Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Thermodynamic phase transition in the rainbow Schwarzschild black hole

JCAP10(2014)003

Yongwan Gim and Wontae Kim

Department of Physics, Sogang University,
Seoul 121-742, Republic of Korea

E-mail: yongwan89@sogang.ac.kr, wtkim@sogang.ac.kr

Received July 2, 2014
Revised August 18, 2014
Accepted September 15, 2014
Published October 1, 2014

Abstract. We study the thermodynamic phase transition in the rainbow Schwarzschild black hole where the metric depends on the energy of the test particle. Identifying the black hole temperature with the energy from the modified dispersion relation, we obtain the modified entropy and thermodynamic energy along with the modified local temperature in the cavity to provide well defined black hole states. It is found that apart from the conventional critical temperature related to Hawking-Page phase transition there appears an additional critical temperature which is of relevance to the existence of a locally stable tiny black hole; however, the off-shell free energy tells us that this black hole should eventually tunnel into the stable large black hole. Finally, we discuss the reason why the temperature near the horizon is finite in the rainbow black hole by employing the running gravitational coupling constant, whereas it is divergent near the horizon in the ordinary Schwarzschild black hole.

Keywords: modified gravity, quantum black holes

ArXiv ePrint: 1406.6475

 doi:10.1088/1475-7516/2014/10/003

Contents

1 Introduction

There has been much attention to modified dispersion relations (MDR) which are associated with the energy of test particle in gravity's rainbow. Since they have given not only experimental explanation for the threshold anomalies in ultra high cosmic rays and Tev photons [1–8] but also have appeared theoretically in the semi-classical limit of loop quantum gravity [9–13], the rainbow gravity has been extensively studied in order to explore various aspects of black holes and cosmology [14–34].

While the ordinary uncertainty principle has been promoted to the generalized uncertainty principle (GUP) in the quantum regime [35–44], the MDR has been essentially based on the deformation of relativity called the doubly special relativity [45–50] which is the extended version of Einstein's special relativity in that the Plank length is also required to be invariant under any inertial frames apart from the invariant speed of light. However, it has been claimed that in this framework a nonlinear Lorentz transformation in the momentum space is needed to keep the double invariant constants accompanying a deformed Lorentz symmetry, so that the ordinary dispersion relation should be modified by the nonlinear Lorentz transformation. On the other hand, instead of this non-linear realization of Lorentz transformation, Magueijo and Smolin [51] proposed that the spacetime background felt by a test particle would depend on its energy such that the energy of the test particle deforms the background geometry and consequently the dispersion relation.

In particular, there have been extensive studies for black hole temperature in rainbow black holes in connection with black hole thermodynamics [28–33]. In fact, the black hole temperature can be easily determined from the MDR and the uncertainty relation once the spacial uncertainty is identified with the size of the black hole [39]. Explicitly, from the uncertainty relation one can relate the particle momentum with the black hole mass by identifying the position uncertainty with the black hole horizon, and then obtain the black hole temperature from the particle energy which is associated with the momentum through the dispersion relation. On the other hand, it could also be defined straightforwardly by the surface gravity in such as the metric of the rainbow black hole, where it would naturally depend on the energy of the test particle [29]. Note that the latter temperature from the surface gravity should be consistent with the former one using the MDR and Heisenberg uncertainty relation. Recently, the thermodynamic quantities were calculated in the rainbow Schwarzschild black hole with a particular choice of rainbow functions and corresponding thermodynamics was studied in ref. [33]. However, the energy was directly regarded as the black hole temperature by invoking the ordinary dispersion relation rather than the MDR in the rainbow gravity.

In this work, we would like to reconsider this issue and calculate the black hole temperature from the definition of the surface gravity in the rainbow Schwarzschild black hole and then the energy dependence of the temperature will be eliminated by employing both the MDR and the uncertainty relation. This procedure consistently determines the black hole temperature and gives rise to a certain different temperature from the previous one in ref. [33]. So it would be interesting to study the thermodynamic quantities of entropy and heat capacity according to this newly defined black hole temperature. Furthermore, we shall investigate thermodynamic phase transition of the rainbow Schwarzschild black hole in order to find out how much the rainbow effect of the metric changes the ordinary phase transition in black hole thermodynamics.

In section 2, the black hole temperature for the rainbow Schwarzschild black hole will be calculated following the definition of the standard surface gravity, and then the energy dependence of the test particle in the rainbow metric will be rephrased by the use of two elements of the MDR and the Heisenberg uncertainty principle. In section 3, from the first law of thermodynamics, the entropy is derived and then local thermodynamic quantities including the thermodynamic energy and heat capacity in the cavity will be presented along the line of local thermodynamic approach in ref. [52]. In section 4, we shall obtain the on-shell free energy and the off-shell free energy for the rainbow black hole and study phase transition between various black hole states and the hot flat space. It will be shown that there exist two kinds of critical temperatures in contrast to the case of the ordinary Schwarzschild black hole. A similar behavior appeared in the exactly soluble quantized Schwarzschild black hole [53]. Apart from the conventional critical temperature called the Hawking-Page phase transition [54–58], there appears an additional critical temperature which has something to do with the existence of a locally stable tiny black hole; however, the off-shell free energy shows that it should eventually tunnel into the stable large black hole. Finally, conclusion and discussion will be given in section 5.

2 Black hole temperature

We are going to define the black hole temperature by employing the definition from the surface gravity of the rainbow metric; however, it depends on the energy of a test particle. So the main purpose in this section is to show how to consistently eliminate the energy dependence from the temperature. Let us start with the MDR given in ref. [51],

$$\omega^2 f(\omega/\omega_p)^2 - p^2 g(\omega/\omega_p)^2 = m^2, \tag{2.1}$$

where ω, p, m are the energy, momentum, mass of a test particle, respectively, and the Planck energy is denoted by ω_p. The functions of $f(\omega/\omega_p)$, $g(\omega/\omega_p)$ are so called the rainbow functions which will be determined depending on the specific models, where they should be reduced to $\lim_{\omega\to 0} f(\omega/\omega_p) = 1$ and $\lim_{\omega\to 0} g(\omega/\omega_p) = 1$ in the absence of the test particles.

On the other hand, one of the interesting MDRs [59, 60] could be found in the high-energy regime as

$$m^2 \approx \omega^2 - p^2 + \eta p^2 \left(\frac{\omega}{\omega_p}\right)^n, \tag{2.2}$$

where η is a positive free parameter and n is assumed to be positive integer. Comparing eq. (2.1) with eq. (2.2), one can determine the specific rainbow functions as [33],

$$f(\omega/\omega_p) = 1, \quad g(\omega/\omega_p) = \sqrt{1 - \eta \left(\frac{\omega}{\omega_p}\right)^n}. \tag{2.3}$$

Next, let us consider the rainbow Schwarzschild black hole described as [51]

$$ds^2 = -\frac{1}{f(\omega/\omega_p)^2}\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{g(\omega/\omega_p)^2}\left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + \frac{r^2}{g(\omega/\omega_p)^2}d\Omega^2, \tag{2.4}$$

then the Hawking temperature T_{H} is calculated as

$$T_{\mathrm{H}} = \frac{\kappa_{\mathrm{H}}}{2\pi} = \frac{1}{8\pi GM}\frac{g(\omega/\omega_p)}{f(\omega/\omega_p)}, \tag{2.5}$$

where κ_H is the surface gravity at the horizon. By the use of the explicit form of rainbow functions (2.3), the black hole temperature can be written as

$$T_{\mathrm{H}} = \frac{1}{8\pi GM}\sqrt{1 - \eta\left(\frac{\omega}{\omega_p}\right)^n}. \tag{2.6}$$

In order to eliminate the dependence of the particle energy in the black temperature of the rainbow gravity (2.6), one can use the Heisenberg uncertainty principle of $\Delta x \Delta p \sim 1$, which yields a relation between the momentum p and the mass of black hole M [39]

$$p = \Delta p \sim \frac{1}{2GM} \tag{2.7}$$

where $\Delta x = 2GM$. In principle, plugging eq. (2.7) into the MDR (2.2), one can determine the energy of ω; however, it is non-trivial to solve the MDR for a general n. If we choose $n = 2$ for simplicity, the energy for the massless particle can be easily solved as

$$\omega = \frac{\omega_p}{\sqrt{\eta + 4G^2\omega_p^2 M^2}}. \tag{2.8}$$

Plugging the above relation (2.8) into the temperature (6) for $n = 2$, we can obtain new temperature as

$$T_{\mathrm{H}} = \frac{1}{8\pi GM}\sqrt{\frac{4GM^2}{4GM^2 + \eta}}, \tag{2.9}$$

with $G = 1/\omega_p^2$. It goes to asymptotically well-known Hawking temperature while it is finite for $M \to 0$. Thus, the temperature (2.6) of the rainbow black hole could be expressed in terms of the mass of the black hole by solving both the MDR (2.2) and the Heisenberg uncertainty relation (2.7).

Now one might wonder what the difference is between our result (2.9) and the previous result in ref. [33], where the temperature was given as

$$T'_{\mathrm{H}} = \frac{1}{4\pi(2GM)^{\frac{n+2}{2}}}\sqrt{(2GM)^n - \frac{\eta}{\omega_p^n}}. \tag{2.10}$$

In this case, the mass of the black hole has a lower bound of $M = \eta^{1/n}/(2G\omega_p)$ due to the negative sign in the square root. In ref. [33], the particle energy was regarded as the particle momentum such as $\omega = p = \Delta p \sim 1/(2GM)$ for a massless particle and then this relation was plugged into eq. (2.6) directly in order to eliminate the ω-dependence. In other words, in ref. [33] the MDR was partially used in the sense that the author used the correct definition (2.6) based on the MDR in the surface gravity but subsequently the ordinary dispersion relation rather than the MDR was used in the course of eliminating the ω-dependence by taking $\omega = p$ along with the uncertainty relation (2.7). Therefore, for the consistent treatment, we used the MDR (2.2) along with the uncertainty relation (2.7) when the energy dependence was eliminated in eq. (2.6).

Note that for the case of highly massive black holes or in the absence of the rainbow effect implemented by $\eta = 0$, the temperature (2.9) is reduced to the ordinary Hawking temperature. The most interesting thing to distinguish from the previous results in ref. [33] is that the temperature (2.9) subject to the MDR has a massless limit rather than the remnant whose mass would be $M_{\rm rem} = \sqrt{\eta}/(2\sqrt{G})$ for $n = 2$ as seen from the temperature (2.10).

Although the massless limit is allowed in our calculations, there still exists the lower bound of the energy as $\omega = \omega_p/\sqrt{\eta}$ as seen from eq. (2.8). Thus, for $M \to 0$, the temperature (2.9) becomes finite while it is divergent in the ordinary Schwarzschild black hole. It implies that the divergent ordinary temperature could be regularized in the regime of the rainbow gravity so that the parameter η plays a role of the cutoff and it becomes finite as $T_{\rm H} = 1/(4\pi\sqrt{G\eta})$ where it is still divergent if the cutoff is removed as $\eta \to 0$. In fact, it has also been shown that the divergent quantities such as entropy and free energy can be regularized by the use of the MDR in the brick wall method in ref. [61]. Thus the regularized finite behavior of the temperature would be better than the divergent one, which will be used in studying thermodynamic quantities in the next section. And, we will discuss whether the vanishing temperature can be realized in this rainbow Schwarzschild black hole or not in the last section.

3 Thermodynamic quantities

In this section, we calculate thermodynamic quantities in the rainbow Schwarzschild black hole (2.4) characterized by the rainbow functions (2.3) fixed as $n = 2$ for exact solubility. For this purpose, the first law of black hole thermodynamics is required to obtain the black hole entropy, and then the local temperature for hot black holes in a cavity [62] will be derived in order for the local thermodynamic energy and heat capacity. The local thermodynamic analysis given by York [52] presents a well-defined thermodynamic partition function which has something to do with the existence of the large stable black hole.

From the first law of black hole thermodynamics of $dS = dM/T$, the entropy associated with the temperature (2.9) can be obtained as

$$S = 4\pi GM^2\sqrt{1+\frac{\eta}{4GM^2}} + \eta\pi\sinh^{-1}\left(\frac{2\sqrt{G}M}{\sqrt{\eta}}\right). \tag{3.1}$$

For $\eta = 0$, the entropy (3.1) respects one-quarter of area law of $S = A/4$, where A is the area of the black hole. The next leading order of correction to the area law is the logarithmic term as $S \approx A/4+\eta\pi/2\ln(A/4)$, which is reminiscent of quantum correction to the entropy [63–67]. In some sense, it is interesting to note that the rainbow metric plays a role of the quantum corrected metric.

Now, the local temperature T_{loc} calculated at a finite distance r outside the black hole is defined as [62],

$$T_{\mathrm{loc}} = \frac{1}{8\pi GM\sqrt{1-\frac{2GM}{r}}}\sqrt{\frac{4GM^2}{4GM^2+\eta}}, \tag{3.2}$$

where it is implemented by the redshift factor of the metric. For a given r, the local temperature (3.2) is divergent as seen from figure 1(a) when the black hole size approaches M_3 where $r = 2GM_3$ since the black hole is very hot near the horizon, whereas it is finite as $T_1 = 1/(4\pi\sqrt{G\eta})$ for $M = 0$ as was discussed in the previous section. As for the black hole states, there are two extrema: one is the local minimum of T_0 at $M = M_2$ and the other is the local maximum of T_2 at $M = M_1$. Note that there are two black hole states for $T_0 < T < T_1$ and three black hole states for $T_1 < T < T_2$, while there appears just one black hole state for $T > T_2$. The details will be studied in the next section together with analysis of the free energy.

Let us calculate the local thermodynamic energy E_{tot} by employing the local thermodynamic first law, which yields

$$\begin{aligned} E_{\mathrm{tot}} &= \int_0^M T_{\mathrm{loc}} dS \\ &= \frac{r}{G}\left(1-\sqrt{1-\frac{2GM}{r}}\right), \end{aligned} \tag{3.3}$$

where we used the entropy (3.1) and the temperature (3.2). It happens to be the same with conventional expression which is independent of the energy of test particles since the choice of rainbow functions (2.3) shows $f = 1$ so that the time-like Killing vector is the same with the ordinary one. To investigate thermodynamic stability of the black hole, we calculate the heat capacity defined as $C = \partial E_{\mathrm{tot}}/\partial T_{\mathrm{loc}}$, and explicitly it reads

$$C = \frac{4\pi(r-2GM)(4GM^2+\eta)^{\frac{3}{2}}}{\sqrt{G}(12GM^2-4Mr+\eta)}. \tag{3.4}$$

For convenience's sake, let us define the black hole states depending on its mass scale as tiny, small, and large lack hole satisfying $M < M_1$, $M_1 < M < M_2$, and $M_2 < M < M_3$, respectively. As seen from figure 1(b), there are two stable regions of $M < M_1$ and $M_2 < M < M_3$, while there is only one unstable region of $M_1 < M < M_2$, and thus the tiny and large black hole are stable and the small black hole is unstable. Note that there was no stable tiny black hole in the ordinary Schwarzschild black hole [52]. What needs to be answered is that whether these locally stable states would undergo tunneling or not by investigating the free energy, which is studied in the next section.

4 Free energy and phase transition

In this section, we first calculate the on-shell free energy defined as $F_{\mathrm{on}} = E_{\mathrm{tot}} - T_{\mathrm{loc}}S$ in the rainbow Schwarzschild black hole in order to study thermodynamic phase transition [52–54, 63, 67]. As is well known in the ordinary Schwarzschild black hole, the hot flat space is more probable than the large stable black hole below T_c, while the large stable black hole is more probable than the hot flat space above T_c. Moreover, the on-shell free energy

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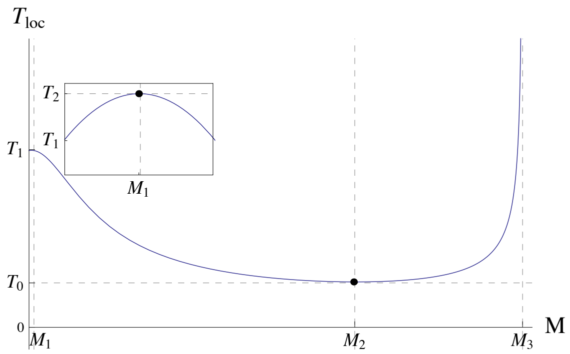
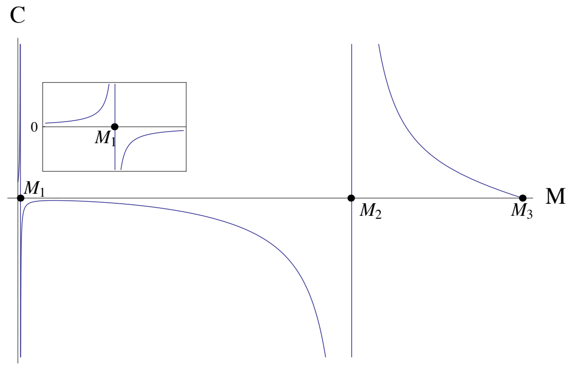


(a) local temperature



(b) heat capacity

Figure 1. The local temperature (a) and heat capacity (b) are plotted for $\eta = 1$, $r = 10$, and $G = 1$. They show that the temperature have two extrema at M_1 and M_2, respectively in figure (a), and the stability changes appear at those extrema in Fig (b). The maximum mass of the black hole is $M_3 = r/(2G)$.

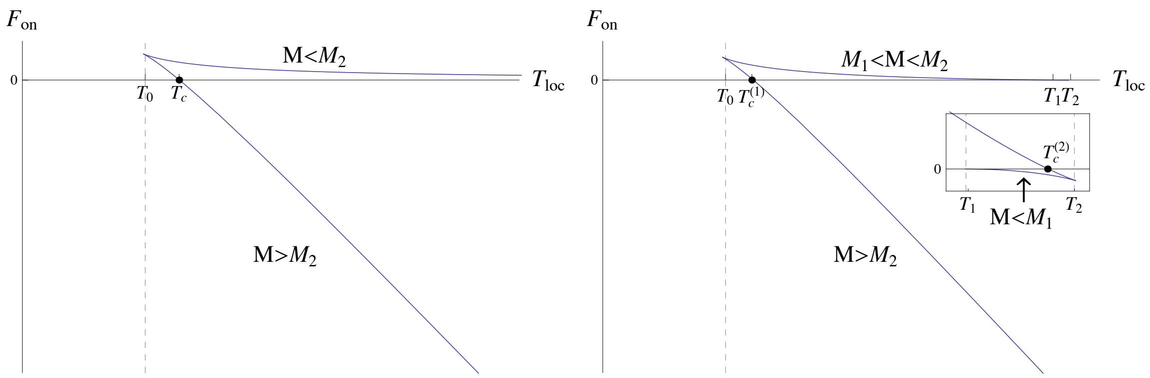
of the unstable small back hole of $M < M_2$ is always positive as seen from figure 2(a), so that the small black hole should decay into the hot flat space as long as $T > T_0$. In the rainbow Schwarzschild black hole, the on-shell free energy is also calculated by the use of the temperature (3.2) and the thermodynamic energy (3.3), which is explicitly written as

$$F_{\rm on} = \frac{r}{G}\left(1 - \sqrt{1 - \frac{2GM}{r}}\right) - \frac{1}{\sqrt{1 - \frac{2GM}{r}}}\left(\frac{M}{2} + \frac{\eta \sinh^{-1}\left(\frac{2\sqrt{G}M}{\sqrt{\eta}}\right)}{4\sqrt{4G^2M^2 + G\eta}}\right), \tag{4.1}$$

where it recovers the on-shell free energy of the Schwarzschild black hole for $\eta = 0$ as expected. And, the on-shell free energy of the hot flat space vanishes since $E_{\rm tot} = S = 0$ as seen from eq. (3.1) and the integral form of the energy (3.3) for $M \to 0$ in any arbitrary temperature, i.e., $F_{\rm on}^{\rm hfs} = 0$. In fact, this is also consistent with the result from eq. (4.1) when we take $M \to 0$ since $sinh^{-1}x \sim x$ for $x \to 0$.

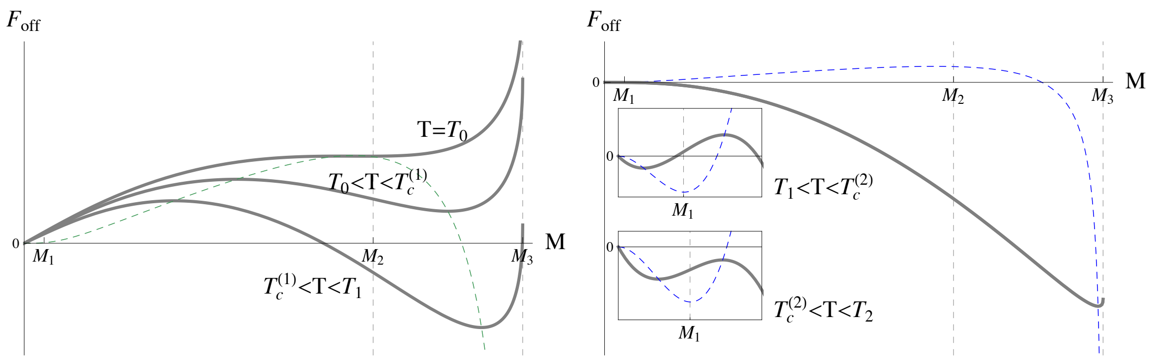
Now, let us discuss mainly three regions in figure 2(b). (i) For $T_0 < T < T_1$, the behavior of the free energy is analogous to the conventional one as shown in figure 2(a), so that the first critical temperature $T_c^{(1)}$ in the rainbow black hole plays a role of $T_{(c)}$ in the ordinary black hole. (ii) For $T_1 < T < T_c^{(2)}$, the hot flat space collapses to form a tiny black hole of $M < M_1$ which is stable as was shown from the positive heat capacity in figure 1(b). (iii) For $T_c^{(2)} < T < T_2$, the hot flat space would collapse to the small black hole of $M_1 < M < M_2$. As shown in figure 2(b), the temperature for the tiny and small black holes should be terminated at T_2, and only the stable large black hole exists above $T > T_2$. From (ii) and (iii), the on-shell free energy of the tiny black hole is still higher than that of the large black hole, so that the tiny black hole undergoes a tunneling and eventually decays into the large black hole. This tunneling effect can be easily understood in terms of the following off-shell free energy.

By using the entropy (3.1) and the energy (3.3) with an arbitrary temperature, the off-shell free energy defined as $F_{\rm off} = E_{\rm tot} - TS$ can be plotted in figure 3 (a) and (b). In figure 3 (a), the overall behaviors of the off-shell free energy are coincident with those of the ordinary Schwarzschild black hole in ref. [52] and those in the anti-de Sitter Schwarzschild black hole in ref. [54] as long as $T_0 < T < T_1$ in that the large black hole tunnels into the hot flat space.



(a) Schwarzschild black hole (b) Rainbow black hole

Figure 2. The on-shell free energy which is a function of the local temperature is plotted by setting $\eta = 1, r = 10$, and $G = 1$. There exists a single critical temperature T_c in the ordinary Schwarzschild black hole in figure 2 (a), while there appear two critical temperatures of $T_c^{(1)}$ and $T_c^{(2)}$ as in figure 2 (b).



(a) $T_0 < T < T_1$ (b) $T_1 < T < T_2$

Figure 3. The off-shell free energy subject to a temperature is plotted by setting $\eta = 1, r = 10$, and $G = 1$, and it is expressed by a solid curve. The dotted curve is for the on-shell free energy which corresponds to the extrema of off-shell free energy. In figure (b), the two small boxes are presented in order to explicitly show the off-shell free energy around the second critical temperature $T_c^{(2)}$.

However, as the temperature is increased, there appears a tiny black hole above T_1 and it decays into the large black hole across the potential barrier with the tunneling probability given as the difference between the free energy of the unstable small black hole and that of the tiny black hole. Note that the difference between the small and tiny free energies are always positive, since the free energy of the small black hole is always higher than that of the tiny black hole even although the free energy of the small black hole is positive in the upper box while it is negative in the lower box depending on the temperature as shown in figure 3 (b).

5 Discussion

We have calculated local thermodynamic quantities in the rainbow Schwarzschild black hole subject to the MDR and study its phase transition in terms of investigating the on-shell and off-shell free energies. First of all, the momentum of the emitted particle from the black

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hole was expressed by the black hole mass based on the Heisenberg uncertainty principle and then the temperature was derived by employing the nontrivial dispersion relation between the energy and the momentum characterized by the specific MDR. According to this modified black hole temperature, we considered the on-shell free energy and the off-shell free energy of the black hole in the finite size by introducing the isothermal surface of the cavity. In contrast to the conventional Schwarzschild black hole, it was shown that there exists an additional stable tiny black hole together with the conventional black hole states above T_1. Apart from the well-known critical temperature in Hawking-Page phase transition, there exists an additional critical temperature which is of relevance to the existence of a locally stable tiny black hole; however, the off-shell free energy shows that it should eventually tunnel into the stable large black hole with the finite transition probability since this tiny black hole is just locally stable.

The temperature in the rainbow Schwarzschild black hole with the rainbow functions (2.3) becomes finite when the black hole evaporates completely, whereas it was divergent in the ordinary Schwarzschild black hole. The reason for this finiteness of the temperature is related to the fact that the Newton constant is running as $G(\omega) = G/g(\omega)$ [51], which can be read off from the rainbow metric (2.4). In our case, it can be explicitly expressed as $G(M) = G\sqrt{1+\eta/(4GM^2)}$ by the use of eq. (2.8), then the temperature (2.9) can be effectively written as $T_{\rm H}(M) = 1/(8\pi G(M)M)$. For $M \to 0$, the gravitational coupling becomes strong at the order of $1/M$ that the temperature becomes finite in this rainbow black hole. Then, one might wonder whether the black hole temperature in the rainbow gravity can be zero or not for $M \to 0$ when we choose different types of rainbow functions. To answer this question, let us redefine arbitrary two rainbow functions as $f(\omega/\omega_p) = 1 + \tilde{f}(\omega/\omega_p)$, $g(\omega/\omega_p) = 1 + \tilde{g}(\omega/\omega_p)$ for convenience, where $\lim_{\omega\to 0}\tilde{f}(\omega/\omega_p) = 0$ and $\lim_{\omega\to 0}\tilde{g}(\omega/\omega_p) = 0$. Without loss of generality, one can solve MDR in eq. (2.1) for the massless case and Heisenberg relation (2.7), then it can be shown that $\omega(1+\tilde{f}(\omega/\omega_p))/(1+\tilde{g}(\omega/\omega_p)) = 1/(2GM)$. Requiring the condition of $T \propto \omega \to 0$ with $M \to 0$, it gives rise to inconsistency in the sense that the right hand side is divergent but the left hand side is zero. Thus, any choices of rainbow functions in the rainbow Schwarzschild black hole can not make the temperature vanish when $M \to 0$.

In the above discussion, it was shown that the ordinary Heisenberg uncertainty relation and the MDR in the rainbow gravity did not make the temperature vanish for $M \to 0$. Now, in order to get the vanishing temperature at the vanishing limit of the black hole mass, one might try to consider some other ways such as (i) GUP consideration [35–44] or (ii) an iteration procedure on eq. (2.2) which amounts to applying iteration procedure to the temperature (2.6) directly. (i) For the first case, the most simple modification of the uncertainty principle is realized as $\Delta x \Delta p \geq 1 + \ell^2(\Delta p)^2$, where it leads to the minimal length of $\Delta x_{\rm min} = 2\ell$. The cutoff ℓ can be chosen as the Planck scale so that it can be fixed as $\ell^2 = 1/(\omega_p)^2$ to make our notations consistent. By setting $\Delta x = 2GM$ [39], the modified temperature improved by using both the MDR for $n = 2$ and the GUP can be obtained as $T = (\omega_p/4\pi)\left[\left(2M^2 - (1-\eta)\omega_p^2 - 2M\sqrt{(M-\omega_p)(M+\omega_p)}\right)\left(4\eta M^2 + (1-\eta)^2\omega_p^2\right)^{-1}\right]^{1/2}$ which is reduced to the well-known temperature based on the conventional GUP, which is $T_{\rm GUP} = (M/4\pi)\left[1 - \sqrt{1-\omega_p^2/M^2}\,\right]$ [39] for $\eta = 0$. Unfortunately, the modified temperature is still finite as $T = \omega_p/(4\pi\sqrt{1+\eta})$ at the minimum mass of $M = \omega_p$. This fact is not surprising since the finite temperature with the remnant is somehow a generic feature in the regime of the GUP [39]. Thus, the above GUP combining the MDR does not likely to give the desired result. (ii) Secondly, apart from the first case subject to the GUP and the MDR, let us solve directly the temperature (2.6) by taking advantage of the relation in ref. [29]. In fact, the particle energy has been regarded as the temperature of the particle, and it is natural to take the relation of $\omega = 4\pi T$ with the calibration factor [39]. Putting this energy-temperature relation into eq. (2.6) directly, we can derive the equation, $\eta(4\pi/\omega_p)^n T^n + (8\pi GM)^2 T - 1 = 0$ [29]. Interestingly for $n = 2$, the solution to this equation is exactly coincident with the temperature (2.9). The temperatures corresponding to other values of $n = 1, 3, 4$ have a single real positive definite solution respectively. The behaviors for $n = 1, 3, 4$ near $M \to 0$ are very similar to the case of $n = 2$ which gives the finite result as was shown in section 2. Note that the above equation was solved for $n \leq 4$ in order for exact solutions without numerical methods. We did not find any evidence to make the temperature vanish when $M \to 0$ even in the second case either. Therefore, it deserves further study in this direction.

Acknowledgments

We would like to thank M. Eune and E.J. Son for exciting discussions. W.K. was supported by the Sogang University Research Grant of 2014 (201410019).

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