Fluid Mechanic

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Lagrangian Description

Follow fluid particles as they move through the flow,

$$\vec{X}(\vec{X}_0,\, t). \tag{1}$$

$\vec{X}$ is the position vector of a fluid particle at time t and $\vec{X}_0$ is the position of that particle at some reference time t_0.

$$\vec{X}_0 = \vec{X}(\vec{X}_0,\, t = t_0).$$

To calculate the **velocity** of a fluid particle, one may derive $\vec{X}$ directly.

$$\vec{V} = \frac{\mathrm{d}\vec{X}}{\mathrm{d}t}. \tag{2}$$

Eulerian Description

Fixates on a particular point in space, record the properties of the fluid elements passing through that point.

$$\vec{v}(\vec{x},\, t). \tag{3}$$

It means the velocity vector at the laboratory coordinate $\vec{x}$ at time t.

Connection to Two Descriptions

The velocity under the descriptions above should be the same.

$$\vec{v}\left(\vec{X}(\vec{X},\, t),\, t\right) = \frac{\mathrm{d}\vec{X}}{\mathrm{d}t}(\vec{X}_0,\, t). \tag{4}$$

Material Derivatives

If we want to get the acceleration of a particle, in Lagrangian description, one may directly derive the velocity, then we find that

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\vec{X}}{\mathrm{d}t}(\vec{X}_0,\, t)\right), \tag{5}$$

changing to the Eulerian description, then *Equation*(5) can be written as

$$\begin{aligned}\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\vec{X}}{\mathrm{d}t}(\vec{X}_0,\, t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}\left(\vec{X}(\vec{X}_0,\, t),\, t\right) \\ &= \frac{\vec{v}}{\partial t} + \sum \frac{\partial \vec{v}}{\partial x_i}\frac{\partial x_i}{\partial t} \\ &= \frac{\vec{v}}{\partial t} + \sum v_i \frac{\partial \vec{v}}{\partial x_i} = \frac{\vec{v}}{\partial t} + \left(\vec{V}\cdot\bigtriangledown\right)\vec{v}.\end{aligned}$$

One may also find the physical explanation behind it, in Eulerian description, what we consider is a fixed point in the laboratory coordinate. However, the position of the particle we study will change, so the second term represents the change of position. In addition, we may use this operator in other situations, so we give a name to it: **material derivative**.

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \vec{v}\cdot\bigtriangledown. \tag{6}$$

The left-hand side of the equation is the *Lagrangian derivative*, which means the change as we follow the path (just the derivative of function under Lagrange description corresponds to t). On the right-hand side of the equation, the first term is the *Eulerian time derivative* and the second term is the *advective derivative*, which means the change due to the change in position.

Pathline

The pathline comes from the Lagrangian description, it is the locus of any particle for a time interval.

$$\vec{X}_0 = \vec{X}(\vec{X}_0,\, t = t_0).$$

It is just as the form of the position under the Lagrange description.

Streamline

The streamline comes from the Eulering description. Fix the time t, and observe the velocity of each position. The streamline is tangential to the velocity. Here is a question: how one can draw it? One may consider $\vec{v}\times \mathrm{d}\vec{l} = 0$ since the line is tangential to the velocity.

$$0 = \vec{v}\times\mathrm{d}\vec{l} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ v_x & v_y & v_z \\ \mathrm{d}x & \mathrm{d}y & \mathrm{d}z\end{vmatrix}$$

Then one can find that

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z} \tag{7}$$

Conservation of Mass

Fix an arbitrary region of space $\mathcal{D}$ with boundary $\partial\mathcal{D}$. Then the change of mass is the mass flowing into the boundary, that is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}} \rho \,\mathrm{d}V = -\int_{\partial\mathcal{D}} \rho\, \vec{v}\cdot\vec{n}\,\mathrm{d}S,$$

where $\vec{n}$ is the outward unit normal vector of $\partial\mathcal{D}$ and $\vec{v}$ is the velocity. Using the divergence theorem, the equation above can be written as

$$\int_{\mathcal{D}} \left(\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\vec{v}\right)\mathrm{d}V = 0$$

Since $\mathcal{D}$ is arbitrary, one may get that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\vec{v} = 0.$$

Expand the divergence $\operatorname{div}\rho\vec{v} = \vec{v}\cdot\triangledown\rho + \rho\operatorname{div}\vec{v}$, we have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\operatorname{div}\vec{v} = 0 \tag{8}$$

There is a special type of fluid which is called **incompressible flow**, its density would not change with respect to time, thus one may write the condition as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 = \operatorname{div}\vec{v} \tag{9}$$

Note that you need to "follow" the particle and make sure its mass does not change due to time, thus you need material derivatives. In fact, we rarely use derivatives directly to a fluid particle.

Stream Function for Incompressible Flow

Suppose the fluid is incompressible, then we have

$$\operatorname{div}\vec{v} = 0$$

It implies that there exists a vector potential $\vec{A}$ such that

$$\vec{v} = \triangledown\times\vec{A}$$

For two dimensional flow, say $\vec{v} = (u(x,\,y,\,t),\,w(x,\,y,\,t),\,0)$. One can immediately know that $\vec{A}$ is of the form $\vec{A} = (0,\,0,\,\psi(x,\,y,\,t))$. Here ψ is called the **stream function**. Consider the curl of $\vec{A}$ and the gradient of the stream function, we have

$$\begin{aligned}\vec{v} &= \triangledown\times\psi = \left(\frac{\partial\psi}{\partial y},\,-\frac{\partial\psi}{\partial x},\,0\right),\\ \vec{n} &= \triangledown\psi = \left(\frac{\partial\psi}{\partial x},\,\frac{\partial\psi}{\partial y},\,0\right).\end{aligned} \tag{10}$$

Note that from the equations above, we can find that

$$\vec{n} \cdot \vec{v} = 0$$

and thus the level surface of the stream function (perpendicular to the gradient of ψ) is just the streamline.

Example 1. *Single Source or Sink*

Example 2. *Dipole*

Velocity Potential for Irrotational flows