Previous Year Questions: PDE (2008-2022)

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Linear Partial Differential Equations of order One

1. Find the general solution of the partial differential equation $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ and also find the particular solution which passes through the lines $x = 1, y = 0$.
2. Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that this equation is satisfied by the surface $yz + zx + xy = 0$.
3. (i) Form the partial differential equation by elimination the arbitrary function f given by: $f(x^2 + y^2, z - xy) = 0$.
   (ii) Find the integral surface of: $x^2p + y^2q + z^2 = 0$ which passes through the curve: $xy = x + y, z = 1$.
4. Solve the PDE $(x + 2z)p + (4zx - y)q = 2x^2 + y$.
5. Solve partial differential equation $px + qy = 3z$
6. Form a partial differential equation by eliminating the arbitrary functions f and g from $z = yf(x) + xg(y)$.
7. Find the surface which intersects the surfaces of the system $z(x + y) = C(3z + 1), (C \text{ being a constant})$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.
8. Solve the partial differential equation: $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$.
9. Find the general equation of surfaces orthogonal to the family of spheres given by $x^2 + y^2 + z^2 = cz$.
10. Find the general integral of the particle differential equation $(y + zx)p - (x + yz)q = x^2 - y^2$.
11. Find the partial differential equation of the family of all tangent planes to the ellipsoid: $x^2 + 4y^2 + 4z^2 = 4$, which are not perpendicular to the xy-plane.
12. Find the general solution of the partial differential equation: $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$ and find its integral surface that passes through the curve: $x = t, y = t^2, z = 1$.

13. From a partial differential equation of the family of surfaces given by the following expression: $\psi(x^2 + y^2 + 2z^2, y^2 - 2zx) = 0$
14. Form a partial differential equation by eliminating the arbitrary functions f and g from $z = yf(x) + xg(y)$ and specify its nature (elliptic, hyperbolic or parabolic) in the region $x > 0, y > 0$.
15. Find the integral surface of the PDE $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$ that contains the curve: $xz = a^3, y = 0 \; on \; it.$
16. Obtain the PDE by eliminating arbitrary function f from the equation $f(x+y+z, x^2+y^2+z^2) = 0$.

Solve the following partial differential equation

$$zp + yq = x$$
$$x_0(s) = s, \qquad y_0(s) = 1, \qquad z_0(s) = 2s$$

by the method of characteristics.

17. Solve the first order quasilinear partial differential equation by the method of characteristics:

$$x\frac{\partial u}{\partial x} + (u - x - y)\frac{\partial u}{\partial x} = x + 2y \; in \; x > 0, -\infty < y < \infty \; with \; u = 1 + y \; on \; x = 1.$$

18. It is given that the equation of any cone with vertex at (a, b, c) is $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$. Find the differential equation of the cone. (10, 2022)

Non-Linear Partial Differential Equations of order One

1. Find complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$ using Charpit's method.

Find a complete integral of the partial differential equation

$$2(pq + yp + qx) + x^2 + y^2 = 0$$

2. Find the solution of the PDE $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$; which passes through the x axis.
3. Find the complete integral of the PDE $p = (z+qy)^2$ by using Charpit's method.
4. Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes though the parabola $4z + x^2 = 0, y = 0$.

Homogeneous Linear PDE with Const Coefficients

1. Find the general solution of the partial differential equation: $(D^2 + DD' - 6D'^2)z = y \cos x$ where $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$

2. Solve: $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy$ where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$.
3. Find the surface satisfying the PDE $(D^2 - 2DD' + D'^2)z = 0$ and the conditions that $bz = y^2$ when $x = 0$ and $az = x^2$ when $y = 0$.
4. Solve partial differential equation $(D - 2D')(D - D')^2 z = e^{x+y}$
5. Solve $(D^2 + DD' - 6D'^2)z = x^2 \sin(x + y)$.
6. Solve the partial differential equation $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.
7. Solve $(D^2 + DD' - 2D'^2)u = e^{x+y}$
8. Solve for the general solution: $p \cos(x + y) + q \sin(x + y) = z$.
9. Solve the partial differential equation $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$.
10. Solve $(D^2 - 2DD' - D'^2)z = e^{x+2y} + x^3 + \sin 2x$
11. Solve the partial differential equation: $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x + y) + 24(y - x) + e^{3x+4y}$.
12. Solve the partial differential equation: $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{2x+y} + \sin(x - 2y)$.
13. Find the general solution of the partial differential equation: $(D^2 + DD' - 6D'^2)z = x^2 \sin(x + y)$ where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$. (15, 2022)

Non-Homogeneous Linear PDE with Const Coefficients

1. Solve the PDE $(D^2 - D')(D - 2D')Z = e^{2x+y} + xy$
2. Solve the PDE $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2 y$
3. Find the surface satisfying $\frac{\partial^2 z}{\partial x^2} = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$.
4. Find the general solution of the PDE $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

Reduction to Canonical Form

1. Reduce $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.
2. Find the characteristics of: $y^2 r - x^2 t = 0$ where r and t have their usual meanings.
3. Reduce the following 2^{nd} order partial differential equation into canonical form and find its general solution. $xu_{xx} + 2x^2 u_{xy} - u_x = 0$.

4. Reduce the equation $y\frac{\partial^2 z}{\partial x^2}+(x+y)\frac{\partial^2 z}{\partial x\partial y}+x\frac{\partial^2 z}{\partial y^2}=0$ to its canonical form when $x \neq y$.
5. Reduce the second-order partial differential equation $x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$ into canonical form.
6. Reduce the equation $y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial x}$ to canonical form and hence solve it.
7. Reduce the following second order partial differential equation to canonical form and find the general solution: $\frac{\partial^2 u}{\partial x^2}-2x\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial y}+12x$.
8. Reduce the following partial differential equation to canonical form and hence solve it: $yu_{xx}+(x+y)u_{xy}+xu_{yy}=0$. (15, 2022)

Application of PDE

1. Find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0, x=a, y=0$ *and* $y=b$. The edges $x=0, x=a$ *and* $y=0$ are kept at temperature zero while the edge $y=b$ is kept at $100°C$.
2. A tightly stretched string has its ends fixed at $x=0$ and $x=1$. At time $t=0$, the string is given a shape defined by $f(x)=\mu x(l-x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at time $t>0$.

Solve the following heat equation

$$u_t - u_{xx} = 0, \quad 0<x<2, \quad t>0$$
$$u(0,t)=u(2,t)=0, t>0$$
$$u(x,0)=x(2-x), 0\le x\le 2$$

3. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0\le x\le a, 0\le y\le b$ satisfying the boundary conditions $u(0,y)=0, u(x,0)=0, u(x,b)=0,\ \frac{\partial u}{\partial x}(a,y)=T\sin^3\frac{\pi y}{a}$.
4. Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length whose one end is kept at $10°C$ and the other end is insulated. Also, it is given that $y(x,0)=1-x,\ 0<x<1$
5. A string of length l is fixed at its ends. The string from the mid-point is pulled up to a height k and then released from rest. Find the deflection $y(x,t)$ of the vibrating string.

6. The edge $r = a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.
7. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in equilibrium position. If it is set vibrating by giving each point a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.
8. Find the deflection of a vibrating string $(length = \pi, ends\ fixed, \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2})$ corresponding to zero initial velocity and initial deflection $\mathrm{f(x)} = \mathrm{k(\sin x - \sin 2x)}$.
9. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1, t > 0$. Given that
   a. $u(x, 0) = 0, 0 \le x \le 1$.
   b. $\frac{\partial u}{\partial t}(x, 0) = x^2, 0 \le x \le 1$
   c. $u(0, t) = u(1, t) = 0$, for all t.
10. Find the solution of the initial-boundary value problem $u_t - u_{xx} + u = 0$, $0 < x < l, t > 0$; $u(0, t) = u(l, t) = 0, t \ge 0$; $u(x, 0) = x(l - x), 0 \le x \le l$.
11. Find the temperature $u(x, t)$ in a bar of silver of length $10\, cm$ and constant cross section of area $1\, cm^2$. Let density $\rho = 10.6 g/cm^3$, thermal conductivity $K = 1.04\, cal\,/(cm\, sec°C)$ and specific heat $\sigma = 0.056\, cal/g\, °C$. The bar is perfectly isolated laterally with ends kept at $0°C$ and initial temperature $f(x) = sin(0.1\pi x)$°C. Note that $u(x, t)$ follows the heat equation $u_t = c^2 u_{xx}$ $where\ c^2 = k\,/(\rho\sigma)$.
12. Let τ be a closed curve in xy-plane and let S denote the region bounded by the $curve\ \tau$. Let $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x, y)\ \forall (x, y) \in S$. If f is prescribed at each point $(x,\ y)$ of S and w is prescribed on the boundary τ of S, then prove that any solution $w = w(x, y)$, satisfying these conditions, is unique.
13. Given the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$; t>0,$where\ c^2 = \frac{T}{m}; T\ is\ constant\ tension\ in\ the\ string$
   a. Find the appropriate solution of the wave equation
   b. Find also the solution under the conditions $y(0, t) = 0, y(l, t) = 0$ for all t and $\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0, y(x, 0) = a \sin\frac{\pi x}{l}, 0 < x < l, a > 0$.
14. A thin annulus occupies the region $0 < a \le r \le b, 0 \le \theta \le 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at $0°$, while along

the outer edge the temperature is held at $T = K\cos\frac{\theta}{2}$ where K is a constant. Determine the temperature distribution in the annulus.

15. One end of tightly stretched flexible thin string of length l is fixed at the origin and the other at $x = l$. It is plucked at $x = \frac{l}{3}$ so that it assumes initially the shape of a triangle of height h in xy plane. Find the displacement y at any distance x and at any time t after the string is released from the rest. Take, $\frac{horizontal\ tension}{mass\ per\ unit\ length} = c^2$.

16. Solve the wave equation $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}; 0 < \mathrm{x} < \mathrm{L}, \mathrm{t} > 0$; subject to the conditions $u(0,t) = 0,\ u(L,t) = 0,\ u(x,0) = \frac{1}{4}x(L-x),\ \left(\frac{\partial u}{\partial t}\right)|_{t=0} = 0$

17. Solve the heat equation $u_t - u_{xx} = 0,\ 0 < x < l,\ t > 0$ subject to the conditions:

$u(0,t) = u(l,t) = 0, t > 0$

$u(x,0) = x(l-x), 0 \le x \le l$. (20, 2022)