Tutorial Sheet I: Linear Algebra

Vector Space

1. Show that the set of all $m \times n$ matrices with their elements as real numbers is a vector space over the field F of real numbers with respect to addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication.
2. The vector space of all ordered n *tuples* over a field F.
3. The vector space of all polynomial over a field F.
4. Let S be any non-empty set and let F be any field. Let V be the set of all functions from S to F *i.e.* , let $V = \{f \mid f: S \to F\}$ .Let us define sum of two elements f and g in V as : $(f + g)(x) = f(x) + g(x)\ \forall\ x \epsilon S$. Also let us define scalar multiplication of an element f in V by an element c in f as : $(cf)(x) = cf(x)\ \forall\ x \epsilon S$. Then *V(F)* is a vector space.
5. The set of all convergent sequence is a vector space over the field of real numbers.
6. Prove that the set of all vectors in a plane over the field of real numbers is a vector space.
7. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define $(x, y) + (x_1, y_1) = (x + x_1, 0)$; $c(x, y) = (cx, 0)$. Is V, with these operations, a vector space over the field of real numbers?
8. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Examine in each of the following cases whether V is a vector space over the field of real numbers or not?
   a. $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$; $c(x, y) = (|c|\, x, |c|\, y)$
   b. $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$; $c(x, y) = (0, cy)$
   c. $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$; $c(x, y) = (c^2 x,\ c^2 y)$.
9. Let R be the field of numbers and let P_n be the set of all polynomials (of degree at most n) over the field R. Prove that P_n is a vector space over the field R.
10. How many elements are there in the vector space of polynomials of degree at most n in which the coefficient are the elements of the field $I(p)$ over the elements of $I(p)$, p being a prime number.
11. Prove that the set of all solutions (a, b, c) of the equation $a + b + 2c = 0$ is a subspace of the vector space $V_3(R)$ *or* R^3.

12. Let V be the real vector space of all functions f from R into R. Which of the following sets of the functions are subspaces of V:
    a. All f such that $f(x^2) = [f(x)]^2$
    b. All f such that $f(0) = f(1)$
    c. All f such that $f(3) = 1 + f(-5)$
13. Let V be vector space of all functions from R into R; let V_e be the set of all even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.
    a. Prove that V_e and V_o are subspaces of V.
    b. Prove that $V_e + V_o = V$
    c. Prove that $V_e \cap V_o = \{0\}$.
14. Show that $S = \{(1,2,4), (1,0,0), (1,1,0), (0,0,1)\}$ is a linearly dependent subset of the vector space $V_3(R)$ where R is the field of real numbers.
15. If F is the field of real numbers, prove that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(F)$ are linearly dependent iff $a_1 b_2 - a_2 b_1 = 0$.
16. Show that the vectors $(1,1,0,0)$, $(0,1,-1,0)$, $(0,0,0,3)$ in R^4 are linearly independent.
17. Find whether the vectors $2x^3 + x^2 + x + 1$, $x^3 + 3x^2 + x - 2$ and $x^3 + 2x^2 - x + 3$ of $R[x]$, are linearly independent or not.
18. Find a maximal linearly independent subsystem of the system of vectors $\alpha_1 = (2,-2,-4), \alpha_2 = (1,9,3), \alpha_3 = (-2,-4,1)$ and $\alpha_4 = (3,7,-1)$.
19. Show that the infinite set $S = \{1, x, x^2, x^3 \ldots\ldots x^n \ldots.\}$ is a basis of the vector space $F[x]$ of polynomials over the field F.
20. Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.
21. Let V be the vector space of ordered pairs of complex numbers over the real field R i.e. let V be the vector space $C^2(R)$. Show that the set $S = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis of V.
22. Determine whether or not the following vectors form a basis of R^3: $(1,1,2), (1,2,5), (5,3,4)$.
23. Show that the set $\{(1, i, 0),\ (2i, 1, 1),\ (0, 1+i, 1-i)\}$ is a basis of $V_3(C)$.
24. Show that the set $S = \{1, x, x^2, x^3 \ldots\ldots x^n\}$ of $n+1$ polynomials in x is a basis of the vector space $P_n(R)$, of all polynomials in x (of degree at most n) over the field of real numbers.
25. Select a basis, if any, of $R^3(R)$ from the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where $\alpha_1 = (1,-3,2)$, $\alpha_2 = (2,4,1)$, $\alpha_3 = (3,1,3)$, $\alpha_4 = (1,1,1)$.