# By Avinash Singh (Ex IES, B.Tech IITR)

## Tutorial Sheet II: Linear Algebra

### Linear Transformations-I

1. Show that the mapping $f: V_3(F) \to V_2(F)$ defined by $f(a_1, a_2, a_3) = (a_1, a_2)$ is a homomorphism of $V_3(F)$ *onto* $V_2(F)$.
2. If $V$ is finite dimensional and $f$ is a homomorphism of $V$ onto $V$, prove that $f$ must be one one and so, an isomorphism.
3. If $V$ is finite dimensional and $f$ is homomorphism of $V$ into itself which is not onto, prove that there is some $\alpha \neq \mathbf{0}$ in V such that $f(\alpha) = \mathbf{0}$.
4. Show that the mapping $T: (a, b) \to (a + 2, b + 3)$ of $V_2(R)$ into itself is not a linear transformation.
5. Let $f$ be a linear transformation from a vector space $U$ into a vector space $V$. If $S$ is a subspace of $U$, prove that $f(S)$ will be subspace of $V$.
6. If $f: U \to V$ is an isomorphism of the vector space of $U$ into the vector space $V$, then a set of vectors $\{f(\alpha_1), f(\alpha_2), f(\alpha_3) \dots \dots \dots, f(\alpha_r)\}$ is linearly independent iff the set $\{\alpha_1, \alpha_2, \dots \dots \dots, \alpha_r\}$ is linearly independent.
7. Show that the mapping $f: V_2(R) \to V_3(R)$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation from $V_2(R)$ *into* $V_3(R)$. Find the range, rank, null space and nullity of $T$.
8. Let $F$ be field of complex numbers and let $T$ be the function from $F^3$ *into* $F^3$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2 - x_3, -x_1 - 2x_2)$. Verify that $T$ is a linear transformation. Describe the null space of $T$.
9. Let $V$ be $n$ dimensional vector space over the field $F$ and let $T$ be a linear transformation from $V$ into $V$ such that the range and null space of $T$ are identical. Prove that $n$ is even. Give an example of such a linear transformation.
10. Let $V$ be a vector space over the field $F$ and $T$ be a linear transformation from $V$ into $V$. Prove that the following two statements about T are equivalent:
    a. The intersection of the range of $T$ and the null space of $T$ is the zero subspace of V *i.e.* $R(T) \cap N(T) = \{0\}$
    b. $T[T(\alpha)] = \mathbf{0} \Rightarrow T(\alpha) = \mathbf{0}$

11. Let *S(R)* be the vector space of all polynomial functions in $x$ with coefficients as the elements of the field $R$ of real numbers. Let $D$ and $T$ be linear operators on $V$ defined by $D(f(x)) = \frac{d}{dx} f(x)$ $and$ $T(f(x)) = \int_0^x f(x)dx$ for every $f(x) \epsilon V$. Then show that $DT = I$ $(identity\ operator)$ and $TD \neq I$
12. Describe explicitly the linear transformation $T: R^2 \to R^2$ such that $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$.
13. Describe explicitly a linear transformation $V_3(R)$ $into$ $V_3(R)$ which has its range and subspace spanned by $(1,0,-1) and$ $(1,2,2)$.
14. Let $T$ be a linear operator on $V_3(R)$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)\ \forall\ (a,b,c)\ \epsilon\ V_3(R)$. Is $T$ invertible? If so, find a rule for $T^{-1}$ like the one which defines $T$
15. A linear transformation $T$ is defined on $V_2(C)$ by $T(a,b) = (\alpha a + \beta b, \gamma a + \delta b)$, where $\alpha, \beta, \gamma, \delta$ are fixed elements of $C$. Prove that T is invertible iff $\alpha\delta - \beta\gamma \neq 0$.
16. Let $V$ be a vector space over the field $F$ and $T$ a linear operator on $V$. If $T^2 = \mathbf{0}$, what can you say about the relation of the range of $T$ to the null space of $T$? Give an example of a linear operator $T$ on $V_2(R)$ such that $T^2 = \mathbf{0}$ but $T \neq 0$.
17. If $A$ and $B$ are linear transformations on the same VS, then necessary and sufficient condition that both $A$ and $B$ be invertible is that both $AB$ and $BA$ be invertible.
18. If $A$ is linear transformation on a VS $V$ such that $A^2 - A + I = \hat{0}$, then $A$ is invertible.
19. Let $T$ be linear Transformation on the VS $V_2(F)$ defined by $T(a,b) = (a,0)$. Write the matrix of $T$ relative to the standard ordered basis of $V_2(F)$.
20. Let *V(R)* be the VS of all polynomials in $x$ with coefficients in $R$ of the form $f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$ $i.e.$ the space of polynomials of degree three or less. The differentiation operator $D$ is a linear transformation on $V$. The set $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where $\alpha_1 = x^0, \alpha_2 = x^1, \alpha_3 = x^2, \alpha_4 = x^3$ is an ordered basis for $V$. Write the matrix of $D$ relative to the ordered basis $B$.
21. Find the matrix of LT $T$ on $V_3(R)$ defined as $T(a,b,c) = (2b+c, a-4b, 3a)$, with respect to the ordered basis $B$ and also with respect to the ordered basis $B'$ where
    a. $B = \{(1,0,0), (0,1,0), (0,0,1)\}$
    b. $B' = \{(1,1,1), (1,1,0), (1,0,0)\}$.
22. Let $T$ be the linear operator on $R^3$ defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$. What is the matrix of T:
    a. In the standard ordered basis $B$ for $R^3$
    b. In the ordered basis $B' = \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1,0,1), \alpha_2 = (-1,2,1)$ $and$ $\alpha_3 = (2,1,1)$?
    c. Find the transition matrix P from $B$ to $B'$.
23. Let $T$ be a linear operator on $R^3$ defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$. Prove that $T$ is invertible and find the formula for $T^{-1}$.

24. Consider the vector space $V(R)$ of all $2 \times 2$ matrices over the field $R$ of real numbers. Let $T$ be linear transformation on $V$ that sends each matrix $X$ onto $AX$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of $T$ with respect to the ordered basis $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ for $V$ where $\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
25. Show that the vectors $\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$ form a basis for $R^3$. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

## Linear Transformations-II

1. Find a linear map $F: R^3 \to R^4$ whose image is spanned by $(1,2,0,-4)$ and $(2,0,-1,-3)$.
2. Suppose that $F: V \to U$ is linear and that $V$ is of finite dimension. Show that $V$ and the image of $F$ have the same dimension if and only of $F$ is nonsingular. Determine all nonsingular linear mappings $T: R^4 \to R^3$.
3. Consider the linear operator $T$ on $R^3$ defined by $T(x,y,z) = (2x, 4x - y, 2x + 3y - z)$. Show that T is invertible Find formulas for $T^{-1}, T^2, T^{-2}$.
4. Consider the following $2 \times 2$ matrix $A$ and basis $S$ of $R^2$: $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$ and $S = \{u_1, u_2\} = \{(1,-2), (3,-7)\}$. The matrix $A$ defines a linear operator on $R^2$. Find the matrix $B$ that represents the mapping $A$ relative to the basis. $\left(\begin{bmatrix} -63 & -235 \\ 19 & 71 \end{bmatrix}\right)$
5. The vectors $u_1 = (1,1,0), u_2 = (0,1,1), u_3 = (1,2,2)$ from a basis $S$ of $R^3$. Find the coordinates of an arbitrary vector $v = (a,b,c)$ relative to the basis $S$. $(b - c, -2a + 2b - c, a - b + c)$.
6. Consider the following bases of $R^2$: $S = \{u_1, u_2\} = \{(1,-2), (3,-4)\}$ *and* $S' = \{v_1, v_2\} = \{(1,3), (3,8)\}$.
   a. Find the coordinates of $v = (a,b)$ relative to the basis $S$.
   b. Find the change of basis matrix $P$ from $S$ to $S'$
   c. Find the coordinates of $v = (a,b)$ relative to the basis $S'$
   d. Find the change of basis matrix $Q$ from $S'$ back to $S$.
7. Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & -2 & 2 \end{bmatrix}$. Find the matrix $B$ that represents the linear operator $A$ relative to the basis $S = (u_1, u_2, u_3) = \{(1,1,0), (0,1,1), (1,2,2)\}$. $\left(\begin{bmatrix} 8 & 1 & 3 \\ 7 & -6 & -11 \\ -5 & 3 & 6 \end{bmatrix}\right)$
8. Let $F: R^3 \to R^2$ be the linear map defined by $F(x,y,z) = (3x + 2y - 4z, x - 5y + 3z)$. Find the matrix of $F$ in the following bases of $R^3$ and $R^2$: $S = \{(1,1,1), (1,1,0), (1,0,0)\}$, $S' = \{(1,3), (2,5)\}$. $\left(\begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}\right)$