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Phase transition of quantum-corrected Schwarzschild black hole

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ABSTRACT

We study the thermodynamic phase transition of a quantum-corrected Schwarzschild black hole. The modified metric affects the critical temperature which is slightly less than the conventional one. The space without black holes is not the hot flat space but the hot curved space due to vacuum fluctuations so that there appears a type of Gross–Perry–Yaffe phase transition even for the very small size of black hole, which is impossible for the thermodynamics of the conventional Schwarzschild black hole. We discuss physical consequences of the new phase transition in this framework.

1. Introduction

Thanks to Hawking radiation based on a Bekenstein's conjecture [1,2], there has been much attention to the thermodynamics of a black hole system [3]. If the black hole is regarded as a thermal object in equilibrium, then it is natural to apply the thermodynamics; however, a crucial difference from the other thermal systems is that it is a gravitational object whose entropy is written by the area law [4,5] which provides intriguing thermodynamic issues.

In particular, a hot flat space without black holes can decay into a black hole state because thermal particles can be a source of gravitational collapse and then the black hole resides in thermal equilibrium with the Hawking radiation called Gross–Perry–Yaffe (GPY) phase transition [6]. From the thermodynamic point of view at the isothermal surface [7,8], one can get a small unstable black hole with the mass M_1 and a large stable black hole with the mass M_2 in the Schwarzschild black hole. In connection with the GPY phase transition, the off-shell free energy of the hot flat space without black holes shows that the GPY phase transition occurs only in the large black hole. Actually, the thermodynamic phase transition and behaviors have been well appreciated in terms of various ways in the modified Schwarzschild black holes [9–15]. To study quantum-mechanical aspects of thermodynamic phase transition, we have to consider the back reaction of the spacetime due to quantum fluctuations. In particular, the deformation of the Schwarzschild metric has been studied in Ref. [16] for the spherically symmetric quantum fluctuations of the metric in detail. It may give some improved thermodynamic properties especially in the UV region although they are expected to be the same with the thermodynamic behaviors at the large black hole.

In this work, we will study the phase transition of the quantum-corrected Schwarzschild black hole in order to uncover quantum-mechanical aspects of thermodynamic behaviors. On general grounds, the vacuum without black holes at a zero temperature can be defined in terms of the Minkowski space. Then, the hot thermal particles in the flat space can decay into black holes. What it means is that the free energy of the black hole is lower than the free energy of the hot flat space. Now, in this quantum-corrected metric, the vacuum without black holes is non-trivial since it is not Ricci flat due to quantum fluctuations even in the absence of the black hole. Hence, it is natural to regard the hot curved space as a counterpart of the hot flat space for the ordinary Schwarzschild black hole. As expected, the hot curved space can also decay into the large stable black hole. For convenience, let us define a tiny black hole whose mass is less than the critical mass, which will be shown in later. Then, even in the UV region, we can show that the hot curved space collapses into the tiny black hole. It can be interpreted as a type of GPY phase transition in the UV region.

In Section 2, the quantum-corrected metric given in Ref. [16] is recapitulated. The spherically symmetric reduction of the Einstein–Hilbert action can be written in terms of a renormalizable two-dimensional dilaton gravity [17,18], which yields the quantum-corrected metric. In Section 3, the relevant thermodynamic quantities will be calculated at a finite isothermal surface. In particular,

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they vanish at the finite distance before $r = 0$ because of quantum fluctuations. To study the phase transition of the quantum-corrected Schwarzschild black hole, we construct the off-shell free energy of the hot curved space and the black hole, and show that the critical temperature to create the black hole is less than the conventional critical temperature in Section 4. Moreover, it turns out that the free energy of the quantum-corrected black hole is negatively shifted near UV region, which lies in a lower state than the free energy of the hot curved space. It is the essential ingredient in the formation of the tiny black hole. Finally, the summary and discussion are given in Section 5.

2. Quantum-corrected Schwarzschild metric

In this section, we would like to introduce the quantum-corrected metric in a self-contained manner for our notations [16]. So, we start with the Einstein–Hilbert action with the matter action given by

$$I = \int d^4x \sqrt{-g^{(4)}} \left[\frac{R}{16\pi G_N} + L_{\mathrm{matter}} \right], \tag{1}$$

where G_N is the Newton constant. From now on, we neglect the classical matter contribution. Now, the spherically symmetric reduction of the four-dimensional metric can be performed by assuming

$$(ds)^2_{(4)} = ds^2_{(2)} + \frac{2G_N}{\pi} e^{-2\phi}\, d\Omega^2, \tag{2}$$

where we express the radial part in terms of the dilaton field ϕ maintaining the two-dimensional diffeomorphism. Then, we get the two-dimensional dilaton-gravity action [18]

$$I = \frac{1}{2\pi} \int d^2x \sqrt{-g^{(2)}} \left[e^{-2\phi} R + 2e^{-2\phi} (\nabla\phi)^2 + \frac{\pi}{G_N} \right]. \tag{3}$$

We assume that the generally renormalizable action takes the following form

$$I = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[e^{-2\phi} R + 2e^{-2\phi} (\nabla\phi)^2 + \frac{\pi}{G_N} U(\phi) \right], \tag{4}$$

with a new general potential $U(\phi)$ for the renormalization. Next, the divergences can be determined by the two-dimensional nonlinear σ-model as

$$I = -\frac{1}{2\pi} \int d^2X \sqrt{-\hat{g}(X)} \left[G_{\alpha\beta}(X) \hat{\nabla} X^\alpha \hat{\nabla} X^\beta + \frac{1}{2} \Phi(X) \hat{R} + T(X) \right] \tag{5}$$

where $\hat{g}_{\mu\nu}$ is a fiducial metric and $G_{\alpha\beta}(X)$ is a target space metric, respectively. After the identification of the coordinate X^α, the dilaton Φ, the tachyon field T, and the target metric $G_{\alpha\beta}(X)$, one can choose the vanishing β-function [17]. Using the renormalization group equation for the potential, $\beta^U = \partial_t U$, $t = \ln(\mu/\mu_0)$ [16], one can get the renormalized potential as

$$U(\phi) = \frac{e^{-\phi}}{\sqrt{e^{-2\phi} - \frac{4}{\pi} G_R}} \tag{6}$$

where $G_R = G_N \ln(\mu/\mu_0)$ and μ is a scale parameter. Then, solving the equations of motion for the action (4), one can obtain the quantum-corrected Schwarzschild metric,

$$g(r) = -\frac{2N}{r} + \frac{1}{r} \int^r U(r)\, dr = -\frac{2M}{r} + \frac{\sqrt{r^2 - a^2}}{r}, \tag{7}$$

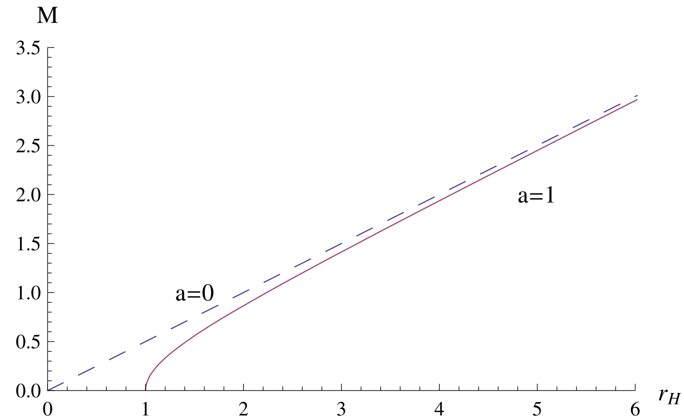

Fig. 1. It shows that the modification of relation between the event horizon and the mass. For a given mass M, the size of the classical black hole ($a = 0$) is smaller than that of the quantum-corrected black hole ($a = 1$), which can be seen clearly in the UV region.

where $a^2 \equiv 4G_R/\pi$. The radial coordinate is restricted to $r > a$ and the four-dimensional quantum-corrected metric is written as

$$(ds)^2 = -g(r)\, dt^2 + \frac{1}{g(r)} dr^2 + r^2\, d\Omega^2, \tag{8}$$

where the event horizon is located at $r_{\mathrm{H}} = \sqrt{(2M)^2 + a^2}$. Note that the size of the quantum-corrected black hole is slightly larger than the classical one as seen from Fig. 1 because of quantum fluctuations.

The metric (8) looks asymptotically like a Reissner–Nordstrom metric $g(r) \approx 1 - 2M/r - a^2/2r^2$, however, it gives completely different behavior because of the negative signature of the third term in the metric. It is interesting to note that the spacetime is not Ricci flat even in spite of the absence of the classical matter contribution,

$$R(a) = \frac{1}{a^2} \left[2\left(\frac{a}{r}\right)^2 \left(1 - \frac{1}{\sqrt{1 - (\frac{a}{r})^2}}\right) + \left(\frac{a}{r}\right)^4 \left(1 - \left(\frac{a}{r}\right)^2\right)^{-\frac{3}{2}} \right]$$
$$= \begin{cases} \infty & r \to a, \\ 0 & r \to \infty \end{cases} \tag{9}$$

where the curvature scalar can be written as asymptotically $R \approx 2a^4/r^6 \neq 0$. The reason why the mass parameter does not appear in the scalar curvature is that the original Schwarzschild metric is Ricci flat. The parameter a appears in such a way that the quantum-mechanical fluctuation breaks the Ricci flatness. Of course, for the vanishing limit of $a = 0$, the curvature scalar is zero as expected. Essentially, the vacuum fluctuation of the flat spacetime induces the virtual particles, which are the source of the present curved spacetime. It means that the vacuum geometry is nontrivial even in spite of the absence of the black hole ($M = 0$). The classical vacuum corresponding to the flat spacetime was deformed by the spherically symmetric quantum correction. After all, the ground state is curved. From the thermodynamic point of view, if one considers the hot particles in this background, then it is natural to consider the instability of the hot curved spacetime, which is an extension of the Gross–Perry–Yaffe instability of the hot flat spacetime.

3. Thermodynamic quantities

We shall calculate thermodynamic quantities in order to study the phase transition from the hot curved space to black holes. Let us first define the Hawking temperature,

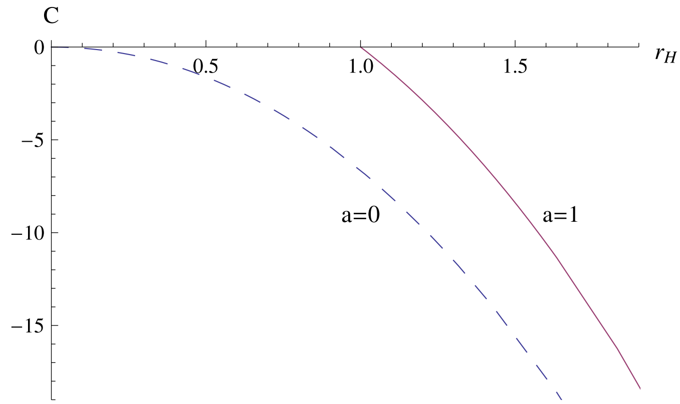

Fig. 2. Plot of the heat capacity for $r=10$ at the UV region which is far from the small black hole. The solid line for the quantum-mechanical one is slightly shifted and the heat capacity approaches zero at the finite size.

$$T_{\rm H}(a)=\frac{1}{4\pi}\left[\sqrt{-g^{tt}g^{rr}}\left(-g'_{tt}\right)\right]\Big|_{r=r_{\rm H}}=\frac{1}{4\pi\sqrt{r_{\rm H}^2-a^2}}. \tag{10}$$

It blows up for $r_{\rm H}=a$. Next, the observer at the finite isothermal surface sees the Tolman temperature [19,20] as

$$T_{\rm loc}(a)=\frac{T_{\rm H}}{\sqrt{g(r)}}=\frac{1}{4\pi\sqrt{r_{\rm H}^2-a^2}}\frac{\sqrt{r}}{\sqrt{\sqrt{r^2-a^2}-\sqrt{r_{\rm H}^2-a^2}}}. \tag{11}$$

Let us assume that the black hole entropy satisfies the area law,

$$S=\frac{A}{4}=\pi r_{\rm H}^2, \tag{12}$$

which is clear since the present quantum correction just modifies the potential term in the action so that the area law is consistent with the Wald entropy [21].

The thermodynamic local energy can be derived from the thermodynamic first law,

$$dE=T\,dS, \tag{13}$$

which is explicitly calculated as

$$E(a)=E_0+\int_{S_0}^{S}T_{\rm loc}(r)\,dS=E_0+\sqrt{r}\left[\sqrt{\sqrt{r^2-a^2}}-\sqrt{\sqrt{r^2-a^2}-\sqrt{r_{\rm H}^2-a^2}}\right], \tag{14}$$

using $dS=2\pi r_{\rm H}\,dr_{\rm H}$ in Eq. (14). Note that for $a=0$, it recovers the well-known local energy of the Schwarzschild black hole. Specifying the boundary condition of $E_0=0$, we get $E=M$ for the infinite cavity. In this case, the thermodynamic energy is nothing but the ADM mass along with the Hawking temperature so that the thermodynamic first law $dM=dS/T_{\rm H}$ is trivially satisfied.

For the thermodynamic stability, one can calculate the heat capacity at the finite boundary,

$$C(a)=\left(\frac{dE}{dT_{\rm loc}}\right)_r=\frac{4\pi}{3}\frac{(r_{\rm H}^2-a^2)[\sqrt{r^2-a^2}-\sqrt{r_{\rm H}^2-a^2}\,]}{3\sqrt{r_{\rm H}^2-a^2}-2\sqrt{r^2-a^2}}. \tag{15}$$

The small black hole is unstable for $r_{\rm H}<(2r/3)\sqrt{1+5a^2/4r^2}$ while the large black hole is stable for $r_{\rm H}>(2r/3)\sqrt{1+5a^2/4r^2}$, which is very similar to the conventional Schwarzschild black hole in the box. The difference comes from the heat capacity in the UV region so that the vanishing heat capacity for the quantum-corrected Schwarzschild black hole appears at the finite size as seen from Fig. 2.

As for the Tolman temperature (11) and the heat capacity (15) in connection with the stability of the black hole, the Schwarzschild black hole without the box gives rise to thermal instability. The essential reason is that Hawking temperature which is measured at the infinity is proportional to the inverse mass, so that the Hawking temperature decreases if the black hole absorbs a small amount of radiation. In other words, it yields the negative heat capacity irrespective of the size of the black hole. Moreover, the density of states for the canonical ensemble is pathological because it is not well-defined in this black hole system of the negative heat capacity [22]. To overcome these difficulties, one can take the advantage of the Tolman temperature by introducing finite thermal bath instead of the infinite thermal bath characterized by the Hawing temperature. The Tolman temperature is defined at the surface gravity in terms of the Killing vectors at the finite surface so that it contains the red-shift factor of the metric $g(r)$. Then, it gives interesting feature that the black hole temperature increases with respect to the mass in the large black hole for the given size of the cavity, and the heat capacity is eventually positive and then the large black hole can be stable. Moreover, the canonical ensemble with the Tolman temperature can be well-defined.

4. Free energy and phase transition

We are going to obtain the off-shell free energy to find the critical temperature of the black hole formation. Then, the phase transition from the hot curved space to the black hole system is studied. Now, the off-shell free energy can be defined as

$$F_{\rm off}^{\rm BH}(a)=E(a)-TS=\sqrt{r}\left[\sqrt{\sqrt{r^2-a^2}}-\sqrt{\sqrt{r^2-a^2}-2M}\,\right]-\pi\left(4M^2+a^2\right)T. \tag{16}$$

For $a=0$, it is reduced to the free energy for the Schwarzschild black hole. However, the free energy (16) for $M=0$ is not zero at any temperatures, which is in contrast to the conventional one. It will affect the phase transition from the hot curved space to the black hole.

By the way, the critical temperature can be calculated as

$$T_c(a)=\frac{3\sqrt{3}}{8\pi r}\left(1+\left(\frac{a}{r}\right)^2\right)^{-\frac{3}{4}} \tag{17}$$

from extrema of the off-shell free energy, $dF_{\rm off}^{\rm BH}(a)/dM|_{T=T_c}=0$. Among three extrema, the physically meaningful two extrema in thermal equilibrium appear at the positive mass region. The small root defined by $M_1(a)$ is for the small unstable black hole and the other one defined by $M_2(a)$ is for the large stable black hole. Note that they are equal root $M_1(a)=M_2(a)$ at the critical temperature. The large black hole can be nucleated above the critical temperature as seen from Fig. 3. After some calculations, we can find the small black hole is less than the conventional one while the large black hole is larger than the conventional one, i.e., $M_1(a)<M_1(0)$ and $M_2(a)>M_2(0)$, where $M_1(0)$ and $M_2(0)$ are just small and large masses for $a=0$. In particular, the quantum-corrected critical temperature is less than the conventional critical temperature, which means that the large stable black hole can be nucleated in equilibrium at a slightly small temperature compared to the classically expected temperature.

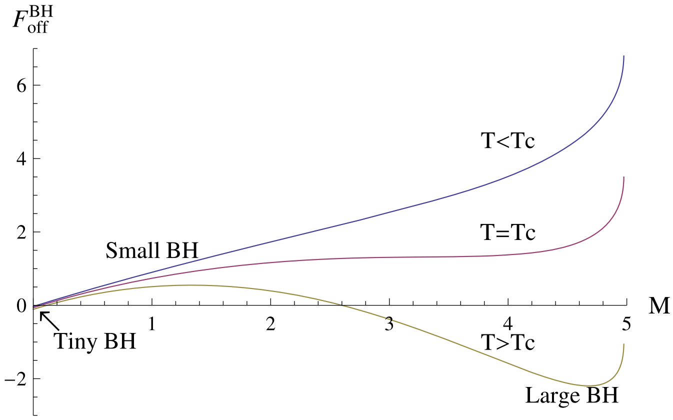

Fig. 3. For $a = 1$, $r = 10$, $T = 0.0350$ and $T_c(1) = 0.0205$, the large black hole of mass $M_2 = 4.680$ can be nucleated in stable equilibrium and the small black hole of the mass $M_1 = 1.330$ can decay into either the large black hole or massless black hole state. Overall behaviors are the same with conventional ones except the UV region. Intriguing thermodynamic properties in the UV region for this tiny black hole will be given in Fig. 4.

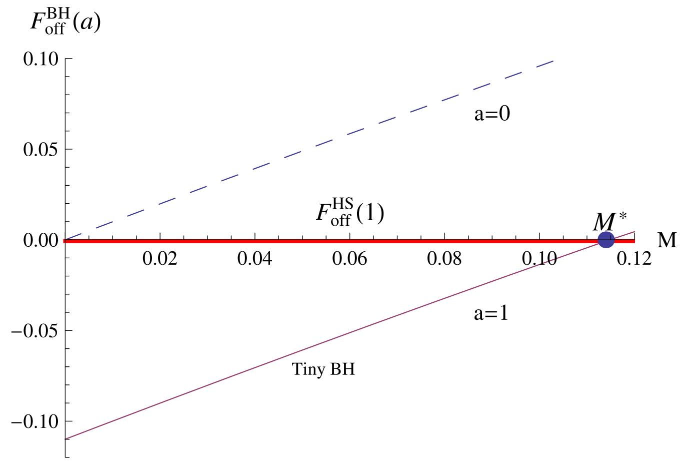

Fig. 4. Plot of the off-shell free energy at $r = 10$, $T = 0.0350$ and $T_c(1) = 0.0205$. The horizontal bold line describes the free energy of the hot curved space which is actually negative. The solid curve is for the off-shell free energy of the quantum-corrected one $F_{\text{off}}^{\text{BH}}(1)$, which is lower than the dotted curve of the classical off-shell free energy $F_{\text{off}}^{\text{BH}}(0)$. $M^* = 0.114$ is a critical mass to form a tiny black hole.

For the completeness of the phase transition, we consider the free energy of the hot curved space at a temperature. For simplicity, the free energy for a single scalar field on the curved space without black holes is given by

$$F_{\text{off}}^{\text{HS}}(a) = -\frac{2}{3\pi}\int_a^r dr\,\frac{r^2}{g(r)}\int_0^\infty dE\,\frac{[E^2 - g(r)m^2]^{\frac{3}{2}}}{(e^{\beta E} - 1)}$$
$$= -\frac{2\pi^3}{135}\sqrt{r^2 - a^2}\left(r^2 + 2a^2\right)T^4 + O\left(m^2\right). \tag{18}$$

For a massless limit, it can be regarded as the free energy for gravitons by adding spin degrees of freedom. Note that the free energy of the hot flat space is greater than that of the hot curved space. The reason why we consider the hot curved space rather than the hot flat space is that our spacetime is already curved due to the quantum fluctuation, which has something to do with the non-Ricci flatness of the quantum-corrected Schwarzschild black hole as shown in the previous section. In other words, the spacetime without black holes is essentially curved because of vacuum fluctuations. The free energy difference between the hot flat space without black hole $F_{\text{off}}^{\text{HS}}(0)$ and the hot curved space without black hole $F_{\text{off}}^{\text{HS}}(a)$ is explicitly given as

$$F_{\text{off}}^{\text{HS}}(0) - F_{\text{off}}^{\text{HS}}(a) = -\frac{2\pi^3}{135}r^3T^4\left[1 - \sqrt{1 - \frac{a^2}{r^2}}\left(1 + \frac{2a^2}{r^2}\right)\right], \tag{19}$$

which is positive for $a^2/r^2 < \sqrt{3/4}$. If the size of the cavity is properly large compared to the parameter a, the free energy of the hot flat space is greater than the free energy of the hot curved space. So, one can naturally imagine that the transition from the hot flat space to the hot curved space $F_{\text{off}}^{\text{HS}}(0) \to F_{\text{off}}^{\text{HS}}(a)$ is possible.

We are now in a position to mention the possibility of phase transition using the off-shell free energies of the hot curved space and the black hole. In fact, the GPY phase transition for the Schwarzschild black hole appears only for the large black hole, $F_{\text{off}}^{\text{HS}}(0) > F_{\text{off}}^{\text{BH}}(0)$. In our case also, the same GPY phase transition occurs for the large black hole, $F_{\text{off}}^{\text{HS}}(a) > F_{\text{off}}^{\text{BH}}(a)$. Moreover, the free energy of the black hole is still lower than the free energy of the hot curved space even in the UV region, $F_{\text{off}}^{\text{HS}}(a) > F_{\text{off}}^{\text{BH}}(a)$ as long as $T_c(a) < T < [135a^2/\{2\pi^2\sqrt{r^2 - a^2}(r^2 + 2a^2)\}]^{1/3}$ whereas $F_{\text{off}}^{\text{HS}}(0) < F_{\text{off}}^{\text{BH}}(0)$ for the conventional case. This is plotted in Fig. 4 at a temperature greater than the critical temperature. Note that the mass of the tiny black hole should be less than the critical mass $M^* = 0.114$ in Fig. 4, which is very small compared to the mass of the small black hole $M_1 = 1.330$ in Fig. 3. Therefore, one can see that the hot curved space can be nucleated into the tiny black hole; however, it is unstable and loses its mass eventually.

5. Discussions

We have shown that the phase transition of the quantum-corrected Schwarzschild black hole is almost the same with the conventional one for the large black hole, which is just the Gross–Perry–Yaffe phase transition; however, the critical temperature is less than that of the Schwarzschild black hole on account of the quantum correction. In the UV region, the hot curved space without black holes can also decay into the tiny black hole, which means that the GPY phase transition occurs with the help of the quantum correction so that the tiny black hole state is more stable than the hot curved space. This tiny black hole is not in thermal equilibrium and subsequently can decay into much lower free energy state.

In connection with this state, we would like to mention the end state of the black hole for $M = 0$. Following the conventional thermodynamic analysis for $T > T_c$, the energy is zero so that the entropy is naturally zero, which yields $F_{\text{off}}^{\text{BH}} = 0$. However, the free energy of the hot flat space is $F_{\text{off}}^{\text{HS}}(0) < 0$. It means that there does not appear the GPY phase transition. However, from the beginning, we have considered the quantum-mechanical deformation of the metric to explore the UV region because the small size of black holes will receive quantum corrections significantly. In this case, the black hole has a minimum size of $r_{\text{H}} = a$ and it has a non-vanishing entropy $S = \pi a^2$ with $E = 0$. Then, the free energy of the black hole becomes negative as $F_{\text{off}}^{\text{BH}}(a) = -\pi a^2 T < 0$. Of course, it is lower than the free energy of the hot curved space. As a result, it happens that $F_{\text{off}}^{\text{HS}}(a) \to F_{\text{off}}^{\text{BH}}(a) \to$ remnant at $M = 0$. Although it suggests that there may be some object which has some degrees of freedom but it is not clear at this stage in the absence of the full quantized theory.

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