

Superradiance and statistical entropy of a hairy black hole in three dimensions

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We calculate the statistical entropy of a rotating hairy black hole by taking into account superradiant modes in the brick wall method. The UV cutoff is independent of the gravitational hair, which gives the well-defined area law of the entropy. It can be shown that the angular momentum and the energy of matter field depend on the gravitational hair. For the vanishing gravitational hair, it turns out that the energy for matter is related to both the black hole mass and the black hole angular momentum whereas the angular momentum for matter field is directly proportional to the angular momentum of the black hole.

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I. INTRODUCTION

There has been much attention to three-dimensional topological massive gravity [1,2] because it has rich structures even though it is lower dimensional gravity. Recently, new massive gravity [3,4] has been also intensively studied; in particular, it can be shown that there is a new type of a rotating black hole solution apart from the rotating Banados-Teitelboim-Zanelli (BTZ) black hole [5]. A new rotating black hole has three hairs: two of them are mass and angular momentum and the other corresponds to the gravitational hair [6–9].

On the other hand, the statistical origin of the entropy for black holes has been studied in terms of the brick wall method [10]. Subsequently, there have been extensive applications of the brick wall method to various black holes [11–13]. In connection with the rotating hairy black hole, there exists a superradiant mode so that the brick wall method is nontrivial, and it should be treated carefully [14]. Moreover, a thin-layer method has been introduced [15,16], where it is in thermal equilibrium locally and the divergent term due to a box with infinite size does not appear anymore. For a thin layer near the horizon, this method is valid if the proper thickness is taken keeping the local equilibrium state, since the degree of freedom is dominant near the horizon.

In this paper, we would like to calculate the entropy of the rotating hairy black hole [6] from new massive gravity [3,4] by using the brick wall method [10] in thin-layer approximations [15,16], where the angular speed of a particle near the horizon can be approximately fixed to a constant. So we will show how to take into account the superradiant mode in the entropy calculation of the rotating hairy black hole which has complicated metric components. The entropy of the hairy black hole in connection

with the superradiant mode for the rotating case deserves to be studied. In Sec. II, a new hairy rotating black hole and its thermodynamic quantities are introduced. Additionally, one can write down the explicit form of the horizons and the radius of the ergosphere analytically. In Sec. III, we calculate the thermodynamic quantities such as the entropy, the angular momentum, and the energy by considering the superradiant and the nonsuperradiant modes simultaneously. A summary is given in Sec. IV.

II. ROTATING HAIRY BLACK HOLE

We consider the rotating black hole described by [6–9]

$$ds^2 = -N^2 f dt^2 + \frac{dr^2}{f} + r^2(d\phi - \Omega_0 dt)^2, \quad (1)$$

where $N(r) = 1 + b\ell^2(1 - \eta)/(4\Sigma)$, $f(r) = (\Sigma^2/r^2) \times [\Sigma^2/\ell^2 + b(1 + \eta)\Sigma/2 + b^2\ell^2(1 - \eta)^2/16 - \mu\eta]$, $\Omega_0(r) = a(\mu - b\Sigma)/(2r^2)$, and $\Sigma(r)^2 = r^2 - \mu\ell^2(1 - \eta)/2 - b^2\ell^4(1 - \eta)^2/16$. a , b , and μ are integration constants and Λ and η have been defined as $\Lambda = -1/(2\ell^2)$ and $\eta \equiv \sqrt{1 - a^2/\ell^2}$. For the limit of $b = 0$, the metric functions reduce to $N = 1$, $f = -\mu + r^2/\ell^2 + \mu^2 a^2/(4r^2)$, and $\Omega_0 = \mu a/(2r^2)$, and Eq. (1) simply describes the rotating BTZ black hole [5]. Note that the range of the rotating parameter a is given by $-\ell \leq a \leq \ell$. Then the Arnowitt-Deser-Misner (ADM) mass and the ADM angular momentum can be obtained as [8]

$$\mathcal{M} = \frac{\mu}{4G} + \frac{b^2\ell^2}{16G}, \quad \mathcal{J} = \mathcal{M}a, \quad (2)$$

respectively. G is a Newton constant in three dimensions, which is set to $G = 1$ for convenience. The entropy has been also obtained as $\mathcal{S} = \pi\ell\sqrt{2\mathcal{M}(1 + \eta)}$ [6,7,9]. Now, the condition of $f(r) = 0$ gives

$$r_{\pm} = \ell\sqrt{2(1 + \eta)}\left(\sqrt{\mathcal{M}} \pm \frac{|b|\ell}{4}\sqrt{\eta}\right), \quad (3)$$

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$$r_0 = \ell \sqrt{2(1-\eta)} \left[\mathcal{M} - \frac{b^2 \ell^2}{32} (1+\eta) \right]^{1/2}, \quad (4)$$

which yields the horizon r_+ . Equation (1) describes the rotating BTZ black hole, which has two horizons of $r_{\pm} = \ell \sqrt{\mu(1+\eta)/2}$ and $r_0 = \ell \sqrt{\mu(1-\eta)/2}$. Especially for $a=0$, the black hole has two horizons $r_{\pm} = 2\ell(\sqrt{\mathcal{M}} \pm |b|\ell/4)$ with $r_0 = 0$. Next, the Hawking temperature of the black hole can be obtained from the surface gravity,

$$T_H = \frac{1}{4\pi} N(r_+) f'(r_+) = \frac{\eta}{\pi \ell} \sqrt{\frac{2\mathcal{M}}{1+\eta}}. \quad (5)$$

$$r_e = 2\ell \sqrt{\mathcal{M} + \frac{b^2 \ell^2}{16} + \frac{|b|\ell}{4} \sqrt{2(1+\eta)} \left[\mathcal{M} + \frac{b^2 \ell^2}{32} (1-\eta) \right]^{1/2}}, \quad (8)$$

which can be calculated from $\Omega_-(r_e) = 0$.

III. STATISTICAL ENTROPY

Now, we consider a scalar field in a thin layer between $r_+ + h$ to $r_+ + h + \delta$ with $h \ll r_+$ and $\delta \ll r_+$, where h is a cutoff parameter and δ is a small constant related to the thickness of the thin layer. It satisfies the massless Klein-Gordon equation, $\square \Phi(t, r, \phi) = 0$. Assuming $\Phi(t, r, \phi) = \Psi_{\omega m}(r) e^{-i\omega t + im\phi}$, we obtain $rN\partial_r(rNf\partial_r\Psi_{\omega m}) + r^2N^2f^2k^2\Psi_{\omega m} = 0$, where $k(r; \omega, m) = N^{-1}f^{-1}\sqrt{(\omega - \Omega_+m)(\omega - \Omega_-m)}$. In the WKB approximation with $\Psi \sim e^{iS(r)}$, k is the radial momentum defined by $k = \partial S / \partial r$. Therefore, the number of states less than the energy ω and the angular momentum m is given by

$$n(\omega, m) = \frac{1}{\pi} \int_{r_h+h}^{r_h+h+\delta} dr "k"(r; \omega, m), \quad (9)$$

where $"k"(r; \omega, m) = k(r; \omega, m)$ if $k^2 > 0$ and $"k"(r; \omega, m) = 0$ if $k^2 < 0$. The free energy of a rotating black hole should be written as [14]

$$F = F_{\text{NS}} + F_{\text{SR}}, \quad (10)$$

where

$$\beta F_{\text{NS}} = \sum_{\lambda \notin \text{SR}} \int d\omega g(\omega, m) \ln[1 - e^{-\beta(\omega - m\Omega_H)}], \quad (11)$$

$$\beta F_{\text{SR}} = \sum_{\lambda \in \text{SR}} \int d\omega g(\omega, m) \ln[1 - e^{\beta(\omega - m\Omega_H)}], \quad (12)$$

where the “NS” and “SR” denote the nonsuperradiant mode with $\omega - m\Omega_H > 0$ and superradiant mode with $\omega - m\Omega_H < 0$, respectively, λ is the set of (ω, m) , and the density of the number of states is given by $g(\omega, m) = dn/d\omega$ for the NS mode and $g(\omega, m) = -dn/d\omega$ for the

Consequently, the first law of thermodynamics $d\mathcal{M} = T_H dS + \Omega_H d\mathcal{J}$ is satisfied.

The maximum Ω_+ and the minimum Ω_- of the angular velocity for a particle are given by

$$\Omega_{\pm}(r) = \Omega_0 \pm \frac{N}{r} \sqrt{f}. \quad (6)$$

Note that the angular velocity of a particle on the event horizon becomes

$$\Omega_H = \Omega_0(r_+) = \frac{1}{\ell} \sqrt{\frac{1-\eta}{1+\eta}}, \quad (7)$$

and the radius r_e of the ergosphere is explicitly written as

SR mode. Substituting Eq. (9) into Eqs. (11) and (12), we obtain

$$\begin{aligned} \beta F_{\text{NS}} = & -\frac{\beta}{\pi} \int dr \sum_m \int d\omega \frac{"k"(r; \omega, m)}{e^{\beta(\omega - \Omega_H m)} - 1} \\ & + \frac{1}{\pi} \int dr \sum_m "k"(r; \omega, m) \\ & \times \ln[1 - e^{-\beta(\omega - \Omega_H m)}]_{\omega_{\min}^{(m)}}^{\omega_{\max}^{(m)}}, \end{aligned} \quad (13)$$

$$\begin{aligned} \beta F_{\text{SR}} = & -\frac{\beta}{\pi} \int dr \sum_m \int d\omega \frac{"k"(r; \omega, m)}{e^{-\beta(\omega - \Omega_H m)} - 1} \\ & - \frac{1}{\pi} \int dr \sum_m "k"(r; \omega, m) \\ & \times \ln[1 - e^{\beta(\omega - \Omega_H m)}]_{\omega_{\min}^{(m)}}^{\omega_{\max}^{(m)}}, \end{aligned} \quad (14)$$

where $\omega_{\max}^{(m)}$ and $\omega_{\min}^{(m)}$ denote the maximum and the minimum of ω for a given m in each mode, respectively. For convenience, Eq. (13) can be rewritten as

$$F_{\text{NS}} \equiv F_{\text{NS}}^{(m>0)} + F_{\text{NS}}^{(m<0)}, \quad (15)$$

where

$$\begin{aligned} \beta F_{\text{NS}}^{(m>0)} = & -\frac{\beta}{\pi} \int_{r_++h}^{r_++h+\delta} \frac{dr}{Nf} \int_0^{\infty} dm \\ & \times \int_{\Omega_+m}^{\infty} d\omega \frac{\sqrt{(\omega - \Omega_+m)(\omega - \Omega_-m)}}{e^{\beta(\omega - \Omega_H m)} - 1}, \end{aligned} \quad (16)$$

$$\begin{aligned}
\beta F_{\text{NS}}^{(m<0)} &= -\frac{\beta}{\pi} \int_{r_++h}^{r_++h+\delta} \frac{dr}{Nf} \int_{-\infty}^0 dm \\
&\times \int_0^\infty d\omega \frac{\sqrt{(\omega - \Omega_+ m)(\omega - \Omega_- m)}}{e^{\beta(\omega - \Omega_H m)} - 1} \\
&- \frac{1}{\pi} \int_{r_++h}^{r_++h+\delta} \frac{dr}{Nf} \int_{-\infty}^0 dm \sqrt{\Omega_+ \Omega_- m^2} \\
&\times \ln(1 - e^{\beta \Omega_H m}). \quad (17)
\end{aligned}$$

From Eq. (14), the free energy of the SR mode is written as

$$\begin{aligned}
\beta F_{\text{SR}} &= -\frac{\beta}{\pi} \int_{r_++h}^{r_++h+\delta} \frac{dr}{Nf} \int_0^\infty dm \\
&\times \int_0^{\Omega_- m} d\omega \frac{\sqrt{(\omega - \Omega_+ m)(\omega - \Omega_- m)}}{e^{-\beta(\omega - \Omega_H m)} - 1} \\
&+ \frac{1}{\pi} \int_{r_++h}^{r_++h+\delta} \frac{dr}{Nf} \int_0^\infty dm \sqrt{\Omega_+ \Omega_- m^2} \\
&\times \ln(1 - e^{-\beta \Omega_H m}). \quad (18)
\end{aligned}$$

Then, the total free energy which consists of the nonsuperradiant and superradiant modes (10) becomes

$$F = -\frac{\zeta(3)}{4\beta^3} \int_{r_++h}^{r_++h+\delta} \frac{dr}{Nf} \frac{(\Omega_+ - \Omega_-)^2}{(\Omega_+ - \Omega_H)^{3/2}(\Omega_H - \Omega_-)^{3/2}}, \quad (19)$$

which leads to

$$F = -\frac{\zeta(3)}{\beta^3} \frac{2r_+}{N(r_+)^2 f'(r_+)^{3/2}} \left(\frac{1}{\sqrt{h}} - \frac{1}{\sqrt{h+\delta}} \right) \quad (20)$$

in the leading order of the cutoff and the thickness. Note that the second term in the free energy for the positive mode in Eq. (17) and the second term in the free energy for the superradiant mode in (18) canceled out. Thus the entropy can be simplified as

$$S = \beta^2 \frac{\partial F}{\partial \beta} \Big|_{\beta=\beta_H} = \frac{3\zeta(3)}{8\pi^2} r_+ \sqrt{f'(r_+)} \left(\frac{1}{\sqrt{h}} - \frac{1}{\sqrt{h+\delta}} \right), \quad (21)$$

where β_H is defined as the inverse of the Hawking temperature T_H . The proper lengths for the UV cutoff parameter and the thickness are defined by $\bar{h} \equiv \int_{r_+}^{r_++h} dr \sqrt{g_{rr}} \simeq 2\sqrt{h}/\sqrt{f'(r_+)}$ and $\bar{\delta} \equiv \int_{r_++h}^{r_++h+\delta} dr \sqrt{g_{rr}} \simeq 2(\sqrt{h+\delta} - \sqrt{h})/\sqrt{f'(r_+)}$. Then, the entropy is written as $S = 3\zeta(3)r_+\bar{\delta}/[4\pi^2\bar{h}(\bar{h} + \bar{\delta})]$. Recovering dimensions, the entropy becomes

$$S = \frac{c^3 A}{4G\hbar} \frac{3\zeta(3)\ell_P \bar{\delta}}{2\pi^3 \bar{h}(\bar{h} + \bar{\delta})}, \quad (22)$$

where $A \equiv 2\pi r_+$ and $\ell_P \equiv \hbar G/c^3$ are the area of the event horizon and the three-dimensional Planck length, respectively. If the cutoff is chosen as $\bar{h}(\bar{h} + \bar{\delta})/\bar{\delta} = [3\zeta(3)/(2\pi^3)]\ell_P$,

the entropy (22) agrees with the Bekenstein-Hawking entropy $S_{\text{BH}} = c^3 A/(4G\hbar)$.

Finally, let us calculate the angular momentum of matter, which becomes

$$\begin{aligned}
J &= -\frac{\partial F}{\partial \Omega_H} \Big|_{\beta=\beta_H} \\
&= \frac{a}{2} \left(\sqrt{\mathcal{M}} + \frac{|b|\ell}{4} \sqrt{\eta} \right) \left[\sqrt{\mathcal{M}} + \frac{\ell}{8} (b + |b|) \left(\sqrt{\eta} + \frac{1}{\sqrt{\eta}} \right) \right], \quad (23)
\end{aligned}$$

and the internal energy of the system is written as

$$\begin{aligned}
E &= F_H + \beta_H^{-1} S + \Omega_H J \\
&= \frac{1}{6} \left(\sqrt{\mathcal{M}} + \frac{|b|\ell}{4} \sqrt{\eta} \right) \\
&\times \left[(3 + \eta) \sqrt{\mathcal{M}} + \frac{3\ell}{8\sqrt{\eta}} (b + |b|)(1 - \eta^2) \right]. \quad (24)
\end{aligned}$$

Note that the angular momentum (23) and the energy (24) of matter have well-defined limits and they are compatible with the results in Ref. [14] for $b = 0$.

On the other hand, it would be interesting to note that a partition function from free energy (20) can be compared with the result for the partition function of the corresponding two-dimensional conformal field theory (CFT) on the boundary of three-dimensional anti-de Sitter (AdS) space-time [17]. For this purpose, we write down the free energy (20) by using Eqs. (5) and (7) along with the proper lengths \bar{h} and $\bar{\delta}$ as

$$F = -\frac{\zeta(3)}{2\pi(\beta_H/\ell)^2[1 - (\ell\Omega_H)^2]} \frac{\bar{\delta}}{\bar{h}(\bar{\delta} + \bar{h})}, \quad (25)$$

where we restricted to the case of $b = 0$ and identified β with β_H . If one chooses the cutoff as $\bar{h}(\bar{\delta} + \bar{h})/\bar{\delta} = [3\zeta(3)/\pi^3]\ell_P$, then the free energy (25) is simplified as

$$F = -\frac{\pi^2}{6(\beta_H/\ell)^2[1 - (\ell\Omega_H)^2]\ell_P}. \quad (26)$$

In order to write down the free energy in terms of dimensionless quantities, we rescale the free energy, the inverse Hawking temperature, and the angular velocity at the horizon by $\ell_P F \rightarrow F$, $\beta_H/\ell \rightarrow \beta$, and $\ell\Omega_H \rightarrow \Omega$, respectively. Then, the free energy becomes $F = -\pi^2/[6\beta^2(1 - \Omega^2)]$. Since the relation between the partition function Z and the free energy is given by $\beta F = -\ln Z$, we can obtain

$$\ln Z = \frac{\pi^2}{6\beta(1 - \Omega^2)}, \quad (27)$$

which agrees with the result given in Ref. [17]. However, it may depend on the cutoff within our brick wall formulation so that the coefficient can be adjusted. As a result, the degrees of freedom near the horizon can be described by

the boundary degrees of freedom. In fact, the bulk degrees of freedom can be read off from the boundary degrees of freedom from the AdS/CFT while the bulk degrees of freedom can be also described by the degrees of freedom near the horizon based on the brick wall formalism. Combining these two notions, the boundary degrees of freedom at both ends can be connected.

IV. SUMMARY

In the course of calculations, the second term of the free energy for the positive mode in Eq. (17) and the second term of the free energy for the superradiant mode in (18) canceled out so that from the simplified resulting free energy we have obtained the statistical entropy satisfying the area law by determining the UV cutoff which is

independent of the hairs of the black hole, and additionally derived the angular momentum and the energy of matter field.

The energy E is always positive and it depends on the mass of the black hole, the angular momentum of the black hole, and the gravitational hair b . For the limit of $b = 0$, the angular momentum can be reduced to $J = \frac{1}{2}\mathcal{J}$ and $E = \frac{1}{2}\mathcal{M} + \frac{1}{6}\sqrt{\mathcal{M}^2 - \mathcal{J}^2/\ell^2}$. It means that the angular momentum of the matter is directly proportional to that of the black hole while the energy is related to the mass and angular momentum of the black hole simultaneously.

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