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HOMOLOGICAL ALGEBRA

2019- 07- 22

Definition 1. A **chain complex** $C_\bullet = C$ of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules with R-module maps $d = d_n : C_n \to C_{n-1}$ such that each composite $d \circ d : C_n \to C_{n-2}$ is zero. The maps $d_n = d$ are called the **differentials** of C. The kernel $\operatorname{Ker} d_n$ is the module of n**-cycles** of C, and denoted $Z_n = Z_n(C)$. The image $\operatorname{Im} d_{n+1}$ is the module of n**-boundaries** of C, and denoted $B_n = B_n(C)$. Since $d \circ d = 0$, $0 \subset B_n \subset Z_n \subset C_n$ for all n. The quotient $H_n(C) = Z_n/B_n$ is called the n**-th homology module** of C. We call a chain complex is **exact** if $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$ for all $n \in \mathbb{Z}$.

Exercise 2. Set $C_n = \mathbb{Z}/8$ for $n \geq 0$ and $C_n = 0$ for $n < 0$; for $n > 0$ let d_n send $x(\mod 8)$ to $4x(\mod 8)$. Show that $C_\bullet$ is a chain complex of $\mathbb{Z}/8$-modules and compute its homology modules.

Solution. Since $d \circ d(x) = 16x(\mod 8) = 0(\mod 8)$ for all $x \in \mathbb{Z}/8$, $C_\bullet$ is a chain complex. For $n < 0$, since $d = 0$, $H_n(C) = 0$. For $n > 0$, since the kernel is $\{0, 2, 4, 8\}$ and the image is $\{0, 4\}$, $H_n(C) = \mathbb{Z}/2$. For $n = 0$, since the kernel is $\mathbb{Z}/8$ and the image is $\{0, 4\}$, $H_n(C) = \mathbb{Z}/4$. Thus we get

$$H_n(C) = \begin{cases} \mathbb{Z}/2, & n > 0 \\ \mathbb{Z}/4, & n = 0 \\ 0, & n < 0 \end{cases} \tag{1}$$

■

Definition 3. The category $\mathsf{Ch}(\mathsf{mod} - R)$ is a category whose objects are chain complexes, and morphism $u : C_\bullet \to D_\bullet$ is the **chain complex map**, which is a family of R-module homomorphisms $u_n : C_n \to D_n$, which satisfies $u_{n-1} \circ d_n = d_n \circ u_n$. That is, such that the following diagram commutes.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \\ \cdots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & \cdots \end{array} \tag{2}$$

Exercise 4. Show that a morphism $u : C_\bullet \to D_\bullet$ of chain complexes sends boundaries to boundaries and cycles to cycles, hence maps $H_n(C_\bullet) \to H_n(D_\bullet)$. Prove that each H_n is a functor from $\mathsf{Ch}(\mathsf{mod}_R)$ to mod_R.

Solution. For boundaries $d(C_n)$, $u \circ d(C_n) = d \circ u(C_n) \subset d(D_n)$, thus $u \circ d(C_n)$ are boundaries of D_n. For cycles Z, $d(Z) = 0$, and $d(u(Z)) = u(d(Z)) = 0$, thus $u(Z)$ are boundaries of D_n. Thus $u :$

$C_\bullet \to D_\bullet$ can be quotiented and gives $u : H_n(C_\bullet) \to H_n(D_\bullet)$, which is R-module map. To show H_n is a functor, we need to show that it takes identity morphism to identity morphism, and preserves the composition. The identity morphism $1_{C_\bullet}$ defines identity R-module map $H_n(C_\bullet) \to H_n(C_\bullet)$ by definition, and for two morphisms $u : C_\bullet \to D_\bullet$ and $v : D_\bullet \to E_\bullet$, $v \circ u$ are quotiented and gives $v \circ u : H_n(C_\bullet) \xrightarrow{u} H_n(D_\bullet) \xrightarrow{v} H_n(E_\bullet)$. ■

Exercise 5 (Split exact sequences of vector spaces)**.** Choose vector spaces $\{B_n, H_n\}_{n\in\mathbb{Z}}$ over a field, and set $C_n = B_n \oplus H_n \oplus B_{n-1}$. Show that the projection-inclusions $C_n \to B_{n-1} = C_{n-1}$ make $\{C_n\}$ into a chain complex, and that every chain complex of vector spaces is isomorphic to a complex of this form.

Solution. Take $(b_n, h_n, b_{n-1}) \in B_n \oplus H_n \oplus B_{n-1}$. Then $d \circ d(b_n, h_n, b_{n-1}) = d(b_{n-1}, 0, 0) = (0, 0, 0)$, thus $C_\bullet$ is a chain complex. Now consider a chain complex $V_\bullet$ of vector spaces. Take B_n, H_n as the boundaries and homology modules of $V_\bullet$. Now if we show that $V_n = B_n \oplus H_n \oplus B'_{n-1}$, then the statement is proven. Notice that $H_n = Z_n/B_n$ thus $Z_n = H_n \oplus B_n$. Now due to the first isomorphism theorem, $V_n/Z_n = B_{n-1}$. Therefore $V_n = Z_n \oplus B_{n-1} = B_n \oplus H_n \oplus B_{n-1}$. ■

Exercise 6. Show that $\{\mathrm{Hom}_R(A, C_n)$ forms a chain complex of abelian groups for every R-module A and every R-module chain complex C. Taking $A = Z_n$, show that if $H_n(\mathrm{Hom}_R(Z_n, C)) = 0$, then $H_n(C) = 0$. Is the converse true?

Solution. Define $d : \mathrm{Hom}_R(A, C_n) \to \mathrm{Hom}_R(A, C_{n-1})$ as $d(f : A \to C_n) = d \circ f$, which is a group homomorphism because $d(f+g)(c) = d \circ (f+g)(c) = d(f(c) + g(c)) = d(f(c)) + d(g(c)) = d(f)(c) + d(g)(c)$. Then $d \circ d(f) = (d \circ d) \circ f = 0$, thus this is a chain complex.

For second question, choose the inclusion $i_n : Z_n \hookrightarrow C_n$. Then we can see that $d_n \circ i_n = 0$, thus there is $u : Z_n \to C_{n+1}$ such that $i_n = d_{n+1} \circ u$. Then $Z_n = i_n(C_n) = d_{n+1} \circ u(C_n) \subset d_{n+1}(C_{n+1}) = B_n$, thus $H_n(C) = 0$.

Now consider the chain complex C as $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ with $Z_n = \mathbb{Z}/2$. Notice that $H_n = 0$. Now $\mathrm{Hom}_R(Z_n, 2\mathbb{Z}) = \mathrm{Hom}_R(Z_n, \mathbb{Z}) = 0$ and $\mathrm{Hom}_R(Z_n, \mathbb{Z}/2) = \mathbb{Z}/2$, thus we get the chain complex $0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$, and so $H_n(\mathrm{Hom}_R(Z_n, C)) = \mathbb{Z}/2 \neq 0$, so the converse is not true. ■

Definition 7. A morphism $C_\bullet \to D_\bullet$ of chain complexes is called a **quasi-isomorphism** if the maps $H_n(C) \to H_n(D)$ are all isomorphisms.

Exercise 8. Show that the following are equivalent for every $C_\bullet$:

1. $C_\bullet$ is exact.

2. $C_\bullet$ is **acyclic**, that is, $H_n(C) = 0$ for all n.

3. The map $0 \to C_\bullet$ is a quasi-isomorphism.

Solution. $(1 \to 2)$ Since $C_\bullet$ is exact, $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$, thus $H_n(C) = 0$.

$(2 \to 3)$ Since $H_n(C) = 0$, all the maps $0 \to H_n(C)$ are isomorphisms.

$(3 \to 1)$ Since the maps $0 \to H_n(C)$ are isomorphisms, $H_n(C) = 0$, thus $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$. ■

All these definitions can be obtained by reindexing the chain complex C_n by $C^n = C_{-n}$.

Definition 9. A **cochain complex** $C^\bullet$ of R-modules is a family $\{C^n\}$ of R-modules with maps $d = d^n : C^n \to C^{n+1}$ such that $d \circ d = 0$. The kernel $\operatorname{Ker} d_n$ is the module of **n-cocycles** of C, and denoted $Z^n = Z^n(C)$. The image $\operatorname{Im} d_{n-1}$ is the module of **n-coboundaries** of C, and denoted $B^n = B_n(C)$. Since $d \circ d = 0$, $0 \subset B^n \subset Z^n \subset C^n$ for all n. The quotient $H^n(C) = Z^n / B^n$ is called the **n-th cohomology module** of C. The morphism $u : C^\bullet \to D^\bullet$ is a family of R-module homomorphisms $u^n : C^n \to D^n$ which satisfies $u^{n+1} \circ d^n = d^n \circ u^n$. A morphism $C^\bullet \to D^\bullet$ of chain complexes is called a **quasi-isomorphism** if the maps $H^n(C) \to H^n(D)$ are all isomorphisms.

Definition 10. A chain complex C is **bounded** if all but finitely many C_n are zero. If $C_n = 0$ unless $a \leq n \leq b$, then we say the complex has **amplitude** in $[a, b]$. A complex $C_\bullet$ is **bounded above(below)** if there is a bound $b(a)$ such that $C_n = 0$ for all $n > b(n < a)$. The bounded, bounded above, bounded below chain complexes form full subcategories of $\mathsf{Ch} = \mathsf{Ch}(R - \mathsf{mod})$ that are denoted $\mathsf{Ch}_b, \mathsf{Ch}_-, \mathsf{Ch}_+$. If a chain complex is bounded below with bound 0, then we call it **non-negative complex**, and its category is denoted as $\mathsf{Ch}_{\geq 0}$. All these definitions works with cochain complex, where all the subscripts are changed into superscripts.

Exercise 11 (Homology of a graph)**.** Let Γ be a finite loopless graph with V vertices $(v_1, \cdots v_V)$ and E edges $(e_1, \cdots e_E)$. If we orient the edges, we can form the **incidence matrix** of the graph. This is a $V \times E$ matrix whose (ij) entry is $+1$ if the edge e_j starts at v_i, -1 if the edge e_j ends at v_j, and 0 otherwise. Let C_0 be the free R-module on the vertices, C_1 the free R-module on the edges, $C_n = 0$ if $n \neq 0, 1$, and $d : C_1 \to C_0$ be the incidence matrix. If Γ is connected, show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions 1 and $E - V + 1$, which is the number of **circuits** of the graph, respectively.

Solution. What we need to find is $\operatorname{Im} d$ and $\operatorname{Ker} d$. For $\operatorname{Im} d$, choose the basis $\{v_0, v_1 - v_0, \cdots, v_V - v_0\}$. Then considering the paths connecting v_i and v_0, and take the edges of the paths. This edges gives $v_i - v_0$ when passing through d, thus the only unachievable basis is v_0, thus $H_0(C)$ is the free R-module with dimension 1. For $\operatorname{Ker} d$, notice that the rank of Γ is $V - 1$ by above, and by the rank-nullity theorem, $\operatorname{Ker} d$ is the free R-module with dimension $E - (V - 1) = E - V + 1$. ■

Example 12 (Simplicial homology). Let K be a geometric simplicial complex, and let K_k, where $0 \leq k \leq n$, are the set of k-dimensional simplices of K. Each k-simplex has $k+1$ faces, which are ordered if the set K_0 of vertices is ordered, so we obtain $k+1$ set maps $\partial_i : K_k \to K_{k-1}$. The **simplicial chain complex** of K with coefficients in R is the chain complex $C_\bullet$ formed as follows. The set C_k is a free R module on the set K_k if $0 \leq k \leq n$, and $C_n = 0$ otherwise. Define $\partial_i : C_k \to C_{k-1}$ using the set map $\partial_i : K_k \to K_{k-1}$, and then their alternating sum $d = \sum(-1)^i \partial_i$ is the map $C_k \to C_{k-1}$ in the chain complex C. Showing $d \circ d = 0$ is equivalent to the fact that each $(k-2)$-dimensional simplex in a fixed k-simplex σ of K lies on exactly two faces of σ. The homology obtained from the chain complex $C_\bullet$ is called the **simplicial homology** of K with coefficients on R.

Exercise 13 (Tetrahedron). The tetrahedron T is a surface with 4 vertices, 6 edges, and 4 2-dimensional faces. Thus its homology is the homology of a chain complex $0 \to R^4 \to R^6 \to R^4 \to 0$. Write down the matrices in this complex and verify computationally that $H_2(T) \simeq H_0(T) \simeq R$ and $H_1(T) = 0$.

Proof. First and last map are trivial. For the second map $R^4 \to R^6$, denoting 4 faces as A, B, C, D, and 6 edges as a, b, c, d, e, f, then we can write

$$A \mapsto a - b + d, B \mapsto b - c + e, C \mapsto c - a + f, D \mapsto -(d + e + f) \quad (3)$$

For the third map $R^6 \to R^4$, denoting 4 vertices as v, w, x, y, then we can write

$$a \mapsto v - w, b \mapsto v - x, c \mapsto v - y, d \mapsto w - x, e \mapsto x - y, f \mapsto y - w \quad (4)$$

Consider $0 \to R^4 \to R^6$. The image is 0, and the kernel is R-module with basis $A + B + C + D$, thus we get $H_2(T) = R$.

Consider $R^4 \to R^6 \to R^4$. The image is R-module with basis $\{a - b + d, b - c + e, c - a + f\}$, and the kernel is R-module with basis $\{a - b + d, b - c + e, c - a + f\}$, thus we get $H_1(T) = 0$.

Consider $R^6 \to R^4 \to 0$. The image is R-module with basis $\{v - w, v - x, v - y\}$, and the kernel is v, w, x, y, thus we get $H_1(T) = R$. □

Example 14 (Singular homology)**.** Let X be a topological space and $S_k = S_k(X)$ be the free R-module on the set of continuous maps from the k-simplex Δ_k to X if $k \geq 0$ and $S_k = 0$ if $k < 0$. Restricting $\Delta_k \to X$ to $\Delta_{k-1} \to X$ gives an R-module homomorphism $\partial_i : S_k \to S_{k-1}$, and the alternating sum $d = \sum(-1)^i \partial_i : S_k \to S_{k-1}$ gives a chain complex $S_\bullet$. The reason why $d \circ d = 0$ is similar with simplicial homology case. The homology obtained from the chain complex $S_\bullet$ is called the **singular homology** of X with coefficients in R, and written $H_n(X; R)$. If X is a geometric simplicial complex, then the inclusion $\mathsf{C}_\bullet(X) \to S_\bullet(X)$ is a quasi-isomorphism, and so the simplicial and singular homology modules of X are isomorphic. For more details, see Algebraic Topology by Allen Hatcher.

Definition 15. A cataegory A is called an Ab**-category** if $\mathsf{A}(a, b)$ is given the structure of abelian group in such a way that composition distributes over addition. That is, if $f : a \to b, g, g' : b \to c, h : c \to d$ are morphisms of A, then $h \circ (g + g') \circ f = h \circ g \circ f + h \circ g' \circ f$.

Consider two Ab-categories A, B. A functor $F : \mathsf{B} \to \mathsf{A}$ is an **additive functor** if $F : \mathsf{B}(b, b') \to \mathsf{A}(F(b), F(b'))$ is a group homomorphism.

Consider an Ab-category A. Then A is an **additive category** if A has an object which is both initial and terminal, which is called a **zero object**, and a product $a \times b$ for objects a, b of A.

In additive category, the finite products are same with the finite coproducts.

Example 16. The category Ch is an Ab-category, since we can add chain maps $\{f_n\}, \{g_n\} : \mathsf{C}_\bullet \to D_\bullet$ degreewise, that is, their sum is a family of maps $\{f_n + g_n\}$. The zero object of Ch is the complex 0 of zero modules and maps. For a family $\{A_\alpha\}$ of complexes of R-modules, the product $\prod A_\alpha$ and coproduct $\oplus A_\alpha$ exist in Ch, and defined degreewise, that is, the differentials are the maps

$$\prod d_\alpha = \prod A_{\alpha,n} \to \prod A_{\alpha,n-1}, \quad \oplus d_\alpha : \oplus A_{\alpha,n} \to \oplus A_{\alpha,n-1} \tag{5}$$

This shows that Ch is an additive category.

Exercise 17. Show that direct sum and direct product commute with homology, that is, $\oplus H_n(A_\alpha) \simeq H_n(\oplus A_\alpha)$ and $\prod H_n(A_\alpha) \simeq H_n(\prod A_\alpha)$ for all n.

Proof. Before showing this, we need to show a small lemma: in category $R - \mathsf{mod}$, the product of epimorphisms is epimorphic. Notice that the product of morphisms is morphism in $R - \mathsf{mod}$, and the product of surjective functions is surjective, this is true. Now, since the direct sum and direct product are in dual relation, we only need to prove it on the direct product. Now consider the following diagram.

$$\begin{tikzcd} B \arrow[r, dotted, "h"] \arrow[rr, bend left, "f"] \arrow[dr, "f_\alpha"'] & \operatorname{Ker}(d) \arrow[r, hook, "i"] \arrow[d, "\pi_\alpha|_{\operatorname{Ker}(d)}"] & \prod A_{\alpha,n} \arrow[r, "d"] \arrow[d, "\pi_\alpha"] & \prod A_{\alpha,n-1} \arrow[d, "\pi_\alpha"] \\ & \operatorname{Ker}(d_\alpha) \arrow[r, hook, "i_\alpha"] & A_{\alpha,n} \arrow[r, "d_\alpha"] & A_{\alpha,n-1} \end{tikzcd} \tag{6}$$

Here B is an R-module. Now due to the definition of the product, the functions $i_\alpha \circ f_\alpha$ and projections π_α defines a unique function $f : B \to \prod A_{\alpha,n}$. Now notice that $\pi_\alpha \circ d \circ f = d_\alpha \circ \pi_\alpha \circ f = d_\alpha \circ i_\alpha \circ f_\alpha = 0$, thus again by the definition of the product, $d \circ f = 0$. Due to the universal property of the kernel, there is a unique $h : B \to \operatorname{Ker}(d)$ which makes the diagram above commutes. Therefore we showed that $\operatorname{Ker}(d) = \prod \operatorname{Ker}(d_\alpha)$.

Now from the short exact sequences

$$0 \to \operatorname{Ker}(d_\alpha) \to A_{\alpha,n+1} \to \operatorname{Im}(d_\alpha) \to 0 \tag{7}$$

we can build a sequence

$$0 \to \prod \operatorname{Ker}(d_\alpha) \to \prod A_{\alpha,n+1} \to \prod \operatorname{Im}(d_\alpha) \to 0 \tag{8}$$

which is left exact due to the above argument. Now since in R-module the product of epimorphisms are epimorphic, the above sequence is right exact, hence exact, and

$$\prod \operatorname{Im}(d_\alpha) \simeq \prod A_{\alpha,n+1} / \prod \operatorname{Ker}(d_\alpha) \simeq \prod A_{\alpha,n+1} / \operatorname{Ker}(d) \simeq \operatorname{Im}(d) \tag{9}$$

Now take the product of following sequence

$$0 \to \operatorname{Ker}(d) \to \operatorname{Im}(d) \to H_n(A_{\alpha,n}) \to 0 \tag{10}$$

which gives the desired result. □

Definition 18. Let C is a category and $f : b \to c$ is a morphism in C. Then f is a **constant morphism** or **left zero morphism** if for any object a in C and any morphisms $g, h : a \to b$, $f \circ g = f \circ h$. Dually, f is a **coconstant morphism** or **right zero morphism** if for any object d in C and any morphisms $g, h : c \to d$, $g \circ f = h \circ f$. If $f : b \to c$ is both a constant and coconstant morphism, we call it **zero morphism**. We often write zero morphism from b to c as 0_{bc}, and if its domain and codomain are obvious, 0. A **category with zero morphisms** is a category C such that for all object pairs $a, b \in$ C there is a morphisms 0_{ab} such that for all objects $a, b, c \in$ C and morphisms $f : b \to c, g : x \to$, the following diagram commutes.

$$\begin{tikzcd} a \arrow[r, "0_{ab}"] \arrow[d, "g"] \arrow[dr, "0_{ac}"] & b \arrow[d, "f"] \\ b \arrow[r, "0_{bc}"] & c \end{tikzcd} \tag{11}$$

Then the morphisms 0_{ab} are zero morphisms.

Example 19. Let a category C has a zero object 0. Then for all objects $b, c \in \mathsf{C}$, there are unique morphisms $f : b \to 0, g : 0 \to c$. Now define $0_{bc} = g \circ f$. Then this is a zero morphism from b to c, due to the definition of the zero object.

Example 20. Let C be an Ab-category. Then every morphism set $\mathsf{C}(x, y)$ is an abelian group, thus have a zero element. Denote it 0_{xy}. Now choose the morphisms $f : y \to z, g : x \to y$. Then since $f \circ 0_{xy} + f \circ 0_{xy} = f \circ (0_{xy} + 0_{xy}) = f \circ 0_{xy}$, thus $f \circ 0_{xy} = 0_{xz}$, and same for $0_{yz} \circ g$. Therefore 0_{xy} are zero morphisms and make C a category with zero morphisms.

Definition 21. In an additive category C, a **kernel** of a morphism $f : b \to c$ is a map $i : a \to b$ such that $f \circ i = 0$ and, for any $i' : a' \to b$ such that $f \circ i' = 0$, there is a unique morphism $u : a' \to a$ such that $i \circ u = i'$.

$$\begin{tikzcd}
& b \arrow[dr, "f"] & \\
& a \arrow[u, "i"] \arrow[r, "0"] & c \\
a' \arrow[uur, "i'"] \arrow[ur, dotted, "u"] \arrow[urr, "0"'] & &
\end{tikzcd} \tag{12}$$

A **cokernel** is a dual of a kernel.

Proposition 22. *Take an additive category* $\mathsf{C} = R - \mathsf{mod}$ *and its morphism* $f : x \to y$. *Show that the followings are equivalent.*

1. f *is monic, that is, for any morphisms* $h, k : w \to x$, $f \circ h = f \circ k$ *implies* $h = k$.
2. *For every map* $j : w \to x$, $f \circ j = 0$ *implies* $j = 0$.
3. f *is a kernel of some morphism* $g : y \to z$.

Dually, the followings are equivalent.

1. f *is epic, that is, for any morphisms* $h, k : y \to z$, $h \circ f = k \circ f$ *implies* $h = k$.
2. *For every map* $j : y \to z$, $j \circ f = 0$ *implies* $j = 0$.
3. f *is a cokernel of some morphism* $g : w \to x$.

Proof. Let f be monic. Due to the definition of zero morphism, $f \circ 0 = 0$, thus $f \circ h = f \circ 0$, thus $h = 0$. Conversely, suppose that for every map $j : w \to x$, $f \circ j = 0$ implies $j = 0$. Choose $f \circ h = f \circ k$ for some $h, k : w \to x$. Then $f \circ h - f \circ k = f \circ (h - k) = 0$, thus $h - k = 0$ and $h = k$.

Now notice that f is a kernel of $g : y \to z \in \mathsf{C} = R - \mathsf{mod}$ if and only if f is the injective morphism $f : \mathrm{Ker}(g) \hookrightarrow y$. Furthermore, f is monic if and only if f is injective, thus if f is a kernel then f is monic, and if f is a monic function then f is a kernel of the function g which sends $\mathrm{Im}(f)$ to 0 and $y - \mathrm{Im}(f)$ to $y - \mathrm{Im}(f)$ as identity function. □

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Exercise 23. For a category Ch, and f be its morphism, show that the complex $\mathrm{Ker}(f)$ is a kernel of f and the complex $\mathrm{coKer}(f)$ is a cokernel of f.

Solution. Let $i_n : \mathrm{Ker}(f)_n \to C_n$ be the kernel of $f_n : C_n \to D_n$. Then we have the universal properties for each components: for any $g_n : B_n \to C_n$ such that $f_n \circ g_n = 0$, there is a unique morphism $u_n : B_n \to C_n$ such that $i_n \circ u_n = g_n$. Now suppose that $\{B_n\}$ is a chain complex and $\{g_n\}$ is a chain map. What we now need to show is $\{u_n\}$ is a chain map, that is, $d \circ u_n = u_{n-1} \circ d$. Now notice that $i_{n-1} \circ u_{n-1} \circ d = g_{n-1} \circ d = d \circ g_n = d \circ i_n \circ u_n = i_{n-1} \circ d \circ u_n$, and by previous proposition we know that the kernel i_{n-1} is monic, thus $u_{n-1} \circ d = d \circ u_n$. ■

Definition 24. An **abelian category** is an additive category A such that

1. every map in A has a kernel and cokernel;
2. every monic in A is the kernel of its cokernel;
3. every epi in A is the cokernel of its kernel.

Example 25. The category $R - \mathsf{mod}$ is an abelian category. Indeed, every morphism $f : c \to d$ has a kernel $\mathrm{Ker}(f) = \{x \in c : f(x) = 0\} \hookrightarrow c$ and a cokernel $\mathrm{coKer}(f) = d \to d/\,\mathrm{Im}(f)$. For monic f, $\mathrm{Im}(f) \simeq c$, thus $\mathrm{coKer}(f) = d \to d/c$ and its kernel is c. For epic f, the cokernel of $\mathrm{Ker}(f)$ is $c \hookrightarrow c/\,\mathrm{Ker}(f)$, which is surjective and has a same structure with $f : c \to d$, thus it is f.

Definition 26. For an abelian category C and its morphism f, the **image** of a map $f : b \to c$ is the subobject of c defined as $\mathrm{Ker}(\mathrm{coKer}(f))$.

This definition is same with our previous definition of Im in $R - \mathsf{mod}$, because $\mathrm{Ker}(\mathrm{coKer}(f)) = \mathrm{Ker}(c \to c/\,\mathrm{Im}(f)) = \mathrm{Im}(f)$.

Proposition 27. *For an abelian category* A, *every morphism* $f : b \to c$ *factors as*

$$b \xrightarrow{e} \mathrm{Im}(f) \xrightarrow{m} c \tag{13}$$

where $e = \mathrm{coKer}(\mathrm{Ker}(f))$ *is an epimorphism and* $m = \mathrm{Ker}(\mathrm{coKer}(f))$ *is a monomorphism.*

Proof. Take $m = \mathrm{Ker}(\mathrm{coKer}(f))$, which is monic since it is a kernel. Since $\mathrm{coKer}(f) \circ f = 0$ by definition, f factors as $f = m \circ e$ for some unique e, which is epic. Now for any $g : a \to b$, $f \circ g = 0$ if and only if $e \circ g = 0$, since m is monic. Thus $\mathrm{Ker}(f) = \mathrm{Ker}(e)$. But since e is epic, $e = \mathrm{coKer}(\mathrm{Ker}(e)) = \mathrm{coKer}(\mathrm{Ker}(f))$. □

The argument of this statement is in Categories for the working mathematician, S. MacLane, p189.

Definition 28. For an abelian category A, a sequence $a \xrightarrow{f} b \xrightarrow{g} c$ is **exact** if $\mathrm{Ker}(g) = \mathrm{Im}(f)$.

Definition 29. For an abelian category A, the category $\mathsf{Ch}(\mathsf{A})$ is a category whose objects are chain complexes in A and morphisms are chain maps in A.

Theorem 30. *For an abelian category* A*, the category* $\mathsf{Ch}(\mathsf{A})$ *is an abelian category.*

Proof. The argument for showing additive category is same with $R - \mathsf{mod}$ case. The argument for the first condition is just same with the case Ch. For the second and third condition, consider the components of the morphism f, which are all monic(epic) if and only if f is monic(epic). Since A is an abelian category, the components of f_n is the kernel of its cokernel(cokernel of its kernel), thus f also is. □

Exercise 31. Show that a sequence $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ of chain complexes is exact in Ch just in case each sequence $0 \to A_n \xrightarrow{f} B_n \xrightarrow{g} C_n \to 0$ is exact in A.

Solution. What we need to show is $\mathrm{Ker}(g_\bullet) = \mathrm{Im}(f_\bullet)$, which is equivalent to $\mathrm{Ker}(g_n) = \mathrm{Im}(g_n)$ for all n. ■

Example 32. A **double complex** or **bicomplex** in A is a family $\{C_{p,q}\}$ of objects in A, together with maps $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v = C_{p,q} \to C_{p,q-1}$ such that $d^h \circ d^h = d^v \circ d^v = d^v \circ d^h + d^h \circ d^h \circ d^v = 0$. If there are finitely many nonzero $C_{p,q}$ along each diagonal line $p + q = n$, then we call C **bounded**.

Due to the anticommutivity, the maps d^v are not maps in Ch, but the chain maps $f_{\bullet,q} : C_{\bullet,q} \to C_{\bullet,q-1}$ can be defined by introducing

$$f_{p,q} = (-1)^p d^v_{p,q} : C_{p,q} \to C_{p,q-1} \tag{14}$$

Example 33 (Total complexes)**.** For a bicomplex C, we define the **total complexes** $\mathrm{Tot}^{\prod}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ as

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q}, \quad \mathrm{Tot}^{\oplus}(C)_n = \oplus_{p+q=n} C_{p,q} \tag{15}$$

Then $d = d^h + d^v$ defines maps $d : \mathrm{Tot}^{\prod,\oplus}(C)_{\to}\mathrm{Tot}^{\prod,\oplus}(C)_{n-1}$

such that $d \circ d = 0$ since $d^h \circ d^v + d^v \circ d^h = 0$, thus they are chain complexes. Notice that the total complexes does not always exists, because the infinite (co)direct products could not exists. An abelian category is **(co)complete** if all (co)direct products exist. Both $R - \mathsf{mod}$ and $\mathsf{Ch}(R - \mathsf{mod})$ are complete and cocomplete.

Exercise 34. For a bounded double complex C with exact rows(columns), show that $\mathrm{Tot}^{\prod}(C) = \mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}(C)$ is acyclic.

Solution. Since C is bounded, we can write the element of $\mathrm{Tot}(C)$ as $c = (\cdots, 0, c_{0,0}, c_{1,-1}, \cdots, c_{k,-k}, 0, \cdots)$, by some shifting of indexes if needed. Suppose that $d(c) = 0$, which means,

$$(\cdots, 0, d^v(c_{0,0}), d^v(c_{1,-1}) + d^h(c_{0,0}), \cdots, d^h(c_{k,-k}), 0, \cdots) = 0 \tag{16}$$

Now we want to find the element b of $\mathrm{Tot}(C)$ such that $d(b) = c$. Without loss of generality, we may let the columns are exact. Then since $d^v(c_{0,0}) = 0$, there is $b_{1,0}$ such that $d^v(b_{1,0}) = c_{0,0}$. Now then we have

$$d^v(c_{1,-1}) + d^h(d^v(b_{1,0})) = d^v(c_{1,-1}) - d^v(d^h(b_{1,0})) = d^v(c_{1,-1} - d^h(b_{1,0})) = 0 \tag{17}$$

and due to the exactness we have $b_{1,-1}$ such that $d^v(b_{1,-1}) = c_{1,-1} - d^h(b_{1,0})$. By doing this inductively, which has finitely many steps because C is bounded. ∎

Exercise 35. Give examples of

1. a second quadrant double complex C with exact columns such that $\mathrm{Tot}^{\prod}(C)$ is acyclic but $\mathrm{Tot}^{\oplus}(C)$ is not;
2. a second quadrant double complex C with exact rows such that $\mathrm{Tot}^{\oplus}(C)$ is acyclic but $\mathrm{Tot}^{\prod}(C)$ is not;
3. a double complex in the entire plane for which every row and every column is exact, yet neither $\mathrm{Tot}^{\prod}(C)$ nor $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Solution.

1. Consider the following double complex.

$$\begin{tikzcd}
\ddots \arrow[d, "1"] & & \\
\mathbb{Z} & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, "1"] & \\
& \mathbb{Z} & \mathbb{Z} \arrow[l, "\times 2"']
\end{tikzcd} \tag{18}$$

   Here all the non-represented objects are zero objects and morphisms are zero morphisms. Notice that the columns are exact. Now notice that this double complex takes

$$(\cdots, a_{-2}, a_{-1}, a_0) \mapsto (\cdots, a_{-2} + 2a_{-1}, a_{-1} + 2a_0) \tag{19}$$

For $\mathrm{Tot}^{\prod}(C)$, take $(\cdots, 4, -2, 1)$. This is in the kernel of above map, but not in the image of zero map. For $\mathrm{Tot}^{\oplus}(C)$, we get the number n such that $a_{-k} = 0$ for all $k > n$. Now since

$$(a_{-n}, \cdots, a_0) \mapsto (2a_{-n}, \cdots, a_{-1} + 2a_0) \tag{20}$$

thus if $(a_{-n}, \cdots, a_0)$ is the kernel of above map then $a_{-n} = 0$, and inductively all $a_0 = 0$.

2. Consider the following double complex.

$$\begin{tikzcd} \ddots & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "1"] & \\ & \mathbb{Z} & \mathbb{Z} \arrow[l, "1"'] \end{tikzcd} \tag{21}$$

Here all the non-represented objects are zero objects and morphisms are zero morphisms. Notice that the rows are exact. Now notice that this double complex takes

$$(\cdots, a_{-2}, a_{-1}, a_0) \mapsto (\cdots, 2a_{-2} + a_{-1}, 2a_{-1} + a_0, a_0) \tag{22}$$

For $\mathrm{Tot}^{\oplus}(C)$, suppose that we have $(\cdots, 2a_{-2} + a_{-1}, 2a_{-1} + a_0, a_0) = (\cdots, 0, 0, 1)$. Then $a_0 = 1$, thus $a_{-1} = -2$, and $a_{-2} = -4$, and so on so we get $a_{-n} = 2^n$, which is not in $\mathrm{Tot}^{\oplus}(C)$, thus $(\cdots, 0, 0, 1)$ is not in the image of the above map, but in the kernel of zero map. For $\mathrm{Tot}^{\prod}(C)$, for $(\cdots, b_{-2}, b_{-1}, b_0)$, we take $a_0 = b_0$ and $a_{-n} = b_{-n} - 2a_{-(n-1)}$, which is well defined for all n.

3. Consider the following double complex.

$$\begin{tikzcd} \ddots \arrow[d, "-1"] & & \\ \mathbb{Z} & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "-1"] & \\ & \mathbb{Z} & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "-1"] \\ & & \ddots \end{tikzcd} \tag{23}$$

Here all the non-represented objects are zero objects and morphisms are zero morphisms. Notice that the rows and columns are exact. Now notice that this double complex takes

$$(\cdots, a_{-1}, a_0, a_{-1}, \cdots) \mapsto (\cdots, -a_{-1} + a_0, -a_0 + a_1, \cdots) \tag{24}$$

For $\mathrm{Tot}^{\prod}(C)$, $(\cdots, 1, 1, \cdots)$ is in the kernel of above map, but not in the image of zero map. For $\mathrm{Tot}^{\oplus}(C)$, if $(\cdots, -a_{-1} + a_0, -a_0 +$

$a_1, -a_1 + a_2, \cdots) = (\cdots, 0, 1, 0, \cdots)$ then we get $\cdots = a_{-1} + 1 = a_0 + 1 = a_1 = a_2 = \cdots$, which is not in $\mathrm{Tot}^{\oplus}(C)$, thus $(\cdots, 0, 1, 0, \cdots)$ is not in the image of the above map, but in the kernel of zero map.

■

Definition 36. Let C be a chain complex and n be an integer. The complex $\tau_{\geq n} C$ defined by

$$(\tau_{\geq n} C)_i = \begin{cases} 0, & i < n \\ Z_n, & i = n \\ C_i, & i > n \end{cases} \tag{25}$$

is called the **truncation of** C **below** n. Notice that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0, & i < n \\ H_i(C), & i \geq n \end{cases} \tag{26}$$

The quotient $\tau_{<n} C = C/(\tau_{\geq n} C)$ is called the **truncation of** C **above** n. Notice that

$$H_i(\tau_{<n} C) = \begin{cases} H_i(C), & i < n \\ 0, & i \geq n \end{cases} \tag{27}$$

The complex $\sigma_{<n} C$ defined by

$$(\sigma_{<n} C)_i = \begin{cases} C_i, & i < n \\ 0, & i \geq n \end{cases} \tag{28}$$

is called the **brutal truncation of** C **above** n. Notice that

$$H_i(\tau_{\geq n} C) = \begin{cases} H_i(C), & i < n \\ 0, & i > n \\ C_n/B_n, & i = n \end{cases} \tag{29}$$

The quotient $\sigma_{\geq n} C = C/(\sigma_{<n} C)$ is called the **brutal truncation of** C **below** n. Notice that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0, & i < n \\ H_i(C), & i > n \\ C_n/B_n, & i = n \end{cases} \tag{30}$$

Definition 37. If C is a chain complex and p is an integer, we take a new complex $C[p]$ defined as

$$C[p]_n = C_{n+p} \tag{31}$$

with differential $(-1)^p d$. If C is a cochain complex, we take

$$C[p]^n = C^{n-p} \tag{32}$$

with differential $(-1)^p d$. This job is called **shifting indices** or **translation**. We call $C[p]$ the **p-th translate of** C. Notice that

$$H_n(C[p]) = H_{n+p}(C), \quad H^n(C[p]) = H^{n-p}(C) \tag{33}$$

for chain and cochain complex respectively.

For a (co)chain map $f : C \to D$, we define $f[p] : C[p] \to D[p]$ as

$$f[p]_n = f_{n+p}, \quad f[p]^n = f^{n-p} \tag{34}$$

for chain and cochain map respectively. This makes translation a functor.

Exercise 38. If C is a complex, show that there are exact sequences of complexes:

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0 \tag{35}$$

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0 \tag{36}$$

Solution. We can expand the first sequence as

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r, hook] \arrow[d, "d_{n+1}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "d_{n+1}"] & B_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, hook] \arrow[d, "d_n"] & C_n \arrow[r, "d_n"] \arrow[d, "d_n"] & B_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, hook] \arrow[d, "d_{n-1}"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "d_{n-1}"] & B_{n-2} \arrow[r] \arrow[d, "d_{n-2}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{37}$$

which commutes, and all the rows are exact by first isomorphism theorem, thus the sequence is exact. Similarly, we can expand the second sequence as the sequence of

$$0 \to H_n \overset{i_n}{\hookrightarrow} C_n/B_n \xrightarrow{d_n} Z_{n-1} \xrightarrow{q_{n-1}} H_{n-1} \to 0 \tag{38}$$

which is exact since $\operatorname{Im} i_n = H_n = Z_n/B_n = \operatorname{Ker} d_n$ and $\operatorname{Im} d_n = B_{n-1} = \operatorname{Ker} q_{n-1}$. ■

2019- 07- 28

Exercise 39 (Mapping cone)**.** Let $f : B \to C$ be a morphism of chain complexes. Form a double chain complex D out of f by thinking of f as a chain complex in Ch and using the sign trick, putting $B[-1]$ in the row $q = 1$ and C in the row $q = 0$. Thinking of C and $B[-1]$ as double complexes in the obvious way, show that there is a short exact sequence of double complexes

$$0 \to C \to D \xrightarrow{\delta} B[-1] \to 0 \tag{39}$$

Solution. We can take D as the following double chain complex.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "-d_B"] & B_{n-1} \arrow[r, "-d_B"] \arrow[d, "f"] & B_n \arrow[r, "-d_B"] \arrow[d, "f"] & B_{n+1} \arrow[r, "-d_B"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r, "d_C"] & C_{n-1} \arrow[r, "d_C"] \arrow[d] & C_n \arrow[r, "d_C"] \arrow[d] & C_{n+1} \arrow[r, "d_C"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{40}$$

The image of $C \to D$ is C in D, which is also the kernel of δ, thus the sequence is exact. ■

Theorem 40. *Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ be a short exact sequence of chain complexes. Then there are natural maps $\partial : H_n(C) \to H_{n-1}(A)$, called **connecting homomorphisms**, such that*

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} \cdots \tag{41}$$

is exact. Similarly, if $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ is a short exact sequence of chain complexes, then there are natural maps $\partial : H^n(C) \to H^{n+1}(A)$ such that

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} \cdots \tag{42}$$

is exact.

Proof. We will come back to the proof of this theorem after we show some small but important lemmas. □

Exercise 41. Let $0 \to A \to B \to C \to 0$ be a short exact sequences of complexes. Show that if two of the three complexes A, B, C are exact, then so is the third.

Solution. Let B, C are exact. From the previous theorem, we have a long exact sequence $0 \to 0 \to H^n(A) \to 0 \to \cdots$. Thus $H^n(A) = 0$ for all n, and so A is exact. The proof is same for the B, C case. ■

Exercise 42. Suppose given a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{43}$$

in an abelian category, such that every column is exact. Show the following:

1. If the bottom two rows are exact, so is the top row.
2. If the top two rows are exact, so is the bottom row.
3. If the top and bottom rows are exact, and the composite $A \to C$ is zero, the middle row is also exact.

Solution. From the previous exercise, what we only need to prove is that the rows above diagram is actually chain complexes.

1. Since the above rectangle commutes and $A \to C$ is zero, $A' \to C' \to C$ is zero. Since $C' \to C$ is monic, $A' \to C'$ is zero.
2. Since the below rectangle commutes and $A \to C$ is zero, $A \to A'' \to C''$ is zero. Since $A \to A''$ is epic, $A'' \to C''$ is zero.
3. The additional condition itself shows the middle row is a chain complex.

■

Lemma 43 (Snake lemma)**.** *Consider a commutative diagram of R-modules of the form*

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C &
\end{tikzcd} \tag{44}$$

If the rows are exact, there is an exact sequence

$$\operatorname{Ker}(f) \to \operatorname{Ker}(g) \to \operatorname{Ker}(h) \xrightarrow{\partial} \operatorname{coKer}(f) \to \operatorname{coKer}(g) \to \operatorname{coKer}(h) \tag{45}$$

where ∂ is defined by the formula $\partial(c') = i^{-1} \circ g \circ p^{-1}(c')$ when $c' \in \operatorname{Ker}(h)$. Moreover, if $A' \to B'$ is monic, then so is $\operatorname{Ker}(f) \to \operatorname{Ker}(g)$, and if $B \to C$ is onto, then so is $\operatorname{coKer}(g) \to \operatorname{coKer}(h)$.

There are bunch of notes for this lemma. First, the term 'snake' comes from the shape the line of exact sequence when we add the kernels above the diagram and cokernels below the diagram. Second, this lemma holds in an arbitrary abelian category. This is the corollary of the Freyd-Mitchell embedding theorem which gives an exact fully faithful embedding of small abelian category into $R-$ mod for some ring

Proof. First we need to show that ∂ is well defined. Since p is surjective, $p^{-1}(c')$ is a nonempty set of elements in B'. Since $h(c') = 0$, $g(p^{-1}(c'))$ is in the kernel of $B \to C$, thus in the image of $A \to B$. Since i is injective, we can take $i^{-1}(g(p^{-1}(c')))$. Now picking $b, b' \in B'$ such that $p(b) = p(b') = c$, we have $p(b - b') = 0$, thus there is $a \in A'$ such that $A' \to B'$ maps a to $b - b'$. Due to the commutativity, $f(a) = i^{-1} \circ g(b - b')$, which is zero in $\mathrm{coKer}(A)$. Thus ∂ is well defined.

Now we need to show that the sequence is exact on $\mathrm{Ker}(h)$ and $\mathrm{coKer}(f)$, since others are trivial by the exactness of rows. Notice that $c \in \mathrm{Ker}(\partial)$, then $i^{-1}(g(p^{-1}(c))) = 0$ implies $g(p^{-1}(c)) = 0$. Choose $b \in p^{-1}(c)$, then $g(b) = 0$ thus $b \in \ker(g)$, and $p(b) = c$. Therefore $c \in \mathrm{Im}(\mathrm{Ker}(g) \to \mathrm{Ker}(h))$. Conversely choose $c \in \mathrm{Im}(\mathrm{Ker}(g) \to \mathrm{Ker}(h))$, and take $b \in \mathrm{Ker}(g)$ such that $p(b) = c$. Then $i^{-1}(g(p^{-1}(c))) = i^{-1}(g(b)) = i^{-1}(0) = 0$. Now take $a \in \mathrm{Im}(\partial)$, and take $c \in \mathrm{Ker}(h)$ such that $\partial(c) = a$. Then $i^{-1}(g(p^{-1}(c))) = a$ implies $g(p^{-1}(c)) = i(a) = 0$ in $\mathrm{coKer}(g)$. Finally, take $a \in \mathrm{Ker}(\mathrm{coKer}(f) \to \mathrm{coKer}(g))$, then there is $b \in B'$ such that $i(a) = g(b)$, thus $a = i^{-1}(g(b)) = i^{-1}(g(p^{-1}(p(b))))$, thus $a \in \mathrm{Im}(\partial)$.

Suppose that $A' \to B'$ is monic, that is, if $a \mapsto 0$ then $a = 0$. Thus for all $a \in \mathrm{Ker}(f)$, $a \mapsto 0$ implies $a = 0 \in \mathrm{Ker}(f)$, thus $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$ is monic.

Suppose that $B \to C$ is epic, that is, for all $c \in C$ there is $b \in B$ which maps to c. Now choose $[c'] \in \mathrm{coKer}(h)$. Taking the representation $c' \in C$ of $[c']$, we have $b' \in B$ which maps to c'. Then $[b']$ maps to $[c']$. □

Exercise 44 (5-lemma)**.** In any commutative diagram

$$\begin{tikzcd} A' \arrow[r, "g'"] \arrow[d, "f_a"] & B' \arrow[r, "h'"] \arrow[d, "f_b"] & C' \arrow[r, "i'"] \arrow[d, "f_c"] & D' \arrow[r, "j'"] \arrow[d, "f_d"] & E' \arrow[d, "f_e"] \\ A \arrow[r, "g"] & B \arrow[r, "h"] & C \arrow[r, "i"] & D \arrow[r, "j"] & E \end{tikzcd} \tag{46}$$

with exact rows in any abelian category, show that if f_a, f_b, f_d, and f_e are isomorphisms, then f_c is also an isomorphism. More precisely, show that if f_b and f_d are monic and f_a is epic, then f_c is monic. Dually, show that if f_b and f_d are epic and f_e is monic, then f_c is epic.

Solution. By the Freyd-Mitchell embedding theorem, it is enough to show this theorem in $R-\mathrm{mod}$ category. By duality, we only need to prove the second statement. Take $c \in C'$ such that $f_c(c) = 0$. Then by commutativity, $f_d(i'(c)) = 0$. Since f_d is monic, $i'(c) = 0$, thus $c \in \mathrm{Ker}(i')$. Since the rows are exact, $c \in \mathrm{Im}(h')$, that is, we have $b \in B'$ such that $h'(b) = c$. By commutativity again, $h(f_b(b)) = 0$, thus $f_b(b) \in \mathrm{Ker}(h)$. Again since the rows are exact, $f_b(b) \in \mathrm{Im}(g)$, that is, we have $a' \in A$ such that $g(a') = f_b(b)$. Since f_a is surjective, there is $a \in A'$ such that $g(f_a(a)) = f_b(g'(a)) = f_b(b)$, and since f_b is monic, $g'(a) = b$. Since $c = h'(b) = h'(g'(a)) = 0$, we get f_c is monic. ■

Proof of long exact sequence with connecting homomorphisms. From the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd} \tag{47}$$

The snake lemma implies the following rows are exact.

$$\begin{tikzcd}
 & A_n/d(A_{n+1}) \arrow[r] \arrow[d, "d"] & B_n/d(B_{n+1}) \arrow[r] \arrow[d, "d"] & C_n/d(C_{n+1}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) &
\end{tikzcd} \tag{48}$$

Notice that the kernel of $d : A_n/d(A_{n+1}) \to Z_{n-1}(A)$ is $Z_n/B_n = H_n(A)$, and the cokernel is $Z_{n-1}/B_{n-1} = H_{n-1}(A)$. Therefore snake lemma implies the sequence

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} \cdots \tag{49}$$

is exact. □

Proposition 45. *The construction of long exact sequence from short exact sequence defined as above is a functor from the category with short exact sequences to long exact sequence. That is, for every short exact sequence there is a long exact sequence, and for every map of short exact sequences there is a corresponding map of long exact sequences.*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd} \tag{50}$$

Proof. To prove this, we only need to show that the diagram above commutes. Since H_n is a functor, the left two squares commute. Due to the Freyd-Mitchell embedding theorem, we only need to work on $R - \mathsf{mod}$ category. Take $z \in H_n(C)$ which is represented by $c \in C_n$. Then the image of z, $z' \in H_n(C')$, is represented by the image of c. Also if $b \in B_n$ maps to c, then its image $b' \in'_n$ maps to c'. Now we observe that the element $d(b) \in B_{n-1}$ belongs to the submodule $Z_{n-1}(A)$ and represents $\partial(z) \in H_{n-1}(A)$, which can be found in the construction of ∂. Thus $\partial(z')$ is represented by the image of $d(b)$, which is the image of a representative of $\partial(z)$, thus $\partial(z')$ is the image of $\partial(z)$. □

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Exercise 46. Consider the boundaries-cycles exact sequence $0 \to Z \to C \to B[-1] \to 0$ associated to a chain complex C. Show that the corresponding long exact sequence of homology breaks up into short exact sequences.

Solution. Notice that $d(B) = 0$ since $d \circ d = 0$. Therefore we get the long exact sequence

$$\cdots \to 0 \to H_n(Z) \to H_n(C) \to 0 \to H_{n-1}(Z) \to \cdots \tag{51}$$

This shows that $H_n(Z) \simeq H_n(C)$. Indeed, since $\mathrm{Im}(d) = B_n$ and $\mathrm{Ker}(d) = Z_n$ in Z, $H_n(Z) = Z_n/B_n = H_n(C)$. ■

Exercise 47. Let f be a morphism of chain complexes. Show that if $\mathrm{Ker}(f)$ and $\mathrm{coKer}(f)$ are acyclic, then f is a quasi-isomorphism. Is the converse true?

Solution. Take $f : B \to C$. Notice that the sequences

$$0 \to \mathrm{Ker}(f) \to B \to \mathrm{Im}(f) \to 0 \tag{52}$$

and

$$0 \to \mathrm{Im}(f) \to C \to \mathrm{coKer}(f) \to 0 \tag{53}$$

are exact. Since $\mathrm{Ker}(f)$ and $\mathrm{coKer}(f)$ are acyclic, the long exact sequence shows that

$$H_n(B) \simeq H_n(\mathrm{Im}(f)) \simeq H_n(C) \tag{54}$$

and thus f is a quasi-isomorphism.

Conversely, consider the following morphism.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "0"] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{55}$$

Both sequences are exact, and hence f is a quasi-isomorphism. But both kernel $0 \to \mathbb{Z} \to 0 \to 0 \to 0$ and cokernel $0 \to 0 \to 0 \to \mathbb{Z} \to 0$ are not acyclic. ■

Exercise 48. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of double complexes of modules. Show that there is a short exact sequence of total complexes, and conclude that if $\mathrm{Tot}(C)$ is acyclic, then $\mathrm{Tot}(A) \to \mathrm{Tot}(B)$ is a quasi-isomorphism.

Solution. The last statement can be proven by using the long exact sequence. Now to make the short exact sequence

$$0 \to \mathrm{Tot}(A) \to \mathrm{Tot}(B) \to \mathrm{Tot}(C) \to 0 \tag{56}$$

define the maps as

$$\prod_{p+q=n} A_{p,q} \to \prod_{p'+q'=n} B_{p',q'}, \quad (\cdots, a_{p,q}, \cdots) \mapsto (\cdots, f_{p,q}(a_{p,q}), \cdots) \tag{57}$$

This is a chain map since f, g are map between double complexes, and short exact because f, g gives short exact sequence. ■

Definition 49. A complex C is called **split** if there are maps $s_n : C_n \to C_{n+1}$ such that $d = d \circ s \circ d$. The maps s_n are called the **splitting maps**. If in addition C an exact sequence, then we say C is **split exact**.

Example 50. Let $R = \mathbb{Z}$ or $\mathbb{Z}/4$, and let C be a complex

$$\cdots \xrightarrow{\times 2} R \xrightarrow{\times 2} R \xrightarrow{\times 2} \cdots \tag{58}$$

This complex is exact but not split exact.

Exercise 51.

1. Show that acyclic bounded below chain complexes of free R-modules are always split exact.

2. Show that an acyclic chain complex of finitely generated free abelian groups is always split exact, even when it is not bounded below.

Solution.

1. First we want to show that if C is a free module, then every exact sequence
$$0 \to A \to B \to C \to 0 \tag{59}$$
has a split $s : C \to B$. Since C is free, there is a basis E of C, and since $B \to C$ is surjective, for every $e_\alpha \in E \subset C$ there is $b_\alpha \in B$ such that $b_\alpha \mapsto e_\alpha$. Now define $s : C \to B$ as $s(e_\alpha) = b_\alpha$. Now consider $d \circ s \circ d(b)$ for some $b \in B$. Since $d(b) \in C$, we may write $d(b) = \sum_i r_i e_i$. Now $d \circ s(d(b)) = d(\sum_i r_i s(e_i)) = \sum_i r_i d(b_\alpha) = \sum_i r_i d(b_\alpha)$. Now denote the chain as
$$\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \tag{60}$$

Indeed this sequence is split exact, since $B = A \oplus C$ thus we may take a map $B \to A$ by taking A to A with identity, C to 0, and define homomorphically for others. To show this, take $b \in B$ which maps to $c \in C$. Now take $b - s(c)$. Since $d(b - s(c)) = d(b) - d(s(c)) = c - c = 0$, $b - s(c) \in A$, thus $B = A + C$. Furthermore, suppose that $b \in B$ is in both A, C. Then b maps to 0, but since $s(b) = 0$, $b = 0$. This statement is related to the fact that the free modules are projective.

Now we have a following exact sequence which is split exact.

$$0 \to \mathrm{Ker}(d_1) \hookrightarrow C_1 \xrightarrow{d_1} C_0 \to 0 \tag{61}$$

Thus we may choose $s_0 : C_0 \to C_1$ such that $d_1 \circ s_0 \circ d_1 = d_1$. Also the following chain is exact.

$$\cdots \xrightarrow{d_3} C_2 \xrightarrow{d_2} \mathrm{Im}(d_2) \xrightarrow{0} \tag{62}$$

Now use induction steps to achieve s_n.

2. Consider the map $f : A \to B$ where A, B are finitely generated free abelian groups. Since the subgroup of free group is free, we may choose the finite generators of $\mathrm{Im}(A)$, and we may choose the orthogonal subgroup B' of B. Now for each generators $b \in \mathrm{Im}(A)$ there is $a \in A$ such that $f(a) = b$. Define $s : B \to A$ as $s(b) = a$ if b is a generator of $\mathrm{Im}(A)$, and $s(b) = 0$ if b is a generator of B'. Then we get $d \circ s \circ d = d$.

■

Lemma 52 (Splitting lemma)**.** *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \tag{63}$$

be a short exact sequence in $R - \mathsf{mod}$ category. Then the followings are equivalent.

1. *The sequence $0 \to A \to B$ splits.*

2. *The sequence $B \to C \to 0$ splits.*

3. *$A \oplus C \simeq B$.*

Proof. ($3 \Rightarrow 1$) Since $A \oplus C \simeq B$, we may identify A with $i(A)$. Define $s : B \to A$ as the projection operator. Then $i \circ s \circ i(a) = i(a)$ for all $a \in A$.

($3 \Rightarrow 2$) Since $A \oplus C \simeq B$, we may define $s : C \to B$ as the inclusion by identifying C with $s(C)$. Suppose that $b = a + c$ for $b \in B, a \in A, c \in C$. Then $j \circ s \circ j(b) = j \circ s(c) = j(c) = j(b)$ for all $b \in B$.

($1 \Rightarrow 3$) First, since i is injective, $i \circ s \circ i = i$ implies $s \circ i = 1_A$. Consider $s : B \to A$ such that $i \circ s \circ s = i$. Choose $b \in B$. Now notice that $b = (b - i \circ s(b)) + i \circ s(b)$. Notice that $i \circ s(b) \in \mathrm{Im}(i)$, and $s(b - i \circ s(b)) = s(b) - s \circ i \circ s(b) = s(b) - s(b) = 0$ thus $b - i \circ s(b) \in \mathrm{Ker}(s)$. Now, suppose that $b \in \mathrm{Im}(i) \cap \mathrm{Ker}(s)$. Then $i(a) = b$ for some $a \in A$ and $s(b) = 0$, thus $s \circ i(a) = 0$. Since $s \circ i = 1_A$, $a = 0$, $b = 0$. Hence $B = \mathrm{Im}(i) \oplus \mathrm{Ker}(s)$. Now since i is injective, $\mathrm{Im}(i) \simeq A$. Finally, consider $j : \mathrm{Ker}(s) \to C$ be the restricted map of j. For any $c \in C$ we have $b \in B$ such that $j(b) = c$, and then $j(b - i(s(b)) = c$. Thus j is injective. If $j(b) = 0$, then $j \in \mathrm{Im}(i)$, and since $\mathrm{Im}(i) \cap \mathrm{Ker}(s) = 0$, $b = 0$. Thus $j : \mathrm{Ker}(s) \to C$ is an isomorphism, and $\mathrm{Ker}(s) \simeq C$.

($2 \Rightarrow 3$) First, since j is surjective, $j \circ s \circ j = j$ implies $j \circ s = 1_C$. Choose $b \in B$. Now notice that $b = (b - s \circ j(b)) + s \circ j(b)$. Notice that $s \circ j(b) \in \mathrm{Im}(s)$, and $j(b - s \circ j(b)) = j(b) - j \circ s \circ j(b) = j(b) - j(b) = 0$ thus $b - s \circ j(b) \in \mathrm{Ker}(j)$. Now, suppose that $b \in \mathrm{Im}(s) \cap \mathrm{Ker}(j)$. Then $s(c) = b$ for some $c \in C$ and $j(b) = 0$, thus $j \circ s(c) = 0$. Since $j \circ s = 1_C$, $c = 0$. Hence $B = \mathrm{Im}(s) \oplus \mathrm{Ker}(j)$. Now since $\mathrm{Im}(i) \simeq \mathrm{Ker}(j)$ and i is injective, $\mathrm{Ker}(j) \simeq A$. Finally, since $j \circ s$ is a bijection, s is an injection, and thus $\mathrm{Im}(s) \simeq C$. □

Exercise 53. Let C be a chain complex, with boundaries B_n and cycles Z_n in C_n. Show that C is split if and only if there are R-module decomposition $C_n \simeq Z_n \oplus B'_n$ and $Z_n \simeq B_n \oplus H'_n$. Show that C is exact if and only if $H'_n = 0$.

Solution. The first statement shows second statement directly.

Suppose that C is split with splitting map s. Consider the map $d : s \circ d(C_n) \to \mathrm{Im}(d) = B_{n-1}$. If $d(c) = 0$ for $c \in s \circ d(C_n)$ then we have $c' \in C_n$ such that $c = s \circ d(c')$, thus $d \circ s \circ d(c') = d(c') = 0$ so $c = 0$. Hence $\mathrm{Ker}(d) = 0$. Also for all $c \in \mathrm{Im}(d)$, i.e. $c = d(c')$, $d \circ s \circ d(c') = d(c') = c$. Thus this map is isomorphic, and $s \circ d(C_n) \simeq B_{n-1}$. Now consider the following short exact sequence.

$$0 \to Z_n \to C_n \to B_{n-1} \to 0 \tag{64}$$

We may take the right splitting map $B_{n-1} \to C_n$ as the inclusion map $s \circ d(C_n) \hookrightarrow C_n$. This shows that $C_n \simeq Z_n \oplus B_{n-1}$ where $B'n \simeq B_{n-1} \simeq s \circ d(C_n)$.

Now consider $c \in \mathrm{Im}(d_{n+1})$, i.e. $c = d(c')$, then $c = d \circ s \circ d(c')$ thus $c \in d \circ s(C_n)$. Conversely if $c \in d \circ s(C_n)$ then $c \in \mathrm{Im}(d_{n+1})$ obviously, therefore $d \circ s(C_n) = \mathrm{Im}(d_{n+1}) = B_n$. Now consider the following short exact sequence.

$$0 \to B_n \to Z_n \to Z_n/B_n \to 0 \tag{65}$$

We may take the left splitting map $Z_n \to B_n \simeq d \circ s(C_n)$ as the map $d \circ s$. This shows that $Z_n \simeq B_n \oplus Z_n/B_n \simeq B_n \oplus H'_n$.

Finally suppose that $C_n \simeq Z_n \oplus B'_n$ and $Z_n \simeq B_n \oplus H_n$. Define the splitting map $s : C_n \to C_{n+1}$ as $s|_{B_n} = 1_{B_n} : B_n \mapsto B'_{n+1} \simeq B_n$, $s|_{H_n} = 0$, and $s|_{B'_n} = 0$. Notice that $d|_{B_n} = 0, d|_{H_n} = 0$, and $d|_{B'_n} = 1_{B'_n} : B'_n \simeq$

$B_{n-1} \to B_{n-1}$. This shows that $d \circ s \circ d = d$.

$$\begin{tikzcd}
C_{n+1} \arrow[d, "d"] & H_{n+1} \arrow[drr, "0"] & \oplus & 0 & \oplus & B_{n+1} \arrow[dll, "0"] & \oplus & B'_{n+1} \arrow[dll, "\simeq"] \\
C_n \arrow[d, "d"] & H_n \arrow[drr, "0"] & \oplus & 0 & \oplus & B_n \arrow[dll, "0"] & \oplus & B'_n \arrow[dll, "\simeq"] \\
C_{n-1} & H_{n-1} & \oplus & 0 & \oplus & B_{n-1} & \oplus & B'_{n-1}
\end{tikzcd} \tag{66}$$

$$\begin{tikzcd}
C_{n+1} & H_{n+1} & \oplus & 0 & \oplus & B_{n+1} & \oplus & B'_{n+1} \\
C_n \arrow[u, "s"] & H_n \arrow[urr, "0"] & \oplus & 0 & \oplus & B_n \arrow[urr, "\simeq"] & \oplus & B'_n \arrow[ullll, "0"] \\
C_{n-1} \arrow[u, "s"] & H_{n-1} \arrow[urr, "0"] & \oplus & 0 & \oplus & B_{n-1} \arrow[urr, "\simeq"] & \oplus & B'_{n-1} \arrow[ullll, "0"]
\end{tikzcd} \tag{67}$$

■

Proposition 54. *For a two chain complexes C, D and maps $s_n : C_n \to D_{n+1}$. Define $f_n : C_n \to D_n$ defined as $f_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n$.*

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$$\begin{tikzcd}
C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "s"] \arrow[d, "f"] & C_{n-1} \arrow[dl, "s"] \\
D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1}
\end{tikzcd} \tag{68}$$

Then f is a chain map from C to D.

Proof. Direct calculation shows $d \circ f = d \circ (d \circ s + s \circ d) = d \circ s \circ d = (d \circ s + s \circ d) \circ d = f \circ d$. □

Definition 55. A chain map $f : C \to D$ is **null homotopic** if there are maps $s_n : C_n \to D_{n+1}$ such that $f = d \circ s + s \circ d$. The maps $\{s_n\}$ are called a **chain construction** of f.

Exercise 56. Show that C is a split exact chain complex if and only if the identity map on C is null homotopic.

Solution. Suppose that C is a split exact chain complex. Choose $s_n : C_n \to C_{n+1}$ as the split maps. From previous exercise, we may decompose C_n into $Z_n \oplus B'_n \simeq B_n \oplus B'_n$. The structure of d and s map can be drawn as below.

$$\begin{tikzcd}
C_{n+1} \arrow[d, "d"] & 0 & \oplus & B_{n+1} \arrow[dll, "0"'] & \oplus & B'_{n+1} \arrow[dll, "\simeq"'] \\
C_n \arrow[d, "d"] & 0 & \oplus & B_n \arrow[dll, "0"'] & \oplus & B'_n \arrow[dll, "\simeq"'] \\
C_{n-1} & 0 & \oplus & B_{n-1} & \oplus & B'_{n-1}
\end{tikzcd} \tag{69}$$

$$\begin{tikzcd}
C_{n+1} & 0 & \oplus & B_{n+1} & \oplus & B'_{n+1} \\
C_n \arrow[u, "s"] & 0 & \oplus & B_n \arrow[urr, "\simeq"] & \oplus & B'_n \arrow[ullll, "0"] \\
C_{n-1} \arrow[u, "s"] & 0 & \oplus & B_{n-1} \arrow[urr, "\simeq"] & \oplus & B'_{n-1} \arrow[ullll, "0"]
\end{tikzcd} \tag{70}$$

Thus $d \circ s$ is the projection on B_n and $s \circ d$ is the projection on B'_n. This shows that $d \circ s + s \circ d$ is identity.

Now suppose that the identity map on C is null homotopic, that is, we have the maps $s_n : C_n \to C_{n+1}$ such that $d \circ s + s \circ d = 1$. Then $d = d \circ s \circ d + s \circ d \circ d = d \circ s \circ d$, thus s_n are splitting maps. ■

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Definition 57. Two chain maps $f, g : C \to D$ are **chain homotopic** if $f - g$ is null homotopic, that is, if there are maps $s_n : C_n \to D_{n+1}$ such that $f - g = d \circ s + s \circ d$. The maps s_n are called a **chain homotopy** from f to g.

Definition 58. For a chain map $f : C \to D$, f is called a **chain homotopy equivalence** if there is a chain map $g : D \to C$ such that $g \circ f$ is chain homotopic to 1_C and $f \circ g$ is chain homotopic to 1_D.

The term homotopy comes from the following fact. For a map $f : X \to Y$ between topological spaces, there is a induced chain map $f_* : S(X) \to S(Y)$ between the corresponding singular chain complexes. If f is topologically null homotopic, then f_* is null homotopic; if f is a homotopy equivalence, then f_* is a chain homotopy equivalence; if f and another map $g : X \to Y$ are topologically homotopic, then f_* and g_* are chain homotopic.

Lemma 59. *If a chain map $f : C \to D$ is null homotopic, then every map $f_* : H_n(C) \to H_n(D)$ is zero. If f and g are chain homotopic, then they induce the same maps $f_* = g_* : H_n(C) \to H_n(D)$.*

Proof. The first statement shows the second statement directly.

Suppose that $f = s \circ d + d \circ s$ for some $s : C_n \to D_{n+1}$. Consider an n-cycle $c \in C_n$. Then $f(c) = s \circ d(c) + d \circ s(c) = d \circ s(c)$, thus $f(c)$ is the boundary in D. Hence $f_*(c) = 0$. □

Exercise 60. Consider the homology $H_n(C)$ of C as a chain complex with zero differentials. Show that if the complex C is split, then there is a chain homotopy equivalence between $C_\bullet$ and $H_\bullet(C)$. Conversely, if $C_\bullet$ and $H_\bullet(C)$ are chain homotopy equivalent, show that C is split.

Solution. Suppose that C is split. Then by previous exercise, we can take the chain map $f : C_\bullet \to H_\bullet(C)$ as the projection map and $g : H_\bullet(C) \to C_\bullet$ as the inclusion map: more precisely, $f = 1_C - s \circ d - d \circ s$ for splitting map s. Then $f \circ g$ is identity map itself. Also, $f \circ g - 1_C = s \circ d + d \circ s$, thus f and g are chain homotopy equivalences.

Now suppose that $C_\bullet$ and $H_\bullet(C)$ are chain homotopy equivalent, that is, take $f : C_\bullet \to H_\bullet(C)$ and $g : H_\bullet(C) \to C_\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to identities. since g is a chain map, $d \circ g = 0$. Also we may take maps $s_n : C_n \to C_{n+1}$ such that $1 - g \circ f = s \circ d + d \circ s$. Then,

$$d \circ s \circ d = d(s \circ d + d \circ s) = d(1 - g \circ f) = d - d \circ g \circ f = d \tag{71}$$

therefore s is a splitting map. ■

Exercise 61. In this exercise we shall show that the chain homotopy classes of maps form a quotient category K of the category Ch of all chain complexes. The homology functors H_n on Ch will factor through the quotient functor $\mathsf{Ch} \to \mathsf{K}$.

1. Show that chain homotopy equivalences is an equivalence relation on the set of all chain maps from C to D. Let $\mathrm{Hom}_{\mathsf{K}}(C, D)$ denote the equivalence classes of such maps. Show that $\mathrm{Hom}_{\mathsf{K}}(C, D)$ is an abelian group.

2. Let f and g be chain homotopic maps from C to D. If $u : B \to C$ and $v : D \to E$ are chain maps, show that $v \circ f \circ u$ and $v \circ g \circ u$ are chain homotopic. Deduce that there is a category K whose objects are chain complexes and whose morphisms are given in 1.

3. Let f_0, f_1, f_0 and g_1 be chain maps from C to D such that f_i is chain homotopic to g_i for $i = 1, 2$. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category, and that $\mathsf{Ch} \to \mathsf{K}$ is an additive functor.

4. Is K an abelian category? Explain.

Solution. 1. For two chain maps $f, g : C \to D$, we say $f \sim g$ if $f - g = d \circ s + s \circ d$ for some maps $s_n : C_n \to D_{n+1}$. To show that $\sim$ is an equivalent relation, notice that $f \sim f$ by zero maps $0_n : C_n \to D_{n+1}$, and $f \sim g$ by s_n implies $g \sim f$ by $-s_n$. Finally, suppose that $f \sim g$ by s_n and $g \sim h$ by t_n. Then $f \sim h$ by $s_n + t_n$, since $f - h = (f - g) + (g - h) = (d \circ s + s \circ d) + (d \circ t + t \circ d) = d \circ (s + t) + (s + t) \circ d$.

Now for the equivalence classes $[f], [g] \in \mathrm{Hom}_{\mathsf{K}}(C, D)$, define $[f] + [g] = [f + g]$. This definition is well defined, since if $f \sim f'$ by s_n and $g \sim g'$ by t_n then $f + g \sim f' + g'$ by $s_n + t_n$. Now we can see that this addition is associative, and the zero map $[0]$ is identity and $[-f] = -[f]$. This gives the result.

2. Suppose that f and g are chain homotopic with s_n. Then $f - g = d \circ s + s \circ d$. Now since $v \circ f \circ u - v \circ g \circ u = v \circ (f - g) \circ u = v \circ (d \circ s + s \circ d) \circ u = d \circ (v \circ s \circ u) + (v \circ s \circ u) \circ d$ since u, v are chain maps, $v \circ f \circ u$ and $v \circ g \circ u$ are chain homotopic by $v \circ s \circ u$. Now define $[g] \circ [f] = [g \circ f]$. This definition is well defined, since if $f \sim f' : C \to D$ by s_n and $g \sim g' : D \to E$ by t, then $g' \circ f' \sim g \circ f$ since $g' \circ f' - g \circ f = (g' \circ f' - g' \circ f) + (g' \circ f - g \circ f) = g' \circ (d \circ s + s \circ d) + (d \circ t + t \circ d) \circ f = d \circ (g' \circ s + t \circ f) + (g' \circ s + t \circ f) \circ d$. Therefore the identity map 1 gives $[1] \circ [f] = [f]$ and $([h] \circ [g]) \circ [f] = [h \circ g] \circ [f] = [(h \circ g) \circ f] = [h \circ (g \circ f)] = [h] \circ ([g \circ f])$. Thus K is a category.

3. This is shown in 1., and this shows K is an Ab-category. Since the objects of K and Ch are same, K is an additive category: the zero objects are zero object, and the product is contained. Also, $[f + g] = [f] + [g]$ implies that $[\bullet] : \mathsf{Ch} \to \mathsf{K}$ is an additive functor.

4. No. Consider the chain map f between two chain complexes $\cdots \to 0 \to \mathbb{Z}/4 \to 0 \to \cdots$ and $\cdots \to 0 \to \mathbb{Z}/2 \to 0 \to \cdots$ defined by natural map $\mathbb{Z}/4 \to \mathbb{Z}/2$. TBD

■

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Definition 62. Let $f : B \to C$ be a chain map. The **mapping cone** of f is a chain complex $\mathrm{cone}(f)$ whose degree n part is $B_{n-1} \oplus C_n$ and the differential is $d(b, c) = (-d(b), d(c) - f(b))$.

Exercise 63. Let $\mathrm{cone}(C)$ denote the mapping cone of the identity map 1_C of C. Show that $\mathrm{cone}(C)$ is split exact, with $s(b, c) = (-c, 0)$ defining the splitting map.

Solution. Notice that $d \circ s \circ d(b, c) = d \circ s(-d(b), d(c) - b) = d(-d(c) + b, 0) = (-d(-d(c) + b), d(c) - b) = (-d(b), d(c) - b) = d(b, c)$. ■

Exercise 64. Let $f : C \to D$ be a chain map. Show that f is null homotopic if and only if f extends to a map $(-s, f) : \mathrm{cone}(C) \to D$.

Solution. Suppose that f is null homotopic, that is, $f = s \circ d + d \circ s$ for some $s_n : C_n \to D_{n+1}$. Then $d \circ (-s, f)(b, c) = d(f(c) - s(b)) = d \circ s \circ d(c) + d \circ d \circ s(c) - d \circ s(b) = d \circ s \circ d(c) - d \circ s(b)$ and $(-s, f) \circ d(b, c) = (-s, f)(-d(b), d(c) - b) = s \circ d(b) + f \circ d(c) - f(b) = -d \circ s(b) + d \circ s \circ d(c)$, thus $(-s, f)$ is a chain map. Conversely, if $(-s, f)$ is a chain map for some s, then we have $d \circ f(c) - d \circ s(b) = s \circ d(b) + f \circ d(c) - f(b)$, which implies $s \circ d + d \circ s = f$, thus f is null homotopic. ∎

Lemma 65. *For a chain map $f : B \to C$, the short exact sequence*

$$0 \to C \to cone(f) \xrightarrow{\delta} B[-1] \to 0 \tag{72}$$

gives the homology long exact sequence

$$\cdots \to H_{n+1}(cone(f)) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(cone(f)) \xrightarrow{\delta_*} \cdots \tag{73}$$

Then $\partial = f_*$.

Proof. For a cycle $b \in B_n$, The element $(-b, 0)$ of cone(f) lifts f via δ, and taking differential gives $d(-b, 0) = (d(b), f(b)) = (0, f(b))$. Thus $\partial[b] = [f(b)]$, which is $f_*([b])$. □

Corollary 66. *A chain map $f : B \to C$ is a quasi-isomorphism if and only if the mapping cone complex **cone**(f) is exact.*

Proof. If f is a quasi-isomorphism then ∂ is isomorphism, thus $H_n(C) \to H_n(\text{cone}(f))$ and δ_* are zero maps. Thus $H_n(\text{cone}(f)) = 0$ and so cone(f) is exact. Conversely, if cone(f) is exact then $H_n(\text{cone}(f)) = 0$, thus $H_n(B) \xrightarrow{\partial = f_*} H_n(C)$ is an isomorphism. □

Definition 67. Let K be a simplicial complex. The **topological cone** CK of K is obtained by adding a new vertex s to K and making the cone of the simplicies to get a new $n+1$ simplex for every old n-simplex of K. Notice that the simplicial chain complex $C_\bullet(s)$ of the one point space $\{s\}$ is R in degree 0 and zero elsewhere. Then $C_\bullet(s)$ is a subcomplex of the simplicial chain complex $C_\bullet(CK)$ of the topological cone CK, and the quotient $C_\bullet(CK)/C_\bullet(s)$ is the chain complex cone$(C_\bullet K)$ of the identity map of $C_\bullet(K)$. The fact that cone$(C_\bullet K)$ is null homotopic reflects the fact that the topological cone CK is contractible.

Samely, if $f : K \to L$ is a simplicial map, the **topological mapping cone** Cf of f is obtained by glueing CK and L together, identifying the subcomplex K of CK with its image in L. If f is an inclusion of simplicial complexes, Cf is a simplicial complex. The quotient chain complex $C_\bullet(Cf)/C_\bullet(s)$ is the mapping cone cone(f_*) of the chain map $f_* : C_\bullet(K) \to C_\bullet(L)$.

Definition 68. For a chain complex map $f : B \to C$, the **mapping cylinder** of f is a chain complex $\mathrm{cyl}(f)$ whose degree n part is $B_n \oplus B_{n-1} \oplus C_n$ and the differential is $d(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b'))$.

Exercise 69. Let $\mathrm{cyl}(C)$ denote the mapping cylinder of the identity map 1_C of C. Show that two chain maps $f, g : C \to D$ are chain homotopic if and only if they extend to a map $(f, s, g) : \mathrm{cyl}(C) \to D$.

Solution. Suppose that f and g are chain homotopic, that is, $f - g = s \circ d + d \circ s$. Then $(f, s, g) \circ d(a, b, c) = (f, s, g)(d(a) + b, -d(b), d(c) - b) = f \circ d(a) + f(b) - s \circ d(b) + g \circ d(c) - g(b) = d \circ f(a) + d \circ s(b) + d \circ g(c) = d \circ (f, s, g)(a, b, c)$. Conversely, if $(f, s, g) \circ d(a, b, c) = d \circ (f, s, g)(a, b, c)$ then $f \circ d(a) + f(b) - s \circ d(b) + g \circ d(c) - g(b) = d \circ f(a) + d \circ s(b) + d \circ g(c)$ implies $(f - g)(b) = (s \circ d + d \circ s)(b)$, thus $f - g = s \circ d + d \circ s$. ■

Exercise 70. If $f : B \to C$, $g : C \to D$, and $e : B \to D$ are chain maps, show that e and $g \circ f$ are chain homotopic if and only if there is a chain map $\gamma = (e, s, g) : \mathrm{cyl}(f) \to D$. Note that e and g factor through γ.

Solution. Suppose that e and $g \circ f$ are chain homotopic, that is, $e - g \circ f = s \circ d + d \circ s$. Then $(e, s, g) \circ d(a, b, c) = (e, s, g)(d(a) + b, -d(b), d(c) - f(b)) = e \circ d(a) + e(b) - s \circ d(b) + g \circ d(c) - g \circ f(b) = d \circ e(a) + d \circ s(b) + d \circ g(c) = d \circ (e, s, g)(a, b, c)$. Conversely, if $(e, s, g) \circ d(a, b, c) = d \circ (e, s, g)(a, b, c)$ then $e \circ d(a) + e(b) - s \circ d(b) + g \circ d(c) - g \circ f(b) = d \circ e(a) + d \circ s(b) + d \circ g(c)$ implies $(e - g \circ f)(b) = (s \circ d + d \circ s)(b)$, thus $e - g \circ f = s \circ d + d \circ s$. ■

Lemma 71. *The subcomplex of elements $(0, 0, c)$ is isomorphic to C, and the corresponding inclusion $\alpha : C \to \mathrm{cyl}(f)$ is a quasi-isomorphism.*

Proof. Notice that $\mathrm{cyl}(f)/\alpha(C) = \mathrm{cone}(-1_B)$, which is split exact. Now from the short exact sequence

$$0 \to C \xrightarrow{\alpha} \mathrm{cyl}(f) \to \mathrm{cone}(-1_B) \to 0 \tag{74}$$

we have a long exact sequence

$$\cdots \to H_{n+1}(\mathrm{cyl}(f)) \to H_{n+1}(\mathrm{cone}(-1_B)) \to H_n(C) \to H_n(\mathrm{cyl}(f)) \to \cdots \tag{75}$$

Now since $\mathrm{cone}(-1_B)$ is exact, $H_n(C) \to H_n(\mathrm{cyl}(f))$ is an isomorphism. □

Exercise 72. Show that $\beta(b, b', c) = f(b) + c$ defines a chain map from $\mathrm{cyl}(f)$ to C such that $\beta \circ \alpha = 1_C$. Then show that the formula $s(b, b', c) = (0, b, 0)$ defines a chain homotopy from the identity of $\mathrm{cyl}(f)$ to $\alpha \circ \beta$. Conclude that α is in fact a chain homotopy equivalence between C and $\mathrm{cyl}(f)$.

Solution. First $d \circ \beta(b, b', c) = d(f(b) + c) = d \circ f(b) + d(c)$ and $\beta \circ d(b, b', c) = \beta(d(b) + b', -d(b'), d(c) - f(b')) = f \circ d(b) + f(b') + d(c) - f(b') = f \circ d(b) + d(c)$, thus β is a chain map. Now $\beta \circ \alpha(c) = \beta(0, 0, c) = c$ thus $\beta \circ \alpha = 1_C$. Finally, $(s \circ d + d \circ s)(b, b', c) = s(d(b) + b', -d(b'), d(c) - f(b')) + d(0, b, 0) = (0, d(b) + b', 0) + (b, -d(b), -f(b)) = (b, b', -f(b)) = (b, b', c) - (0, 0, f(b) + c)$, and since $\alpha \circ \beta(b, b', c) = \alpha(f(b) + c) = (0, 0, f(b) + c)$, $1 - \alpha \circ \beta = s \circ d + d \circ s$. Therefore α is a chain homotopy equivalence between C and $\mathrm{cyl}(f)$. ∎

Definition 73. Let X be a cellular complex and $I = [0, 1]$. The space $I \times X$ is the **topological cylinder** of X, which is also a cell complex. If $C_\bullet(X)$ is the cellular chain complex of X, then the cellular chain complex $C_\bullet(I \times X)$ of $I \times X$ can be identified with the mapping cylinder chain complex of the identity map on $C_\bullet(X)$, $\mathrm{cyl}(1_{C_\bullet(X)})$.

Samely, if $f : X \to Y$ is a cellular map, then the **topological mapping cylinder** $\mathrm{cyl}(f)$ is obtained by glueing $I \times X$ and Y together, identifying $0 \times X$ with the image of X under f, which is also a cell complex. Then the cellular chain complex $C_\bullet(\mathrm{cyl}(f))$ can be identified with the mapping cylinder of the chain map $C_\bullet(X) \to C_\bullet(Y)$.

Lemma 74. *The subcomplex of elements $(b, 0, 0)$ in cyl(f) is isomorphic to B, and cyl$(f)/B$ is the mapping cone of f. The composite $B \to cyl(f) \xrightarrow{\beta} C$ is the map f, where $\beta(b, b', c) = f(b) + c$ is the chain homotopy equivalence. Thus the map $f_* : H(B) \to H(C)$ factors through $H(B) \to H(cyl(f))$. Thus we may construct a commutative diagram of chain complexes with exact rows as following:*

$$\begin{tikzcd} & & C & & & \\ 0 \arrow[r] & B \arrow[r] \arrow[ur, "f"] & cyl(f) \arrow[r] \arrow[u, "\beta"] & cone(f) \arrow[r] & 0 & \\ & 0 \arrow[r] & C \arrow[r] \arrow[u, "\alpha"] & cone(f) \arrow[r, "\delta"] \arrow[u, equal] & B[-1] \arrow[r] & 0 \end{tikzcd} \tag{76}$$

and the homology long exact sequences can be drawn as following:

$$\begin{tikzcd} \cdots \arrow[r, "-\partial"] & H_n(B) \arrow[r] \arrow[d, equal] \arrow[dr, "f"] & H_n(cyl(f)) \arrow[r] \arrow[d, equal] & H_n(cone(f)) \arrow[r, "-\partial"] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & H_{n+1}(B[-1]) \arrow[r, "\partial"] & H_n(C) \arrow[r] & H_n(cone(f)) \arrow[r, "\delta"] & H_n(B[-1]) \arrow[r, "\partial"] & \cdots \end{tikzcd} \tag{77}$$

This diagram commutes.

Proof. It suffices to show that the right square commutes. Let (b, c) be an n-cycle in cone(f), thus $d(b, c) = (-d(b), d(c) - f(b)) = 0$ implies

$d(b) = 0$ and $f(b) = d(c)$. Lifting (b, c) to $(0, b, c)$ in $\text{cyl}(f)$ and taking differential gives $d(0, b, c) = (b, -d(b), d(c) - f(b) = (b, 0, 0)$. thus ∂ maps the class of (b, c) to the class of $b = -\delta(b, c)$ in $H_{n-1}(B)$. Thus the right square commutes. □

Proposition 75. *For any short exact sequence of complexes*

$$0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0 \tag{78}$$

the following natural isomorphism of long exact sequences holds.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] \arrow[d, equal] & H_n(cyl(f)) \arrow[r] \arrow[d, "\simeq"] & H_n(cone(f)) \arrow[r, "\partial"] \arrow[d, "\simeq"] & H_{n-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] & H_n(C) \arrow[r] & H_n(D) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd} \tag{79}$$

Proof. Consider a chain map $\phi : \text{cone}(f) \to D$ defined by $\phi(b, c) = g(c)$. Then the following diagram commutes with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & \text{cone}(f) \arrow[r, "\delta"] \arrow[d, equal] & B[-1] \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \text{cyl}(f) \arrow[r] \arrow[d, "\beta"] & \text{cone}(f) \arrow[r] \arrow[d, "\phi"] & 0 & \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & D \arrow[r] & 0 &
\end{tikzcd} \tag{80}$$

Now since β is a quasi-isomorphism, by 5-lemma and the functority of long exact sequence, ϕ is a quasi-isomorphism. Thus the naturality of ∂ gives the diagram above commutes. □

Exercise 76. Considering B and C as modules considered as chain complexes concentrated in degree zero, $\text{cone}(f)$ is the complex $0 \to B \xrightarrow{-f} C \to 0$. Show that ϕ defined in above proposition is a chain homotopy equivalence if and only if $f : B \hookrightarrow C$ is a split injection.

Proof. Since B and C are modules, D is also a module. Thus, ϕ is a chain homotopy equivalence if and only if there is a map $\alpha : D \to C$ such that there is $r : C \to B$ with $\alpha \circ \phi = 1 - f \circ r$ and $\phi \circ \alpha = 1$. Now since $\phi \circ f = 0$, $f - f \circ r \circ f = \alpha \circ \phi \circ f = 0$, thus $f = f \circ r \circ f$ and thus f is a split injection. Conversely, if $f = f \circ r \circ f$ for some r, then since f is injective $1 = r \circ f$. Now define $\alpha = (1 - f \circ r) \circ \phi^{-1}$. Notice that ϕ is surjective. Now if $a, b \in \phi^{-1}(c)$, then $\phi(a) - \phi(b) = \phi(a - b) = 0$, thus $a - b \in \operatorname{Ker} \phi = \operatorname{Im} f$. Thus there is e such that $f(e) = a - b$, and $(1 - f \circ r)(a - b) = (a - b) - f \circ r \circ f(e) = (a - b) - f(e) = 0$, thus this map is well defined. Finally, since $\phi \circ f = 0$, $\phi \circ \alpha = \phi \circ \phi^{-1} = 1$. □

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Exercise 77. Show that the composite

$$H_n(D) \simeq H_n(\mathrm{cone}(f)) \xrightarrow{-\delta_*} H_n(B[-1]) \simeq H_{n-1}(B) \tag{81}$$

is the connecting homomorphism ∂ in the homology long exact sequence for

$$0 \to B \to C \to D \to 0 \tag{82}$$

Solution. The result is obvious due to the result of previous proposition and lemma. ■

Exercise 78. Show that there is a quasi-isomorphism $B[-1] \to \mathrm{cone}(g)$ dual to ϕ. Then dualize the previous exercise, by showing that the composite

$$H_n(D) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{\simeq} H_n(\mathrm{cone}(g)) \tag{83}$$

is the usual map induced by the inclusion of D in $\mathrm{cone}(g)$.

Solution. This is basically the dual version of previous statements.

■

Exercise 79. Given a map $f : B \to C$ of complexes, let v denote the inclusion of C into $\mathrm{cone}(f)$. Show that there is a chain homotopy equivalence $\mathrm{cone}(v) \to B[-1]$. This equivalence is the algebraic analogue of the topological fact that for any map $f : K \to L$ of (topological) cell complexes the cone of the inclusion $L \hookrightarrow Cf$ is homotopy equivalent to the suspension of K.

Solution. First notice that $\mathrm{cone}(v)_n = C_{n-1} \oplus \mathrm{cone}(f)_n = C_{n-1} \oplus B_{n-1} \oplus C_n$ with differential operator $d(c_{n-1}, b_{n-1}, c_n) = (-d(c_{n-1}), d(b_{n-1}, c_n) - v(c_{n-1})) = (-d(c_{n-1}), -d(b_{n-1}), d(c_n) - f(b_{n-1}) - c_{n-1})$. Now consider $\psi : \mathrm{cone}(v) \to B[-1]$ defined as $\psi(c_{n-1}, b_{n-1}, c_n) = (-1)^n b_{n-1}$ and $\phi : B[-1] \to \mathrm{cone}(v)$ defined as $\phi(b_{n-1}) = ((-1)^{n-1} f(b_{n-1}), (-1)^n b_{n-1}, 0)$. Then $\psi \circ \phi = 1$ and $\phi \circ \psi(c_{n-1}, b_{n-1}, c_n) = \phi((-1)^n b_{n-1}) = (-f(b_{n-1}), b_{n-1}, 0) = (d(c_n), 0, c_n) + (-d(c_n) + f(b_{n-1}) + c_{n-1}, 0, 0) = d(-c_n, 0, 0) + s(-d(c_{n-1}), -d(b_{n-1}), d(c_n) - f(b_{n-1}) - c_{n-1}) = (d \circ s + s \circ d)(c_{n-1}, b_{n-1}, c_n)$ with $s(c_{n-1}, b_{n-1}, c_n) = (-c_n, 0, 0)$. ■

Exercise 80. Let $f : B \to C$ be a morphism of chain complexes. Show that the natural maps $\mathrm{Ker}(f)[-1] \xrightarrow{\partial} \mathrm{cone}(f) \xrightarrow{\beta} \mathrm{coKer}(f)$ give rise to a long exact sequence:

$$\cdots \xrightarrow{\partial} H_{n-1}(\mathrm{Ker}(f)) \xrightarrow{\alpha} H_n(\mathrm{cone}(f)) \xrightarrow{\beta} H_n(\mathrm{coKer}(f)) \xrightarrow{\partial} H_{n-2}(\mathrm{Ker}(f)) \xrightarrow{\alpha} \cdots \tag{84}$$

Solution. ■

Exercise 81. Let C and C' be split complexes, with splitting maps s, s'. If $f : C \to C'$ is a morphism, show that $\sigma(c, c') = (-s(c), s'(c') - s' \circ f \circ s(c))$ defines a splitting of $\mathrm{cone}(f)$ if and only if the map $f_* : H_*(C) \to H_*(C')$ is zero.

Solution. ■

Lemma 82. *Let* $\mathsf{C} \subset \mathsf{A}$ *be a full subcategory of an abelian category* A.

1. C *is additive* $\Leftrightarrow 0 \in \mathsf{C}$ *and* C *is closed under* $\oplus$.

2. C *is abelian and* $\mathsf{C} \subset \mathsf{A}$ *is exact* $\Leftrightarrow \mathsf{C}$ *is additive, and* C *is closed under* Ker *and* coKer.

Proof. 1. One direction is obvious. Suppose that $0 \in \mathsf{C}$ and C is closed under $\oplus$. Since C is a full subcategory, the morphism set are same, thus we can give the exactly same abelian group structure to the morphism set. Thus C is an Ab-category, and so an additive category.

2. For one direction, since C is abelian, C is additive, ans since $\mathsf{C} \subset \mathsf{A}$ is additive, C is closed under Ker and coKer. For the other direction, since C is closed under Ker and coKer, an exact chain in C is still an exact chain in A, thus $\mathsf{C} \subset \mathsf{A}$ is exact and A is an abelian category.

□

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A functor $F : \mathsf{C} \to \mathsf{D}$ is **full** if for each objects $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is surjective; **faithful** if for each objects $x, y \in \mathsf{C}$ the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is injective; **embedding** if it is faithful and the map $F : \mathsf{ob}(\mathsf{C}) \to \mathsf{ob}(\mathsf{D})$ is also injective; **fully embedding** of C into D if it is full and embedding. If $F : \mathsf{C} \to \mathsf{D}$ is a full embedding, we call C a **full subcategory** of D. A functor $F : \mathsf{D} \to \mathsf{E}$ from an

Let A be an abelian group category and C be a torsion-free abelian group category. Consider $f : \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$ in C. This map has a cokernel $\mathsf{Z} \to 0$, which is not a cokernel in A. Indeed, there is no map $0 \to \mathbb{Z}/n$ which makes $\mathbb{Z} \to 0 \to \mathbb{Z}/n = \mathbb{Z} \to \mathbb{Z}/n$.

Example 83. 1. Inside $R - \mathsf{mod}$ we have an additive subcategory consists of the finitely generated R-modules, which is abelian if and only if R is noetherian.

2. Inside Ab, the torsion-free groups form an additive category, the p-groups form an abelian category, finite p-groups form an abelian category, and $\mathbb{Z}/p - \mathsf{mod}$ of vector spaces over the field $\mathbb{Z}/p$ is a full subcategory of Ab.

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This is because R is noetherian if and only if all the ideals are finitely generated.

Definition 84. Let C be any category and A be an abelian category. The **functor category** A^{C} is the abelian category whose objects are functors $F : \mathsf{C} \to \mathsf{A}$, and the maps are natural transformations.

Example 85. 1. If C is the discrete category of integers, then Ab^{C} contains the abelian category of **graded abelian groups** as a full subcategory.

2. If C is the poset category of integers, then the abelian category $\mathsf{Ch}(\mathsf{A})$ of cochain complexes is a full subcategory of A^{C}.

3. If R is a ring considered as a one-object category, then $R-\mathsf{mod}$ is the full subcategory of all additive functors in Ab^R.

Definition 86. Let X be a topological space and U the poset of open subsets of X. A contravariant functor $F : \mathsf{U} \to \mathsf{A}$ such that $F(\emptyset) = \{0\}$ is called a **presheaf** on X with values in A, and the presheaves are the objects of the abelian category $\mathsf{A}^{\mathsf{U}^{\mathrm{op}}} = \mathrm{Presheaves}(X)$.

Example 87. Consider $C^0(U) = \{\text{continuous functions } f : U \to \mathbb{R}\}$. If $U \subset V$, then the maps $C^0(V) \to C^0(U)$ are given by restricting the domain of a function from V to U. Thus a functor C^0 from U to the category with continuous functions $C^0(U)$ is a presheaf.

Definition 88. A **sheaf** on X with values in A is a presheaf F satisfying the **sheaf axiom**: Let $\{U_i\}$ be an open covering of an open subset U of X. If $\{f_i \in F(U_i)\}$ are such that each f_i and f_j agree in $F(U_i \cap U_j)$, then there is a unique $f \in F(U)$ that maps to every f_i under $F(U) \to F(U_i)$. That is, the following sequence is exact.

$$0 \to F(U) \to \prod F(U_i) \xrightarrow{\text{diff}} \prod_{i<j} F(U_i \cap U_j) \tag{85}$$

Exercise 89. Let M be a smooth manifold. For each open $U \subset M$, let $C^\infty(U)$ be the set of smooth functions from U to $\mathbb{R}$. Show that $C^\infty(U)$ is a sheaf in M.

Solution. For the collection of maps $\{f_i \in F(U_i)\}$ such that f_i, f_j agree in $F(U_i \cap U_j)$, define f on U as $f(x) = f_i(x)$ if $x \in U_i$. This definition is well defined, since if $x \in U_i \cap U_j$ then $f_i(x) = f_j(x)$. Now since continuity and differentiability on point is determined by its open neighborhood, f is smooth since f_i are smooth. Finally, if f, g are both such maps, then $f(x) - g(x) = 0$, thus $f = g$. ∎

Exercise 90. Let A be any abelian group. For every open subset U of X, let $A(U)$ denote the set of continuous maps from U to the discrete topological space A. Show that A is a sheaf on X.

Solution. Define the map as above, and prove the continuity in same way. ∎

Example 91. The category $\mathrm{Sheaves}(X)$ is an abelian category, but not an abelian subcategory of $\mathrm{Presheaves}(X)$. For any space X, let O be the sheaf such that $\mathsf{O}(U)$ is the group of continuous maps from U into $\mathbb{C}$. Define O^* with $\mathbb{C}^*$. Then there is a short exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathsf{O} \xrightarrow{\exp} \mathsf{O}^* \to 0 \tag{86}$$

But this map is not an exact sequence of presheaves if $X = \mathbb{C}^*$, because the map $\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*$ is not surjective; notice that there is no global log function, thus there is no preimage of $z : \mathbb{C}^* \to \mathbb{C}^*$. Indeed, $H^1(\mathbb{C}^*, \mathbb{Z}) \simeq \mathbb{Z}$, and the contour integral $\frac{1}{2\pi i} \oint f'(z)/f(z)dz$ gives the image of $f(z)$ in the cokernel.

Definition 92. Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories. F is called **left(right) exact** if for every short exact sequences $0 \to A \to B \to C \to 0$ in A, the sequence $0 \to F(A) \to F(B) \to F(C)(F(A) \to F(B) \to F(C) \to 0)$ is exact in B. F is called **exact** if it is both left and right exact. A covariant functor F is called (left, right) exact if the corresponding covariant functor $F' : \mathsf{A}^{op} \to \mathsf{B}$ is (left, right) exact.

Example 93. The inclusion of Sheaves(X) into Presheaves(X) is a left exact functor. The **Sheafification** Presheaves$(X) \to$ Sheaves(X) is an exact functor. The proof will given later.

Exercise 94. Show that the above definitions are equivalent to the following, which are often given as the definitions. A (covariant) functor F is left(right) exact if exactness of the sequence $0 \to A \to B \to C(A \to B \to C \to 00)$ implies exactness of the sequence $0 \to F(A) \to F(B) \to F(C)(F(A) \to F(B) \to F(C) \to 0)$.

Solution. On direction is obvious. For the other direction, first consider a monic $i : A \to B$. Then $0 \to A \xrightarrow{i} B \to \mathrm{coKer}(i) \to 0$ is exact, thus $0 \to F(A) \xrightarrow{F(i)} F(B) \to F(\mathrm{coKer}(i))$ is exact. Therefore $F(i)$ is monic. Now consider $f : B \to C$ such that $0 \to A \xrightarrow{i} B \xrightarrow{f} C$ is exact. Then $0 \to A \xrightarrow{i} B \xrightarrow{f} \mathrm{Im}(f) \to 0$ is exact, therefore $0 \to F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(f)} F(\mathrm{Im}(f))$ is exact. Since $\mathrm{Im}(f) \to C$ is monic, $F(\mathrm{Im}(f)) \to F(C)$ is monic, thus $0 \to F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(f)} F(C)$ is also exact. ■

Proposition 95. *Let* A *be an abelian category. Then* $Hom_{\mathsf{A}}(M, -)$ *is a left exact functor from* A *to* Ab *for every* M *in* A*. Thus, given an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* A*, the following sequence of abelian groups is also exact:*

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \tag{87}$$

Proof. Suppose that $\alpha \in \mathrm{Hom}(M, A)$. First $g_* \circ f_*(\alpha) = g \circ f \circ \alpha = 0$, this is a chain. If $f_*(\alpha) = f \circ \alpha = 0$ then $\alpha = 0$ since f is monic, thus f_* is monic. Finally, suppose that $\beta \in \mathrm{Hom}(M, B)$ satisfies $g_*(\beta) = g \circ \beta = 0$. Then $\beta(M) \subset \mathrm{Im}(f)$ due to the exactness, thus considering $A \rightarrowtail \mathrm{Im}(f) \rightarrowtail B \to C$, we have a map $\alpha : M \to A$ satisfying $\beta = f \circ \alpha$. □

Corollary 96. $\mathrm{Hom}_{\mathsf{A}}(-, M)$ *is a left exact contravariant functor.*

Proof. $\mathrm{Hom}_{\mathsf{A}}(A, M) = \mathrm{Hom}_{\mathsf{A}^{\mathrm{op}}}(M, A)$. □

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Definition 97. A (co)homological **δ-functor** between A and B is a collection of of additive functors $T_n : \mathsf{A} \to \mathsf{B}$ ($T^n : \mathsf{A} \to \mathsf{B}$) for $n \geq 0$, together with morphisms $\delta_n : T_n(C) \to T_{n-1}(A)(\delta^n : T^n(C) \to T^{n+1}(A))$ defined for each short exact sequence $0 \to A \to B \to C \to 0$ in A, which satisfies:

1. For each short exact sequences $0 \to A \to B \to C \to 0$, there is a long exact sequence

$$\cdots \to T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} \cdots \tag{88}$$

$$(\cdots \to T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} \cdots) \tag{89}$$

2. For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the morphisms δ give a commutative diagram

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \end{array} \tag{90}$$

$$\left(\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \end{array} \right) \tag{91}$$

Example 98. Homology gives a homological δ-functor H_8 from $\mathsf{Ch}_{\geq 0}(\mathsf{A})$ to A; cohomology gives a cohomological δ-functor H^* from $\mathsf{Ch}^{\geq 0}(\mathsf{A})$ to A.

Exercise 99. Let S be a category of short exact sequences $0 \to A \to B \to C \to 0$ in A. Show that δ_i s a natural transformation from the functor sending the sequence to $T_i(C)$ to the functor sending the sequence to $T_{i-1}(A)$.

Solution. The final commutating square shows the desired property. ■

Example 100 (p-torsion)**.** If p is an integer, the functors $T_0(A) = A/pA$, $T_1(A) = {}_pA := \{a \in A : pa = 0\}$, and $T_n(A) = 0$ for all $n \geq 2$, fit together to form a homological δ functor; taking $T^0 = T_1$ and $T^1 = T_0$ gives a cohomological δ functor. To show this, apply the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "p"] & B \arrow[r] \arrow[d, "p"] & C \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \tag{92}$$

then we get

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0 \tag{93}$$

The same proof shows that if $r \in R$ for a ring R, then $T_0(M) = M/rM$ and $T_1(M) = {}_rM := \{m \in M : rm = 0\}$ fit together to form a homological δ functor from $R - \mathsf{mod}$ to Ab; taking $T^0 = T_1$ and $T^1 = T_0$ gives a cohomological δ functor.

Definition 101. A **morphism** $S \to T$ of δ functors is a system of natural transformations $S_n \to T_n$ that commute with δ. A homological δ functor T is **universal** if, given another δ functor S and a natural transformation $S_0 \to T_0$, there exists a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors that extends f_0. The dual statement defines cohomological δ functor T.

Example 102. We will show later that homology $H_\bullet : \mathsf{Ch}_{\geq 0}(\mathsf{A}) \to \mathsf{A}$ and cohomology $H_\bullet : \mathsf{Ch}^{\geq 0}(\mathsf{A}) \to \mathsf{A}$ are universal δ functors.

Exercise 103. If $F : \mathsf{A} \to \mathsf{B}$ is an exact functor, show that $T_0 = F$ and $T_n = 0$ for $n \neq 0$ defines both homological and cohomological universal δ functor.

Solution. Since F is exact functor, T is a δ functor with $\delta = 0$. Thus defining $\alpha_n : S_n \to T_n = 0$ as a zero map makes the diagram commutes; defining $\beta^n : T^n = 0 \to S^n$ as a zero map makes the diagram commutes. ∎

Definition 104. If $F : \mathsf{A} \to \mathsf{B}$ is an additive functor, we call the functors T_n of (co)homological δ functor T as the **left(right) satellite functors** of F if $T_0 = F(T^0 = F)$.

Definition 105. Let A be an abelian category. An object $P \in \mathsf{A}$ is **projective** if it satisfies the following **universal lifting property**: given an epimorphism $g : B \to C$ and a morphism $\gamma : P \to C$, there is a morphism $\beta : P \to B$ such that $\gamma = g \circ \beta$.

$$\begin{tikzcd} & P \arrow[dl, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd} \tag{94}$$

Proposition 106. *An R-module is projective if and only if it is a direct summand of a free R-module.*

Proof. Let $F(A)$ be a free module based on the module A. Then we have a natural surjection $\pi : F(A) \to A$, thus the sequence

$$0 \to \mathrm{Ker}(\pi) \to F(A) \xrightarrow{\pi} A \to 0 \tag{95}$$

is exact. Now if A is projecttive, then there is a following lifting.

$$\begin{tikzcd} & A \arrow[dl, "\exists i"'] \arrow[d, "1_A"] & \\ F(A) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \tag{96}$$

This makes the short exact sequence splits, thus $F(A) = A \oplus \mathrm{Ker}(\pi)$. Conversely, if A is a direct summand of a free R-module, then by lifting the image of basis using the surjectivity of $g : B \to C$, we can show that P is projective. $\square$

Example 107. 1. Consider $R = R_1 \times R_2 = R_1 \times 0 \oplus 0 \times R_2$. Then $P = R_1 \times 0$ is projective, but not free.

2. Considering $R = M_n(F)$ and $V = F^n$, V can be considered as a left R-module, and $R = \underbrace{V \oplus \cdots \oplus V}_{n}$. But for every free modules of R their dimension on F must be dn^2 for some cardinal d, V is not free over R.

3. The finite abelian group category A is an abelian category without projective objects, since there is no nontrivial free object due to the finiteness.

Definition 108. For an abelian category A, we say that A has **enough projectives** if for every object A there is an epimorphism $P \to A$ with projective P.

Lemma 109. *Let P be an object of abelian category A. P is projective if and only if $\mathrm{Hom}_{\mathsf{A}}(P, -)$ is an exact functor.*

Proof. Suppose that $\mathrm{Hom}(P, -)$ is exact. Choose epic $g : B \to C$ and $\gamma \in \mathrm{Hom}(P, C)$. Since g_* is epic, we can find $\beta \in \mathrm{Hom}(P, B)$ satisfying $g_*(\beta) = g \circ \beta = \gamma$. Conversely, let P be projective. Since $\mathrm{Hom}(P, -)$ is left exact in general, what we need to show is $g_* : \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C)$ is epic. But for $\gamma \in \mathrm{Hom}(P, C)$, the universal lifting property shows we have $\beta \in \mathrm{Hom}(P, B)$ such that $\gamma = g_*(\beta)$. Thus $\mathrm{Hom}(P, -)$ is exact. $\square$

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Definition 110. A chain complex P in abelian category is called a **chain complex of projectives** if all P_n are projective.

Exercise 111. Show that a chain complex P is a projective object in Ch if and only if it is a split exact complex of projectives. Their brutal truncations $\sigma_{\geq 0}$ form the projective objects in $\mathsf{Ch}_{\geq 0}$.

Solution. Notice that $\text{cone}(1_P)$ is split exact, and the following sequence is short exact.

$$0 \to P \to \text{cone}(1_P) \to P[-1] \to 0 \tag{97}$$

Since $\text{cone}(1_P)$ is exact, P is exact. ■

Exercise 112. Show that if an abelian category A has enough projectives, then so does the category $\mathsf{Ch}(\mathsf{A})$ of chain complexes over A.

Solution. ■

Definition 113. Let M be an object of abelian category A. A **left resolution** of M is a complex $P_\bullet$ with $P_i = 0$ for $i < 0$ with a map $\epsilon : P_0 \to M$ so that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0 \tag{98}$$

is exact. If each P_i is projective, we call it a **projective resolution**.

Lemma 114. *Every R-module M has a projective resolution. Generally, if an abelian category A has enough projectives, then every object M in A has a projective resolution.*

Since we may find the surjective map $F(A) \to A$ and $F(A)$ is free object, which is itself the summand of a free object, $F(A)$ is projective, thus $R-\mathsf{mod}$ has enough projectives.

Proof. Consider a surjection $\epsilon_0 : P_0 \to M$ with projective P_0. Define $M_0 = \operatorname{Ker} \epsilon_0$. Now inductively, for a module M_{n-1}, choose $\epsilon_n : P_n \to M_{n-1}$ and take $M_n = \operatorname{Ker} \epsilon_n$.

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & & & 0 \\
& M_1 \arrow[ur] \arrow[dr] & & & & M \arrow[ur] & \\
P_2 \arrow[ur] \arrow[rr, "d_2"] & & P_1 \arrow[rr, "d_1"] \arrow[dr] & & P_0 \arrow[ur, "\epsilon_0"] & & \\
& & & M_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{99}$$

Define d_n as the composition of maps $P_n \to M_n \to P_{n-1}$. Now since $d_n(P_n) \simeq M_{n-1}$ and $\operatorname{Ker}(d_{n-1}) = \operatorname{Ker}(P_{n-1} \to M_0) = \operatorname{Im}(M_{n-1} \to P_1) \simeq M_{n-1}$, we get the sequence is exact, hence the sequence is a projective resolution of M. □

Exercise 115. Show that if $P_\bullet$ is a complex of projectives with $P_i = 0$ for $i < 0$, then a map $\epsilon : P_0 \to M$ giving a resolution for M is the same thing as a quasi-isomorphism $\epsilon : P_\bullet \to M$, where M is considered as a complex concentrated in degree zero.

Solution. Since M has zero homology groups except $H_0(M) = M$, P_i is exact for $n > 0$ and $P_0 / \operatorname{Im}(d_1) \simeq M$. Now since ϵ must induce the isomorphism to M, ϵ must be surjective, and considering $0 \to \operatorname{Ker}(\epsilon) \to P_0 \to M \to 0$ and $0 \to \operatorname{Im}(d_1) \to P_0 \to M \to 0$, which are both exact and there is a chain map between them, by 5-lemma $\operatorname{Ker}(\epsilon) \simeq \operatorname{Im}(d_1)$. ■

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Theorem 116 (Comparison Theorem)**.** *Let $P_\bullet \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' : M \to N$ in abelian category A. Then for every resolution $Q_\bullet \to \eta N$ of N, there is a chain map $f : P_\bullet \to Q_\bullet$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. The chain map is unique up to chain homotopy equialence.*

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\exists"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "\exists"] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd} \tag{100}$$

Proof. Denoting $f' = f_{-1}$, we will use induction. We can cut the chain into two chains as following.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & P_{n+1} \arrow[r, "d"] \arrow[d, "\exists f_{n+1}"] & Z_n(P) \arrow[r] \arrow[d, "f'_n"] & 0 & 0 \arrow[r] & Z_n(P) \arrow[r, "d"] \arrow[d, "f'_n"] & P_n \arrow[r, "d"] \arrow[d, "f_n"] & \cdots \\
\cdots \arrow[r, "d"] & Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0 & 0 \arrow[r] & Z_n(Q) \arrow[r, "d"] & Q_n \arrow[r, "d"] & \cdots
\end{tikzcd} \tag{101}$$

Here all the rows are exact. Since P_{n+1} is projective, we can lift $f'_n \circ d = f_n \circ d$ to f_{n+1} satisfying $f_n \circ d = d \circ f_{n+1}$. Now consider another lift $g : P_\bullet \to Q_\bullet$ and let $h = f - g$. To construct $s_n : P_n \to Q_{n+1}$, for $n < 0$ define $s_n = 0$; for $n = 0$, since $\eta \circ h_0 = \epsilon(f' - f') = 0$, h_0 sends P_0 to $Z_0(Q) = d(Q_1)$, thus we may lift h_0 to $s_0 : P_0 \to Q_1$ such that $h_0 = d \circ s_0 = d \circ s_0 + s_{-1} \circ d$. Now for given h_{n-1} satisfying $h_{n-1} = d \circ s_{n-1} + s_{n-2} \circ d$, then $d(h_n - s_{n-1} \circ d) = d \circ h - h \circ d + s \circ d \circ d = 0$. Thus $h_n - s_{n-1} \circ d$ takes P_n to $Z_n(Q)$, so we can lift it to $s_n : P_n \to Q_{n+1}$ such that $d \circ s_n = h_n - s_{n-1} \circ d$. □

Lemma 117 (Horseshoe Lemma)**.** *Suppose we have a diagram*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} \tag{102}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the P_n assemble to form a projective resolution P of A, and the column lifts to an exact sequence of complexes $0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$, where $i_n : P_n' \to P_n$ and $\pi_n : P_n \to P_n''$ are the natural inclusion and projection, respectively.

Proof. Lift ϵ'' to a map $P_0'' \to A$, then the direct sum of the lifted map and $i_A \circ \epsilon : P_0' \to A$ gives a map $\epsilon : P_0 \to A$. Now we have the following commuting diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker}(\epsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\epsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\epsilon"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\epsilon'') \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{103}$$

Since the right two columns are short exact sequences, the snake lemma shows that the left column is exact and $\mathrm{coKer}(\epsilon) = 0$. Thus P_0 maps onto A. Now this process finishes the initial step and gives the following diagram, which can be filled up by induction.

$$\begin{tikzcd} & & 0 \arrow[d] & \\ \cdots \arrow[r] & P_1' \arrow[r] & \mathrm{Ker}(\epsilon') \arrow[r] \arrow[d] & 0 \\ & & \mathrm{Ker}(\epsilon) \arrow[d] & \\ \cdots \arrow[r] & P_1'' \arrow[r] & \mathrm{Ker}(\epsilon'') \arrow[r] \arrow[d] & 0 \\ & & 0 & \end{tikzcd} \tag{104}$$

□

Exercise 118. Show that there are maps $\lambda_n : P_n'' \to P_{n-1}'$ so that $d(p', p'') = (d'(p') + \lambda(p''), d''(p''))$.

Solution. Suppose that $d(p', p'') = (f_1(p') + f_2(p''), g_1(p') + g_2(p''))$. To commute with d', we have $(d'(p'), 0) = d'(p', 0) = (f_1(p'), g_1(p')$, thus $g_1 = 0$ and $f_1 = d$. To commute with d'', we have $g_1(p') + g_2(p'') = g_2(p'') = d''(p'')$, thus $g_2 = 0$. Therefore we get $d(p', p'') = (d(p') + f_2(p''), d(p''))$. ■

Definition 119. Let A be an abelian category. An object $I \in A$ is **injective** if it satisfies the following **universal lifting property**: given a monomorphism $f : A \to B$ and a map $\alpha : A \to I$, there is a morphism $\beta : B \to I$ such that $\alpha = \beta \circ f$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, "\exists\beta"] \\ & I & \end{tikzcd} \tag{105}$$

We say that A has **enough injectives** if for every object $A \in$ A there is a monomorphism $A \to I$ with injective I.

Theorem 120 (Baer's Criterion)**.** *A right R-module E is injective if and only if for every right ideal J of R, every map $J \to E$ can be extended to a map $R \to E$.*

Proof. One direction is obvious. Consider an R-module B, its submodule A, and a map $\alpha : A \to E$. Let $\mathcal{E}$ be the poset of all extensions $\alpha' : A' \to E$ of α to an intermediate submodule $A \subset A' \subset B$. We give the partial order $\alpha' \leq \alpha''$ if α'' extends α'. Then the Zorn's lemma shows that there is a maximal extension $\alpha' : A' \to E$ in $\mathcal{E}$. Now suppose that $b \in B - A'$. The set $J = \{r \in R : br \in A'\}$ is a right ideal of R, and by assumption the map $J \xrightarrow{b} A' \xrightarrow{\alpha'} E$ extends to a map $f : R \to E$. Let A'' be the submodule $A' + bR$ of B, and define $\alpha'' : A'' \to E$ by $\alpha(a + br) = \alpha'(a) + f(r)$ for all $a \in A', r \in R$. Since $\alpha'(br) = f(r)$ for $br \in A' \cap bR$, and since α'' extends α', this contradicts the maximality of α'. Thus there is no such b, and so $A' = B$. □

Exercise 121. Let $R = \mathbb{Z}/m$. Use Baer's criterion to show that R is an injective R-module. Then show that $\mathbb{Z}/d$ is not an injective R-module when $d \mid m$ and some prime p divides both d and m/d. (The hypothesis ensures that $\mathbb{Z}/m \neq \mathbb{Z}/d \oplus \mathbb{Z}/e$.)

Solution. The solution can be given by the following corollary, since $R = \mathbb{Z}/m$ is a principal ideal domain and $\mathbb{Z}/d$ is not divisible: if so, then for all $r \in \mathbb{Z}/m$ and $a \in \mathbb{Z}/d$, there is $b \in \mathbb{Z}/d$ such that $a = br$ mod d, that is, $a = br + dn$. This implies $\gcd(r, d)$ divides a. Take $a = 1$ and $r = m/d$ gives p divides 1, which is contradiction. ■

Corollary 122. *Suppose that R is a principal ideal domain. An R-module is injective if and only if it is divisible, that is, for every $r \neq 0 \in R$ and every $a \in A$, $a = br$ for some $b \in A$.*

Proof. By Baer's criterion, A is injective if and only if for every right ideal J of R, every map $J \to A$ can be extended to a map $R \to A$. Since R is a principal ideal domain, all J can be represented by (r). Each maps can be uniquely determined by the pair (r, a), where $r \in R$ and $a \in A$. Thus, if A is divisible, then there is $b \in A$ such that $a = br$, thus we can define $R \to A$ as $r \mapsto br$. Conversely, if A is injective, then there is an extension of the map f determined by (r, a), and taking $f(1) = b$ gives $a = br$. □

Example 123. The divisible abelian groups $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty} = \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$ are injective. Indeed, every injective abelian group is a direct sum of these. In particular, the injective abelian group Q/Z is isomorphic to $\oplus_p \mathbb{Z}_{p^\infty}$.

$\mathbb{Z}\left[\frac{1}{p}\right]$ is the group of rational numbers of the form $a/p^n, n \in \mathbb{N}$.

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Exercise 124. For an abelian group A, denote $I(A)$ as the product of copies of the injective group $\mathbb{Q}/\mathbb{Z}$ indexed by the set $\mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) - 0$, where 0 is the zero map. Then $I(A)$ is injective since it is a product of injectives, and there is a canonical map $e_A : A \to I(A)$. Show that e_A is an injection, and thus, show that Ab has enough injectives.

Solution. Take $a \in A$. Notice that $f \in \mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}$-th component of $e_A(a)$ is $f(a)$. Define $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a)$ as some nonzero value in $\mathbb{Q}/\mathbb{Z}$, for example, $\left[\frac{1}{2}\right]$. Now since we have a map $a\mathbb{Z} \to A$ and $\mathbb{Q}/\mathbb{Z}$ is injective, we can extend f to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$, which has $f'(a) \neq 0$. Therefore e_A is injective. ■

Exercise 125. Show that an abelian group A is zero if and only if $\mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) = 0$.

Solution. If A is zero then there is one trivial homomorphism. Suppose that $\mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) = 0$. We have defined an injection $e_A : A \to I(A)$ from previous exercise, but in this case $I(A) = 0$, thus $A = 0$. ■

Lemma 126. *The following are equivalent for an object I in an abelian category A:*

1. *I is injective in A;*
2. *I is projective in A;*
3. *The contravariant functor $\mathrm{Hom}_{\mathsf{A}}(-, I)$ is exact.*

Proof. Since the opposite category of abelian category is abelian, and the dual of injective object is project object, the lemma is shown by duality. □

Definition 127. Let M be an object of A. A **right resolution** of M is a cochain complex $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} \cdots \tag{106}$$

is exact, which is same as a cochain map $M \to I^\bullet$, where M is considered as a complex concentrated in degree 0. If each I^i is injective, the right resolution is called an **injective resolution**.

Lemma 128. *If the abelian category A has enough injectives, then every object in A has an injective resolution.*

Proof. This lemma is the dual version of projective resolution. □

Theorem 129 (Comparison Theorem.). *Let $N \to I^\bullet$ be an injective resolution of N and $f' : M \to N$ be a map in A. Then for every resolution $M \to E^\bullet$, there is a cochain map $f : E^\bullet \to I^\bullet$ lifting f'. The map f is unique up to cochain homotopy equivalence.*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f'"] & E^0 \arrow[r] \arrow[d, "\exists"] & E^1 \arrow[r] \arrow[d, "\exists"] & \cdots \\ 0 \arrow[r] & N \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd} \tag{107}$$

Proof. This theorem is the dual version of projective comparison theorem. □

Exercise 130. Show that I is an injective object in the category of chain complexes if and only if I is a split exact complex of injectives. Then show that if A has enough injectives, so does the category $\mathsf{Ch}(\mathsf{A})$.

Proof. Taking the dual of projective chain complexes, we get the first statement. Now since $\mathsf{Ch}(\mathsf{A})^{\mathrm{op}} \simeq \mathsf{Ch}(\mathsf{A}^{\mathrm{op}})$, we get the second statement. □

Definition 131. A pair of functors $L : \mathsf{A} \to \mathsf{B}$ and $R : \mathsf{B} \to \mathsf{A}$ are **adjoint functors** if for all $A \in \mathsf{A}$ and $B \in \mathsf{B}$, there is a natural bijection

$$\tau = \tau_{AB} : \mathrm{Hom}_{\mathsf{B}}(L(A), B) \to \mathrm{Hom}_{\mathsf{A}}(A, R(B)) \tag{108}$$

that is, for all $f : A \to A' \in \mathsf{A}$ and $g : B \to B' \in \mathsf{B}$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{B}}(L(A'), B) \arrow[r, "L(f)^*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathsf{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathsf{B}}(L(A), B') \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathsf{A}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathsf{A}}(A, R(B)) \arrow[r, "R(g)_*"] & \mathrm{Hom}_{\mathsf{A}}(A, R(B'))
\end{tikzcd} \tag{109}$$

We call L the **left adjoint functor** and R the **right adjoint functor**.

Lemma 132. *For every right R-module M, the natural map*

$$\tau : \mathrm{Hom}_{\mathsf{Ab}}(M, A) \to \mathrm{Hom}_{\mathsf{mod}-R}(M, \mathrm{Hom}_{\mathsf{Ab}}(R, A)) \tag{110}$$

is an isomorphism, where $\tau(f)(m)$ is the map $r \mapsto f(mr)$. Thus, forgetful functor and $\mathrm{Hom}_{\mathsf{Ab}}(R, -)$ are left and right adjoint functor pair.

Proof. Take $g : M \to \mathrm{Hom}_{\mathsf{Ab}}(R, A)$. Define $\mu : \mathrm{Hom}_{\mathsf{mod}-R}(M, \mathrm{Hom}_{\mathsf{Ab}}(R, A)) \to \mathrm{Hom}_{\mathsf{Ab}}(M, A)$ as $\mu(g)(m) = g(m)(1)$. Now $(\tau \circ \mu(g))(m)(r) = \mu(g)(mr) = g(mr)(1) = g(m)(1)r = g(m)(r)$ and $(\mu \circ \tau(f))(m) = \tau(f)(m)(1) = f(m)$, thus τ is an isomorphism. □

Proposition 133. *If an additive functor $R : \mathsf{B} \to \mathsf{A}$ is right adjoint to an exact functor $L : \mathsf{A} \to \mathsf{B}$ and I is an injective object of B, then $R(I)$ is an injective object of A.*

In other words, right adjoint functor of exact functor preserves injectives.

Dually, if an additive functor $L : \mathsf{A} \to \mathsf{B}$ is left adjoint to an exact functor $R : \mathsf{B} \to \mathsf{A}$ and P is a projective object of A, then $L(P)$ is a projective object of B.

In other words, left adjoint functor of exact functor preserves projectives.

Proof. For an injection $f : A \to A'$ in A, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{B}}(L(A'), I) & \xrightarrow{L(f)^*} & \mathrm{Hom}_{\mathsf{B}}(L(A), I) \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ \mathrm{Hom}_{\mathsf{A}}(A', R(I)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathsf{A}}(A, R(I)) \end{array} \tag{111}$$

Since L is exact and I is injective, $L(f^*)$ is onto. Hence f^* is onto, and so $\mathrm{Hom}_{\mathsf{A}}(-, R(I))$ is exact. Therefore $R(I)$ is injective. □

Corollary 134. *If I is an injective abelian group, then* $\mathrm{Hom}_{\mathsf{Ab}}(R, I)$ *is an injective R-module.*

Proof. This can be proven directly by previous lemma and proposition. □

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Exercise 135. Let M be an R-module and $Z(M)$ be a product of copies $I_0 = \mathrm{Hom}_{\mathsf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$, indexed by the set $\mathrm{Hom}_R(M, I_0) - 0$. Then $Z(M)$ is injective since it is a product of injectives, and there is a canonical map $e_M : M \to Z(M)$. Show that e_M is an injection, and thus, show that $R - \mathsf{mod}$ has enough injectives.

Solution. Take $r \in R$. Notice that $f \in \mathrm{Hom}_{\mathsf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$-th component of $e_M(r)$ is $f(r)$. Define $f : r\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(r)$ as some nonzero value in $\mathbb{Q}/\mathbb{Z}$, for example, $\left[\frac{1}{2}\right]$. Now since we have a map $r\mathbb{Z} \to R$ and $\mathbb{Q}/\mathbb{Z}$ is injective, we can extend f to a map $f' : R \to \mathbb{Q}/\mathbb{Z}$, which has $f'(r) \neq 0$. Therefore e_M is injective. ■

Definition 136. A set I is a **directed set** if there is a relation $\leq$ such that:

1. $i \leq i$ for all $i \in I$;
2. $i \leq j, j \leq k$ then $i \leq k$ for all $i, j, k \in I$;
3. For any $i, j \in I$, there is $k \in I$ such that $i \leq k$ and $j \leq k$.

For a directed set I, let $\{A_i : i \in I\}$ be a family of objects of category A indexed by I. Let $f_{ij} : A_i \to A_j$ be a morphism for all $i \leq j$ with:

1. f_{ii} is the identity of A_i;
2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

Then the pair $\langle A_i, f_{ij} \rangle$ is called a **direct system over** I.

Let $\langle A_i, f_{ij} \rangle$ be a direct system in A. A **target** of $\langle A_i, f_{ij} \rangle$ is a pair $\langle A, \phi_i \rangle$, where A is an object in A and $\phi_i : A_i \to A$ are morphisms satisfying $\phi_i = \phi_j \circ f_{ij}$ for all $i \leq j$. A **direct limit** of $\langle A_i, f_{ij} \rangle$ is a target $\langle A, \phi_i \rangle$ such that for all target $\langle B, \psi_i \rangle$, there is a unique morphism $u : X \to Y$ such that $u \circ \phi_i = \psi_i$ for all i.

$$\begin{tikzcd}
A_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & A_j \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& A \arrow[d, dotted, "u"] & \\
& B &
\end{tikzcd} \tag{112}$$

We write $A = \lim_{\to} A_i$.

Definition 137. For a topological space X and a sheaf F, a **stalk** of a sheaf F at a point $x \in X$ is the abelian group $F_x : \lim_{\to}\{F(U) : x \in U\}$.

Definition 138. For an abelian group A and a topological space X, the **skyscraper sheaf** x_*A at the point $x \in X$ is the presheaf

$$(x_*A)(U) = \begin{cases} A & x \in U \\ 0 & x \notin U \end{cases} \tag{113}$$

Exercise 139. Show that x_*A is a sheaf and that

$$\mathrm{Hom}_{\mathsf{Ab}}(F_x, A) \simeq \mathrm{Hom}_{\mathsf{Sheaves(X)}}(F, x_*A) \tag{114}$$

for every sheaf F. Thus if A_x is an injective abelian group, then x_*A_x is an injective object in Sheaves(X) for each x, and that $\prod_{x \in X} x_*A_x$ is also injective.

Solution. Restricting $x_*A(U)$ to U_i gives 0 if $x \notin U_i$ and A if $x \in U_i$, which shows that x_*A is a sheaf. To show the isomorphism, take $\tau : \mathrm{Hom}_{\mathsf{Ab}}(F_x, A) \to \mathrm{Hom}_{\mathsf{Sheaves(X)}}(F, x_*A)$ as $\tau(f)(U)$ is a zero map if $x \notin U$ and $\tau(f)(U)$ is a composition map $F(U) \to F_x \xrightarrow{f} A$. Now take $\mu : \mathrm{Hom}_{\mathsf{Sheaves(X)}}(F, x_*A) \to \mathrm{Hom}_{\mathsf{Ab}}(F_x, A)$ as the map which is uniquely generated by the direct limit. Since each construction gives the same commuting diagram, $\tau \circ \mu$ and $\mu \circ \tau$ are identity maps. Thus the stalk functor and skyscraper sheaf functor are left and right adjoints respectively. Since the stalk functor is exact, the previous proposition shows the last statement. ■

Stalk functor is exact since the direct limit functor is exact.

Example 140. Sheaves(X) has enough injectives. Indeed, given a fixed sheaf F, choose an injection $F_x \to I_x$ with I_x injective in Ab for each $x \in X$. Combining the natural maps $F \to x_*F_x$ with $x_*F_x \to x_*I_x$ gives a map from F to the injective sheaf $I = \prod_{x \in X} x_*I_x$. The map $F \to I$ can be shown that it is an injection.

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Example 141. Let I be a small category and A be an abelian category. If the product of any set of objects exists in A, which is, if A is complete, and A has enough injectives, then the functor category A^I has enough injectives. Indeed, for each $k \in I$, the coordinate functor $\mathsf{A}^I \to \mathsf{A}$ mapping $A \mapsto A(k)$ is an exact functor. Now for an object $A \in \mathsf{A}$, define the functor $k_*A : I \to \mathsf{A}$ as $k_*A(i) = \prod_{\mathrm{Hom}_I(i,k)} A$. Now if $\eta : i \to j$ is a map in I, then the map $k_*A(i) \to k_*A(j)$ is determined by the index map $\eta^* : \mathrm{Hom}(j,k) \to \mathrm{Hom}(i,k)$, that is, $\phi \in \mathrm{Hom}(i,k)$-th component becomes $\eta^*(\phi) = \phi \circ \eta$-th component. Now for a morphism $f : A \to B$, there is a corresponding morphism $k_*f : k_*A \to k_*B$ which is defined slotwise. This shows that $k_* : \mathsf{A} \to \mathsf{A}^I$ is an additive functor. The following exercise then shows that A^I has enough injectives.

Exercise 142. From the previous example, show that k_* is right adjoint to the k-th coordinate functor, so that k_* preserves injectives. Now for each $F \in \mathsf{A}^I$, embed $F(k)$ in an injective object $A_k \in \mathsf{A}$, and so let $F \to k_*A_k$ be the corresponding adjoint map. Show that $E = \prod_{k \in I} k_*A_k$ exists in A^I, that E is an injective object, and that $F \to E$ is an injection.

Solution. Choose $A \in \mathsf{A}$ and $F \in \mathsf{A}^I$. Then we have to show the isomorphism between $\mathrm{Hom}(F, k_*(A))$ and $\mathrm{Hom}(F(k), A)$. Let $f \in \mathrm{Hom}(F, k_*(A))$, then [LATER] ∎

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Definition 143. Let $F : \mathsf{A} \to \mathsf{B}$ be a right exact functor between two abelian categories. If A has enough projectives, then we define the **left derived functors** L_iF for $i \geq 0$ of F as, for an object $A \in \mathsf{A}$, choose a projective resolution $P \to A$ and define

$$L_iF(A) = H_i(F(P)) \tag{115}$$

Since F is right exact, $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, thus $L_0F(A) \simeq F(A)$.

Lemma 144. *The objects $L_iF(A)$ of B are well defined up to natural isomorphisms. That is, if $Q \to A$ is another projective resolution, then there is a canonical isomorphism*

$$L_iF(A) = H_i(F(P)) \simeq H_i(F(Q)) \tag{116}$$

In particular, a different choice of the projective resolutions would yield new functors $\hat{L}_iF$, which are naturally isomorphic to the functors L_iF.

Proof. Due to the comparison theorem, there is a chain map $f : P \to Q$ lifting the identity map 1_A, which gives a map $f_* : H_iF(P) \to$

$H_iF(Q)$. Due to the uniqueness of the map f up to chain homotopy equivalence, f_* is canonical. Samely, there is a chain map $g : Q \to P$ lifting 1_A, and corresponding map $g_* : H_iF(Q) \to H_iF(P)$. Since $g \circ f$ and 1_P are both chain maps $P \to P$ lifting 1_A, and due to the uniqueness, we have $g_* \circ f_* = 1_{H_iF(P)}$. Samely, $f_* \circ g_* = 1_{H_iF(Q)}$. Thus f_*, g_* are isomorphisms. □

Corollary 145. *If A is projective, the $L_iF(A) = 0$ for $i \neq 0$.*

Proof. Consider the projective resolution $\cdots \xrightarrow{1_A} A \xrightarrow{0} A \xrightarrow{1_A} A \to 0$. This gives $L_iF(A) = 0$ for $i \neq 0$. □

Definition 146. Let $F : \mathsf{A} \to \mathsf{B}$ be a right exact functor between abelian categories. Then an object $Q \in \mathsf{A}$ is F**-acyclic** if $L_iF(Q) = 0$ for all $i \neq 0$. For an object $A \in \mathsf{A}$, a left resolution $Q \to A$ for which each Q_i is F-acyclic is an F**-acyclic resolution**.

Lemma 147. *If $f : A' \to A$ is any map in A, then there is a natural map $L_iF(f) : L_iF(A') \to L_iF(A)$ for each i.*

Proof. Let $P' \to A'$ and $P \to A$ be the projective resolutions. Then the comparison theorem gives a lift of f to a chain map $\tilde{f} : P' \to P$, hence a map $L_iF(f) := \tilde{f}_* : H_iF(P') \to H_iF(P)$, which is independent of the choice of $\tilde{f}$. □

Exercise 148. Show that $L_0F(f) = F(f)$ under the identification $L_0F(A) \simeq F(A)$.

Proof. $L_0F(f) : L_0F(A') \to L_0F(A) \simeq F(A') \to F(A)$, since the lifting of f should satisfy the commutativity of the diagram, which takes kernel to kernel. □

Theorem 149. *Each $L_iF : \mathsf{A} \to \mathsf{B}$ is an additive functor.*

Proof. First the identity map on P lifts the identity on A, thus $L_iF(1_A)$ is the identity map. Now consider the maps $A' \xrightarrow{f} A \xrightarrow{g} A''$ and chain maps $\tilde{f}, \tilde{g}$ lifting f, g respectively. Then $\tilde{g} \circ \tilde{f}$ lifts $g \circ f$, thus $g_* \circ f_* = (g \circ f)_*$, and so L_iF is a functor. Finally, if $f_1, f_2 : A' \to A$ are two maps with lifts $\tilde{f}_1, \tilde{f}_2$ respectively, then the sum $\tilde{f}_1 + \tilde{f}_2$ lifts $f_1 + f_2$, thus $f_{1*} + f_{2*} = (f_1 + f_2)_*$, thus L_iF is additive. □

Exercise 150. If $U : \mathsf{B} \to \mathsf{C}$ is an exact functor, then show that $U(L_iF) \simeq L_i(U \circ F)$.

Solution. For an object A, choose a projective resolution P, then

$$L_i(U \circ F)(A) = H_i(U \circ F(P)), \quad U(L_iF)(A) = U(H_i(F(P))) \tag{117}$$

Thus what we need to show is that if there is a chain complex B, then there is an isomorphism between $H_i(U(B))$ and $U(H_i(B))$. Now, consider the exact sequence $0 \to B_i(B) \xrightarrow{r} Z_i(B) \xrightarrow{q} H_i(B) \to 0$. Taking the exact functor U gives an exact sequence $0 \to U(B_i(B)) \xrightarrow{U(r)} U(Z_i(B)) \xrightarrow{U(q)} U(H_i(B)) \to 0$, thus $U(H_i(B)) \simeq U(Z_i(B))/U(B_i(B))$ with $B_i(U(B)) \simeq U(B_i(B))$ and $Z_i(U(B)) \simeq U(Z_i(B))$. This shows the desired result. ■

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Theorem 151. *Let $F : \mathsf{A} \to \mathsf{B}$ be a right exact functor between two abelian categories. The derived functors L_*F form a homological δ-functor.*

Proof. For a short exact sequence $0 \to A' \to A \to A'' \to 0$, choose projective resolutions $P' \to A'$ and $P'' \to A''$. Then by the horseshoe lemma, there is a projective resolution $P \to A$ which fits into a short exact sequence $0 \to P' \to P \to P'' \to 0$. Since P''_n are projective, each $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. Now since F is additive, it preserves the addition and zero map, thus $0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$ is split exact. This shows that $0 \to F(P') \to F(P) \to F(P'') \to 0$ is a short exact sequence of chain complexes. From this, we can take the corresponding long exact homology sequence, which gives

$$\cdots \xrightarrow{\partial} L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial} L_{i-1}F(A') \to \cdots \tag{118}$$

To show the naturality of ∂, consider a commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "i_B"] & B \arrow[r, "\pi_B"] & B'' \arrow[r] & 0 \end{tikzcd} \tag{119}$$

Take projective resolutions of the corners as $\epsilon' : P' \to A', \epsilon'' : P'' \to A'', \eta' : Q' \to B', \eta'' : Q'' \to B''$. By the horseshoe lemma, we get the corresponding projective resolutions $\epsilon : P \to A, \eta : Q \to B$. Also by the comparison theorem, we have chain maps $F' : P' \to Q'$ and $F'' : P'' \to Q''$ lifting the maps f', f'' respectively. Our aim is to show that there is a chain map $F : P \to Q$ lifting f, and giving a following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"] & 0 \\ 0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0 \end{tikzcd} \tag{120}$$

Since H_* is a homological δ-functor, this gives the naturality of ∂. If we define the maps $\gamma_n : P''_n \to Q'_n$ such that F_n are defined as

$$F_n(p', p'') = (F'(p') + \gamma(p''), F''(p'')) \tag{121}$$

then, if F is a chain map over f, this gives the commutative diagram. Now to make F a lifting of f, the map $\eta \circ F_0 - f \circ \epsilon : P_0 = P'_0 \oplus P''_0 \to B$ must be vanish. This implies

$$i_B \circ \eta' \circ \gamma_0 = f \circ \lambda_P - \lambda_Q F''_0 : P''_0 \to B \tag{122}$$

where λ_P, λ_Q are the restrictions of ϵ and η to P''_0 and Q''_0, respectively. Now since

$$\pi_B(f \circ \lambda - \lambda \circ F''_0) = f'' \circ \pi_A \circ \lambda - \pi_B \circ \lambda \circ F''_0 = f'' \circ \epsilon'' - \eta'' \circ F''_0 = 0 \tag{123}$$

thus there is $\beta : P''_0 \to B'$ so that $i_B \circ \beta = f \circ \lambda - \lambda \circ F''_0$. Now using projectivity, define γ_0 to be any lift of β to Q'_0, satisfying $\beta = \eta' \circ \gamma_0$. To make F a chain map, we have

$$(d \circ F - F \circ d)(p', p'') = ((d' \circ F' - F' \circ d')(p') + (d' \circ \gamma - \gamma \circ d'' + \lambda \circ F'' - F' \circ \lambda)(p''), (d'' \circ F'' - F'' \circ d'')(p'')) = 0 \tag{124}$$

This means the map $d' \circ \gamma_n : P''_n \to Q'_{n-1}$ must equal

$$g_n = \gamma_{n-1} \circ d'' - \lambda_n F'_n + F''_{n-1} \circ \lambda_n \tag{125}$$

Now use induction. Suppose γ_i defined for $i < n$, so that g_n exists. Then $d' \circ g_n = 0$ due to the inductive definition. Since Q' is exact, the map g_n factors through a map $\beta : P''_n \to d(Q'_n)$, and we take γ_n any lift of β to Q'_n. This constructs the desired chain map F. $\square$