Charged Particle and Strong Cosmic Censorship in Reissner–Nordström–de Sitter black holes

Yongwan Gim[a1] and Bogeun Gwak[b2]

[a]*Department of Physics and Research Institute for Basic Science, Sogang University, Seoul 04107, Republic of Korea*

[b]*Department of Physics and Astronomy, Sejong University, Seoul 05006, Republic of Korea*

Abstract

We investigate the instability of the unstable circular orbit of a charged null particle to test the strong cosmic censorship conjecture in the near-extremal Reissner–Nordström–de Sitter black hole. The instability is estimated as the Lyapunov exponent and found to depend on the mass and charge of the black hole. Then, we explicitly show that the charged null particle in the unstable circular orbits corresponds to the charged massless scalar field in the eikonal limit. This provides a compact relation representing the quasinormal frequency in terms of the characteristics of unstable circular orbits. According to the relation, the strong cosmic censorship conjecture is valid.

[1]yongwan89@sogang.ac.kr

[2]rasenis@sejong.ac.kr

1 Introduction

Black holes are intriguing astronomical objects in the universe. The event horizon covers the inside of a black hole; accordingly to classical mechanics, none of the massless and massive particles can escape from the gravity of the black hole by crossing the event horizon. In other words, the classical black holes only absorb matter. However, black holes can lose energy through Hawking radiation through a quantum effect [1, 2]. Thus, the black holes are considered as thermodynamic systems with the Hawking temperature, expressed in terms of surface gravity, and the Bekenstein–Hawking entropy, expressed in terms of the area of the black hole [3, 4]. Recently, the existence of black holes was experimentally proven by the Laser Interferometer Gravitational-Wave Observatory (LIGO) by detecting the signals of the gravitational wave from the collisions between black holes.

At the center of some black holes, there exists a curvature singularity enclosed by the event horizon or the Cauchy horizon. Without horizons, the naked singularity results in the failure of the predictability of the Einstein equations. Hence, all singularities, except the Big Bang singularity, should be hidden inside the horizons of the black holes such that they are invisible to the observers; this is the cosmic censorship conjecture [5–7]. There are two kinds of conjectures—weak conjecture and strong conjecture. The weak cosmic censorship (WCC) conjecture indicates that the singularities inside the black holes should be hidden from distant observers such that the event horizon is stable against perturbations. The WCC conjecture was first investigated in a rotating black hole, where addition of a particle cannot overspin the black hole beyond the extremal condition [8]. Because there is no general theorem to prove the validity of the WCC conjecture, it has been tested in various black holes through particle absorption [9–31] and scattering of the test field [32–42].

A strong cosmic censorship (SCC) conjecture suggests that the singularities inside the black holes should be spacelike singularities [43,44]; the historical review of the SCC conjecture can be found in [45]. Actually, infalling observers can find spacelike singularities, but they cannot escape from inside the black holes due to its spacelike geometry. However, if the black hole possesses an inner horizon, namely, the Cauchy horizon, the singularity inside the black hole becomes a timelike singularity. It leads to the failure of the SCC conjecture. Therefore, the instability of the Cauchy horizon is an important issue in the context of SCC conjecture. For an asymptotic flat black hole such as the Reissner–Nordström black hole and the Kerr black hole, the inner horizon is unstable if the horizon is perturbed by external fields; the perturbation field is infinitely blue-shifted at the Cauchy horizon. Thus, the singularity is in the spacelike geometry, which proves the validity of the SCC conjecture [46–49]. However, for the Reissner–Nordström-de Sitter (RNdS) black hole, the perturbation field can be red-shifted at the Cauchy horizon due to the existence of a cosmological horizon. Nevertheless, it was suggested that relevant modes can provide the instability of the inner horizon; therefore, the RNdS black hole should not be considered a counterexample to the SCC conjecture [50]. Recently, the authors in [51] claimed that the decay rates of the neutral perturbation fields are determined by quasinormal modes (QNMs) of the RNdS black hole, and that the SCC conjecture is violated in the near-extremal RNdS black hole. Since then, the validity of the SCC conjecture in the RNdS black hole is controversial [52–56]. In particular, this argument has been extended to the charged perturbation fields in the RNdS black hole [57–62]. Recently, the validity of the SCC conjecture was investigated using circular null geodesics of a neutral particle in the near-extremal Kerr–Newman–de Sitter black hole [63].

Interestingly, the unstable circular orbits of a null particle were recently studied under the SCC conjecture. The unstable circular orbits of the neutral null particles are directly related to the QNMs of neutral scalar fields [64, 65]. The scalar field with the QNMs satisfies the boundary conditions of purely ingoing waves at the event horizon and purely outgoing waves at the cosmological horizon of the de Sitter black holes. For the neutral scalar field, the real and imaginary parts of the QMNs in the eikonal limit are given as a compact relationship with the angular velocity and the Lyapunov exponent of the null particle in the unstable circular geodesics [65]. The Lyapunov exponent determines the stability of the particles along the circular null geodesics; the particle starts to leak out of the geodesics at this instability time scale, which is an inverse of the Lyapunov exponent [65]. The imaginary part of the QNMs determine the characteristic time scale for the decay of the neutral scalar fields. According to the compact relationship in [65], the Lyapunov exponent of a neutral massless particle in the null circular geodesic is connected with the imaginary part of the QNM of neutral scalar fields in the eikonal limit. Therefore, one can understand the instability of the QNMs of the neutral scalar fields in the eikonal limit in terms of the instability of the charged particles leaking out of the unstable circular null orbits.

In this work, we investigate the instability of the unstable circular null geodesics of the *charged null particle* around the RNdS black hole. Then, the Lyapunov exponent is obtained and related to the QNMs of the *charged massless scalar field* in the eikonal limit to understand and discuss the SCC conjecture. Here, we find the *compact relationship* between the charged particle and scalar field in the near-extremal RNdS black hole. The compact relationship for the charged case is first obtained in our analysis. Before that, the compact relationship for the *neutral particle* was discussed in the RNdS black holes [65], higher-dimensional RNdS black holes [54], Kerr–dS black holes [66], higher-dimensional Kerr–dS black holes [54], and Kerr–Newman–dS black holes [63]. Therefore, our compact relationship is the generalization to that of the neutral cases. Further, we study the tendency of the critical exponent that indicates the instability of the circular null geodesics of the charged null particle. According to the critical exponent, the instability of the unstable circular geodesics depends on the ratio between the mass and charge of the black hole. Finally, the validity of the SCC conjecture in the near-extremal RNdS black hole will be investigated by the Lyapunov exponent of the circular geodesics of the charged null particles.

The remainder of this paper is organized as follows: In Sec. 2, we introduce the geometry of the RNdS Black Holes. In Sec. 3, we derive the geodesic motion of the charged particle from the Hamilton–Jacobi equations in the RNdS black hole, and subsequently, we define the effective potential for the radial motion of a charged particle in the RNdS black holes. In Sec. 4, we calculate the instability time scale, typical orbital time scale, and the critical exponent of the circular null geodesics of the charged null particle. In Sec. 5, we show that the compact relation is valid even between the charged null particle and the charged scalar field on the near-extremal RNdS black hole. In Sec. 6, using the result of Sec. 5, we investigate the validity of the SCC conjecture in the case of the near-extremal RNdS black hole. Finally, the summary is given in Sec. 7.

2 Reissner–Nordström–de Sitter Black Holes

The RNdS black hole is the solution to the four-dimensional Einstein–Maxwell gravity with a positive cosmological constant. The action is

$$\mathcal{S}_{\mathrm{G}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda\right), \tag{1}$$

where R, $F_{\mu\nu}$, and Λ denote the curvature, Maxwell field strength, and cosmological constant, respectively. The Maxwell field strength is given by a gauge field A_μ with an electric charge Q as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A = -\frac{Q}{r}dt. \tag{2}$$

The RNdS black hole is a static solution. The metric with the mass M and charge Q is

$$ds^2 = -\frac{\Delta(r)}{r^2}dt^2 + \frac{r^2}{\Delta(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \quad \Delta(r) = -\frac{\Lambda r^4}{3} + r^2 - 2Mr + Q^2. \tag{3}$$

Since the RNdS black hole is asymptotically the dS spacetime, there are three horizons: a cosmological horizon r_{c}, an outer horizon r_{o}, and an inner horizon r_{i}. Their radial locations range as $r_{\mathrm{i}} < r_{\mathrm{o}} < r_{\mathrm{c}}$. In this regard, the metric function Δ in Eq. (3) can be rewritten in terms of these horizons as

$$\Delta(r) = \frac{\Lambda}{3}(r_{\mathrm{c}} - r)(r - r_{\mathrm{o}})(r - r_{\mathrm{i}})(r + r_{\mathrm{c}} + r_{\mathrm{o}} + r_{\mathrm{i}}). \tag{4}$$

Comparing the metric functions in Eq. (3) with Eq. (4), we can rewrite the cosmological constant Λ, mass M, and electric charge Q of the RNdS black hole in terms of the horizons as

$$\Lambda = \frac{3}{r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})}, \; M = \frac{1}{2}\frac{(r_{\mathrm{c}} + r_{\mathrm{o}})(r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})}{r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})}, \; Q^2 = \frac{r_{\mathrm{c}} r_{\mathrm{i}} r_{\mathrm{o}}(r_{\mathrm{c}} + r_{\mathrm{o}} + r_{\mathrm{i}})}{r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})}. \tag{5}$$

Surface gravities are associated with the amplification and decay rates of the QNMs that are closely related to the SCC conjecture. Hence, they play an important role in our analysis. The surface gravities of the RNdS black hole are at the inner horizon κ_{i}, outer horizon κ_{o}, and cosmological horizon κ_{c}

$$\begin{aligned}
\kappa_{\mathrm{i}} &= \frac{(r_{\mathrm{c}} - r_{\mathrm{i}})(r_{\mathrm{o}} - r_{\mathrm{i}})(r_{\mathrm{c}} + r_{\mathrm{o}} + 2r_{\mathrm{i}})}{2r_{\mathrm{i}}^2\left(r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})\right)}, \quad \kappa_{\mathrm{o}} = \frac{(r_{\mathrm{c}} - r_{\mathrm{o}})(r_{\mathrm{o}} - r_{\mathrm{i}})(r_{\mathrm{c}} + 2r_{\mathrm{o}} + r_{\mathrm{i}})}{2r_{\mathrm{o}}^2\left(r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})\right)}, \\
\kappa_{\mathrm{c}} &= \frac{(r_{\mathrm{c}} - r_{\mathrm{o}})(r_{\mathrm{c}} - r_{\mathrm{i}})(2r_{\mathrm{c}} + r_{\mathrm{o}} + r_{\mathrm{i}})}{2r_{\mathrm{c}}^2\left(r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}})\right)}, \quad \kappa_{\mathrm{n}} = \frac{(r_{\mathrm{c}} + r_{\mathrm{o}} + 2r_{\mathrm{i}})(r_{\mathrm{c}} + 2r_{\mathrm{o}} + r_{\mathrm{i}})(2r_{\mathrm{c}} + r_{\mathrm{o}} + r_{\mathrm{i}})}{2(r_{\mathrm{c}} + r_{\mathrm{o}} + r_{\mathrm{i}})^2(r_{\mathrm{c}}^2 + r_{\mathrm{o}}^2 + (r_{\mathrm{c}} + r_{\mathrm{i}})(r_{\mathrm{o}} + r_{\mathrm{i}}))},
\end{aligned} \tag{6}$$

where $\kappa_{\rm n}$ is the surface gravity-like value at the negative solution of $\Delta(r)=0$: $r_{\rm n}=-(r_{\rm i}+r_{\rm o}+r_{\rm c})$. Hence, it has no physical meaning and is only used for simplifying equations. It is worth noting that there are two kinds of extremal black holes—one is the case where the inner horizon coincides with the outer horizon, and the other is that the outer horizon coincides with the cosmological horizon. We call the former one as the extremal black hole in $r_{\rm i}=r_{\rm o}$ and the latter one as the extremal black hole in $r_{\rm o}=r_{\rm c}$, for convenience.

3 Effective Potential of Charged Particle

The unstable circular orbit of the particle in the eikonal limit is found to be associated with the decay rate of the QNM. Hence, we investigate the effective potential of the charged particle to demonstrate the unstable circular orbit in the spacetime of the RNdS black hole in Eq. (3). Here, the radial equation, including the effective potential, is obtained by the Hamilton–Jacobi method. The Hamiltonian is given as

$$\mathcal{H}=\frac{1}{2}g^{\mu\nu}(p_\mu-qA_\mu)(p_\nu-qA_\nu), \tag{7}$$

in which the particle with the electric charge q and four momenta p_μ is coupled with a gauge field. Then, the Hamilton–Jacobi action of the charged particle with the mass m is written as

$$\mathcal{S}=\frac{1}{2}m^2\tau-Et+\mathcal{S}_r(r)+\mathcal{S}_\theta\left(\theta\right)+L\phi, \tag{8}$$

where the affine parameter denotes τ. From the metric in Eq. (3), the translation symmetries can be read in time t and azimuthal angle ϕ. Then, the conserved quantities with respect to these translation symmetries are defined as the energy E and angular momentum L of the particle. According to the Hamilton–Jacobi method, the momenta of the particle is in terms of the action of Eq. (8) by $\mathcal{S}$ as $p_\mu=\partial_\mu S$. Then, substituting Eq. (8) into Eq. (7), the Hamilton–Jacobi equation is obtained as

$$-2\frac{\partial\mathcal{S}}{\partial\tau}=-\frac{r^2}{\Delta}\left(-E+\frac{qQ}{r}\right)^2+\frac{\Delta}{r^2}(\partial_r\mathcal{S}_r(r))^2+\frac{1}{r^2}\left(\partial_\theta\mathcal{S}_\theta(\theta)\right)^2+\frac{L^2}{r^2\sin^2\theta}=m^2. \tag{9}$$

Here, the Hamilton–Jacobi equation can be separated into radial and θ-directional equations by the separate constant $\mathcal{K}$ as

$$\mathcal{K}=-m^2r^2+\frac{r^4}{\Delta}\left(-E+\frac{qQ}{r}\right)^2-\Delta(\partial_r\mathcal{S}_r)^2,\quad \mathcal{K}=(\partial_\theta\mathcal{S}_\theta)^2+\frac{L^2}{\sin^2\theta}. \tag{10}$$

Then, we can rewrite the Hamilton–Jacobi action as

$$\mathcal{S}=\frac{1}{2}m^2\tau-Et+\int dr\sqrt{\rho(r)}+\int d\theta\sqrt{\Theta(\theta)}+L\phi, \tag{11}$$

where the functions $\rho(r)$ and $\Theta(\theta)$ are given by

$$\Sigma(r)=\frac{1}{\Delta}\left(-K-m^2r^2+\frac{r^4}{\Delta}\left(-E+\frac{qQ}{r}\right)^2\right),\qquad \Xi(\theta)=K-\frac{1}{\sin^2\theta}L^2. \tag{12}$$

Then, full geodesic equations to the charged particle can be obtained from Eq. (11) by the Hamilton–Jacobi method. Here, our concern is about the unstable circular orbits on the equator; therefore, we can impose $\theta = \pi/2$ and $\dot{\theta} = 0$ without the loss of generality, because the RNdS black hole has a spherical symmetric geometry. On the equator, the geodesics are obtained as

$$\dot{t} = \frac{r^2}{\Delta}\left(E - \frac{qQ}{r}\right), \quad \dot{r} = \sqrt{\frac{\Delta}{r^2}\left(-m^2 + \frac{r^2}{\Delta}\left(-E + \frac{qQ}{r}\right)^2 - \frac{1}{r^2}\frac{L^2}{\sin^2\theta}\right)}, \quad \dot{\phi} = \frac{L}{r^2}, \tag{13}$$

where a dot denotes a derivative with respect to the affine parameter τ. Finally, the effective potential on the equator can be read from Eq. (13) as

$$V_{\text{eff}} \equiv \dot{r}^2 = \frac{\Delta}{r^2}\left(-m^2 + \frac{r^2}{\Delta}\left(-E + \frac{qQ}{r}\right)^2 - \frac{L^2}{r^2}\right), \tag{14}$$

where $m^2 > 0$ for a timelike and $m^2 = 0$ for null. We can estimate possible orbits of the particle from the effective potential in Eq. (14). The unstable circular orbit is obtained when the radius of the orbit coincides with the location of the maximum in the effective potential for the null case.

4 Instability of Circular Orbits of Charged Null Particles

We can now estimate the motions of the particle from the effective potential in Eq. (14). The unstable circular orbits of a null particle that is connected to the Lyapunov exponent are also obtained from the effective potential satisfying the expression

$$V_{\text{eff}}(r_{\text{cir}}) = 0, \quad V'_{\text{eff}}(r_{\text{cir}}) = 0, \quad V''_{\text{eff}}(r_{\text{cir}}) > 0, \tag{15}$$

where the prime denotes derivatives with respect to the coordinate r. Further,

$$\begin{aligned}
V_{\text{eff}}(r_{\text{cir}}) &= \frac{r_{\text{cir}}^2(-qQ + Er_{\text{cir}})^2 - L^2\Delta(r_{\text{cir}})}{r_{\text{cir}}^4}, \\
V'_{\text{eff}}(r_{\text{cir}}) &= \frac{2qQr_{\text{cir}}^2(-qQ + Er_{\text{cir}}) + L^2(4\Delta(r_{\text{cir}}) - r_{\text{cir}}\Delta'(r_{\text{cir}}))}{r_{\text{cir}}^5}, \\
V''_{\text{eff}}(r_{\text{cir}}) &= \frac{2qQr_{\text{cir}}^2(3qQ - 2Er_{\text{cir}}) - L^2(20\Delta(r_{\text{cir}}) + r_{\text{cir}}(-8\Delta'(r_{\text{cir}}) + r_{\text{cir}}\Delta''(r_{\text{cir}}))}{r_{\text{cir}}^6},
\end{aligned} \tag{16}$$

where

$$\Delta'(r_{\text{cir}}) = -2M + 2r_{\text{cir}} - \frac{4\Lambda}{3}r_{\text{cir}}^3, \quad \Delta''(r_{\text{cir}}) = 2 - 4\Lambda r_{\text{cir}}^2. \tag{17}$$

According to the first and second conditions in Eq. (15) with $m = 0$, the energy and angular momentum of the charged null particle are in the unstable circular orbits

$$\frac{E}{q} = \frac{Q}{r_{\text{cir}}}\left(1 + \frac{2\Delta(r_{\text{cir}})}{r_{\text{cir}}\Delta'(r_{\text{cir}}) - 4\Delta(r_{\text{cir}})}\right), \qquad \frac{L}{q} = \frac{2Qr_{\text{cir}}\sqrt{\Delta(r_{\text{cir}})}}{r_{\text{cir}}\Delta'(r_{\text{cir}}) - 4\Delta(r_{\text{cir}})}, \tag{18}$$

Then, we can substitute the radius of the circular orbit $r_{\rm cir}$ into the expressions for energy and angular momentum of the particle. The second derivative of the effective potential for the circular null orbits is thus rewritten as

$$V''_{\rm eff}(r_{\rm cir}) = \frac{2q^2Q^2\left(-8\Delta(r_{\rm cir})^2 + r^2\Delta'(r_{\rm cir})^2 + 4r_{\rm cir}\Delta(r_{\rm cir})\Delta'(r_{\rm cir}) - 2r_{\rm cir}^2\Delta(r_{\rm cir})\Delta''(r_{\rm cir})\right)}{r_{\rm cir}^4(r_{\rm cir}\Delta'(r_{\rm cir}) - 4\Delta(r_{\rm cir}))^2}, \tag{19}$$

Further, the first derivative of the coordinate time t with respect to the affine parameter τ at the radius of the circular motion is rewritten as

$$\dot{t}\,\Big|_{r=r_{\rm cir}} = \frac{2qQr_{\rm cir}}{r\Delta'(r_{\rm cir}) - 4\Delta(r_{\rm cir})}. \tag{20}$$

By combining Eqs. (19) and (20), we obtain the Lyapunov exponent for the circular null geodesic [65] as follows

$$\lambda_{\rm cir} = \sqrt{\frac{V''_{\rm eff}(r_{\rm cir})}{2\dot{t}^2}} = \frac{1}{2r_{\rm cir}^3}\sqrt{r_{\rm cir}^2\Delta'(r_{\rm cir})^2 - 8\Delta(r_{\rm cir})^2 + 4r_{\rm cir}\Delta(r_{\rm cir})\Delta'(r_{\rm cir}) - 2r_{\rm cir}^2\Delta(r_{\rm cir})\Delta''(r_{\rm cir})}. \tag{21}$$

The Lyapunov exponent is closely related to the instabilities of the orbits. This can be shown in terms of the time scale named the instability time scale as $\mathcal{T}_\lambda \equiv 1/\lambda_{\rm cir}$. The instability time scale represents the time period for which the particle on the circular orbit leaks out of the geodesics. Since the particle on the unstable orbit cannot fill one period of the orbit entirely, the instability time is shorter than its typical orbital time $\mathcal{T}_\Omega \equiv 2\pi/\Omega_{\rm cir}$, which is the time period of the orbit for one turn. Then, we can expect $\mathcal{T}_\lambda < \mathcal{T}_\Omega$, and this can be simply represented by the critical exponent [67] as

$$\gamma = \frac{\mathcal{T}_\lambda}{\mathcal{T}_\Omega}. \tag{22}$$

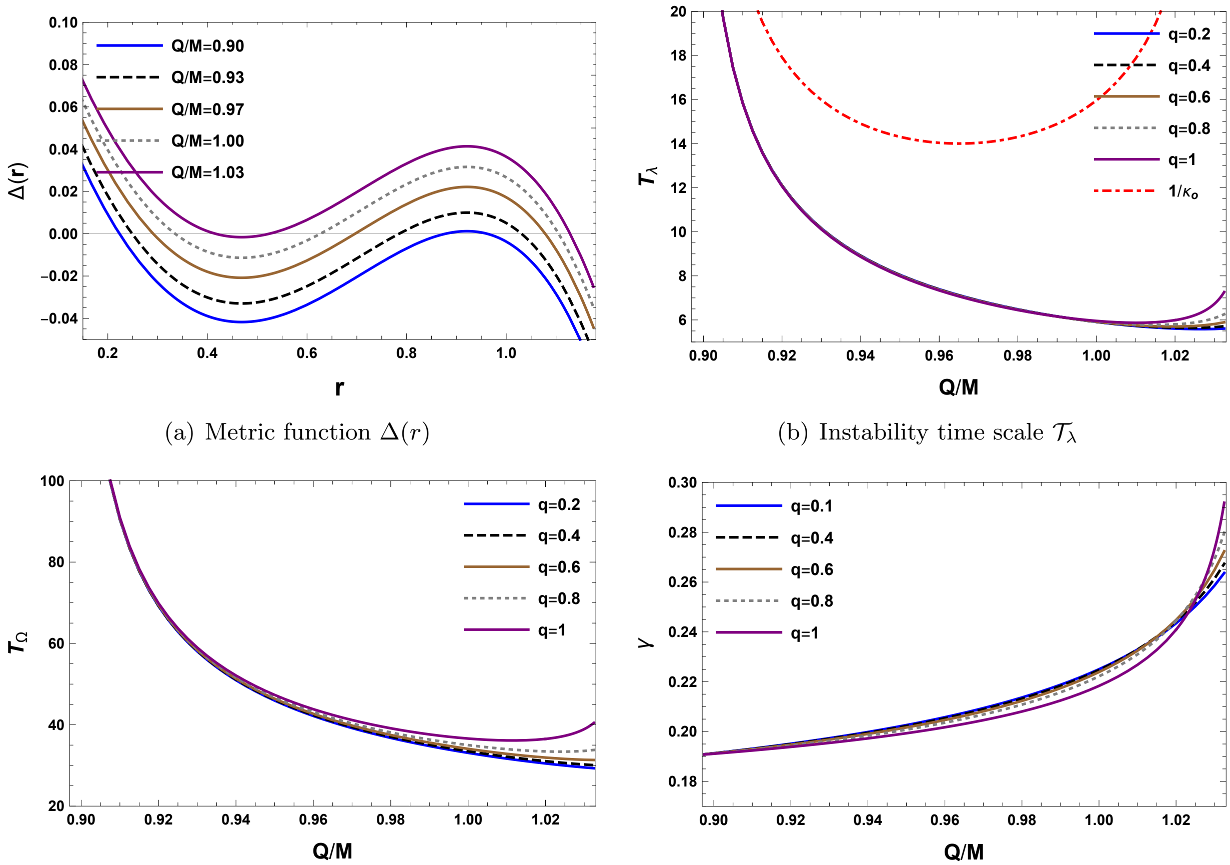
If the critical exponent with respect to the inequality $\gamma < 1$, it implies that the particle leaks out of the circular null geodesic before finishing a turn. Thus, the instability increases for a small critical exponent. In the RNdS black hole, the instability and typical orbital time scales are obtained as

$$\mathcal{T}_\lambda = \frac{2r_{\rm cir}^3}{\sqrt{r_{\rm cir}^2\Delta'(r_{\rm cir})^2 - 8\Delta(r_{\rm cir})^2 + 4r_{\rm cir}\Delta(r_{\rm cir})\Delta'(r_{\rm cir}) - 2r_{\rm cir}^2\Delta(r_{\rm cir})\Delta''(r_{\rm cir})}}, \quad \mathcal{T}_\Omega = \frac{2\pi r_{\rm cir}^2}{\sqrt{\Delta(r_{\rm cir})}}, \tag{23}$$

where we use $\Omega_{\rm cir} = \sqrt{\Delta(r_{\rm cir})}/r_{\rm cir}^2$. Then, the critical exponent is found to be

$$\gamma \equiv \frac{\mathcal{T}_\lambda}{\mathcal{T}_\Omega} = \frac{r_{\rm cir}\sqrt{\Delta(r_{\rm cir})}}{\pi\sqrt{r_{\rm cir}^2\Delta'(r_{\rm cir})^2 - 8\Delta(r_{\rm cir})^2 + 4r_{\rm cir}\Delta(r_{\rm cir})\Delta'(r_{\rm cir}) - 2r_{\rm cir}^2\Delta(r_{\rm cir})\Delta''(r_{\rm cir})}}. \tag{24}$$

The instabilities of the circular orbits can be estimated from Eq. (24) with respect to the radius $r_{\rm cir}$. Note that there is a universal upper bound for the Lyapunov exponents of chaotic motions, i.e., $\lambda_{\rm cir} < \kappa_{\rm o}$, as discussed in [68]. This implies that the lower bound of the instability time scale is $\kappa_{\rm o}^{-1} < \mathrm{T}_\lambda$. However, the inequality is found to be violated for the charged particle in the RNdS black hole [69]. Thus, the instability time scale has no lower bound, which is consistent with our results.



(a) Metric function $\Delta(r)$

(b) Instability time scale $\mathcal{T}_\lambda$

(c) Typical orbital time scale $\mathcal{T}_\Omega$

(d) Critical exponent γ

Figure 1: Function $\Delta(r)$, instability time scale $\mathcal{T}_\lambda$, typical orbital time scale $\mathcal{T}_\Omega$, and critical exponent γ with respect to the ratio of the charge to mass of the RNdS black hole for $E = 1$, $\Lambda = 1$, $M = 0.4$.

Since Eqs. (23) and (24) are given in terms of the radii of the unstable circular orbits, we depict their behaviors in the numerical analysis in Fig. 1. The states of RNdS black holes depend on the rate between Q and M, and the black holes only exist within specific rates, as shown in Fig. 1(a). The minimum value of the rate is $Q/M = 0.9$, where the RNdS black hole is the extremal black hole in $r_{\rm o} = r_{\rm c}$. Then, as the rate increases, the timelike spacetime becomes large. Finally, the rate reaches the maximum value $Q/M = 1.03$, where the black hole becomes the extremal black hole in $r_{\rm i} = r_{\rm o}$. The instability and typical orbital time scales in Eq. (23) are represented in Figs. 1(b) and 1(c) with respect to the rate Q/M. The instabilities of the orbits depend on Q/M rather than q, and we can observe that $T_\lambda > T_\Omega$. Further, there is no lower bound on the instability time scale, according to the red dashed line in Fig. 1(b). This is consistent with the observations in [69]. The instability of the orbit increases as the rate Q/M increases. This can be observed in Fig. 1(d) for the critical exponent. The critical exponent increases and is maximum at the extremal black hole in $r_{\rm i} = r_{\rm o}$. This implies that the orbit is most stable at the extremal black hole in $r_{\rm i} = r_{\rm o}$. Furthermore, the instability slightly decreases as the charge of the particle increases at the extremal black hole.

5 Lyapunov Exponent in QNM of Charged Scalar Field

The linearized relaxation of the scalar field is governed by the QNM in the spacetime of a black hole. Here, we prove that the Lyapunov exponent of the charged null particle is associated with the decay of the QNM under the charged scalar field in the near-extremal RNdS black hole in $r_{\rm o} \approx r_{\rm c}$. Thus, from this point on, we focus on the near-extremal case.

5.1 QNM of Charged Scalar Field in Near-Extremal Black Hole

We consider the massless scalar field with the electric charge in the RNdS black hole. Since the electric charge of the scalar field is coupled with the gauge field in the background, the covariant derivative should be introduced. Then, the field equation is given as

$$\left((\nabla^{\mu} - iqA^{\mu})(\nabla_{\mu} - iqA_{\mu}) - \mu^2\right)\Psi(t,r,\theta,\phi) = 0, \tag{25}$$

where μ and q are the mass and charge of the scalar field. Since the field equation can be separated for each coordinate, the solution to the scalar field is written in the form

$$\Psi(t,r,\theta,\phi) = \frac{\Phi(r)}{r} Y_{lm}(\theta) e^{-i\omega t} e^{im\phi}, \tag{26}$$

where $\Phi(r)$ and $Y_{lm}(\theta)$ are the radial solution and spherical harmonics, respectively. The eigenvalues are ω, m, and l corresponding to the frequency and angular momenta of the scalar field, respectively. The radial equation is separately obtained according to Eqs. (25) and (26). Further, we assume the tortoise coordinate

$$r^* = \int \frac{r^2}{\Delta} dr = -\frac{\log(r_{\rm c} - r)}{2\kappa_{\rm c}} + \frac{\log(r - r_{\rm o})}{2\kappa_{\rm o}} - \frac{\log(r - r_{\rm i})}{2\kappa_{\rm i}} + \frac{\log(r + r_{\rm c} + r_{\rm o} + r_{\rm i})}{2\kappa_{\rm n}}, \tag{27}$$

where the range is $-\infty < r^* < \infty$. Then, the radial equation is simplified into the Schrödinger-like equation as

$$\frac{d^2\Phi}{dr^{*2}} + \left(\left(\omega - \frac{qQ}{r}\right)^2 - \mathcal{V}_{\rm eff}(r)\right)\Phi = 0, \tag{28}$$

whose potential is

$$\mathcal{V}_{\rm eff}(r) = \frac{\Delta(r)}{r^2}\left(\frac{l(l+1)}{r^2} + \mu^2 + \frac{1}{r}\frac{d}{dr}\left(\frac{\Delta(r)}{r^2}\right)\right). \tag{29}$$

At the outer and cosmological horizons, the solutions to the radial equation in Eq. (28) are obtained as

$$\Phi(r^*) \sim e^{\pm i(\omega - qQ/r_{\rm o})r^*}, \quad r^* \to -\infty\,(r \to r_{\rm o}); \quad \Phi(r^*) \sim e^{\pm i(\omega - qQ/r_{\rm c})r^*}, \quad r^* \to \infty\,(r \to r_{\rm c}). \tag{30}$$

The QNMs in dS black holes are characterized to choose the boundary condition as [70]

$$\Phi(r^*) \sim \begin{cases} e^{-i(\omega - qQ/r_\mathrm{o})r^*} & r^* \to -\infty \, (r \to r_\mathrm{o}) \\ e^{i(\omega - qQ/r_\mathrm{c})r^*} & r^* \to \infty \, (r \to r_\mathrm{c}) \end{cases}, \tag{31}$$

which implies that we only consider incoming flux of the scalar field at the outer horizon and outgoing flux at the cosmological horizon. For the boundary condition in Eq. (31), we impose the near-extremal condition such that the outer horizon is located close to the cosmological horizon. This can be written as

$$\epsilon = \frac{r_\mathrm{c} - r_\mathrm{o}}{r_\mathrm{o}} \ll 1. \tag{32}$$

We herein obtain the QNMs of the scalar field by the approximation to the Pöschl–Teller potential, because the decay rate can be read in the imaginary part of the quasinormal frequency. In the approximation, the peak of the effective potential (29) has an important role. The location of the peak satisfies the condition

$$\left.\frac{d\mathcal{V}_\mathrm{eff}}{dr^*}\right|_{r^*=r^*_\mathrm{p}} = 0. \tag{33}$$

Further, since we already assume the near-extremal condition, the peak has to be at the near-outer horizon; thus,

$$\delta = \frac{r_\mathrm{p} - r_\mathrm{o}}{r_\mathrm{o}} \ll 1. \tag{34}$$

According to Eq. (27), the location of the peak is approximately found in terms of the relationship between the radial and tortoise coordinates as

$$r_\mathrm{p} = \frac{r_\mathrm{c} + r_\mathrm{o} e^{-2\kappa_\mathrm{o} r^*_\mathrm{p}}}{1 + e^{-2\kappa_\mathrm{o} r^*_\mathrm{p}}}. \tag{35}$$

Then, by Eq. (35) and with our assumptions, the dimensionless parameters ϵ and δ are connected as

$$\delta = \frac{1}{2}(1 + \tanh(\kappa_\mathrm{o} r^*_\mathrm{p}))\epsilon. \tag{36}$$

Further, the function Δ can be rewritten under the peak of the effective potential in Eq. (29) as

$$\Delta(r_\mathrm{p}) = \frac{r_\mathrm{o}^2 (r_\mathrm{o} - r_\mathrm{i})(r_\mathrm{i} + 2r_\mathrm{o} + r_\mathrm{c})\Lambda\epsilon^2}{3} \frac{1}{4\cosh^2(\kappa_\mathrm{o} r^*_\mathrm{p})}. \tag{37}$$

Hence, the effective potential is rewritten in the form of the Pöschl–Teller potential at the near-extremal RNdS black hole as

$$\mathcal{V}_\mathrm{eff} = \frac{\mathcal{V}_0}{\cosh^2(\kappa_\mathrm{o} r^*_\mathrm{p})}, \quad \mathcal{V}_0 = \frac{\Lambda\epsilon^2}{12}\left(\mu^2 + \frac{l(l+1)}{r_\mathrm{o}^2}\right)(r_\mathrm{o} - r_\mathrm{i})(r_\mathrm{i} + 3r_\mathrm{o}). \tag{38}$$

The quasinormal frequency in the Pöschl–Teller potential can be generally read according to [71]. Therefore,

$$\omega = \sqrt{\mathcal{V}_0 - \frac{\kappa_\mathrm{o}^2}{4}} + \frac{qQ}{r_\mathrm{o}} - \frac{qQ}{2r_\mathrm{o}}\left(1 + \tanh\left(\kappa_\mathrm{o} r^*_\mathrm{p}\right)\right)\epsilon - i\left(n + \frac{1}{2}\right)\kappa_\mathrm{o}, \tag{39}$$

where the real part of the frequency includes the coupling term between charge and gauge field. Since the scalar field is associated with the unstable circular orbit of the particle in the eikonal limit, if we assume the condition of the eikonal limit, the quasinormal frequency is

$$\omega=\frac{l\epsilon}{2r_{\rm o}}\sqrt{\frac{(r_{\rm o}-r_{\rm i})(r_{\rm i}+3r_{\rm o})}{3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2}}-i\left(n+\frac{1}{2}\right)\kappa_{\rm o}, \tag{40}$$

where the real and imaginary parts of the quasinormal frequency are given as

$$\mathrm{Re}(\omega)=\frac{l\epsilon}{2r_{\rm o}}\sqrt{\frac{(r_{\rm o}-r_{\rm i})(r_{\rm i}+3r_{\rm o})}{3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2}},\quad \mathrm{Im}(\omega)=-\left(n+\frac{1}{2}\right)\kappa_{\rm o}. \tag{41}$$

This is in agreement with the quasinormal mode in the eikonal limit in [58]. Thus, the decay rate is obtained in the dominant QNM of the scalar field

$$\mathrm{Im}(\omega)=-\frac{1}{2}\kappa_{\rm o}. \tag{42}$$

Here, we expect that the decay rate is associated with the Lyapunov exponent of the charged null particle. Hence, the Lyapunov exponent should be rewritten under the same assumptions.

5.2 Decay Rate of Dominant QNM and Lyapunov Exponent

We already obtained the decay rate of the dominant QNM of the massless scalar field in Eq. (41) under the near-extremal black hole in $r_{\rm o}=r_{\rm c}$. Further, the Lyapunov exponent was also obtained for unstable circular orbits in an arbitrary state of the RNdS black hole. Here, we impose the same assumptions of the QNM to the Lyapunov exponent. Then, the Lyapunov exponent can be clearly related to the QNM. First, we assume the near-extremal limit in Eq. (32), which implies that the radius of the circular orbit is located in the near-horizon regime, $r_{\rm cir}\approx r_{\rm o}$. According to Eq. (15) for the circular orbit, the radius is written in terms of $r_{\rm o}$ and ϵ.

$$r_{\rm cir}=r_{\rm o}+\frac{r_{\rm o}}{2}\epsilon+\mathcal{O}(\epsilon)^2. \tag{43}$$

Further, for the radius of the circular orbit, the function Δ and its first derivative are rewritten as

$$\Delta(r_{\rm cir})=\frac{r_{\rm o}^2(r_{\rm o}-r_{\rm i})(3r_{\rm o}+r_{\rm i})\epsilon^2}{4(3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2)},\quad \Delta'(r_{\rm cir})=\frac{r_{\rm o}^3\epsilon^2}{3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2}, \tag{44}$$

which are the second order of ϵ, so their contributions are subtle. Then, the cosmological constant, mass, and electric charge of the black hole can be expressed in terms of the first order of ϵ owing to the near-extremal condition. From Eq. (5),

$$\begin{aligned}\Lambda&=\frac{3}{3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2}-\frac{9r_{\rm o}^2+3r_{\rm o}r_{\rm i}}{(3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2)^2}\epsilon+\mathcal{O}(\epsilon^2),\\ Q&=\frac{r_{\rm o}\sqrt{r_{\rm i}}\sqrt{2r_{\rm o}+r_{\rm i}}}{\sqrt{3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2}}+\frac{r_{\rm o}\sqrt{r_{\rm i}}(3r_{\rm o}+r_{\rm i})(r_{\rm o}^2+r_{\rm o}r_{\rm i}+r_{\rm i}^2)}{2\sqrt{2r_{\rm o}+r_{\rm i}}(3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2)^{\frac{3}{2}}}\epsilon+\mathcal{O}(\epsilon^2),\\ M&=\frac{r_{\rm o}(r_{\rm o}+r_{\rm i})^2}{3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2}+\frac{r_{\rm o}(r_{\rm o}+r_{\rm i})(3r_{\rm o}+r_{\rm i})(r_{\rm o}^2+r_{\rm i}^2)\epsilon}{2(3r_{\rm o}^2+2r_{\rm o}r_{\rm i}+r_{\rm i}^2)^2}+\mathcal{O}(\epsilon^2),\end{aligned} \tag{45}$$

where the charge is assumed to be positive for convenience. Further, the surface gravity on the outer horizon in Eq. (6) is also rewritten as

$$\kappa_{\mathrm{o}} = \frac{(r_{\mathrm{o}} - r_{\mathrm{i}})(3r_{\mathrm{o}} + r_{\mathrm{i}})\epsilon}{2r_{\mathrm{o}}(3r_{\mathrm{o}}^2 + 2r_{\mathrm{o}}r_{\mathrm{i}} + r_{\mathrm{i}}^2)} + \mathcal{O}(\epsilon^2), \tag{46}$$

Then, in combination with Eq. (43) and Eq. (45), the angular velocity for the circular orbit on the near-extremal RNdS black hole is obtained as

$$\Omega_{\mathrm{cir}} = \frac{\epsilon}{2r_{\mathrm{o}}}\sqrt{\frac{(r_{\mathrm{o}} - r_{\mathrm{i}})(3r_{\mathrm{o}} + r_{\mathrm{i}})}{3r_{\mathrm{o}}^2 + 2r_{\mathrm{o}}r_{\mathrm{i}} + r_{\mathrm{i}}^2}}. \tag{47}$$

Moreover, the Lyapunov exponent in Eq. (21) can be approximated as

$$\lambda_{\mathrm{cir}} = \frac{\epsilon(r_{\mathrm{o}} - r_{\mathrm{i}})(3r_{\mathrm{o}} + r_{\mathrm{i}})}{2r_{\mathrm{o}}(3r_{\mathrm{o}}^2 + 2r_{\mathrm{o}}r_{\mathrm{i}} + r_{\mathrm{i}}^2)}. \tag{48}$$

Then, the quasinormal frequency in Eq. (41) is clearly associated with the charged null particle in the first order of ϵ. The quasinormal frequency is thus obtained as

$$\mathrm{Re}(\omega) = \frac{l\epsilon}{2r_{\mathrm{o}}}\sqrt{\frac{(r_{\mathrm{o}} - r_{\mathrm{i}})(r_{\mathrm{i}} + 3r_{\mathrm{o}})}{3r_{\mathrm{o}}^2 + 2r_{\mathrm{o}}r_{\mathrm{i}} + r_{\mathrm{i}}^2}} = \ell\Omega_{\mathrm{cir}}, \quad \mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right)\kappa_{\mathrm{o}} = -\left(n + \frac{1}{2}\right)\lambda_{\mathrm{cir}}. \tag{49}$$

This implies that the QNMs of the massless scalar field with electric charge correspond to the unstable circular orbits of the charged null particle in the near-extremal RNdS black hole. In particular, the Lyapunov exponent is twice the decay rate of the dominant QNM in Eq. (42) as

$$\lambda_{\mathrm{cir}} = 2\,|\mathrm{Im}(\omega)|\,. \tag{50}$$

Therefore, the instabilities of the circular orbits for the charged null particle are proportional to the decay rates of the dominant QNMs for the charged massless scalar field.

6 Strong Cosmic Censorship Conjecture

Here, we investigate the validity of the SCC conjecture in the near-extremal RNdS black hole by the Lyapunov exponent of the charged null particle. According to Eqs. (49) and (50), the Lyapunov exponent of the charged null particle corresponds to the decay rate of the dominant QNM of the charged massless scalar field. In the SCC conjecture, destabilizing the Cauchy horizon plays a significant role, and the destabilizing depends on the decay rate of the dominant QNM because the decay rate determines whether the dominant QNM is infinitely blue-shifted or not. Since the blueshift occurs owing to the Cauchy horizon, the amplification rate is fixed to the surface gravity of the inner horizon. On the contrary, the decay occurs exponentially as

$$|\Psi - \Psi_0| \sim e^{-\eta t}, \tag{51}$$

where the spectral gap is η [72–75]. Hence, we have to find the decay rate of the QNMs to determine the dominant one between amplification and decay. At the Cauchy horizon, the divergence of the scalar field energy is related to the ratio $\beta \equiv \eta/\kappa_{\rm i}$ [62] in the SCC conjecture given by [76]. When $\beta > \frac{1}{2}$, the decay of the charged massless scalar field is dominant, and the singularity is timelike because the Cauchy horizon is still stable. Then, the SCC conjecture is violated in this case. However, if $\beta < \frac{1}{2}$, the Cauchy horizon can be destabilized by the infinitely blue-shifted scalar field. Thus, the SCC conjecture is valid. In our analysis, we observe that the decay rate of the dominant QNM is half of the Lyapunov exponent of the charged null particle in the near-extremal black hole. Hence,

$$\frac{\lambda_{\rm cir}}{2} = |{\rm Im}(\omega)| = \kappa_{\rm o} < |\kappa_{\rm i}|. \tag{52}$$

This implies that the amplification of the charged massless scalar field is dominant in the near-extremal black hole, and $\beta < 1/2$. Therefore, the SCC conjecture is valid in our analysis based on the compact relationships in Eqs. (49) and (50). This result is in agreement with the result in [58,61], while [59,60]

7 Summary

We investigated the validity of the SCC conjecture in the near-extremal RNdS black hole using the Lyapunov exponent of the charged null particle in the circular null geodesic. Since the SCC conjecture has to be discussed in QNMs, the relationship between the Lyapunov exponent and QNMs is generalized to the case of the charged massless scalar field corresponding to the charged null particle. Then, we compute the Lyapunov exponent representing the instabilities of orbits in RNdS black holes. By using the instability time scale associated with the Lyapunov exponent, the circular null orbits of the charged null particle around the extremal black hole in $r_{\rm i} = r_{\rm o}$ are more stable than those around the extremal black hole in $r_{\rm o} = r_{\rm c}$. Further, since the decay rate in the QNM with the eikonal limit can be written in the analytical form in the case of the near-extremal RNdS black hole, we imposed the same assumption on the Lyapunov exponent of the charged null particle. Thus, we found the compact relationship between the Lyapunov exponent and QNMs in Eq. (49) for the charged case. According to the compact relationship in the charged case, the decay rate in the near-extremal black hole of $r_{\rm o} = r_{\rm c}$ represents $\beta < 1/2$. This implies that the SCC conjecture is valid in our analysis based on the Lyapunov exponent.

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2018R1C1B6004349) and the faculty research fund of Sejong University in 2018. B.G. appreciates APCTP for its hospitality during completion of this work.

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