} \frac{\pi^2}{(sin\pi z)^2}.$$

For $z = n \in \mathbb{Z}$, we have

$$\operatorname{Res}(f,\, n) = \lim_{z \to n} (z-n) f(z) = \lim_{z \to n} \frac{\pi(z-n)\sum_{k=0}^{infty} (-1)^k \frac{(\pi z - n)^{2k}}{(2k)!}}{(z+u)^2 \sum_{k=0}^{\infty} (-1)^k \frac{(\pi z - n)^{2k+1}}{(2k+1)!}} = \frac{1}{(n+u)^2}.$$

Also, we may try to compute the inetgral directly. Consider

$$\int_{\gamma_R} \frac{\pi \cot z}{(u+z)^2}\, dz = \int_0^{2\pi} \frac{\pi \cot\left(\pi n + \frac{\pi}{2} e^{i\theta}\right)}{(u + (n+\frac{1}{2})e^{i\theta})^2} i(n+\frac{1}{2}) e^{i\theta}\, d\theta.$$

Take the norm of the inetgral, we may find that

$$\left| \int_{\gamma_R} \frac{\pi \cot z}{(u+z)^2}\, dz \right| \leq \frac{\pi(n+\frac{1}{2})}{(n+\frac{1}{2})^2 - u^2} \int_0^{2\pi} \left| \cot\left(\pi n + \frac{\pi}{2}\right) \right| d\theta. \tag{1}$$

For $\left|\cot\left(\pi n + \frac{\pi}{2}\right)\right|$, we know that

$$\begin{aligned} |\cot(\pi z)| &= \left| \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \right| = \left| \frac{1 + e^{-2\pi y} \cos 2\pi x + i e^{-2\pi y} \sin 2\pi x}{1 - e^{-2\pi y} \cos 2\pi x - i e^{-2\pi y} \sin 2\pi x} \right| \\ &= \sqrt{\frac{1 + 2e^{-2\pi y} \cos 2\pi x + e^{-4\pi y} \cos^2 2\pi x + e^{-4y} \sin^2 2x}{1 - 2e^{-2\pi y} \cos 2\pi x + e^{-4\pi y} \cos^2 2\pi x + e^{-4y} \sin^2 2x}} \\ &= \sqrt{\frac{1 + 2e^{-2\pi y} \cos 2\pi x + e^{4\pi y}}{1 - 2e^{-2\pi y} \cos 2\pi x + e^{4\pi y}}} = \sqrt{\frac{\cosh w\pi y + cos 2\pi x}{\cosh w\pi y - cos 2\pi x}} \\ &= \sqrt{1 + \frac{2\cos 2\pi x}{\cosh 2\pi y - \cos 2\pi x}}. \end{aligned}$$

Now, let $|z| = n + \frac{1}{2}$, in this case, we know that $|x| \leq n + \frac{1}{2}$ and $|y| \leq n + \frac{1}{2}$. Observing that $\cos 2\pi x \leq 0$ when $n + \frac{1}{4} \leq |x| \leq n + \frac{1}{2}$, thus we analyze the two cases: $n + \frac{1}{4} \leq |x| \leq n + \frac{1}{2}$ and $|z| \leq n + \frac{1}{4}$.

(i) When $n + \frac{1}{4} \leq |z| \leq n + \frac{1}{2}$, we know that $\cos 2\pi x \leq 0$. Since $\cosh 2\pi y \geq 1$ and $\cos 2\pi x \leq 1$, we know that

$$-1 \leq \frac{2\cos 2\pi x}{\cosh 2\pi y - \cos 2\pi x} \leq 0,$$

where the left inequality comes from the square of the norm.

(ii) When $|z| \leq n + \frac{1}{4}$, consider cosh$2\pi y$,

$$\begin{aligned}\cosh 2\pi y = \frac{e^{2\pi y} + e^{-2\pi y}}{2} &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(2\pi y)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-2\pi y)^k}{k!}\right) \\ &= \sum_{k=0}^{\infty} \frac{(2\pi y)^{2k}}{(2k)!} \leq 1 + \frac{1}{2}(2\pi y)^2.\end{aligned}$$

Note that $x^2 + y^2 = (n + \frac{1}{2})^2$, then

$$\begin{aligned}1 + \frac{1}{2}(2\pi y)^2 &= 1 + 2\pi^2\left((n + \frac{1}{2})^2 - x^2\right) \\ &\geq 1 + 2\pi^2\left((n + \frac{1}{2})^2 - (n + \frac{1}{4})^2\right) \\ &\geq 1 + \frac{3}{8}\pi^2.\end{aligned}$$

In this case, we can find that

$$\frac{2\cos 2\pi x}{\cosh 2\pi y - \cos 2\pi x} \leq \frac{2}{1 + \frac{3}{8}\pi^2 + 1} = \frac{1}{1 + \frac{3}{16}\pi^2} \leq \frac{16}{3\pi^2}.$$

Thus $\frac{2\cos 2\pi x}{\cosh 2\pi y - \cos 2\pi x}$ is bounded, we may state that $|\cos \pi z| \leq M$ for some constant M. In this case, Equation (1) inplies that

$$\left|\int_{\gamma_R} \frac{\pi \cot z}{(u+z)^2}\, dz\right| \leq \frac{\pi(n + \frac{1}{2})}{(n + \frac{1}{2})^2 - u^2}\, 2\pi M.$$

Let $n \to \infty$, one may find that $\int_{\gamma_R} f(z)\, dz \to 0$. Hence

$$\begin{aligned}0 &= \lim_{n\to\infty} \sum_{k=-n}^{n} \mathrm{Res}(f,\, k) + \mathrm{Res}(f,\, -u) \\ \sum_{k=-\infty}^{\infty} \frac{1}{(u+n)^2} &= \frac{\pi^2}{(\sin \pi u)^2}\end{aligned}$$