By Avinash Singh (Ex IES, B.Tech IITR)

Tutorial Sheet III: Matrices

1. Every square matrix can be expressed in one and only one way as the sum of symmetric and a skew symmetric matrix.

2. If A is Hermitian matrix, show that iA is skew-Hermitian matrix.

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $A^k = \begin{bmatrix} 1+2k & -4k \\ 2 & 1-2k \end{bmatrix}$, where k is any positive integer.

4. Show that the possible square roots of the two rowed unit matrix I are $\pm I$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$, where $1 - \alpha^2 = \beta\gamma$.

5. Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

6. Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

7. Show that $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involuntary.

8. Determine the values of α, β, γ when $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

9. If A is idempotent and $A + B = I$, then show that B is idempotent and $AB = BA = O$.

10. If A is real skew symmetric matrix such that $A^2 + I = O$, show that A is orthogonal and is of even order.

11. If A be square matrix, then show that $adj.A' = (adj.A)'$.

12. If A and B are square matrices of the same order, then $adj.(AB) = (adj.B).(adj\ A)$

13. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}$.

14. Find the inverse of A by Gauss Jordan Method where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

15. If $adj\, B = A$, and P, Q are two unimodular matrices, $i.e. |P| = 1,\ |Q| = 1$, then show that $adj.(Q^{-1}BP^{-1}) = PAQ$.

16. If the product of two non-zero square matrices is a zero matrix, show that both of them must be singular matrices.

17. Find the rank of matrix $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$. (2)

18. Show that no skew symmetric matrix can be of rank 1.

19. **Reduce** A to Echelon Form and then to it's row canonical form where $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$, Hence find the rank of A.

20. Reduce the matrix A to its normal form where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, hence find the rank of A.

21. Find the non-singular matrices P and Q such that the normal form of A is PAQ where $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$, hence find its rank.

22. Compute the inverse of $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 1 & 0 \end{bmatrix}$.

23. Discuss for all values of k , the system of equations: $\begin{aligned} 2x + 3ky + (3k+4)z &= 0 \\ x + (k+4)y + (4k+2)z &= 0 \\ x + 2(k+1)y + (3k+4)z &= 0 \end{aligned}$

24. Solve the following equations with the help of matrices: $\begin{aligned} x + 2y + 3z &= 14 \\ 3x + y + 2z &= 11 \\ 2x + 3y + z &= 11 \end{aligned}$

25. **Solve** the system $\begin{aligned} 2x_1 + 5x_2 + 2x_3 - 3x_4 &= 3 \\ 3x_1 + 6x_2 + 5x_3 + 2x_4 &= 2 \\ 4x_1 + 5x_2 + 14x_3 + 14x_4 &= 11 \\ 5x_1 + 10x_2 + 8x_3 + 4x_4 &= 4 \end{aligned}$ by

    a. Gaussian Elimination Method

    b. Gauss Jordan Method

26. Write down 2×2 matrix A which corresponds to a counterclockwise rotation of $60°$ about the origin.

27. **Determine** the algebraic and geometric multiplicity of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Characteristic Roots and Characteristic Vectors of a Matrix

1. Find the characteristic roots and corresponding characteristic vectors for each of the following matrix: $\begin{matrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 3 & -1 & 3 \end{matrix}$

2. If $a + b + c = 0$, find the characteristic roots of matrix $A = \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$.

3. Find the latent roots and latent vectors of the matrix $A = \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$.

4. Show that the two matrices A, $P^{-1}AP$ have the same characteristic roots.

5. If A and B are two square matrices, then the matrices AB and BA have the same characteristic roots.

6. Show that the characteristic roots of A^{Θ} are the conjugates of the characteristic roots of A.

7. If A is nonsingular, prove that the eigen values of A^{-1} are reciprocals of the eigen values of A.

8. If α is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is a characteristic root of adj. A.

9. Show that if $\lambda_1, \lambda_2, \dots \dots \dots . \lambda_n$ are n eigen values of a square matrix A of order n then the eigen values of the matrix A^2 be $\lambda_1^2, \lambda_2^2, \dots \dots \dots . \lambda_n^2$.

10. Show that the characteristic roots of an idempotent matrix are either zero or unity.

11. If A is both real symmetric and orthogonal, prove that all it's eigen values are $+1$ or -1.

12. If S is a skew-Hermitian matrix, show that the matrices $I - S$ and $I + S$ are both non-singular. Also show that $A = (I + S)(I - S)^{-1}$ is unitary matrix.

13. **Show** that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is Skew-Hermitian and also unitary. Find the eigen values and eigen vectors.

14. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1}.

15. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 4$, $A^n = A^{n-2} + A^3 - A$. Hence evaluate A^{20}.

16. Show that the matrix $A = \begin{bmatrix} 7 & 4 & -1 \\ 5 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.

17. A square matrix is said to be idempotent if $A^2 = A$. Show that if A is idempotent, then all eigen values of A are equal to *1* or *0*.

18. Prove that if A is similar to a diagonal matrix, then A' is similar to A.

19. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable.

20. If $X_1 = \frac{1}{3}[2 \quad -1 \quad 2]^T$ and $X_2 = k[3 \quad -4 \quad -5]^T$ where $= 1/\sqrt{50}$, construct an orthogonal matrix $A = [X_1 \quad X_2 \quad X_3]$.