Jacobian Matrix

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Consider a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^n$,

$$F(x^1, x^2, \dots, x^n) = \begin{bmatrix} F^1(x^1, \dots, x^n) \\ \vdots \\ F^n(x^1, \dots, x^n) \end{bmatrix}.$$

Note that for conventions, the indices on coordinates are superscripts, and the indices for partial derivative are subscripts. However, for indices for unit coordiante basis, we still use subscripts. Take the differentiation of $F(x^1, ., x^n)$, we get

$$\begin{aligned} dF(x^1, ., x^n) = \begin{bmatrix} dF^1 \\ \vdots \\ dF^n \end{bmatrix} &= \begin{bmatrix} F^1_{x^1}\, dx^1 + \dots + F^1_{x^n}\, dx^n \\ \vdots \\ F^n_{x^1}\, dx^1 + \dots + F^n_{x^n}\, dx^n \end{bmatrix} \\ &= \begin{bmatrix} F^1_{x^1} & \dots & F^1_{x^n} \\ \vdots & \ddots & \vdots \\ F^n_{x^1} & \dots & F^n_{x^n} \end{bmatrix} \begin{bmatrix} dx^1 \\ \vdots \\ dx^n \end{bmatrix} \end{aligned}$$

Here the matrix is called the Jacobian matrix of F, denoted by $\frac{\partial(F^1, \dots, F^n)}{\partial(x^1, \dots x^n)}$ (or simply J). Then we get an interesting formula:

$$dF = J\, dx.$$

For functionals in Euclidean space, we know that the "derivative" is gradient. Here we can define a similar form of "derivative" for $F : \mathbb{R}^n \to \mathbb{R}^n$: $\nabla F = J$. Then the directional derivative can be written as

$$\frac{\partial F}{\partial v} = \nabla F v = Jv.$$

In addition, there is another perspective to approach to Jacobian matrix. Consider the conversion of coordiante $(x^1, \dots, x^n) \to (u^1, \dots, u^n)$. Then the position can be written as

$$\vec{r}(u^1, \dots, u^n) = x^1 \hat{e}_{x^1} + \dots + x^n \hat{e}_{x^n}$$

Take the differentiation, we get

$$d\vec{r}(u^1, \dots, u^n) = \sum_{i=1}^{n} \frac{\partial \vec{r}}{\partial u^i}\, du^i.$$

We hope that $\{\frac{\partial \vec{r}}{\partial u^i}\}$ forms a basis. To check whether the new coordiante system $(u^1, \dots, u^n)$ is acceptable (the vectors above is a basis), we need to check the determinant

$$\begin{vmatrix} \frac{\partial \vec{r}}{\partial u^1} & \dots & \frac{\partial \vec{r}}{\partial u^n} \end{vmatrix}$$

If the determinant is not 0, then the vectors $\{\frac{\partial \vec{r}}{\partial u^i}\}$ forms a basis for $\mathbb{R}^n$. One may find that this determinant is just the determinant of Jacobian matrix corresponding to the map $T(u^1,\,\ldots,\,u^n) = x^1\hat{e}_{x^1} + \ldots + x^n\hat{e}_{x^n}$ where the unit vectors

$$\hat{e}_{x^i} = \frac{\frac{\partial \vec{r}}{\partial u^i}}{\left|\frac{\partial \vec{r}}{\partial u^i}\right|}.$$

The determinant being non-zero also means that the map preserve the linear independence.