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CATEGORY THEORY

Definition 1. A **category** C is a collection of

- a collection of **objects**, $\mathsf{ob}(\mathsf{C})$, containing $X, Y, Z, \cdots$
- a collection of **morphisms**, $\mathsf{mor}(\mathsf{C})$, containing $f, g, h, \cdots$

which satisfies:

- for morphism f, there is a **domain** X and **codomain** Y in objects, and we write $f : X \to Y$;
- for any two morphisms f, g where the codomain of f is equal to the domain of g, the **composite morphism** $g \circ f : X \to Z$ exists;
- for each object X, there is a **identity morphism** 1_X such that for any $f : X \to Y$, $1_Y f = f 1_X = f$;
- for three morphisms f, g, h where $h \circ g$ and $g \circ f$ are well defined, $h \circ (g \circ f) = (h \circ g) \circ f$, and written as $h \circ g \circ f$.

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Notice that we did not used here the word 'set'. Bertrand Russell showed that there is no sets of all sets(Russell's paradox). Therefore, by using the word set, we cannot treat the category of sets, groups, or lots of concepts we want. Therefore we used the word 'collection': the definition of this word depends on the context, even sometimes this word is informal.

Sometimes we write gf rather then $g \circ f$, if there is no ambiguity.

Example 2.

1. Set is the category which has sets as objects and functions as morphisms.
2. Group is the category which has groups as objects and homomorphisms as morphisms. Ab, Ring, Mod_R, and Field are also defined in the same sense for abelian groups, rings, R-modules, and fields.
3. Meas is the category which has measurable spaces as objects and measurable functions as morphisms.
4. Top is the category which has topological spaces as objects and continuous functions as morphisms. Man is also defined in the same sense for smooth manifolds.
5. Poset is the category which has partially ordered sets as objects and order-preserving functions as morphisms.

Due to the existence of the identity morphisms, it is possible to reconstruct the data of objects by using the data of morphisms. Indeed, in the set theory we have focused on the elements of set, but in the category theory we focus on the morphisms. This concept becomes clearer when we think the group as one object category where the elements of groups are morphisms, which will be discussed later, which is indeed the Cayley's theorem. Despite of this fact, it is quite common to name the category following after the objects, not the morphisms.

Example 3.

1. A group G defines a category $\mathsf{B}G$ with one object, where the morphisms are the group elements.
2. A poset P itself is a category with its elements as objects and $x \leq y$ implies there is a unique morphism $f : x \to y$.
3. A set S itself is a category with its elements as objects and all morphisms are identity morphisms. A category which has only identity morphisms is called **discrete category**.

Partially ordered set is the set P with binary operation $\leq$ satisfying: for all $x, y, z \in P$, $x \leq x$, $x \leq y \leq x$ then $x = y$, and $x \leq y \leq z$ then $x \leq z$. **Order-preserving function** $f : P \to Q$ for partially ordered set P, Q is the map satisfying $x \leq y$ implies $f(x) \leq f(y)$. This definition shows that Poset is the category.
The objects of the categories above are all set-like: if we forget the special structures, we get the category Set. These kind of categories are called *concrete categories*, which will be defined exactly later.

Definition 4. A category is **small** if it has only a set's worth of morphisms.

Category theory uses larger concept then set, **class**. The exact construction of the class needs the extension of Zermelo-Fraenkel axioms, which is not the topic of this paper.

For small category, the identity morphisms are the subset of the set of morphisms, thus it has a set's worth of objects.

Definition 5. A category is **locally small** if between any pair of objects there is only a set's worth of morphisms.

For locally small category, the set of morphisms with domain X and codomain Y is often written as $\mathrm{Hom}(X, Y)$, or $\mathsf{C}(X, Y)$ to emphasize which category we are working in.

Definition 6. The morphism $f : X \to Y$ is called **isomorphism** if there is $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$. If there is an isomorphism between X and Y, then we call X and Y are **isomorphic**, and write $X \simeq Y$. If a morphism has same domain and codomain, then we call it **endomorphism**; if an endomorphism is isomorphism, then we call it **automorphism**.

Example 7. The isomorphisms of Set are bijections; the isomorphisms of $\mathsf{Group}, \mathsf{Ring}, \mathsf{Mod}_R, \mathsf{Field}$ are isomorphisms(which sound quite trivial); the isomorphisms of Top are homeomorphisms; the isomorphisms of partially ordered set-generated category P is the identity.

Lemma 8. *A morphism can have at most one inverse isomorphism.*

Proof. Let $f : X \to Y$ has two inverse isomorphisms g, h. Then $gfh = g(fh) = g1_Y = g$ and $gfh = (gf)h = 1_X h = h$, thus $g = h$. □

Definition 9. A **groupoid** is a category where every morphism is isomorphism.

In abstract algebra, groupoid is defined as a set G with inverse g^{-1} and partial function $* : G \times G \to G$, satisfying 1. if $g * h, h * k$ are defined then $(g * h) * k$ and $g * (h * k)$ are defined and equal, and conversely if $(g * h) * k$ and $g * (h * k)$ are defined then they are equal and $g * h, h * k$ are defined, 2. $g^{-1} * g$ and $g * g^{-1}$ are always defined, 3. $g * h$ is defined then $g * h * h^{-1} = h$ and $g^{-1} * g * h = h$. This definition and category theoretic definition are same in the range of set.

Example 10.

1. A **group** is a groupoid with one object.
2. For any space X, the **fundamental groupoid** $\Pi_1(X)$ is a category whose objects are the points of X and the morphism between two points are the endpoint-preserving homotopy classes of paths.
3. For the group G acting on the set X, the **action groupoid** is the category where the objects are the elements of X and the morphisms from x to y is the group element g satisfying $y = gx$.

We already have the algebraic definition of group. However, in category theory, this becomes the definition of group.

Definition 11. For category C, a category D is called a **subcategory** if $\mathsf{ob}(\mathsf{D})$ and $\mathsf{mor}(\mathsf{D})$ is the subcollection of $\mathsf{ob}(\mathsf{C})$ and $\mathsf{mor}(\mathsf{C})$ respectively.

Of course the morphisms of D must have domain and codomain in $\mathsf{ob}(\mathsf{D})$.

Lemma 12. *Any category* C *contains a* ***maximal groupoid****, the subcategory containing all of the objects and only those morphisms that are isomorphisms.*

Proof. what we need to show is that the composition of two isomorphisms is isomorphism. For isomorphisms $f : X \to Y$, $g : Y \to Z$, there is $f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$ such that $f^{-1}f = 1_X, ff^{-1} = 1_Y, g^{-1}g = 1_Y$ and $gg^{-1} = 1_Z$. Now notice that $gff^{-1}g^{-1} = g(ff^{-1})g^{-1} = gg^{-1} = 1_Z$ and $f^{-1}g^{-1}gf = f^{-1}(g^{-1}g)f = f^{-1}f = 1_X$, hence gf is isomorphism. □

Proposition 13. *Consider a morphism $f : x \to y$. If there exists a pair of morphisms $g, h : y \to x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.*

Proof. $gfh = (gf)h = 1_x h = h$ and $g(fh) = g1_y = g$ thus $g = h$ and so f is an isomorphism. □

Proposition 14. *For any category C and any object $c \in \mathsf{C}$,*

1. *There is a category c/C whose objects are morphisms $f : c \to x$ with domain c and in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that $g = hf$.*

2. *There is a category C/c whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the domains so that $f = gh$.*

Proof.

1. What we need to prove is the composition rule: the morphism from f to g, F, and the morphism from g to h, G, satisfies $g = Ff$ and $h = Gg$. Then $h = G(Ff) = (GF)f$, which says that GF is exactly the morphism from f to h.

2. This is very similar with above, except the arrow direction is opposite. The morphism from f to g, F, and the morphism from g to h, G, satisfies $f = gF$ and $g = hG$. Then $f = (hG)F = h(GF)$, which says that GF is exactly the morphism from f to h.

□

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Definition 15. For category C, the **opposite category** C^{op} has the same objects in C, and for each morphism $f : x \to y$ in C we take $f^{\mathrm{op}} : y \to x$ as a morphism in C^{op}. The identity becomes 1_x^{op}, and the composition of morphisms becomes $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

Notice that the opposite category is also a category.

Definition 16. For a theorem, if we take the opposite category, we get a theorem, which is called a **dual theorem**, and is proven by the dual statement of the proof.

Lemma 17. *The following are equivalent:*

1. *$f : x \to y$ is an isomorphism in C.*
2. *For all objects $c \in \mathsf{C}$, post-composition with f defines a bijection $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$.*
3. *For all objects $c \in \mathsf{C}$, pre-composition with f defines a bijection $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$.*

Most of categories we will treat from now are locally small categories. However, lots of them can be proven in general categories also, using similar statement: for example, we may change the word bijection to isomorphism in the sense of general category.

Post-composition means that for morphism $g : c \to x$, $f_*(g) = f \circ g : c \to y$. **Pre-composition** means that for morphism $g : y \to c$, $f^*(g) = g \circ f : x \to c$.

Proof. Noticeable point is that the second statement is exact dual of third statement, and vice versa. Therefore, it is sufficient to prove that $1 \Leftrightarrow 2$, and then $1 \Leftrightarrow 3$ is proven automatically by dual theorem. For $1 \Rightarrow 2$, take g be the isomorphic inverse of f, and g_* be the post-composition with g. Then, for all $h \in \mathsf{C}(c, x)$,

$$g_* f_*(h) = g_*(fh) = gfh = (gf)h = 1_x h = h \tag{1}$$

thus $g_* f_* = 1_{\mathsf{C}(c,x)}$. Also, for all $h \in \mathsf{C}(c, y)$,

$$f_* g_*(h) = f_*(gh) = fgh = (fg)h = 1_y h = h \tag{2}$$

thus $f_* g_* = 1_{\mathsf{C}(c,y)}$ and so f_* is bijection.

Conversely, for $2 \Rightarrow 1$, since f_* is bijection, there must exists $g \in \mathsf{C}(y, x)$ such that $f_*(g) = fg = 1_y$. Also, the function $gf \in \mathsf{C}(x, x)$ satisfies $f_*(gf) = fgf = 1_y f = f \in \mathsf{C}(x, y)$, and since $f_*(1_x) = f 1_x = f$ and f_* is bijection, $1_x = gf$. Therefore f is an isomorphism. □

Definition 18. A morphism $f : x \to y$ in a category C is

1. a **monomorphism** or **monic** if for any parallel morphisms $h, k : w \to x$, $fh = fk$ implies that $h = k$, or for locally small category, $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is injective;
2. an **epimorphism** or **epic** if for any parallel morphisms $h, k : y \to z$, $hf = kf$ implies that $h = k$, or for locally small category, $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is injective.

In general, we use $\rightarrowtail$ for monomorphisms and $\twoheadrightarrow$ for epimorphisms.

For the category Set, if $f : X \to Y$ is monomorphism, then for any $x \in X$, define $1_x : \bullet \to X$ where $1_x(\bullet) = x$: then $f 1_x = f 1_{x'}$ means $f(x) = f(x')$, and $1_x = 1_{x'}$ means $x = x'$, which coincides with the definition of injectivity. Also for epimorphism, if $f : X \mapsto Y$ is epimorphism, suppose that $y \in Y - f(X)$. Now for two point set $\{0, 1\}$, define a map $h, k : Y \to \{0, 1\}$ as $h(Y) = 0$ and $k(Y - \{y\}) = 0, k(y) = 1$. Then $hf = kf$ since $y \notin f(X)$, but $h \neq k$, contradiction, thus f is surjective.

Example 19. For Ring, the inclusion mapping $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is monic and epic, but there is no nontrivial morphism from $\mathbb{Q}$ to $\mathbb{Z}$. This can be shown as following: for some $\mathbb{Z}$-ring R and parallel morphisms $h, k : R \to \mathbb{Z}$, obviously $ih = ik$ implies $h = k$ since i is inclusion. Also for parallel morphisms $h, k : \mathbb{Q} \to R$, suppose that $hi = ki$ but $h \neq k$. Then there is $q \in \mathbb{Q}$ such that $h(q) \neq k(q)$. Since $hi = ki$, $q \notin \mathbb{Z}$. Thus there is prime p and integer r such that $q = r/p$. Now $h(q) \neq k(q)$ implies $p \cdot h(q) \neq p \cdot k(q)$, but since the morphisms are ring homomorphisms, we get $h(r) \neq k(r)$, which is contradiction. Therefore $h = k$. Now suppose that there is a nontrivial ring homomorphism f from $\mathbb{Q}$ to $\mathbb{Z}$. Then we have $f(q) = n \neq 0$ for some $q = r/p \in \mathbb{Q}$ and $n \in \mathbb{Z}$. Then $2n \cdot f(r/2pn) = n$, but there is no integer k which satisfies $2nk = n$ for $n \neq 0$.

Definition 20. Suppose that $s : x \to y$ and $r : y \to x$ are morphisms such that $rs = 1_x$. Then we call s a **section**, **split monomorphism**, or **right inverse** of r, and r a **retraction**, **split epimorphism**, or **left inverse** of s. We call x a **retract** of y.

Notice that section is always monomorphism and retraction is always epimorphism, which is easily proven using definition and associativity.

Lemma 21.

1. *If $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$ are monic, then $gf : x \rightarrowtail z$ is monic.*

2. *If $f : x \to y$ and $g : y \to z$ gives monic composition $gf : x \rightarrowtail z$, then f is monic.*

Dually,

1′ *If $f : x \twoheadrightarrow y$ and $g : y \twoheadrightarrow z$ are epic, then $gf : x \twoheadrightarrow z$ is epic.*

2′ *If $f : x \to y$ and $g : y \to z$ gives epic composition $gf : x \twoheadrightarrow z$, then g is epic.*

These results shows that monomorphisms or epimorphisms define a subcategory of given category.

Proof. First we will show first two statements, and then we will show the other statements are dual of above.

For 1., take parallel morphisms $h, k : w \to x$. Since g is monic, $g(fh) = g(fk)$ implies $fh = fk$, and since f is monic, $fh = fk$ implies $h = k$. Therefore $gfh = gfk$ implies $h = k$ and so gf is monic.

For 2., take parallel morphisms $h, k : w \to x$. Then $fh = fk$ implies $gfh = gfk$, which implies $h = k$ since gf is monic. Therefore f is monic.

For 1′, noticing the dual of monic is epic, the dual statement of 1. becomes: If $f : y \twoheadrightarrow x$ and $g : z \twoheadrightarrow y$ are epic, then $fg : z \twoheadrightarrow x$ is epic. Changing notation $f \leftrightarrow g$ and $x \leftrightarrow z$ gives 1′.

For 2′, the dual statement of 2. becomes: If $f : y \to x$ and $g : z \to y$ gives epic composition $fg : z \twoheadrightarrow x$, then f is monic. Changing notation $f \leftrightarrow g$ and $x \leftrightarrow z$ gives 2′. □

Proposition 22. $\mathsf{C}/c \simeq (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$.

Proof. The category $c/(\mathsf{C}^{\mathrm{op}})$ has objects as morphisms $f : x \to c$ and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : y \to x$ between the domains so that $g = fh$. Taking the opposite category in whole changes the morphism direction in the sense that now h is a morphism from g to f. Changing notation $f \leftrightarrow g$ and $x \leftrightarrow y$ gives: $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ has objects as morphisms $f : x \to c$ and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the domains so that $f = gh$. This statement is exactly the definition of category C/c. □

Theorem 23.

1. *A morphism $f : x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, the post-composition function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective.*
2. *A morphism $f : x \to y$ is a split monomorphism in a category C if and only if for all $c \in \mathsf{C}$, the pre-composition function $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective.*

Solution.

1. Suppose that f is a split epimorphism. Then there exists a morphism $g : y \to x$ such that $fg = 1_y$. Now, for $k \in \mathsf{C}(c, y)$, $f(gk) = (fg)k = 1_y k = k$, therefore f_* is surjective. Conversely, suppose that f_* is surjective for all $c \in \mathsf{C}$. Then by taking c as y, we get $g : y \to x$ such that $fg = 1_y$, which shows f is a split epimorphism.
2. Taking the dual of the statement above, we get: a morphism $f : y \to x$ is a split monomorphism in a category C if and only if for all $c \in \mathsf{C}$, the pre-composition function $f^* : \mathsf{C}(x, c) \to \mathsf{C}(y, c)$ is surjective. (Note that the surjectivity does not changes its arrow direction, because this is not the morphism in C but the function of sets of morphisms.) Changing $x \leftrightarrow y$ gives the desired result.

■

Theorem 24. *A morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Therefore, by duality, a morphism that is both an epimorphism and a split monomorphism is necessarily an isomorphism.*

Proof. Suppose that $f : x \to y$ is monomorphism and a split epimorphism. Then we have $g : y \to x$ such that $fg = 1_y$, and for any parallel morphisms $h, k : w \to x$, $fh = fk$ implies $h = k$. Now, since $fgf = 1_y f = f = f1_x$, $gf = 1_x$. Therefore f is isomorphism and g is its inverse isomorphism. □

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Definition 25. A **functor** $F : \mathsf{C} \to \mathsf{D}$ consists of the following data:

- an object $F(c) \in \mathsf{D}$ for each object $c \in \mathsf{C}$;
- a morphism $F(f) : F(c) \to F(c') \in \mathsf{D}$ for each morphism $f : c \to c' \in \mathsf{C}$,

which satisfies the following **functoriality axioms**:

- for any composable morphism pair $f, g \in \mathsf{C}$, $F(g) \circ F(f) = F(g \circ f)$;

- for each object $c \in \mathsf{C}$, $F(1_c) = 1_{F(c)}$.

Example 26.

1. Let C be the category which is one of $\mathsf{Group}, \mathsf{Ring}, \mathsf{Mod}_R, \mathsf{Field}, \mathsf{Meas}, \mathsf{Top}$, or Poset. We have a **forgetful functor** $F : \mathsf{C} \to \mathsf{Set}$ which sends each object to base set and each morphism to base function. Since it forgets all the algebraic properties they have and becomes a set, this functor is called forgetful. There are some partially forgetful functors like $\mathsf{Mod}_R \to \mathsf{Ab}$ or $\mathsf{Ring} \to \mathsf{Ab}$, which forgets some of the algebraic properties but not all.

Ab is the category of abelian groups.

2. The fundamental group defines a functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Group}$. A continuous function $f : (X, x) \to (Y, y)$ induces a group homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$, which can be easily proven that this satisfies functoriality axioms.

Top_* means a pair of topological space with its one element.

3. For each $n \in \mathbb{Z}$, there are functors $Z_n, B_n, H_n : \mathsf{Ch}_R \to \mathsf{Mod}_R$ Z_n, called n**-cycles**, is defined as $Z_n C_\bullet = \ker(d : C_n \to C_{n-1})$; B_n, called n**-boundary**, is defined as $B_n C_\bullet = \mathrm{im}(d : C_{n+1} \to C_n)$, and H_n, called n**th homology**, is defined as $H_n C_\bullet = Z_n C_\bullet / B_n C_\bullet$. All these satisfies the functoriality axioms.

Ch_R is the category of **chain complex**: a collection $(C_n)_{n\in\mathbb{Z}}$ of R-modules equipped with R-module homomorphisms $d : C_n \to C_{n-1}$ with $d^2 = 0$. The morphism $f_n : C_n \to C'_n$ satisfies $df_n = f_{n-1}d$ for all $n \in \mathbb{Z}$.

4. We have a functor $F : \mathsf{Set} \to \mathsf{Group}$ which sends a set X to the **free group** on X. Remember that the free groups can be defined by using the universal property: For a set X, we have unique free group $F(X)$ (up to isomorphism) which satisfies that for every group G and function $f : X \to G$, there is a unique group homomorphism $\varphi : F(X) \to G$ which satisfies $\varphi \circ i = f$, where $i : X \to F(X)$ is the inclusion. This kind of definition repetitively appears when we say about free module. Indeed, this definition is the categorical definition of free objects, which will be seen later.

Definition 27. A **covariant functor** F from C to D is a functor $F : C \to D$. A **contravariant functor** F from C to D is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

Proposition 28. *Due to the definition, contravariant functor satisfies:*

- $F(c) \in \mathsf{D}$ *for each object* $c \in \mathsf{C}$;
- $F(f) : F(c') \to F(c) \in \mathsf{D}$ *for each morphism* $f : c \to c' \in \mathsf{C}$,

and the functoriality axioms becomes:

- *for any composable pair* $f, g \in \mathsf{C}$, $F(f) \circ F(g) = F(g \circ f)$;
- *for each object* $c \in \mathsf{C}$, $F(1_c) = 1_{F(c)}$.

Proof. Since the dual of category has same objects with original category, the conditions for objects does not changes. The only changes happens on the morphisms on C, which effects on second statement, and it also changes the composition of the morphisms, which effects on third statement. □

Example 29.

1. The functor $* : \mathsf{Vect}_{\mathbb{K}}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{K}}$ which carries a vector space V to its **dual space** $V^* = \mathrm{Hom}(V, \mathbb{K})$ is a covariant functor. For the linear map $\phi : V \to W$, the functor gives the dual map $\phi^* : W^* \to V^*$, in the sense that for $f : W \to \mathbb{K}$ and $g : V \to \mathbb{K}$, $f \circ \phi = g$.

2. The functor Spec : $\mathsf{CRing}^{\mathrm{op}} \to \mathsf{Top}$ which carries a commutative ring R to the set of prime ideals $\mathrm{Spec}(R)$ with Zariski topology is a covariant functor. Consider a ring homomorphism $\phi : R \to S$ and prime ideal $P \subset S$. The inverse image $\phi^{-1}(P) \subset R$ is the prime ideal of R, and therefore the inverse image function $\phi^{-1} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is well defined; indeed it is easy to show that this is a continuous map.

3. A **presheaf** is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. For example, take topological space X and take a category $\mathcal{O}(X)$, the poset of open subsets of X. Since the poset has morphism $V \to U$ if $V \subset U$, we can see that the presheaf satisfies that if $V \subset U$, then we have a function $\mathrm{res}_{V,U} : F(U) \to F(V)$.

The **Zariski topology** is the set of prime ideals $\mathrm{Spec}(R)$ whose closed sets are $V_I = \{P \in \mathrm{Spec}(R) : I \subset P\}$ for all ideal I.

The domain of functor does not needs to be Set: Ab or Ring is also possible, but compositing forgetful functor we get same result. For the example of presheaf, we think the functor F so that $F(U)$ is the ring of bounded functions on U. If $V \subset U$ then we take the ring homomorphism $\mathrm{res}_{V,U} : F(U) \to F(V)$ satisfying $\mathrm{res}_{V,U}(f) = f|_V$, which shows that F is presheaf. If F satisfies some more conditions, we call F *sheaf*, which will be discussed later.

Lemma 30. *Functors preserve isomorphisms.*

Proof. Consider $F : \mathsf{C} \to \mathsf{D}$ a functor and $f : x \to y$ an isomorphism in C with inverse $g : y \to x$. Then we have

$$F(g)F(f) = F(gf) = F(1_x) = 1_{F(x)} \tag{3}$$

and similar for inverse, thus $F(f)$ is isomorphism. □

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Example 31. Remember that the group G defines a one-object category $\mathsf{B}G$ whose morphisms are identified with the elements of g. For a category C, think a functor $X : \mathsf{B}G \to \mathsf{C}$, which sends single object $\bullet \in \mathsf{B}G$ to $X \in \mathsf{C}$, and a morphism g to $g_* : X \to X$. Then the endomorphisms(indeed automorphisms, because functors preserve isomorphisms) g_* satisfies $g_*h_* = (gh)_*$ for all $g, h \in \mathsf{B}G$ and $e_* = 1_X$ for identity $e \in \mathsf{B}G$. This functor X is called a **left action**, or just **action**, of the group G on the object $X \in \mathsf{C}$. If $\mathsf{C} = \mathsf{Set}$ then X is called a G**-set**, if $\mathsf{C} = \mathsf{Vect}$ then a G**-representation**, and if $\mathsf{C} = \mathsf{Top}$ then a G**-space**.

If X is a covariant functor $X : \mathsf{B}G^{\mathrm{op}} \to \mathsf{C}$, then we call X a **right action**. If so, then the endomorphism $g^* : X \to X$ has a composition $g^*h^* = (hg)^*$. If we do not need to specify, then we call X an **action**.

Example 32. Consider a category C with two objects $\bullet, \circ$ and has one nontrivial morphism $\bullet \to \circ$. This is monomorphic and epimorphic. However, take a functor $F : \mathsf{C} \to \mathsf{Mod}_{\mathbb{Z}}$ where $F(\bullet) = F(\circ) = \mathbb{Z}$ and $F(\to) : \mathbb{Z} \to \mathbb{Z}$ is a trivial map $n \mapsto 0$. This is neither monomorphic nor epimorphic.

Proposition 33. *The split monomorphisms and split epimorphisms are preserved by functors.*

Proof. The proof is very same with the proof for isomorphisms. □

Definition 34. If C is locally small, then for any object $c \in \mathsf{C}$, we call a pair of covariant and contravariant functors represented by c as **functors represented by** c and define as following:

- for covariant functor, $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$, $x \mapsto \mathsf{C}(c, x)$, and $f : x \to y$ maps to $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ by post-composition;
- for contravariant functor, $\mathsf{C}(-, c) : \mathsf{C} \to \mathsf{Set}$, $x \mapsto \mathsf{C}(x, c)$, and $f : x \to y$ maps to $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ by pre-composition.

Definition 35. For any categories $\mathsf{C} \times \mathsf{D}$, there is a category $\mathsf{C} \times \mathsf{D}$, which is called the **product category**, defined as following:

- the objects are ordered pairs (c, d) for objects $c \in \mathsf{C}, d \in \mathsf{D}$;
- the morphisms are ordered pairs $(f, g) : (c, d) \to (c', d')$ where $f : c \to c' \in \mathsf{C}, g : d \to d' \in \mathsf{D}$,
- the identities and compositions are defined componentwise.

Definition 36. If C is locally small, then there is a **two-sided functor** $\mathsf{C}(-, -) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$, which is defined as following:

- a pair of objects x, y maps to $\mathsf{C}(x, y)$;
- a pair of morphisms $f : w \to x, h : y \to z$ maps to the function $(f^*, h_*) : C(x, y) \to C(w, z)$ defined as $g \mapsto hgf$.

Definition 37. The category Cat is the category which has small categories as its objects and functors as its morphisms. For two small categories, the collection of functors between them is actually a set, thus this is locally small category, but since Set or all the other concrete categories are the proper subcategory of Cat, this is not a small category, and thus we do not have the Russell's paradox. Notice that none of the concrete categories are the *object* of Cat.

Samely, the category CAT is the category which has locally small categories as its objects and functors as is morphisms. Since Set is not small, CAT is not locally small, thus we also need not to worry about Russell's paradox. We have an inclusion functor $\mathsf{Cat} \hookrightarrow \mathsf{CAT}$.

Definition 38. The functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ satisfying $F \circ G = 1_{\mathsf{D}}$ and $G \circ F = 1_{\mathsf{C}}$ are the **isomorphisms of categories**, and then the categories C, D are **isomorphic categories**.

Example 39.

1. The functor op : $\mathsf{CAT} \to \mathsf{CAT}$ is a non-trivial automorphism of the category.

2. For any group G, the functor $-1 : \mathsf{B}G \to \mathsf{B}G^{\text{op}}$ defined by $g \to g^{-1}$ is isomorphic. This shows that every right action and left action are equivalent. This is true for groupoid also.

3. Not every category is isomorphic with its opposite category. Consider $\mathbb{N}$ as a partially ordered set category. Then $\mathbb{N}$ has minimal operator, but $\mathbb{N}^{\text{op}}$ does not, which shows that they are not isomorphic.

4. One final, nontrivial, and important isomorphism between two categories is given below. Let E/F be a finite Galois extension and $G := \text{Aut}(E/F)$ the Galois group.

   Now consider the **orbit category** $\mathcal{O}_G$ for group G, whose objects are cosets G/H for subgroup $H \leq G$. The morphisms $f : G/H \to G/K$ are defined as the G**-equivariant maps**, which means the functions that commute with the left G-action: $g' f(gH) = f(g'gH)$. We may show that all the G-equivariant maps can be represented as $gH \mapsto g\gamma K$, for $\gamma \in G$ with $\gamma^{-1}H\gamma \subset K$.

   Also consider the category Field_F^E whose objects are intermediate fields $F \subset K \subset E$, and the morphisms $K \to L$ is a field homomorphism that fixes the elements on F pointwise. Notice that the group of automorphisms of the object $E \in \mathsf{Field}_F^E$ is the Galois group $G = \text{Aut}(E/F)$.

   Finally we define a functor $\Phi : \mathcal{O}_G^{\text{op}} \to \mathsf{Field}_F^E$ which sends G/H to the subfield of E whose elements are fixed by H under the action of Galois group, and if $G/H \to G/K$ is induced by γ then the field homomorphism $x \mapsto \gamma x$ sends an element $x \in E$ which is fixed by K to an element $\gamma x \in E$ which is fixed by H. The **Fundamental theorem of Galois theory** says that Φ is bijection; indeed, Φ is isomorphism between $\mathcal{O}_G^{\text{op}}$ and Field_F^E.

A field extension E/F is a **finite Galois extension** if F is a finite-index subfield of E and the size of the group of automorphisms of E fixing F, $\text{Aut}(E/F)$ is same with the index $[E : F]$.

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Example 40. Take a category C with objects $\{a, b, c, d\}$ and nontrivial morphisms $a \to b, c \to d$. Take another category D with objects x, y, z and nontrivial morphisms $x \to y, y \to z, x \to z$. Now take a functor F such that $F(a) = x, F(b) = F(c) = y, F(d) = z$ for objects and works accordingly on morphisms. Then the image only has nontrivial morphisms $x \to y$ and $y \to z$ but no $x \to z$, which has no composition. Thus, we have an example that the objects and morphisms in the image of the functor $F : \mathsf{C} \to \mathsf{D}$ do not define a subcategory of D.

Proposition 41. *Given functors $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{E} \to \mathsf{C}$, there is a category $F \downarrow G$, the* ***comma category****, which has:*

- *triples $(d \in \mathsf{D}, e \in \mathsf{E}, f : F(d) \to G(e) \in \mathsf{C})$ as objects;*
- *a pair of morphisms $(h : d \to d', k : e \to e')$ so that $f' \circ F(h) = G(k) \circ f$ as morphisms $(d, e, f) \to (d', e', f')$*

We define a pair of projection functors dom $: F \downarrow G \to \mathsf{D}$ *and* cod $: F \downarrow G \to \mathsf{E}$.

$$\begin{tikzcd}
d \arrow[r, "F"] \arrow[d, "h"'] & F(d) \arrow[r, "f"] \arrow[d, "F(h)"'] & G(e) \arrow[d, "G(k)"'] & e \arrow[l, "G"'] \arrow[d, "k"'] \\
d' \arrow[r, "F"] & F(d') \arrow[r, "f'"] & G(e') & e' \arrow[l, "G"']
\end{tikzcd} \tag{4}$$

Slice categories, c/C and C/c, are the special cases of this comma category: write the functor from one object category to C whose image is $c \in \mathsf{C}$ as c, and the identity functor on C as 1_{C}. Then we get $c/\mathsf{C} = c \downarrow 1_{\mathsf{C}}$ and $\mathsf{C}/c = 1_{\mathsf{C}} \downarrow c$.

Proof. We can take a pair of identity morphisms for identity morphism, so what we need to show is the composition rule. Suppose we have two morphisms $(d, e, f) \xrightarrow{(h,k)} (d', e', f') \xrightarrow{(h',k')} (d'', e'', f'')$. The composition of pair of morphisms will be taken as $(h' \circ h, k' \circ k)$, so what we need to show is $f'' \circ F(h' \circ h) = G(k' \circ k) \circ f$. But due to the property of functor, we can re-write this as $f'' \circ F(h') \circ F(h) = G(k') \circ G(k) \circ f$. Now, $f'' \circ F(h') \circ F(h) = G(k') \circ f' \circ F(h) = G(k') \circ G(k) \circ f$ due to the definition. □

Example 42. The functors need not reflect isomorphisms, that is, we have a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism $f \in \mathsf{C}$ such that $F(f)$ is an isomorpihsm in D but f is not an isomorphism in C. Let C, D are categories with two objects $\bullet, \circ$, where $\bullet \to \circ \in \mathsf{C}, \mathsf{D}$ and $\circ \to \bullet \in \mathsf{D}$. Take functor $F : \mathsf{C}, \mathsf{D}$ as $F(\bullet) = \bullet, F(\circ) = \circ$, and $F(\bullet \to \circ) = \bullet \to \circ$. Then $F(\bullet \to \circ)$ is isomorphism because we have $\circ \to \bullet$ in D, but $\bullet \to \circ$ is not an isomorphism in C.

Definition 43. Given categories C, D and functors $F, G : \mathsf{C} \to \mathsf{D}$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_c : F(c) \to G(c)$ in D for each object $c \in \mathsf{C}$, the collection of which define the **components** of the natural transformation, so that for any morphism $f : c \to c' \in \mathsf{C}$, $G(f) \circ \alpha_c = \alpha_{c'} \circ F(f)$ holds. A **natural isomorphism**

is a natural transformation $\alpha : F \Rightarrow G$ in which every component α_c is an isomorphism.

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & G(c') \end{tikzcd} \tag{5}$$

Example 44. Consider a category Set^{∂} whose objects are sets and morphisms are **partial functions**: $f : X \to Y$ is a function from $X' \subset X$ to Y. The composition of two partial functions is defined as the composition of functions.

Now we take the functor $(-)_+ : \mathsf{Set}^{\partial} \to \mathsf{Set}_*$ which sends X to the pointed set X_+, the disjoint union of X and freely-added basepoint: we may take set as $X_+ := X \cup \{X\}$ and the basepoint as X due to the axiom of regularity. The partial function $f : X \to Y$ becomes the pointed function $f_+ : X_+ \to Y_+$ where all the elements outside of the domain of definition of f maps to the basepoint of Y_+. Conversely, we take the inverse functor $U : \mathsf{Set}_* \to \mathsf{Set}^{\partial}$ discarding the basepoint and following functional inverse.

The **axiom of regularity** is the axiom of ZF(Zermelo-Fraenkel) set theory, which says that the set does not contains itself as its element. This shows that X and $\{X\}$ are disjoint.

The construction says that $U(-)_+$ is the identity endofunctor of Set^{∂}, but $(U-)_+$ sends $(X, x) \to (X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$, which is isomorphic but not identical, hence the functor is not isomorphic. But the structure of these are very same.

This is the reason why we do not use the condition $GF = 1_D, FG = 1_C$ for the isomorphism for category.

Example 45.

1. For vector space of any dimension over the field $\mathbb{K}$, the map $\mathrm{ev} : V \to V^{**}$ that sends $v \in V$ to $\mathrm{ev}_v : V^* \to \mathbb{K}$ defines the components of a natural transformation from the identity endofunctor on $\mathsf{Vect}_{\mathbb{K}}$ to the double dual functor. The map $V \xrightarrow{\phi} W$ becomes $V \xrightarrow{\phi} W$ by the identity endofunctor and $V^{**} \xrightarrow{\phi^{**}} W^{**}$ by the double dual functor. What now we need to show is $\mathrm{ev}_{\phi v} = \phi^{**}(\mathrm{ev}_v)$. The first one

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Example 46.

1. For vector space of any dimension over the field $\mathbb{K}$, the map $\mathrm{ev} : V \to V^{**}$ that sends $v \in V$ to $\mathrm{ev}_v : V^* \to \mathbb{K}$ defines the components of a natural transformation from the identity endofunctor on $\mathsf{Vect}_{\mathbb{K}}$ to the double dual functor. The map $V \xrightarrow{\phi} W$ becomes $V \xrightarrow{\phi} W$ by the identity endofunctor and $V^{**} \xrightarrow{\phi^{**}} W^{**}$

by the double dual functor. Since $\mathrm{ev}_{\phi(v)} = \phi^{**}(\mathrm{ev}_v)$, this is natural transformation. However, there is no natural isomorphism between the identity functor and its dual functor on finite-dimensional vector spaces, which is because the identity functor is covariant but the dual functor is contravariant.

2. Consider cHaus as a category of compact Hausdorff spaces and continuous maps, and Ban as a category of Banach spaces and continuous linear maps. Consider a finite signed measure $\mu : \mathrm{Baire}(X) \to \mathbb{R}$ where $\mathrm{Baire}(X)$ is a Baire algebra of X, a σ-algebra generated by closed G_δ sets. The Jordan decomposition of μ gives $\mu = \mu_+ + \mu_-$, which gives the norm $\|\mu\| = \mu_+(X) + \mu_-(X)$, and this gives the Banach space of a finite signed Baire measure $\Sigma(X)$. Then we can define a functor $\Sigma : \mathsf{cHaus} \to \mathsf{Ban}$, which takes a continuous map $f : X \to Y$ to the map $\Sigma(f)(\mu) = \mu \circ f^{-1} : \Sigma(X) \to \Sigma(Y)$. Also, consider a functor $C^* : \mathsf{cHaus} \to \mathsf{Ban}$, which takes X to the linear dual $C(X)^*$ of the Banach space $C(X)$ of continuous real-valued functions on X.

   Now for each $\mu \in \Sigma(X)$, there is a linear functional $\phi_\mu : C(X) \to \mathbb{R}$, which is defined as $\phi_\mu(g) = \int_X g d\mu$ for $g \in C(X)$. Now for each $\mu \in \Sigma(X), f : X \to Y, h \in C(Y)$, since $\int_X h \circ f d\mu = \int_Y hd(\mu \circ f^{-1})$, which shows that the morphisms $\mu \mapsto \phi_\mu$ are the components of the natural transformation $\eta : \Sigma \to C^*$. Furthermore, the **Riesz representation theorem** says that this is a natural isomorphism.

3. Consider a category of commutative ring cRing and a category of group Group. For a commutative ring K, consider the general linear group $GL_n K$ and the group of units K^*. Then GL_n and $(-)^*$ are functors. Now for each general linear group M consider the determinant $\det_n M$. Since M is invertible, $\det_n M \in K^*$. Furthermore, for any ring homomorphism $\phi : K \to K'$, $\det_{K'} \circ GL_n(\phi) = \phi^* \circ \det_K$, thus the morphisms $\det_K$ are the components of the natural transformation $\det : GL_n \to (-)^*$.

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Lemma 47. *Let 2 be a category with two objects $0, 1$ and one nontrivial morphism $0 \to 1$. Consider two categories C, D, two functors $F, G : \mathsf{C} \to \mathsf{D}$, and natural transformations $\alpha : F \Rightarrow G$. This correspond bijectively to functors $H : \mathsf{C} \times 2 \to D$ such that, considering the projection functor i_0, i_1, the following diagram commutes.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathsf{C} \times 2 \arrow[d, "H"] & \mathsf{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathsf{D} & \end{tikzcd} \tag{6}$$

Proof. For a natural transformation α, define H as, for $c, c' \in \mathsf{Ob}(\mathsf{C})$ and $f : c \to c'$, $H(c, 0) = F(c), H(c, 1) = G(c), H(f, 0 \to 0) = F(f), H(f, 1 \to 1) = G(f)$, and $H(f, 0 \to 1) = G(f) \circ \alpha_c = \alpha_{c'} \circ F(f) : F(c) \to G(c')$, then H is a functor. Conversely, for such functor $H : \mathsf{C} \times \mathsf{2} \to D$, define a collection of natural transformations α_c as $H(1_c, 0 \to 1)$, then since H is a functor, $G(f) \circ \alpha_c = H(f, 1 \to 1) \circ H(1_c, 0 \to 1) = H(1_{c'}, 0 \to 1) \circ H(f, 0 \to 0) = \alpha_{c'} \circ F(f)$ where $f : c \to c'$. □

Definition 48. An **equivalence of categories** is the functors $F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}$ with natural isomorphisms $\eta : 1_{\mathsf{C}} \simeq G \circ F$, $\epsilon : F \circ G \simeq 1_{\mathsf{D}}$. If so, we call categories C and D are **equivalent**, and write $\mathsf{C} \simeq \mathsf{D}$.

Proposition 49. *The equivalence of categories is indeed a equivalence relation.*

Proof. Suppose that $\mathsf{C} \simeq \mathsf{D} \simeq \mathsf{E}$. Then there are functors $F : \mathsf{C} \leftrightarrow \mathsf{D} : G, H : \mathsf{D} \leftrightarrow \mathsf{E} : K$ such that $1_{\mathsf{C}} \simeq G \circ F, 1_{\mathsf{D}} \simeq F \circ G, 1_{\mathsf{D}} \simeq K \circ H, 1_{\mathsf{E}} \simeq H \circ K$. Now consider $H \circ F : C \leftrightarrow E : G \circ K$. Then $H \circ F \circ G \circ K \simeq H \circ 1_{\mathsf{D}} \circ K = H \circ K \simeq 1_{\mathsf{E}}$ and $G \circ K \circ H \circ F \simeq G \circ 1_{\mathsf{D}} F = G \circ F \simeq 1_{\mathsf{C}}$, thus $\mathsf{C} \simeq \mathsf{E}$. □

Example 50. Consider a category Set^{∂} whose objects are sets and morphisms are **partial functions**: $f : X \to Y$ is a function from $X' \subset X$ to Y. The composition of two partial functions is defined as the composition of functions.

Now we take the functor $(-)_+ : \mathsf{Set}^{\partial} \to \mathsf{Set}_*$ which sends X to the pointed set X_+, the disjoint union of X and freely-added basepoint: we may take set as $X_+ := X \cup \{X\}$ and the basepoint as X due to the axiom of regularity. The partial function $f : X \to Y$ becomes the pointed function $f_+ : X_+ \to Y_+$ where all the elements outside of the domain of definition of f maps to the basepoint of Y_+. Conversely, we take the inverse functor $U : \mathsf{Set}_* \to \mathsf{Set}^{\partial}$ discarding the basepoint and following functional inverse.

The **axiom of regularity** is the axiom of ZF(Zermelo-Fraenkel) set theory, which says that the set does not contains itself as its element. This shows that X and $\{X\}$ are disjoint.

The construction says that $U(-)_+$ is the identity endofunctor of Set^{∂}, but $(U-)_+$ sends $(X, x) \to (X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$, which is isomorphic but not identical, hence the functor is not isomorphic. But the structure of these are very same.

This is the reason why we do not use the condition $GF = 1_D, FG = 1_C$ for the isomorphism for category. But we have a natural isomorphism $\eta : 1_{\mathsf{Set}_*} \simeq (U-)_+$ with $\eta_{(X,x)} : (X, x) \to (X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$, thus the categories $\mathsf{Set}^{\partial}, \mathsf{Set}_*$ are equivalent.

Definition 51. A functor $F : \mathsf{C} \to \mathsf{D}$ is

- **full** if for each objects $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is surjective;
- **faithful** if for each objects $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is injective;
- **essentially surjective on objects** if for every object $d \in \mathsf{D}$ there is an object $c \in \mathsf{C}$ such that d is isomorphic to $F(c)$;
- **embedding** if it is faithful and the map $F : \mathsf{ob}(\mathsf{C}) \to \mathsf{ob}(\mathsf{D})$ is also injective;
- **fully faithful** if it is full and faithful;
- **full embedding** of C into D if it is full and embedding, and then C is a **full subcategory** of D.

Lemma 52. *Consider a morphism $f : a \to b$ and isomorphisms $a \simeq a', b \simeq b'$. Then there is a unique morphism $f' : a' \to b'$ so that any of, or equivalently all of, the following diagrams commute.*

$$\begin{tikzcd} a \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "f"] & a' \arrow[d, "f'"] \\ b \arrow[r, leftrightarrow, "\simeq"] & b' \end{tikzcd} \tag{7}$$

Proof. The diagram with arrows $a \leftarrow a', b \to b'$ defines the function $f' : a' \to b'$ uniquely. Now denote the isomorphisms as $\phi_{aa'} : a \leftrightarrow a' : \phi_{a'a}$ and $\phi_{bb'} : b \leftrightarrow b' : \phi_{b'b}$. Then the followings are equivalent: $\phi_{bb'} \circ f \circ \phi_{a'a} = f'$, $f \circ \phi_{a'a} = \phi_{b'b} \circ f'$, $f = \phi_{b'b} \circ f' \circ \phi_{aa'}$, $\phi_{bb'} \circ f = f' \circ \phi_{aa'}$. Each equations represents that the commutativity of four diagrams are equivalent. □

Lemma 53. *Consider the following diagram where the outer rectangle commutes.*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[r, "j"] \arrow[d, "h"] & c \arrow[d, "l"] \\ a' \arrow[r, "k"] & b' \arrow[r, "m"] & c' \end{tikzcd} \tag{8}$$

Then above diagram commute if either:

1. *the right square commutes and m is a monomorphism; or*
2. *the left square commutes and f is an epimorphism.*

Proof. Notice that two statements are dual, so we need to prove first one only. By the condition, we have $m \circ k \circ g = l \circ j \circ f = m \circ h \circ f$. Since m is a monomorphism, $k \circ g = h \circ f$. □

Theorem 54 (characterizing equivalences of categories)**.** *A functor defining an equivalence of categories is fully faithful and essentially surjective on objects. Assuming the axiom of choice, any fully faithful functor which is essentially surjective on objects defines an equivalence of categories.*

Proof. Consider $F : \mathsf{C} \leftrightarrow \mathsf{D} : G$ such that $\eta : 1_{\mathsf{C}} \simeq G \circ F$ and $\epsilon : 1_{\mathsf{D}} \simeq F \circ G$. For every object $d \in \mathsf{D}$, since $F(G(d)) \simeq d$, F is essentially surjective on objects. Now take two morphisms $f, g : c \to c'$ in C. If $F(f) = F(g)$, then $G(F(f)) = G(F(g))$. Now, due to the natural isomorphism, for every $f : c \to c'$, $G(F(f)) \circ \eta_c = \eta_{c'} \circ f$, thus $\eta_{c'} \circ f = \eta_{c'} \circ g$. Since $\eta_{c'}$ is isomorphism, taking its inverse to the left of above equality gives $f = g$. Therefore F, and symmetrically G, is faithful.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f=g"'] & G(F(c)) \arrow[d, "G(F(f))=G(F(g))"] \\ c' \arrow[r, "\eta_{c'}"] & G(F(c')) \end{tikzcd} \tag{9}$$

Finally, consider a morphism $k : F(c) \to F(c')$. Then $G(k) : G(F(c)) \to G(F(c'))$. Using lemma above, we have a unique morphism $h : c \to c'$ satisfying $\eta_{c'} \circ h = G(k) \circ \eta_c$. This commutation relation says that $G(k) = G(F(h))$, thus due to the faithfulness of G, $k = F(h)$, thus F is full.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "h"'] & G(F(c)) \arrow[d, "G(k)=G(F(h))"] \\ c' \arrow[r, "\eta_{c'}"] & G(F(c')) \end{tikzcd} \tag{10}$$

Now suppose that $F : \mathsf{C} \to \mathsf{D}$ is a fully faithful functor which is essentially surjective on objects. For each objects $d \in \mathsf{D}$, using the essentially surjectivity and the axiom of choice, take an object $G(d) \in \mathsf{C}$ and an isomorphism $\epsilon_d : F(G(d)) \simeq d$. Then by lemma above, for each morphism $l : d \to d'$ there is a unique morphism $m : F(G(d)) \to F(G(d'))$ satisfying $l \circ \epsilon_d = \epsilon_{d'} \circ m$. Since F is fully faithful, there is a unique morphism $G(d) \to G(d')$, which defines $G(l)$. This definition makes $\epsilon : F \circ G \Rightarrow 1_{\mathsf{D}}$ a natural transformation.

To show that this G is actually a functor, notice that since ϵ is a natural transformation, $1_d \circ \epsilon_d = \epsilon_d \circ F(G(1_d))$. Also, since $F(1_{G(d)})$ is an identity morphism on $F(G(d))$, $1_d \circ \epsilon_d = \epsilon_d \circ F(1_{G(d)})$, thus by above lemma, $F(1_{G(d)}) = F(G(1_d))$, and since F is fully faithfull, $1_{G(d)} = G(1_d)$.

$$\begin{tikzcd} F(G(d)) \arrow[r, "\epsilon_d"] \arrow[d, "F(G(1_d))=F(1_{G(d)})"'] & d \arrow[d, "1_d"] \\ F(G(d)) \arrow[r, "\epsilon_d"] & d \end{tikzcd} \tag{11}$$

Similarly, for morphisms $l : d \to d'$ and $l' : d' \to d''$, both $F(G(l') \circ$

$G(l))$ and $F(G(l' \circ l))$ satisfies the commutation relation, and thus $G(l') \circ G(l) = G(l' \circ l)$.

$$\begin{tikzcd} F(G(d)) \arrow[r, "\epsilon_d"] \arrow[d, "F(G(l')\circ G(l))=F(G(l'\circ l))"'] & d \arrow[d, "l'\circ l"] \\ F(G(d'')) \arrow[r, "\epsilon_d"] & d'' \end{tikzcd} \tag{12}$$

Finally, define $\eta_c : c \to F(G(c))$ by using the equation $\epsilon^{-1}_{F(c)} = F(\eta_c) : F(c) \to F(G(F(c)))$ and the fully faithfulness of F. Then for any $f : c \to c'$, consider the following diagram.

$$\begin{tikzcd} F(c) \arrow[r, "F(\eta_c)"] \arrow[d, "F(f)"] & F(G(F(c))) \arrow[r, "\epsilon_{F(c)}"] \arrow[d, "F(G(F(f)))"] & F(c) \arrow[d, "F(f)"] \\ F(c') \arrow[r, "F(\eta_{c'})"] & F(G(F(c'))) \arrow[r, "\epsilon_{F(c')}"] & F(c') \end{tikzcd} \tag{13}$$

By the definition of η, the outer rectangle commutes. Also, since ϵ is a natural transformation, the right square commutes. Since $\epsilon_{F(c')}$ is an isomorphism, the left square commutes, and fully faithfulness of F makes possible to drop the initial F on the commuting diagram. Thus η is a natural transformation. □

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Definition 55. A category C is **connected** if for any objects $c, c' \in \mathsf{C}$ there is a finite chain of morphisms $c \to c_1 \to \cdots \to c_n \to c'$.

Proposition 56. *A connected groupoid is equivalent to the automorphism group of any of its objects as a category.*

Proof. Choose an object g in a connected groupoid G, and take a group $G = \mathsf{G}(g, g)$. Consider the inclusion $\mathsf{B}G \hookrightarrow \mathsf{G}$. Then this inclusion functor is fully faithful, and for every $g' \in G$, g is isomorphic to g', thus it is essentially surjective on objects. Therefore, by the theorem above, this functor defines an equivalence of category. □

Corollary 57. *In a path-connected space X, any choice of basepoint $x \in X$ gives an isomorphic fundamental group $\pi_1(X, x)$.*

Proof. Any space X has a fundamental groupoid $\Pi_1(X)$, and fixing a basepoint x, the group of automorphisms of the object $x \in \Pi_1(X)$ is a fundamental group $\pi_1(X, x)$. Thus $\pi_1(X, x) \simeq \Pi_1(X)$, and since the equivalence of category is equivalence relation, for any $x, x' \in X$, $\pi_1(X, x) \simeq \pi_1(X, x')$. Since these are one object category, there is a functor which is bijective on functors, and this gives the isomorphism between groups. Therefore, $\pi_1(X, x) \simeq \pi_1(X, x')$ in the sense of group theory also. □

Definition 58. A category C is **skeletal** if it contains just one objects in each isomorphism classes. The **skeleton** skC of a category C is the unique skeletal category up to isomorphism that is equivalent to C.

This category skC can be constructed from C by choosing one object in each isomorphism class in C and defining skC as a full subcategory of C. This gives an equivalence of categories since the inclusion functor is fully faithful and essentially surjective on objects, but the concept $\mathsf{sk} : \mathsf{CAT} \to \mathsf{CAT}$ is not a functor.

Example 59. Consider a left G-set $X : \mathsf{B}G \to \mathsf{Set}$. The **translation groupoid** $\mathsf{T}_G X$ is a category whose objects are the points of X and morphisms are $g : x \to y$ for $g \in G$ with $g \cdot x = y$. The objects of the skeleton $\mathsf{skT}_G X$ are the **orbits** of the group action. For $x \in X$, write its orbit O_x. Then since $\mathsf{skT}_G X \simeq \mathsf{T}_G X$, $\mathsf{skT}_G X(O_x, O_x) \simeq \mathsf{T}_G X(x, x) = G_x$, where G_x is the **stabilizer** of x, which is the set of group elements $g \in G$ satisfying $g \cdot x = x$. Now, since we may choose other elements from O_x, thus all the morphism sets $\mathsf{T}_G X(x, y) = G_x$ if $x, y \in O_x$. Also, the set of all morphisms with domain x is isomorphic to G. Therefore, $|G| = |O_x||G_x|$, which is the **orbit-stabilizer theorem**.

Definition 60. A category is **essentially small** if it is equivalent to a small category. A category is **essentially discrete** if it is equivalent to a discrete category.

Lemma 61. *Consider functors $F, G, H : \mathsf{C} \to \mathsf{D}$ and natural transformations $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$. Then there is a natural transformation $\beta \circ \alpha : F \Rightarrow H$ whose components are $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$. This is called a* ***vertical composition****.*

Proof. For any morphism $f : c \to c'$ in C, two squares of the following diagram commutes, because α, β are natural transformations.

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"] & G(c) \arrow[r, "\beta_c"] \arrow[d, "G(f)"] & H(c) \arrow[d, "H(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & G(c') \arrow[r, "\beta_{c'}"] & H(c') \end{tikzcd} \tag{14}$$

Thus the outer rectangle commutes, hence the composition $\beta_c \circ \alpha_c$ gives the natural transformation. ☐

Corollary 62. *For a pairs of categories C, D, there is a* ***functor category*** *D^{C} whose elements are functors and morphisms are natural transformations.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "F", ""{name=F1, below}] \arrow[r, "G" description, ""{name=G1}] \arrow[r, bend right=60, "H"', ""{name=H1, above}] \arrow[Rightarrow, from=F1, to=G1, "\alpha"] \arrow[Rightarrow, from=G1, to=H1, "\beta"] & \mathsf{D} & \rightarrow & \mathsf{C} \arrow[r, bend left=60, "F", ""{name=F2, below}] \arrow[r, bend right=60, "H"', ""{name=H2, above}] \arrow[Rightarrow, from=F2, to=H2, "\beta\circ\alpha"] & \mathsf{D} \end{tikzcd} \tag{15}$$

Proof. The lemma above shows the composition of natural transformations, and we only need to prove the associativity and existence of identity natural transformation. For associativity, since the natural transformation α is composed by the morphisms α_c, which has associativity, the composition of natural transformation also has the morphisms. For identity natural transformation between F and F, take α_c as the identity maps $F(c) \to F(c)$, which gives the natural transformation and whose composition with other natural transformation $\beta : F \Rightarrow G$ and $\gamma : H \Rightarrow F$ gives $\beta \circ \alpha = \beta$ and $\alpha \circ \gamma = \gamma$. □

Lemma 63. *Consider functors $F, G : \mathsf{C} \to \mathsf{D}, H, K : \mathsf{D} \to \mathsf{E}$ and natural transformations $\alpha : F \Rightarrow G, \beta : H \Rightarrow K$. Then there is a natural transformation $\beta * \alpha : H \circ F \Rightarrow K \circ G$, which is defined as $(\beta * \alpha)_c = K(\alpha_c) \circ \beta_{F(c)} = \beta_{G(c)} \circ H(\alpha_c)$. This is called a **horizontal composition**.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left=60, "H"] \arrow[r, bend right=60, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E} \end{tikzcd} \to \begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "H\circ F"] \arrow[r, bend right=60, "K\circ G"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathsf{D} \end{tikzcd} \tag{16}$$

$$\begin{tikzcd} H(F(c)) \arrow[r, "\beta_{F(c)}"] \arrow[d, "H(\alpha_c)"] \arrow[dr, "(\beta*\alpha)_c"] & K(F(c)) \arrow[d, "K(\alpha_c)"] \\ H(G(c)) \arrow[r, "\beta_{G(c)}"] & K(G(c)) \end{tikzcd} \tag{17}$$

Proof. The square in above diagram commutes due to the naturality of $\beta : H \Rightarrow K$ applied on $\alpha_c : F(c) \to G(c)$. To show $\beta * \alpha$ satisfies the naturality, we need to show that $K(G(f)) \circ (\beta * \alpha)_c = (\beta * \alpha)_{c'} \circ H(F(f))$ for any morphism $f : c \to c'$ in C. Now consider the following diagram.

$$\begin{tikzcd} H(F(c)) \arrow[r, "H(\alpha_c)"] \arrow[d, "H(F(f))"] & H(G(c)) \arrow[r, "\beta_{G_c}"] \arrow[d, "H(G(f))"] & K(G(c)) \arrow[d, "K(G(f))"] \\ H(F(c')) \arrow[r, "H(\alpha_{c'})"] & H(G(c')) \arrow[r, "\beta_{G(c')}"] & K(G(c')) \end{tikzcd} \tag{18}$$

The right square commutes by the naturality of β, and the left square is the commutative diagram of α passed after the functor H, which hence commutes again. Therefore the outer rectangle commutes, which shows that $\beta * \alpha$ is a natural transformation.' □

Lemma 64 (Middle four interchange)**.** *Consider functors $F, G, H : \mathsf{C} \to \mathsf{D}$, $J, K, L : \mathsf{D} \to \mathsf{E}$, and natural transformations $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$, $\gamma : J \Rightarrow K, \delta : K \Rightarrow L$. Then $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=70, "F"] \arrow[r, "G" description] \arrow[r, bend right=70, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathsf{D} \arrow[r, bend left=70, "J"] \arrow[r, "K" description] \arrow[r, bend right=70, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathsf{E} \end{tikzcd} \to \begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathsf{D} \arrow[r, bend left=60, "J"] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, "\Downarrow\delta\circ\gamma"] & \mathsf{E} \end{tikzcd} \to \begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "J\circ F"] \arrow[r, bend right=60, "L\circ H"'] \arrow[r, phantom, "\Downarrow(\delta\circ\gamma)*(\beta\circ\alpha)"] & \mathsf{E} \end{tikzcd} \tag{19}$$

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, "G" description] \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathsf{D} \arrow[r, "K" description] \arrow[r, bend left=60, "J"] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathsf{E}
\end{tikzcd}
\to
\begin{tikzcd}
\mathsf{C} \arrow[r, "K\circ G" description] \arrow[r, bend left=60, "J\circ F"] \arrow[r, bend right=60, "L\circ H"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma*\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\delta*\beta"] & \mathsf{E}
\end{tikzcd}
\to
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "J\circ F"] \arrow[r, bend right=60, "L\circ H"'] \arrow[r, phantom, "\Downarrow(\delta*\beta)\circ(\gamma*\alpha)"] & \mathsf{E}
\end{tikzcd}
\tag{20}
$$

Proof. First, $((\delta \circ \gamma) * (\beta \circ \alpha))_c = L(\beta_c \circ \alpha_c) \circ (\delta \circ \gamma)_{F(c)} = L(\beta_c) \circ L(\alpha_c) \circ \delta_{F(c)} \circ \gamma_{F(c)}$ and $((\delta * \beta) \circ (\gamma * \alpha))_c = L(\beta_c) \circ \delta_{G(c)} \circ K(\alpha_c) \circ \gamma_{F(c)}$. Now $L(\alpha_c) \circ \delta_{F(c)} = \delta_{G(c)} \circ K(\alpha_c)$ because of the naturality of α, therefore we get the desired result.

$$
\begin{tikzcd}
J(F(c)) \arrow[r, "\gamma_{F(c)}"] \arrow[d, "J(\alpha_c)"] \arrow[dr, "(\gamma*\alpha)_c"] & K(F(c)) \arrow[r, "\delta_{F(c)}"] \arrow[d, "K(\alpha_c)"] & L(F(c)) \arrow[d, "L(\alpha_c)"] \\
J(G(c)) \arrow[r, "\gamma_{G(c)}"] \arrow[d, "J(\beta_c)"] & K(G(c)) \arrow[r, "\delta_{G(c)}"] \arrow[d, "K(\beta_c)"] \arrow[dr, "(\delta*\beta)_c"] & L(G(c)) \arrow[d, "L(\beta_c)"] \\
J(H(c)) \arrow[r, "\gamma_{H(c)}"] & K(H(c)) \arrow[r, "\delta_{H(c)}"] & L(H(c))
\end{tikzcd}
\tag{21}
$$

□

Definition 65. A **2-category** is a collection of

- objects, for example the categories C,
- 1-morphisms between pair of objects, for example the functors $F : \mathsf{C} \to \mathsf{D}$,
- 2-morphisms between parallel pairs of 1-morphisms, for example the natural transformations $\alpha : F \Rightarrow G$ with $F : \mathsf{C} \to \mathsf{D}$

which satisfies

- the objects and 1-morphisms form a category;
- the 1-morphisms and 2-morphisms form a category under vertical composition;
- the 1-morphisms and 2-morphisms form a category under horizontal composition;
- the middle four interchange law between vertical and horizontal composition holds.

Definition 66. An object $c \in \mathsf{C}$ is **initial** if the covariant functor $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$ is naturally isomorphic to the constant functor $* : \mathsf{C} \to \mathsf{Set}$ taking every objects to a singleton set. An object $c \in \mathsf{C}$ is **terminal** if the contravariant functor $\mathsf{C}(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is naturally isomorphic to the constant functor $* : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ taking every objects to a singleton set.

Definition 67. A covariant or contravariant functor F from a locally small category C to Set is **representable** if there is an object $c \in \mathsf{C}$ such that F is naturally isomorphic to $\mathsf{C}(c,-)$ or $\mathsf{C}(-,c)$. A **representation** of a functor F is a choice of object $c \in \mathsf{C}$ and, natural isomorphism $\mathsf{C}(c,-) \simeq F$ if F is covariant, and $\mathsf{C}(-,c) \simeq F$ if F is contravariant.

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Definition 68. A **universal property** of an object X in category C is a description of the covariant functor $\mathsf{C}(X,-)$ or of the contravariant functor $\mathsf{C}(-,X)$.

Example 69.

1. Consider the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$. This functor is represented by the group $\mathbb{Z}$. Indeed, there is a natural isomorphism $\mathsf{Group}(\mathbb{Z},-) \simeq U$ which takes the homomorphism $\phi \in \mathsf{Group}(\mathbb{Z},G)$ to an element $g \in U(G)$ where $g = \phi(1)$ bijectively. We thus say $\mathbb{Z}$ is the free group on a single generator.
2. For any unital ring R, consider the forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$. This functor is represented by the R-module R. The construction of a natural isomorphism $\mathsf{Mod}_R(R,-) \simeq U$ is very similar with above. We thus say R is the free R-module on a single generator.
3. Consider the forgetful functor $U : \mathsf{Ring} \to \mathsf{Set}$. This functor is represented by the ring $\mathbb{Z}[x]$. We thus say $\mathbb{Z}[x]$ is the free unital ring on a single generator.
4. Consider a functor $U(-)^n : \mathsf{Group} \to \mathsf{Set}$ which sends a group G to the set of n-tuples of elements of G. This functor is represented by the free group F_n on n generators.
5. Consider a functor $U(-)^n : \mathsf{Ab} \to \mathsf{Set}$ which sends an abelian group G to the set of n-tuples of elements of G. This functor is represented by the free abelian group $\oplus_n \mathbb{Z}$ on n generators.

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Theorem 70 (Yoneda lemma)**.** *Consider a locally small category* C. *For any functor* $F : \mathsf{C} \to \mathsf{Set}$ *and any object* $c \in \mathsf{C}$, *there is a bijection*

$$\mathsf{Nat}(\mathsf{C}(c,-),F) \simeq F(c) \tag{22}$$

which associates a natural transformation $\alpha : \mathsf{C}(c,-) \Rightarrow F$ *to the element* $\alpha_c(1_c) \in F(c)$. *This correspondence is natural in both* c *and* F.

Proof. Take a function $\Phi : \mathsf{Nat}(\mathsf{C}(c, -), F) \to F(c)$ which maps a natural transformation $\alpha : \mathsf{C}(c, -) \Rightarrow F$ to $\alpha_c(1_c)$ where $\alpha_c : \mathsf{C}(c, c) \to F(c)$. Now we want to define an inverse function $\Psi : F(c) \to \mathsf{Nat}(\mathsf{C}(c, -), F)$ which constructs a natural transformation $\Psi(x_c) : \mathsf{C}(c, -) \Rightarrow F$ for $x_c \in F(c)$. Define $\Psi(x_c)_d : \mathsf{C}(c, d) \to F(d)$ as $\Psi(x_c)_d(f) = F(f)(x_c)$ for $f : c \to d$. Now to show that $\Psi(x_c)$ is a natural transformation, we need to show that for some morphism $g : d \to e$ in C, $\Psi(x_c)_e \circ \mathsf{C}(c, g) = F(g) \circ \Psi(x_c)_d$. Take $f : c \to d$, then $\Psi(x_c)_e \circ \mathsf{C}(c, g)(f) = \Psi(x_c)_e(g \circ f) = F(g \circ f)(x_c)$ and $F(g) \circ \Psi(x_c)_d(f) = F(g) \circ F(f)(x_c) = F(g \circ f)(x_c)$, thus they are same. Now, $\Phi \circ \Psi(x_c) = \Psi(x_c)_c(1_c) = F(1_c)(x_c) = 1_c(x_c) = x_c$, Ψ is a right inverse of Φ. Choose a natural transformation $\alpha : \mathsf{C}(c, -) \Rightarrow F$. Then $\Psi \circ \Phi(\alpha)_d(f) = \Psi(\alpha_c(1_c))_d(f) = F(f)(\alpha_c(1_c))$. Now since α is natural, $\alpha_d \circ \mathsf{C}(c, f) = F(f) \circ \alpha_c$, thus $\Psi \circ \Phi(\alpha)_d(f) = \alpha_d \circ \mathsf{C}(c, f)(1_c) = \alpha_d(f)$, thus $\Psi \circ \Phi(\alpha) = \alpha$.

For the naturality of functor, we need to show that the following diagram commutes.

$$\begin{array}{ccc} \mathsf{Nat}(\mathsf{C}(c,-),F) & \xrightarrow{\Phi_F} & F(c) \\ \Big\downarrow{\scriptstyle \mathsf{Nat}(\mathsf{C}(c,-),\beta)} & & \Big\downarrow{\scriptstyle \beta_c} \\ \mathsf{Nat}(\mathsf{C}(c,-),G) & \xrightarrow{\Phi_G} & G(c) \end{array} \tag{23}$$

Choose $\alpha \in \mathsf{Nat}(\mathsf{C}(c, -), F)$. Then the above statement is equivalent to $\beta_c(\Phi_F(\alpha)) = \Phi_G(\beta \circ \alpha)$. Now $\beta_c(\alpha_c(1_c)) = \beta_c \circ \alpha_c(1_c) = (\beta \circ \alpha)_c(1_c) = \Phi_G(\beta \circ \alpha)$.

For the naturality of object, we need to show that the following diagram commutes.

$$\begin{array}{ccc} \mathsf{Nat}(\mathsf{C}(c,-),F) & \xrightarrow{\Phi_c} & F(c) \\ \Big\downarrow{\scriptstyle \mathsf{Nat}(\mathsf{C}(f,-),F)} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathsf{Nat}(\mathsf{C}(d,-),F) & \xrightarrow{\Phi_d} & F(d) \end{array} \tag{24}$$

Choose $\alpha \in \mathsf{Nat}(\mathsf{C}(c, -), F)$. Then the above statement is equivalent to $F(f)(\Phi_c(\alpha)) = \Phi_d(\alpha \circ f)$. Now $F(f)(\Phi_c(\alpha)) = F(f)(\alpha_c(1_c))$ and $\Phi_d(\alpha \circ f) = (\alpha \circ f)_d(1_d) = (\alpha_d \circ f)(1_d) = \alpha_d(f) = F(f)(\alpha_c(1_c))$ due to the naturality of α. □

Corollary 71. *The functor $y : \mathsf{C} \hookrightarrow \mathsf{Set}^{\mathsf{C}^{op}}$ defined as $y(c) = \mathsf{C}(-, c)$ and $y(f : c \to d) = f_* : \mathsf{C}(-, c) \to \mathsf{C}(-, d)$ is a full embedding, and called a **covariant embedding**. The functor $y : \mathsf{C}^{op} \hookrightarrow \mathsf{Set}^{\mathsf{C}}$ defined as $y(c) = \mathsf{C}(c, -)$ and $y(f : c \to d) = f^* : \mathsf{C}(d, -) \to \mathsf{C}(c, -)$ is a full embedding, and called a **contravariant embedding**.*

Proof. The injectivity of object is trivial, thus we need to show the functors give the bijections $(C)(c, d) \simeq \mathsf{Nat}(\mathsf{C}(-, c), \mathsf{C}(-, d))$

and $\mathsf{C}(c,d) \simeq \mathsf{Nat}(\mathsf{C}(d,-),\mathsf{C}(c,-))$. Now since different morphisms $f,g : c \to d$ define distinct natural transformations $f_*, g_* : \mathsf{C}(-,c) \Rightarrow \mathsf{C}(-,d)$ and $f^*, g^* : \mathsf{C}(d,-) \Rightarrow \mathsf{C}(c,-)$, thus the injection is shown. For surjection, take a natural transformation $\alpha : \mathsf{C}(d,-) \Rightarrow \mathsf{C}(c,-)$. The Yoneda lemma says that this natural transformation corresponds to morphisms $f : c \to d$ where $f = \alpha_d(1_d)$. Now the natural transformation $f^* : \mathsf{C}(d,-) \Rightarrow \mathsf{C}(c,-)$ also takes $f^*_d(1_d) = f$, which shows that $f^* = \alpha$ by the bijectivity of Yoneda lemma. □

Corollary 72 (Cayley's theorem)**.** *Any group is isomorphic to a subgroup of a permutation group.*

Proof. Take a group G and consider its category form $\mathsf{B}G$. The image of the covariant Yoneda embedding $\mathsf{B}G \hookrightarrow \mathsf{Set}^{\mathsf{B}G^{\mathrm{op}}}$ is the right G-set G, acting by right multiplication. Then the Yoneda embedding gives the isomorphism between G and the endomorphism group of the right G-set G. Take the forgetful functor $\mathsf{Set}^{\mathsf{B}G^{\mathrm{op}}} \to \mathsf{Set}$. This identifies G with the subgroup of the automorphism group $\mathrm{Sym}(G)$ of the set G. □