A Solution to Question 3 of Exercise 6

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Proposition 1. *Let μ be a spectral radius of G, then*

$$\delta(G) \leq \mu \leq \Delta(G).$$

Proof. First we show that $\mu \leq \Delta(G)$. Let A be the adjacent matrix of G, since μ is an eigenvector, thus

$$\exists\, v\, (Av = \mu v). \tag{1}$$

Consider each element of Av and μv in the Equation (1), it gives that μv_i is just a partial sum of $\{v_1, v_2, \dots, v_n\}$.

$$\mu v_i = v_{n_1} + v_{n_2} + \dots + v_{n_m}.$$

Consider the largest v_i, denoted by $v_{max} = \max\{v_1, v_2, \dots, v_n\}$, then we may find that

$$\mu v_{max} = v_{n_1} + v_{n_2} + \dots + v_{n_m} \leq \Delta(G) v_{max}.$$

Thus $\mu \leq \Delta(G)$.

Then we try to show that $\delta(G) \leq \mu$. One may find that the previous idea does not help, so we need try other ways. Notice that the sum of elements in each row is the degree of a vertix. Let $e = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$, then we have $Ae = \begin{bmatrix} \deg(v_1) & \dots & \deg(v_n) \end{bmatrix}^T$. Remember that the sum of degrees is twice of the size of the graph, thus

$$e^T A e = \sum_{i=1}^{n} \deg(v_i) = 2\,|E(G)|. \tag{2}$$

Also, we can regard e as a linear combination of an eigenbasis. Since A is symmetric, we know that it can be diagonalized, which means that we can choose an orthonormal eigenbasis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then

$$\exists r_1, r_2, \dots, r_n\, (e = \sum_{i=1}^{n} r_i \alpha_i).$$

Now we compute $e^T A e$ and compare it with Equation (2).

$$\begin{aligned} e^T A e &= (\sum_{i=1}^{n} r_i \alpha_i^T) \left(A (\sum_{i=1}^{n} r_i \alpha_i) \right) = (\sum_{i=1}^{n} r_i \alpha_i^T)(\sum_{i=1}^{n} \lambda_i r_i \alpha_i) = \sum_{i=1}^{n} \lambda_i r_i^2 \\ &\leq \mu \sum_{i=1}^{n} r_i^2 = \mu e^T e = \mu n. \end{aligned} \tag{3}$$

By Equation (2) and Equation (3), one may find that

$$\mu n \geq 2|E(G)| \geq n\delta(G).$$

Thus $\mu \geq \delta(G)$. □

Proposition 2. *Let G be a connected graph, and S be the spectum of G. Show that*

(a) $\Delta(G) \in S$ *if and only if G is regular.*

(b) $-\Delta(G) \in S$ *if and only if G is regular and bipartite.*

Proof. (a) "$\Leftarrow$" is not difficult since by the previous proposition we know that $\delta(G) \leq \mu \leq \Delta(G)$ and $\delta(G) = \Delta(G)$ inplies that $\Delta(G) = \mu \in S$.

Now we try to prove "$\Rightarrow$". Suppose $\Delta(G) \in S$. Let p_m be the vertex corresponds to v_{max} where v_{max} is as defined in the previous proof. By the previous assertion that

$$\Delta(G)v_{max} \leq \mu v_{max} = v_{n_1} + v_{n_2} + \ldots + v_{n_m} \leq \text{degree}(p_m)v_{max} \leq \Delta(G)v_{max},$$

we know that three "$\leq$" should be "$=$", thus $degree(p_m) = \Delta(G)$ and $v_{n_k} = v_{max}$. Apply the above inplication to neighborhoods of p_m and repeat to neighborhoods of neighborhoods. Since the graph is connected, we may say that each vertex has degree $\Delta(G)$. Thus G is a regular.

(b) First, we prove "$\Leftarrow$". The adjacent matrix of a regular bipartite graph can be written as

$$A = \begin{bmatrix} O_m & J_{m,n} \\ J_{n,m} & O_n, \end{bmatrix}$$

where O_m is an $m \times m$ matrix with entries all be 0 and $J_{m,n}$ is an $m \times n$ matrix with entries all be 1. To find its eigenvalue, we may first try some special vectors. Let $v_0 = \begin{bmatrix} a & \ldots & a & b & \ldots & b \end{bmatrix}^T$ whose first m entries being a and others being b. One may find that when $a = \sqrt{n}$ and $b = \sqrt{m}$, v is an eigenvector corresponding to $\sqrt{mn}$, but it is not what we want, got stuck.

□

Additional things:

$\mu v_i = \sum v_j$ where i and j are adjacent, then consider $\max\{|v_i|\}$, we may find those $v_j = -v_i$, and thus those v_j are also in $\max\{|v_i|\}$. Repeat the argument and find that for any two elements in v, they are either equal or add up to 0. Conside those elements having the same sign, applying A, they would be mapped to all other rows, and thus it is bipartite.