Unit 1→Set Theory: Solved and unsolved practice problems

Question 1: Let A = {a, b, c} and the relation R be defined on A as follows:
R = {(a, a), (b, c), (a, b)}.
Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Solution: In order to make R reflexive, (b, b) and (c, c) will be added to R.

And in order to make R transitive, (a, c) will be added to R.

Therefore, the minimum number of order pair to be added to R will be (b, b), (c, c) and (a, c) - Answer

Question 2: Let D be the domain of real valued function f defined by $f(x) = \sqrt{25 - x^2}$ then, write D.

Solution: Here given D is the domain of $f(x) = \sqrt{25 - x^2}$

Therefore,

$$25 - x^2 \geq 0$$
$$\Rightarrow 25 \geq x^2$$
$$\Rightarrow x^2 \leq 25$$
$$\Rightarrow -5 \leq x \leq 5$$

Therefore, D = [– 5, 5] - Answer

Question 3: $Let\ f, g: R \to R$ be defined by $f(x)2x + 1\ and\ g(x) = x^2 - 2, \forall x \in R,$ respectively. Then find g o f.

Solution:

$$Given, f(x) = 2x + 1\ and\ g(x) = x^2 - 2, \forall x \in R$$
$$Therefore, g\ o\ f(x) = g(f(x))$$
$$= g(2x + 1)$$
$$= (2x + 1)^2 - 2$$
$$= 4x^2 + 1 + 4x - 2$$
$$Thus, g\ of(x) = 4x^2 + 4x - 1 \quad Answer$$

Question 4: $Let\ f, g: R \to R$ be the function defined by $f(x) = 1x - 3 \forall x \in R. Write\ f^{-1}$

Solution:

$$Given, f(x) = 1x - 3 \forall x \in R$$
$$Now, let\ y = f(x)$$
$$Therefore, y = 2x - 3$$
$$\Rightarrow 2x = y + 3$$
$$\Rightarrow x = \frac{y+3}{2}$$
$$Therefore, f^{-1}(x) = \frac{x+3}{2}\ Answer$$

Question 5: If A = {a, b, c, d} and the function f = {(a, b), (b, d), (c, a), (d, c)}, write f^{-1}

Solution: Given, f = {(a, b), (b, d), (c, a), (d, c)}

Therefore, f^{-1} ={(b, a), (d, b), (c, a), (c, d)} Answer

Question 6: If $f: R \rightarrow R$ is defined by $f(x) = x^2 - 3x + 2, wrtie\ f(f(x)).$

Solution: Given,

$$f(x) = x^2 - 3x + 2$$
$$Therefore, f(f(x)) = f(x^2 - 3x + 2)$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$$
$$= x^4 - 6x^3 + 9x^2 + 4x^2 - 3x^2 - 12x + 9x + 4 - 6 + 2$$
$$\Rightarrow f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x \quad Answer$$

Question 7: Is g = {(1, 1), (2, 3, (3, 5), (4, 7)} a function? If g is described by g(x) = αx + β, then what value should be assigned to α and β?

Solution: Given, g = {(1, 1), (2, 3, (3, 5), (4, 7)}

Therefore, each of the element of domain will be have unique image.

Consequently, g is a function.

$Now, since\ g(x) = \alpha x + \beta\ (as\ given\ in\ question)$

$Therefore, g(1) = \alpha \times 1 + \beta$

$Thus, 1 = \alpha + \beta\ ------(i)$

$Similarly, g(2) = \alpha \times 2 + \beta$

$Therefore, \quad 3 = 2\alpha + \beta ----(ii)$

$Now, from\ equation(i) and\ (ii) we\ get$

$$\begin{array}{l} 1 = \alpha + \beta \\ \underline{\ 3 = 2\alpha + \beta} \\ -\ \ -\ \ - \\ \hline -2 = -\alpha \end{array}$$

$Therefore, \alpha = 2$

Now, after substituting the value of α in equation (i), we get

$1 = 2 + \beta$

$\Rightarrow \beta = 1 - 2$

$\Rightarrow \beta = -1$

$Thus, \alpha = 2\ and\ \beta = -1 \quad Answer$

Question 1: Determine whether each of the following relations are reflexive, symmetric and transitive.

(i) Relation R in the set A = {1, 2, 3, ..., 13, 14} defined as $R = \{(x, y): 3x - y = 0\}$

Solution:

$Given, A = \{1, 2, 3, \ldots\ldots, 13, 14\}$

$R = \{(x, y): 3x - y = 0\}$

$\therefore R = \{(1,3)(2,6), (3,9), (4,12)\}$

$Here, (x, x) \notin R,$

Thus, R is not reflexive

$Now, as\ (x, y) \in R$

$But, (y, x) \notin R,$

Thus, R is not symmetric

$Now, again, (1,3) \in and\ (3,9) \in R$

$But, here, (1,9) \notin R$

Therefore, R is not transitive

Thus, R is neither reflexive nor symmetric and nor transitive.

(ii) Relation of R in the set N of natural numbers defined as $R = \{(x, y): y = x + 5\ and\ x < 4\}$

Solution:

$Given, R = \{(x, y): y = x + 5\ and\ x < 4\}$

$Thus, R = \{(1,6), (2.7), (3,8)\}$

$Here, (x, x) \notin R$

Thus, R is not reflexive relation

$Now, as\ (x, y) \in R, \quad But, (y, x) \notin R$

Thus, R is not symmetric

$Again, (1,6) \in Rand\ (2,7) \in R$

$But, (1,7) \notin R$

Thus, R is not transitive.

Therefore, R is neither Reflexive, nor symmetric and nor transitive.

(iii) Relation R in the set A = {1, 2, 3, 4, 5, 6} $R = \{(x, y): y\ is\ divisible\ by\ x\}$

Solution: Given,R={(x,y):y is divisible by x} in A={1,2,3,4,5,6}

Here, R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3,3), (3, 6), (4,4), 5, 5), (6, 6)}

$Now, Since, (1, 1) \in R$

$(2,2) \in R\ and\ (3,3) \in R$

Thus, R is reflexive.

$Again, Since, (1,2) \in R$

$But, (2,1) \notin R$

Thus, R is not symmetric.

$Again, since, (1, 4) \in R$

$And\ (4,4) \in R$

$\Rightarrow (1, 4) \in R$

Thus, R is transitive.

Therefore, R is reflexive and transitive but not symmetric.

(iv) Relation R in the set Z of all integers defined as $R = \{(x, y): x - y \text{ is an integer}\}$

Solution:

$Given, R = \{(x, y): x - y \text{ is an integer}\}$
In set Z of all integer.

$Here, (x, x), i.e. (1, 1) = 1 - 1 = 0 \in Z$
Therefore, R is reflexive relation.

$Now, (x, y) \in R$

$\Rightarrow (y, x) \in R, i.e. x - y \text{ is an integer}$

$\Rightarrow y - x \text{ is also an integer}$
Therefore, R is symmetric.

$Again, (x_1, y_1) = x_1 - y_1 \in Z$

$And, (y_1, z_1) = y_1 - z_1 \in Z$

$\Rightarrow (x_1, z_1) \in R$
Therefore, R is transitive.

Thus, R is reflexive, symmetric and transitive.

(v) Relation R in the set A of human beings in a town at a particular time given by

$(a) R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$
Solution:

$Here, since, x \text{ and } x \text{ work at the same place},$

$\therefore (x, x) \in R$
And thus, R is reflexive.

$Again, since\ x \text{ and } y \text{ work at same place}$

$\therefore (x, y) \in R \quad \Rightarrow (y, x) \in R$

$And\ thus, y \text{ and } x \text{ work at same place}$
Thus, R is symmetric.

$Now, again, (x, y) \in R$

$And\ (y, z) \in R$

$\Rightarrow (x, z) \in R$

Therefore, R is transitive

Thus, R is reflexive, symmetric and transitive.

$(b) R = \{(x, y): x\ and\ y\ live\ in\ the\ same\ locality\}$

Solution: Since, x and x both live in the same locality,

$Thus, (x, x) \in R$

Therefore, R is reflexive.

Since, x and y both live in the same locality

$Thus, (x, y) \in R$

This means y and x both also live in the same locality

Therefore, R is symmetric

$Now, since, (x, y) \in Rand\ (y, z) \in R$

$\Rightarrow (x, z) \in R$

Thus, R is transitive.

Thus, R is reflexive, symmetric and transitive.

$(c) R = \{(x, y): x\ is\ exactly\ 7\ cm\ taller\ than\ y\}$

Solution: Here, x is not exactly 7 cm taller than x

$Thus, (x, x) \notin R$

Thus, R is not reflexive

Now, x is exactly 7 cm taller than y, thus, y is not exactly 7 cm taller than x.

$Thus, (x, y) \in R\ but\ (y, x) \notin R$

Thus, R is not symmetric,

Again, x is exactly 7 cm taller than y and y is exactly 7 cm taller than z, then x would not be 7 cm taller than z

Thus, R is not transitive relation.

Therefore, R is neither reflexive nor transitive and nor symmetric.

$(d) R = \{(x, y): x\ is\ wife\ of\ y\}$

Solution: Here, it is clear that x is not the wife of x

$Thus, (x, x) \notin R$

$Thus, R\ is\ not\ reflexive$

Again, here x is wife of y, but y is not wife of x

$Thus, (x, y) \in R but\ (y, x) \notin R$

Thus, R is not symmetric

$Again, since\ (x, y) \in R\ \ and\ (y, z) \in R$

$\therefore (x, z) \notin R$

Thus, R is not transitive.

Therefore, R is neither reflexive nor transitive and nor symmetric.

$(e)\ R = \{(x, y): x\ is\ father\ of\ y\}$

Solution: Here, it is clear that x is not the father of x

$Thus, (x, x) \notin R$

$Thus, R\ is\ not\ reflexive$

Again, here x is father of y, but y is not father of x

$Thus, (x, y) \in R\ \ but\ (y, x) \notin R$

Thus, R is not symmetric

$Again, since\ (x, y) \in R\ \ and\ (y, z) \in R$

$\therefore (x, z) \notin R$

Thus, R is not transitive.

Therefore, R is neither reflexive nor transitive and nor symmetric.

Question 2: Show that the relation in the set R of real number, defined as $R = \{(a, b): a \leq b^2\},$ Is neither reflexive nor symmetric nor transitive.

Solution:

$If\ (a, a) \in R$

$\therefore a \leq a^2$

But this relation is contradictory, and hence false

Thus, R is not reflexive

$Again, if\ (a, b) = (b, a)$

$\therefore a \leq b^2\ and\ b \leq a^2$

This relation is also not possible and hence false

Thus, R is not symmetric

$Now, if\ a \leq b^2\ and\ b \leq c^2$

$\Rightarrow a \leq c^2$

This relation is also false

Thus, R is not transitive

Hence, R is neither reflexive nor transitive and nor symmetric.

Question 3: Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a,b) : b = a + 1} is reflexive, symmetric or transitive.

Solution:

Let A = {1, 2, 3, 4, 5, 6}

A relation R is defined on set A as:

R = {(a,b) : b = a + 1}

Thus, R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

$Now, 1, 2, 3, 4, 5, 6 \in R$

$But, (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \notin R$

Thus, R is not reflexive relation

$Again, (1,2) \in R\ but\ (2,1) \notin R$ Again,(1,2)∈R but (2,1)∉R

Thus, R is not symmetric

$Now, again, (3,4) \in R\ and\ (4,5) \in R$

$But, (3,5) \notin R$

Now,again,(3,4)∈R and (4,5)∈R

But,(3,5)∉R

Thus, R is not transitive.

Hence, R is neither reflexive nor transitive and nor symmetric.

Question 4: Show that the relation R in R defined as $R = \{(a, b) : a \leq b\},$ is reflexive and transitive but not symmetric.

Solution:

$Given, R = \{(a, b): a \leq b\}$

$Let\ b = a$

$\therefore (a, b) \in R\ as\ a \leq b$

Thus, R is reflexive relation

$Again, let\ (a, b) \in R$

$\Rightarrow a \leq b$

$\Rightarrow (b, a) \notin R$

$As\ a \leq b\ and\ b \leq a, both\ cannot\ be\ true$

Thus, R is not symmetric

$Again, let\ (a, b) \in R\ and\ (b, c) \in R$

$\Rightarrow a \leq b\ and\ b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$

Thus, R is transitive.

Therefore, R is reflexive and transitive but not symmetric.

Question 5: Check whether the relation R in R defined by $R = \{(a, b): a < b^3\}$ is reflexive, symmetric or transitive.

Solution:

$Here, given\ R = \{(a, b): a < b^3\}$

$Let\ a = \frac{1}{2}$

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right) \notin R$

$Because\ \frac{1}{2} \nleq \left(\frac{1}{2}\right)^3$

Thus, R is not reflexive

$Now, let\ (1,2) \in R$

$\therefore (2,1) \notin R$

$Because, 2 \nleq (1)^3$

Thus, R is not symmetric

$Let\ \left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R$

$as\ 3 < \left(\frac{3}{2}\right)^3\ and\ \frac{3}{2} < \left(\frac{6}{6}\right)^3$

$\therefore \left(3, \frac{6}{5}\right) \notin R\ \ since, 3 > \left(\frac{6}{5}\right)^3$

Thus, R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

Question 6: Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Solution:

$Given, R\ in\ the\ set\ \{1, 2, 3\}\ given\ by$

$R = \{(1, 2), (2, 1)\}$

$Here, (a, a), i.e. (1, 1) \notin R$

Thus, R is not reflexive

$Now, when\ (a, b) \in R\ and\ (b, a) \in R$

Therefore, R is symmetric

$Again, (a, b) \in Rand\ (b, c) \in R$

$\Rightarrow (a, c) \notin R$

Thus, R is not transitive.

Hence, R is symmetric but not reflexive or transitive.

Question 7: Show that the relation R in the set of all the books in a library of a college, given by R = {(x, y): x and y have same number of pages}, is an equivalence relation.

Solution: Given, by R = {(x, y): x and y have same number of pages}

$Thus, (x, x) \in R$

And hence, R is reflexive.

Again, since x an y have same number of pages

$Thus, (x, y) \in R\ and\ (y, x) \in R$

Thus, R is symmetric

$Now, (x, y) \in R\ \Rightarrow (x, z) \in R$

Because number of pages in x and z is same.

Thus, R is transitive.

Hence, R is reflexive as well as symmetric and transitive.

Thus, R is an equivalence realtion.

Question 8: Show that the relation R in the set A = {1, 2, 3, 4, 5}, given by R = {(a, b): |a – b| is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

Solution:

$Here, Let\ A = \{1, 2, 3, 4, 5\}$

$Given, R = \{(a, b): |a - b|\ is\ even\}$

$Thus, R = \{(1, 3), (1, 5), (3, 5), (2, 4)\}$

$Now, let\ an\ element\ a\ of\ set\ of\ A,$

$Therefore, |a - a| = 0$

Thus, R is reflexive

$Again, if\ |a - b| is\ even$

$\therefore |b - a|\ would\ also\ be\ even$

Therefore, R is symmetric.

$And\ when\ (a, b) \in R\ and\ (b, c) \in R$

$\Rightarrow |a - b| is\ even\ |b - c| is\ even$

$Let\ a - b = 2m_1\ and\ |b - c| 2m_2$

$Where, m_1\ and\ m_2\ are\ integers$

$Thus, a - c = (a - b) + (b - c)$

$\Rightarrow\ a - c = 2m_1 + 2m_2$

$\Rightarrow\ a - c = 2(m_1 + m_2)$

$\Rightarrow |a - c| is\ even$

$\Rightarrow (a, c) \in R$

Thus, R is transitive.

Now, the elements of {1, 3, 5} are related to each other.

Because |1 – 3| = 2;

| 3 – 5 | = 1, and | 1- 5 | = 4

And all numbers are even numbers.

Similarly, elements of (2, 4) are related to each other.

Because, |2 – 4| = 2, which is even number.

But, no element of set, {1, 3, 5} is related to any element of {2, 4}

Because, | 1 – 2| = 1; |3 – 2| = 1; |5 – 2|= 3; |3 – 4| = 1 and | 5 – 4| =1, which are not even numbers.

Hence, no element of {1, 3, 5} is related to any element of {2, 4}

Question 9: Show that the relation R in the set $A = \{x \in Z: 0 \ \leq x \leq 12\}, given\ by$
$(i) R = \{(a, b): |a - b| is\ a\ multiple\ of\ 4\}$

$(ii) R = \{(a, b): a = b\}$
is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution:

$Given, A = \{x \in Z: 0 \ \leq x \leq 12\}$

$(i) R = \{(a, b): |a - b| is\ a\ multiple\ of\ 4\}$
= {(0, 0), (0, 4), (0, 8), (0, 12), (1, 1), (1, 5), (1, 9), (2, 2), 2, 6), 2, 10), 3, 3), (3, 7), (3, 11), (4, 4), (4, 8), (4, 12), 5, 5), (5, 9), (6, 6), (6, 10), (7, 7), (7, 11), (8, 8), (8, 12), (9, 9), (10, 10), (11, 11), (12, 12)}

$Here, 0, 1, 2, 3, \dots \dots 12, \in A$

$\therefore (0, 0), (1, 1), (2, 2), \dots \dots (12, 12) \in R$

$\therefore R\ is\ reflexive$

$Again, (1, 5) \in R\ as |5 - 1| = |1 - 5|$

$\therefore R\ is\ symmetric.$

$Again, (2, 6) \in R and\ (6, 10) \in R$

$\therefore (2, 10) \in R$
Therefore, R is transitive.

Thus, R is reflexive, symmetric and transitive. Thus, R is an equivalence relation.

The set of elements related to 1 is equal to {1, 5, 9}

$(ii) R = \{(a, b): a = b\}$

$\therefore R = \{(0, 0), (1, 1), (2, 2), \dots . . (12, 12)\}$

$Here, 01, 2, 3, \dots \dots \dots . 12 \ \in A$

$\therefore (0, 0), (1, 1), (2, 2), \dots \dots \dots (12, 12) \in R$
Thus, R is reflexive.

$Again, (2,2) \in R \quad \therefore (2,2) \in R$

Thus, R is symmetric

$Again, (5,5) \text{ and } (5,5) \in R$

$\therefore (5,5) \in R$

Thus, R is transitive.

Therefore, R is equivalence relation.

The set of elements related to 1 = {1}

Question 10: Give an example of a relation, which is

i. Symmetric but neither reflexive nor transitive
ii. Transitive but neither reflexive nor symmetric
iii. Reflexive and symmetric but not transitive
iv. Reflexive and transitive but not symmetric
v. Symmetric and transitive but not reflexive.

Solution:

(i) Let R = {(1, 2), (2, 1)}

$Here, since, (1,1) \notin R$

$Thus, R \text{ is not reflexive}$

$Here, (1,2) \in R \text{ and } (2,1) \in R$

$Thus, R \text{ is symmetric}$

$Again, (1,2) \in R \text{ and } (2,1) \in R$

$But \ (1,1) \notin R$

$Thus, R \text{ is not transitive.}$

(ii) Let R = {(1, 2), (2, 3), (1, 3)}

$Here, (2,2) \notin R$

$(1,1) \notin R \text{ and } (3,3) \notin R$

$Thus, R \text{ is not reflexive}$

$Now, since, (1,2) \in R \text{ but } (2,1) \notin R$

$Thus, R \text{ is not symmetric}$

$Again, (1,2) \in R \text{ and } (2,3) \in R$

$\Rightarrow (1,3) \in R$

$Thus, R \text{ is transitive.}$

(iii) R = {(1, 1) (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)}

$Here, (1,1) \in R\ (2,2) \in R\ and\ (3,3) \in R$

$Thus, R\ is\ reflexive.$

$Now, (1,2) \in R \Rightarrow (2,1) \in R$

$Thus, R\ is\ symmetric$

$Again, (1,2) and\ (2,3) \in R$

$But, (1,3) \notin R$

$Thus, R\ is\ not\ transitive.$

(iv) R ={(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (, 2), (2, 3), (3, 2)}

$Here, (1,1), (2,2), (3,3), (4,4), (5,5) \in R$

$Thus, R\ is\ reflexive$

$Now, Since, (1,2) \in R\ but\ (2,1) \notin R$

$Thus, R\ is\ not\ symmetric$

$Again, (1,2) and\ (2,3) \in R$

$\Rightarrow (1,3) \in R$

$Thus, R\ is\ transitive.$

(v) R = {(2, 3), (3, 2), (1, 2), (1, 3), (3, 1)}

$Here, (3,3) \notin R$

$Thus, R\ is\ not\ reflexive$

$Since, (2,3) \in R \Rightarrow (3,2) \in R$

Thus, R is symmetric.

$Again, since, (1,3) \in R\ and\ (3,1) \in R$

$\Rightarrow (1,1) \in R$

Thus, R is transitive.

Question 11: Show that the relation R in the set A of points in a plane given by R = {(P, Q): distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Solution: Let O is the origin

Given, R = {(P, Q): distance of the point P from the origin is same as the distance of the point Q from the origin}

Therefore, R = {(P, Q): OP = PQ}

Now, let OP = y

$Therefore, (y, y) \in R \; because \; OP = OP$
Thus, R is reflexive relation.

Let OP = OQ = y

$Therefore, (y, y) \in R$

$\Rightarrow (y, y) \in R$
Thus, R is symmetric.

Again, Let OP = OQ = y and OQ = OR = y

$Therefore, (y, y) \in R \; and \; (y, y) \in R$

$\Rightarrow (y, y) \in R$
Thus, R is transitive.

Thus, all distance related to P from the origin is same as OP. As a circle is the locus of all points having same distance from a point, in the given case from O, therefore, the set of the points related to P is a circle passing through P with O as the centre, a fixed point.

Question 12: Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2): T_1 is similar to T_2}, is an equivalence relation. Consider three right angled triangles T_1 with sides 3, 4, 5 : T_2 with sides 5, 12, 13: and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Solution:

$Given, R = \{(T_1, T_2): T_1 \; is \; similar \; to \; T_2\}$ and T_1 and T_2 are triangles.

Since, every triangle is similar to itself.

$Thus, T_1, T_2 \in R$

$\therefore R \; is \; reflexive$
Similarly, two triangles are similar

$\therefore T_1 \cong T_2 \;\Rightarrow\; T_2 \cong T_1$

$\therefore R \; is \; symmetric.$

$Now, if \; T_1 \cong T_2 \; and \; T_2 \cong T_3$

$\therefore T_1 \cong T_3$

$Thus, R \; is \; transitive$
Thus, R shows equivalence relation.

Again, as given in question,

Three right angled triangles

T_1 with sides 3, 4, 5

T_2 with sides 5, 12, 13 and

T_3 with sides 6, 8, 10.

Now, in triangle, T_1 and T_3 proportion of sides is

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{1}{2}$$

Since, corresponding sides of triangles, T_1 and T_3 are proportional, thus, there triangle T_1 and T_3 are similar.

Hence, triangles T1 and T3 are related.

Question 13: Show that the relation R defined in the set A of all polygons R = {(P_1, P_2) : P_1 and P_2 have same number of sides} is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with side 3, 4 and 5?

Solution: Given, R = {(P_1, P_2) : P_1 and P_2 have same number of sides}

$Where, \quad P_1, P_2 \in A$

Since, P_1 and P_2 have same number of sides.

$\therefore\ (P_1, P_1) \in R\ for\ all\ P_1 \in A$

$\therefore R\ is\ reflexive$

$Now, (P_1, P_2) \in R\ \ for\ \ P_1, P_2 \in A$

Since number of sides in P_1 and P_2 are equal

$\therefore (P_2, P_1) \in R$

Thus, R is symmetric.

$Again, (P_1, P_2) \in R\ \ \&\ \ (P_2, P_3) \in R$

$Where, P_1, P_2, P_3 \in A$

Since, number of sides in P_1, P_2 and P_3 are equal

$\therefore (P_1, P_3) \in R$

$Thus, R\ is\ transitive$

Thus, R is an equivalence relation.

Now, since 3, 4 and 5 are the sides of given triangles T, which is a Pythagoras triplet, thus, given triangle is a right angled triangle.

Thus, the set A is set of right angled triangles.

Question 14: Let L be the set of all lines in XY – plane and R be the relation to L defined as R = {(L_1, L_2): L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line y = 2x + 4.

Solution: Here given,

$R = \{(L_1, L_2): L_1\ is\ parallel\ to\ L_2\}$

It is clear that

$L_1 \parallel L_1 \quad \Rightarrow (L_1, L_1) \in R$

Thus, R is reflexive.

$Since, L_1 \parallel L_2 \quad \therefore L_2 \parallel L_1$

$i.e.\ (L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$

Thus, R is symmetric.

$Now, if\ L_1 \parallel L_2\ and\ L_2 \parallel L_3$

$\therefore L_1 \parallel L_3$

Thus, R is transitive.

Since, R is reflexive, symmetric and transitive, thus, R is equivalence relation.

Now, the set of parallel lines related to the line y = 2x + 4, is y = 2x + C where C is any arbitrary constant.

Question 15: Let R be the relation in the set {1, 2, 3, 4} is given by R={(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric

(C) R is symmetric and transitive but not reflexive

(D) R is an equivalence relation.

Answer: (B) R is reflexive and transitive but not symmetric

Explanation: Here, A = {1, 2, 3, 4}

R ={(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}

$Since, (1,1), (2,2), (3,3), (4,4) \in R$

$Thus, R \text{ is reflexive}$

$Again, since\ (1,2) \in R but\ (2,1) \notin R$

$Thus, R \text{ is not symmetric}$

$Now, if\ (1,3) \in R\ and\ (3,2) \in R$

$\Rightarrow (1,2) \in R$

$\therefore R \text{ is transitive}$

Thus, Option (B) is correct

Question 16: Let R be the relation in the set N given by R= {(a, b) : a = b – 2, b > 6} Choose the correct answer

$(A)(2,4) \in R$ $\quad$ $(B)\ (3,8) \in R$

$(C)(6,8) \in R$ $\quad$ $(D)\ (8,7) \in R$

$Answer: (C)(6,8) \in R$

Explanation: Given, a = b -2, b > 6

In the case of option (A), a = 2, and b = 4

Here, since b < 6 thus, Option (A) is not correct

In the case of option (B)

a = 3, b = 8, which does not satisfy the equation a = b – 2

Thus, option (B) is not correct.

In the case of option (C)

a = 6 and b = 8

This satisfies the equation a = b – 2

Thus, option (C) is correct

Similarly, option (D) also not satisfies the equation a = b – 2

Question 1: Let A = {a, b, c} and the relation R be defined on A as follows:
R = {(a, a), (b, c), (a, b)}.
Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Solution: In order to make R reflexive, (b, b) and (c, c) will be added to R.

And in order to make R transitive, (a, c) will be added to R.

Therefore, The minimum number of order pair to be added to R will be (b, b), (c, c) and (a, c) - Answer

Question 2: Let D be the domain of real valued function f defined by $f(x) = \sqrt{25 - x^2}$ then, write D.

Solution: Here given D is the domain of $f(x) = \sqrt{25 - x^2}$

Therefore,

$$25 - x^2 \geq 0$$
$$\Rightarrow 25 \geq x^2$$
$$\Rightarrow x^2 \leq 25$$
$$\Rightarrow -5 \leq x \leq 5$$

Therefore, D = [– 5, 5] - Answer

Question 3: $Let\ f, g: R \to R$ be defined by $f(x)2x + 1\ and\ g(x) = x^2 - 2, \forall x \in R,$ respectively. Then find g o f.

Solution:

$$Given, f(x) = 2x + 1\ and\ g(x) = x^2 - 2, \forall x \in R$$
$$Therefore, g\ o\ f(x) = g(f(x))$$
$$= g(2x + 1)$$
$$= (2x + 1)^2 - 2$$
$$= 4x^2 + 1 + 4x - 2$$
$$Thus, g\ of(x) = 4x^2 + 4x - 1 \quad Answer$$

Question 4: $Let\ f, g: R \to R$ be the function defined by $f(x) = 1x - 3 \forall x \in R. Write\ f^{-1}$

Solution:

$$Given, f(x) = 1x - 3 \forall x \in R$$
$$Now, let\ y = f(x)$$
$$Therefore, y = 2x - 3$$
$$\Rightarrow 2x = y + 3$$
$$\Rightarrow x = \frac{y+3}{2}$$
$$Therefore, f^{-1}(x) = \frac{x+3}{2}\ Answer$$

Question 5: If A = {a, b, c, d} and the function f = {(a, b), (b, d), (c, a), (d, c)}, write f^{-1}

Solution: Given, f = {(a, b), (b, d), (c, a), (d, c)}

Therefore, f^{-1} ={(b, a), (d, b), (c, a), (c, d)} Answer

Question 6: If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2, wrtie\ f(f(x)).$

Solution: Given,

$$f(x) = x^2 - 3x + 2$$
$$Therefore, f(f(x)) = f(x^2 - 3x + 2)$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$$
$$= x^4 - 6x^3 + 9x^2 + 4x^2 - 3x^2 - 12x + 9x + 4 - 6 + 2$$
$$\Rightarrow f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x \quad Answer$$

Question 7: Is g = {(1, 1), (2, 3, (3, 5), (4, 7)} a function? If g is described by g(x) = αx + β, then what value should be assigned to α and β?

Solution: Given, g = {(1, 1), (2, 3, (3, 5), (4, 7)}

Therefore, each of the element of domain will be have unique image.

Consequently, g is a function.

$Now, since\ g(x) = \alpha x + \beta\ (as\ given\ in\ question)$

$Therefore, g(1) = \alpha \times 1 + \beta$

$Thus, 1 = \alpha + \beta\ ------(i)$

$Similarly, g(2) = \alpha \times 2 + \beta$

$Therefore, \quad 3 = 2\alpha + \beta ----(ii)$

$Now, from\ equation (i) and\ (ii) we\ get$

$$\begin{array}{l} 1 = \alpha + \beta \\ \underline{\ 3 = 2\alpha + \beta} \\ - \quad - \quad - \\ \hline -2 = -\alpha \end{array}$$

$Therefore, \alpha = 2$

Now, after substituting the value of α in equation (i), we get

$1 = 2 + \beta$

$\Rightarrow \beta = 1 - 2$

$\Rightarrow \beta = -1$

$Thus, \alpha = 2\ and\ \beta = -1 \quad Answer$

Question 16: If A = {1, 2, 3, 4}, define relations on A which have properties of being

(a) Reflexive, transitive but not symmetric

Solution:

$Let\ R_1 = \{(1,2), (2,1), (1,1), (2,2)\}$

$Thus, it\ is\ clear\ that\ (1,2) \in R_1 and\ (2,1) \in R_1$

$Thus, R_1\ is\ reflexive$

$Again, (1,2) \in R_1, (2,1) \in R_1$

$\Rightarrow (1,1) \in R_1$

$Similarly, (2,1) \in R_2, (1,2) \in R_2$

$\Rightarrow (2,2) \in R_2$

$Therefore, R_2\ is\ transitive$

(b) Symmetric but neither reflexive nor transitive

Solution:

$Let\ R_2 = \{(1,2),(2,1)\}$

$Thus, it\ is\ clear\ that\ (1,2) \in R_2\ and\ (2,1) \in R_2$

$Therefore, R_2\ is\ symmetric$

$But\ here, (1,1) \notin R_2$

$Thus, R_2\ is\ neither\ reflexive\ nor\ transitive$

(c) Reflexive, symmetric and transitive.

Solution:

$Let\ R_3 = \{(1,2),(2,1),(1,1),(2,2),(3,3),(4,4)\}$

$Thus, it\ is\ clear\ that\ R_3\ is\ Reflexive, symmetric\ and\ transitive$

Question: 17 – Let R be relation defined on the set of natural number N as follows: $R = \{(x,y): x \in N, y \in N, 2x + y = 41\}$ Find the domain and range of the relation R. Also, verify whether R is reflexive, symmetric and transitive.

Solution:

$Given, R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$

$Thus, domain\ of\ R = \{1,23,4, \dots \dots \dots \dots .. ,20\}$

$And\ range\ of\ R = (1,3,5,7,9, \dots \dots \dots \dots \dots ,39\}$

$Here, since\ (2,2) \notin R$

$because\ 2 \times 2 + 2 = 6 \neq 41$

$Thus, R\ is\ not\ reflexive$

$Again, since\ (1,39) \in R$

$Because, 2 \times 1 + 39 = 41$

$But, (39,1) \notin R$

$Because, 2 \times 39 + 1 \neq 41$

$Thus, R\ is\ not\ symmetric$

$Now, because\ 2 \times 11 + 19 = 41$

$Thus, (11,19) \in R$

$And\ since, 2 \times 19 + 3 = 41$

$Thus, (19,3) \in R$

$But, since, 2 \times 11 + 3 = 25 \neq 41$

$Thus, (11,3) \notin R$

$Therefore, R\ is\ not\ transitive$

Thus, R is neither reflexive, nor symmetric and nor transitive.

Question – 18 – Given A {2, 3, 4}, B = {2, 5, 6, 7}. Construct an example of each of the following:

(a) an injective mapping from A to B

Solution:

$Let\ f: A \to B\ denotes\ a\ mapping\ such\ that$

$f = \{(x, y): y = x + 3\}$

$It\ can\ be\ written\ as\ follows\ in\ roster\ form$

$f = \{(2,5), (3,6), (4,7)\}$

But this is an injective mapping.

(b) a mapping from A to B which is not injective

Solution:

$Let\ g: A \rightarrow B\ denotes\ a\ mapping\ such\ that$

$g = \{(2,2): (3,5), (4,5)\}$

Here it is clear that it is not an injective mapping.

(c) a mapping from B to A

Solution:

$Let\ h: B \rightarrow A\ denots\ a\ mapping\ such\ that$

$h = \{(2,2), (5,3)(6,4), (7,4)\}$

Here it is clear that every first component is from B and second component is from A, thus h is a mapping from B to A.

Question – 19: Give an example

(i) Which is one-one but not onto

Solution:

Let A be the set of all 100 students in a school in a particular class say ninth. $Let\ f: A \rightarrow N$ be the mapping defined by $f(x) =\ roll\ number\ of\ the\ student\ x.$

Here it is clear that f is one-one because no two students of the same class can have the same roll number.

Let roll number of student start from 1 and ends on 100.

This implies that 101 in N is not the roll number of any of the student of the class, so that 101 is not an image of any element of A under f.

Therefore, f is not onto.

(ii) Which is not one-one but onto

Solution:

$Let\ f: N \rightarrow N, given\ by\ f(1) = f(2)$

$and\ f(x) =\ x - 1, for\ every\ x > 2$

This is onto but not one-one.

(iii) Which is neither one-one nor onto

Solution:

$Let\ f: R \rightarrow R, defined\ as\ f(x) = x^2$

Here it is neither one-one nor onto.

Question 20: Let A = R – {3}, B = R – {1}. Let $f: A \to B$ be defined by $f(x) = \frac{x-2}{x-3} \forall x \in A.$ Then show that f is bijective.

Solution: Given,

$$A = R - \{3\}, \qquad B = R - \{1\}$$

$$f: A \to B \text{ be defined by } f(x) = \frac{x-2}{x-3} \forall x \in A$$

Now, for injectivity:

$$Let\ f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

After cross multiplication, we get

$$(x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$

$$\Rightarrow \cancel{x_1x_2} - 3x_1 - 2x_2 + \cancel{6} - \cancel{x_1x_2} + 2x_1 + 3x_2 - \cancel{6} = 0$$

$$\Rightarrow -3x_1 + 2x_1 - 2x_2 + 3x_2 = 0$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$\Rightarrow x_2 = x_1$$

Thus, f(x) is an injective function.

Now, for surjectivity:

$$Let\ y = \frac{x-2}{x-3}$$

$$After\ cross\ multiplication\ we\ get$$

$$y(x-3) = x - 2$$

$$\Rightarrow xy - 3y = x - 2$$

$$\Rightarrow xy - x = 3y - 2$$

$$\Rightarrow x(y-1) = 3y - 2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

$$\Rightarrow x = \frac{3y-2}{y-1} \in A\ \forall\ y \in B$$

Therefore, f(x) is a surjective function.

Here, we can see that f(x) is a surjective and injective both funtion.

Thus, f(x) is bijective.

Question 21: Let A = [– 1, 1]. Then, discuss whether the following functions defined on A are one-one, onto or bijective:

$(i) f(x) = \frac{x}{2}$

Solution:

$Given, f(x) = \frac{x}{2}$

$Let\ f(x_1) = f(x_2)$

$Therefore, \quad \frac{x_1}{2} = \frac{x_2}{2}$

$\Rightarrow x_1 = x_2$

This shows that f(x) is one-one

$Now, Let\ y = \frac{x}{2}$

$i.e.\ \ x = 2y \notin A\ \forall\ y \in A$

$as\ for\ y = 1\ \in A\ \ and\ x = 2 \notin A$

Clearly, f(x) is not onto.

Thus, f(x) is not bijective as it is one-one and not onto.

$(ii)\ g(x) = |x|$

Solution:

$Given, \qquad g(x) = |x|$

$Let\ g(x_1) = g(x_2)$

$\Rightarrow |x_1| = |x_2|$

$\Rightarrow x_1 = \pm\, x_2$

Clearly, g(x) is not one-one

$Now, let\ y = |x|$

$\Rightarrow x = \pm y \notin A\ \forall\ y \in A$

Here, it is also clear that g(x) is not onto.

Since, g(x) is neither one-one nor onto, thus g(x) is not bijective.

$(iii)\ h(x) = \ x|\,x\,|$

Solution:

$Given, \quad h(x) = x\,|\,x\,|$

$Let, h(x_1) = h(x_2)$

$\Rightarrow x_1\,|\,x_1| = x_2\,|\,x_2|$

$\Rightarrow x_1 = x_2$

Hence, h(x) is a surjective function.

There h(x) is bijective.

$(iv)\ k(x) = x^2$

Solution:

$Given, \quad k(x) = x^2$

$Let, \quad k(x_1) = k\,(x_2)$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm\, x_2$

Here, it is clear that k(x) is not one-one.

$Now, let\ \ y = x^2$

$\Rightarrow x = \sqrt{y} \in A\ \forall\ y \in A$

$Here, since\ y = -1, x = \sqrt{-1} \notin A$

$Therefore, k(x) = x^2\ is\ neither\ one - one\ not\ onto.$

Question 22: Each of the following defines a relation on N:

$(x) x\ is\ greater\ than, y, x, y \in N$

$(ii) x + y = 10, x, y \in A$

$(iii)\ xy\ is\ square\ of\ an\ integer\ x, y \in N$

$(iv) x + 4y = 10, \quad x, y \in N$

Determine which of the above relations are reflexive, symmetric and transitive.

Solution:

(i) *Given,* $\quad x$ *is greater than* $y, x, y \in N$

This implies that $x > y$

Now, for $(x, x) \in R$

Therefore, $x > y$ *is not true for any* $x \in N$

Thus, R *is not reflexive.*

Now, let $(x, y) \in R$ *and* $(y, z) \in R$

$\Rightarrow x > y$ *and* $y > z$

Therefore, $\quad x > z$

Clearly, R *is transitive.*

Hence, the given relation is only transitive.

(ii) *Given,* $x + y = 10, \quad x, y \in N$

Therefore, $R = \{(x, y): x + y = 10, x, y \in N\}$

$= \{(1,9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1)\}$

Here, since, $(1,1) \notin N$

Therefore, R is not reflexive.

Again, as for all $(x, y) \in R$ *there is* $(y, x) \in R$

Therefore, R is symmetric.

Again, since $(1, 9) \in R$ *and* $(9,1) \in R$

But, here $(1, 1) \notin R$

Therefore, R is not transitive.

Thus, R is only symmetric.

(iii) *Given,* xy *is square of an integer* $x, y \in N$

$\Rightarrow R = \{(x, y): xy$ *is square of an integer* $x, y \in N\}$

Here it is clear that $(x, x) \in R$ *for all* $x \in N$

Since, x^2 *is square of an integer for any* $x \in N$

Therefore, R is reflexive.

If $(x, y) \in R$

this, $\Rightarrow xy$ *is a square of an integer*

$\Rightarrow yx$ *is a square of an integer*

$\Rightarrow (y, x) \in R$

Therefore, it is clear that R is symmetric.

$Now, if\ (x,y) \in R\ and\ (y,z) \in R$

$\Rightarrow xy\ is\ a\ square\ of\ an\ integer\ and\ yz\ is\ a\ square\ of\ an\ integer$

$Let\ xy = m^2\ and\ yz = n^2\ for\ some\ m,n \in Z$

$$Therefore, \quad x = \frac{m^2}{y}\ and\ z = \frac{n^2}{y}$$

$$\Rightarrow xz = \frac{m^2\,n^2}{y^2}, which\ is\ a\ square\ of\ an\ integer$$

Therefore, R is transitive.

$(iv)\ Given\ x + 4y = 10, x, y \in N$

$Let\ R = \{(x,y): x + 4y = 10, x, y \in N$

$= \{(2,3), (6,1)\}$

$Thus, it\ is\ clear\ that\ (1,1), (3,3), \ldots \ldots \ldots . \notin R$

Therefore, R is not reflexive.

$Thus, (6,1) \in R\ \ but\ (1,6) \notin R$

Therefore, R is not symmetric.

Since, there is no element which begins with $y\ for\ any\ (x,y) \in R$

Therefore, R is a transitive.

Question 23: Let A = {1, 2, 3, ………, 9} and R be the relation in A x A defined by (a, b) R (c, b) if a + d = b + c for (a, b), (c, d) in A × A. Prove that R is an equivalence relation and also obtain the equivalent class [(2, 5)].

Solution:

$Let\ A = \{1,2,3, \ldots \ldots .., 9\}\ and\ R\ (a,b)R\ (c,d)$

$if\ \ a + d = b + c\ for\ (a,b) \in A\ \times A$

$and\ (c,d) \in A \times A$

$Let\ (a,b)R\ (a,b)$

$\Rightarrow a + b = b + a, \forall\ a, b\ \in A$

$Which\ is\ rue\ for\ any\ a, b\ \in A$

Therefore, R is reflexive.

Let (a, b) R (c, d)

$\Rightarrow a + d = b + c$

$\Rightarrow c + b = d + a$

$\Rightarrow (c,d)R\ (a,b)$

Therefore, R is symmetric.

Let (a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow a + d = b + c \text{ and } c + f = d + e$

$a + d = b + c \text{ and } d + e = c + f$

$\Rightarrow (a + d) - (d + e) = (b + c) - (c + f)$

$\Rightarrow a - e = b - f$

$\Rightarrow a + f = b + e$

$\Rightarrow (a + b) R \ (e + f)$

Therefore, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation.

Equivalence class containing {(2, 5)} is {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}.

Question 24: Using the definition, prove that the function $f: A \rightarrow B$ is invertible if and only if f is both one-one and onto.

Solution: By the definition of an invertible function:

A function $f: x \rightarrow y$ is defined to be and invertible function, if there exists a function $g: y \rightarrow x \text{ such that } gof = I_x \text{ and } fog = I_y$

The function g is called the inverse of f and is denoted by f^{-1}.

$For\ gof = I_x \text{ and } fog = I_y\ f(x)$ has to one-one and onto.

Therefore, f(x) should be both one-one and onto.

Question 25: Functions $f, g: R \rightarrow R$ are defined respectively, by $f(x) = x^2 + 3x + 1, \quad g(x) = 2x - 3,$ find:

(i) fog

Solution:

$Given\ f(x) = x^2 + 3x + 1, and\ g(x) = 2x - 3$

$Now, fog\ (x) = f(g(x))$

$= f(2x - 3)$

$= (2x - 3)^2 + 3(2x - 3) + 1$

$= 4x^2 + \cancel{9} - 12x + 6x - \cancel{9} + 1$

$Thus, fog\ (x) = 4x^2 - 6x + 1$

(ii) gof

Solution:

$Given\ f(x) = x^2 + 3x + 1, and\ g(x) = 2x - 3$

$Now, \quad gof(x) = g(f(x))$

$= g(x^2 + 3x + 1)$

$= 2(x^2 + 3x + 1) - 3$

$= 2x^2 + 6x + 2 - 3$

$Thus, \quad gof(x) = 2x^2 + 6x - 1$

(iii) fof

Solution:

$Given\ f(x) = x^2 + 3x + 1, and\ g(x) = 2x - 3$

$Now, \quad fof(x) = f(f(x))$

$= f(x^2 + 3x + 1)$

$= (x^2 + 3x + 1)^2 - 3(x^2 + 3x + 1) + 1$

$= x^4 + 9x^2 + 1 + 6x^3 + 2x^2 + 6x + 3x^2 + 9x + 3 + 1$

$= x^4 + 6x^3 + (9x^2 + 2x^2 + 3x^2) + (6x + 9x) + 3 + 1 + 1$

$Thus, fof = x^4 + 6x^3 + 14x^2 + 15x + 5 \quad Answer$

(iv) gog

Solution:

$Given\ f(x) = x^2 + 3x + 1, and\ g(x) = 2x - 3$

$Now, gog(x) = g(g(x))$

$= g(2x - 3)$

$= 2(2x - 3) - 3$

$= 4x - 6 - 3$

$= 4x - 9$

$Thus, gog = 4x - 9\ \ Answer$

Question: 26 – Let * be the binary operation defined on Q. Find which of the following binary operations are commutative.

$(i)\ a * b = a - b\ \forall\ a, b\ \in Q$

Solution:

$Here\ given\ a * b = a - b\ \forall\ a, b\ \in Q$

$thus, b * a = b - a\ \forall\ a, b\ \in Q$

$Thus, it\ is\ clear\ a * b\ \neq b * a$

$Therefore, here * is\ not\ commutative.$

$(ii)\ a * b = a^2 + b^2\ \forall\ a, b\ \in Q$

$Solution:$

$Here, given\ \ a * b = a^2 + b^2\ \forall, a, b\ \in Q$

$Thus, \quad b * a = b^2 + a^2$

$\Rightarrow b * a = a^2 + b^2$

$\Rightarrow b * a = a * b$

$Clearly, \quad * is\ commutative.$

$(iii)\ a * b = a + ab\ \forall\ a, b\ \in Q$

$Solution:$

$Here, given, \quad a * b = a + ab\ \forall\ a, b\ \in Q$

$Thus, b * a = b + ab$

$Thus, a + ab \neq b + ab$

$Therefore, \quad a * b \neq b * a$

$Thus, * is\ not\ commutative.$

$(iv)\ a * b = (a-b)^2\ \forall\, a, b \in Q$

$Solution:$

$Here, given\ a * b = (a-b)^2\ \forall\, a, b \in Q$

$Thus, a * b = (b-a)^2\ \forall\, a, b \in Q$

$\Rightarrow a * b = b * a$

$Thus, \quad * \ is\ commutative.$

Question: 27 – Let * be binary operation defined on R by $a * b = 1 + ab, \forall\, a, b \in R.$ Then the operation is:

(i) Commutative but not associative

(ii) associative but not commutative

(iii) neither commutative nor associative

(iv) both commutative and associative.

Answer: (i) is commutative but not associative.

Explanation:

$given\ \ a * b = 1 + ab$

$\Rightarrow a * b = 1 + ba$

$\Rightarrow a * b = b * a$

Therefore, * is commutative.

$Now, also\ a * (b * c) = a * (1 + bc)$

$\Rightarrow a * (b * c) = 1 + a(1 + bc)$

$\Rightarrow a * (b * c) = a + a + abc - - - -(i)$

$Now, again\ (a * b) * c = (1 + ab) * c$

$\Rightarrow (a * b) * c = 1 + (1 + ab)c$

$\Rightarrow (a * b) * c = 1 + c + abc - - - (ii)$

Now, from equation (i) and (ii) it is clear that

$a * (b * c) \neq (a * b) * c$

$Thus, this\ is\ not\ associative$

Example 1: Identify the range and domain the relation below: {(-2, 3), {4, 5), (6, -5), (-2,3)}

Solution: Since the x values are the domain, the answer is, therefore,
⟹ {-2, 4, 6}
The range is {-5, 3, 5}.

Example 2: Check whether the following relation is a function: B = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)}

Solution: B = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)}
Though a relation is not classified as a function if there is a repetition of x – values, this problem is a bit tricky because x values are repeated with their corresponding y-values.

Example 3: Determine the domain and range of the following function: Z = {(1, 120), (2, 100), (3, 150), (4, 130)}.

Solution: Domain of z = {1, 2, 3, 4 and the range is {120, 100, 150, 130}

Example 4: Check if the following ordered pairs are functions:

1. W= {(1, 2), (2, 3), (3, 4), (4, 5)
2. Y = {(1, 6), (2, 5), (1, 9), (4, 3)}

Solution:

1. All the first values in W = {(1, 2), (2, 3), (3, 4), (4, 5)} are not repeated, therefore, this is a function.
2. Y = {(1, 6), (2, 5), (1, 9), (4, 3)} is not a function because the first value 1 has been repeated twice.

Example 5: Determine whether the following ordered pairs of numbers are a function. R = (1,1); (2,2); (3,1); (4,2); (5,1); (6,7)

Solution: There is no repetition of x values in the given set of ordered pairs of numbers. Therefore, R = (1,1); (2,2); (3,1); (4,2); (5,1); (6,7) is a function.

Example 6

Identify the range and domain the relation below:

{(-2, 3), {4, 5), (6, -5), (-2, 3)}

<u>Solution</u>

Since the x values are the domain, the answer is, therefore,

⟹ {-2, 4, 6}

The range is {-5, 3, 5}.

Example 7

Check whether the following relation is a function:

B = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)}

<u>Solution</u>

B = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)}

Though a relation is not classified as a function if there is a repetition of x – values, this problem is a bit tricky because x values are repeated with their corresponding y-values.

Example 8

Determine the domain and range of the following function: Z = {(1, 120), (2, 100), (3, 150), (4, 130)}.

<u>Solution</u>

Domain of z = {1, 2, 3, 4 and the range is {120, 100, 150, 130}

Example 9

Check if the following ordered pairs are functions:

1. W= {(1, 2), (2, 3), (3, 4), (4, 5)
2. Y = {(1, 6), (2, 5), (1, 9), (4, 3)}

<u>Solution</u>

1. All the first values in W = {(1, 2), (2, 3), (3, 4), (4, 5)} are not repeated, therefore, this is a function.
2. Y = {(1, 6), (2, 5), (1, 9), (4, 3)} is not a function because, the first value 1 has been repeated twice.

Example 10

Unit 1→Set Theory: Solved and unsolved practice problems

Determine whether the following ordered pairs of numbers are a function.

R = (1,1); (2,2); (3,1); (4,2); (5,1); (6,7)

Solution

There is no repetition of x values in the given set of ordered pairs of numbers.

Therefore, R = (1,1); (2,2); (3,1); (4,2); (5,1); (6,7) is a function.

Practice Questions

1. Check whether the following relation is a function:

a. A = {(-3, -1), (2, 0), (5, 1), (3, -8), (6, -1)}

b. B = {(1, 4), (3, 5), (1, -5), (3, -5), (1, 5)}

c. C = {(5, 0), (0, 5), (8, -8), (-8, 8), (0, 0)}

d. D = {(12, 15), (11, 31), (18, 8), (15, 12), (3, 12)}

2. The Cartesian product B x B has 9 elements among which are found (–1, 0) and (0,1). Find the set B and the remaining elements of B x B.

3. Redefine the function: f(x) = |x – 1| – |x + 4|. Write its domain also.

4. Find the domain and range of the real function f(x) = x/1+x2.

5. If A = {a, b, c, d} & B = {e, f, g}. Is R = {(a, e) (a, f) (a, g) (b, e) (b, f) (b, g) (c, e) (c, f) (d, g)} a function from A to B. Give reasons to support your answer.

6. Let A = {a, b, c} and the relation R be defined on A as follows:

R = {(a, a), (b, c), (a, b)}.

Then, write the minimum number of ordered pairs to be added in R to make R reflexive and transitive.

7. Is g = {(1, 1), (2, 3, (3, 5), (4, 7)} a function? If g is described by g(x) = αx + β, then what value should be assigned to α and β?

11. Determine the range and domains of the relation R defined by R = {(x – 1), (x + 2) : x ∈ (2, 3, 4, 5)}

12.Let A = {3, 4, 5} and B = {6, 8, 9, 10, 12}. Let R be the relation 'is a factor of' from A to B. Find R.

Solved Examples (Functions):

1) Find the domain of $f(x) = 5x - 3$

Solution

The domain of a linear function is all real numbers, therefore,

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

2) Write $y = x^2 + 4x + 1$ using function notation and evaluate the function at $x = 3$.

Solution

Given, $y = x^2 + 4x + 1$

By applying function notation, we get

$f(x) = x^2 + 4x + 1$

Evaluation:

Substitute x with 3

$f(3) = 3^2 + 4 \times 3 + 1 = 9 + 12 + 1 = 22$

Example 2

Find the domain of the function $f(x)=-2x^2 + 12x + 5$

Solution

The function $f(x) = -2x^2 + 12x + 5$ is a quadratic polynomial, therefore, the domain is $(-\infty, \infty)$

How to find the domain for a rational function with a variable in the denominator?

To find the domain of this type of function, set the denominator to zero and calculate the variable's value.

Let's see a few examples below to understand this scenario.

Example 3

Determine the domain of $x-4/ (x^2 -2x-15)$

<u>Solution</u>

Set the denominator to zero and solve for x

$\Longrightarrow x^2 - 2x - 15 = (x - 5) (x + 3) = 0$

Hence, $x = -3$, $x = 5$

For the denominator not to be zero, we need to avoid the numbers −3 and 5. Therefore, the domain is all real numbers except −3 and 5.

Example 4

Calculate the domain and the range of the function $f(x) = -2/x$.

<u>Solution</u>

Set the denominator to zero.

$\Longrightarrow x = 0$

Therefore, domain: All real numbers except 0.

The range is all real values of x except 0.

Example : Evaluate the function $f(x) = 3(2x+1)$ when $x = 4$.

<u>Solution</u>

Plug $x = 4$ in the function $f(x)$.

$f(4) = 3[2(4) + 1]$

$f(4) = 3[8 + 1]$

$f(4) = 3 \times 9$

$f(4) = 27$

Example: *Write* the function $y = 2x^2 + 4x – 3$ in function notation and find $f(2a + 3)$.

<u>Solution</u>

$y = 2x^2 + 4x – 3 \Longrightarrow f(x) = 2x^2 + 4x – 3$

Substitute x with $(2a + 3)$.

$f(2a + 3) = 2(2a + 3)^2 + 4(2a + 3) – 3$

$= 2(4a^2 + 12a + 9) + 8a + 12 – 3$
$= 8a^2 + 24a + 18 + 8a + 12 – 3$
$= 8a^2 + 32a + 27$

Example : Represent $y = x^3 – 4x$ using function notation and solve for y at $x = 2$.

<u>Solution</u>

Given the function $y = x^3 – 4x$, replace y with $f(x)$ to get;

$f(x) = x^3 – 4x$

Now evaluate $f(x)$ when $x = 2$

⟹ $f(2) = 2^3 - 4 \times 2 = 8 - 8 = 0$

Therefore, the value of y at x=2 is 0

Example :Find f (k + 2) given that, $f(x) = x^2 + 3x + 5$.

Solution

To evaluate f (k + 2), substitute x with (k + 2) in the function.

⟹ $f(k + 2) = (k + 2)^2 + 3(k + 2) + 5$

⟹ $k^2 + 2^2 + 2k(2) + 3k + 6 + 5$

⟹ $k^2 + 4 + 4k + 3k + 6 + 5$

$= k^2 + 7k + 15$

Example: Given the function notation $f(x) = x^2 - x - 4$. Find the value of x when f (x) = 8

Solution

$f(x) = x^2 - x - 4$

Substitute f(x) by 8.

$8 = x^2 - x - 4$

$x^2 - x - 12 = 0$

Solve the quadratic equation by factoring to get;

⟹ $(x - 4)(x + 3) = 0$

⟹ $x - 4 = 0$; $x + 3 = 0$

Therefore, the values of x when f (x) = 8 are;

$x = 4$; $x = -3$

Unit 1→Set Theory: Solved and unsolved practice problems

Example: Evaluate the function $g(x) = x^2 + 2$ at $x = −3$

<u>Solution</u>

Substitute x with -3.

$g(−3) = (−3)^2 + 2 = 9 + 2 = 11$

Real life examples of function notation

Function notation can be applied in real life to evaluate mathematical problems as shown in the following examples:

Example: To manufacture a certain product, a company spends x dollars on raw materials and y dollars on the labour. If the production cost is described by the function $f(x, y) = 36000 + 40x + 30y + xy/100$. Calculate cost of production when the firm spends 10,000 and1,000 on raw materials and labour respectively.

Substitute the values of x and y in the production cost function

$\Longrightarrow f(10000, 1000) = 36000 + 40(10000) + 30(1000) + (10000)(1000)/100.$

$\Longrightarrow f(10000, 1000) = 36000 + 4000000 + 30000 + 100000$

$\Longrightarrow$ \$4136000.

Example : Mary is saves 100 weeklies for her an upcoming birthday party. If she already has1000, how much will she have after 22 weeks.

<u>Solution:</u>

Let x = number of weeks, and f(x) = total amount. We can write this problem in function notation as;

$f(x)=100x + 1000$
Now evaluate the function when $x =22$
$f(22) =100(22) +1000$
$f(22) =3200$

Therefore, the total amount is \$3200.

Example: The rate of talk-time of two mobile networks A and B charges is 34 plus 0.05/min and 40 plus 0.04/min respectively.

1. Represent this problem in function notation.
2. Which mobile network is affordable given that average number of minutes used each month is 1,160.
3. When is the monthly bill of the two networks equal?

Solution

1. Let x be the number of minutes used in each network.

Therefore, the function of network A is $f(x) = 0.05x + 34$ and network B is $f(x) = 0.04x+\$40$.

1. To determine which network is affordable, substitute $x = 1160$ in each function

$A \Longrightarrow f(1160) = 0.05(1160) + 34$

$= 58 + 34 = \$ 92$

$B \Longrightarrow f(1160) = 0.04(1160) + 40$

$= 46.4 + 40$

$= \$ 86.4$

Therefore, network B is affordable because its total talk-time cost is less than that of A.

1. Equate the two functions and solve x

$\Longrightarrow 0.05x + 34 = 0.04x + 40$

$\Longrightarrow 0.01x = 6$

$x = 600$

The monthly bill of A and B will be equal when the average number of minutes is 600.

Proof:

A $\Longrightarrow$ 0.05(600) +34 = $64

B $\Longrightarrow$ 0.04(600) + 40 = $64

Example : A certain number is such that when its added to 142, the result is 64 more than thrice the original number. Find the number.

<u>Solution</u>

Let x = the original number and f(x) be the resultant number after adding 142.

$f(x) = 142 + x = 3x + 64$

$2x = 78$

$x = 39$

Example : If the product of two consecutive positive integers is 1122, find the two integers.

<u>Solution</u>

Let x be the first integer;

second integer = x + 1

Now form the function as;

$f(x) = x (x + 1)$

find the value of x if f(x) = 1122

Replace the function f(x) by 1122

$1122 = x (x + 1)$

$1122 = x^2 + 1$

$x^2 = 1121$

Find the square of both sides of the function

$x = 33$

$x + 1 = 34$

The integers are 33 and 34.