Cauchy Problem for 1st Order Quasilinear PDE

Yifan Li

March 21, 2024

These is my personal thought to Cauchy problem for 1st order quasilinear PDE. They are only one of the ways to look at the problem, and in particular, all errors are almost surely mine.

Consider the equation

$$a(x,\, y,\, u,)u_x + b(x,\, y,\, u)u_y = c(x,\, y,\, u), \tag{1}$$

where $u = u(x,\, y)$ and $a,\ b,\ c$ are *continuously differentiable functions* (C^1).

Observing the Eq.(1), if we regard it as a dot product of two vector, we have

$$(a,\, b,\, c) \cdot (u_x,\, u_y,\, -1) = 0 \tag{2}$$

One may find that $(u_x,\, u_y,\, -1)$ is the normal vector of the surface $z = u(x,\, y)$, which is the solution, called **integral surface**. In addition, the dot product being 0 means that $(a,\, b,\, c)$ is tangential to the integral surface. Consider the "stream line" of the vector field $(a,\, b,\, c)$:

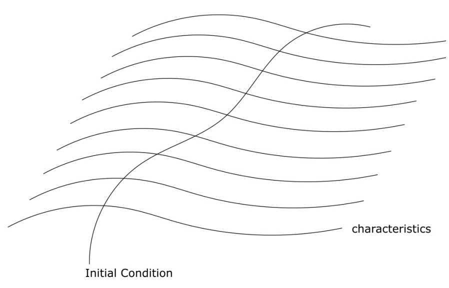
$$\frac{dx}{dt} = a(x,\, y,\, z),\ \frac{dy}{dt} = b(x,\, y,\, z),\ \frac{dz}{dt} = c(x,\, y,\, z), \tag{3}$$

they are called **characteristics**. Since we construct the characteristics by the idea of stream line, we may know that once a point of characteristics is in the integral surface, the whole curve would be in. Moreover, we may know that the integral surface is the union of characteristics.

Now consider the differential equation with initial condition:

$$\begin{cases} a(x,\, y,\, u,)u_x + b(x,\, y,\, u)u_y = c(x,\, y,\, u), \\ u(f(s),\, g(s)) = h(s). \end{cases} \tag{4}$$

The first equation of Eq.(4) would give a family of curves satisfying Eq.(3). We now study the second equation. One may find that the second equation represents a curve. In this case, both the initial consition and characteristics should embedded in the solution.



Note that what we want is a surface, which needs two variable to determine. Consider a map $(X(s,\, t),\, Y(s,\, t))$ such that

$$X(s,\, 0) = f(s),\ Y(s,\, 0) = g(s),\ \forall\, s_0,\, (X(s_0,\, t),\, Y(s_0,\, t)) \text{ is a characteristics.}$$

Now we consider whether these curves are tangential. We may use Jacobian near the initial condition for help (For the details of Jacobian, you may refer to the note for *Jacobian Matrix*).

$$J(0,\, 0) = \begin{vmatrix} X_s & X_t \\ Y_s & Y_t \end{vmatrix} = \begin{vmatrix} f'(s) & a(x(s,\, 0),\, y(s,\, 0),\, z(s,\, 0)) \\ g'(s) & b(x(s,\, 0),\, y(s,\, 0),\, z(s,\, 0)) \end{vmatrix}$$

If $J(s,\, 0) \neq 0$, then the situation would be similar to the figure above and one can find a C^1 solution. However, the world would not be always so beautiful. It is possible that $J(s, 0) = 0$ for some s, then we need to be careful. There are several subcases for the situation.

1. The initial curve is exactly a characteristics. Then there are infinitely many solutions for the PDE, since we can add an arbitrary smooth curve which is independent and intersecting to the initial curve as a new initial curve to form a furface.

2. The initial curve is not a characteristics, but it could be parallal to the characteristics at the point that $J(s, 0) = 0$. In this case, a C^1 solution may exists.

3. However, there is a worst situation: the initial curve intersects the characteristics and their tangential vectors are on a vertical plane. For example, as the figure below shows, the red line is a characteristics and the lower boundary of the surface is the initial curve. In this case, the solution would not be C^1 anymore, though the solution may exists.

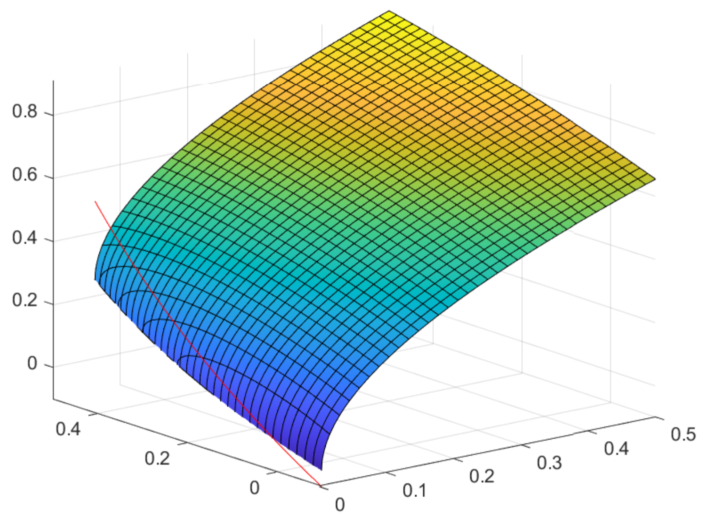


Figure 1: