Some Interesting Properties in System of Linear First-order ODE

Yifan Li

November 2023

1 Linear Independence of Solutions

Remark: We say that a set of vector functions $\{X_1, \ldots, X_n\}$ to be **linearly dependent** on an interval I if

$$\exists\, c_1, \ldots, c_k \text{ not all zero, } \forall t \in I\ (c_1 X_1(t) + \ldots + c_k X_k(t) = 0).$$

If a set of vector functions is not linearly dependent on I, then it is said to be **linearly independent**.

Theorem 1.1. *If a set of solution vectors $X_1, \ldots, X_n$ of a homogeneous system of order n is linearly dependent at some $t_0 \in I$, then it is linearly dependent at any point $t \in I$. Also, if it is linearly independent at some $t_0 \in I$ then it is linearly independent at any $t \in I$*

Proof. We know that any solution vector can be written as $X_i = e^{tA} X_i(0)$. In fact, e^{tA} is just an invertible matrix, which can be regarded as a bijective linear transformation, which conserves the linear independence.

There is another way to think about it. If $X_1, \ldots, X_n$ are linearly dependent at $t_0 \in I$, then

$$\exists c_1, \ldots, c_n\ (c_1 X_1(t_0) + c_2 X_2(t_0) + \ldots + c_n X_n(t_0) = 0)$$

Let $X(t) = c_1 X_1(t) + \ldots + c_n X_n(t)$, both $X(t)$ and 0 are solutions to $X' = AX$ satisfying the initial value condition $X(t_0) = 0$. Thus they are equal by the uniqueness of the solution, i.e. $\forall t \in I\ (c_1 X_1(t) + \ldots + c_n X_n(t) = 0)$. □

2 Laplace Transform of e^{tA}

Before we introduce the Laplace transform of e^{tA}, we need to know what is e^{tA}. We may define it by a Taylor series.

Definition 2.1.

$$e^{tA} := I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \ldots := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!} A^k \qquad (1)$$

From the definition of the exponential of a matrix, we may find

Theorem 2.1.

$$\frac{de^{tA}}{dt} = Ae^{tA} = e^{tA}A \tag{2}$$

Theorem 2.2.

$$\mathcal{L}\{e^tA\} = (sI - A)^{-1} \tag{3}$$

Proof.

$$\begin{aligned}
\mathcal{L}\{e^{tA}\}(sI - A) &= s\mathcal{L}\{e^{tA}\} - A\} \\
&= s\mathcal{L}\{e^{tA}\} - \mathcal{L}\{Ae^{tA}\} \\
&= s\mathcal{L}\{e^{tA}\} - \mathcal{L}\{(e^{tA})'\} \\
&= s\mathcal{L}\{e^{tA}\} - (s\mathcal{L}\{e^{tA}\} - e^{0\cdot A}) \\
&= I
\end{aligned}$$

Thus $\mathcal{L}\{e^{tA}\} = (sI - A)^{-1}$ □

3 Fundamental Matrices

For the homogeneous system $X' = AX$, we can find its general solutions, which can be determined by a fundamental set of solutions for the system.

Definition 3.1. *Fundamental Matrix*

Let $X_1(t), \ldots, X_n(t)$ be fundamental set of solutions of $X' = AX$. Then the matrix

$$\Phi(t) = \begin{bmatrix} X_1(t) & X_2(t) & \ldots & X_n(t) \end{bmatrix} \tag{4}$$

is said to be a ***fundamental matrix*** *for the system $X' = AX$. In addition, the general solution can be determined by $\Phi(t)\,C$ where C is a constant vector.*

Theorem 3.1. *If $\Phi(t)$, $\Psi(t)$ are both fundamental matrices for the same system, then there exists a constant matrix C such that $\Phi(t) = \Psi(t)\,C$. In particular, $\Phi(t) = e^{tA}\Psi(0)$.*

Proof. Let $\Phi(t) = \begin{bmatrix} X_1(t) & \ldots & X_n(t) \end{bmatrix}$ and $\Psi(t) = \begin{bmatrix} Y_1(t) & \ldots & Y_n(t) \end{bmatrix}$, since $\Psi(t)$ is a fundamental matrix, any solution to the system can be written as a linear combination of $\{X_i(t)\}$, then we have

$$\begin{bmatrix} x_{1,j}(t) \\ x_{2,j}(t) \\ \vdots \\ x_{n,j}(t) \end{bmatrix} = X_j(t) = \sum_{k=1}^{n} c_{k,j} Y_k(t) = \begin{bmatrix} \sum_{k=1}^{n} c_{k,j} y_{1,k}(t) \\ \sum_{k=1}^{n} c_{k,j} y_{2,k}(t) \\ \vdots \\ \sum_{k=1}^{n} c_{k,j} y_{n,k}(t) \end{bmatrix}$$

The $i - th$ component of $X_j(t)$ is $\sum_{k=1}^{n} c_{k,j} y_{i,k}(t)$ Thus we may find that

$$\Phi(t) = \left[\sum_{k=1}^{n} y_{i,k}(t)\,c_{k,j}\right] = \Psi(t)\,C$$

□

4 Some Interesting Exercises

Application of Laplace Transform of e^{tA}

Example 4.1. *Let* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. *Compute* e^{tA}

Solution 4.1. *We first consider the Laplace transform of* e^{tA}.

$$\begin{aligned}
\mathcal{L}\{e^{tA}\} = (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 1 & S \end{bmatrix}^{-1} \\
&= \frac{1}{s^2+1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \\
&= \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{L}\{\cos t\} & \mathcal{L}\{\sin t\} \\ \mathcal{L}\{-\sin t\} & \mathcal{L}\{\cos t\} \end{bmatrix} \\
&= \mathcal{L}\{\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\}.
\end{aligned}$$

Thus

$$e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Example 4.2. *Find the general solution for the system*

$$\begin{cases} x_1' = -x_1 \\ x_2' = x_1 - x_2 \end{cases}$$

Solution 4.2. *The system above can be written as* $X' = AX$ *where*

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

Then

$$\begin{aligned}
\mathcal{L}\{e^{tA}\} = (sI - A)^{-1} &= \begin{bmatrix} s+1 & 0 \\ -1 & s+1 \end{bmatrix}^{-1} \\
&= \frac{1}{(s+1)^2} \begin{bmatrix} s+1 & 0 \\ 1 & s+1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)^2} & \frac{1}{s+1} \end{bmatrix} = \mathcal{L}\{\begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix}\}
\end{aligned}$$

Thus the fundamental matrix can be written as

$$\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t}, \end{bmatrix}$$

and the general solution is $X(t) = \Phi(t)\, C = c_1 \begin{bmatrix} e^{-t} \\ te^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$.

Application of Fundamental Matrices

Example 4.3. *Prove that $AB = BA$ implies that $Be^{tA} = e^{tA}B$.*

Proof. Let $\Phi = Be^{tA}$ and $\Psi = e^{tA}B$, then we may find that $\Phi' = BAe^{tA} = ABe^{tA} = A\Phi$ and $\Psi' = Ae^{tA}B = A\Psi$. Thus Φ and Ψ are fundamental matrices of $X' = Ax$. When $t = 0$ one could find that $\Phi(0) = \Psi(0) = B$, thus $\Phi(t) = \Psi(t)$. $\square$

Example 4.4. *Prove that $e^{tA}\,e^{tB} = e^{t(A+B)}$ if $AB = BA$.*

Proof. Let $\Phi(t) = e^{tA}\,e^{tB}$ and $\Psi(t) = e^{t(A+B)}$. Notice that the $\Psi(t)$ is just a fundamental matrix for the system

$$X' = (A+B)X.$$

For the $\Phi(t)$, we have

$$\begin{aligned}(e^{tA}\,e^{tB})' &= Ae^{tA}\,e^{tB} + e^{tA}\,Be^{tB} \\ &= (A+B)e^{tA}\,e^{tB}\end{aligned}$$

Thus it is also a fundamental matrix for $X' = (A+B)X$. Let $t = 0$, we may find that the $\Phi(0) = \Psi(0) = I$, thus they are the same. $\square$