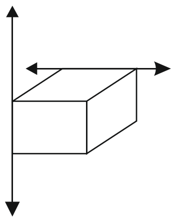
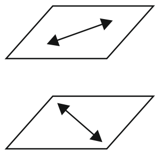
SKEW LINES

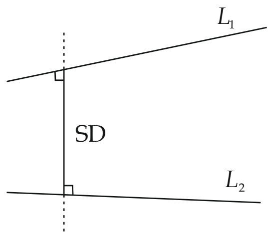
Two lines are said to be skew lines if they do not lie in the same plane.

$$\left.\begin{aligned} \frac{x}{l} &= \frac{y}{m} = \frac{z-c}{0} \\ \frac{x}{l} &= \frac{y}{-m} = \frac{z+c}{0} \end{aligned}\right\}$$





Shortest Distance Line:



The straight line which is perpendicular to each of these two non intersecting lines is called line of shortest distance.

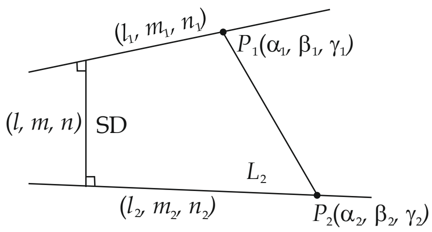
The length of intercept is called the length of shortest distance

Lengths and Equations of SD:

There are (4) ways to find → Depends on forms of the given lines and the question asked.

Method I:

L_1 & L_2 are in point form



Let $L_1 = \dfrac{x-\alpha_1}{l_1} = \dfrac{y-\beta_1}{m_1} = \dfrac{z-\gamma_1}{n_1}$ & $L_2 : \dfrac{x-\alpha_2}{l_2} = \dfrac{y-\beta_2}{m_2} = \dfrac{z-\gamma_2}{n_2}$

SD :

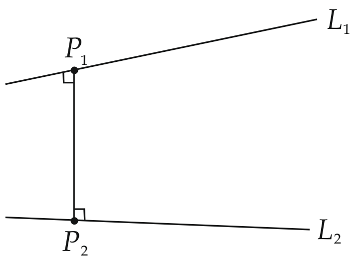
Equation of SD: Will be in plane form

Method II :

Point form

P_1 is general point on L_1

P_2 is general point on L_2



$\boxed{P_1P_2 \perp L_1 \ \& \ L_2}$

$\downarrow$

get P_1 and P_2

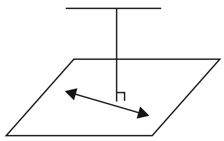
$\downarrow$

Equation is easily known as we know the two points.

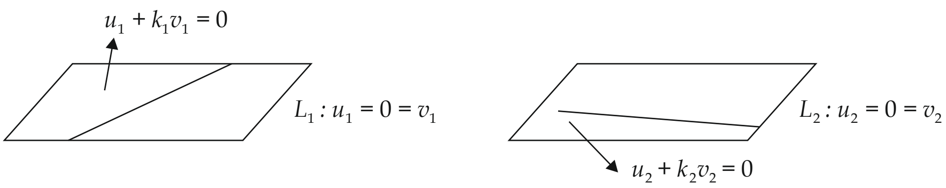
Method III:

L_1 : Point form or symmetric form

L_2 : General form



Method IV:



L_1 & L_2 : both are in general form.

Q. Find the SD between the lines:

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-2}{1} \quad \ldots\text{(i)}$$

$$\frac{x-1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Ans. $\frac{34}{\sqrt{29}}$

Q. Find the SD between the 'z' axis and the line $x + y + 2z = 3$, $2x + 3y + 4z = 4$.

Ans. 2

Locus of a line intersecting two given lines and satisfying one more condition:

Let the equation of given lines

$L_1 : u_1 = 0 = v_1$

$L_2 : u_2 = 0 = v_2$ ...(i)

We know that equation of any line intersecting L_1 & L_2 is given by $\boxed{u_1 + \lambda_1 v_1 = 0,\ u_2 + \lambda_2 v_2 = 0}$

→ Use the given one more condition to find the relation between λ_1 and λ_2.

→ Eliminate λ_1 and λ_2 by using (1)

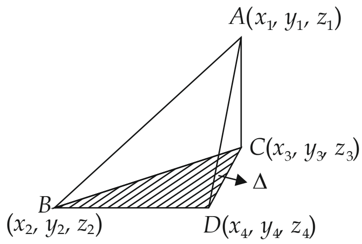
→ The relation thus obtained is the required locus.

Q. A, B are variable points on two given non intersecting lines and AB is of constant length $2k$. Find the surface generated by AB.

Ans. $c^2(mzx - cy)^2 + c^2m^2(yz - mcx)^2 = m^2(z^2 - c^2)^2 (k^2 - c^2)$

TETRAHEDRON

Total 4 planes:



$\rightarrow$ ABC $\rightarrow$ ACD

$\rightarrow$ ABD $\rightarrow$ BCD (Base)

A: Vertex

Basic formula of volume of Tetrahedron:

$$V = \frac{1}{3}p\Delta$$

p :

Δ :

$$\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$$

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} ; \Delta_y = \frac{1}{2}\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} ; \Delta_z = \frac{1}{2}\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$$

Equation of plane BCD ~ $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix} = 0$

Result V = $\frac{1}{6}\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}$

If one vertex at origin : V = $\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Q. *A, B, C* are (3, 2, 1), (–2, 0, –3), (0, 0, –2). Find the locus of P if the volume of the tetrahedron PABC is 5.

Ans. $2x + 3y - 4z = 38$

THE SPHERE

Locus of a point which moves such that its distance (*radius*) from a fixed point (*centre*) is always constant.

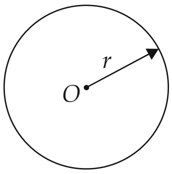
Equations

Central form

$(x-a)^2 + (y-b)^2 + (z-c)^2 = \lambda^2$

Centre = (a, b, c)

Radius = r



General form

$$\boxed{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$$

Centre = $(-u, -v, -w)$

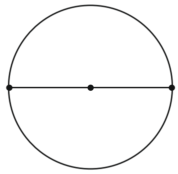
Radius = $\sqrt{u^2 + v^2 + w^2 - d}$

Four unknowns

Notice: no terms of xy, yz, zx

Diametric form

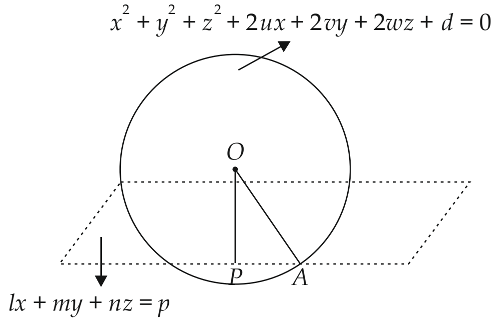
$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

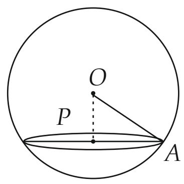


Q. Equation of sphere which passes through (0, 0, 0) and which has its centre at $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Ans. $x^2 + y^2 + z^2 - x - y = 0$

Plane Section of a Sphere





Q. (i) Foot of perpendicular.

(ii) Distance of plane.

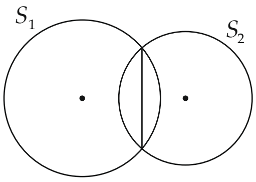
(iii) Radius of circle, equation of circle.

Q. Find the radius and centre of the circle.

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0, x - 2y + 2z = 3.$

Ans. $\left(\frac{13}{8}, \frac{-8}{3}, \frac{-10}{3}\right), 4\sqrt{5}$

Intersection of Two Spheres



$S_1 = 0, S_2 = 0$

$S_1 - S_2 = 0$? (Linear)

$\boxed{\text{Circle}}$ $S_1 = 0$ or $S_2 = 0$ and $S_1 - S_2 = 0$

Sphere through a given circle

Sphere, $S = 0 \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$P \equiv lx + my + nz - p = 0$

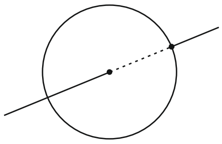
Required sphere : $S + \lambda P = 0$ (why?)

On the similar logic, sphere through circle of intersection of two spheres $S_1 = 0 = S_2$ will be $S_1 + \lambda S_2 = 0$

Q. Find the equations of the circle lying on the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$ and having centre at (2, 3, –4).

Ans. $S = 0, x + 5y - 7z - 45 = 0$

Intersection of a straight line and a sphere



General point on the line $\rightarrow$ put it on the sphere (r_1 and r_2)

Equation of the Tangent Plane

Let the equation of sphere

$S = x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0$

Point = (α, β, γ)

Tangent plane: $x\alpha + y\beta + z\gamma + u(\mathrm{x} + \mathrm{a}) + v(y + \beta) + w(z + \gamma) + \mathrm{d} = 0$

Proof/Derivation

What is a tagent plane?

General Rule (x_1, y_1, z_1)

$x^2 \to xx_1$

$y^2 \to yy_1$

$z^2 \to zz_1$

$xy \to$

$yz \to$

$zx \to$

$x \to (x + x_1)/2$

$y \to (y + y_1)/2$

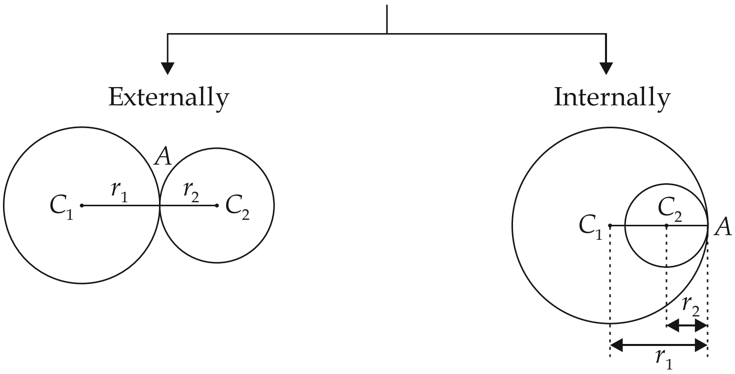
$z \to (z + z_1)/2$

Note: (1) Tangent plane is perpendicular to the radius through that point.

Q. Find the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, which are parallel to the plane $2x + y - z = 0$.

Ans. $2x + y - z \pm 3\sqrt{6} = 0$

TOUCHING SPHERES

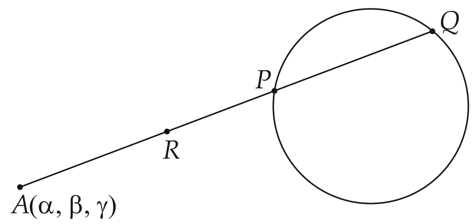


Q. Show that the spheres $x^2 + y^2 + z^2 - 2x - 3 = 0$ and $x^2 + y^2 + z^2 + 6x + 6y + 9 = 0$ touch externally.

Polar Plane

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Line: $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$



$$\boxed{\frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AR}}$$

Locus of R is the polar of $A(\alpha, \beta, \gamma)$ w.r.t. the sphere.

Equation : $x\alpha + y\beta + z\gamma + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0$

Observation:

Properties of Pole and Polar

(1) Let P and Q be two points, O is centre of sphere.

$$\frac{\text{Distance of } P \text{ from the polar plane of } Q}{\text{Distance of } Q \text{ from the polar plane of } P} = \frac{OP}{OQ}$$

(2) **Conjugate points :** (P, Q)

$$\text{Conjugate planes} \begin{cases} \text{Polar Plane of } P \rightarrow \text{Passes through point } Q \\ \text{Polar Plane of } Q \rightarrow \text{Passes through point } P \end{cases}$$

(3) Polar plane of a point w.r.t. a sphere is perpendicular to the line joining the point to the centre of the sphere.

(4) OP.OQ = $(\text{radius})^2$, where Q is POI of line OP with polar plane of P.

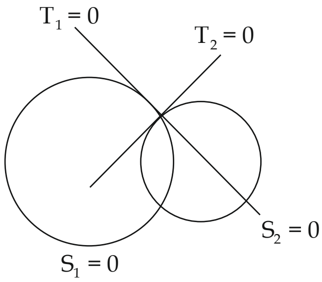
(5) **Polar Lines:** If polar of any point on any one passes through the other line, such lines are polar lines.

Q. Find the pole of the plane $lx + my + nz = p$ w.r.t. sphere $x^2 + y^2 + z^2 = a^2$.

Ans. $\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$

Angle of Intersection of Two Spheres

→ This is the angle between the tangent planes to them at their point of intersection. i.e. Angle between T_1 and T_2.



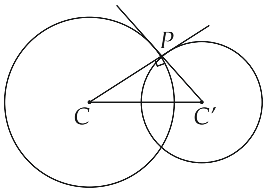
Condition of orthogonality

$$\boxed{uu' + vv' + ww' = \frac{d+d'}{2}}$$

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

$$[CC']^2 = [CP]^2 + [C'P]^2$$



Q. Two points P and Q are conjugate with respect to a sphere S; Prove that the sphere on PQ as diameter cuts S orthogonally.

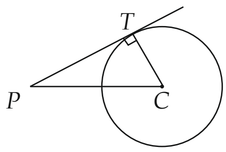
Length of Tagent (PT): P is an outside point.

$PT = \sqrt{S_1}$

(Power of point P w.r.t. sphere $S = 0$)

Proof: $PT^2 = -CT^2 + PC^2$

$= (PC)^2 - r^2$



Radical Plane (2 Spheres)

It is the locus of a point from where the square of the lengths of the tangents to the two spheres are equal.

$\boxed{S_1 - S_2 = 0}$ How?

Radical axis : 3 spheres $S_1 = S_2 = S_3$

Radical centre : 4 spheres $S_1 = S_2 = S_3 = S_4$

Coaxial System Family of Spheres

Any two spheres of this system have the same radical plane.

e.g., (i) $s_1 + \lambda s_2 = 0 \rightarrow$ Radical plane $s_1 - s_2 = 0$

(ii) $s + \lambda P = 0 \rightarrow$ Radical plane $P = 0$

(iii) $x^2 + y^2 + z^2 + 2\lambda x + d = 0 \rightarrow$ Radical plane of this system is $x = 0$.

Limiting Points : Centre of sphere of zero radii of a coaxial system of sphere.

e.q. Consider the system $x^2 + y^2 + z^2 + 2\lambda x + d = 0$

Centre = $(-\lambda, 0, 0)$

radius = $\sqrt{\lambda^2 - d}$

$\therefore$ If the radius = $0 \Rightarrow \lambda = \pm\sqrt{d}$

$\therefore$ Limiting points of the system will be $\left(\pm\sqrt{d}, 0, 0\right)$

Q. Prove that every sphere that passes through the limiting points of a coaxial system cuts every sphere of that system orthogonally.

Q. Find the equation of the radical axis in the symmetric form of the spheres

$S_1 \equiv x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$

$S_2 \equiv x^2 + y^2 + z^2 + 4x + 4z + 4 = 0$

$S_3 \equiv x^2 + y^2 + z^2 + x + +6y - 4z - 2 = 0$

Ans. $\frac{x}{2} = \frac{y-1}{5} = \frac{z-0}{3}$

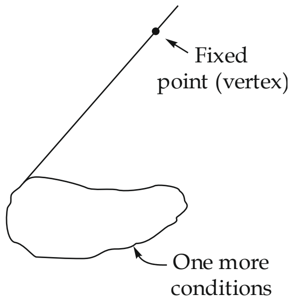
Q. Find the limiting points of the coaxial systems defined by the spheres
$x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$ and $x^2 + y^2 + z^2 + x + y + 2z + 2 = 0$

Ans. $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}\right)$

THE CONE

A cone is a surface generated by straight line passing through a fixed point and satisfying one more condition.

Quadratic Cone: Equation is of second degree.



Cone with Vertex at Origin

Homogenous second degree equation in x, y, z

i.e., $$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

Proof: Let $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ be the cone whose vertex is origin 'O'

Let $P(x_1, y_1, z_1)$ be any point on this cone.

$\therefore$ Equation of generator OP:

$$\frac{x-0}{x_1} = \frac{y-0}{y_1} = \frac{z-0}{z_1}$$

General point on this line: (rx_1, ry_1, rz_1)

It will be on the equation of the cone for every value of 'r'

$$r^2 (ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) + 2r(ux_1 + vy_1 + wz_1) + d = 0 \equiv \text{identity}$$

$$\Rightarrow \quad ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \quad \text{...(i)}$$

$$ux_1 + vy_1 + wz_1 = 0 \quad \text{...(ii)}$$

$$d = 0 \quad \text{...(iii)}$$

(recall that x_1, y_1, z_1 was a general point)

⇒ Every homogeneous equation of second degree in x, y, z represents a cone with its vertex at the origin.

⇒ If the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of the cone given by $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, then its direction cosines viz. l, m, n, satisfy the equation of the cone.

⇒ General equation of a cone of second degree which passes through the coordinate axes.

$$fyz + gzx + hxy = 0$$

Q. Find the equation of the cone with vertex at the origin and which passes through the curve $ax^2 + by^2 + cz^2 = 1$; $\alpha x^2 + \beta y^2 = 2z$.

Let the equations of the curve be written in the homogeneous form at

$$ax^2 + by^2 + cz^2 = t^2$$

...(i)

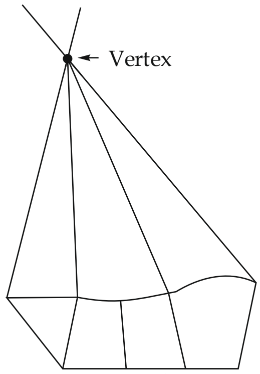
$$\alpha x^2 + \beta y^2 = 2zt$$

...(ii)

New variable 't' has been introduced to make them homogeneous.

⇒ $$t = \frac{(\alpha x^2 + \beta y^2)}{zx}$$, substitute this.

Equation of cone with a given vertex and a given conic for its base



Let vertex = (α, β, γ)

Basic cone: $ax^2 + by^2 + cy^2 + 2gx + 2fy + c = 0, z = 0$

Steps:

Equation of generator

$\downarrow$

General point

$\downarrow$

This will lie on the given base

$\downarrow$

Put the point in the given base and eliminate l, m, n.

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{...(i)}$$

It meets the plane $z = 0$,

$$\therefore \quad \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{\overset{\text{put } 0}{z} - \gamma}{n}$$

$$\Rightarrow \qquad (x, y, 0) = \left\{\alpha - \frac{\lambda\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right\}$$

This point lies on the given conic.

$$\therefore \quad a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$$

Use (i) to eliminate l, m, n and you are done.

i.e.,
$$\frac{l}{n} = \frac{x-\alpha}{z-\gamma}, \quad \frac{m}{n} = \frac{y-\beta}{z-\gamma}$$

Condition for the general equation of the second degree to represent a cone and to find vertex

General equation of second degree

F: $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

Represents a cone with its vertex at (α, β, γ)

Condition,
$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

Centre: $\boxed{\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial t} = 0}$

where 't' is replaced by unity after differentiation.

$\Rightarrow$ 't' is a variable introduced to make $F(x, y, z)$ homogeneous in x, y, z, t.

Funda:

Shift origin $\rightarrow (\alpha, \beta, \gamma)$

$\downarrow$

Now if this is cone, it should be homogeneous.

$\downarrow$

You will get few conditions, use them to eliminate (α, β, γ)

$\downarrow$

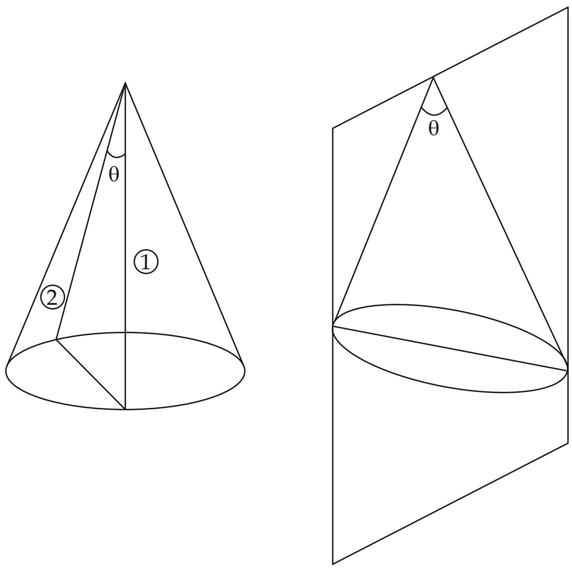
Condition obtained is the required one.

Q. Find the equation of the cone whose vertex is the point (1, 1, 0) and whose guiding curve is $x^2 + z^2 = 4$, $y = 0$

Ans. $x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$

Angle between the lines in which a plane cuts a cone.

Plane: $$ux + vy + wz = 0 \quad \ldots\text{(i)}$$



Cone $\equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ (notice that this cone has vertex at (0, 0, 0) and the plane is also passing through origin) ...(ii)

Let the line in which the plane cuts the cone is

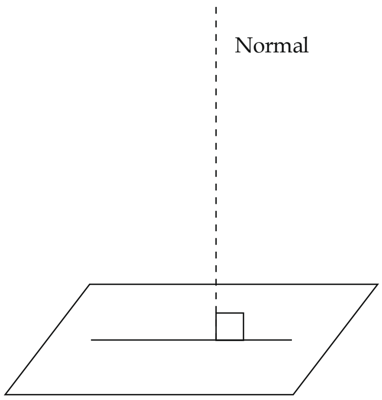
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{...(iii)}$$

This (iii) is also the generator.

So its DC's will satisfy the (ii)

$$\therefore \quad al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \qquad \text{...(iv)}$$

Also, as line lies in the plane:



$$\therefore \quad ul + vm + wn = 0 \qquad \text{...(v)}$$

Eliminating 'n' between (iv) and (v) we get

$$\left(\frac{l^2}{m^2}\right)\left(aw^2 + cu^2 - 2guw\right) + 2\left(\frac{l}{m}\right)\left(cuv - fuw) - gvw + hw^2\right) + \left(bw^2 + cv^2 - 2fvw\right) = 0$$

Look carefully — its quadratic.

$\therefore$ Two line are possible (obvious too)

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\Rightarrow \quad \frac{l_1 l_2}{m_1 m_2} = \text{Product of roots of (vi)}$$

$$= \frac{bw^2 + cv^2 - 2fvw}{aw^2 + cu^2 - 2guw}$$

$$\Rightarrow \quad \frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{cu^2 + aw^2 - 2guw} = \frac{n_1 n_2}{av^2 + bu^2 - 2hvw} \qquad \text{...(vii)}$$

Also, $$\frac{l_1}{m_1} + \frac{l_2}{m_2} = \text{Sum of roots}$$

$$= \frac{-2\left(cuv - fuw - gvw + hw^2\right)}{aw^2 + cu^2 - 2guw}$$

Condition for these lines to be perpendicular:

1. $\boxed{(b+c)u^2+(c+a)v^2+(a+b)w^2=0}$

   Remember that the plane is $\boxed{ux+vy+wz=0}$ and cone is $\boxed{ax^2+by^2+cz^2=0}$

2. If the cone is $\boxed{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}\{f(x,y,z)\}$

   $\Rightarrow$ $\boxed{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)=0}$

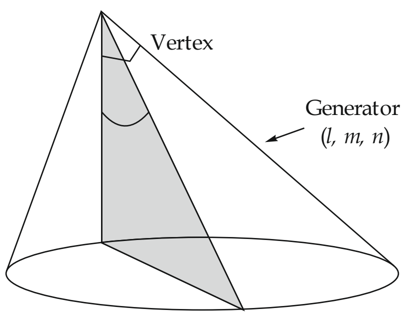
Q. Find the equation of the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Also find the angle between them.

Ans. $\dfrac{x}{-1} = \dfrac{y}{4} = \dfrac{z}{2}$ and $\dfrac{x}{-1} = \dfrac{y}{2} = \dfrac{z}{0}$

Q. Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Condition for the cone to have three mutually perpendicular generator

Let cone: $f(x, y, z)$ $\quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$



Let one of its generator be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, its DC's will satisfy (i)

$$\Rightarrow \qquad f(l, m, n) = 0 \qquad \text{...(ii)}$$

Now, a plane through (0, 0, 0) and perpendicular to this generator ≡

$$lx + my + nz = 0 \quad \text{...(iii)}$$

Now, this plane cuts the cone in two lines and these lines will be ⊥ , if

$$(a + b + c)\,(l^2 + m^2 + n^2) - f\,(l, m, n) = 0$$

$$\Rightarrow \quad (a + b + c)\,(l^2 + m^2 + n^2) = 0$$

$$\Rightarrow \quad (a + b + c) = 0$$

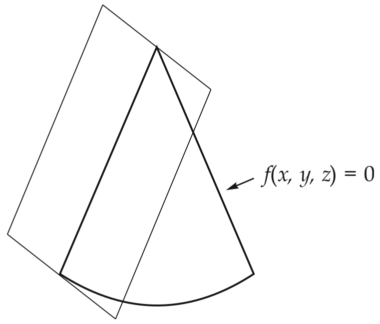
Notice that the condition is independent of (l, m, n), its means there can be infinite set of three mutually perpendicular generators of the cone.

Q. If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equation of the other two.

Ans. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$

Tangent Lines and Tangent Plane

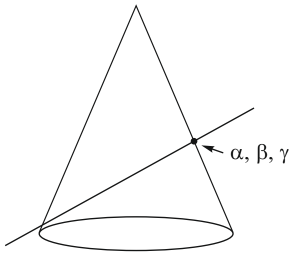
Let cone $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$



Line through (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(ii)}$$

General point $= (\alpha + lr, \beta + mr, \gamma + hr)$



$\because$ Since this point meet the cone

$\therefore a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2) + 2f(\beta + mr)(\gamma + nr) + 2g(\gamma + nr)(\alpha + lr) + 2h(\alpha + lr)(\beta + mr) = 0$

Quadratic in 'r'

$$D = 0$$

$\Rightarrow l(a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma) + n(g\alpha + f\beta + c\gamma) = 0 \qquad \text{...(iii)}$

Now, tangent plane is the locus of tangent line

$\therefore$ Eliminate l, m, n from (iii) using (ii)

$\therefore$ Tangent plane:

$a\alpha x + b\beta y + c\gamma z + f(\gamma y + \beta z) + g(\alpha z + \gamma x) + h(\beta x + \alpha y) = 0$

Now, recall the general method of finding the tangent plane at point (α, β, γ)

Condition of Tangency

Plane $\equiv ux + vy + wz = 0$

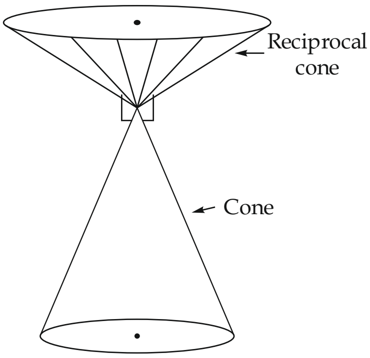
Cone $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0$$

How?

Reciprocal Cone

It is the locus of lines through the vertex at right angle to the tangent planes of the given cone.



Let $f(x, y, z)$ cone $\equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hyx = 0$

Tangent plane $\equiv ux + vy + wz = 0$

Eq. of RC: $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

where, A is cofactor corresponding to 'a' in

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{i.e., } bc - f^2$$

Similarly, $H = g - ch.....$

How?

1. Use of condition of tangency

2. Locus of $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$

Cone $\rightarrow$ Tangent plane $\rightarrow$ Normal $\rightarrow$ RC

Note: Condition for the cone to have three mutually perpendicular tangent planes is same as that of its reciprocal cone to have three mutually perpendicular generators (Think !!)

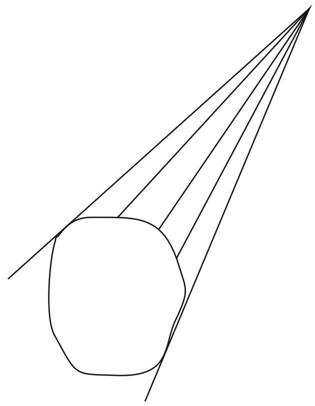
Q. Find the reciprocal cone of the cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$

Ans. $a^2x^2 + b^2y^2 + c^2z^2 = 0$

Q. Prove that $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ represents a cone which touches the coordinate planes. And that the equation of the RC is $fyz + gzx + hxy = 0$.

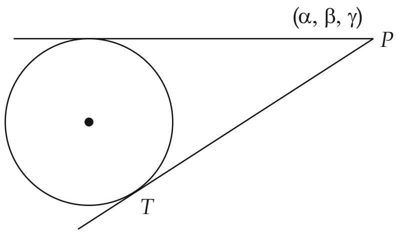
Enveloping Cone

It is the locus of tangent lines drawn from a given point to a given surface.



$$SS_1 = T^2$$

e.g., find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ with vertex at the point (α, β, γ).



What we need is the locus of line PT:

Let eq. of PT ⇒ $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$
...(i)

General point $(\alpha + lr, \beta + mr, \gamma + nr)$

This line meets the sphere.

∴ Put the point on the sphere.

⇒ Quadratic

⇒ $D = 0$

⇒ Condition in l, m, n

⇒ Eliminate l, m, n using (i)

You will get $SS_1 = T^2$

Where,

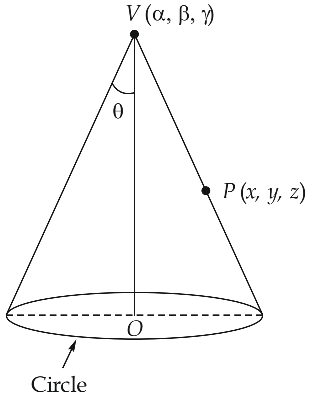
$$S = x^2 + y^2 + z^2 - a^2$$

$$S_1 = \alpha^2 + \beta^2 + \gamma^2 - a^2$$

$$T = \alpha x + \beta y + \gamma z - a^2$$

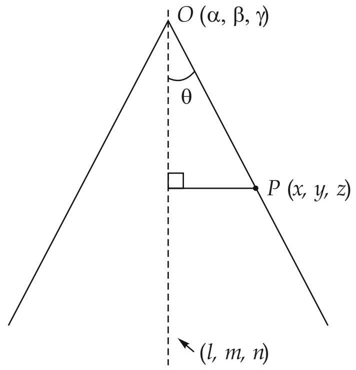
Right Circular Cone

It is the surface generated by a line passing through a fixed point (vertex) and making a constant angle with a fixed line.



Equation:

Equation is locus of OP.



$$\text{DC'S OF OP} = \frac{(x-\alpha)}{\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}, \frac{y-\text{b}}{\sqrt{\ }}, \frac{z-\gamma}{\sqrt{\ }}$$

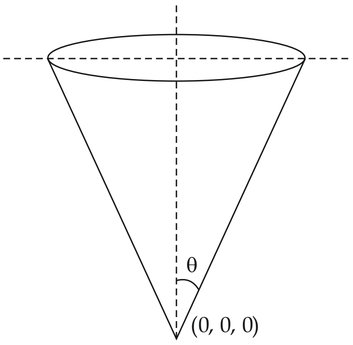
Angle between axis and OP is θ.

$$\therefore \qquad \cos\theta, l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Hence the required equation of RCC is

$$\left[l(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2 = \cos^2\theta\left(l^2+m^2+n^2\right)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]$$

Special case:



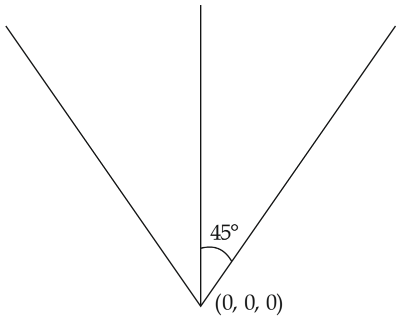
Axis of cone = z-axis

Vertex = (0, 0, 0)

Semi V angle = θ

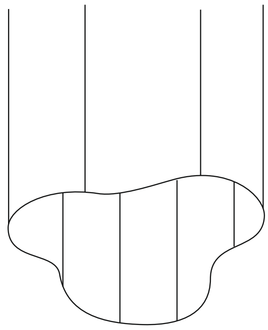
Equation of cone $\equiv x^2 + y^2 = z^2 \tan^2 \theta$

Q. Find the equation of the right circular cone whose axis is $x = y = z$, vertex is origin and whose semi vertical angle is 45°.



CYLINDER

It is the locus of a line moving parallel to a fixed line and satisfying one more condition.



Right Circular Cylinder: If the generating line is always at a constant distance from the fixed line, then the cylinder generated is called RCC.

Equation of a cylinder through a given conic:

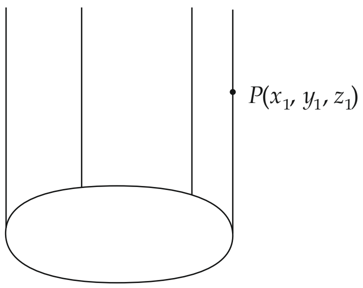
Equation of the conic

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, \quad z = 0$$

Let the generator of the cylinder be paralle to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Equation of any generators through P:



$$\Rightarrow \quad \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

The generator meets the plane $z = 0$ at the point $\left(x_1 - \frac{lz_1}{n},\ y_1 - \frac{mz_1}{n},\ 0\right)$

This generator will meet the conic if

$$a\left(x_1 - \frac{lz_1}{n}\right)^2 + 2h\left(x_1 - \frac{lz_1}{n}\right)\left(y_1 - \frac{mz_1}{n}\right) + b\left(y_1 - \frac{mz_1}{n}\right)^2 + 2g\left(x_1 - \frac{lz_1}{n}\right) + zf\left(y_1 - \frac{mz_1}{n}\right) + c = 0$$

The required locus will be

$$x_1 \to x$$
$$y_1 \to y$$
$$z_1 \to z$$

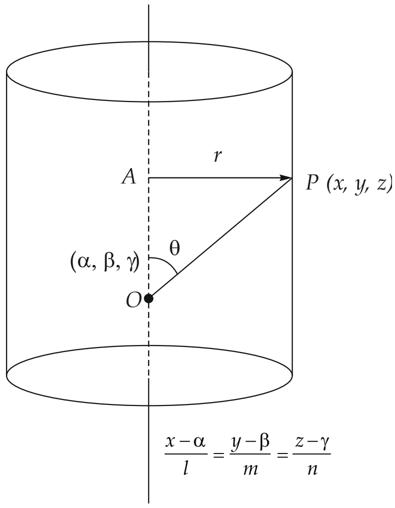
Note:

1. Every equation of the form $f(x, y) = 0$ represents a cylinder through the curve $f(x, y) = 0$, $z = 0$ and whose generators are parallel to 'z' axis.
2. Equation of cylinder which intersect the curve $f_1(x, y, z) = 0 = f_2(x, y, z)$ and whose generators are parallel to x-axis is obtained by eliminating the x between f_1 and f_2.

Q. **Find the equations of the quadric cylinder which intersect the curve $ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$ and whose generators are parallel to the axis of z.**

Ans. $(an^2 + cl^2)\, x^2 + (bn^2 + cm^2)\, y^2 + 2lcmxy - 2cplx - zcpmy + (cp^2 - n^2 = 0$

Equation of *RCC*



$$\sin\theta = \frac{r}{\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

$$\sin^2\theta = \frac{r^2}{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2} \quad \text{...(i)}$$

Also note that, $\cos\theta = l_1 l_2 + m_1 m_2, n_1 n_2$

$(l_1, m_1, n_1) \rightarrow$ DC's of axis

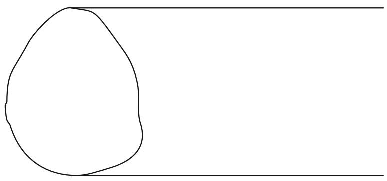
$(l_2, m_2, n_2) \rightarrow$ DC's of OP.

Then eq. (i) represents the equation of RCC.

Q. **Find the equation of RCC whose axis is $x = 2y = -z$ and radius is 4.**

Ans. $5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy = 144.$

- **Equation of tangent plane $\rightarrow$ use general method or go by fundamentals.**
- **Enveloping Cylinder**



It is the locus of the tangents to a surface drawn in a given direction.

Line $\rightarrow$ Parallel to some line $\rightarrow$ It is tangent.

Q. **Equation of the enveloping cylinder of sphere $x^2 + y^2 + z^2 = a^2$ and whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.**

Ans. Let P be any point on a generator.

$\therefore$ Its equation $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

General point $(\alpha + rl, \beta + mr, \gamma + nr)$

Now this generator touches the sphere:

$$(\alpha + rl)^2 + (\beta + rm)^2 + (\gamma + rn)^2 = a^2$$

$$D = 0$$

$$[2(l\alpha + m\beta + n\gamma)]^2 = 4(l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$$

$\therefore$ Locus $(\alpha, \beta, \gamma) \rightarrow (x, y, z)$

THE CONICOID

General equation of second degree in two variables represents a conic.

General equation of 2nd degree in three variables $\rightarrow$ Conicoids.

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ $\rightarrow$ Conicoid

Known conicoids:

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

(iii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$] Hyperboloid of two sheets

(i)–(iii)] Central Conicoid

(iv) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2z}{c}$] Elliptic paraboloid

(v) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{2z}{c}$] Hyperbolic paraboloid

Centre:

Principal plane:

"STANDARD" EQUATION OF THE CONICOID

$$ax^2 + by^2 + cz^2 = 1$$

Topics:
- Tangent plane
- Director sphere
- Polar plane
- Locus of chords with a given mid-point
- Normal
- Diametral plane

Tangent plane

Recall general method:

$$\text{Conicoid} \equiv ax^2 + by^2 + cz^2 = 1$$

$$\text{Point} = (\alpha, \beta, \gamma)$$

$$\downarrow$$

$$\text{Eq. of line } \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

$$\downarrow$$

General point

$$\downarrow$$

If the line touches conicoid ($D = 0$)

$$\downarrow$$

Eliminate l, m, n

Condition of Tangency (CoT):

Again recall the general method

$$\text{Let plane} = lx + my + nz = p$$

$$\text{and conicoid} = ax^2 + by^2 + cz^2 = 1$$

CoT: $$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

POC: $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

$\therefore$ Plane $$lx + my + nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

Always touch the conicoid $ax^2 + by^2 + cz^2 = 1$.

Director sphere: It is the locus of point of intersection of three mutually perpendicular tangent planes to the conicoid.

For conicoid $= ax^2 + by^2 + cz^2 = 1$

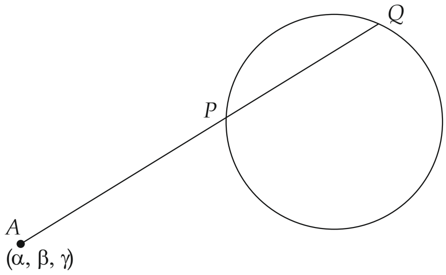
Director sphere is $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Q. Find the equation of the tangent planes to $7x^2 + 5y^2 + 3z^2 = 60$ which pass through the line $7x + 10y = 30$, $5y - 3z = 0$.

Ans. $7x + 5y + 3z = 30$; $14x + 10y + 9z = 50$.

Polar Plane

Recall:



$$\frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ}$$

It is the locus of R.

Equation of polar of $A \Rightarrow a\alpha x + b\beta y + c\gamma z = 1$

Note: Same as the equation of tangent plane at A when A is on the surface.

Conjugate Points:

$$\begin{tikzcd} \text{Point } P \arrow[r] \arrow[dr, dashed] & \text{Polar plane } P \\ \text{Point } Q \arrow[r] \arrow[ur, dashed] & \text{Polar plane } Q \end{tikzcd}$$

Conjugate Plane:

$$\begin{tikzcd} \text{Point } A \arrow[r] \arrow[dr, dashed] & \text{Polar plane } A \\ \text{Point } B \arrow[r] \arrow[ur, dashed] & \text{Polar plane } B \end{tikzcd}$$

Polar lines:

$$\text{Point } AB \longleftrightarrow PQ$$

Q. Show that the equation of the polar of the line $\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$ w.r.t. $x^2-2y^2+3z^2=4$ are $\frac{x+6}{3}=\frac{y-2}{3}=\frac{z-2}{1}$

Ans. $\frac{x+6}{3}=\frac{y-2}{3}=z-2$

Locus of chords with given mid point

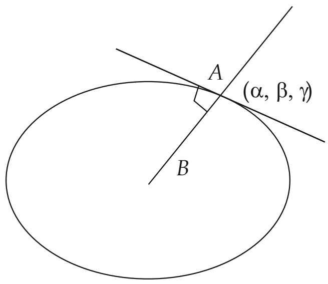
$$T = S_1$$

Q. Find the equation to the plane which cuts $2x^2 - 3y^2 + 5z^2 = 1$ in a conic whose centre is at the point (2, 1, 3).

Ans. $4x - 3y + 15z = 50$

Normals

$$\text{Conicoid} \equiv ax^2 + by^2 + cz^2 = 1$$



$$\text{Point} = (\alpha, \beta, \gamma)$$

Normal is $\perp$ to the tangent plane at that point.

$\therefore$ **Equation of normal AB:**

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma} \quad \text{...(i)}$$

If p is length of perpendicular from origin to tangent plane:

$$p = \frac{1}{\left[(a\alpha)^2+(b\beta)^2+(c\gamma)^2\right]^{1/2}}$$

The DC's of normal will be $(a\alpha p, b\beta p, c\gamma p)$

Now, suppose the normal (i) pass through (x_1, y_1, z_1) then

$$\frac{x_1-a}{a\alpha} = \frac{y_1-\beta}{b\beta} = \frac{z_1-\gamma}{c\gamma} = r(\text{say})$$

We get $$\alpha = \frac{x_1}{1+ar},\ \beta = \frac{y_1}{1+br},\ \gamma = \frac{z_1}{1+cr}$$

Also recall (α, β, γ) is the point on conicoid.

$$\therefore \quad a\left(\frac{x_1}{1+ar}\right)^2 + b\left(\frac{y_1}{1+br}\right)^2 + c\left(\frac{z_1}{1+cr}\right)^2 = 1$$

Observation: 6 degree in 'r'

$\Rightarrow$ Six normals can be drawn from any point.

Q. Find the distance of the points of intersection of the coordinate planes and the normal at P (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. The normal at any point $P(\alpha, \beta, \gamma)$ to the conicoid meets the these principal plane at G_1, G_2 and G_3 show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$.

Ans. Equation of normal at $P(\alpha, \beta, \gamma)$

$$\frac{x-\alpha}{p\frac{\alpha}{a^2}} = \frac{y-\beta}{p\frac{\beta}{b^2}}=\frac{z-\gamma}{p\frac{\gamma}{c^2}}=r \text{ (say)}$$

($\because$ DC's are used)

Let this normal meets the coordinate planes viz. yz, zx, xy planes at G_1, G_2, and G_1, then putting $x = 0$, $y = 0$ and $z = 0$ in succession in the equation of normal.

$$\Rightarrow \quad PG_1 = -\frac{a^2}{p};\ PG_2=-\frac{b^2}{p};\ PG_3=-\frac{c^2}{p}$$

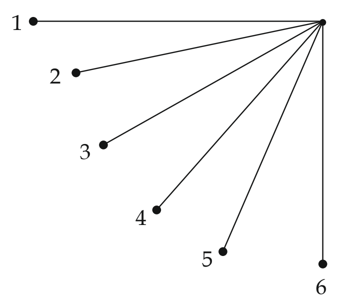
$$\Rightarrow \quad PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$$

Note:

1. Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\downarrow$

Six normals from any point (x_1, y_1, z_1)



$\downarrow$

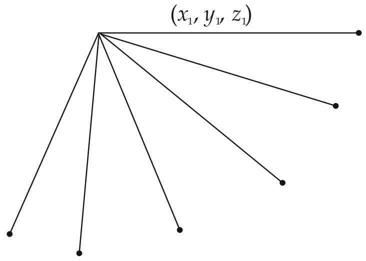
Parametric form of feet of these six normals

$$x = \frac{a^2x_1}{a^2+\lambda},\ y=\frac{b^2y_1}{b^2+\lambda},\ z=\frac{c^2z_1}{c^2+\lambda}$$

(Obviously λ can take 6 values) (remember)

Q. Find the equation of cone through the six normals drawn to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Equation of any line through (x_1, y_1, z_1)



$$\Rightarrow \quad \frac{x-x_1}{l} = \frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{...(i)}$$

Now, if this line is normal, $\quad l = \frac{p\alpha}{a^2},\ m=\frac{p\beta}{b^2},\ n=\frac{p\gamma}{c^2} \quad$ ...(ii)

[Recall (α, β, γ)]

Also,

$$\Rightarrow \quad \alpha = \frac{x_1a^2}{a^2+\lambda},\quad \beta=\frac{b^2y_1}{b^2+\lambda},\quad \gamma=\frac{c^2z_1}{c^2+\lambda}$$

...(iii)

From (ii) and (iii) $\quad l = \frac{p}{a^2}\left(\frac{x_1a^2}{a^2+\lambda}\right)=\frac{px_1}{a^2+\lambda}$

$$\Rightarrow \quad \frac{px_1}{l} = a^2+\lambda$$

Simplify $\quad \frac{py_1}{m} = b^2+\lambda$

$$\frac{pz_1}{n} = c^2+\lambda$$

$$\therefore \quad \frac{px_1}{l}\left(b^2-c^2\right)+\frac{py_1}{m}\left(c^2-a^2\right)+\frac{pz_1}{n}\left(a^2-b^2\right) = 0$$

...(iv)

Using eq. (i), eliminate (l, m, n)

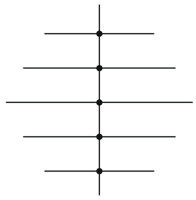
$$\Rightarrow \quad \frac{px_1\left(b^2-c^2\right)}{x-x_1}+\frac{py_1\left(c^2-a^2\right)}{y-y_1}+\frac{pz_1\left(a^2-b^2\right)}{z-z_1} = 0$$

$$\Rightarrow \quad \frac{x_1\left(b^2-c^2\right)}{x-x_1}+\frac{y_1\left(c^2-a^2\right)}{y-y_1}+\frac{z_1\left(a^2-b^2\right)}{z-z_1} = 0 \quad \text{...(v)}$$

$\Rightarrow$ Clearly the six normals lie on the cone given by (v)

Diamental Plane

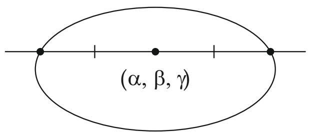
It is the locus of the mid points of a system of parallel chords.



Let conicoid: $ax^2 + by^2 + cz^2 = 1$

Consider a line. $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

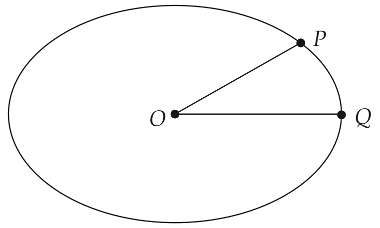
Any general point = $(\alpha + lr, \beta + mr, \gamma + nr)$



Q. Show that every plane through centre is a diametral plane.

Conjugate Diameter and Conjugate Diametral Planes

Ellipsoid: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$



Let, $$P = (x_1, y_1, z_1)$$

$$\text{DR's of } OP = (x_1, y_1, z_1)$$

$$\downarrow$$

Diameter plane of line OP: $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2} = 0$$

...(i)

Let $Q = (x_2, y_2, z_2)$ be any other point on ellipsoid and on the diametral plane (i)

Then, $$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2} = 0$$

Notice the symmetry: Diametral plane of OQ will also pass through P.

Thus we conclude that diametral plane of OP passes through Q and diametral plane of OQ will pass through P.

Now, let's say the line of intersection of these two diameter plane cuts the surface of ellipsoid at $R(x_3, y_3, z_3)$.

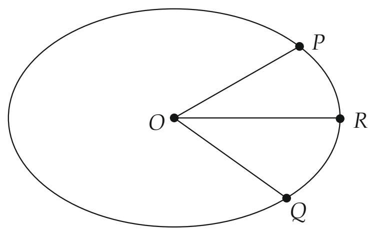
So, we can say R is lying on the diametral plane of OP and OQ.

Hence, P and Q will also lie on the diameter plane of OR.

Also notice that diameter plane passes through (0, 0, 0).

Hence we can conclude that diametral plane of R will be plane OPQ.

Similarly, diametral plane of OP and OQ will be OQR and OPR respectively.



Conjugate semidiameters: OP, OQ, OR

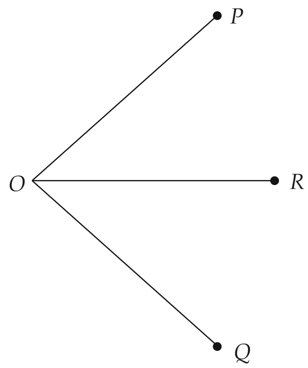
Conjugate diametral planes: POR, QOR, ROP

Relation between the coordinates of P, Q, R

$\because$ P, Q, R lie on the ellipsoid

$$\left.\begin{aligned} \therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2} &= 1 \\ \therefore \quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2} &= 1 \\ \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2} &= 1 \end{aligned}\right\} \quad \text{...(i)}$$

Also diametral planes related results:



$$\left.\begin{aligned} \frac{x_1 x_2}{a^2}+\frac{y_1 y_2}{b^2}+\frac{z_1 z_2}{c^2} &= 0 \\ \frac{x_2 x_3}{a^2}+\frac{y_2 y_3}{b^2}+\frac{z_2 z_3}{c^2} &= 0 \\ \frac{x_3 x_1}{a^2}+\frac{y_3 y_1}{b^2}+\frac{z_3 z_1}{c^2} &= 0 \end{aligned}\right\} \quad \text{...(ii)}$$

If we assume $\left(\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}\right)$, $\left(\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}\right)$ and $\left(\frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}\right)$ as DC's of some lines and combine these with (ii), we can conclude that these three lines are mutually perpendicular.

Now, use lagrange's identify to get various results:

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2} = 1 \text{ etc.}$$

Similarly,

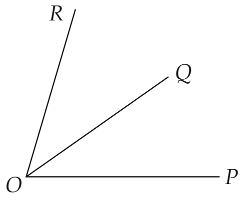
$$y_1^2+y_2^2+y_3^2 = b^2$$

$$z_1^2+z_2^2+z_3^3 = c^2$$

and
$$\begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 &= 0 \\ y_1z_1 + y_2z_2 + y_3z_3 &= 0 \\ z_1x_1 + z_2x_2 + z_3x_3 &= 0 \end{aligned}$$

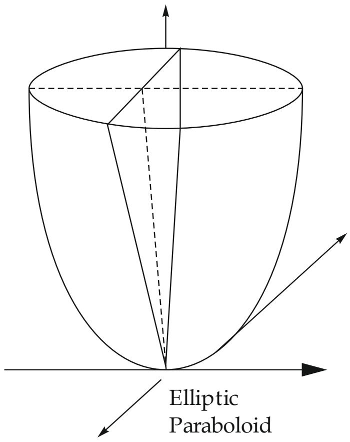
Properties of conjugate semi diameter:

1. $OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$

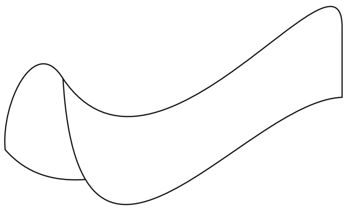
2. 

Volume of parallelopiped whose edges are OP, OQ, $OR \Rightarrow V = abc$

THE PARABOLOID



Hyperbolic paraboloid (saddle)



General Equation of Paraboloid

- Tangent planes
- Polar plane
- $T = S_1$
- Normals
- Diametral plane

$$ax^2 + by^2 = 2cz$$

a & b same sign – Elliptic

a & b opposite—Hyperbolic Paraboloid

Tangent Plane

[Recall the definition) Point $= (\alpha, \beta, \gamma)$

$$a\alpha x + b\beta y = c(z + \gamma)$$

CoT: Let plane $lx + my + nz = p$

Paraboloid $\equiv ax^2 + by^2 = 2cz$

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{2np}{c} = 0$$

Q. Show that the plane $8x - 6y - z = 5$ touches the paraboloid $\left(\frac{x^2}{2}\right) - \left(\frac{y^2}{3}\right) = z$ and find the PoC.

Ans. (8, 9, 5)

Locus of the point of intersection of three mutually perpendicular tangent planes

Let Paraboloid: $ax^2 + by^2 = 2cz$

$\therefore$ Three tangent planes:

$$2n_1(l_1x + m_1y + n_1z) + c\left[\left(\frac{l_1^2}{a}\right) + \left(\frac{m_1^2}{b}\right)\right] = 0$$

$$2n_2(l_2x + m_2y + n_2z) + c\left[\left(\frac{l_2^2}{a}\right) + \left(\frac{m_2^2}{b}\right)\right] = 0$$

$$2n_3(l_3x + m_3y + n_3z) + c\left[\left(\frac{l_3^2}{a}\right) + \left(\frac{m_3^2}{b}\right)\right] = 0$$

Simply add above three and use lagrange's identity $2z + c\left[\frac{1}{a} + \frac{1}{b}\right] = 0$

i.e., locus is $\perp$ to 'z' axis.

- Polar plane
- Pole of a given plane
- Locus of chords with given mid point : $T = S_1$

Q. Prove that the centre of the conic $ax^2 + by^2 = 2z,\ lx + my + nz = p$ is the point $\left(-\frac{l}{an}, -\frac{m}{bn}, \frac{k^2}{n^2}\right)$ where $k^2 = \frac{l^2}{a} + \frac{m^2}{b} + np$.

NORMALS TO THE PARABOLOID

$$ax^2 + by^2 = 2cz \text{ at the point } (\alpha, \beta, \gamma)$$

Eq. $$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c}$$

Number of normals from a point (x_1, y_1, z_1)

We know that eq. of normal

$$ax^2 + by^2 = 2cz$$

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c}$$

If this passes through (x_1, y_1, z_1), then

$$\frac{x_1-\alpha}{a\alpha} = \frac{y_1-\beta}{b\beta} = \frac{z_1-\gamma}{-c} = r \text{ (say)}$$

$$\Rightarrow \quad \frac{x_1-\alpha}{a\alpha} = r$$

$$\Rightarrow \quad \alpha = \frac{x_1}{1+ar}$$

Similarly, $$\beta = \frac{y_1}{1+br}$$

and $$\gamma = z_1 + cr$$

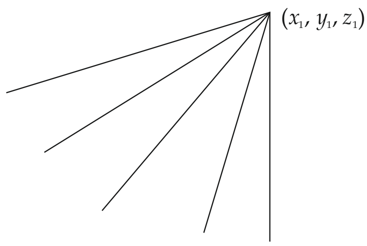
Also (α, β, γ) being a point on the given paraboloid, we have $a\alpha^2 + b\beta^2 = 2c\gamma$.

$$\Rightarrow \quad a\left(\frac{x_1}{1+ar}\right)^2 + b\left(\frac{y_1}{1+br}\right)^2 = 2c(z_1 + cr)$$

$\Rightarrow$ This equation is of five degree in 'r', hence five normals can be drawn from any fixed point.

Cone Through Five Concurrent Normals

We have paraboloid ≡ $$ax^2 + by^2 = 2cz \quad \text{...(i)}$$



Any line through (x_1, y_1, z_1)

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{...(ii)}$$

If this line is a normal to (i) at some (α, β, γ) then

$$l = a\alpha, \quad m = b\beta \quad \text{and} \quad n = -c \qquad \text{...(iii)}$$

Where (α, β, γ) are also given by the parametric form:

$$\alpha = \frac{x_1}{1+ar}, \ \beta = \frac{y_1}{1+br}, \ \gamma = z_1 + cr \qquad \text{...(iv)}$$

From eq. (iii) and (iv)

$$l = a\left(\frac{x_1}{1+ar}\right); \ m = b\left(\frac{y_1}{1+br}\right), \ n = -c$$

$$\frac{ax_1}{l} = 1+ar, \ \frac{by_1}{m} = 1+br, \ \frac{c}{n} = -1$$

Multiply these by b, $-a$ and $(b-a)$ respectively, adding:

$$\frac{ax_1}{l}b + \frac{by_1}{m}(-a) + \frac{c}{n}(b-a) = (1+ar)b + (1+br)(-a) - (b-a)$$

or $$\frac{abx_1}{l} - \frac{aby_1}{m} + \frac{c(b-a)}{n} = 0 \qquad \text{...(iv)}$$

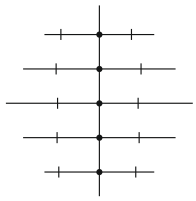
Eliminate l, m, n between (ii) and (iv)

$$\frac{abx_1}{(x-x_1)} - \frac{aby_1}{y-y_1} + \frac{c(b-a)}{z-z_1} = 0$$

⇒ Equation of cone:

Diamental Plane

$$alx + bmy - cn = 0$$



Conjugate diametral plane

The diametral planes are said to be conjugate if each bisect chords parallel to the other.

Let paraboloid be $$ax^2 + by^2 = 2cz$$

...(i)

And let us take two diametral planes:

$$l_1x + m_1y + n_1 = 0 \quad \text{and} \quad l_2x + m_2y + n_2 = 0$$

The first plane is bisecting the chords whose direction ratios are $\frac{l_1}{a}, \frac{m_1}{b}, -\frac{n_1}{c}$

Now, these chords are parallel to second plane i.e., $\perp$ to its normal.

$$\Rightarrow \quad \frac{l_1}{a} \times l_2 + \frac{m_1}{b} \times m_2 - \frac{n_1}{c} \times 0 = 0$$

$$\Rightarrow \quad \frac{l_1 l_2}{a} + \frac{m_1 m_2}{b} = 0$$

This is the condition for planes $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate diametral plane of the paraboloid $ax^2 + by^2 = 2cz$.

Q. Show that the diametral plane $2x + 3y = 4$ and $3x - 4y = 7$ are conjugate for the paraboloid $x^2 + 2y^2 = 4z$.

Enveloping Cone

Enveloping Cylinder

THE GENERATING LINES

Ruled surface: Generated by motion of straight line. E.g. cone, cylinder, hyperboloid of one sheet, hyperbolic paraboloid.

GL of Hyperboloid of one Sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2}-\frac{z^2}{c^2} = 1-\frac{y^2}{b^2}$$

$$\Rightarrow \quad \frac{x}{a}-\frac{z}{c} = \lambda\left(1-\frac{y}{b}\right)$$

$$\Rightarrow \quad \frac{x}{a}+\frac{z}{c} = \frac{1}{\lambda}\left(1+\frac{y}{b}\right)$$

(where λ is constant)

These two equations $\rightarrow$ eq. of planes $\rightarrow$ eq. of line $\rightarrow$ Locus $\rightarrow$ Hyperboloid of one sheet.

Similarly consider

$$\left(\frac{x}{a}-\frac{z}{c}\right) = \mu\left(1+\frac{y}{b}\right)$$

$$\frac{x}{a}+\frac{z}{c} = \frac{1}{\mu}\left(1-\frac{y}{b}\right)$$

Line $\downarrow$ Locus $\downarrow$ Same

$\therefore$ We have two families of straight lines such that every member of each family lies wholly on the hyperboloid of one sheet.

Properties of Generating Lines of HOS

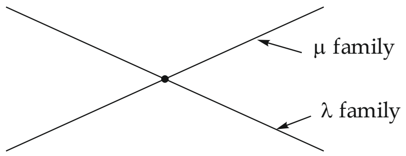
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$$

λ family:

$$\frac{x}{a}-\frac{z}{c} = \lambda\left(1-\frac{y}{b}\right); \quad \frac{x}{a}+\frac{z}{c} = \frac{1}{\lambda}\left(1+\frac{y}{b}\right)$$

and μ family:

$$\frac{x}{a}-\frac{z}{c} = \mu\left(1+\frac{y}{b}\right); \quad \frac{x}{a}+\frac{z}{c} = \frac{1}{\mu}\left(1-\frac{y}{b}\right)$$

1. 

2. No two generators of same system intersect.

3. Any two generators of the different system intersect.

$$\left\{\frac{a(1+\lambda\mu)}{\lambda+\mu}, \frac{b(\lambda-\mu)}{\lambda+\mu}, \frac{c(1-\lambda\mu)}{\lambda+\mu}\right\}$$

Note:

1. If three points of any straight line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ lie on the conicoid then the whole line lies on the conicoid.

2. Generator lies wholly on its surface.

Q. Find the equations of the generating lines of the hyperboloid $yz+2zx+3xy+6=0$ which passes through the point (–1, 0, 3).

Ans. $x + 1 = 0$, $z - 3 = 0$ and $\frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$

GENERATING LINES OF A HYPERBOLIC PARABOLOID

Equations:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$$

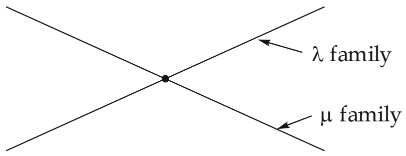
$$\frac{x}{a}-\frac{y}{b}=\lambda z \qquad \frac{x}{a}+\frac{y}{b}=\mu z,\ \frac{x}{a}-\frac{y}{b}=\frac{2}{\mu}$$

$$\frac{x}{a}+\frac{y}{b}=\frac{2}{\lambda}\times z$$

Two planes
one line
λ family

Two planes
One line
μ family

Properties:

1. 

2. No two generators of the same system intersect.
3. Any two generators of the different system intersect.

$$\left\{\frac{a(\lambda-\mu)}{\lambda\mu},\ \frac{b(\mu-\lambda)}{\lambda\mu},\ \frac{2}{\lambda\mu}\right\}$$

4. The tangent planes at any point meet the hyperboloid at two generators through that point.

Q. Find the equation to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z,$, which pass through the point (1, 1, 1).

Ans. $\frac{x-1}{4}=\frac{y-1}{-5}=\frac{z-1}{-1}$ and $\frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-1}{-1}$

GENERAL SECOND DEGREE EQUATIONS

We have considered so for special forms of the equations of second degree in order to discuss geometrical properties of various types of quadrics.

We will see in this chapter, how the general equation of a second degree can be reduced to simpler forms.

Here, $F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

$$= \underbrace{\sum\left(ax^2 + 2fyz\right)} + \underbrace{2\sum ux} + \underbrace{d}$$

Principal direction and principal planes: A direction l, m, n is said to be principal, if it is perpendicular to the diametral plane conjugate to the same. Also then the corresponding conjugate diametral plane is called a principal plane.

Discriminating cubic:

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

This cubic is known as discriminating cubic and each root of the same is called a characteristics root.

Process of Reducing A General Equation to the Standard Form

(Provided the terms of second degree do not form a perfect square)

1. Form the discriminating cubic and solve it.

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

2. If all the characteristics roots $(\lambda_1, \lambda_2, \lambda_3)$ are different from zero then find the coordinates (α, β, γ) of the centre by solving the following equations $\frac{\partial F}{\partial x} = 0; \quad \frac{\partial F}{\partial y} = 0; \quad \frac{\partial F}{\partial z} = 0$

3. After shifting the origin to the centre (α, β, γ) and then rotating the axes the reduced equation becomes $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = \mu$ where $\mu = -(u\alpha + v\beta + w\gamma + d)$.

   The direction cosine l, m, n of the principal direction corresponding to the value λ, are given by the equation.

   $$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n} = \lambda_1$$

4. If one root (say) $\lambda_3 = 0$, then corresponding to this value find the principal direction l_3, m_3, n_3 from any two of the following equations:

   $$al_3 + hm_3 + gn_3 = 0$$

$$hl_3 + bm_3 + f(n_3) = 0$$

$$gl_3 + fm_3 + cn_3 = 0$$

And evaluate $k = ml_3 + um_3 + wn_3$

$k \neq 0$

$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0 \text{ (Paraboloid)}$$

Vertex:

$$ax + hy + gz - l_3 R + u = 0$$

$$hx + by + fz + m_3 k + v = 0$$

$$gx + fy + cz - n_3 k + w = 0$$

and $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

$k = 0$ (line of centres)

$$\frac{\partial F}{\partial x} = 0, \ \frac{\partial F}{\partial y} = 0; \ \frac{\partial F}{\partial z} = 0$$

Choose any point (α, β, γ)

$$\lambda_1 x^2 + \lambda_2 y^2 = \mu$$

$$\mu = -(u\alpha + v\beta + w\gamma + d)$$

Note: When two roots of discriminating cubic are equal, then the surface $F(x, y, z) = 0$ represents a surface of revolution and the axis of rotation is obtained by taking into consideration that value of λ which is different from the equal values of λ.

Q. Reduce the equation:

$2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$ to the standard form.

1. Discriminating cubic : $\lambda = 3, 6, -12$

   $\lambda^3 - (-3)\lambda^2 + [(-14 - 25)] - c$

2. Centre : $\left(\frac{1}{3}, \frac{-1}{3}, \frac{4}{3}\right)$

3. $\mu = -(u\alpha + v\beta + w\gamma + d)$

4. $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = \mu$

Principal direction: $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} \begin{vmatrix} l \\ m \\ n \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$

$\therefore$ Equation of axis

Q. Prove that the equation $x^2+y^2+z^2+yz+zx+xy+3x+y+4z+4=0$ represents an ellipsoid, the square of whose semi axis are $2, 2, \frac{1}{2}$. Show that its principal axis of rotation is given by $x+1=y-1=z+2$.

$\lambda = 1, 1, 4$; centre; $(-1, 1, -2)$, $\mu = 2$

Q. Prove that $x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$ represents a paraboloid of revolution and find the equations of its axis.

$$\lambda=\left(0,\frac{3}{2},\frac{3}{2}\right);\quad (1,1,1);\quad k=-3\sqrt{3}$$

$$(l,m,n)$$

Vertex = (0, 1, 2)
Focus = (1, 2, 3)

Q. Reduce the following equation to the standard form

$$2x^2 + 5y^2 + 2z^2 - 2yz + 4zx - 2xy + 14x - 16y + 14z + 26 = 0$$

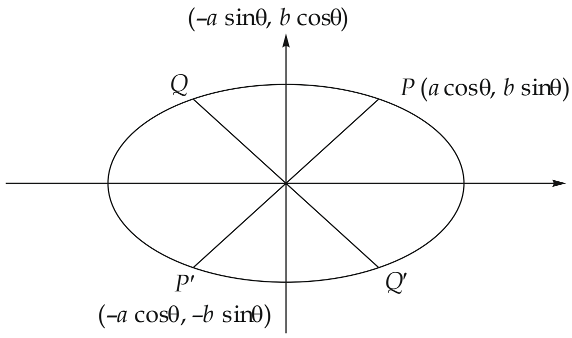
$\lambda = (0, 3, 6)$;

$k = 0$

$\therefore$ (No unique centre (line of centre).

ELLIPSE

Conjugate diameter: If each bisects chords parallel to other.



The eccentric angles of the ends of a pair of conjugate diameter of an ellipse differ by a right angle.