

# Black hole complementarity with the generalized uncertainty principle in Gravity's Rainbow

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**Abstract.** When gravitation is combined with quantum theory, the Heisenberg uncertainty principle could be extended to the generalized uncertainty principle accompanying a minimal length. To see how the generalized uncertainty principle works in the context of black hole complementarity, we calculate the required energy to duplicate information for the Schwarzschild black hole. It shows that the duplication of information is not allowed and black hole complementarity is still valid even assuming the generalized uncertainty principle. On the other hand, the generalized uncertainty principle with the minimal length could lead to a modification of the conventional dispersion relation in light of Gravity's Rainbow, where the minimal length is also invariant as well as the speed of light. Revisiting the gedanken experiment, we show that the no-cloning theorem for black hole complementarity can be made valid in the regime of Gravity's Rainbow on a certain combination of parameters.

**Keywords:** modified gravity, quantum black holes

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## 1 Introduction

The discovery of Hawking radiation emitted from a black hole [1, 2] would lead to the information loss paradox [3]; however, it could be solved for a distant observer outside the horizon by assuming that the Hawking radiation carries the black hole information. In this assumption, the local observer (Bob) outside the horizon gathers the information of the infalling matter state through the Hawking radiation after a certain time which amounts to at least the Page time [4], and then he jumps into the black hole. If the infalling observer (Alice) who has the information of the infalling matter state sends the message with the information to him inside the black hole, then he may have the duplicated information, which is the violation of the no-cloning theorem in quantum theory. The so-called black hole complementarity has been proposed as a solution to this cloning problem to reconcile general relativity and quantum mechanics [5–7]. According to black hole complementarity, the cloning problem never occurs since the observer inside the horizon is not in the causal past of any observer who measures the information through the Hawking radiation outside the horizon [5]. The specific gedanken experiment [6] on the Schwarzschild black hole proves that the required energy to correlate the observations of both sides of the horizon exceeds the mass of the black hole. In other words, the information has to be encoded into the message with super-Planckian frequency to duplicate it. Thus, it turns out that the no-cloning theorem for black hole complementarity is safe for the Schwarzschild black hole.

On the other hand, it has been claimed that the notion of quantum theory may need a revision if one attempts to combine gravitation and quantum theory. One of the possibilities is the generalized uncertainty principle (GUP) which is the extended version of the Heisenberg uncertainty principle by adding a term of uncertainty in position due to the gravitational interaction [8–11]. It could be derived from not only general considerations of quantum mechanics and gravity [12] but also string theory [13–16], which gives rise to a minimal length of the order of the Planck length,  $(\Delta X)_{\min} \sim \sqrt{\alpha_{\text{GUP}}} L_P$ . Many efforts have been devoted to studying various aspects of the GUP [17–24]. Especially, the role of GUP was discussed in the context of the information loss problem [25]. Subsequently, it was also shown that while black hole complementarity on the Heisenberg uncertainty principle could be violated if the black hole evaporated by emitting a sufficiently large number  $N$  of species of massless scalar fields as the Hawking radiation, the GUP can prevent the violation of black hole complementarity assuming that the GUP parameter is proportional to the number of

fields,  $\alpha_{\text{GUP}} \sim N$  [26]. However, the large number of species makes the GUP parameter very large, so that the GUP effect should be too significant.

Now, we note that the existence of the minimal length in the GUP would necessarily lead to important modifications such as the black hole temperature and the Stefan-Boltzmann law. The temperature of the black hole could be modified by the GUP [27], and so there have been many applications to black hole systems [28–32]. Moreover, the Stefan-Boltzmann law should also be corrected in order to describe the black hole evaporation consistently in the regime of the GUP along with the modification of the temperature [33, 34]. So, one might wonder how black hole complementarity can be made valid even for a single scalar field of  $N = 1$ .

On the other hand, the system governed by the GUP does not allow the length scale below the minimal length, which means that the GUP is combined with the doubly special relativity of the extended version of Einstein’s special relativity [35, 36], where there are two observer-independent scales such as the minimal length and the speed of light. The most common illustration was presented to keep the relativistic energy-momentum relation in the framework of the doubly special relativity [37], which gives rise to the modified dispersion relation (MDR) [38]. In the framework of the doubly special relativity, the modification of the measure of integration in momentum space should be considered under the deformed symmetries, otherwise the MDR is only valid in one reference frame, implying a breakdown of the relativistic symmetries. The notion could be promoted to the curved spacetime, where the energy of the test particle deforms the general spacetime of the background geometry, which is named Gravity’s Rainbow [39]. For the MDR [40–47] as well as Gravity’s Rainbow [48–67], there have been extensive studies in order for exploring various aspects for black holes and cosmology. So, it seems to be natural to address the issues related to black hole complementarity with the GUP in the context of Gravity’s Rainbow.

In section 2, in a self-contained manner we recapitulate the well-established gedanken experiment to determine the required energy for the duplication of the infalling information on the Schwarzschild black hole with the Heisenberg uncertainty relation and the ordinary dispersion relation along the line of ref. [6]. In section 3, we calculate the required energy for cloning the information for the Schwarzschild black hole by using the GUP with the corresponding modified Stefan-Boltzmann law and the black hole temperature. It turns out that black hole complementarity is still valid with the GUP for  $N = 1$ . Furthermore, in section 4, the duplication of information with the GUP can be evaded in the framework of Gravity’s Rainbow. Finally, conclusion and discussion will be given in section 5.

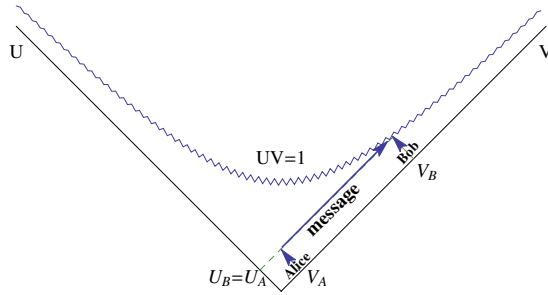
## 2 Black hole complementarity

Let us encapsulate the gedanken experiment in the Schwarzschild black hole by assuming the Heisenberg uncertainty principle of  $\Delta x \Delta p \geq 1$  and the ordinary dispersion relation for massless particles,  $E^2 - p^2 = 0$  [6]. In the Kruskal-Szekeres coordinates, the metric of the Schwarzschild black hole is given by

$$ds^2 = -\frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} dU dV, \quad (2.1)$$

where  $U = \pm e^{-\frac{(t-r^*)}{4GM}}$ ,  $V = e^{\frac{(t+r^*)}{4GM}}$ , and  $r^* = r + 2GM \ln(|r - 2GM|/2GM)$ . The plus and minus signs in  $U$  coordinate are for the inside and outside of the horizon, respectively. To obtain the Page time for the old black hole [4], we consider the Stefan-Boltzmann law,

$$\frac{dM}{dt} = -A\sigma T^4, \quad (2.2)$$



**Figure 1.** For the Schwarzschild black hole (later the rainbow Schwarzschild black hole), the wiggly curve in  $UV = 1$  means the curvature singularity at the origin,  $r = 0$ . Alice passes through the horizon at  $V_A$ , and then, after the Page time, Bob will jump into the horizon at  $V_B$ . Alice should send the message with information to Bob at least at  $U_A$  before Bob hits the singularity.

where  $\sigma$  denotes the Stefan-Boltzmann constant, and  $A$  and  $T$  are the area and temperature of the black hole identified with  $A = 16\pi G^2 M^2$  and  $T = 1/(8\pi GM)$  for the Schwarzschild black hole, respectively. Then, the Page time  $t_P$  can be calculated from the Stefan-Boltzmann law (2.2) as

$$t_P \sim G^2 M^3, \quad (2.3)$$

when the initial Bekenstein-Hawking entropy shrinks in half.

Now, we suppose that Alice first jumps into the horizon at  $V_A$ , and then Bob passes through the horizon with a record of his measurements of information from the Hawking radiation after the Page time  $t_P$  at  $V_B$  as shown in figure 1. Alice should send the message with information to Bob before he hits the curvature singularity,  $U_A = U_B = V_B^{-1} = e^{-t_P/(4GM)} \sim e^{-GM^2}$ . So, the proper time  $\Delta\tau$  for Alice to send the message to Bob at least at  $U_A$  can be calculated from the metric (2.1) near the horizon  $r = r_H$  as  $\Delta\tau^2 \sim G^2 M^2 e^{-GM^2}$ , where  $\Delta V_A$  is a nonvanishing finite value near  $V_A$  for the free-fall [6]. Then, the energy-time uncertainty principle of  $\Delta E \Delta\tau \geq 1$  gives the required energy  $\Delta E$  as

$$\Delta E \sim \frac{1}{GM} e^{GM^2}, \quad (2.4)$$

which is definitely larger than the black hole mass, i.e.,  $\Delta E \gg M$ , so that information must be encoded into the message with super-Planckian frequency. Therefore, the duplication of information is impossible and black hole complementarity can be well-defined.

### 3 Black hole complementarity with GUP

To find out the validity of the no-cloning theorem, we calculate the required energy to duplicate information by employing the modified temperature of the Schwarzschild black hole and the modified Stefan-Boltzmann law which are commensurate with the GUP [33]. Let us start with the GUP defined by [8–16],

$$\Delta x \Delta p \geq 1 + \alpha_{\text{GUP}} L_p^2 \Delta p^2, \quad (3.1)$$

where  $\alpha_{\text{GUP}}$  is the GUP parameter and the Planck length is denoted by  $L_{\text{p}} = \sqrt{G}$ . The first modification is that the black hole temperature from the GUP (3.1) is given as [27],

$$T = \frac{GM}{4\pi\alpha_{\text{GUP}}L_{\text{p}}^2} \left( 1 - \sqrt{1 - \frac{\alpha_{\text{GUP}}L_{\text{p}}^2}{G^2M^2}} \right) \simeq \frac{1}{8\pi GM} + \frac{\alpha_{\text{GUP}}}{32\pi G^2 M^3}. \quad (3.2)$$

Next, let us derive the Stefan-Boltzmann law consistent with the GUP along the line of ref. [33] in order to get the Page time corrected by the GUP. The wave lengths of photons in a cubical box with edges of length  $L$  are subject to the boundary condition  $1/\lambda = n/(2L)$  with a positive integer  $n$ . For oscillators in the box, the energy density is written in an integral form as

$$\begin{aligned} \rho &= \frac{1}{V} \int \bar{E} g(\nu) d\nu \\ &= 2 \int \bar{E} d^3\nu, \end{aligned} \quad (3.3)$$

where  $g(\nu)d\nu$  is the number of modes in an infinitesimal frequency interval  $[\nu, \nu + d\nu]$  and  $\bar{E}$  means the average energy per oscillator given by

$$\bar{E} = \frac{E}{e^{\frac{E}{T}} - 1}. \quad (3.4)$$

The relation for GUP (3.1) should be reflected in the modification of the de Broglie relation as [33]

$$\lambda = \frac{1}{p} (1 + \alpha_{\text{GUP}} L_{\text{p}}^2 p^2), \quad (3.5)$$

and then one can read off the relation between the energy and the frequency by using the conventional dispersion relation,  $\nu = E (1 - \alpha_{\text{GUP}} L_{\text{p}}^2 E^2 + \mathcal{O}(L_{\text{p}}^4 E^4))$ . Thus the energy density at a given temperature  $T$  is calculated from eq. (3.3) as

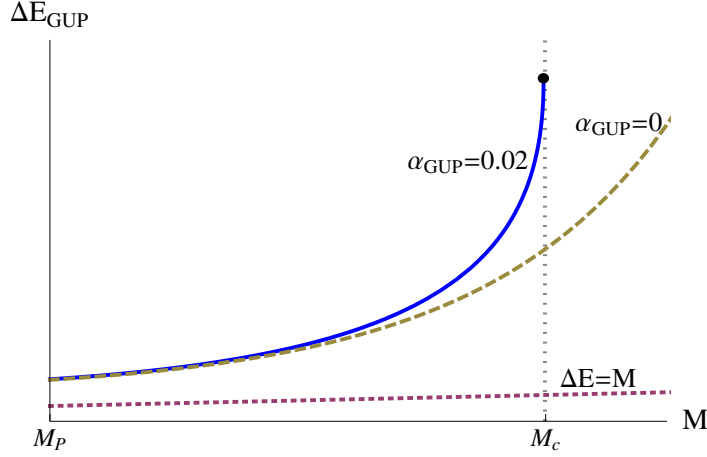
$$\rho = 8\pi \int dE \frac{E^3}{e^{\frac{E}{T}} - 1} (1 - 5\alpha_{\text{GUP}} L_{\text{p}}^2 E^2 + \mathcal{O}(L_{\text{p}}^4 E^4)) \quad (3.6)$$

$$\simeq 8\pi T^4 \int d\xi \frac{\xi^3}{e^\xi - 1} - 40\pi\alpha_{\text{GUP}} L_{\text{p}}^2 T^6 \int d\xi \frac{\xi^5}{e^\xi - 1} \quad (3.7)$$

$$\simeq \frac{8\pi^5}{15} T^4 - \frac{320\pi^7}{63} \alpha_{\text{GUP}} L_{\text{p}}^2 T^6, \quad (3.8)$$

where  $\xi = E/T$  at the finite temperature  $T$ .

It is worth noting that the expression (3.6) is actually valid as long as  $E < E_{\text{M}} = (\sqrt{\alpha_{\text{GUP}}} L_{\text{p}})^{-1}$ , where the integral involved in eq. (3.6) should be integrated up to the finite value of  $E_{\text{M}}$  rather than infinity due to the constraint of the GUP (3.1). In fact, the values of the two respective integrals in eq. (3.7) up to the cutoff of  $\xi_{\text{M}} = (\sqrt{\alpha_{\text{GUP}}} L_{\text{p}} T)^{-1}$  are expected to be slightly less than those values integrated up to infinity since the integrands are positive definite. For the sake of our neat calculation, we release the upper bound up to infinity, and then obtain the larger exact coefficients for each power of the temperature in eq. (3.8).



**Figure 2.** The standard result (2.4) with Heisenberg uncertainty principle ( $\alpha_{\text{GUP}} = 0$ ) and eq. (3.12) for the GUP ( $\alpha_{\text{GUP}} = 0.02$ ) are plotted, respectively. There are upper bounds at  $M_c$  for the case of GUP. As the black hole mass  $M$  increases, the energy denoted by  $\Delta E = M$  linearly increases with a very small slope depending on our scale while the required energy for the GUP increases exponentially. So, the required energy  $\Delta E_{\text{GUP}}$  is always larger than the black hole mass represented by the dotted line of  $\Delta E_{\text{GUP}} = M$ . The Planck mass is chosen as  $M_P = 1/L_P = 1$  for simplicity.

However, for the large black hole, these coefficients are insensitive to the final results. This kind of approximation will also be used in the later calculations.

Next, the Stefan-Boltzmann law improved by the GUP for the evaporating black hole is obtained as

$$\frac{dM}{dt} \simeq -A \left( \frac{8\pi^5}{15} T^4 - \frac{320\pi^7}{63} \alpha_{\text{GUP}} L_P^2 T^6 \right) \quad (3.9)$$

up to the linear order of  $\alpha_{\text{GUP}}$ , where  $A$  denotes the area of the black hole. The entropy can also be calculated by use of the first law of black hole thermodynamics as  $S = \int 1/T dM = 4\pi G M^2 - 2\pi \alpha_{\text{GUP}} \ln(\sqrt{G} M)$  [33].

Using the temperature (3.2) and the modified Stefan-Boltzmann law (3.9), one can get the Page time when the black hole has emitted half of its initial entropy and the information of the black hole starts to be emitted by the Hawking radiation as

$$t_P(M) = \int_{M_{\text{Page}}}^M dM \frac{1}{16\pi G^2 M^2 \left( \frac{8\pi^5}{15} T^4 - \frac{320\pi^7}{63} \alpha_{\text{GUP}} L_P^2 T^6 \right)}, \quad (3.10)$$

where  $M_{\text{Page}}$  is the mass of the black hole at the Page time, which is smaller than the initial mass  $M$  as  $M_{\text{Page}} \ll M$ . Following the argument of ref. [4], we rewrite the page time in terms of the black hole mass up to the subdominant term for the large black hole as

$$t_P(M) \sim G^2 M^3 - \alpha_{\text{GUP}} G M. \quad (3.11)$$

From the Schwarzschild metric (2.1), the interval of the proper time  $\Delta\tau$  which is nothing but the free-fall time for Alice near the horizon of  $r = 2GM$  is given as  $\Delta\tau \sim G M e^{-t_P(GM)^{-1}}$ . One can find the appropriate energy-time uncertainty principle as  $\Delta\tau \Delta E \geq 1 + \alpha_{\text{GUP}} L_P^2 \Delta E^2$  from the GUP (3.1). Finally, the required energy is read off from the generalized energy-time

uncertainty principle as

$$\Delta E_{\text{GUP}} \sim \frac{M}{2\alpha_{\text{GUP}}} e^{-GM^2 + \alpha_{\text{GUP}}} \left( 1 - \sqrt{1 - \frac{4\alpha_{\text{GUP}}}{GM^2} e^{2(GM^2 - \alpha_{\text{GUP}})}} \right), \quad (3.12)$$

where it nicely reduces to eq. (2.4) for  $\alpha_{\text{GUP}} \rightarrow 0$ .

Even though the required energy (3.12) has an upper bound  $M_c$ , the energy is larger than the mass of the black hole as shown in figure 2. It shows that the GUP effect improves the no-cloning theorem in the sense that the required energy for a given black hole mass is larger than that without the GUP correction, i.e.,  $\Delta E_{\text{GUP}} \geq \Delta E \gg M$ , so that black hole complementarity is still valid even for  $N = 1$  when the appropriate temperature and the Stefan-Boltzmann law are employed. However, this is not the whole story since this approach is incomplete in the sense that the minimal length should be treated as the invariant scale, so that the issue should be discussed in the regime of Gravity's Rainbow.

#### 4 Black hole complementarity with GUP in Gravity's Rainbow

In order to avoid the length contraction of the minimal length due to the FitzGerald-Lorentz contraction in Einstein's relativity theory, we introduce the MDR in the doubly special relativity [35, 36] which makes the Planck length invariant as a minimal length. Under the deformed symmetries, the measure of integration in momentum space should be modified in order for the relativistic properties not to be spoiled. However, in our case, it will turn out that the measure is invariant.

By using the non-linear Lorentz transformation in the momentum space, the MDR can be compactly written as [37, 38]

$$f(E)^2 E^2 - g(E)^2 p^2 = m^2, \quad (4.1)$$

where the rainbow functions  $f(E)$  and  $g(E)$  satisfy  $\lim_{E \rightarrow 0} f = 1$  and  $\lim_{E \rightarrow 0} g = 1$ , and  $E$  and  $m$  denote the energy and the mass of the test particle, respectively. The metric tensor associated with the MDR (4.1) is expressed in terms of a one-parameter family of orthonormal frame fields based on the modified equivalence principle as  $g^{\mu\nu}(E) = \eta^{ab} e_a^\mu(E) e_b^\nu(E)$ , where  $e_0(E) = f^{-1}(E) \tilde{e}_0$  and  $e_i(E) = g^{-1}(E) \tilde{e}_i$ , and  $\tilde{e}$  is the ordinary energy-independent vielbein. Then, the energy-dependent Schwarzschild metric is obtained as [39]

$$ds^2 = -\frac{1}{f(E)^2} \left( 1 - \frac{2GM}{r} \right) dt^2 + \frac{1}{g(E)^2} \frac{1}{\left( 1 - \frac{2GM}{r} \right)} dr^2 + \frac{r^2}{g(E)^2} d\Omega^2, \quad (4.2)$$

which is called the rainbow Schwarzschild black hole. From now on, we will employ the rainbow functions proposed in ref. [33],

$$f(E) = \left( 1 + \frac{\beta_{\text{MDR}}}{2} L_{\text{p}} E + \left( \frac{1}{2} \gamma_{\text{MDR}} - \frac{1}{8} \beta_{\text{MDR}} \right) L_{\text{p}}^2 E^2 \right), \quad g(E) = 1, \quad (4.3)$$

where  $\beta_{\text{MDR}}$  and  $\gamma_{\text{MDR}}$  are the MDR parameters. Then, the MDR (4.1) is rewritten for a massless particle as

$$p = E \left( 1 + \frac{\beta_{\text{MDR}}}{2} L_{\text{p}} E + \left( \frac{1}{2} \gamma_{\text{MDR}} - \frac{1}{8} \beta_{\text{MDR}} \right) L_{\text{p}}^2 E^2 \right). \quad (4.4)$$



Now, we are in a position to derive the black hole temperature and the modified Stefan-Boltzmann law by considering not only the GUP (3.1) but also the MDR (4.4). Combining the modified de Broglie relation (3.5) and the MDR (4.4) gives the relation between the energy  $E$  and the frequency  $\nu$ ,

$$\nu = E \left( 1 + \frac{1}{2} \beta_{\text{MDR}} L_{\text{P}} E + \left( \frac{1}{2} \gamma_{\text{MDR}} - \frac{1}{8} \beta_{\text{MDR}}^2 - \alpha_{\text{GUP}} \right) L_{\text{P}}^2 E^2 \right) + \mathcal{O}(L_{\text{P}}^3 E^3) \quad (4.5)$$

with the assumptions of  $\Delta p = p$  and  $\Delta E = E$  since the momentum and the energy uncertainties will be of order of the momentum  $p$  and the energy  $E$ , respectively [61].

The measure of the energy density (3.3) is rewritten by the momentum  $p$  in terms of the de Broglie relation (3.5) as

$$\rho = 2 \int \bar{E} d^3 \left( \frac{g(E)p}{(1 + \alpha_{\text{GUP}} L_{\text{P}}^2 g(E)^2 p^2)} \right), \quad (4.6)$$

where the measure of integration in momentum space should be modified under the deformed symmetries, for example, in the context of modified thermodynamics in ref. [68]. However, the measure of eq. (4.6) includes the three-dimensional momentum  $p$ , so that the measure of integration for the deformed symmetries is invariant since the rainbow function corresponding to the momentum is  $g(E) = 1$ .

Then, the energy density (4.6) is calculated with the average energy (3.4) per oscillator and the MDR(4.4),

$$\begin{aligned} \rho &= 8\pi \int dE \frac{E^3}{e^{\frac{E}{T}} - 1} \left( 1 + 2\beta_{\text{MDR}} L_{\text{P}} E + \left( \frac{5}{2} \gamma_{\text{MDR}} + \frac{5}{8} \beta_{\text{MDR}}^2 - 5\alpha_{\text{GUP}} \right) L_{\text{P}}^2 E^2 + \mathcal{O}(L_{\text{P}}^3 E^3) \right) \\ &\simeq \frac{8\pi^5}{15} T^4 + 384\pi\zeta(5)\beta_{\text{MDR}} L_{\text{P}} T^5 + \left( \frac{1}{2} \gamma_{\text{MDR}} + \frac{1}{8} \beta_{\text{MDR}}^2 - \alpha_{\text{GUP}} \right) \frac{320\pi^7}{63} L_{\text{P}}^2 T^6, \end{aligned} \quad (4.7)$$

where the similar approximations to eq. (3.8) are used, so that the modified Stefan-Boltzmann law induced by the GUP and the MDR is given as

$$\frac{dM}{dt} \simeq -A \left( \frac{8\pi^5}{15} T^4 + 384\pi\zeta(5)\beta_{\text{MDR}} L_{\text{P}} T^5 + \left( \frac{1}{2} \gamma_{\text{MDR}} + \frac{1}{8} \beta_{\text{MDR}}^2 - \alpha_{\text{GUP}} \right) \frac{320\pi^7}{63} L_{\text{P}}^2 T^6 \right), \quad (4.8)$$

where  $A$  is the area of the black hole.

Next, plugging the MDR (4.4) into the GUP (3.1), one can get the following relation,

$$E \left( 1 + \beta_{\text{MDR}} L_{\text{P}} E + \left( \frac{1}{2} \gamma_{\text{MDR}} - \frac{1}{8} \beta_{\text{MDR}} \right) L_{\text{P}}^2 E^2 \right) \Delta x \simeq 1 + \alpha_{\text{GUP}} L_{\text{P}}^2 E^2 + \mathcal{O}(L_{\text{P}}^3 E^3) \quad (4.9)$$

by assuming that  $\Delta p = p$  since the momentum is of order of the momentum  $p$  [27, 61]. Next, by use of  $\Delta x = 2GM$  and  $E = 4\pi T$  [27], the black hole temperature associated with the GUP with the MDR is obtained as

$$T \left( 1 + \beta_{\text{MDR}} L_{\text{P}} (4\pi T) + \left( \frac{1}{2} \gamma_{\text{MDR}} - \frac{1}{8} \beta_{\text{MDR}} \right) L_{\text{P}}^2 (4\pi T)^2 \right) \simeq \frac{1}{8\pi GM} (1 + \alpha_{\text{GUP}} L_{\text{P}}^2 (4\pi T)^2), \quad (4.10)$$

where we neglected the higher-order correction terms above the square of  $L_{\text{P}}^2 T^2$  in eq. (4.9) since the Planck length and the black hole temperature of the large black hole are very



small. And the black hole temperature (4.10) gives rise to the entropy of the black hole as  $S \sim A/4 + \beta_{\text{MDR}}\sqrt{A} + (\alpha_{\text{GUP}} - \gamma_{\text{MDR}}/2)\ln A$ .

Note that the expression (4.10) is a non-linear closed form, so that we will choose one of the simplest but non-trivial combination of parameters,

$$\beta_{\text{MDR}} = 0, \quad \alpha_{\text{GUP}} = \frac{1}{2}\gamma_{\text{MDR}}, \quad (4.11)$$

which makes the entropy of the black hole to be the Bekenstein-Hawking entropy. For the special choice of parameters (4.11), the rainbow functions (4.3) are rewritten as  $f(E) = 1 + \alpha_{\text{GUP}}L_{\text{p}}^2E^2$  and  $g(E) = 1$ . And the modified Stefan-Boltzmann law (4.8) and the temperature (4.10) are simplified as  $dM/dt \simeq -A(8\pi^5/15)T^4$  and  $T \simeq (8\pi GM)^{-1}$ . They are nothing but the Hawking temperature and the conventional Stefan-Boltzmann law, so that the Page time is simply written as  $t_{\text{p}} \sim G^2M^3$ .

Now, let us calculate the required energy for the duplication of information in the rainbow Schwarzschild black hole along the argument in ref. [6]. First, the rainbow metric (4.2) is written in terms of the rainbow Kruskal-Szekeres coordinates defined as [61]

$$ds^2 = -\frac{4r_{\text{H}}^3}{g(E)^2r}e^{-\frac{r}{r_{\text{H}}}}dUdV, \quad (4.12)$$

where  $U = \pm e^{-(g/f)t-r^*/(2GM)}$ ,  $V = e^{((g/f)t+r^*)/(2GM)}$  and  $r^* = r + 2GM \ln(|r - 2GM|/2GM)$ . The plus sign in  $U$  coordinate will be selected to describe the inside of the horizon. As shown in figure 1, Alice should send her information encoded into a message before  $U_{\text{A}} = U_{\text{B}} = V_{\text{B}}^{-1} \sim e^{-(g/f)GM^2}$ , so that the proper time measured by Alice near the horizon  $r = 2GM$  is obtained from the metric (4.12) as

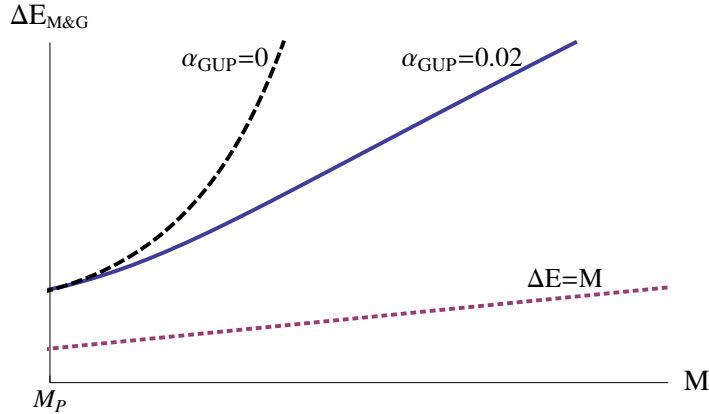
$$\Delta\tau \sim GM e^{-\frac{GM^2}{1+\alpha_{\text{GUP}}L_{\text{p}}^2\Delta E^2}}, \quad (4.13)$$

where we assumed that  $\Delta V_{\text{A}}$  is a nonvanishing finite value [6].

Next, one can find the appropriate energy-time uncertainty principle from the GUP and the MDR as  $\Delta\tau\Delta E \geq 1 + \alpha_{\text{GUP}}L_{\text{p}}^2\Delta E^2 + \mathcal{O}(\alpha_{\text{GUP}}^2L_{\text{p}}^4\Delta E^4)$  by use of the definition of the group velocity as  $v_{\text{G}} = \Delta E/\Delta p$ . Then, the required energy  $\Delta E_{\text{M\&G}}$  for duplication of information is finally obtained as

$$\Delta E_{\text{M\&G}} \sim \frac{1 + \alpha_{\text{GUP}}L_{\text{p}}^2\Delta E_{\text{M\&G}}^2}{GM} e^{\frac{GM^2}{1+\alpha_{\text{GUP}}L_{\text{p}}^2\Delta E_{\text{M\&G}}^2}}, \quad (4.14)$$

which goes to the ordinary relation for  $\alpha_{\text{GUP}} \rightarrow 0$ . Since it is non-trivial to solve eq. (4.14) with respect to  $\Delta E_{\text{M\&G}}$ , and so we solve it for  $M$  as  $M = \sqrt{-(L_{\text{p}}^{-2}/2)(1 + \alpha_{\text{GUP}}L_{\text{p}}^2\Delta E_{\text{M\&G}}^2)W(Y(\Delta E_{\text{M\&G}}))}$ , where the Lambert W-function is defined as  $Y = W(Y)e^{W(Y)}$  with the variable  $Y = -2(1 + \alpha_{\text{GUP}}L_{\text{p}}^2\Delta E_{\text{M\&G}}^2)\Delta E_{\text{M\&G}}^{-2}L_{\text{p}}^{-2}$ . Then, we can demonstrate the behavior of  $\Delta E_{\text{M\&G}}$  with respect to  $M$  by a parametric plot of a curve for the points  $(M, \Delta E_{\text{M\&G}})$  in figure 3. The required energy  $\Delta E_{\text{M\&G}}$  to send the message from Alice to Bob before he hits the singularity always exceeds the black hole mass  $M$ . Thus it indicates that the no-cloning theorem in quantum theory for black hole complementarity can be made valid in the extended regime of the GUP in Gravity's Rainbow.



**Figure 3.** We set the Planck length to  $L_P = 1$ . The upper thick dotted curve is for  $\alpha_{\text{GUP}} = 0$  corresponding to the standard result using the Heisenberg uncertainty principle and the ordinary dispersion relation, whereas the solid curve is for  $\alpha_{\text{GUP}} = 0.02$  corresponding to the required energy corrected by the GUP with the MDR. As the black hole mass  $M$  increases, the energy described by  $\Delta E = M$  linearly increases with a very small slope while the required energy corrected by the GUP and the MDR increases exponentially. So, the required energy  $\Delta E_{\text{M\&G}}$  is larger than the mass of the black hole. Note that we cut off the other branch of the Lambert W-function since it describes the small mass region below the Planck mass  $M_P$ .

## 5 Conclusion and discussion

The required energy for Alice to send the message to Bob who jumped into the black hole at the Page time was calculated in the presence of the minimal length defined by the GUP. We showed that the required energy becomes the super-Planckian scale, so that it turned out that the unitarity in quantum mechanics is maintained and black hole complementarity is safe. Furthermore, we revisited the above issue by employing MDR for the rainbow Schwarzschild black hole. The required energy also exceeds the mass of black hole, so that the no-cloning theorem for black hole complementarity can be made valid. Unfortunately, it is not a general proof since the present calculation is based on the particular rainbow functions and the special choice of parameters.

So, we mention the reason why the specific combination (4.11) of the GUP and MDR parameters was adopted. The parameter  $\beta_{\text{MDR}}$  in the MDR (4.4) gives rise to the square root of the black hole area  $A$  as the leading correction of the black hole entropy of  $S \sim A/4 + \beta_{\text{MDR}}\sqrt{A} + \mathcal{O}(\ln A)$  [33]. But it has been shown that a logarithmic correction appears as the leading correction of the entropy-area relation in various quantum gravity scenarios [69–72]. Note that in loop quantum gravity, such a  $\sqrt{A}$  correction to the entropy of the black hole was excluded [73–75], so that the presence of linear-in- $L_P$  contributions related to  $\beta_{\text{MDR}}$  in the MDR was also eliminated [33, 76]. In these respects, we took the parameter  $\beta_{\text{MDR}}$  to vanish in our work. Finally, we obtained our results by assuming the additional choice of parameters of  $\alpha_{\text{GUP}} = \gamma_{\text{MDR}}/2$  where the entropy of the black hole respects the area law eventually. We hope that the present study will be extended to generic models in the near future.

Finally, we would like to discuss whether the required energies (3.12) and (4.14) are still larger than the mass of the black hole when the Hawking radiation consists of a large number of scalar fields. Our results can be extended to the case that the black hole emits the large number  $N$  of species of massless scalar fields under the large  $N$  rescaling scheme

as  $M \rightarrow M' \equiv \sqrt{N}M$  along the lines of ref. [26]. In this scheme, all plots in figure 2 and figure 3 are just rescaled along the  $M$ -axis without changing any physical deformation, so that we can show that the required energies (3.12) and (4.14) with a large  $N$  still exceed the mass of the black hole. Consequently, despite the large number of scalar fields, it turns out that the violation of black hole complementarity can be evaded.

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