Exercise 4 for Metric Space

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March 23, 2024

These is my personal solution to the exercises 4 of Metric Space. They are only one of the ways to look at the problems, and in particular, all errors are almost surely mine.

Exercise 1. For a subset $W \subseteq (X,\, d)$ of a metric space, prove that

1. $W^\circ = X \backslash \left(\overline{X \backslash W}\right)$;
2. $\overline{W} = X \backslash \left(X \backslash W\right)^\circ$.

Proof. Before we start, we need to know about two properties of the closure and interior of a set: the closure of a set is the smallest closed set containing the set, and the interior of a set is the largest open set contained in the set.

Given a set W, for any open set $U \subseteq W$, we have

$$\forall\, x \in U,\ \exists\, r > 0\ \ (B_r\,(x) \subseteq U \subseteq W)\,.$$

Thus, $U \subseteq W^\circ$. In this case, W° is the largest open set contained in W.

Similarly, for any closed set $V \supseteq W$, we have

$$\forall\, x \in \overline{W},\ \forall\, r > 0\ \ (B_r\,(x) \cap V \supseteq B_r\,(x) \cap W \neq \varnothing)\,.$$

Thus, $\overline{W} \subseteq V$. In this case, $\overline{W}$ is the smallest closed set containing W.

Now we prove the two equations. Since $\overline{X \backslash W}$ is the smallest closed set containing $X \backslash W$, we know that its complement is the largest open set contained in W. Thus, $W^\circ = X \backslash \left(\overline{X \backslash W}\right)$.

Also, since $\left(X \backslash W\right)^\circ$ is the largest open set contained in $X \backslash W$, we know that its complement is the smallest closed set containing W. Thus, $\overline{W} = X \backslash \left(X \backslash W\right)^\circ$. $\square$

Exercise 2. Is $\mathbb{Q}^n$ dense in $\mathbb{R}^n$?

Proof. Note that there is a equivalent definition of dense set: a set is dense in a space if every open ball contains at least one point of the set.

In $\mathbb{R}^n$, consider d_1 metric, we know that its open balls have the form $B_{r_1}\,(x) = \prod (x_i - r_1,\, x_i + r_1)$. For each dimension, we can find a rational number p_i such that $x_i - r_1 < p_i < x_i + r_1$. Thus, we can find a point $p = (p_1,\, \cdots,\, p_n) \in \mathbb{Q}^n$ such that $p \in B_{r_1}\,(x)$. Thus, $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. $\square$