Integration of Euler's equations of motion. Bernoulli's equation. Pressure equation.

[I.A.S. 2005; Kanpur 2002, 04, 05, 09; Meerut 2000, 02, 08]

When a velocity potential exists (so that the motion is irrotational) and the external forces are derivable from a potential function, the equations of motion can always be integrated. Let ϕ be the velocity potential and V be the force potential. Then, by definition, we get

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z, \qquad \text{...(1)}$$

$$X = -\partial V/\partial x, \qquad Y = -\partial V/\partial y, \qquad Z = -\partial V/\partial y, \qquad \text{...(2)}$$

and
$$\partial u/\partial y = \partial v/\partial x, \qquad \partial v/\partial z = \partial w/\partial y, \qquad \partial w/\partial x = \partial u/\partial z. \qquad \text{...(3)}$$

Then well known Euler's dynamical equation are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using (1) (2) and (3), these can be re-written as

$$\left.\begin{aligned} -\frac{\partial^2\phi}{\partial t\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} &= -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ -\frac{\partial^2\phi}{\partial t\partial y} + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} &= -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ -\frac{\partial^2\phi}{\partial t\partial z} + u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} &= -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad \text{...(4)}$$

Re-writing equations (4), we get

$$-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(5)$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2)=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(6)$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+w^2)=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(7)$$

Now
$$d\left(\frac{\partial \phi}{\partial t}\right)=\frac{\partial}{dx}\left(\frac{\partial \phi}{\partial t}\right)dx+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy+\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz \qquad ...(8)$$

$$dV=(\partial V/\partial x)\,dx+(\partial V/\partial y)\,dy+(\partial V/\partial z)\,dz \qquad ...(9)$$

$$dp=(\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy+(\partial p/\partial z)\,dz \qquad ...(10)$$

$$d(u^2+v^2+w^2)=\frac{\partial}{\partial x}(u^2+v^2+w^2)\,dx+\frac{\partial}{\partial y}(u^2+v^2+w^2)\,dy+\frac{\partial}{\partial z}(u^2+v^2+w^2)\,dz \qquad ...(11)$$

Multiplying (5), (6) and (7) by dx, dy and dz respectively, then adding and using (8), (9), (10) and (11), we have

$$-d\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}d(u^2+v^2+w^2)=-dV-\frac{1}{\rho}dp$$

or
$$-d\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}dq^2+dV+\frac{1}{\rho}dp=0 \qquad ...(12)$$

where $q^2=u^2+v^2+w^2=$ (velocity of fluid particle)^2

If ρ is a function of p, integration of (12) gives

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=F(t), \qquad ...(13)$$

where F(t) is an arbitrary function of t arising from integration in which t is regarded as constant. (13) is *Bernoulli's equation* in its most general form. Equation (13) is also known as *pressure equation.*

Special Case I. Let the fluid be homogeneous and inelastic (so that ρ = constant *i.e.*, fluid is incompressible). Then Bernoulli's equation for unsteady and irrotational motion is given by

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\frac{p}{\rho}=F(t) \qquad ...(14)$$

Special Case II. If the motion be steady $\partial\phi/\partial t=0,$ the Bernoulli's equation for steady irrotational motion of an incompressible fluid is given by

$q^2/2+V+p/\rho=C$, where C is an absolute constant. **(Kanpur 2010)** ...(15)

4.2. Bernoulli's theorem. (Steady motion with no velocity potential and conservative field of force).

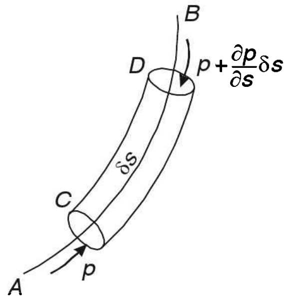
[Agra 2009; Meerut 2009, 2010; Kanpur 2004; Purvanchel 2005; G.N.D.U. Amritsar 2002, 05]

When the motion is steady and the velocity potential does not exist, we have

$$\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C,$$

where V is the force potential from which the external forces are derivable. **[Meerut 2011]**

Proof. Consider a streamline AB in the fluid. Let δs be an element of this stream line and CD be a small cylinder of cross-sectional area α and δs as axis. If q be the velocity and S be the component of external force per unit mass in direction of the streamline, then by Newton's second law of motion, we have



$$\rho\alpha\,\delta s.\frac{Dq}{Dt}=\rho\alpha\,\delta s\cdot S+p\alpha-\left(p+\frac{\partial p}{\partial s}\delta s\right)\alpha$$

or
$$\frac{Dq}{Dt}=S-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

or
$$\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial s}=S-\frac{1}{\rho}\frac{\partial p}{\partial s} \qquad \text{...(1)}$$

If the motion be steady $\partial q/\partial t=0$, and if the external forces have a potential function V such that $S=-\partial V/\partial s$, (1) reduces to

$$\frac{1}{2}\frac{\partial q^2}{\partial s}+\frac{\partial V}{\partial s}+\frac{1}{\rho}\frac{\partial p}{\partial s}=0 \qquad \text{...(2)}$$

If ρ is a function of p, integration of (2) along the streamline AB yields

$$\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C, \qquad \text{...(3)}$$

where C is constant whose value depends on the particular chosen streamline.

Special Case I. If the fluid be homogeneous and incompressible, ρ = constant and hence (3) reduces to

$$q^2/2+V+p/\rho=C. \qquad \textbf{(Kanpur 2008)} \qquad \text{...(4)}$$

Special Case II. Let S be a graviational force per unit mass. Let δh be the vertical distance between C and D. Then we have

$$S=-g\frac{\partial h}{\partial s}=-\frac{\partial}{\partial s}(gh), \qquad \text{as} \qquad V=gh$$

Hence, if the fluid be incompressible, (3) reduces to

$$q^2/2+gh+p/\rho=C. \qquad \text{...(5)}$$

4.3. Illustrative solved examples.

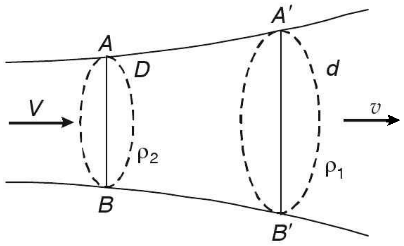
Ex. 1. *A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion be supposed to be that of the divergence from the vertex of the cone, prove that*

$$v/V=(D^2/d^2)e^{(v^2-V^2)/2k} \qquad \textbf{[I.A.S. 1993, 98]}$$

where k is the pressure divided by the density and supposed constant.

Sol. Let AB and $A'B'$ be the ends of the conical pipe such that $A'B'$ $=d$ and $AB=D$. Let ρ_1 and ρ_2 be densities of the stream at $A'B'$ and AB. *By principle of conservation of mass, the mass of the streem that entres the end AB and leaves at the end A'B'* must be the same. Hence the equation of continuity is

$$\pi(d/2)^2 v\rho_1=\pi(D/2)^2 V\rho_2$$



so that $$\frac{v}{V} = \frac{D^2}{d^2} \times \frac{\rho_2}{\rho_1} \qquad ...(1)$$

By Bernoulli's theorem (in absence of external forces like gravity), we have

$$\int \frac{dp}{\rho} + \frac{1}{2} q^2 = C \qquad ...(2)$$

Given that $\quad p/\rho = k \quad$ so that $\quad dp = k\, d\rho$. $\qquad$ ...(3)

$\therefore$ (2) reduces to $\qquad k \int \frac{d\rho}{\rho} + \frac{1}{2} q^2 = C$, using (3)

Integrating, $\qquad k \log \rho + q^2/2 = C$, C being an arbitrary constant $\qquad$ ...(4)

When $\quad q = v, \quad \rho = \rho_1 \quad$ and $\quad$ when $\quad q = V, \quad \rho = \rho_2$. Hence, (4) yields

$$k \log \rho_1 + v^2/2 = C \qquad \text{and} \qquad k \log \rho_2 + V^{2/} 2 = C$$

Subtracting, $\qquad k(\log \rho_2 - \log \rho_1) + (V^2 - v^2)/2 = 0$

or $\qquad \log(\rho_2/\rho_1) = (v^2 - V^2)2k \quad$ or $\quad \rho_2/\rho_1 = e^{(v^2 - V^2)2k}$. $\qquad$ ...(5)

Using (5), (1) reduces to $\qquad v/V = (D^2/d^2) \times e^{(v^2 - V^2)/2k}$.