NEXT IAS

FOUNDATION COURSE FOR CSE
MATHEMATICS OPTIONAL

ANALYTIC GEOMETRY

- PREVIOUS YEAR QUESTIONS (2008-2022)
- ASSIGNMENTS

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PREVIOUS YEAR QUESTIONS ANALYTIC GEOMETRY

PLANE

1. The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 4z - 5 = 0$. Find the equation of the plane in its new position.

2. Find the equation of the plane which passes through the points (0, 1, 1) and (2, 0, –1) and is parallel to the line joining the points (–1, 1, –2), (3, –2, 4). Find also the distance between the line and the plane.

3. Obtain the equation of the plane passing through the point (2, 3, 1) and (4, 5, 3) parallel to x-axis.

4. Find the equation of the plane parallel to $3x - y + 3z = 8$ and passing through the point (1, 1, 1).

STRAIGHT LINE

1. A line is drawn through a variable point on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$ to meet two fixed lines $y = mx$, $z = c$ and $y = -mx$, $z = -c$. Find the locus of the line.

2. Find the equations of the straight line through the point (3, 1, 2) to intersect the straight line $x + 4 = y + 1 = 2(z + 2)$ and parallel to the plane $4x + y + 5z = 0$.

3. Prove that two of the straight lines represented by the equation $x^3 + bx^2y + cxy^2 + y^3 = 0$ will be right angles if $b + c = -2$.

4. Find the shortest distance between the line $\frac{x-1}{2} = \frac{y-2}{4} = z - 3$ and $y - mx = z = 0$. For what values of 'm', will the two lines intersect.

5. Find the surface generated by a line which intersects the lines $y = a = z$, $x + 3z = a = y + z$ and parallel to the plane $x + y = 0$.

6. Find the shortest distance between the skew lines:

$$\frac{x-3}{3} = \frac{8-y}{1} = \frac{z-3}{1} \text{ and } \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

7. Find the shortest distance between the lines $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ and the z-axis.

8. Find the projection of straight line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1}$ on the plane $x + y + 2z = 6$.

9. Show that the lines intersect $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$. Find the intersection point and the equation of the plane containing them.

SPHERE

1. A sphere S has points (0, 1, 0), (3, –5, 2) at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x - 2y + 4z + 7 = 0$ as great circle. **(20)**

2. Find the equations (in symmetric form) of the tangent line to the circle $x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0, 3x - 2y + 4z + 3 = 0$ at the point (–3, 5, 4). **(12)**

3. Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle $x^2 + y^2 + z^2 - 12z - 3y + 4z + 8 = 0, 3x + 4y - 5z + 3 = 0$. **(12)**

4. Show that every sphere through the circle $x^2 + y^2 - 2ax + r^2 = 0, z = 0$ cuts orthogonally every sphere through the circle $x^2 + z^2 = r^2, y = 0$ **(20)**

5. Show that the plane $x + y - 2z = 3$ cuts the sphere $x^2 + y^2 + z^2 - x + y = 2$ in a circle of radius 1 and find the equation of the sphere which has this circle as a great circle.

6. Show that the equation of the sphere which touches the sphere $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ at the point (1, 2, –2) and passes through the point (–1, 0, 0) is $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$. **(10)**

7. A sphere S has points (0, 1, 0), (3, –5, 2) at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane. $5x - 2y + 4z + 7 = 0$ as a great circle.

8. Find the coordinates of the points on the sphere $x^2 + y^2 + z^2 - 4x + 2y = 4$, the tangent planes at which are parallel to the plane $2x - y + 2z = 1$. **(10)**

9. For what positive value of 'a', the plane $ax - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and hence find the point of contact. **(10)**

10. Find the equation of the sphere which passes through the circle $x^2 + y^2 = 4$; $z = 0$ and is cut by the plane. $x + 2y + 2z = 0$ in a circle of radius 3. **(10)**

11. A plane passes through a fixed point (a, b, c) and cuts the axes at the points A, B, C respectively. Find the locus of the centre of the sphere which passes through the origin O and A, B, C. **(15)**

12. Find the equation of the sphere in xyz plane passing through the points (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3). **(12)**

13. **(i)** The plane $x + 2y + 3z = 12$ cuts the axes of coordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC.

    (ii) Prove that the plane $z = 0$ cuts the enveloping conc. of the sphere $x^2 + y^2 + z^2 = 11$ which has the vertex at (2, 4, 1) is a rectangular hyperbola.

CONE + CYLINDER

1. Find the length of the normal chord through a point P of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and prove that if it is equal to $4PG_3$ where G_3 is the point where the normal chord through P meets the xy plane, then P lies on the cone:

$$\frac{x}{a^6}(2c^2 - a^2) + \frac{y^2}{b^6}(2c^2 - b^2) + \frac{z^2}{c^4} = 0. \quad \textbf{(15)}$$

2. Find the equation of the cone with (0, 0, 1) as the vertex and $2x^2 - y^2 = 4, z = 0$ as the guiding curve. **(13)**

3. Show that the cone $3yz - 2zy - 2xy = 0$ has an infinite set of three mutually perpendicular generators.

4. If $6x = 3y = 2z$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$ then obtain the equations of the other two generators. **(13)**

5. Examine whether the plane $x + y + z = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines. **(13)**

6. Prove that the equation, $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$, represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c} = d$. **(13)**

7. A cone has for its guiding curve the circle $x^2 + y^2 + 2ax + 2by = 0, z = 0$ and passes through a fixed point (0, 0, c). If the section of the cone by the plane y = 0 is a rectangular hyperbola, prove that the vertex lies on the fixed circle $x^2 + y^2 + z^2 + 2ax + 2by = 0, 2ax + 2by + cz = 0$. **(15)**

8. A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$ **(20)**

9. Show that the cone $yz + zx + xy = 0$ cuts the sphere $x^2 + y^2 + z^2 = a^2$ in two equal circles, and find their area.

10. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. **(20)**

CONICOID + PARABOLOID

1. Prove that in general, three normals can be drawn from a given point to the paraboloid $x^2 + y^2 = 2az$, but if the point lies on the surface $27a(x^2 + y^2) + 8(a - z)^2 = 0$ then two of the three normals coincide. **[15]**

2. Find the equation of the tangent plane at point (1, 1, 1) to the conicoid $3x^2 - y^2 = 2z$. **[10]**

3. Find the locus of the point of intersection of three mutually perpendicular tangent planes to $ax^2 + by^2 + cz^2 = 1$. **[10]**

4. Two perpendicular tangent planes to the paraboloid $x^2 + y^2 = 2z$ intersect in a straight line in the plane $x = 0$. Obtain the curve to which this straight line touches. **[15]**

5. Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2 + by^2 + cz^2 = 1$, at its point of intersection with the plane $lx + my + nz = p$ generate the cone $p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$ **[15]**

6. Show that the locus of a point from which the three mutually tangent lines can be drawn to the paraboloid $x^2 + y^2 + 2z = 0$ is $x^2 + y^2 + 4z = 1$. **[20]**

. Three points P, Q, R are taken on the elliploid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere. **[20]**

8. Show that the plane $3x+4y+7z+\frac{5}{2}=0$ touches the paraboloid $3x^2+4y^2=10z$ and find the point of contact.

9. Prove that the normals from the points (α, β, γ) to the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ lies on the cone $\frac{\alpha}{x-\alpha}+\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0$.

10. Show that the enveloping cylinders of the ellipsoid $ax^2+by^2+cz^2=1$ with generators perpendicular to z-axis meet the plane $z = 0$ in parabolas.

GENERATING LINES

1. Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point (1, 1, 1). **[13]**

2. Find the equations of the two generating lines through any point $(a \cos\theta, b \sin\theta, 0)$ of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0,$ of the hyperboloid by the plane $z = 0$.

3. A variable generator meets two generators of the system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ in P and P'. Prove that $BP\,.\,B'P'=a^2+c^2$. **[20]**

4. Show that the generators through any one of the ends of an equi-conjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2 + b^2 = 6c^2$. Find also the condition for the generators to be perpendicular to each other. **[20]**

REDUCTION OF GENERAL 2nd DEGREE EQUATION

1. Reduce the following equation to the standard form and hence determine the nature of the conicoid:

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$$

ASSIGNMENT
ANALYTIC GEOMETRY

BASICS

1. Find the locus of a point which moves so that sum of its distances from the points $(a, 0, 0)$ and $(-a, 0, 0)$ is constant.

Ans. $x^2\left(1-\frac{a^2}{k^2}\right)+y^2+z^2=k^2-a^2$

2. Prove that the four points whose coordinates are $(5, -1, 1)$, $(7, -4, 7)$, $(1, -6, 10)$, $(-1, -3, 4)$ are the vertices of a rhombus.

3. Find the distance of the point whose spherical polar coordinates are $\left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{6}\right)$ from the point whose Cartesian coordinates are $\left(2\sqrt{3}, -1, -4\right)$.

Ans. $\sqrt{43}$

4. Show that the points $A(1, 2, 3)$, $B(4, 0, 4)$ and $C(-2, 4, 2)$ are collinear.

5. Find the ratio in which the coordinate plane divide the line joining the points $(-2, 4, 7)$, $(3, -5, 8)$.

Ans. xy plane: $-7 : 8$; yz plane: $2 : 3$; zx plane: $4 : 5$

6. Find the direction cosine of a line that makes equal angles with the axes.

Ans. $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

7. If A, B, C, D are the points $(3, 4, 5)$, $(4, 6, 3)$, $(-1, 2, 4)$ and $(1, 0, 5)$, find the projection of CD on AB.

Ans. $4/3$

8. Show that the straight line whose dc's are given by the equation: $ul + vm + wn = 0$, $al^2 + bm^2 + cn^2 = 0$ are

(i) perpendicular if $u^2(b + c) + v^2(c + a) + w^2(a + b) = 0$

(ii) parallel if $\left(\frac{u^2}{a}\right)+\left(\frac{v^2}{b}\right)+\left(\frac{w^2}{c}\right)=0$

9. Prove that the straight line whose direction cosines are given by relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel if $\sqrt{af} \pm\sqrt{bg}+\pm\sqrt{ch}=0$.

10. If l_1, m_1, n_1 and l_2, m_2, n_2 are the dc's of two lines, then the direction ratio of another which is perpendicular to both the given lines are $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1m_2 - l_2m_1)$. Prove further if the given line are at right anlges to each other then there dr's are the actual dc's.

11. A line makes angles α, β, γ and δ with the four diagonals of a cube. Prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$.

12. If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair is also perpendicular.

13. Prove that three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are coplanar if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0 \cdot$$

14. A plane makes intercepts OA, OB, OC whose measures are a, b, c on the axes OX, OY, OZ. Find the area of the triangle ABC.

Ans. $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$

15. (l_1, m_1, n_1) and (l_2, m_2, n_2) are the dc's of two concurrent lines, show that the dc's of two lines bisecting the angles between them are proportional to $(l_1 \pm l_2)$, $(m_1 \pm m_2)$, $(n_1 \pm n_2)$.

16. The direction cosines of a variable line in two adjacent positions are l, m, n; $l + \delta l$, $m + \delta m$, $n + \delta n$. Show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

PLANE

1. A variable plane moves such that the sum of reciprocals of its intercepts on the three coordinate axes is constant. Show that it passes through a fixed point.

2. A plane meets the coordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r). Show that the equation of the plane is $\left(\frac{x}{p}\right) + \left(\frac{y}{q}\right) + \left(\frac{z}{r}\right) = 3$.

3. A plane makes intercepts –6, 3, 4 upon the coordinate axes. What is the length of perpendicular from the origin on it.

Ans. $\frac{12}{\sqrt{29}}$

4. Show that the four points (0, –1, 0), (2, 1, –1), (1, 1, 1) and (3, 3, 0) are coplanar.

5. Find the equation of the plane passing through the lines of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$.

Ans. $28x - 17y + 9z = 0$

6. Find the equation of the plane perpendicular to yz plane and passing through the points (1, –2, 4) and (3, –4, 5).

Ans. $y + 2z - 6 = 0$

7. Find the equation of the plane which passes through the point (–1, 3, 2) and is perpendicular to each of the two planes $x + 2y - 2z = 5$ and $3x + 3y + 2z = 8$.

Ans. $2x - 4y + 3z + 8$

8. Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$

Ans. 1/6

9. The sum of the distances of any number of fixed points from a plane is zero. Show that the plane always passes through a fixed point.

10. Show that the plane $14x - 8y + 13 = 0$ bisects the obtuse angle between planes $3x + 4y - 5z + 1 = 0$ and $5x + 12y - 13z = 0$.

11. The plane $lx + my = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Prove that the equation of the plane in its new position is $lx + my \pm z\sqrt{l^2 + m^2} \tan\alpha = 0$.

12. Prove that $\frac{3}{y-z} + \frac{4}{z-x} + \frac{5}{x-y} = 0$ represents a pair of planes.

13. Through a point $P(\alpha, \beta, \gamma)$ a plane is drawn at right angles to OP to meet the axes in A, B, C. Prove that the area of the triangle ABC is $\frac{p^5}{(2\alpha\beta\gamma)}$ where $OP = p$.

14. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron $OABC$ is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

15. A triangle, the length of whose sides are a, b and c is places so that the middle points of the sides are on the axes. Show that the lengths α, β, γ intercepted on the axes are given by $8\alpha^2 = b^2 + c^2 - a^2$, $8\beta^2 = c^2 + a^2 - b^2$, $8\gamma^2 = a^2 + b^2 - c^2$. Find the coordinates of its vertices.

Ans. $(-\alpha, \beta, \gamma)$, $(\alpha, -\beta, \gamma)$, $(\alpha, \beta, -\gamma)$

STRAIGHT LINE-I

1. Find the ratio in which the join of (2, 3, 1) and (–2, 1, –3) is cut by the plane $x - 2y + 3z + 4 = 0$. Find also the coordinates of the point of intersection.

2. Find the image of the point $P(3, 5, 7)$ in the plane $2x + y + z = 6$.

Ans. (–1, 3, 5)

3. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{-z}{6}$

Ans. 1

4. Find the equation of the line through (α, β, γ) at right angles to the lines $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$ and $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$

Ans. $\frac{x-\alpha}{m_1 n_2 - m_2 n_1} = \frac{y-\beta}{n_1 l_2 - n_2 l_1} = \frac{z-\gamma}{l_1 m_2 - l_2 m_1}$

5. Find the incentre of the tetrahedron formed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$.

6. P is a point on the plane $lx + my + nz = p$. A point Q is taken on the line OP such that $OP.OQ = p^2$, prove that the locus of Q is $p(lx + my + nz) = x^2 + y^2 + z^2$.

7. A variable plane makes intercepts on the coordinate axes, the sum of whose squares is constant and equal to k^2. Show that the locus of the foot of the perpendicular from the origin to the plane is $(x^{-2} + y^{-2} + z^{-2})(x^2 + y^2 + z^2)^2 = k^2$.

8. Find the equation of the line through the points (a, b, c) and (a', b', c') and prove that it passes through the origin, if $aa' + bb' + cc' = rr'$, where r and r' are the distances of these points from the origin.

9. Find the symmetric form of the equation of line given by $x = ay + b$, $z = cy + d$.

Ans. $\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$

10. Find the symmetric form of the line $3x + 2y + z = 5$, $x + y - 2z = 3$.

Ans. $\frac{x+1}{-5} = \frac{y-4}{7} = \frac{z}{1}$

11. Find the equation to the plane through the points (2, –1, 0), (3, –4, 5) parallel to the line $2x = 3y = 4z$.

Ans. $29x - 27y - 22z - 85 = 0$

12. Find the equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$, $a'x + b'y + c'z + d' = 0$ and parallel to x-axis.

Ans. $(ba' - ab')y + (ca' - c'a)z + (da' - d'a) = 0$

13. Prove that the plane through the point (α, β, γ) and the line $x = py + q = rz + s$ is given by $\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0$

14. Prove that the equation of the two planes inclined at an angle α to the xy plane and containing the line y = 0, z $\cos\beta = x\sin\beta$ is $(x^2+y^2)\tan^2\beta + z^2 - 2zx\tan\beta = y^2\tan^2\alpha$.

15. Find the equation of a system of planes perpendicular to the line with direction ratios, a, b, c.

Ans. $ax + by + cz + k = 0$

16. Find the foot and hence length of the perpendicular from (5, 7, 3) to the line $\frac{1}{3}(x-15) = \frac{1}{8}(y-29) = -\frac{1}{5}(z-5)$. Find also the equation of the plane in which the perpendicular and the given straight line lie.

17. Find the equations of the perpendicular from the origin to the line $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$.

18. Find the distance of the point (3, 8, 2) from the line $\frac{1}{2}(x-1) = \frac{1}{4}(y-3) = \frac{1}{3}(z-2)$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$.

Ans. 7

19. Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Also find their point of intersection.

Ans. (–1, –1, –1)

20. Show that the lines $x + y + z - 3 = 0 = 2x + 3y + 4z - 5$ and $4x - y + 5z - 7 = 0 = 2x - 5y - z - 3$ are coplanar and find the plane in which they lie.

Ans. $x + 2y + 3z = 2$

21. Find the equation of the plane through the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and perpendicular to the plane containing the lines $\frac{x}{m} = \frac{y}{n} = \frac{z}{l}$ and $\frac{x}{n} = \frac{y}{l} = \frac{z}{m}$.

Ans. $(m - n)x + (n - l)y + (l - m)z = 0$

STRAIGHT LINE-II

1. Find the equation of the line which intersect the lines $2x + y - 4 = 0 = y + 2z$ and $x + 3z = 4$, $2x + 5z = 8$ and passes through the point $(2, -1, 1)$.

Ans. $x + y + z = 2$, $x + 2z = 4$

2. A line with DR's $(7, -5, 2)$ is drawn to intersect the line $\frac{x-7}{-1}=\frac{y+2}{1}=\frac{z-5}{3}$, $\frac{x-3}{2}=\frac{y-5}{4}=\frac{z+3}{-3}$. Find the coordinates of the points of intersection and the length intercepted on it.

3. Show that the equation of the line through (a, b, c) which is parallel to the plane $lx + my + nz = 0$ and intersects the line $A_1x + B_1y + C_1z + D_1 = 0 = A_2x + B_2y + C_2z + D_2$ is

$$l(x-a)+m(y-b)+n(z-c)=0,\ \frac{A_1x+B_1y+C_1z+D_1}{A_1a+B_1b+C_1c+D_1}=\frac{A_2x+B_2y+C_2z+D_2}{A_2a+B_2b+C_2c+D_2}$$

4. Show that the equation of the straight line through the origin cutting each of the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \text{ is } \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix}=\begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

INTERSECTION OF THREE PLANES

5. Examine the nature of intersection of planes: $2x - y + z = 4$, $5x + 7y + 2z = 0$, $3x + 4y - 2z + 3 = 0$

Ans. Point $(1, -1, 1)$

6. Show that the planes $2x + 4y + 2z = 7$, $5x + y - z = 9$; $x - y - z = 6$ form a triangular prism.

7. Prove that the planes $2x - 3y - 7z = 0$, $2x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through one line and find its equation.

8. Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ pass through one line if $a^2 + b^2 + c^2 + 2abc = 1$ and find its equations.

Ans. $\frac{x}{\sqrt{1-a^2}}=\frac{y}{\sqrt{1-b^2}}=\frac{z}{\sqrt{1-c^2}}$

9. For what value of λ do the planes $x-y+z+1=0,\ \lambda x+3y+2z-3=0,\ 3x+\lambda y+z-2=0$

(i) Intersect in a point

(ii) Intersect along a line

(iii) Form a triangular prism

Ans. (i) Point: $\lambda \neq 4, \lambda \neq -3$; Line: $\lambda = -3$; (iii) $\lambda = 4$

10. How far is the point $(4, 1, 1)$ from the line of intersection of $x + y + z - 4 = 0 = x - 2y - z - 4$.

Ans. $\frac{3}{14}\sqrt{42}$

11. Find the equation of the two planes through the origin which are parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-1}{-2}$ are distance $\frac{5}{3}$ from it.

Ans. $x - 2y + 2z = 0$ and $2x + 2y + z = 0$

12. Find the length and equations of the perpendicular from the origin to the line $x+2y+3z+4=0=2x+3y+4z+5$. Also find the coordinates of the foot of the perpendicular.

Ans. $\frac{\sqrt{21}}{3},\left(\frac{2}{3},\frac{-1}{3},\frac{-4}{3}\right),\frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}$

13. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction cosines proportional to (2, 3, 6).

Ans. $196 = (3x + 2y - 7)^2 + 9(2y - z - 7)^2 + 4(3x - z)^2$

SKEW LINES

14. Find the SD between the lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-2}{1}$ and $\frac{x-1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$.

Ans. $\frac{34}{\sqrt{29}}$

15. Find the length and the equation common perpendicular to the two lines $\frac{x+3}{-4}=\frac{y-1}{3}=\frac{z}{2}$ and $\frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$.

Ans. $9,\begin{bmatrix}32x+34y+13z-108=0\\4x+11y+5z-27=0\end{bmatrix}$

16. Find the SD between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. Find also its equation and the points where it meets the given lines.

Ans. $3\sqrt{30},\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{1}$

17. Show that the SD between any two opposite edges of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$, $x + y + z = a$ is $\frac{2a}{\sqrt{6}}$ and the three lines of SD intersect at the point $x = y = z = -a$.

18. Two straight lines $\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1},\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ are cut by a third line whose dc's are λ,μ,ν.

Show that 'd' the length intercepted on the third line is given by $d\begin{vmatrix}l_1 & m_1 & n_1\\l_2 & m_2 & n_2\\\lambda & \mu & \nu\end{vmatrix}=\begin{vmatrix}\alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2\\l_1 & m_1 & n_1\\l_2 & m_2 & n_2\end{vmatrix}$. Deduce the length of SD between the first two lines.

Ans. $SD=d=\dfrac{\begin{vmatrix}\alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2\\l_1 & m_1 & n_1\\l_2 & m_2 & n_2\end{vmatrix}}{\sqrt{\left(\Sigma\left(m_1n_2-m_2n_1\right)^2\right)}}$

STRAIGHT LINE

1. Prove that the locus of a variable line which intersect the three given lines $y = mx$, $z = c$; $y = -mx$, $z = -c$; $y = z$, $mx = -c$ is the surface $y^2 - m^2x^2 = z^2 - c^2$.

2. Find the surface generated by a line which intersects the line $y = a = z$ and $x + 3z = a = y + z$ and is parallel to the plane $x + y = 0$.

3. Find the surface generated by a straight line which intersects the line $x + y = 0 = z$, $x - y - z = 0 = x + y - 2a$ and the parabola $y = 0 = x^2 - 2az$.

4. Prove that the locus of a line which meets the lines $y = \pm mx$, $z = \pm c$ and the circle $x^2 + y^2 = a^2$, $z = 0$ is

   $c^2m^2(cy - mxz)^2 + c^2(yz - cmx)^2 = a^2m^2(z^2 - c^2)^2$

5. A straight line is drawn through a variable point on the ellipse $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)=1,\ z=0$ to meet two fixed line $y = mx$, $z = c$ and $y = -mx$, $z = -c$. Find the locus of the straight line.

Ans. $(cmx - yz)^2c^2b^2 + (mxz - cy)^2c^2a^2m^2 = a^2b^2m^2(z^2 - c^2)^2$

SPHERE

1. Find the centre and the radius of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$.

Ans. (1, –2, 3), radius = $\sqrt{3}$

2. Find the equation of the sphere which passes through (a, 0, 0), (0, b, 0), (0, 0, c) and (0, 0, 0).

Ans. $x^2 + y^2 + z^2 - ax - by - cz = 0$

3. Obtain the equation of sphere having its centre on the line 5x + 2z = 0 = 2x – 3y and passing through the points (0, –2, –4) and (2, –1, –1).

Ans. $x^2 + y^2 + z^2 - 6x - 4y - 10z + 12 = 0$

4. A sphere of radius 'k' passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

5. A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C, show that the locus of the centre of the sphere OABC is

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$$

6. Find the equation of the sphere that passes through the points (4, 1, 0), (2, –3, 4), (1, 0, 0) and touches the plane $2x + 2y - z = 11$.

Ans. $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$

7. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of the tetrahedron $OABC$.

Ans. $x^2 + y^2 + z^2 = k^2$

8. Find the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) and has its radius as small as possible.

9. OA, OB, OC are three mutually perpendicular lines through the origin having direction cosines l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3. If $OA = a$, $OB = b$, $OC = c$. Find the equation of sphere $OABC$.

(10) Find the radius and centre of the circle $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$, $x - 2y + 2z = 3$

Ans. $4\sqrt{5}, \left(\frac{13}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$

11. Find the equation of the sphere whose centre is the point (1, 2, 3) and which touches the plane $3x + 2y + z + 4 = 0$. Find also the radius of the circle in which the sphere is cut by the plane $x + y + z = 0$.

Ans. $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$

12. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + y - 2z + 4 = 0$ and the origin.

Ans. $4x^2 + 4y^2 + 4z^2 + 9x + 9y - 18z = 0$

13. Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z + 2 = 0$ in a circle of radius unity and find the equations of the sphere which has this circle for one of its great circles.

Ans. $x^2 + y^2 + z^2 - 2x - 2y + 2 = 0$

14. Prove that the circle $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lies on the same sphere and find its equation. Also find the value of 'a' for which $x + y + z = a\sqrt{3}$ touches the sphere.

Ans. $a = \sqrt{3} \pm 3$

15. Find the equations of the sphere which pass through circle $x^2 + y^2 + z^2 = 5$, $x + 2y + 3z = 3$ and touch the plane $4x + 3y = 15$.

Ans. $x^2 + y^2 + z^2 - \frac{4}{5}x - \frac{8}{5}y - \frac{12}{5}z - \frac{13}{5} = 0$

16. P is the variable point on the given line and A, B, C are its projections on the axes. Show that the sphere O, ABC passes through a fixed circle.

Ans. $x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0$, $lx + my + nz = 0$.

17. A variable is parallel to the given plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$

18. Find the equation of the sphere which passes through the point (α, β, γ) and the circle $x^2 + y^2 = a^2$, $z = 0$.

19. Find the plane, the centre and the radius of the circle common to the two spheres $x^2 + y^2 + z^2 - 4z + 1 = 0$ and $x^2 + y^2 + z^2 - 4x - 2y - 1 = 0$

Ans. $2x + y - 2z + 1 = 0, \left(\frac{2}{3}, \frac{1}{3}, \frac{4}{3}\right), \frac{1}{3}$

20. POP' is a variable diameter of the ellipse $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a circle is described in the plane $PP'\ zz'$ on PP' as diameter, prove that as PP' varies the circle generates the surface $(x^2 + y^2 + z^2)\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right] = x^2 + y^2$.

21. A sphere whose center lies in the positive octant passes through the origin and cuts the planes $x = 0$, $y = 0$, $z = 0$ in circles of radii $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$ respectively. Find the equation of this sphere.

Ans. $x^2 + y^2 + z^2 - 2x\sqrt{(b^2 + c^2 - a^2)} - 2y\sqrt{c^2 + a^2 - b^2} - 2z\sqrt{a^2 + b^2 - c^2} = 0$

22. A is point on OX and B on OY, so that the angle OAB is constant and equal to α. On AB as diameter a circle is drawn whose plane is parallel to OZ. Prove that as AB varies the circle generates the cone $2xy - z^2\sin 2\alpha = 0$.

23. Sphere are described to contain the circle $z = 0$, $x^2 + y^2 = a^2$. Prove that the locus of the extremities of their diameters which are parallel to the x-axis is the rectangular hyperbola $x^2 - z^2 = a^2$, $y = 0$.

Tangent Planes

24. Show that the plane $2x + y - z = 12$ touches the sphere $x^2 + y^2 + z^2 = 24$ and find its point of contact.

Ans. (4, 2, –2)

25. Find the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, which are parallel to the plane $2x + y - z = 0$.

Ans. $2x + y - z \pm 3\sqrt{6} = 0$

26. If three mutually perpendicular chords of lengths d_1, d_2, d_3 be drawn through the point (α, β, γ) to the sphere $x^2 + y^2 + z^2 = a^2$, prove that $d_1^2 + d_2^2 + d_3^2 = 12a^2 - 8(\alpha^2 + \beta^2 + \gamma^2)$.

27. Find the equations of the tangent line to the circle $3x^2 + 3y^2 + 3z^2 - 2x - 3y - 4z - 22 = 0$, $3x + 4y + 5z - 26 = 0$ at the point (1, 2, 3).

28. Find the equation of a sphere touching the three coordinate planes. How many such spheres can be drawn.

29. A sphere touches the three coordinate planes and passes through the point (2, 1, 5). Find its equation.

Ans. $x^2 + y^2 + z^2 - 10(x + y + z) + 30 = 0$

30. Prove that the centres of the spheres which touch the lines $y = mx$, $z = c$, $y = -mx$, $z = -c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0$

31. Find the locus of the centres of spheres of spheres of constant radius which pass through a given point and touch a given line.

Ans. $x^2 - 2az + a^2 = 0$ and $y^2 + z^2 = k^2$

32. Find the locus of the centres of spheres which pass through a given point and touch a given plane.

Ans. $x^2 + y^2 - 2az + a^2 = 0$

Touching Sphere

33. Show that the spheres $x^2 + y^2 + z^2 = 100$ and $x^2 + y^2 + z^2 - 24x - 30y - 32z + 400 = 0$ touch externally and find their point of contact.

Ans. $\left(\frac{24}{5}, 6, \frac{32}{5}\right)$

34. Show that the spheres $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch internally and find their point of contact.

Ans. $\left(\frac{48}{7}, \frac{-1}{7}, \frac{24}{7}\right)$

Pole/Polar

35. Find the pole of the plane $lx + my + nz = p$ w.r.t. the sphere $x^2 + y^2 + z^2 = a^2$.

Ans. $\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$

36. Prove that the polar plane of any point on the line $\frac{x}{2} = \frac{y-1}{3} = \frac{z+3}{4}$ with respect to the sphere $x^2 + y^2 + z^2 = 1$ passes through the line $\left(\frac{1}{13}\right)(2x+3) = \left(\frac{-1}{3}\right)(y-1) = -z$.

Angle of Intersection

37. Show that the two spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ are orthogonal. Find their plane of intersection.

Ans. $3x + y + z + 6 = 0$.

38. Two points P and Q are conjugate with respect to a sphere S; prove that the sphere on PQ as diameter cuts S orthogonally.

39. Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$

Ans. $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$.

40. Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\frac{r_1 r_2}{\sqrt{(r_1^2 + r_2^2)}}$.

41. Find the equation of a sphere which cuts the four given spheres orthogonally.

Ans. $$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ d_1 & u_1 & v_1 & w_1 & -1 \\ d_2 & u_2 & v_2 & w_2 & -1 \\ d_3 & u_3 & v_3 & w_3 & -1 \\ d_4 & u_4 & v_4 & w_4 & -1 \end{vmatrix} = 0$$

42. Find the length of the tangent drawn from the point (1, 2, 3) to the sphere

$5(x^2 + y^2 + z^2) - x + 10y + 20z + 8 = 0$

Ans. PT = $\sqrt{\frac{157}{5}}$

Coaxial System

43. Prove that every sphere that passes through the limiting points of a coaxial system cuts every sphere of that system orthogonally.

44. Find the limiting points of coaxial systems defined by the spheres

$x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$ and $x^2 + y^2 + z^2 + x + y + 2z + 2 = 0$

Ans. $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$

CONE

1. Find the equation of the cone whose vertex is at the origin and base is the circle $x = a$, $y^2 + z^2 = b^2$ and show that the section of the cone by a lane parallel to the plane X-Y is a hyperbola.

Ans. $b^2x^2 + a^2y^2 - a^2z^2 = 0$

2. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinate axes is A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

3. Planes through OX, OY include on angle α.

Show that their line of intersection lies on the cone $z^2(x^2 + y^2 + z^2) = x^2y^2\tan^2\alpha$.

4. Find the equation of the cone which passes through three coordinate axes and the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$; $\frac{x}{3} = \frac{y}{2} = \frac{z}{-1}$

Ans. $3yz + 10zx + 6xy = 0$

5. OP and OQ are two lines which remain perpendicular and move so that the plane OPQ passes through OZ. If OP describes the cone $f\left(\frac{y}{x}, \frac{z}{x}\right) = 0$, prove that OQ describes the cone $f\left\{\frac{y}{x}, \left(-\frac{x}{z} - \frac{y^2}{zx}\right)\right\} = 0$

6. Find the equation of a conc whose vertex is (α, β, γ) and base $y^2 = 4ax$, $z = 0$.

Ans. $(\beta z - y\gamma)^2 = 4a(\alpha z - x\gamma)\,(z - \gamma)$

7. A cone has as base the circle $x^2 + y^2 + 2ax + 2by = 0$, $z = 0$ and passes through the fixed point $(0, 0, c)$. If the section of the cone by zx plane is a rectangular hyperbola, prove that the vertex lies on fixed circle.

8. Prove that the equation $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$ represents a cone. Hence find its vertex.

Ans. $(-1, -2, -3)$

9. Prove that the angle between the lines given by $x + y + z = 0$, $ayz + bzx + cxy = 0$ is $\frac{\pi}{2}$ if $a + b + c = 0$ and $\frac{\pi}{3}$ if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0$.

10. If the plane $2x - y + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines, find the value of 'c'.

Ans. 2

11. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two.

Ans. $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ & $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$

12. Show that the locus of points from which three mutually perpendicular lines can be drawn to intersect a given circle $x^2 + y^2 = a^2$, $z = 0$ is a surface of revolution.

Ans. $x^2 + y^2 + 2z^2 = a^2$

13. Find the locus of points from which three mutually perpendicular lines can be drawn to intersect the conic $z = 0$, $ax^2 + by^2 = 1$

Ans. $ax^2 + by^2 + (a + b)z^2 = 1$

14. Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$. So that line joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere.

Ans. $x^2 + y^2 + z^2 = \lambda^2$

15. Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 0$ are reciprocal to each other.

16. A line OP is such that the two planes through OP each of which cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators are perpendicular, prove that the locus of OP is a cone and find it.

Ans. $(2a + b + c)x^2 + (2b + c + a)y^2\ (2c + a + b)z^2 = 0$

17. Show that the general equation to a cone which touches the coordinate planes is $a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0$

18. Prove that the tangent lines from the origin of coordinates to the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = k^2$ lie on the cone given by the equation $(a^2 + b^2 + c^2 - k^2)\ (x^2 + y^2 + z^2) = (ax + by + cz)^2$.

19. Show that the three mutually perpendicular tangent lines can be drawn to the sphere $x^2 + y^2 + z^2 = r^2$ from any point on the sphere $x^2+y^2+z^2=\frac{3}{2}r^2$.

20. Find the equation to the right circular cone whose vertex is $(2, -3, 5)$, axis makes equal angles with the coordinate axes and semi vertical angle is 30°.

Ans. $5(x^2 + y^2 + z^2) - 8(xy + yz + zx) - 4x + 86y - 58z + 278 = 0$

21. Find the equation of the cone formed by rotating the line $2x + 3y = 6$, $z = 0$ about the y axis.

CYLINDER

1. Find the equation of the cylinder with generators parallel to z-axis and passing through the curve $ax^2 + by^2 = 2cx$, $lx + my + nz = p$.

2. Find the equation of the surface generated by a straight line which is parallel to the line $y = mx$, $z = nx$ and intersect the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0.$

Ans. $b^2(nx-z)^2 + a^2(ny-mz)^2 = a^2b^2n^2$

3. Find the equation of right circular cylinder whose axis is $x = 2y = -z$ and radius is 4.

Ans. $5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy = 144$

4. Find the equation of right circular cylinder whose axis is $x - 2 = z$, $y = 0$ and passes through the point (3, 0, 0).

Ans. $x^2 + 2y^2 + z^2 - 2zx - 4x + 2z + 3 = 0$

5. Find the equation of the right circular cylinder which passes the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.

Ans. $x^2 + y^2 + z^2 + xy - xz + yz - 9 = 0$

6. Show that the equation of the right circular cylinder described on the circle through the three points $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$ as the guiding curve is $x^2 + y^2 + z^2 - yz - zx - xy = 1$.

7. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ whose generators are parallel to the line $x = y = z$.

Ans. $x^2 + y^2 + z^2 - yz - zx - xy - 4x + 5y - z - 2 = 0$

8. Show that the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to z-axis meets the plane $z = 0$ in parabolas.

Ans. $ab(mx-ly)^2 = ab^2 + bm^2$, $z = 0$

9. Find the equation of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and deduce from it the equation of the enveloping cylinder whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Ans. $\left(\sum\frac{x^2}{a^2}-1\right)\left(\sum\frac{l^2}{a^2}\right)=\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{cz}\right)^2$

10. Find the equation of the enveloping cylinder of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ whose generators are parallel to the line $x = y = z$.

Ans. $(b+c)x^2 + (c+a)y^2 + (a+b)z^2 - 2abxy - 2bcyz - 2cazx - (a+b+c) = 0$

CONICOID

1. Find the equation of the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$.

2. Tangent planes are drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$ through (α, β, γ). Show that the perpendicular from the centre to the conicoid to these planes generate the cone.

$$(\alpha x + \beta y + \gamma z)^2 = \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}$$

3. A tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the coordinate axis in the points P, Q and R. Find the locus of the centroid of the triangle PQR.

4. Find the locus of the foot of the central perpendicular on varying tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

5. If $2r$ is the distance between the parallel tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that a line through the origin perpendicular to the planes lies on the cone $x^2\left(a^2-r^2\right)+y^2\left(b^2-r^2\right)+z^2\left(c^2-r^2\right)=0$.

6. Show that the tangent planes at the extremities of any diameter of an ellipsoid are parallel.

7. Through a fixed point $(k, 0, 0)$ pairs of perpendicular lines are drawn to the conicoid $ax^2+by^2+cz^2=1$. Show that the planes through any pair touches the cone $\frac{(x-k)^2}{(b+c)\left(ak^2-1\right)}+\frac{y^2}{c\left(ak^2-1\right)-a}+\frac{z^2}{b\left(ak^2-1\right)-a}=0$.

8. Find the surface generated by straight lines drawn through a fixed point (α, β, γ) at right angles to their polar with respect to the conicoid $ax^2+by^2+cz^2=1$.

9. Find the locus of straight lines through a fixed point (α, β, γ) whose polar lines with respect to the quadrics $ax^2+by^2+cz^2=1$ and $a'x^2+b'y^2+c'z^2=1$ are coplanar.

10. Prove that the centres of sections of the $ax^2+by^2+cz^2=1$, by the planes which are at a constant distance p from the origin lie on the surface $\left(ax^2+by^2+cz^2\right)=p^2\left(a^2x^2+b^2y^2+c^2z^2\right)$

11. Show that a line joining a point P to the centre of a conicoid $ax^2+by^2+cz^2=1$ passes through the centre of the section of the conicoid by the polar plane of P.

12. Find the locus of the centres of the sections $ax^2+by^2+cz^2=1$ which touches $\alpha x^2+\beta y^2+\gamma z^2=1$.

13. Prove that the middle points of the chords of $ax^2+by^2+cz^2=1$, which are parallel to $x=0$ and touch $x^2+y^2+z^2=r^2$ lie on the surface $by^2\left(bx^2+by^2+cz^2-br^2\right)+cz^2\left(cx^2+by^2+cz^2-cr^2\right)=0$.

14. Find the length of the normal chord through P of the ellipsoid $\sum\frac{x^2}{a^2}=1$ and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord though P meets the XY plane, then P lies on the cone $\frac{x^2}{a^6}\left(2c^2-a^2\right)+\frac{y^2}{b^6}\left(2c^2-b^2\right)+\frac{z^2}{c^4}=0$.

15. The normal at a variable point P of the ellipsoid $\sum\left(\frac{x^2}{a^2}\right)=1$ meets the xy plane in G_3 and G_3Q is drawn parallel to z-axis and equal to G_3P. Prove that the locus of Q is given by $\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1$. Find the locus of R, if OR is drawn from the centre equal and parallel to G_3P.

16. Normals at P and P', points of the ellipsoid $\sum\left(\frac{x^2}{a^2}\right)=1$, meet the xy plane in G_2 and G_3 and make angles θ and θ' with PP'. Prove that $PG_3\cos\theta+P'G'_3\cos\theta'=0$.

17. Prove that the lines drawn from the origin parallel to the normal of $ax^2+by^2+cz^2=1$ at its point of intersection with the plane $lx+my+nz=p$ generate the cone.

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

18. If P, Q, R, P', Q', R' are the feet of the six normals from a point to the ellipsoid $\sum \frac{x^2}{a^2} = 1$, and the plane PQR is given by $lx + my + nz = p$, prove that the plane $P'Q'R'$ is given by $\frac{x}{a^2 l} + \frac{y}{b^2 m} + \frac{z}{c^2 n} = \frac{1}{p} = 0$.

19. If OP, OQ and OR be the conjugate semi-diameters of the ellipsoid $\sum \frac{x^2}{a^2} = 1$ and P, Q, R be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively, then

(i) Find the equation of the plane PQR.

(ii) Prove that if the plane $lx + my + nz = p$, passes through the points P, Q, R then $a^2l^2 + b^2m^2 + c^2n^2 = 3p^2$.

(iii) Prove that the pole of the plane PQR lies on the ellipsoid.

20. If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid $\sum \frac{x^2}{a^2} = 1$.

21. Prove that the locus of the section of the ellipsoid $\sum \frac{x^2}{a^2} = 1$ by the plane PQR is the ellipsoid $\sum \frac{x^2}{a^2} = \frac{1}{3}$.

22. Find locus of the asymptotic line drawn from the origin to the conicoid $ax^2 + by^2 + cz^2 = 1$.

PARABOLOID

1. Show that the plane $8x - 6y - z = 5$ touches the paraboloid $\left(\frac{x^2}{2}\right) - \left(\frac{y^2}{3}\right) = z$, and find the point of contact.

Ans. (8, 9, 5)

2. Find the condition that $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 2\left(\frac{z}{c_1}\right)$, $\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = 2\left(\frac{z}{c_2}\right)$; $\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} = \frac{2z}{c_3}$ have a common tangent plane.

Ans. $\begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix} = 0$

3. Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ intersect in a line lying on the plane $x = 0$. Prove that the line touches the parabola $x = 0$, $y^2 = (a + b)(2z + a)$.

4. Find the equation of the plane which cuts the paraboloid $x^2 - 2y^2 = 3z$ in the conic with centre (1, 2, 3).

Ans. $2x - 8y - 3z + 23 = 0$

5. Show that the feet of the normals from the point (α, β, γ) on the paraboloid $x^2 + y^2 = 2az$ lie on a sphere.

Ans. $x^2 + y^2 + z^2 - (\gamma + a)z - \left\{\frac{(\alpha^2 + \beta^2)}{2\beta}\right\} y = 0$

6. Prove that the equations of the chord through the point (1, 2, 3) which is bisected by the diametral plane $10x - 24y = 21$ of the paraboloid $5x^2 - 6y^2 = 7z$ are $(x - 1) = \frac{1}{2}(y - 2) = \frac{1}{3}(z - 3)$.

7. Find the locus of the point from which three mutually perpendicular tangents can be drawn to the paraboloid.

GENERATING LINES

1. Find the equations of the generators of the hyperboloid $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right)=1$ which pass through the point $(a\cos\theta, b\sin\theta, 0)$.

Ans. $\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}$ (learn this result)

2. CP, CQ are any two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=c$, CP', CQ' are the conjugate diameters of the ellipse $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)=1,\ z=-c,$ drawn in the same directions as CP and CQ. Prove that the hyperboloid $\left(\frac{2x^2}{a}\right)+\left(\frac{2y^2}{b^2}\right)-\frac{z^2}{c^2}=1$ is generated by either PQ' or $P'Q$.

3. Prove that in general two generators of the hyperboloids $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ can be drawn to cut a given generator at right angles.

4. Find the locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet.

Ans. $x^2+y^2+z^2=a^2+b^2-c^2$

5. If A and A' are the extremities of the major axis of the principal elliptic section and any generator meets the two generators of the same system through A and A' in P and P' respectively, then prove that $AP \,.\, A'P' = b^2 + c^2$.

6. Show that the equations $y-\lambda z+\lambda+1=0,\ (\lambda+1)x+y+\lambda=0$ represent for different values of λ generators of one system of the hyperboloid $yz+zx+xy+1=0$ and find the equations to the generators of the other system.

7. Find the locus of the point of intersection of perpendicular generators of the hyperbolic paraboloid.

Ans. $\left[a^2-b^2+2z=0\right]$

8. Planes are drawn through the origin O and the generators through any point P of the paraboloid $x^2 - y^2 = az$. Prove that the angle between them is $\tan^{-1}\left(\frac{2r}{a}\right)$, where 'r' is the length of 'OP'.

9. Find the vertices of the skew quadrilateral formed by the four generators of the hyperboloid $\frac{x^2}{4}+y^2-z^2=49$ passing through (10, 5, 1) and (14, 2, – 2).

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