Review for PDE

Yifan Li

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These is my personal viewpoint to the partial differential equations. They are only one of the ways to look at them, and in particular, all errors are almost surely mine.

1 Bsic Definitions

Before we step into the world of partial differential equations, we need to know some basic definitions.

1.1 Linearity

Definition. i) **Linear** A function f is linear if it is linear with respect to u and its derivatives and the coefficients are only depends on x.

ii) **Semilinear** A function f is semilinear if it is linear with respect to u and its derivatives but the coefficients may depend on u.

iii) **Quasilinear** A function f is quasilinear if it is linear with respect to the highest order derivatives of u and the coefficients may depend on u and lower order derivatives.

1.2 Initial and Boundary Conditions

Definition. i) **Dirichlet Boundary Condition**

$$u(x,t) = g(x) \quad \text{on} \quad \partial\Omega.$$

ii) **Neumann Boundary Condition**

$$\frac{\partial u}{\partial n} = g(x) \quad \text{on} \quad \partial\Omega.$$

iii) **Robin Boundary Condition** $\frac{\partial u}{\partial n} + au$ is specified.

1.3 Conservation Laws

If u is conserved, then let N denote the whole quantity of u in an arbitrary domain Ω, then we have the conservation law

$$\frac{dN}{dt} = \int_\Omega \frac{\partial u}{\partial t} dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS.$$

using the divergence theorem, we have

$$0 = \int_\Omega \frac{\partial u}{\partial t} dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = \int_\Omega \left(\frac{\partial u}{\partial t} - \nabla \cdot \nabla u \right) dx.$$

Since the domain Ω is arbitrary, we have the conservation law

$$\frac{\partial u}{\partial t} - \nabla \cdot \nabla u = 0.$$

In one dimensional case, we have

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0.$$

where f is the flux of u.

2 First Order PDEs

In this section, we will discuss the first order partial differential equations of the form

$$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = c(x,t,u).$$

The solution of the equation is a function of a surface in the space of (x,t,u), whose normal is given by the vector $\vec{n} = (u_x, u_y, -1)$. If we write the equation(2) as the vector form, we have

$$\vec{n} \cdot \vec{a} = 0,$$

where $\vec{a} = (a,b,c)$. This means that the normal vector is orthogonal to the vector $\vec{a}$, i.e. the solution surface is tangential to the vector firld $\vec{a}$. This is the geometric interpretation of the first order PDEs.

For the vector field $\vec{a}$, we may draw its "streamlines". The streamlines are also called the characteristics of the PDEs. The method to find the characteristics will be discussed later. Before it we need to know the following definitions about the staff related to the characteristics.

Definition (First Integral)**.** A union of the characteristics which forms a C^1 surface is called a first integral of the PDEs.

Definition (Independence of the First Integral)**.** Two first integrals are independent if the tangent planes of the two surfaces are not parallel, that is,

$$\nabla u_1 \times \nabla u_2 \neq 0 \quad \text{(everywhere)}.$$

Now we can discuss the method to find the characteristics of the first order PDEs.

2.1 Lagrange's Method

Since the characteristics are the streamlines of the vector field $\vec{a}$, we know that they are tangential to the vector field $\vec{a}$. Thus the direction of the characteristics is

$$d\vec{v} = \begin{bmatrix} dx \\ dt \\ du \end{bmatrix} // \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

write the above relation by each component, we have

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

Solving the system above, we would get two first integrals of the PDEs.

$$\begin{cases} \Phi(x,\, y,\, u) = C_1, \\ \Psi(x,\, y,\, u) = C_2. \end{cases}$$

Lagrange said that we can use these two independent first integrals to find the solution of the PDEs.

Theorem (Lagrange)**.** If the PDEs is of the form

$$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = c(x,t,u),$$

and the characteristics are given by

$$\begin{cases} \Phi(x,\, y,\, u) = C_1, \\ \Psi(x,\, y,\, u) = C_2. \end{cases}$$

then the "general" solution of the PDEs is given by

$$u(x,t) = F(\Phi(x,\, y,\, u),\, \Psi(x,\, y,\, u)).$$

Proof. Let $\omega = F(\Phi, \Psi)$, then we have

$$\nabla\omega \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(\frac{\partial F}{\partial \Phi}\nabla\Phi + \frac{\partial F}{\partial \Psi}\nabla\Psi\right) \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0,$$

since $\nabla\Phi$ and $\nabla\Psi$ are first integrals. Thus the level surface of ω is tangential to the vector field $\vec{a}$, which means that the solution of the PDEs is given by $0 = \omega = F(\Phi, \Psi)$. □

2.2 Cauchy's Method

There is another way to preperesent a curve, which is parametric representation. In this method, we can write the characteristics as

$$\begin{cases} x = x(t), \\ y = y(t), \\ u = u(t). \end{cases}$$

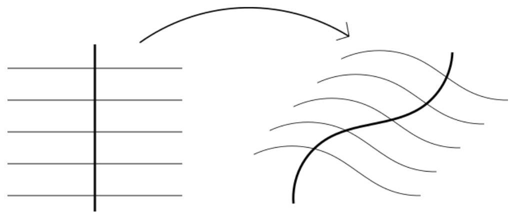
In this case, we have the relation

$$\begin{cases} \frac{dx}{dt} = a(x,y,u), \\ \frac{dy}{dt} = b(x,y,u), \\ \frac{du}{dt} = c(x,y,u). \end{cases}$$

If the initial condition is a curve in the space of (x, y, u), parametrized as

$$\begin{cases} x(s,\, 0) = f(s), \\ y(s,\, 0) = g(s), \\ u(s,\, 0) = h(s). \end{cases}$$

then the ideal situation is that for each point on the curve, there is a unique characteristic passing through it. In this case, we can those curves to form the solution surface. One variable of the surface determine the initial point that the characteristic passes through, and the other two variables determine the point on the characteristic. This is the geometric interpretation of the Cauchy's method.



However, the above situation is just the ideal case. In the real world, the initial curve may be tangential to the characteristics at some points. To illustrate this, we need to consider something more. The curves in 3-dimensional space maight be complicated, but we can look at their projection to the plane $x-y$. To determine whether the projections are tangential, we need to consider the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} f'(s) & a(f(s),\, g(s),\, h(s)) \\ \\ g'(s) & b(f(s),\, g(s),\, h(s)) \end{vmatrix}.$$

If $J \neq 0$, then we are in the ideal case. If $j = 0$ somewhere, then we need to consider whether the initial curve is really tangential to the characteristics. If it is not, then the surface would have a singularity at that point and we would not have a C^1 solution (*May be it can be regaded as a shock*). If it is, then it is possible that we can still have a C^1 solution (*but it seems that we are not sure with informations we have now*). In addition, there is another situation that the initial curve is tangential to the characteristics everywhere. In this case, we have infinitely many solutions (*can only determine one characteristics curve and still have infinitely many possibility*).

2.3 Shock

Consider the PDE from the conservation law

$$\begin{cases} q(u)u_x + u_t = 0, \\ u(x,\, 0) = f(x). \end{cases}$$

The characteristics are given by

$$x = q(f(s))t + s.$$

and the general solution is

$$u(x,\, t) = f(x - q(u)t).$$

Shock means that u locally "jumps" from one value to another. There are two perspectives to look at the shock. One is that the partial derivatives u_t and u_x "blow up" at some point, then u would perform a jump. The other is that there are two characteristics intersect at some point, then the solution is multi-valued at that point (geometrically, it is a jump).

We first consider the first perspective. Consider the partial derivatives of u

$$\begin{cases} u_t = \frac{-qf'}{1+q'f't}, \\ \\ u_x = \frac{f'}{1+q'f't}. \end{cases}$$

The partial derivatives blow up at the point where $1 + q'f't = 0$. This means that the solution u jumps at that point.

Now we consider the second perspective. The characteristics are given by

$$x = q(f(s))t + s.$$

Along the characteristics, we know that u is constant. Thus the solution is multi-valued at the point where two characteristics intersect. That is,

$$X := s_1 + q(f(s_1))t = s_2 + q(f(s_2))t.$$

By mean value theorem, we have

$$0 = X'(\xi) = 1 + q'(f(\xi))f'(\xi)t,$$

for some ξ between s_1 and s_2.

Observing the two viewpoint, we may find that the denominator of the partial derivatives shows the density of the characteristics. The infinity of the density means the intersection of the characteristics, which means the shock.

3 Linear Partial Differential Equations

3.1 Linear Partial Differential Operators

There is a simple notation for the partial derivatives.

$$\begin{cases} D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}, \\ \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n). \end{cases}$$

Then a linear partial differential operator can be written as

$$P(x,\, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha.$$

Note that here $a_\alpha(x)$ is just a function of x, α here is just an index, not a action as it acts on D. When we solve the PDEs, we usually ignore the lower order terms since they do not have significant influence on the solution. So there is another important notation for the linear partial differential operator, which is called the **principle part**.

$$P_m(x,\, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha.$$

3.2 Canonical Form

First Order PDEs

The general form of the first order PDEs is

$$a(x,y)u_x + b(x,y)u_t + c(x,y)u + d.$$

It is possible that the above PDE is hard to solve using previous methods. However, we can always transform it into a simpler form. To achieve this, we may try to change the coordinate system to eliminate one of the first order terms. That form is called the **canonical form**.

Let $\xi = \xi(x,\, y)$ and $\eta = \eta(x,\, y)$, then we have

$$\begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x, \\ u_y = u_\xi \xi_y + u_\eta \eta_y. \end{cases}$$

and the PDE becomes

$$(a\xi_x + b\xi_y)\, u_\xi + (a\eta_x + b\eta_y)\, u_\eta + cu + d = 0.$$

To make our life easier, we may choose η such that the coefficient of u_η is zero. That is

$$a\eta_x + b\eta_y = 0.$$

Solving this equation, we can easily get η. Now we need to determine ξ. To make the conversion meaningful, we need to make sure that the Jacobian of the transformation is not zero in the domain. (Normally the simplest choice $\xi = x$ or $\xi = y$ is enough.)

In this case, we can finally transform the PDE into the canonical form

$$Au_\xi + Cu + E = 0,$$

which can be regarded as a first order ODE and can be solved easily.