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CATEGORY THEORY

Definition 1. A **category** C is a collection of

- a collection of **objects**, $\mathsf{ob}(\mathsf{C})$, containing $X, Y, Z, \cdots$
- a collection of **morphisms**, $\mathsf{mor}(\mathsf{C})$, containing $f, g, h, \cdots$

which satisfies:

- for morphism f, there is a **domain** X and **codomain** Y in objects, and we write $f : X \to Y$;
- for any two morphisms f, g where the codomain of f is equal to the domain of g, the **composite morphism** $g \circ f : X \to Z$ exists;
- for each object X, there is a **identity morphism** 1_X such that for any $f : X \to Y$, $1_Y f = f 1_X = f$;
- for three morphisms f, g, h where $h \circ g$ and $g \circ f$ are well defined, $h \circ (g \circ f) = (h \circ g) \circ f$, and written as $h \circ g \circ f$.

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Notice that we did not used here the word 'set'. Bertrand Russell showed that there is no sets of all sets(Russell's paradox). Therefore, by using the word set, we cannot treat the category of sets, groups, or lots of concepts we want. Therefore we used the word 'collection': the definition of this word depends on the context, even sometimes this word is informal.

Sometimes we write gf rather then $g \circ f$, if there is no ambiguity.

Due to the existence of the identity morphisms, it is possible to reconstruct the data of objects by using the data of morphisms. Indeed, in the set theory we have focused on the elements of set, but in the category theory we focus on the morphisms. This concept becomes clearer when we think the group as one object category where the elements of groups are morphisms, which will be discussed later, which is indeed the Cayley's theorem. Despite of this fact, it is quite common to name the category following after the objects, not the morphisms.

Example 2.

1. Set is the category which has sets as objects and functions as morphisms.
2. Group is the category which has groups as objects and homomorphisms as morphisms. Ab, Ring, Mod_R, and Field are also defined in the same sense for abelian groups, rings, R-modules, and fields.
3. Meas is the category which has measurable spaces as objects and measurable functions as morphisms.
4. Top is the category which has topological spaces as objects and continuous functions as morphisms. Man is also defined in the same sense for smooth manifolds.
5. Poset is the category which has partially ordered sets as objects and order-preserving functions as morphisms.

Partially ordered set is the set P with binary operation $\leq$ satisfying: for all $x, y, z \in P$, $x \leq x$, $x \leq y \leq x$ then $x = y$, and $x \leq y \leq z$ then $x \leq z$. **Order-preserving function** $f : P \to Q$ for partially ordered set P, Q is the map satisfying $x \leq y$ implies $f(x) \leq f(y)$. This definition shows that Poset is the category.
The objects of the categories above are all set-like: if we forget the special structures, we get the category Set. These kind of categories are called *concrete categories*, which will be defined exactly later.

Example 3.

1. A group G defines a category $\mathsf{B}G$ with one object, where the morphisms are the group elements.
2. A poset P itself is a category with its elements as objects and $x \leq y$ implies there is a unique morphism $f : x \to y$.
3. A set S itself is a category with its elements as objects and all morphisms are identity morphisms. A category which has only identity morphisms is called **discrete category**.

Definition 4. A category is **small** if it has only a set's worth of morphisms.

Category theory uses larger concept then set, **class**. The exact construction of the class needs the extension of Zermelo-Fraenkel axioms, which is not the topic of this paper.

For small category, the identity morphisms are the subset of the set of morphisms, thus it has a set's worth of objects.

Definition 5. A category is **locally small** if between any pair of objects there is only a set's worth of morphisms.

For locally small category, the set of morphisms with domain X and codomain Y is often written as $\mathrm{Hom}(X, Y)$, or $\mathsf{C}(X, Y)$ to emphasize which category we are working in.

Definition 6. The morphism $f : X \to Y$ is called **isomorphism** if there is $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$. If there is an isomorphism between X and Y, then we call X and Y are **isomorphic**, and write $X \simeq Y$. If a morphism has same domain and codomain, then we call it **endomorphism**; if an endomorphism is isomorphism, then we call it **automorphism**.

Example 7. The isomorphisms of Set are bijections; the isomorphisms of $\mathsf{Group}, \mathsf{Ring}, \mathsf{Mod}_R, \mathsf{Field}$ are isomorphisms(which sound quite trivial); the isomorphisms of Top are homeomorphisms; the isomorphisms of partially ordered set-generated category P is the identity.

Lemma 8. *A morphism can have at most one inverse isomorphism.*

Proof. Let $f : X \to Y$ has two inverse isomorphisms g, h. Then $gfh = g(fh) = g1_Y = g$ and $gfh = (gf)h = 1_X h = h$, thus $g = h$. □

Definition 9. A **groupoid** is a category where every morphism is isomorphism.

In abstract algebra, groupoid is defined as a set G with inverse g^{-1} and partial function $* : G \times G \to G$, satisfying 1. if $g * h, h * k$ are defined then $(g * h) * k$ and $g * (h * k)$ are defined and equal, and conversely if $(g * h) * k$ and $g * (h * k)$ are defined then they are equal and $g * h, h * k$ are defined, 2. $g^{-1} * g$ and $g * g^{-1}$ are always defined, 3. $g * h$ is defined then $g * h * h^{-1} = h$ and $g^{-1} * g * h = h$. This definition and category theoretic definition are same in the range of set.

Example 10.

1. A **group** is a groupoid with one object.
2. For any space X, the **fundamental groupoid** $\Pi_1(X)$ is a category whose objects are the points of X and the morphism between two points are the endpoint-preserving homotopy classes of paths.
3. For the group G acting on the set X, the **action groupoid** is the category where the objects are the elements of X and the morphisms from x to y is the group element g satisfying $y = gx$.

We already have the algebraic definition of group. However, in category theory, this becomes the definition of group.

Definition 11. For category C, a category D is called a **subcategory** if $\mathsf{ob}(\mathsf{D})$ and $\mathsf{mor}(\mathsf{D})$ is the subcollection of $\mathsf{ob}(\mathsf{C})$ and $\mathsf{mor}(\mathsf{C})$ respectively.

Of course the morphisms of D must have domain and codomain in $\mathsf{ob}(\mathsf{D})$.

Lemma 12. *Any category C contains a **maximal groupoid**, the subcategory containing all of the objects and only those morphisms that are isomorphisms.*

Proof. what we need to show is that the composition of two isomorphisms is isomorphism. For isomorphisms $f : X \to Y$, $g : Y \to Z$, there is $f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$ such that $f^{-1}f = 1_X, ff^{-1} = 1_Y, g^{-1}g = 1_Y$ and $gg^{-1} = 1_Z$. Now notice that $gff^{-1}g^{-1} = g(ff^{-1})g^{-1} = gg^{-1} = 1_Z$ and $f^{-1}g^{-1}gf = f^{-1}(g^{-1}g)f = f^{-1}f = 1_X$, hence gf is isomorphism. □

Proposition 13. *Consider a morphism $f : x \to y$. If there exists a pair of morphisms $g, h : y \to x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.*

Proof. $gfh = (gf)h = 1_x h = h$ and $g(fh) = g1_y = g$ thus $g = h$ and so f is an isomorphism. □

Proposition 14. *For any category C and any object $c \in \mathsf{C}$,*

1. *There is a category c/C whose objects are morphisms $f : c \to x$ with domain c and in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that $g = hf$.*

2. *There is a category C/c whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the domains so that $f = gh$.*

Proof.

1. What we need to prove is the composition rule: the morphism from f to g, F, and the morphism from g to h, G, satisfies $g = Ff$ and $h = Gg$. Then $h = G(Ff) = (GF)f$, which says that GF is exactly the morphism from f to h.

2. This is very similar with above, except the arrow direction is opposite. The morphism from f to g, F, and the morphism from g to h, G, satisfies $f = gF$ and $g = hG$. Then $f = (hG)F = h(GF)$, which says that GF is exactly the morphism from f to h.

□

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Definition 15. For category C, the **opposite category** C^{op} has the same objects in C, and for each morphism $f : x \to y$ in C we take $f^{\mathrm{op}} : y \to x$ as a morphism in C^{op}. The identity becomes 1_x^{op}, and the composition of morphisms becomes $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

Notice that the opposite category is also a category.

Definition 16. For a theorem, if we take the opposite category, we get a theorem, which is called a **dual theorem**, and is proven by the dual statement of the proof.

Lemma 17. *The following are equivalent:*

1. *$f : x \to y$ is an isomorphism in C.*

2. *For all objects $c \in \mathsf{C}$, post-composition with f defines a bijection $f_* : \mathsf{C}(c,x) \to \mathsf{C}(c,y)$.*

3. *For all objects $c \in \mathsf{C}$, pre-composition with f defines a bijection $f^* : \mathsf{C}(y,c) \to \mathsf{C}(x,c)$.*

Most of categories we will treat from now are locally small categories. However, lots of them can be proven in general categories also, using similar statement: for example, we may change the word bijection to isomorphism in the sense of general category.

Post-composition means that for morphism $g : c \to x$, $f_*(g) = f \circ g : c \to y$. **Pre-composition** means that for morphism $g : y \to c$, $f^*(g) = g \circ f : x \to c$.

Proof. Noticeable point is that the second statement is exact dual of third statement, and vice versa. Therefore, it is sufficient to prove that $1 \Leftrightarrow 2$, and then $1 \Leftrightarrow 3$ is proven automatically by dual theorem. For $1 \Rightarrow 2$, take g be the isomorphic inverse of f, and g_* be the post-composition with g. Then, for all $h \in \mathsf{C}(c,x)$,

$$g_* f_*(h) = g_*(fh) = gfh = (gf)h = 1_x h = h \tag{1}$$

thus $g_* f_* = 1_{\mathsf{C}(c,x)}$. Also, for all $h \in \mathsf{C}(c,y)$,

$$f_* g_*(h) = f_*(gh) = fgh = (fg)h = 1_y h = h \tag{2}$$

thus $f_* g_* = 1_{\mathsf{C}(c,y)}$ and so f_* is bijection.

Conversely, for $2 \Rightarrow 1$, since f_* is bijection, there must exists $g \in \mathsf{C}(y,x)$ such that $f_*(g) = fg = 1_y$. Also, the function $gf \in \mathsf{C}(x,x)$ satisfies $f_*(gf) = fgf = 1_y f = f \in \mathsf{C}(x,y)$, and since $f_*(1_x) = f1_x = f$ and f_* is bijection, $1_x = gf$. Therefore f is an isomorphism. □

Definition 18. A morphism $f : x \to y$ in a category C is

1. a **monomorphism** or **monic** if for any parallel morphisms $h,k : w \to x$, $fh = fk$ implies that $h = k$, or for locally small category, $f_* : \mathsf{C}(c,x) \to \mathsf{C}(c,y)$ is injective;

2. an **epimorphism** or **epic** if for any parallel morphisms $h,k : y \to z$, $hf = kf$ implies that $h = k$, or for locally small category, $f^* : \mathsf{C}(y,c) \to \mathsf{C}(x,c)$ is injective.

In general, we use $\rightarrowtail$ for monomorphisms and $\twoheadrightarrow$ for epimorphisms.

For the category Set, if $f : X \to Y$ is monomorphism, then for any $x \in X$, define $1_x : \bullet \to X$ where $1_x(\bullet) = x$: then $f1_x = f1_{x'}$ means $f(x) = f(x')$, and $1_x = 1_{x'}$ means $x = x'$, which coincides with the definition of injectivity. Also for epimorphism, if $f : X \mapsto Y$ is epimorphism, suppose that $y \in Y - f(X)$. Now for two point set $\{0,1\}$, define a map $h,k : Y \to \{0,1\}$ as $h(Y) = 0$ and $k(Y - \{y\}) = 0, k(y) = 1$. Then $hf = kf$ since $y \notin f(X)$, but $h \neq k$, contradiction, thus f is surjective.

Example 19. For Ring, the inclusion mapping $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is monic and epic, but there is no nontrivial morphism from $\mathbb{Q}$ to $\mathbb{Z}$. This can be shown as following: for some $\mathbb{Z}$-ring R and parallel morphisms $h,k : R \to \mathbb{Z}$, obviously $ih = ik$ implies $h = k$ since i is inclusion. Also for parallel morphisms $h,k : \mathbb{Q} \to R$, suppose that $hi = ki$ but $h \neq k$. Then there is $q \in \mathbb{Q}$ such that $h(q) \neq k(q)$. Since $hi = ki$, $q \notin \mathbb{Z}$. Thus there is prime p and integer r such that $q = r/p$. Now $h(q) \neq k(q)$ implies $p \cdot h(q) \neq p \cdot k(q)$, but since the morphisms are ring homomorphisms, we get $h(r) \neq k(r)$, which is contradiction. Therefore $h = k$. Now suppose that there is a nontrivial ring homomorphism f from $\mathbb{Q}$ to $\mathbb{Z}$. Then we have $f(q) = n \neq 0$ for some $q = r/p \in \mathbb{Q}$ and $n \in \mathbb{Z}$. Then $2n \cdot f(r/2pn) = n$, but there is no integer k which satisfies $2nk = n$ for $n \neq 0$.

Definition 20. Suppose that $s : x \to y$ and $r : y \to x$ are morphisms such that $rs = 1_x$. Then we call s a **section**, **split monomorphism**, or **right inverse** of r, and r a **retraction**, **split epimorphism**, or **left inverse** of s. We call x a **retract** of y.

Notice that section is always monomorphism and retraction is always epimorphism, which is easily proven using definition and associativity.

Lemma 21.

1. *If $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$ are monic, then $gf : x \rightarrowtail z$ is monic.*

2. *If $f : x \to y$ and $g : y \to z$ gives monic composition $gf : x \rightarrowtail z$, then f is monic.*

Dually,

1′ *If $f : x \twoheadrightarrow y$ and $g : y \twoheadrightarrow z$ are epic, then $gf : x \twoheadrightarrow z$ is epic.*

2′ *If $f : x \to y$ and $g : y \to z$ gives epic composition $gf : x \twoheadrightarrow z$, then g is epic.*

These results shows that monomorphisms or epimorphisms define a subcategory of given category.

Proof. First we will show first two statements, and then we will show the other statements are dual of above.

For 1., take parallel morphisms $h, k : w \to x$. Since g is monic, $g(fh) = g(fk)$ implies $fh = fk$, and since f is monic, $fh = fk$ implies $h = k$. Therefore $gfh = gfk$ implies $h = k$ and so gf is monic.

For 2., take parallel morphisms $h, k : w \to x$. Then $fh = fk$ implies $gfh = gfk$, which implies $h = k$ since gf is monic. Therefore f is monic.

For 1′, noticing the dual of monic is epic, the dual statement of 1. becomes: If $f : y \twoheadrightarrow x$ and $g : z \twoheadrightarrow y$ are epic, then $fg : z \twoheadrightarrow x$ is epic. Changing notation $f \leftrightarrow g$ and $x \leftrightarrow z$ gives 1′.

For 2′, the dual statement of 2. becomes: If $f : y \to x$ and $g : z \to y$ gives epic composition $fg : z \twoheadrightarrow x$, then f is monic. Changing notation $f \leftrightarrow g$ and $x \leftrightarrow z$ gives 2′. □

Proposition 22. $\mathsf{C}/c \simeq (c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$.

Proof. The category $c/(\mathsf{C}^{\mathrm{op}})$ has objects as morphisms $f : x \to c$ and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : y \to x$ between the domains so that $g = fh$. Taking the opposite category in whole changes the morphism direction in the sense that now h is a morphism from g to f. Changing notation $f \leftrightarrow g$ and $x \leftrightarrow y$ gives: $(c/(\mathsf{C}^{\mathrm{op}}))^{\mathrm{op}}$ has objects as morphisms $f : x \to c$ and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the domains so that $f = gh$. This statement is exactly the definition of category C/c. □

Theorem 23.

1. *A morphism $f : x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, the post-composition function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective.*

2. *A morphism $f : x \to y$ is a split monomorphism in a category C if and only if for all $c \in \mathsf{C}$, the pre-composition function $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective.*

Solution.

1. Suppose that f is a split epimorphism. Then there exists a morphism $g : y \to x$ such that $fg = 1_y$. Now, for $k \in \mathsf{C}(c, y)$, $f(gk) = (fg)k = 1_y k = k$, therefore f_* is surjective. Conversely, suppose that f_* is surjective for all $c \in \mathsf{C}$. Then by taking c as y, we get $g : y \to x$ such that $fg = 1_y$, which shows f is a split epimorphism.

2. Taking the dual of the statement above, we get: a morphism $f : y \to x$ is a split monomorphism in a category C if and only if for all $c \in \mathsf{C}$, the pre-composition function $f^* : \mathsf{C}(x, c) \to \mathsf{C}(y, c)$ is surjective. (Note that the surjectivity does not changes its arrow direction, because this is not the morphism in C but the function of sets of morphisms.) Changing $x \leftrightarrow y$ gives the desired result.

■

Theorem 24. *A morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Therefore, by duality, a morphism that is both an epimorphism and a split monomorphism is necessarily an isomorphism.*

Proof. Suppose that $f : x \to y$ is monomorphism and a split epimorphism. Then we have $g : y \to x$ such that $fg = 1_y$, and for any parallel morphisms $h, k : w \to x$, $fh = fk$ implies $h = k$. Now, since $fgf = 1_y f = f = f 1_x$, $gf = 1_x$. Therefore f is isomorphism and g is its inverse isomorphism. □

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Definition 25. A **functor** $F : \mathsf{C} \to \mathsf{D}$ consists of the following data:

- an object $F(c) \in \mathsf{D}$ for each object $c \in \mathsf{C}$;
- a morphism $F(f) : F(c) \to F(c') \in \mathsf{D}$ for each morphism $f : c \to c' \in \mathsf{C}$,

which satisfies the following **functoriality axioms**:

- for any composable morphism pair $f, g \in \mathsf{C}$, $F(g) \circ F(f) = F(g \circ f)$;

- for each object $c \in \mathsf{C}$, $F(1_c) = 1_{F(c)}$.

Example 26.

1. Let C be the category which is one of $\mathsf{Group}, \mathsf{Ring}, \mathsf{Mod}_R, \mathsf{Field}, \mathsf{Meas}, \mathsf{Top}$, or Poset. We have a **forgetful functor** $F : \mathsf{C} \to \mathsf{Set}$ which sends each object to base set and each morphism to base function. Since it forgets all the algebraic properties they have and becomes a set, this functor is called forgetful. There are some partially forgetful functors like $\mathsf{Mod}_R \to \mathsf{Ab}$ or $\mathsf{Ring} \to \mathsf{Ab}$, which forgets some of the algebraic properties but not all.

   Ab is the category of abelian groups.

2. The fundamental group defines a functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Group}$. A continuous function $f : (X, x) \to (Y, y)$ induces a group homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$, which can be easily proven that this satisfies functoriality axioms.

   Top_* means a pair of topological space with its one element.

3. For each $n \in \mathbb{Z}$, there are functors $Z_n, B_n, H_n : \mathsf{Ch}_R \to \mathsf{Mod}_R$ Z_n, called n**-cycles**, is defined as $Z_n C_\bullet = \ker(d : C_n \to C_{n-1})$; B_n, called n**-boundary**, is defined as $B_n C_\bullet = \mathrm{im}(d : C_{n+1} \to C_n)$, and H_n, called n**th homology**, is defined as $H_n C_\bullet = Z_n C_\bullet / B_n C_\bullet$. All these satisfies the functoriality axioms.

   Ch_R is the category of **chain complex**: a collection $(C_n)_{n\in\mathbb{Z}}$ of R-modules equipped with R-module homomorphisms $d : C_n \to C_{n-1}$ with $d^2 = 0$. The morphism $f_n : C_n \to C'_n$ satisfies $df_n = f_{n-1}d$ for all $n \in \mathbb{Z}$.

4. We have a functor $F : \mathsf{Set} \to \mathsf{Group}$ which sends a set X to the **free group** on X. Remember that the free groups can be defined by using the universal property: For a set X, we have unique free group $F(X)$ (up to isomorphism) which satisfies that for every group G and function $f : X \to G$, there is a unique group homomorphism $\varphi : F(X) \to G$ which satisfies $\varphi \circ i = f$, where $i : X \to F(X)$ is the inclusion. This kind of definition repetitively appears when we say about free module. Indeed, this definition is the categorical definition of free objects, which will be seen later.

Definition 27. A **covariant functor** F from C to D is a functor $F : C \to D$. A **contravariant functor** F from C to D is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

Proposition 28. *Due to the definition, contravariant functor satisfies:*

- $F(c) \in \mathsf{D}$ *for each object* $c \in \mathsf{C}$;
- $F(f) : F(c') \to F(c) \in \mathsf{D}$ *for each morphism* $f : c \to c' \in \mathsf{C}$,

and the functoriality axioms becomes:

- *for any composable pair* $f, g \in \mathsf{C}$, $F(f) \circ F(g) = F(g \circ f)$;
- *for each object* $c \in \mathsf{C}$, $F(1_c) = 1_{F(c)}$.

Proof. Since the dual of category has same objects with original category, the conditions for objects does not changes. The only changes happens on the morphisms on C, which effects on second statement, and it also changes the composition of the morphisms, which effects on third statement. □

Example 29.

1. The functor $* : \mathsf{Vect}_{\mathbb{K}}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{K}}$ which carries a vector space V to its **dual space** $V^* = \mathrm{Hom}(V, \mathbb{K})$ is a covariant functor. For the linear map $\phi : V \to W$, the functor gives the dual map $\phi^* : W^* \to V^*$, in the sense that for $f : W \to \mathbb{K}$ and $g : V \to \mathbb{K}$, $f \circ \phi = g$.

2. The functor Spec : $\mathsf{CRing}^{\mathrm{op}} \to \mathsf{Top}$ which carries a commutative ring R to the set of prime ideals $\mathrm{Spec}(R)$ with Zariski topology is a covariant functor. Consider a ring homomorphism $\phi : R \to S$ and prime ideal $P \subset S$. The inverse image $\phi^{-1}(P) \subset R$ is the prime ideal of R, and therefore the inverse image function $\phi^{-1} : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is well defined; indeed it is easy to show that this is a continuous map.

3. A **presheaf** is a functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. For example, take topological space X and take a category $\mathcal{O}(X)$, the poset of open subsets of X. Since the poset has morphism $V \to U$ if $V \subset U$, we can see that the presheaf satisfies that if $V \subset U$, then we have a function $\mathrm{res}_{V,U} : F(U) \to F(V)$.

The **Zariski topology** is the set of prime ideals $\mathrm{Spec}(R)$ whose closed sets are $V_I = \{P \in \mathrm{Spec}(R) : I \subset P\}$ for all ideal I.

The domain of functor does not needs to be Set: Ab or Ring is also possible, but compositing forgetful functor we get same result. For the example of presheaf, we think the functor F so that $F(U)$ is the ring of bounded functions on U. If $V \subset U$ then we take the ring homomorphism $\mathrm{res}_{V,U} : F(U) \to F(V)$ satisfying $\mathrm{res}_{V,U}(f) = f|_V$, which shows that F is presheaf. If F satisfies some more conditions, we call F *sheaf*, which will be discussed later.

Lemma 30. *Functors preserve isomorphisms.*

Proof. Consider $F : \mathsf{C} \to \mathsf{D}$ a functor and $f : x \to y$ an isomorphism in C with inverse $g : y \to x$. Then we have

$$F(g)F(f) = F(gf) = F(1_x) = 1_{F(x)} \tag{3}$$

and similar for inverse, thus $F(f)$ is isomorphism. □

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Example 31. Remember that the group G defines a one-object category $\mathsf{B}G$ whose morphisms are identified with the elements of g. For a category C, think a functor $X : \mathsf{B}G \to \mathsf{C}$, which sends single object $\bullet \in \mathsf{B}G$ to $X \in \mathsf{C}$, and a morphism g to $g_* : X \to X$. Then the endomorphisms(indeed automorphisms, because functors preserve isomorphisms) g_* satisfies $g_* h_* = (gh)_*$ for all $g, h \in \mathsf{B}G$ and $e_* = 1_X$ for identity $e \in \mathsf{B}G$. This functor X is called a **left action**, or just **action**, of the group G on the object $X \in \mathsf{C}$. If $\mathsf{C} = \mathsf{Set}$ then X is called a G**-set**, if $\mathsf{C} = \mathsf{Vect}$ then a G**-representation**, and if $\mathsf{C} = \mathsf{Top}$ then a G**-space**.

If X is a covariant functor $X : \mathsf{B}G^{\mathrm{op}} \to \mathsf{C}$, then we call X a **right action**. If so, then the endomorphism $g^* : X \to X$ has a composition $g^* h^* = (hg)^*$. If we do not need to specify, then we call X an **action**.

Example 32. Consider a category C with two objects $\bullet, \circ$ and has one nontrivial morphism $\bullet \to \circ$. This is monomorphic and epimorphic. However, take a functor $F : \mathsf{C} \to \mathsf{Mod}_{\mathbb{Z}}$ where $F(\bullet) = F(\circ) = \mathbb{Z}$ and $F(\to) : \mathbb{Z} \to \mathbb{Z}$ is a trivial map $n \mapsto 0$. This is neither monomorphic nor epimorphic.

Proposition 33. *The split monomorphisms and split epimorphisms are preserved by functors.*

Proof. The proof is very same with the proof for isomorphisms. □

Definition 34. If C is locally small, then for any object $c \in \mathsf{C}$, we call a pair of covariant and contravariant functors represented by c as **functors represented by** c and define as following:

- for covariant functor, $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$, $x \mapsto \mathsf{C}(c, x)$, and $f : x \to y$ maps to $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ by post-composition;
- for contravariant functor, $\mathsf{C}(-, c) : \mathsf{C} \to \mathsf{Set}$, $x \mapsto \mathsf{C}(x, c)$, and $f : x \to y$ maps to $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ by pre-composition.

Definition 35. For any categories $\mathsf{C} \times \mathsf{D}$, there is a category $\mathsf{C} \times \mathsf{D}$, which is called the **product category**, defined as following:

- the objects are ordered pairs (c, d) for objects $c \in \mathsf{C}, d \in \mathsf{D}$;
- the morphisms are ordered pairs $(f, g) : (c, d) \to (c', d')$ where $f : c \to c' \in \mathsf{C}, g : d \to d' \in \mathsf{D}$,
- the identities and compositions are defined componentwise.

Definition 36. If C is locally small, then there is a **two-sided functor** $\mathsf{C}(-, -) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$, which is defined as following:

- a pair of objects x, y maps to $\mathsf{C}(x, y)$;
- a pair of morphisms $f : w \to x, h : y \to z$ maps to the function $(f^*, h_*) : C(x, y) \to C(w, z)$ defined as $g \mapsto hgf$.

Definition 37. The category Cat is the category which has small categories as its objects and functors as its morphisms. For two small categories, the collection of functors between them is actually a set, thus this is locally small category, but since Set or all the other concrete categories are the proper subcategory of Cat, this is not a small category, and thus we do not have the Russell's paradox. Notice that none of the concrete categories are the *object* of Cat.

Samely, the category CAT is the category which has locally small categories as its objects and functors as is morphisms. Since Set is not small, CAT is not locally small, thus we also need not to worry about Russell's paradox. We have an inclusion functor $\mathsf{Cat} \hookrightarrow \mathsf{CAT}$.

Definition 38. The functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ satisfying $F \circ G = 1_{\mathsf{D}}$ and $G \circ F = 1_{\mathsf{C}}$ are the **isomorphisms of categories**, and then the categories C, D are **isomorphic categories**.

Example 39.

1. The functor $\mathrm{op} : \mathsf{CAT} \to \mathsf{CAT}$ is a non-trivial automorphism of the category.

2. For any group G, the functor $-1 : \mathsf{B}G \to \mathsf{B}G^{\mathrm{op}}$ defined by $g \to g^{-1}$ is isomorphic. This shows that every right action and left action are equivalent. This is true for groupoid also.

3. Not every category is isomorphic with its opposite category. Consider $\mathbb{N}$ as a partially ordered set category. Then $\mathbb{N}$ has minimal operator, but $\mathbb{N}^{\mathrm{op}}$ does not, which shows that they are not isomorphic.

4. One final, nontrivial, and important isomorphism between two categories is given below. Let E/F be a finite Galois extension and $G := \mathrm{Aut}(E/F)$ the Galois group.

   Now consider the **orbit category** $\mathcal{O}_G$ for group G, whose objects are cosets G/H for subgroup $H \leq G$. The morphisms $f : G/H \to G/K$ are defined as the G**-equivariant maps**, which means the functions that commute with the left G-action: $g' f(gH) = f(g' gH)$. We may show that all the G-equivariant maps can be represented as $gH \mapsto g\gamma K$, for $\gamma \in G$ with $\gamma^{-1} H \gamma \subset K$.

   Also consider the category Field_F^E whose objects are intermediate fields $F \subset K \subset E$, and the morphisms $K \to L$ is a field homomorphism that fixes the elements on F pointwise. Notice that the group of automorphisms of the object $E \in \mathsf{Field}_F^E$ is the Galois group $G = \mathrm{Aut}(E/F)$.

   Finally we define a functor $\Phi : \mathcal{O}_G^{\mathrm{op}} \to \mathsf{Field}_F^E$ which sends G/H to the subfield of E whose elements are fixed by H under the action of Galois group, and if $G/H \to G/K$ is induced by γ then the field homomorphism $x \mapsto \gamma x$ sends an element $x \in E$ which is fixed by K to an element $\gamma x \in E$ which is fixed by H. The **Fundamental theorem of Galois theory** says that Φ is bijection; indeed, Φ is isomorphism between $\mathcal{O}_G^{\mathrm{op}}$ and Field_F^E.

A field extension E/F is a **finite Galois extension** if F is a finite-index subfield of E and the size of the group of automorphisms of E fixing F, $\mathrm{Aut}(E/F)$ is same with the index $[E : F]$.

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Example 40. Take a category C with objects $\{a, b, c, d\}$ and nontrivial morphisms $a \to b, c \to d$. Take another category D with objects x, y, z and nontrivial morphisms $x \to y, y \to z, x \to z$. Now take a functor F such that $F(a) = x, F(b) = F(c) = y, F(d) = z$ for objects and works accordingly on morphisms. Then the image only has nontrivial morphisms $x \to y$ and $y \to z$ but no $x \to z$, which has no composition. Thus, we have an example that the objects and morphisms in the image of the functor $F : \mathsf{C} \to \mathsf{D}$ do not define a subcategory of D.

Proposition 41. *Given functors $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{E} \to \mathsf{C}$, there is a category $F \downarrow G$, the **comma category**, which has:*

- *triples $(d \in \mathsf{D}, e \in \mathsf{E}, f : F(d) \to G(e) \in \mathsf{C})$ as objects;*
- *a pair of morphisms $(h : d \to d', k : e \to e')$ so that $f' \circ F(h) = G(k) \circ f$ as morphisms $(d, e, f) \to (d', e', f')$*

We define a pair of projection functors $\mathrm{dom} : F \downarrow G \to \mathsf{D}$ *and* $\mathrm{cod} : F \downarrow G \to \mathsf{E}$.

$$\begin{tikzcd} d \arrow[r, "F"] \arrow[d, "h"'] & F(d) \arrow[r, "f"] \arrow[d, "F(h)"'] & G(e) \arrow[d, "G(k)"'] & e \arrow[l, "G"'] \arrow[d, "k"'] \\ d' \arrow[r, "F"] & F(d') \arrow[r, "f'"] & G(e') & e' \arrow[l, "G"'] \end{tikzcd} \tag{4}$$

Slice categories, c/C and C/c, are the special cases of this comma category: write the functor from one object category to C whose image is $c \in \mathsf{C}$ as c, and the identity functor on C as 1_C. Then we get $c/\mathsf{C} = c \downarrow 1_\mathsf{C}$ and $\mathsf{C}/c = 1_\mathsf{C} \downarrow c$.

Proof. We can take a pair of identity morphisms for identity morphism, so what we need to show is the composition rule. Suppose we have two morphisms $(d, e, f) \xrightarrow{(h,k)} (d', e', f') \xrightarrow{(h',k')} (d'', e'', f'')$. The composition of pair of morphisms will be taken as $(h' \circ h, k' \circ k)$, so what we need to show is $f'' \circ F(h' \circ h) = G(k' \circ k) \circ f$. But due to the property of functor, we can re-write this as $f'' \circ F(h') \circ F(h) = G(k') \circ G(k) \circ f$. Now, $f'' \circ F(h') \circ F(h) = G(k') \circ f' \circ F(h) = G(k') \circ G(k) \circ f$ due to the definition. $\square$

Example 42. The functors need not reflect isomorphisms, that is, we have a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism $f \in \mathsf{C}$ such that $F(f)$ is an isomorpihsm in D but f is not an isomorphism in C. Let C, D are categories with two objects $\bullet, \circ$, where $\bullet \to \circ \in \mathsf{C}, \mathsf{D}$ and $\circ \to \bullet \in \mathsf{D}$. Take functor $F : \mathsf{C}, \mathsf{D}$ as $F(\bullet) = \bullet, F(\circ) = \circ$, and $F(\bullet \to \circ) = \bullet \to \circ$. Then $F(\bullet \to \circ)$ is isomorphism because we have $\circ \to \bullet$ in D, but $\bullet \to \circ$ is not an isomorphism in C.

Definition 43. Given categories C, D and functors $F, G : \mathsf{C} \to \mathsf{D}$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_c : F(c) \to G(c)$ in D for each object $c \in \mathsf{C}$, the collection of which define the **components** of the natural transformation, so that for any morphism $f : c \to c' \in \mathsf{C}$, $G(f) \circ \alpha_c = \alpha_{c'} \circ F(f)$ holds. A **natural isomorphism**

is a natural transformation $\alpha : F \Rightarrow G$ in which every component α_c is an isomorphism.

$$\begin{array}{ccc} F(c) & \xrightarrow{\alpha_c} & G(c) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\alpha_{c'}} & G(c') \end{array} \tag{5}$$

Example 44. Consider a category Set^{∂} whose objects are sets and morphisms are **partial functions**: $f : X \to Y$ is a function from $X' \subset X$ to Y. The composition of two partial functions is defined as the composition of functions.

Now we take the functor $(-)_+ : \mathsf{Set}^{\partial} \to \mathsf{Set}_*$ which sends X to the pointed set X_+, the disjoint union of X and freely-added basepoint: we may take set as $X_+ := X \cup \{X\}$ and the basepoint as X due to the axiom of regularity. The partial function $f : X \to Y$ becomes the pointed function $f_+ : X_+ \to Y_+$ where all the elements outside of the domain of definition of f maps to the basepoint of Y_+. Conversely, we take the inverse functor $U : \mathsf{Set}_* \to \mathsf{Set}^{\partial}$ discarding the basepoint and following functional inverse.

The **axiom of regularity** is the axiom of ZF(Zermelo-Fraenkel) set theory, which says that the set does not contains itself as its element. This shows that X and $\{X\}$ are disjoint.

The construction says that $U(-)_+$ is the identity endofunctor of Set^{∂}, but $(U-)_+$ sends $(X, x) \to (X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$, which is isomorphic but not identical, hence the functor is not isomorphic. But the structure of these are very same.

This is the reason why we do not use the condition $GF = 1_D, FG = 1_C$ for the isomorphism for category.

Example 45.

1. For vector space of any dimension over the field $\mathbb{K}$, the map $\mathrm{ev} : V \to V^{**}$ that sends $v \in V$ to $\mathrm{ev}_v : V^* \to \mathbb{K}$ defines the components of a natural transformation from the identity endofunctor on $\mathsf{Vect}_{\mathbb{K}}$ to the double dual functor. The map $V \xrightarrow{\phi} W$ becomes $V \xrightarrow{\phi} W$ by the identity endofunctor and $V^{**} \xrightarrow{\phi^{**}} W^{**}$ by the double dual functor. What now we need to show is $\mathrm{ev}_{\phi v} = \phi^{**}(\mathrm{ev}_v)$. The first one

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Example 46.

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1. For vector space of any dimension over the field $\mathbb{K}$, the map $\mathrm{ev} : V \to V^{**}$ that sends $v \in V$ to $\mathrm{ev}_v : V^* \to \mathbb{K}$ defines the components of a natural transformation from the identity endofunctor on $\mathsf{Vect}_{\mathbb{K}}$ to the double dual functor. The map $V \xrightarrow{\phi} W$ becomes $V \xrightarrow{\phi} W$ by the identity endofunctor and $V^{**} \xrightarrow{\phi^{**}} W^{**}$

by the double dual functor. Since $\mathrm{ev}_{\phi(v)} = \phi^{**}(\mathrm{ev}_v)$, this is natural transformation. However, there is no natural isomorphism between the identity functor and its dual functor on finite-dimensional vector spaces, which is because the identity functor is covariant but the dual functor is contravariant.

2. Consider cHaus as a category of compact Hausdorff spaces and continuous maps, and Ban as a category of Banach spaces and continuous linear maps. Consider a finite signed measure $\mu : \mathrm{Baire}(X) \to \mathbb{R}$ where $\mathrm{Baire}(X)$ is a Baire algebra of X, a σ-algebra generated by closed G_δ sets. The Jordan decomposition of μ gives $\mu = \mu_+ + \mu_-$, which gives the norm $\|\mu\| = \mu_+(X) + \mu_-(X)$, and this gives the Banach space of a finite signed Baire measure $\Sigma(X)$. Then we can define a functor $\Sigma : \mathsf{cHaus} \to \mathsf{Ban}$, which takes a continuous map $f : X \to Y$ to the map $\Sigma(f)(\mu) = \mu \circ f^{-1} : \Sigma(X) \to \Sigma(Y)$. Also, consider a functor $C^* : \mathsf{cHaus} \to \mathsf{Ban}$, which takes X to the linear dual $C(X)^*$ of the Banach space $C(X)$ of continuous real-valued functions on X.

   Now for each $\mu \in \Sigma(X)$, there is a linear functional $\phi_\mu : C(X) \to \mathbb{R}$, which is defined as $\phi_\mu(g) = \int_X g d\mu$ for $g \in C(X)$. Now for each $\mu \in \Sigma(X), f : X \to Y, h \in C(Y)$, since $\int_X h \circ f d\mu = \int_Y hd(\mu \circ f^{-1})$, which shows that the morphisms $\mu \mapsto \phi_\mu$ are the components of the natural transformation $\eta : \Sigma \to C^*$. Furthermore, the **Riesz representation theorem** says that this is a natural isomorphism.

3. Consider a category of commutative ring cRing and a category of group Group. For a commutative ring K, consider the general linear group GL_nK and the group of units K^*. Then GL_n and $(-)^*$ are functors. Now for each general linear group M consider the determinant $\det_n M$. Since M is invertible, $\det_n M \in K^*$. Furthermore, for any ring homomorphism $\phi : K \to K'$, $\det_{K'} \circ GL_n(\phi) = \phi^* \circ \det_K$, thus the morphisms $\det_K$ are the components of the natural transformation $\det : GL_n \to (-)^*$.

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Lemma 47. *Let 2 be a category with two objects $0, 1$ and one nontrivial morphism $0 \to 1$. Consider two categories C, D, two functors $F, G : \mathsf{C} \to \mathsf{D}$, and natural transformations $\alpha : F \Rightarrow G$. This correspond bijectively to functors $H : \mathsf{C} \times 2 \to D$ such that, considering the projection functor i_0, i_1, the following diagram commutes.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathsf{C} \times 2 \arrow[d, "H"] & \mathsf{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathsf{D} & \end{tikzcd} \tag{6}$$

Proof. For a natural transformation α, define H as, for $c, c' \in \mathsf{Ob}(\mathsf{C})$ and $f : c \to c'$, $H(c,0) = F(c), H(c,1) = G(c), H(f, 0 \to 0) = F(f), H(f, 1 \to 1) = G(f)$, and $H(f, 0 \to 1) = G(f) \circ \alpha_c = \alpha_{c'} \circ F(f) : F(c) \to G(c')$, then H is a functor. Conversely, for such functor $H : \mathsf{C} \times 2 \to D$, define a collection of natural transformations α_c as $H(1_c, 0 \to 1)$, then since H is a functor, $G(f) \circ \alpha_c = H(f, 1 \to 1) \circ H(1_c, 0 \to 1) = H(1_{c'}, 0 \to 1) \circ H(f, 0 \to 0) = \alpha_{c'} \circ F(f)$ where $f : c \to c'$. □

Definition 48. An **equivalence of categories** is the functors $F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}$ with natural isomorphisms $\eta : 1_{\mathsf{C}} \simeq G \circ F$, $\epsilon : F \circ G \simeq 1_{\mathsf{D}}$. If so, we call categories C and D are **equivalent**, and write $\mathsf{C} \simeq \mathsf{D}$.

Proposition 49. *The equivalence of categories is indeed a equivalence relation.*

Proof. Suppose that $\mathsf{C} \simeq \mathsf{D} \simeq \mathsf{E}$. Then there are functors $F : \mathsf{C} \leftrightarrow \mathsf{D} : G, H : \mathsf{D} \leftrightarrow \mathsf{E} : K$ such that $1_{\mathsf{C}} \simeq G \circ F, 1_{\mathsf{D}} \simeq F \circ G, 1_{\mathsf{D}} \simeq K \circ H, 1_{\mathsf{E}} \simeq H \circ K$. Now consider $H \circ F : C \leftrightarrow E : G \circ K$. Then $H \circ F \circ G \circ K \simeq H \circ 1_{\mathsf{D}} \circ K = H \circ K \simeq 1_{\mathsf{E}}$ and $G \circ K \circ H \circ F \simeq G \circ 1_{\mathsf{D}} F = G \circ F \simeq 1_{\mathsf{C}}$, thus $\mathsf{C} \simeq \mathsf{E}$. □

Example 50. Consider a category Set^{∂} whose objects are sets and morphisms are **partial functions**: $f : X \to Y$ is a function from $X' \subset X$ to Y. The composition of two partial functions is defined as the composition of functions.

Now we take the functor $(-)_+ : \mathsf{Set}^{\partial} \to \mathsf{Set}_*$ which sends X to the pointed set X_+, the disjoint union of X and freely-added basepoint: we may take set as $X_+ := X \cup \{X\}$ and the basepoint as X due to the axiom of regularity. The partial function $f : X \to Y$ becomes the pointed function $f_+ : X_+ \to Y_+$ where all the elements outside of the domain of definition of f maps to the basepoint of Y_+. Conversely, we take the inverse functor $U : \mathsf{Set}_* \to \mathsf{Set}^{\partial}$ discarding the basepoint and following functional inverse.

The **axiom of regularity** is the axiom of ZF(Zermelo-Fraenkel) set theory, which says that the set does not contains itself as its element. This shows that X and $\{X\}$ are disjoint.

The construction says that $U(-)_+$ is the identity endofunctor of Set^{∂}, but $(U-)_+$ sends $(X, x) \to (X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$, which is isomorphic but not identical, hence the functor is not isomorphic. But the structure of these are very same.

This is the reason why we do not use the condition $GF = 1_D, FG = 1_C$ for the isomorphism for category. But we have a natural isomorphism $\eta : 1_{\mathsf{Set}_*} \simeq (U-)_+$ with $\eta_{(X,x)} : (X, x) \to (X - \{x\} \cup \{X - \{x\}\}, X - \{x\})$, thus the categories $\mathsf{Set}^{\partial}, \mathsf{Set}_*$ are equivalent.

Definition 51. A functor $F : \mathsf{C} \to \mathsf{D}$ is

- **full** if for each objects $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is surjective;
- **faithful** if for each objects $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is injective;
- **essentially surjective on objects** if for every object $d \in \mathsf{D}$ there is an object $c \in \mathsf{C}$ such that d is isomorphic to $F(c)$;
- **embedding** if it is faithful and the map $F : \mathsf{ob}(\mathsf{C}) \to \mathsf{ob}(\mathsf{D})$ is also injective;
- **fully faithful** if it is full and faithful;
- **full embedding** of C into D if it is full and embedding, and then C is a **full subcategory** of D.

Lemma 52. *Consider a morphism $f : a \to b$ and isomorphisms $a \simeq a', b \simeq b'$. Then there is a unique morphism $f' : a' \to b'$ so that any of, or equivalently all of, the following diagrams commute.*

$$\begin{tikzcd} a \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "f"] & a' \arrow[d, "f'"] \\ b \arrow[r, leftrightarrow, "\simeq"] & b' \end{tikzcd} \tag{7}$$

Proof. The diagram with arrows $a \leftarrow a', b \to b'$ defines the function $f' : a' \to b'$ uniquely. Now denote the isomorphisms as $\phi_{aa'} : a \leftrightarrow a' : \phi_{a'a}$ and $\phi_{bb'} : b \leftrightarrow b' : \phi_{b'b}$. Then the followings are equivalent: $\phi_{bb'} \circ f \circ \phi_{a'a} = f'$, $f \circ \phi_{a'a} = \phi_{b'b} \circ f'$, $f = \phi_{b'b} \circ f' \circ \phi_{aa'}$, $\phi_{bb'} \circ f = f' \circ \phi_{aa'}$. Each equations represents that the commutativity of four diagrams are equivalent. □

Lemma 53. *Consider the following diagram where the outer rectangle commutes.*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[r, "j"] \arrow[d, "h"] & c \arrow[d, "l"] \\ a' \arrow[r, "k"] & b' \arrow[r, "m"] & c' \end{tikzcd} \tag{8}$$

Then above diagram commute if either:

1. *the right square commutes and m is a monomorphism; or*
2. *the left square commutes and f is an epimorphism.*

Proof. Notice that two statements are dual, so we need to prove first one only. By the condition, we have $m \circ k \circ g = l \circ j \circ f = m \circ h \circ f$. Since m is a monomorphism, $k \circ g = h \circ f$. □

Theorem 54 (characterizing equivalences of categories)**.** *A functor defining an equivalence of categories is fully faithful and essentially surjective on objects. Assuming the axiom of choice, any fully faithful functor which is essentially surjective on objects defines an equivalence of categories.*

Proof. Consider $F : \mathsf{C} \leftrightarrow \mathsf{D} : G$ such that $\eta : 1_{\mathsf{C}} \simeq G \circ F$ and $\epsilon : 1_{\mathsf{D}} \simeq F \circ G$. For every object $d \in \mathsf{D}$, since $F(G(d)) \simeq d$, F is essentially surjective on objects. Now take two morphisms $f, g : c \to c'$ in C. If $F(f) = F(g)$, then $G(F(f)) = G(F(g))$. Now, due to the natural isomorphism, for every $f : c \to c'$, $G(F(f)) \circ \eta_c = \eta_{c'} \circ f$, thus $\eta_{c'} \circ f = \eta_{c'} \circ g$. Since $\eta_{c'}$ is isomorphism, taking its inverse to the left of above equality gives $f = g$. Therefore F, and symmetrically G, is faithful.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f=g"'] & G(F(c)) \arrow[d, "G(F(f))=G(F(g))"] \\ c' \arrow[r, "\eta_{c'}"] & G(F(c')) \end{tikzcd} \tag{9}$$

Finally, consider a morphism $k : F(c) \to F(c')$. Then $G(k) : G(F(c)) \to G(F(c'))$. Using lemma above, we have a unique morphism $h : c \to c'$ satisfying $\eta_{c'} \circ h = G(k) \circ \eta_c$. This commutation relation says that $G(k) = G(F(h))$, thus due to the faithfulness of G, $k = F(h)$, thus F is full.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "h"'] & G(F(c)) \arrow[d, "G(k)=G(F(h))"] \\ c' \arrow[r, "\eta_{c'}"] & G(F(c')) \end{tikzcd} \tag{10}$$

Now suppose that $F : \mathsf{C} \to \mathsf{D}$ is a fully faithful functor which is essentially surjective on objects. For each objects $d \in \mathsf{D}$, using the essentially surjectivity and the axiom of choice, take an object $G(d) \in \mathsf{C}$ and an isomorphism $\epsilon_d : F(G(d)) \simeq d$. Then by lemma above, for each morphism $l : d \to d'$ there is a unique morphism $m : F(G(d)) \to F(G(d'))$ satisfying $l \circ \epsilon_d = \epsilon_{d'} \circ m$. Since F is fully faithful, there is a unique morphism $G(d) \to G(d')$, which defines $G(l)$. This definition makes $\epsilon : F \circ G \Rightarrow 1_{\mathsf{D}}$ a natural transformation.

To show that this G is actually a functor, notice that since ϵ is a natural transformation, $1_d \circ \epsilon_d = \epsilon_d \circ F(G(1_d))$. Also, since $F(1_{G(d)})$ is an identity morphism on $F(G(d))$, $1_d \circ \epsilon_d = \epsilon_d \circ F(1_{G(d)})$, thus by above lemma, $F(1_{G(d)}) = F(G(1_d))$, and since F is fully faithfull, $1_{G(d)} = G(1_d)$.

$$\begin{tikzcd} F(G(d)) \arrow[r, "\epsilon_d"] \arrow[d, "F(G(1_d))=F(1_{G(d)})"'] & d \arrow[d, "1_d"] \\ F(G(d)) \arrow[r, "\epsilon_d"] & d \end{tikzcd} \tag{11}$$

Similarly, for morphisms $l : d \to d'$ and $l' : d' \to d''$, both $F(G(l') \circ$

$G(l))$ and $F(G(l' \circ l))$ satisfies the commutation relation, and thus $G(l') \circ G(l) = G(l' \circ l)$.

$$\begin{array}{ccc} F(G(d)) & \xrightarrow{\epsilon_d} & d \\ {\scriptstyle F(G(l') \circ G(l)) = F(G(l' \circ l))} \downarrow & & \downarrow {\scriptstyle l' \circ l} \\ F(G(d'')) & \xrightarrow{\epsilon_d} & d'' \end{array} \tag{12}$$

Finally, define $\eta_c : c \to F(G(c))$ by using the equation $\epsilon_{F(c)}^{-1} = F(\eta_c) : F(c) \to F(G(F(c)))$ and the fully faithfulness of F. Then for any $f : c \to c'$, consider the following diagram.

$$\begin{array}{ccccc} F(c) & \xrightarrow{F(\eta_c)} & F(G(F(c))) & \xrightarrow{\epsilon_{F(c)}} & F(c) \\ \downarrow {\scriptstyle F(f)} & & \downarrow {\scriptstyle F(G(F(f)))} & & \downarrow {\scriptstyle F(f)} \\ F(c') & \xrightarrow{F(\eta_{c'})} & F(G(F(c'))) & \xrightarrow{\epsilon_{F(c')}} & F(c') \end{array} \tag{13}$$

By the definition of η, the outer rectangle commutes. Also, since ϵ is a natural transformation, the right square commutes. Since $\epsilon_{F(c')}$ is an isomorphism, the left square commutes, and fully faithfulness of F makes possible to drop the initial F on the commuting diagram. Thus η is a natural transformation. □

Definition 55. A category C is **connected** if for any objects $c, c' \in \mathsf{C}$ there is a finite chain of morphisms $c \to c_1 \to \cdots \to c_n \to c'$.

2019- 07- 17

Proposition 56. *A connected groupoid is equivalent to the automorphism group of any of its objects as a category.*

Proof. Choose an object g in a connected groupoid G, and take a group $G = \mathsf{G}(g, g)$. Consider the inclusion $\mathsf{B}G \hookrightarrow \mathsf{G}$. Then this inclusion functor is fully faithful, and for every $g' \in G$, g is isomorphic to g', thus it is essentially surjective on objects. Therefore, by the theorem above, this functor defines an equivalence of category. □

Corollary 57. *In a path-connected space X, any choice of basepoint $x \in X$ gives an isomorphic fundamental group $\pi_1(X, x)$.*

Proof. Any space X has a fundamental groupoid $\Pi_1(X)$, and fixing a basepoint x, the group of automorphisms of the object $x \in \Pi_1(X)$ is a fundamental group $\pi_1(X, x)$. Thus $\pi_1(X, x) \simeq \Pi_1(X)$, and since the equivalence of category is equivalence relation, for any $x, x' \in X$, $\pi_1(X, x) \simeq \pi_1(X, x')$. Since these are one object category, there is a functor which is bijective on functors, and this gives the isomorphism between groups. Therefore, $\pi_1(X, x) \simeq \pi_1(X, x')$ in the sense of group theory also. □

Definition 58. A category C is **skeletal** if it contains just one objects in each isomorphism classes. The **skeleton** skC of a category C is the unique skeletal category up to isomorphism that is equivalent to C.

This category skC can be constructed from C by choosing one object in each isomorphism class in C and defining skC as a full subcategory of C. This gives an equivalence of categories since the inclusion functor is fully faithful and essentially surjective on objects, but the concept $\mathsf{sk} : \mathsf{CAT} \to \mathsf{CAT}$ is not a functor.

Example 59. Consider a left G-set $X : \mathsf{B}G \to \mathsf{Set}$. The **translation groupoid** $\mathsf{T}_G X$ is a category whose objects are the points of X and morphisms are $g : x \to y$ for $g \in G$ with $g \cdot x = y$. The objects of the skeleton $\mathsf{skT}_G X$ are the **orbits** of the group action. For $x \in X$, write its orbit O_x. Then since $\mathsf{skT}_G X \simeq \mathsf{T}_G X$, $\mathsf{skT}_G X(O_x, O_x) \simeq \mathsf{T}_G X(x, x) = G_x$, where G_x is the **stabilizer** of x, which is the set of group elements $g \in G$ satisfying $g \cdot x = x$. Now, since we may choose other elements from O_x, thus all the morphism sets $\mathsf{T}_G X(x, y) = G_x$ if $x, y \in O_x$. Also, the set of all morphisms with domain x is isomorphic to G. Therefore, $|G| = |O_x||G_x|$, which is the **orbit-stabilizer theorem**.

Definition 60. A category is **essentially small** if it is equivalent to a small category. A category is **essentially discrete** if it is equivalent to a discrete category.

Lemma 61. *Consider functors $F, G, H : \mathsf{C} \to \mathsf{D}$ and natural transformations $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$. Then there is a natural transformation $\beta \circ \alpha : F \Rightarrow H$ whose components are $(\beta \circ \alpha)_c = \beta_c \circ \alpha_c$. This is called a* ***vertical composition****.*

Proof. For any morphism $f : c \to c'$ in C, two squares of the following diagram commutes, because α, β are natural transformations.

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"] & G(c) \arrow[r, "\beta_c"] \arrow[d, "G(f)"] & H(c) \arrow[d, "H(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & G(c') \arrow[r, "\beta_{c'}"] & H(c') \end{tikzcd} \tag{14}$$

Thus the outer rectangle commutes, hence the composition $\beta_c \circ \alpha_c$ gives the natural transformation. $\square$

Corollary 62. *For a pairs of categories C, D, there is a* ***functor category*** *D^{C} whose elements are functors and morphisms are natural transformations.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=50, "H"'{name=H}] \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] & \mathsf{D} \end{tikzcd} \to \begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "H"'{name=H}] \arrow[from=F, to=H, Rightarrow, "\beta\circ\alpha"] & \mathsf{D} \end{tikzcd} \tag{15}$$

Proof. The lemma above shows the composition of natural transformations, and we only need to prove the associativity and existence of identity natural transformation. For associativity, since the natural transformation α is composed by the morphisms α_c, which has associativity, the composition of natural transformation also has the morphisms. For identity natural transformation between F and F, take α_c as the identity maps $F(c) \to F(c)$, which gives the natural transformation and whose composition with other natural transformation $\beta : F \Rightarrow G$ and $\gamma : H \Rightarrow F$ gives $\beta \circ \alpha = \beta$ and $\alpha \circ \gamma = \gamma$. □

Lemma 63. *Consider functors $F, G : \mathsf{C} \to \mathsf{D}, H, K : \mathsf{D} \to \mathsf{E}$ and natural transformations $\alpha : F \Rightarrow G, \beta : H \Rightarrow K$. Then there is a natural transformation $\beta * \alpha : H \circ F \Rightarrow K \circ G$, which is defined as $(\beta * \alpha)_c = K(\alpha_c) \circ \beta_{F(c)} = \beta_{G(c)} \circ H(\alpha_c)$. This is called a **horizontal composition**.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E} & \to & \mathsf{C} \arrow[r, bend left=50, "H\circ F"] \arrow[r, bend right=50, "K\circ G"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathsf{D} \end{tikzcd} \tag{16}$$

$$\begin{tikzcd} H(F(c)) \arrow[r, "\beta_{F(c)}"] \arrow[d, "H(\alpha_c)"] \arrow[dr, "(\beta*\alpha)_c"] & K(F(c)) \arrow[d, "K(\alpha_c)"] \\ H(G(c)) \arrow[r, "\beta_{G(c)}"] & K(G(c)) \end{tikzcd} \tag{17}$$

Proof. The square in above diagram commutes due to the naturality of $\beta : H \Rightarrow K$ applied on $\alpha_c : F(c) \to G(c)$. To show $\beta * \alpha$ satisfies the naturality, we need to show that $K(G(f)) \circ (\beta * \alpha)_c = (\beta * \alpha)_{c'} \circ H(F(f))$ for any morphism $f : c \to c'$ in C. Now consider the following diagram.

$$\begin{tikzcd} H(F(c)) \arrow[r, "H(\alpha_c)"] \arrow[d, "H(F(f))"] & H(G(c)) \arrow[r, "\beta_{G_c}"] \arrow[d, "H(G(f))"] & K(G(c)) \arrow[d, "K(G(f))"] \\ H(F(c')) \arrow[r, "H(\alpha_{c'})"] & H(G(c')) \arrow[r, "\beta_{G(c')}"] & K(G(c')) \end{tikzcd} \tag{18}$$

The right square commutes by the naturality of β, and the left square is the commutative diagram of α passed after the functor H, which hence commutes again. Therefore the outer rectangle commutes, which shows that $\beta * \alpha$ is a natural transformation.‘ □

Lemma 64 (Middle four interchange)**.** *Consider functors $F, G, H : \mathsf{C} \to \mathsf{D}$, $J, K, L : \mathsf{D} \to \mathsf{E}$, and natural transformations $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$, $\gamma : J \Rightarrow K, \delta : K \Rightarrow L$. Then $(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathsf{D} \arrow[r, bend left=60, "J"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathsf{E} & \to & \mathsf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathsf{D} \arrow[r, bend left=50, "J"] \arrow[r, bend right=50, "L"'] \arrow[r, phantom, "\Downarrow\delta\circ\gamma"] & \mathsf{E} & \to & \mathsf{C} \arrow[r, bend left=50, "J\circ F"] \arrow[r, bend right=50, "L\circ H"'] \arrow[r, phantom, "\Downarrow(\delta\circ\gamma)*(\beta\circ\alpha)"] & \mathsf{E} \end{tikzcd} \tag{19}$$

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathsf{D} \arrow[r, bend left=60, "J"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathsf{E} \arrow[r] & \mathsf{C} \arrow[r, bend left=60, "J\circ F"] \arrow[r, "K\circ G" description] \arrow[r, bend right=60, "L\circ H"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma*\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\delta*\beta"] & \mathsf{E} \arrow[r] & \mathsf{C} \arrow[r, bend left=60, "J\circ F"] \arrow[r, bend right=60, "L\circ H"'] \arrow[r, phantom, "\Downarrow(\delta*\beta)\circ(\gamma*\alpha)"] & \mathsf{E}
\end{tikzcd}
\tag{20}
$$

Proof. First, $((\delta \circ \gamma) * (\beta \circ \alpha))_c = L(\beta_c \circ \alpha_c) \circ (\delta \circ \gamma)_{F(c)} = L(\beta_c) \circ L(\alpha_c) \circ \delta_{F(c)} \circ \gamma_{F(c)}$ and $((\delta * \beta) \circ (\gamma * \alpha))_c = L(\beta_c) \circ \delta_{G(c)} \circ K(\alpha_c) \circ \gamma_{F(c)}$. Now $L(\alpha_c) \circ \delta_{F(c)} = \delta_{G(c)} \circ K(\alpha_c)$ because of the naturality of α, therefore we get the desired result.

$$
\begin{tikzcd}
J(F(c)) \arrow[r, "\gamma_{F(c)}"] \arrow[d, "J(\alpha_c)"] \arrow[dr, "(\gamma*\alpha)_c"] & K(F(c)) \arrow[r, "\delta_{F(c)}"] \arrow[d, "K(\alpha_c)"] & L(F(c)) \arrow[d, "L(\alpha_c)"] \\
J(G(c)) \arrow[r, "\gamma_{G(c)}"] \arrow[d, "J(\beta_c)"] & K(G(c)) \arrow[r, "\delta_{G(c)}"] \arrow[d, "K(\beta_c)"] \arrow[dr, "(\delta*\beta)_c"] & L(G(c)) \arrow[d, "L(\beta_c)"] \\
J(H(c)) \arrow[r, "\gamma_{H(c)}"] & K(H(c)) \arrow[r, "\delta_{H(c)}"] & L(H(c))
\end{tikzcd}
\tag{21}
$$

□

Definition 65. A **2-category** is a collection of

- objects, for example the categories C,
- 1-morphisms between pair of objects, for example the functors $F : \mathsf{C} \to \mathsf{D}$,
- 2-morphisms between parallel pairs of 1-morphisms, for example the natural transformations $\alpha : F \Rightarrow G$ with $F : \mathsf{C} \to \mathsf{D}$

which satisfies

- the objects and 1-morphisms form a category;
- the 1-morphisms and 2-morphisms form a category under vertical composition;
- the 1-morphisms and 2-morphisms form a category under horizontal composition;
- the middle four interchange law between vertical and horizontal composition holds.

Definition 66. An object $c \in \mathsf{C}$ is **initial** if the covariant functor $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$ is naturally isomorphic to the constant functor $* : \mathsf{C} \to \mathsf{Set}$ taking every objects to a singleton set. An object $c \in \mathsf{C}$ is **terminal** if the contravariant functor $\mathsf{C}(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is naturally isomorphic to the constant functor $* : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ taking every objects to a singleton set.

Definition 67. A covariant or contravariant functor F from a locally small category C to Set is **representable** if there is an object $c \in \mathsf{C}$ such that F is naturally isomorphic to $\mathsf{C}(c,-)$ or $\mathsf{C}(-,c)$. A **representation** of a functor F is a choice of object $c \in \mathsf{C}$ and, natural isomorphism $\mathsf{C}(c,-) \simeq F$ if F is covariant, and $\mathsf{C}(-,c) \simeq F$ if F is contravariant.

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Definition 68. A **universal property** of an object X in category C is a description of the covariant functor $\mathsf{C}(X,-)$ or of the contravariant functor $\mathsf{C}(-,X)$.

Example 69.

1. Consider the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$. This functor is represented by the group $\mathbb{Z}$. Indeed, there is a natural isomorphism $\mathsf{Group}(\mathbb{Z},-) \simeq U$ which takes the homomorphism $\phi \in \mathsf{Group}(\mathbb{Z},G)$ to an element $g \in U(G)$ where $g = \phi(1)$ bijectively. We thus say $\mathbb{Z}$ is the free group on a single generator.
2. For any unital ring R, consider the forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$. This functor is represented by the R-module R. The construction of a natural isomorphism $\mathsf{Mod}_R(R,-) \simeq U$ is very similar with above. We thus say R is the free R-module on a single generator.
3. Consider the forgetful functor $U : \mathsf{Ring} \to \mathsf{Set}$. This functor is represented by the ring $\mathbb{Z}[x]$. We thus say $\mathbb{Z}[x]$ is the free unital ring on a single generator.
4. Consider a functor $U(-)^n : \mathsf{Group} \to \mathsf{Set}$ which sends a group G to the set of n-tuples of elements of G. This functor is represented by the free group F_n on n generators.
5. Consider a functor $U(-)^n : \mathsf{Ab} \to \mathsf{Set}$ which sends an abelian group G to the set of n-tuples of elements of G. This functor is represented by the free abelian group $\oplus_n \mathbb{Z}$ on n generators.

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Theorem 70 (Yoneda lemma)**.** *Consider a locally small category C. For any functor $F : \mathsf{C} \to \mathsf{Set}$ and any object $c \in \mathsf{C}$, there is a bijection*

$$\mathsf{Nat}(\mathsf{C}(c,-),F) \simeq F(c) \tag{22}$$

which associates a natural transformation $\alpha : \mathsf{C}(c,-) \Rightarrow F$ to the element $\alpha_c(1_c) \in F(c)$. This correspondence is natural in both c and F.

Proof. Take a function $\Phi : \mathsf{Nat}(\mathsf{C}(c,-),F) \to F(c)$ which maps a natural transformation $\alpha : \mathsf{C}(c,-) \Rightarrow F$ to $\alpha_c(1_c)$ where $\alpha_c : \mathsf{C}(c,c) \to F(c)$. Now we want to define an inverse function $\Psi : F(c) \to \mathsf{Nat}(\mathsf{C}(c,-),F)$ which constructs a natural transformation $\Psi(x_c) : \mathsf{C}(c,-) \Rightarrow F$ for $x_c \in F(c)$. Define $\Psi(x_c)_d : \mathsf{C}(c,d) \to F(d)$ as $\Psi(x_c)_d(f) = F(f)(x_c)$ for $f : c \to d$. Now to show that $\Psi(x_c)$ is a natural transformation, we need to show that for some morphism $g : d \to e$ in C, $\Psi(x_c)_e \circ \mathsf{C}(c,g) = F(g) \circ \Psi(x_c)_d$. Take $f : c \to d$, then $\Psi(x_c)_e \circ \mathsf{C}(c,g)(f) = \Psi(x_c)_e(g \circ f) = F(g \circ f)(x_c)$ and $F(g) \circ \Psi(x_c)_d(f) = F(g) \circ F(f)(x_c) = F(g \circ f)(x_c)$, thus they are same. Now, $\Phi \circ \Psi(x_c) = \Psi(x_c)_c(1_c) = F(1_c)(x_c) = 1_c(x_c) = x_c$, Ψ is a right inverse of Φ. Choose a natural transformation $\alpha : \mathsf{C}(c,-) \Rightarrow F$. Then $\Psi \circ \Phi(\alpha)_d(f) = \Psi(\alpha_c(1_c))_d(f) = F(f)(\alpha_c(1_c))$. Now since α is natural, $\alpha_d \circ \mathsf{C}(c,f) = F(f) \circ \alpha_c$, thus $\Psi \circ \Phi(\alpha)_d(f) = \alpha_d \circ \mathsf{C}(c,f)(1_c) = \alpha_d(f)$, thus $\Psi \circ \Phi(\alpha) = \alpha$.

For the naturality of functor, we need to show that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{Nat}(\mathsf{C}(c,-),F) \arrow[r, "\Phi_F"] \arrow[d, "\mathsf{Nat}(\mathsf{C}(c,-),\beta)"] & F(c) \arrow[d, "\beta_c"] \\
\mathsf{Nat}(\mathsf{C}(c,-),G) \arrow[r, "\Phi_G"] & G(c)
\end{tikzcd} \tag{23}$$

Choose $\alpha \in \mathsf{Nat}(\mathsf{C}(c,-),F)$. Then the above statement is equivalent to $\beta_c(\Phi_F(\alpha)) = \Phi_G(\beta \circ \alpha)$. Now $\beta_c(\alpha_c(1_c)) = \beta_c \circ \alpha_c(1_c) = (\beta \circ \alpha)_c(1_c) = \Phi_G(\beta \circ \alpha)$.

For the naturality of object, we need to show that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{Nat}(\mathsf{C}(c,-),F) \arrow[r, "\Phi_c"] \arrow[d, "\mathsf{Nat}(\mathsf{C}(f,-),F)"] & F(c) \arrow[d, "F(f)"] \\
\mathsf{Nat}(\mathsf{C}(d,-),F) \arrow[r, "\Phi_d"] & F(d)
\end{tikzcd} \tag{24}$$

Choose $\alpha \in \mathsf{Nat}(\mathsf{C}(c,-),F)$. Then the above statement is equivalent to $F(f)(\Phi_c(\alpha)) = \Phi_d(\alpha \circ f)$. Now $F(f)(\Phi_c(\alpha)) = F(f)(\alpha_c(1_c))$ and $\Phi_d(\alpha \circ f) = (\alpha \circ f)_d(1_d) = (\alpha_d \circ f)(1_d) = \alpha_d(f) = F(f)(\alpha_c(1_c))$ due to the naturality of α. ☐

Corollary 71. *The functor $y : \mathsf{C} \hookrightarrow \mathsf{Set}^{\mathsf{C}^{op}}$ defined as $y(c) = \mathsf{C}(-,c)$ and $y(f : c \to d) = f_* : \mathsf{C}(-,c) \to \mathsf{C}(-,d)$ is a full embedding, and called a* ***covariant embedding****. The functor $y : \mathsf{C}^{op} \hookrightarrow \mathsf{Set}^{\mathsf{C}}$ defined as $y(c) = \mathsf{C}(c,-)$ and $y(f : c \to d) = f^* : \mathsf{C}(d,-) \to \mathsf{C}(c,-)$ is a full embedding, and called a* ***contravariant embedding****.*

Proof. The injectivity of object is trivial, thus we need to show the functors give the bijections $(C)(c,d) \simeq \mathsf{Nat}(\mathsf{C}(-,c),\mathsf{C}(-,d))$

and $\mathsf{C}(c,d) \simeq \mathsf{Nat}(\mathsf{C}(d,-),\mathsf{C}(c,-))$. Now since different morphisms $f,g : c \to d$ define distinct natural transformations $f_*,g_* : \mathsf{C}(-,c) \Rightarrow \mathsf{C}(-,d)$ and $f^*,g^* : \mathsf{C}(d,-) \Rightarrow \mathsf{C}(c,-)$, thus the injection is shown. For surjection, take a natural transformation $\alpha : \mathsf{C}(d,-) \Rightarrow \mathsf{C}(c,-)$. The Yoneda lemma says that this natural transformation corresponds to morphisms $f : c \to d$ where $f = \alpha_d(1_d)$. Now the natural transformation $f^* : \mathsf{C}(d,-) \Rightarrow \mathsf{C}(c,-)$ also takes $f^*_d(1_d) = f$, which shows that $f^* = \alpha$ by the bijectivity of Yoneda lemma. □

Corollary 72 (Cayley's theorem)**.** *Any group is isomorphic to a subgroup of a permutation group.*

Proof. Take a group G and consider its category form $\mathsf{B}G$. The image of the covariant Yoneda embedding $\mathsf{B}G \hookrightarrow \mathsf{Set}^{\mathsf{B}G^{\mathrm{op}}}$ is the right G-set G, acting by right multiplication. Then the Yoneda embedding gives the isomorphism between G and the endomorphism group of the right G-set G. Take the forgetful functor $\mathsf{Set}^{\mathsf{B}G^{\mathrm{op}}} \to \mathsf{Set}$. This identifies G with the subgroup of the automorphism group $\mathrm{Sym}(G)$ of the set G. □

HYUNJUN PARK

QUANTUM FIELD THEORY

2019- 03- 04

The topic of those courses are the interaction picture(QFT) and cubit(QIT); those contents will be summarized later.

2019- 03- 05

The ones we discussed on the Quantum Field Theory lecture which was on 2/27 and 3/4(on 2/25 we just had orientation) was about the perturbation theory. If the Hamiltonian has interaction term, then it frequently becomes very hard to calculate. For example, we consider the free Lagrangian

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m_0^2\phi^2 \tag{25}$$

which could be thought as the infinite number of harmonic oscillators, hence exactly solvable. Now put ϕ^4 interaction term:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 \tag{26}$$

gives the Hamiltonian

$$\mathcal{H} = \underbrace{\frac{1}{2}\pi^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{1}{2}m_0^2\phi^2}_{H_0} + \underbrace{\frac{\lambda_0}{4!}\phi^4}_{H_I}. \tag{27}$$

As we have done in quantum mechanics, we will assume λ_0 very small and take a series expansion to get the result(and use only finite terms if needed).

If we have a free Hamiltonian, then the two solutions ϕ_1 and ϕ_2 does not interacts: if they are solutions then $\phi_1 + \phi_2$ is also the solution. This shows that even though these solutions 'collides', those solution does not change their form. The existence of interaction term breaks this law, which is more realistic picture which might happen in experiment.

When we think the collision experiment, it is good to think the asymptotic states: at time $\pm\infty$. Suppose that $|\psi\rangle_{\text{in/out}}$ and $\phi\rangle_{\text{in/out}}$ are the same state at asymptotic limit $t = \mp\infty$: i.e.

$$e^{-iH(t-t_0)}|\psi\rangle_{\text{in/out}} \xleftrightarrow{t=\mp\infty} e^{-iH_0(t-t_0)}|\phi\rangle_{\text{in/out}} \tag{28}$$

for some reference time t_0. Then we can write as

$$|\psi\rangle_{\text{in/out}} = \lim_{t\to\mp\infty} e^{iH(t-t_0)}e^{-iH_0(t-t_0)}|\phi\rangle_{\text{in/out}} \tag{29}$$

Defining $\Omega(t) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}$ gives

$$|\psi\rangle_{\text{in/out}} = \Omega(\mp\infty)|\phi\rangle_{\text{in/out}} \tag{30}$$

Now $\Omega(\mp\infty)$ does not depends on t_0, therefore

Here for concreteness, it is important to assume that $\Omega(\mp\infty)$ exists. This kind of assumption could be nontrivial(and obviously there are enormous counterexamples) in mathematical sense, but we accept this fact in physical sense.

$$0 = \frac{\partial}{\partial t_0}\Omega(\mp\infty) = -iH\Omega(\mp\infty) + i\Omega(\mp\infty)H_0 \tag{31}$$

This implies

$$H\Omega(\mp\infty) = \Omega(\mp\infty)H_0 \tag{32}$$

Now,

$$H|\psi\rangle_{\text{in/out}} = H\Omega(\mp\infty)|\phi\rangle_{\text{in/out}} = \Omega(\mp\infty)H_0|\phi\rangle_{\text{in/out}} \tag{33}$$

If $|\phi\rangle_{\text{in/out}}$ is the eigenstate of H_0 with eigenvalue E, then we get

$$H|\psi\rangle_{\text{in/out}} = E|\psi\rangle_{\text{in/out}} \tag{34}$$

We might think this result as following sense. Since the energy spectrum of QFT is continuous unlike in QM, we can find the eigenstates of interacting and noninteracting system, which has equal energy. The operator $\Omega(\mp\infty)$ is the transformation operator between those states.

Now we want to solve the equation

$$(H_0 + V)|\psi\rangle = E|\psi\rangle \tag{35}$$

where E satisfies

$$H_0|\phi\rangle = E|\phi\rangle \tag{36}$$

Direct addition gives

$$(E - H_0)|\psi\rangle = (E - H_0)|\phi\rangle + V|\psi\rangle \tag{37}$$

and dividing both side by $E - H_0$ gives the solution. But since $E - H_0$ has singular point, this is impossible. To avoid this, we put the infinitesimal imaginary constant $i\epsilon$: then,

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon}V|\psi\rangle \tag{38}$$

Here $\epsilon > 0$. This gives the correct boundary condition: detector detects only "after" given a fire. This is called **Lippmann-Schwinger Equation**.

Define a transfer matrix T as

$$T|\phi\rangle = V|\psi\rangle \tag{39}$$

then

$$\begin{aligned} T|\phi\rangle &= V|\psi\rangle = V|\phi\rangle + V\frac{1}{E - H_0 + i\epsilon}V|\psi\rangle \\ &= V|\phi\rangle + V\frac{1}{E - H_0 + i\epsilon}V|\phi\rangle + V\frac{1}{E - H_0 + i\epsilon}V\frac{1}{E - H_0 + i\epsilon}V|\psi\rangle \\ &= \cdots \end{aligned} \tag{40}$$

so

$$T = V + V\frac{1}{E - H_0 + i\epsilon}V + V\frac{1}{E - H_0 + i\epsilon}V\frac{1}{E - H_0 + i\epsilon}V + \cdots \quad (41)$$

Transfer matrix basically checks the probability of the scattering for the non-interacting eigenstate basis. Therefore, the expansion of T by V and green's function $\frac{1}{E-H_0+i\epsilon}$ implies that the transfer occurs as an interaction, or an interaction then propagation then an interaction, or interaction, propagation, interaction, propagation, then interaction, and so on.

HYUNJUN PARK

QUANTUM PHASE TRANSITION

2019- 03- 11

Here we are treating the following hamiltonian:

$$H_I = \underbrace{-Jg\sum_i \hat{\sigma}_i^x}_{H_0} \underbrace{-J\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z}_{H_1} . \tag{42}$$

For the case $g \gg 1$, the fist term dominates, hence we found the ground state as

$$|0\rangle = \prod_i |\rightarrow\rangle_i, \tag{43}$$

where

$$|\rightarrow\rangle_i = \frac{|\uparrow\rangle_i + |\downarrow\rangle_i}{\sqrt{2}}, \quad |\leftarrow\rangle_i = \frac{|\uparrow\rangle_i - |\downarrow\rangle_i}{\sqrt{2}} \tag{44}$$

and for the case $g \ll 1$, the second term dominates, hence we found the ground state as

$$|\uparrow\rangle = \prod_i |\uparrow\rangle_i, \quad |\downarrow\rangle = \prod_i |\downarrow\rangle_i.$$

First we will treat the $g \gg 1$ case, using the perturbation calculation. On the system of M sites with periodic boundary condition, we get

$$\begin{aligned}
E_0^{(0)} &= -MJg \\
E_0^{(1)} &= -J\sum_{\langle ij\rangle} \langle 0|\hat{\sigma}_i^z \hat{\sigma}_j^z|0\rangle = 0 \\
|\psi_0^{(1)}\rangle &= J\sum_{\langle ij\rangle}\sum_{m\neq n} \frac{1}{E_m^{(0)} - E_0^{(0)}} |\psi_m^{(0)}\rangle\langle\psi_m^{(0)}|\hat{\sigma}_i^z \hat{\sigma}_j^z|0\rangle \\
&= \frac{1}{4g}\sum_{\langle ij\rangle} |i,j\rangle \\
E_0^{(2)} &= -J\sum_{\langle ij\rangle} \langle 0|\hat{\sigma}_i^z \hat{\sigma}_j^z|\psi_0^{(1)}\rangle \\
&= -\frac{J}{4g}\sum_{\langle ij\rangle}\sum_{\langle i'j'\rangle} \langle 0|\hat{\sigma}_i^z \hat{\sigma}_j^z|i',j'\rangle \\
&= -\frac{MJ}{4g}
\end{aligned} \tag{45}$$

Here,

$$|i,j\rangle = |\leftarrow\rangle_i|\leftarrow\rangle_j \prod_{n\neq i,j} |\rightarrow\rangle_n. \tag{46}$$

Therefore,

$$E_0 = -MJg\left(1 + \frac{J}{4g^2} + \mathcal{O}\left(\frac{1}{g^3}\right)\right). \tag{47}$$

But what about the excited states? For the lowest excited states,

$$|i\rangle = |\leftarrow\rangle_i \prod_{n\neq i} |\rightarrow\rangle_i, \tag{48}$$

there are M of them, degenerated with energy $E_0 + 2Jg$, if $g = \infty$. We call this *single(one)-particle state*. Similarly, for the second lowest excited states,

$$|i, j\rangle = | \leftarrow\rangle_i | \leftarrow\rangle_j \prod_{n \neq i,j} | \rightarrow\rangle_n, \tag{49}$$

there are $M(M-1)/2$ of them, degenerated with energy $E_0 + 4Jg$, if $g = \infty$. We call this *two-particle state*. In general there are $M!/(M-p)!p!$ many of p-particle states with energy $E_0 + 2pJg$.

Now for one-particle state, one-dimensional ring case, We need to calculate the degenerated perturbation theory, which means, we need to diagonalize the matrix

$$\langle i|H_1|j\rangle = -J \sum_{\langle nm\rangle} \langle i|\hat{\sigma}^z_n \hat{\sigma}^z_m|j\rangle. \tag{50}$$

But only

$$\langle i|H_1|i+1\rangle = -J \tag{51}$$

and all the other terms are zero. This is actually 1-dimensional tight-binding model, which can be diagonalized by taking the basis as

$$|k\rangle = \frac{1}{\sqrt{M}} \sum_j e^{ikx_j}|j\rangle. \tag{52}$$

which satisfies

$$\langle k|H_1|k\rangle = -2J\cos(ka) \tag{53}$$

where a is the lattice constant. Now we can calculate the energy perturbation term.

$$\begin{aligned}
E_1^{(0)} &= -MJg + 2Jg \\
E_1^{(1)} &= \langle k|H_1|k\rangle = -2J\cos(ka) \\
E_1^{(2)} &= -\sum_{m\neq 1} \frac{|\langle \psi_m^{(0)}|H_1|k\rangle|^2}{E_m^{(0)} - E_1^{(0)}} \\
&= -\frac{J^2}{4Jg} \sum_{l<m<n} \left| \sum_{\langle ij\rangle} \langle l, m, n|\hat{\sigma}_i \hat{\sigma}_j|k\rangle \right|^2 \\
&= -\frac{J}{4g} \left(\sum_{\substack{l,m=l+1,\\ n=l+2}} |\langle l+2|k\rangle + \langle l|k\rangle|^2 + \sum_{\substack{l,m=l+1,\\ n\neq l+2, n\neq l-3}} |\langle n|k\rangle|^2 \right) \\
&= -\frac{J}{4g} \left(4\cos^2(ka) + (M-4) \right) \\
&= -\frac{J}{4g} \left(M - 4\sin^2(ka) \right) \\
&= -\frac{J}{4g} \left(M - 2\left(1 - \cos(2ka)\right)\right)
\end{aligned}$$

Therefore we get

$$E_1 = Jg\left(-M + 2 - \frac{2}{g}\cos(ka) + \frac{1}{2g^2}(1 - \cos(2ka)) - \frac{M}{4g^2} + \mathcal{O}\left(\frac{1}{g^3}\right)\right) \tag{54}$$

The *quasiparticle residue*, $\mathcal{A}$, is the overlap between the actual one-particle state at momentum $k = 0$, and that obtained by creating a particle in the ground state by the particle creation operator:

$$\mathcal{A} := |\langle k = 0|\hat{\sigma}^z(k)|0\rangle| \,. \tag{55}$$

Here we set

$$\hat{\sigma}^z(k) = \frac{1}{\sqrt{M}}\sum_j e^{ikx_j}\hat{\sigma}^z_j. \tag{56}$$

What is quasiparticle residue? In the book *Quantum Phase Transition* by Subir Sachdev, the quasiparticle residue is the residue (as the function of ω) of the response function $\chi(k, \omega)$. This function can be directly observed by ARPES. Physically, we can think this concept as the concept of "effective mass": the interaction modifies the fermion into the quasiparticle, which changes its physical properties, and thus also the quasiparticle residue.

Notice that

$$|0\rangle = |0^{(0)}\rangle + \frac{1}{4g}\sum_{\langle ij\rangle}|i, j\rangle + \mathcal{O}\left(\frac{1}{g^2}\right) \tag{57}$$

Here $|0^{(0)}\rangle$ is the original ground state where all the spins pointing right. Now,

$$\begin{aligned}\hat{\sigma}^z(k)|0^{(0)}\rangle &= \frac{1}{\sqrt{M}}\sum_j e^{ikx_j}\hat{\sigma}^z_j|0^{(0)}\rangle = \frac{1}{\sqrt{M}}\sum_j e^{ikx_j}|j\rangle \\ \hat{\sigma}^z(k)|i, j\rangle &= \frac{1}{\sqrt{M}}\sum_n e^{ikx_n}\hat{\sigma}^z_n|i, j\rangle\end{aligned} \tag{58}$$

But notice that the three-particle state has g^{-1} order, and thus the three-particle state does not contributes to g^{-1} order term. Therefore we only need to think about the one-particle state:

$$\hat{\sigma}^z(k)|i, j\rangle = \frac{1}{\sqrt{M}}\left(e^{ikx_i}|j\rangle + e^{ikx_j}|i\rangle\right) \tag{59}$$

Now,

$$\begin{aligned}\langle k|\hat{\sigma}^z(k)|0\rangle &= 1 + \frac{1}{4gM}\sum_{\langle ij\rangle, n} e^{-ikx_n}\langle n|\left(e^{ikx_i}|j\rangle + e^{ikx_j}|i\rangle\right) + \mathcal{O}\left(\frac{1}{g^2}\right) \\ &= 1 + \frac{1}{4gM}\sum_{\langle ij\rangle}\left(e^{ika} + e^{-ika}\right) + \mathcal{O}\left(\frac{1}{g^2}\right) \\ &= 1 + \frac{\cos(ka)}{2g} + \mathcal{O}\left(\frac{1}{g^2}\right)\end{aligned}$$

Tending $k \to 0$ gives

$$\mathcal{A} = 1 + \frac{1}{2g} + \mathcal{O}\left(\frac{1}{g^2}\right) \tag{60}$$

HYUNJUN PARK

BOSONIZATION THEORY

2019- 03- 18

There is a four-day lecture series from professor Masaki Oshikawa.

The Nobel prize on 2016 was given to J. Kosterlitz, D. Thouless and D. Haldane, due to the discovery of topological phenomenons: Kosterlitz-Thouless transition, Haldane gap, and the TKNN formula, which is the topological quanta respective to quantum Hall effect. In this talk, the relation between Haldane gap and K-T transition is discussed.

In most of the cases, phase transition occurs due to the spontaneous symmetry breaking, or order-disorder transition in the other words. For example we have 2-dimensional Ising model. In high temperature regime the system has Z_2 symmetry, but in low temperature the Z_2 symmetry.

For XY model we expect the same explanation. However there exists a **Mermin-Wigner's theorem**, which says that in $d \leq 2$ dimensional system, there is no spontaneous symmetry breaking of continuous symmetries at $T > 0$. Since XY symmetry has $U(1) \simeq O(2)$ symmetry, it cannot be broken on finite temperature. However there possibly exists the **topological phase transition**. J. Kosterlitz and D. Thouless discovered that on low temperature, the correlation $\langle \vec{s}_i \cdot \vec{s}_j \rangle \sim \left(\frac{1}{r}\right)^\eta$ at low temperature and $\langle \vec{s}_i \cdot \vec{s}_j \rangle \sim \exp\left(-\frac{r}{\xi}\right)$.

On the XY model, there exist some structures which has nonzero winding number: vortex and antivortex. Due to the nontrivial winding number, it needs a massive energy to generate those structure. Furthermore, if a vortex has winding number 1 and an antivortex has winding number -1, then they emerges and become nothing. In this sense we can say that the vortex and antivortex attracts each other.

See `https://johncarlosbaez.wordpress.com/2016/10/07/kosterlitz-thouless-transition/` for some animations.

In low temperature, there is not enough energy to separate a vortex and antivortex pair, so each are paired. In high temperature, however, they moves freely, and there could be more vortices then antivortices (or converse) since we have enough energy fluctuation to create number of (anti)vortex individually.

The calculation of RG flow also shows the result. Drawing the RG flow on the T-μ, where μ is the (anti)vortex fugacity, we also get the critical value $\eta_c = 1/4$. Thus we get the non symmetry breaking type phase transition, which is now called the topological phase transition.

Realization of classical XY model was done with the thin film of Helium-4, which has the superfluid-normal fluid transition at certain temperature. Here the phase of wavefunction works as the vectors on XY model. The correlation function of superfluid follows the power low, and there exists a superfluid density, $\rho_s = \frac{m^2 k_B T}{2\pi\eta\hbar^2}$. Since for the low temperature η is finite but at high temperature $\eta = 0$, we get a universal jump $\frac{\rho_c(T_c - 0)}{T_c} = \frac{m^2 k_B}{2m\eta_c\hbar^2}$. It looks like a first order transition

Prof. Oshikawa mentioned that someone thinks the superfluid happens because of long range Bose-Einstein condensation, but this is not true because there is no symmetry breaking of 2-dimensional system on finite temperature.

due to the discontinuous drop of superfluid density, it is actually not a first order transition.

Now we move on to the 1-dimensional quantum Heisenberg antiferromagnet. For $S = 1/2$ case, the result is well-known as the **Bethe ansatz**, which gives the exact solutions using the magnon excitation. Even though these solutions are hard to treat, now we know that there is no long-range order on even $T = 0$ case, due to the quantum fluctuation, even though it has gapless excitations. Also the correlation function is written as $\langle \vec{s}_i \cdot \vec{s}_j \rangle \sim \left(\frac{1}{r}\right)^{\eta}$.

Long-range order implies that there exists a gapless excitation spectrum, due to the **Goldstone theorem**.

Most of the physicists believed that this is true for any spin S. However, Haldane "conjectured" that if $S \in \mathbb{Z} + \frac{1}{2}$ then there is a gapless spectrum with $\langle \vec{s}_i \cdot \vec{s}_j \rangle \sim \left(\frac{1}{r}\right)^{\eta}$, but if $S \in \mathbb{Z}$ then there is an excitation gap, called **Haldane gap**, with $\langle \vec{s}_i \cdot \vec{s}_j \rangle \sim \exp\left(-\frac{r}{\xi}\right)$. Here it is noticable that $S \in \mathbb{Z} + \frac{1}{2}$ case is very similar with low T case on classical 2-dimensional XY model, where $S \in \mathbb{Z}$ case is very similar with high T case.

Prof. Oshikawa mentioned that this is called a conjecture not because it is not proven but because it is quite unexpected result: at least Prof. Haldane thinks that it is proven, in the sense of condensed matter physics. He also mentioned that mentioning Haldane's conjecture a 'conjecture' in front of Haldane is not a good choice - he does not like the naming of the 'conjecture', because he knew he was right.

The common understanding of the Haldane's conjecture is from the $O(3)$ nonlinear sigma model, which was quite common concept for particle physicists but not for the condensed matter physicists. We write down the action

$$\mathcal{A} = \mathcal{A}_0 + i\theta\mathcal{Q}, \tag{61}$$

where

$$\begin{aligned} \mathcal{A}_0 &= \frac{1}{2g}\int dxd\tau \left(\partial_\mu \vec{n}\right)^2 \\ \mathcal{Q} &= \frac{1}{8\pi} \in dxd\tau\epsilon_{\mu\nu}\vec{n}\left(\partial_\mu\vec{n} \times \partial_\nu\vec{n}\right) \end{aligned} \tag{62}$$

Here we call $i\theta\mathcal{Q}$ term a **topological term**. If we write down the partition function

$$\mathcal{Z}_0 = \int \mathcal{D}\vec{n}\, e^{-\mathcal{A}_0}, \tag{63}$$

then we get the 2-dimensional classical Heisenberg model. For quantum properties, we need to put the $\mathcal{Q}$ term. Indeed, we will see that $\theta = 2\pi S$ is the topological angle for the 1-dimensional quantum antiferromagnetic Heisenberg model with spin S, and thus $e^{i\theta\mathcal{Q}} = (-1)^{2S\mathcal{Q}}$. Here we can see that $S \in \mathbb{Z}$ does not have any topological term and always disordered, which is equivalent to 2-dimensional class Heisenberg model. In $S \in \mathbb{Z} + \frac{1}{2}$ case, however, we have the quantum critical point, which means that the gap is closed.

But indeed, the actual origin of the Haldane's conjecture is the Tomonaga-Luttinger theory.

First we see the XXZ chain model with spin half,

$$H = \sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z s_{j+1}^z. \tag{64}$$

Using the Bethe ansatz gives the exact solution for this model. This solution shows that this model is gapped for $\Delta < -1$(anti-ferromagnetic phase, has $\eta = 0$ for $\Delta = -1$) and $\Delta > 1$(ferromagnetic phase, has $\eta = 1$ for $\Delta = 1$), and gapless for $-1 \leq \Delta \leq 1$(has $\eta = 1 - \frac{\cos^{-1}(\Delta)}{\pi}$). Since quantum spin is also compact as classical spin, there exists vortices on 1+1 dimension, which makes us possible to observe the BKT transition.

This model has Tomonaga-Luttinger universality, where the Fermi velocity is $v_F = \frac{\pi\sqrt{1-\Delta^2}}{2\cos^{-1}(\Delta)}$ and the Luttinger parameter is $K = \frac{\pi}{2\cos^{-1}(\Delta)}$.

Now, in 1+1 dimensional picture, if there is a single vortex, and if it moves, then one spin changes its motion direction, for example, from clockwise to anticlockwise. This makes the phase difference, which is determined by the system itself, which is called the **Berry phase**. This makes arguable for Haldane's conjecture. Here we have a concept of the argue. For $S = 1/2$ case, the Berry phase is $\pi \mod 2\pi$, hence the sign of $|\psi\rangle$ must be flipped as the vertex moves, for one spin. This does not allows the single vortex. For integer spin case, the Berry phase is $0 \mod 2\pi$, which makes different nature. Indeed, we have phase transition at $\eta = 1/4$, and $\Delta = 1$ thus belongs to high T phase, and gapless.

In the afternoon session, we have tried to follow up what Haldane did in his paper, which have written about the Haldane's conjecture with Tomonaga-Luttinger liquid theory. Tomonaga-Luttinger liquid(TLL) is the universal description of 1-dimension quantum manybody problems, which is equivalent to the relativistic field theory of free bosons in 1+1D. For example, 1-dimensional Hubbard model of electrons can be changed into TLL, by using the low energy bosonization.

Quantum spin chain, which we are curious in, can be thought as the interacting many boson system by following argument. We have operators S_j^z, S_j^+, S_j^- for each site j. Then we may think that $S^{+(-)}$ is the creation(annihilation) operator for S_j^z eigenvalues, $-S, -S+1, \cdots, S$, which can be also thought as the state with $0, 1, \cdots, 2S$ boson particles on the site. Indeed, by taking $S_j^{+(-)}$ as $\phi(x)^{+(-)}$, the field creation operator, then because $[S_j^+, S_k^+] = [\psi^+(x), \psi^+(y)] = 0$, we get the bosonic operators. Notice that indeed the number of particles on each sites has upper bound: $n_j \leq S$.

Now, to make the bosonic 1-dimensional chain with interaction to TLL, we need to take the low energy limit, and do the process which is called **"bosonisation of bosons"**. We now forget the lattice and take continuous 1-dimensional space, and describe the collective motion of bosons. If there are bosons in the positions x_j, then we think the **labelling field**, $\phi_l(x)$, which is a monotonically increasing function with $\phi_l(x_j) = 2\pi j$. Then the density can be written as

$$\rho(x) = \sum_j \delta(x - x_j) = \sum_n [\partial_x \phi_l(x)]\, \delta(\phi_l(x) - 2\pi n) \tag{65}$$

Now using the Poisson summation formula,

$$\sum_{p\in\mathbb{Z}} e^{ip\phi_l} = 2\pi \sum_n \delta(\phi_l - 2\pi n) \tag{66}$$

we get

$$\rho(x) = \frac{\partial_x \phi_l(x)}{2\pi} \sum_{p\in\mathbb{Z}} e^{ip\phi_l(x)}. \tag{67}$$

Now, notice that we can write

$$\phi_l(x) = 2\pi\rho_0 - 2\phi(x), \tag{68}$$

where ρ_0 is the average density in the ground state and $\phi(x)$ is the fluctuation of the density. Then

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\partial_x\phi\right] \sum_{p\in Z} e^{2ip[\pi\rho_0 x - \phi(x)]} \tag{69}$$

Averaging the density over length scale $L \gg \rho_0^{-1}$ gives, due to the fast oscillation,

$$\bar{\rho}(x) \sim \rho_0 - \frac{1}{\pi}\partial_x\phi \tag{70}$$

But we have some arguments using $p \neq 0$.

Now we write down the annihilation operator of boson, $\psi(x)$. Then we have

$$\rho(x) = \psi^\dagger(x)\psi(x) \tag{71}$$

and therefore we may write

$$\psi(x) = e^{i\theta(x)}\sqrt{\rho(x)}. \tag{72}$$

From the boson commutation relation, $[\psi(x), \psi^\dagger(x')] = \delta(x - x')$, we have

$$[\rho(x), e^{-i\theta(x')}] = \delta(x - x')e^{-i\theta(x')} \tag{73}$$

this gives

$$[\rho(x), \theta(x')] = i\delta(x - x') \tag{74}$$

and using

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\partial_x\phi\right] \sum_{p\in\mathbb{Z}} e^{2ip[\pi\rho_0 - \phi(x)]} \tag{75}$$

we get

$$[\partial_x\phi(x), \theta(x')] = -i\pi\delta(x - x') \tag{76}$$

and so

$$[\phi(x), \theta(x')] = -\frac{i\pi}{2}\,\mathrm{sgn}(x - x') \tag{77}$$

Thus, in this sense, we can think that $\phi(x)$, the charge-density wave phase, and $\theta(x)$, the quantum mechanical phase of microscopic wave function, are in dual relation.

Considering the Free boson model, we need to write down the Hamiltonian as

$$\begin{aligned}\mathcal{H}_0 &= \sum_j \frac{p_j^2}{2m} \\ &= \frac{1}{2m}\int (\partial_x\psi^\dagger)(\partial_x\psi)dx \\ &\simeq \frac{\rho_0}{2m}\int (\partial_x\theta)^2 dx + \cdots\end{aligned}$$

Thus Hamiltonian of the system depends on θ term.

Now we consider the interaction. Assuming the interaction is δ-function like, we get

$$\mathcal{H}_I = \frac{u}{2}\int \rho(x)^2 dx \tag{78}$$

where

$$\rho(x) \simeq \rho_0 - \frac{1}{\pi}(\partial_x\phi) + \cdots \tag{79}$$

and thus

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \simeq \int \frac{\rho_0}{2m}(\partial_x\theta)^2 + \frac{u}{2\pi^2}(\partial_x\phi)^2 + \cdots \tag{80}$$

considering boundary condition and ignoring constant term.

2019- 03- 19

Yesterday, we have considered the Hamiltonian, which can be transformed into the Lagrangian form as

$$\mathcal{L} = \frac{1}{2\pi k}(\partial_\mu\phi)^2 \tag{81}$$

which is same with the Lagrangian of vibration of the string, which is actually a gapless system. But in the real system, most of the systems are gapped, and the gaplessness comes out on a critical point. Therefore we often say the gapless point as "critical point". Therefore, in some sense, the gaplessness of spin 1/2 chain is rather surprising then the gaplessness of spin 1 chain.

Now we come back to the original model, which is the spin chain model. In this model, S_j^α is defined on lattice points, whose lattice constant is a. Then, from previous argument,

$$S_j^z = -S + n_j = -S + \int_{(j-\frac{1}{2})a}^{(j+\frac{1}{2})a} \rho(x)dx \tag{82}$$

where

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\partial_x\phi\right]\sum_{p\in\mathbb{Z}} e^{2ip[\pi\rho_0 x - \phi(x)]} \tag{83}$$

Now $p = 0$ term gives the term

$$\rho_0 a - \frac{1}{\pi}[\phi((j+\frac{1}{2})a) - \phi((j-\frac{1}{2})a)] \tag{84}$$

What we get when $p \neq 0$? Considering $\phi(x)$ varying very linearly and slowly, we may consider $\partial_x\phi$ and $\phi(x)$ be a small constant when we take integral. Then we get

$$\frac{1}{2\pi pi}\left[e^{2pi\left[\pi\rho_0\left(j+\frac{1}{2}\right)a-\phi\left(\left(j+\frac{1}{2}\right)a\right)\right]} - e^{2pi\left[\pi\rho_0\left(j-\frac{1}{2}\right)a-\phi\left(\left(j-\frac{1}{2}\right)a\right)\right]}\right] \tag{85}$$

Now notice that the ground state of spin-S antiferromagnetic chain must have $\langle S_j^z\rangle = 0$, considereing Néel state. This implies $\langle n_j\rangle = S$, i. e. $\rho_0 a = S$. Here the spin quantum number enters. Putting this value, we write $p \neq 0$ term as

$$\frac{1}{2\pi pi}\left[e^{2pi\left[\pi S\left(j+\frac{1}{2}\right)-\phi\left(\left(j+\frac{1}{2}\right)a\right)\right]} - e^{2pi\left[\pi S\left(j-\frac{1}{2}\right)-\phi\left(\left(j-\frac{1}{2}\right)a\right)\right]}\right] \tag{86}$$

From this, we can calculate the S_j^z operator using $\phi(x)$ field. We have two possible cases: $S \in \mathbb{Z}$ and $S \in \mathbb{Z}+\frac{1}{2}$, which gives quite different results: if $S \in \mathbb{Z}$, then

$$S_j^z = A\partial_x\phi\left[1 + \sum_{p\geq 0} C_p\cos(2p\phi(x))\right] \tag{87}$$

and if $S \in \mathbb{Z}+\frac{1}{2}$, then

$$\begin{aligned} S_j^z = B\partial_x\phi\Bigg[1 &+ \sum_{p=2,4,6,\cdots} D_p\cos(2p\phi(x)) \\ &+ (-1)^j\sum_{p=1,3,5,\cdots} D_p\sin(2p\phi(x))\Bigg] \end{aligned} \tag{88}$$

where the $\mathcal{O}((\partial_x\phi)^2)$ terms are ignored here.

Using this, we can represent the operator $S_j^z S_{j+1}^z$. Here, it is good method to use the **operator product expansion**, which is the expansion of the product of two operators in short distance to the expansion by a single point operator. Then we get

This job can be done by writing down the field into creation and annihilation operators, or by using the Wick's theorem.

$$S_j^z S_{j+1}^z \sim \underbrace{A + B(\partial_x\phi)^2}_{\mathcal{H}_{\text{TLL}}} + \underbrace{\sum_{p\geq 1} C_p\cos(2p\phi(x)) + \cdots}_{\mathcal{H}'} \tag{89}$$

where $S \in \mathbb{Z}$ and

$$S_j^z S_{j+1}^z \sim \underbrace{A + B(\partial_x\phi)^2}_{\mathcal{H}_{\text{TLL}}} + \underbrace{(-1)^j\sum_{p=1,3,5,\cdots} C_p\sin(2p\phi(x))}_{\text{Vanishes by }\sum_j}$$

$$+ \underbrace{\sum_{p=2,4,6,\cdots} C_p \sin(2p\phi(x)) + \cdots}_{\mathcal{H}'}$$

where $S \in \mathbb{Z} + \frac{1}{2}$. Here $\mathcal{H}_{\text{TLL}}$ is the Tomonaga-Luttinger liquid Hamiltonian.

Now the Hamiltonian-included term $\cos(2p\phi(x))$ can be written as $\frac{1}{2}\left[e^{2ip\phi(x)} + e^{-2ip\phi(x)}\right]$. Thus we are curious about the fact that what $e^{2ip\phi(x)}$ does on the state. To do this, first we know

$$[\phi(x), \theta(x')] = -\frac{i\pi}{2}\,\text{sgn}(x - x') \tag{90}$$

thus

$$[e^{2ip\phi(x)}, \theta(x')] = \pi p\,\text{sgn}(x - x')e^{2ip\phi(x)} \tag{91}$$

Now suppose $e^{2ip\phi(y)}$ is applied on $|\theta(x) = 0\rangle$ state. Then

$$\theta(x)\left[e^{2ip\theta(y)}|\theta(x) = 0\rangle\right] = -\pi p\,\text{sgn}(y - x)[e^{2ip\theta(y)}|\theta = 0\rangle] \tag{92}$$

This means we get the state whose phase factor $\theta(x)$ changes oppositely with the boundary point y. Thus by the time evolution, we get the vortex-like structure.

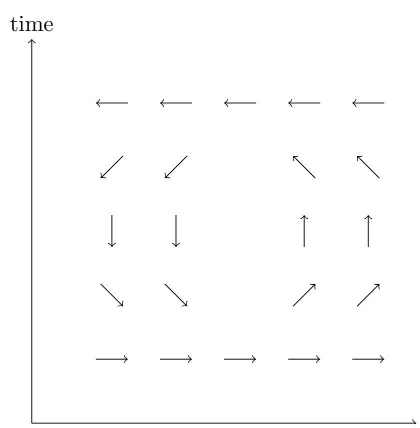

Figure 1: The vortex-like structure appearing when $p = 1$

Notice that this vortex-like structure only appears when $p = 1$, thus for $S \in \mathbb{Z} + \frac{1}{2}$ case, there is no such a vortex structure.

Now come back to the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{TLL}} + \sum_p \lambda_p \cos(2p\phi) \tag{93}$$

If we calculate the correlation of interaction, then we get

$$\langle\cos(2p\phi(x))\cos(2p\phi(y))\rangle_{\text{TLL}} \sim |x - y|^{-2p^2\kappa} \tag{94}$$

where κ is the Luttinger parameter which is determined by the interaction. This is because the scaling parameter of $e^{\pm 2ip\phi}$ is $p^2\kappa$. Thus, if we calculate the Renormalization of the action

$$S = \int dx d\tau \mathcal{L} \tag{95}$$

then the λ_p becomes $\lambda_p l^{2-p^2\kappa}$. Here 2 is the spacetime dimension. Thus, if $p^2\kappa > 2$ then $\lambda_p \to 0$, which is called **irrelevant**, and if $p^2\kappa < 2$ then $\lambda_p \to \infty$, which is called **relevant**.

Now consider the Heisenberg antiferromagnetic chain, which is $\Delta = 1$ case for XXZ model. This system has $\kappa = \frac{1}{2}$, Now if $p = 1$ then $p^2\kappa = \frac{1}{2} < 2$, thus $e^{2i\phi}$ is relevant and $\cos(2\phi)$ term dominates. Therefore the lowest value of Lagrangian is fixed at $\phi = \pi$, which implies the gapped system. However, if $p = 2$, then $p^2\kappa = 2$, which is the marginal case: $e^{4i\phi}$ can be relevant or not. If it is relevant, then the system is gapped; if it is irrelevant, then the system is gapless.

The Tomonaga-Luttinger liquid with $k = \frac{1}{2}$ is equivalent to the $SU(2)$ Wess-Zumino-Witten model with level 1.

However, by using the Lieb-Schultz-Mattis theorem, we can show that the $p = 2$ case, i.e. $S \in \mathbb{Z} + \frac{1}{2}$ case, is gapless.

Until here, we only thought the operators $e^{2ip\phi}$ terms, but $e^{iq\theta}$ terms also exists. To satisfy the boundary condition $\theta \sim \theta + 2\pi$, we must have $q \in \mathbb{Z}$. In this case, the scaling dimension of $e^{iq\theta}$ is $\frac{q^2}{4\kappa}$, which can be relevant. However, if the original system has $U(1)$ symmetry, i.e. conservation of S^z, then $e^{iq\theta}$ term cannot appear because it breaks $U(1)$ symmetry.

We can play the similar argument for ϕ field. Recall

$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x) \tag{96}$$

then

$$\phi_l(x + \delta x) = 2\pi\rho_0(x + \delta x) - 2\phi(x + \delta x) \tag{97}$$

Now by translation $x \mapsto x + \delta x$ if $\phi_l(x)$ changes to $\phi_l(x + \delta x)$, then we get

$$\phi(x) \mapsto \phi(x) + \phi'(x)\delta x - \pi\rho_0\delta x = \phi(x) - \pi\rho_0\delta x \tag{98}$$

where $\phi' \to 0$.

From the boundary condition of ϕ_l, $\phi_l \sim \phi_l + 2\pi$, thus $\phi \sim \phi + \pi$. Now if we consider the operator $e^{2ip\phi}$, then this is well defined only if $p \in \mathbb{Z}$.

Now we need to consider the lattice translation symmetry, $x \mapsto x + a$. Then $\phi \mapsto \phi - \pi\rho_0 a$. Now using $\rho_0 = S/a$ for the ground state of spin chain, $\phi \mapsto \phi - \pi S$, and $e^{2ip\phi} \mapsto e^{2ip\phi}e^{2\pi i S}$. Thus, $e^{2ip\phi}$ is translational invariant if $S \in \mathbb{Z}$ or $p \in 2\mathbb{Z}$, and not translational invariant if $S \in \mathbb{Z} + \frac{1}{2}$ and $p \in 2\mathbb{Z} + 1$. This result can be compared with the allowed interaction terms calculated above. Furthermore, this result also implies that if translational symmetry is broken, then $e^{2ip\phi}$ with odd p are allowed, which may open the gap. the good example is the following Hamiltonian:

$$\mathcal{H} = J \sum_j \left[1 + (-1)^j \delta\right] (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z) \tag{99}$$

The gap of this Hamiltonian with $S \in \mathbb{Z} + \frac{1}{2}$ and $S \in \mathbb{Z}$ are following, which shows that breaking the translational symmetry opens the gap.

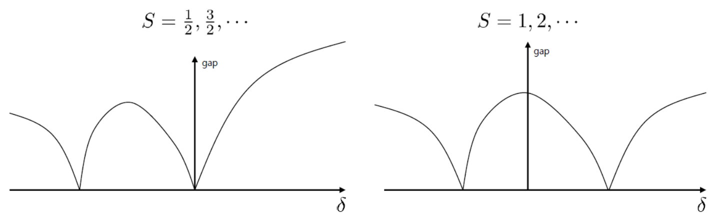

Figure 2: The gap of translation-broken Hamiltonian

Now we think the spin chain with magnetic field, whose Hamiltonian is

$$\mathcal{H} = J \sum (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) - H \sum_j S_j^z \tag{100}$$

This Hamiltonian still contains $U(1)$ symmetry and translational symmetry. Then we still have same analysis, except we may have $\langle S_j^z \rangle = m \neq 0$ possibly. Now since $-S_j^z = S - n_j$ where n_j is the number of 'bosonic' particles, we have

$$m = \langle S_j^z \rangle = -S + \langle n_j \rangle \tag{101}$$

thus $\langle n_j \rangle = S + m$. We often call $\langle n_j \rangle = \nu$ a "filling factor", because it represents how many particles are in one unit cell. From this we get the average density, $\rho_0 = \frac{S+m}{a}$.

Lattice translation changes $\phi \mapsto \phi - \pi\rho_0 a = \phi - \pi(S+m) = \phi - \pi\nu$, thus

$$e^{2ip\phi} \mapsto e^{2ip\phi} e^{-2\pi i p\nu} \tag{102}$$

Now we write down $\nu = p'/q'$ where p', q' are coprimes. Then we can say that $e^{2ip\phi}$ is translation invariant only if $p = nq'$ for some $n \in \mathbb{Z}$, and all $e^{2ip\phi}$ with $p \neq nq'$ are forbidden.

Furthermore, considering the boundary condition $\phi \sim \phi + \pi$, if $\nu = p'/q'$, then q' times of translation gives $\phi \mapsto \phi - \pi p' \sim \phi$. Thus with filling factor $\nu = p'/q'$, we may say that we have effectively $\mathbb{Z}/q'\mathbb{Z}$ symmetry. Mixing this with $U(1)$ symmetry, the system has $U(1) \times \mathbb{Z}/q'\mathbb{Z}$ symmetry. But if q' is large enough, then we may consider $\mathbb{Z}/q'\mathbb{Z}$ as $U(1)$ group, and the symmetry becomes $U(1) \times U(1)$, which can be considered as the chiral symmetry.

2019- 03- 20

Yesterday, we have considered the system with filling factor $\nu = p'/q'$, where p', q' are coprimes. In this case, the Lagrangian allows only the $\cos(2p\phi)$ terms when $p = nq'$ for some $n \in \mathbb{Z}$. Now suppose that $cos(2q'\phi)$ is allowed and $q'^2\kappa < 2$. Then the term is relevant, and ϕ is pinned to the potential minimum, which shows the gap of the system. Furthermore, we can see that when the 1-dimensional system with $\nu = p'/q'$ gains a gap, then there are q' degenerate ground states, which are related by $\phi \mapsto \phi + n\frac{\pi}{q'}$ for $n = 0, 1, \cdots, q' - 1$. This is related to the spontaneous symmetry breaking of the lattice translational symmetry.

Indeed, if we consider $\nu = p'/q'$ with $q' > 1$, then the particles tend to move more freely, which implies the gapless energy spectrum. To open a gap, the particles must be "locked" on the lattice, so that they cannot move freely. Indeed, consider $\nu = 1/3$ case. Then we can lock the particles

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what happened?

2019- 03- 21

2019- 07- 31

Definition 1. Consider a set of fermion creation and annihilation operators

$$\{c_{k,\eta}, c^\dagger_{k',\eta'}\} = \delta_{\eta\eta'}\delta_{kk'} \tag{103}$$

$$\{c_{k,\eta}, c_{k',\eta'}\} = 0 \tag{104}$$

where $\eta = 1, \cdots, M$ is a **species index**, and k is a discrete, unbounded **momentum index** with

$$k = \frac{2\pi}{L}\left(n_k - \frac{1}{2}\delta_b\right), \quad n_k \in \mathbb{Z}, \delta_b \in [0, 2) \tag{105}$$

Here L is the **length** of the system and δ_b is called a **boundary condition parameter**.

Now the fermion fields can be defined as following.

$$\psi_\eta(x) := \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{-ikx} c_{k\eta} \tag{106}$$

whose inverse is

$$c_{k\eta} = \sqrt{2\pi L} \int_{-L/2}^{L/2} dx\, e^{ikx}\psi_\eta(x) \tag{107}$$

Proposition 2. *The fermion field ψ_η obeys the following* ***periodicity condition****.*

$$\psi_\eta(x + L/2) = e^{i\pi\delta_b}\psi_\eta(x - L/2) \tag{108}$$

Proof. Notice that

$$k(x + L/2) = k(x - L/2) + 2\pi\left(n_k - \frac{1}{2}\delta_b\right) \tag{109}$$

taking exponential of this term, $2\pi n_k$ vanishes and only $\pi\delta_b$ lefts, which gives the desired result. □

Proposition 3. *The fermion field ψ_η obeys the following anti-commutation relations.*

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'} 2\pi \sum_{n\in\mathbb{Z}} \delta(x - x' - nL) e^{i\pi n\delta_b} \tag{110}$$

$$\{\psi_\eta(x), \psi_{\eta'}(x')\} = 0 \tag{111}$$

Proof. For the first one

$$\begin{aligned}
\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} &= \frac{2\pi}{L} \sum_{k,k'=-\infty}^{\infty} e^{-ikx+ik'x'} \{c_{k\eta}, c^\dagger_{k'\eta'}\} \\
&= \frac{2\pi}{L}\delta_{\eta\eta'} \sum_{k=-\infty}^{\infty} e^{-ik(x-x')} \\
&= \frac{2\pi}{L}\delta_{\eta\eta'} \sum_{n=-\infty}^{\infty} e^{-2\pi i n(x-x')/L} e^{\pi i\delta_b(x-x')/L} \\
&= \delta_{\eta\eta'} \sum_{n=-\infty}^{\infty} \delta(x - x' - nL) e^{\pi i\delta_b(x-x')/L} \\
&= \delta_{\eta\eta'} \sum_{n=-\infty}^{\infty} \delta(x - x' - nL) e^{\pi i\delta_b n}
\end{aligned}$$

Second one is true since the anticommutator between annihilation operators are zero. □

Definition 4. A **vacuum state** $|\vec{0}\rangle_0$ is a state which is defined by the properties

$$\begin{cases} c_{k\eta}|\vec{0}\rangle_0 := 0, & k > 0(\leftrightarrow n_k > 0) \\ c^\dagger_{k\eta}|\vec{0}\rangle_0 := 0, & k \leq 0(\leftrightarrow n_k \leq 0) \end{cases} \tag{112}$$

A **normal ordering with respect to the vacuum state** of a function of $c, c^\dagger$'s is a re-ordering of $c, c^\dagger$ with all $c_{k\eta}$ with $k > 0$ and all $c^\dagger_{k\eta}$ with $k \leq 0$ are moved to the right of all other operators, which gives

$$: ABC\cdots := ABC\cdots - {}_0\langle\vec{0}|ABC\cdots|\vec{0}\rangle_0, \quad A, B, C \in \{c_{k\eta}; c^\dagger_{k\eta}\} \tag{113}$$

Definition 5. The operator $\hat{N}_\eta$ is the operator that counts the number of η-electrons relative to $|\vec{0}\rangle_0$, which is defined as

$$\hat{N}_\eta := \sum_k : c^\dagger_{k\eta} c_{k\eta} := \sum_k \left[c^\dagger_{k\eta} c_{k\eta} - {}_0\langle\vec{0}|c^\dagger_{k\eta} c_{k\eta}|\vec{0}\rangle_0 \right] \tag{114}$$

The set of all states with the same $\hat{N}_\eta$ eigenvalues $\vec{N} = (N_1, \cdots, N_M)$ is called the $\vec{N}$-particle Hilbert space $H_{\vec{N}}$.

The $\vec{N}$**-particle ground state** is a state defined as following.

$$|\vec{N}_\eta\rangle_0 := (C_1)^{N_1} \cdots (C_M)^{N_M} |\vec{0}\rangle_0 \tag{115}$$

Here,

$$(C_\eta)^{N_\eta} := \begin{cases} c^\dagger_{N_\eta \eta} c^\dagger_{(N_\eta - 1)\eta} \cdots c^\dagger_{1\eta}, & N_\eta > 0 \\ 1, & N_\eta = 0 \\ c_{(N_\eta+1)\eta} c_{(N_\eta+2)\eta} \cdots c_{0\eta}, & N_\eta < 0 \end{cases} \tag{116}$$

Definition 6. For $q := \frac{2\pi}{L} n_q > 0$ with $n_q \in \mathbb{Z}^+$, the **bosonic creation and annihilation operators** are defined as

$$b^\dagger_{q\eta} := \frac{i}{\sqrt{n_q}} \sum_k c^\dagger_{(k+q)\eta} c_{k\eta}, \quad b_{q\eta} := \frac{-i}{\sqrt{n_q}} \sum_k c^\dagger_{(k-q)\eta} c_{k\eta} \tag{117}$$

Proposition 7. *The bosonic creation and annihilation operators satisfies the bosonic commutation relations.*

$$[b_{q\eta}, b_{q'\eta'}] = [N_{q\eta}, b_{q'\eta'}] = 0, \quad [b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{\eta\eta'} \delta_{qq'} \tag{118}$$

2019- 08- 17

Free electron gas. For the free electron gas with temperature $T = 0$, the free electron occupation can be written as

$$n_k = \begin{cases} 1, & 0 \le k \le k_F \\ 0, & k_F < k \end{cases} \tag{119}$$

which has a discontinuity at the **fermi surface** k_F with amplitude 1. The excitations of the state consist in adding particles with a well-defined momentum k, which has energy $\epsilon(k)$. The probability to find a state with a frequency ω and a momentum k can be written as the **spectral function** $A(k, \omega)$, which is $\delta(\omega - \xi(k))$ in free electron gas where $\xi(k) = \epsilon(k) - \mu$.

Fermi liquid theory.

HYUNJUN PARK

ALGEBRAIC TOPOLOGY

2019- 03- 12

Here every map is continuous.

Definition 1. A **deformation retraction** of a space X onto a subspace A is a family of maps $f_t : X \to X, t \in I = [0,1]$, such that $f_0 = 1_X$, $f_1(X) = A$, and $f_t|_A = 1_A$ for all $t \in I$.

Definition 2. For a map $f : X \to Y$, the **mapping cylinder** M_f is the quotient space $(X \times I) \sqcup Y / \sim$, where $(x,1) \sim f(x)$.

Proposition 3. *The mapping cylinder M_f with map $f : X \to Y$ deformation retracts to Y.*

Proof. We need to take a map $f_t : M_f \to M_f$ satisfying deformation retraction conditions. It is easy to take the map as $f_t|_Y = 1_Y$ and $f_t(x,s) = (x, s(1-t)+t)$, and which we need to check is continuity now. Since M_f is a quotient space, each f_t is determined by a map $g_t : (X \times I) \sqcup Y \to M_f$, which respects the relation $g_t(x,1) = g_t(f(x))$. Thus the map $G(a,t) = g_t(a)$ is continuous on $((X \times I) \sqcup Y)$. Now define a new relation $\sim'$ on $((X \times I) \sqcup Y) \times I$ as $(a,t) \sim' (a',t')$ if $a \sim a'$ and $t = t'$. Since G is continuous, the map $F : ((X \times I) \sqcup Y) \times I / \sim' \to M_f$ induced by G is continuous. *Because I is locally compact*, $((X \times I) \sqcup Y) \times I / \sim' \simeq ((X \times I) \sqcup Y / \sim) \times I$, therefore the map F is continuous. □

Definition 4. A family of maps $f_t : X \to Y$ is called **homotopy** if $F(x,t) = f_t(x)$ is continuous on $X \times I$. Two maps $f_0, f_1 : X \to Y$ is called **homotopic** if there exists homotopy f_t between them. If f, g are homotopic, then we write $f \simeq g$.

Example 5. Deformation retraction of X onto a subspace A is a homotopy from 1_X to the **retraction** of X onto A, which is the map $r : X \to X$ such that $r(X) = A$ and $r|_A = 1_A$.

Every space X retracts to a one-point set $\{x_0\}$, where $x_0 \in X$. However there exists some spaces that does not deformation retracts to one-point subset: for example, S^1.

Definition 6. A homotopy $f_t : X \to Y$ where $f_t|_A$ is the constant function on t is called a **homotopy relative to** A, or a homotopy rel$A$.

Definition 7. A map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Then the spaces X and Y are **homotopy equivalent** or have the same **homotopy type**, and write $X \simeq Y$.

Example 8. If a space X deformation retracts onto a subspace A by $f_t : X \to X$, then if $r : X \to A$ is the resulting retraction and $i : A \hookrightarrow X$ is the inclusion, then $r \circ i = 1_A$ and $i \circ r \simeq 1_X$ by deformation retraction, thus X and A are homotopy equivalent.

Definition 9. A space X is called a **cell complex** or **CW complex** if X is constructed as follow.

The **real projective n-space** $\mathbb{R}P^n$ is the space of all lines through the origin in $\mathbb{R}^{n+1}$, which is equivalent to the space $S^n/(v \sim -v)$, the antipodal quotient of the n-sphere. This is *also* equivalent with the space $D^n/(v \sim -v)$, the antipodal quotient of the n-hemisphere. Notice that the quotiented space here is $\partial D^n \simeq S^{n-1}$, which gives that after quotienting we get $\mathbb{R}P^{n-1}$. Therefore basically $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell.

1. Starting with a discrete set X^0, whose points are called **0-cells**;
2. Inductively, generate the n-**skeleton** X^n from X^{n-1} by attaching n-**cells** e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$, i.e. X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha \simeq S^{n-1}$, where D^n_α is an n-disk.
3. One can stop this at finite stage, taking $X = X^n$, or take a limit, setting $X = \cup_n X^n$. In latter case, we give a weak topology: $A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n.

If $X = X^n$ for some n, we call n the **dimension** of X.

Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map ϕ_α and is a homeomorphism from the interior of D^n_α onto e^n_α.

A **subcomplex** of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X. A pair (X, A) consisting of a cell complex X and its subcomplex A is called a **CW pair**.

CW complex can be defined by not using the inductive definition, which uses the characteristic maps. The hausdorff space X is called the **CW complex** if there is a set of maps $\mathbb{D}^n \to X$, Φ_n, which satisfies: For each n-**dimensional cells** $\phi \in \Phi_n$, $\phi|_{\text{int}\mathbb{D}^n}$ is homeomorphic to its image; For each $x \in X$, there exists a unique $(n \in \mathbb{N}, \phi \in \Phi_n)$ such that $x \in \phi(\text{int}\mathbb{D}^n)$; For each $\phi \in \Phi_n$, $\phi(\partial\mathbb{D}^N)$ intersects with finitely many cells with dimension $< n$; $C \subset X$ is closed iff for every $n \in \mathbb{N}$ and $\phi \in \Phi_n$, $\phi^{-1}(C) \subset \mathbb{D}^n$ is closed.

Definition 10. If X, Y are cell complexes, then $X \times Y$ has the structure of a cell complex with cells as the products $e^m_\alpha \times e^n_\beta$, where e^m_α ranges over X and e^n_β ranges over Y.

The topology on $X \times Y$ is sometimes finer then the product topology, however if either X or Y has finitely many cells, or if both X and Y has countably many cells, then they has same topology.

Definition 11. If (X, A) is a CW pair, then the quotient space X/A has the cells as the cells of $X - A$ with one new 0-cell, and those attaching maps are $S^{n-1} \xrightarrow{\phi_\alpha} X^{n-1} \to X^{n-1}/A^{n-1}$.

Definition 12. For a space X, the space $CX = (X \times I)/(X \times \{0\}$ is called the **cone**, and the space $SX = (X \times I)/(X \times \{0\}/(X \times \{1\})$ is called the **suspension**.

If X is CW complex, then also CX and SX are.

Definition 13. For two spaces X, Y, the space $X * Y$ defined by $X \times Y \times I/(x, y_1, 0) \sim (x, y_2, 0)/(x_1, y, 1) \sim (x_2, y, 1)$. A join of $n+1$-points is a convex polyhedron of dimension n, which is called a **simplex**, and written as Δ^n.

If X, Y are CW complex, then there is a natural CW complex structure on $X * Y$ with X, Y as subcomplexes, which may have finer topology then the quotient of $X \times Y \times I$ as it was in product space.

Definition 14. For $x_0 \in X$ and $y_0 \in Y$, the **wedge sum** $X \vee Y$ is the space $X \sqcup Y/x_0 \sim y_0$. The **smash product** $X \wedge Y$ is the space $X \times Y/X \times y_0 \vee Y \times x_0$, where wedge is taken as (x_0, y_0).

Example 15. $\mathbb{S}^n \wedge \mathbb{S}^m \simeq \mathbb{S}^{n+m}$.

2019- 03- 13

Definition 16. For the spaces $A \subset X$, if for every map $f_0 : X \to Y$ and for every homotopy $f^A_t : A \to Y$ with $f_0|_A = f^A_0$, there is a homotopy $f_t : X \to Y$ with $f_t|_A = f^A_t$, then we call (X, A) has the **homotopy extension property**.

This definition implies that every pair of maps $X \times \{0\} \to Y$ and $A \times I \to Y$ which agrees on $A \times \{0\}$ can be extended to a map $X \times I \to Y$.

Proposition 17. *If A is a closed subspace of X, then a pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

This is true for all space A, but the proof is quite long. I'll put the proof of it if I have enough time.

Proof. Suppose that (X, A) has the homotopy extension property. Then for an identity map $X \times \{0\} \cup A \times \{0\} \to X \times \{0\} \cup A \times \{0\}$, we have an extension $X \times I \to X \times \{0\} \cup A \times \{0\}$, which gives the retraction. For the inverse, take two maps $X \times \{0\} \to Y$ and $A \times I \to Y$ which agrees on $A \times \{0\}$. Since $X \times \{0\}$ and $A \times I$ are both closed, we can use *pasting lemma*, so that $X \times \{0\} \cup A \times I \to Y$ is continuous. Using the retraction $X \times I \to X \times \{0\} \cup A \times I$, we get the extension $X \times I \to Y$. □

Pasting lemma says that if $X, Y \subset A$ are *both closed or both open* with $X \cup Y = A$, and if $f : A \to B$ is a function where $f|_X, f|_Y$ are continuous, then f is continuous. *Proof.* If $U \subset B$ is closed, then $f^{-1}(U) \cap X$ and $f^{-1}(U) \cap Y$ are closed, so their union $f^{-1}(U)$ is closed. Same for open case. *Counterexample.* Take $f : (-\infty, 0] \cup \{1\} \to 1$ with $f(x) = 1$ and $g : (0, \infty) \to \mathbb{R}$ with $g(x) = x$.

Example 18. Take (I, A) with $A = \{0\} \cup \bigcup_{n=1}^{\infty} \{\frac{1}{n}\}$. Since there is no retraction $I \times I \to I \times \{0\} \cup A \times I$, (I, A) does not have the homotopy extension property.

Proposition 19. *Take $I = [0, 1]$ and $A = \{0\} \cup \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\} \subset I$. Then there is no retraction of $I \times I$ into $I \times \{0\} \cup A \times I$.*

Proof. Suppose there exists the retraction f. Then $f(0, 1) = (0, 1)$, and since f is continuous, for a small ball B around $(0, 1)$ with radius $\epsilon < 1$, $f^{-1}(B)$ is open and so contains a small open ball U around $(0, 1)$. Then U contains $(\frac{1}{n}, 1)$ for some n, and thus $(\frac{t}{n}, 1)$ for all $t \in [0, 1]$. Notice that this is path connecting $(0, 1)$ and $(\frac{1}{n}, 1)$. Thus there must be exist a path connecting $(0, 1)$ and $(\frac{1}{n}, 1)$ in B, which is not true. □

Proposition 20. *A pair (X, A) has the homotopy extension property if A has a* ***mapping cylinder neigborhood*** *in X, which means, there is a closed boundary N of A, which gives $N - \partial N$ as an open boundary of A, such that there is a map $f : \partial N \to A$ and a homeomorphism $h : M_f \to N$ with $h|_{A \cup \partial N} = 1_{A \cup \partial N}$.*

Proof. Since $I \times I$ retracts on $I \times \{0\} \cup \partial I \times I$, $\partial N \times I \times I$ retracts on $\partial N \times I \times \{0\} \cup \partial N \times \partial I \times I$. This retraction induces a retraction of $M_f \times I$ onto $M_f \times \{0\} \cup (A \cup \partial N) \times I$, hence $(M_f, A \cup \partial N)$ has the homotopy extension property. Since $M_f \simeq N$, $(N, A \cup \partial N)$ also has the homotopy extension property. Now for any map $f_0 : X \to Y$ and a homotopy $f_t^A : A \to Y$ with $f_0|_A = f_0^A$, take the constant homotopy $f_t^{X-(N-\partial N)} : X - (N - \partial N) \to Y$ which is same with $f_0|_{X-(N-\partial N)}$. By using these, we now have the homotopy $f_t^{A \cup \partial N} : A \cup \partial N \to Y$. By the homotopy extension property of $(N, A \cup \partial N)$, we get the extension homotopy $f_t^N : N \to Y$, which agrees with $f_t^{X-(N-\partial N)}$ on $(N - \partial N) \times N$. This is closed set, so by pasting lemma, we get the total homotopy. □

Proposition 21. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.*

Proof. There is a deformation retraction $r : D^n \times I \to D^n \times 0 \cup \partial D^n \times I$, defined by the projection from $(0, 2) \in D^n \times \mathbb{R}$ for example. Thus there is a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$. Now we deformation retract $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ during the t-interval $[1/2^{n+1}, 1/2^n]$, then the infinite concatenation of a homotopies is a deforamtion retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$: at $t = 0$ it is continuous on $X^n \times I$, and since X has weak topology, the given map is continuous. □

2019- 03- 26

Proposition 22. *If the pair (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Proof. Since A is contractible, there is a map $g_t : A \to A$ which says $g_0 = 1_A$ and $g_1(A) = a \in A$. Since (X, A) satisfies the homotopy extension property, we have $f_t : X \to X$ which is the extension of g_t and $f_0 = 1_X$. Since $f_t(A) \subset A$, $q \circ f_t : X \to X/A$ can be factorized into $\bar{f}_t : X/A \to X/A$, which satisfies $q \circ f_t = \bar{f}_t \circ q$. Also, the map $f_1 : X \to X$ can be factorized into $g : X/A \to X$ satisfying $f \circ q = f_1$, because $f_1(A) = a \in A$. Finally, since

$$q \circ g(\bar{x}) = q \circ g \circ q(x) = q \circ f_1(x) = \bar{f}_1 \circ q(x) = \bar{f}_1(\bar{x}), \tag{120}$$

we have $q \circ g = \bar{f}_1$. Since $g \circ q = f_1 \simeq f_0 = 1_X$ and $q \circ g = \bar{f}_1 \simeq \bar{f}_0 = 1_{X/A}$, we get g and q are inverse homotopy equivalences. □

Corollary 23. *If (X, A) is a CW pair of a CW complex X and a contractible subcomplex A, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Example 24. $S^2/S^0 \simeq S^1 \vee S^2$. Indeed, consider a space X, which has a sphere S^2 attached with arc A on two different points, and denote the arc connecting those two points as B. Then X can be thought as CW complex and A, B can be thought as its subcomplex. Also, $X/A \simeq S^2/S^0$ and $X/B \simeq S^1 \vee S^2$, which gives desired result.

Definition 25. For a CW complex X and a 0-cell $x_0 \in X$, $SX/(\{x_0\} \times I)$ is called **reduced suspension** and written as ΣX.

Proposition 26. *For CW complexes X, Y, $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$.*

Proof.

$$\begin{aligned}
\Sigma(X \vee Y) &= [((X \sqcup Y)/(x_0 \sim y_0)) \times I] / S_1/S_2/(x_0 \times I) \\
&= [((X \sqcup Y)/(x_0 \sim y_0)) \times I] / (x_0 \times I)/S_1/S_2 \\
&= ((X \times I) \sqcup (Y \times I))/(x_0 \times I \sim y_0 \times I)) / (x_0 \times I)/S_1/S_2 \\
&= ((X \times I)/(x_0 \times I) \sqcup (Y \times I)/(y_0 \times I))/(x_0 \sim y_0)) / S_1/S_2 \\
&= \big((X \times I)/(x_0 \times I)/S_{1x}/S_{2x} \sqcup (Y \times I)/(y_0 \times I)/S_{1y}/S_{2y})\big) \\
&\quad /(x_0 \sim y_0) \\
&= (\Sigma X \sqcup \Sigma Y)/(x_0 \sim y_0) \\
&= \Sigma X \vee \Sigma Y
\end{aligned}$$

□

2019-03-28

Definition 27. For two spaces X_0, X_1 and a map $f : A \to X_0$ where $A \subset X_1$, $X_0 \sqcup_f X_1 := X_0 \sqcup X_1 / a \sim f(a)$ is called a space X_0 with X_1 **attached along** A **via** f.

Example 28. For spaces X, Y and map $f : X \to Y$, a mapping cylinder M_f is a space $X \times I$ with Y attached along $X \times \{1\}$ via $\tilde{f} : X \times \{1\} \to Y, \tilde{f}(x,1) = f(x)$.

Example 29. For spaces X, Y and map $f : X \to Y$, a **mapping cone** C_f is a space $Y \sqcup_f CX$.

CX is the cone $(X \times I)/(X \times \{0\})$.

Proposition 30. *Take CW complexes X_0, X_1. If (X_1, A) is a CW pair and we have attaching maps $f, g : A \to X_0$ which are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0.*

Proof. Suppose that $F : A \times I \to X_0$ is a homotopy between f and g. Now we may consider $X_0 \sqcup_F (X_1 \times I)$, which contains $X_0 \sqcup_{f,g} X_1$. Since (X_1, A) is a CW pair, deformation retracting $X_1 \times I$ to $X_1 \times \{0\} \cup A \times I$ is possible, we can deformation retract $X_0 \sqcup_F (X_1 \times I)$ to $X_0 \sqcup_F (X_1 \times \{0\} \cup A \times I)$. Now deformation retracting $A \times I$ to $A \times \{0\}$ gives $X_0 \sqcup_f X_1$ and to $A \times \{1\}$ gives $X_0 \sqcup_g X_1$. Since all of these deformation retracts is identity restricted on X_0, therefore we get the desired result. □

2019-04-02

Example 31. Again, $S^2/S^0 \simeq S^1 \vee S^2$. Indeed, consider $A \subset S^2$ be the arc connecting north and south pole, and define $f, g : A \to S^1$ as $f(\theta) = (\cos\theta, \sin\theta)$ and $g(\theta) = (1, 0)$, where A is parametrized by θ. Since f, g are homotopic by

$$F(\theta, t) = (\cos(t\theta), \sin(t\theta)), \tag{121}$$

and $S^2 \sqcup_f S^1 \simeq S^2/S^0$ and $S^2 \sqcup_g S^1 \simeq S^1 \vee S^2$, we get the desired result.

Proposition 32. *Suppose (X, A) and (Y, A) satisfies the homotopy extension property, and $f : X \to Y$ is a homotopy equivalence with $f|_A = 1_A$. Then f is a homotopy equivalence rel A.*

Proof. Let $g : Y \to X$ be a homotopy inverse of f.

First, let $h_t : X \to X$ be a homotopy from $g \circ f = h_0$ to $1_X = h_1$. Then restricting h_t to A, we can say $h_t|_A$ is the homotopy from $g|_A$ to 1_A, since $f|_A = 1_A$. Now using the homotopy extension property of

(Y, A), we may construct a homotopy $g_t : Y \to X$ from $g = g_0$ to g_1 where $g_1|_A = 1_A$. Now we want to show that $g_1 \circ f \simeq 1_X$ rel A.

We have homotopy

$$k_t = \begin{cases} g_{1-2t} \circ f, & 0 \le t \le \frac{1}{2} \\ h_{2t-1}, & \frac{1}{2} \le t \le 1 \end{cases}$$

between $g_1 \circ f$ and $1_X = h_1$, which is not needed to have $k_t|_A = 1_A$, but $k_0|_A = k_1|_A = 1_A$. Also since $g_t|_A = h_t|_A$, $k_t|_A = k_{1-t}|_A$. Now we define $k_{t,u} : A \to X$ as

$$k_{t,u} = \begin{cases} k_t|_A, & u \le 2t-1 \text{ or } u \le -2t+1 \\ k_{\frac{u+1}{2}}|_A, & -2t+1 \le u \text{ and } -2t+1 \le u \end{cases}$$

Then, along the line $\{0\} \times [0,1] \cup [0,1] \times \{1\} \cup \{1\} \times [0,1]$, we get $k_{t,u} = 1_A$. Since $k_{t,u}$ can be thought as the homotopy from $k_{t,0} = k_t|_A : A \times I \to X$ to $k_{t,1} = k_1|_A = 1_A : A \times I \to X$, by homotopy extension property of $(X \times I, A \times I)$, we can extend this to $\tilde{k}_{t,u} : X \times I \times I \to X$ which satisfies $\tilde{k}_{t,0} = k_t$. Finally define

Since (X, A) has homotopy extension property, $X \times I$ can be deformation retracted to $X \times \{0\} \cup A \times I$, and thus $X \times I \times I$ can be deformation retracted to $X \times I \times \{0\} \cup A \times I \times I$, thus $(X \times I, A \times I)$ has homotopy extension property.

$$\tilde{h}_t = \begin{cases} k_{0,3t}, & t \in [0, \frac{1}{3}] \\ k_{3t-1,1}, & t \in [\frac{1}{3}, \frac{2}{3}] \\ k_{1,3-3t}, & t \in [\frac{2}{3}, 1] \end{cases} \tag{122}$$

which is continuous. since $\tilde{h}_0 = g_1 \circ f$ and $\tilde{h}_1 = h_1 = 1_X$, we have homotopy $g_1 \circ f \simeq 1_X$ rel A.

Now, since $f \circ g \simeq f \circ g_1 \simeq 1_Y$, we may redo the above argument, which gives a map f_1 which is homotopic with f, $f_1|_A = 1_A$, and $f_1 \circ g_1 \simeq 1_Y$ rel A. Since $g_1 \circ f \simeq 1_X$ rel A, we get $f_1 \simeq f_1 \circ g_1 \circ f \simeq f$ rel A. Therefore $f_1 \circ g_1 \simeq f \circ g_1 \simeq 1_Y$ rel A. □

2019- 04- 04

Corollary 33. *If (X, A) satisfies the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.*

Proof. By the proposition above, inclusion $i : A \hookrightarrow X$ is a homotopy equivalence rel A, whose homotopy is deformation retraction. □

Corollary 34. *A map $f : X \to Y$ is a homotopy equivalence if and only if X is a deformation retract of the mapping cylinder M_f. Hence, two spaces X and Y are homotopy equivalent if and only if there is a third space containing both X and Y as deformation retracts.*

Proof. Notice that we have inclusions $i : X \hookrightarrow M_f$, $j : Y \hookrightarrow M_f$, and a canonical retraction $r : M_f \to Y$ satisfying $r \circ j = 1_Y$ and $j \circ r \simeq 1_{M_f}$. Then $f = r \circ i$ and $i \simeq j \circ f$ by the definition of M_f. Now, if i is homotopy equivalence, then since r is homotopy equivalence, f is homotopy equivalence; if f is homotopy equivalence, then since j is homotopy equivalence, i is homotopy equivalence. Since X has a mapping cylinder neighborhood, $X \times [0,1]$, in M_f, (M_f, X) satisfies homotopy extension property, and so by the corollary above, if i is homotopy equivalence then X is a deformation retract of M_f. Conversely, if X is a deformation retract of M_f, then the inclusion $i : X \hookrightarrow M_f$ is homotopy equivalence, thus f is homotopy equivalence. □

2019- 04- 05

From now, $I = [0,1]$.

Definition 35. A **path** in a space X is a continuous map $f : I \to X$. We call $f(0), f(1) \in X$ the **endpoints** of f. For two paths $f, g : I \to X$, if $f(0) = g(0)$ and $f(1) = g(1)$, then we call these paths have same endpoints.

Definition 36. A **homotopy of paths** in X is a homotopy $F : I \times I \to X$, which is also written as $f_t(s) = F(s,t)$, between paths f_0 and f_1 such that $f_t(0) = x_0, f_t(1) = x_1$ for $x_0, x_1 \in X$. If so, then we call f_0 and f_1 **path homotopic**, and write $f_0 \simeq f_1$.

Proposition 37. *Every paths $f_0, f_1 : I \to X \subset \mathbb{R}^n$, whose endpoints are same, are path homotopic if X is convex.*

Proof. Define homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$, which is well defined at $t = 0, 1$ obviously, and also well defined on $t \in (0,1)$ since $X \subset \mathbb{R}^n$ is a convex set. Since f_0, f_1 are continuous, $f_t(x)$ is continuous. Also, taking $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1) = x_1$, we get $f_t(0) = tf_0(0) + (1-t)f_1(0) = tx_0 + (1-t)x_0 = x_0$, and same for x_1. □

This path homotopy is called **linear homotopy**.

Proposition 38. *The relation of homotopy of paths is an equivalence relation.*

The **relation** on a set X is a subset $\sim \subset X \times X$. If $(x_0, x_1) \in \sim$, then we write $x_0 \sim x_1$.

A relation $\sim$ is an **equivalence relation** if, for all $x_0, x_1, x_2 \in X$,

1. $x_0 \sim x_0$,
2. $x_0 \sim x_1$ if and only if $x_1 \sim x_0$,
3. $x_0 \sim x_1 \sim x_2$ then $x_0 \sim x_2$.

Proof. Let X be a set.

1. For a path $f_0 : I \to X$, define $F : I \times I \to X$ as $F(s,t) = f_0(s)$. This is continuous and $F(0,t) = f_0(0), F(1,t) = f_0(1)$, thus have fixed endpoints. Finally, $F(s,0) = F(s,1) = f_0(s)$, therefore this is a homotopy of paths between f_0 and f_0. Therefore $f_0 \simeq f_0$.

2. For paths $f_0, f_1 : I \to X$ with same endpoints x_0, x_1, if $f_0 \simeq f_1$, then we may take $F : I \times I \to X$ as a path homotopy between f_0, f_1,

i.e. $F(s,0) = f_0(s), F(s,1) = f_1(s)$, and $F(0,t) = x_0, F(1,t) = x_1$. Now take $G : I \times I \to X$ as $G(s,t) = F(s,1-t)$. Then G is continuous, $G(s,0) = F(s,1) = f_1(s), G(s,1) = F(s,0) = f_0(s)$, and $G(0,t) = F(0,1-t) = x_0, G(1,t) = F(1,1-t) = x_1$. Therefore $f_1 \simeq f_0$.

3. For paths $f_0, f_1, f_2 : I \to X$ with same endpoints x_0, x_1, if $f_0 \simeq f_1 \simeq f_2$, then we may take $F : I \times I \to X$ and $G : I \times I \to X$ as a path homotopy between f_0, f_1 and f_1, f_2, respectively. Now take $H : I \times I \to X$ as

$$H(s,t) = \begin{cases} F(s,2t), & t \in [0,\frac{1}{2}] \\ G(s,2t-1), & t \in [\frac{1}{2},1] \end{cases} \tag{123}$$

Since $F(s,1) = f_1(s) = G(s,0)$, the map above is well defined and continuous. Since F, G fix endpoints, H fixes endpoints. Finally, since $H(s,0) = f_0(s)$ and $H(s,1) = f_2(s)$, H is homotopy of paths between f_0 and f_2, thus $f_0 \simeq f_2$.

□

Definition 39. The equivalence class of a path $f : I \to X$ under the equivalence relation of homotopy of path is called the **homotopy class** of f and denoted as $[f]$.

Definition 40. Given two paths $f, g : I \to X$ with $f(1) = g(0)$, the **composition**, or **product path**, is defined as

$$f \cdot g(s) = \begin{cases} f(2s), & s \in [0,\frac{1}{2}] \\ g(2s-1), & s \in [\frac{1}{2},1] \end{cases} \tag{124}$$

Proposition 41. *For two paths $f, g : I \to X$ with $f(1) = g(0)$, the product path $f \cdot g(s)$ is a path.*

Proof. Since $f(1) = g(0)$, by pasting lemma, $f \cdot g : I \to X$ is continuous. □

Lemma 42. *For the paths $f_0, f_1, g_0, g_1 : I \to X$, where f_0, f_1 has same endpoints x_0, x_1 and g_0, g_1 has same endpoints x_1, x_2, if $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.*

Proof. For path homotopies $F : I \times I \to X$ of $f_0 \simeq f_1$ and $G : I \times I \to X$ of $g_0 \simeq g_1$, define $H : I \times I \to X$ as

$$H(s,t) = \begin{cases} F(2s,t), & s \in [0,\frac{1}{2}] \\ G(2s-1,t), & s \in [\frac{1}{2},1] \end{cases} \tag{125}$$

This is well defined since $F(1,t) = G(0,t) = x_1$, and by pasting lemma this is continuous. Finally, $H(s,0) = f_0 \cdot g_0(s)$ and $H(s,1) = f_1 \cdot g_1(s)$. □

Definition 43. Take a path $f : I \to X$. The **reparametrization** of f is the composition map $f \circ \phi$ where $\phi : I \to I$ is a continuous map with $\phi(0) = 0$ and $\phi(1) = 1$.

Lemma 44. *For a path $f : I \to X$ and its reparametrization g, $f \simeq g$.*

Proof. Since g is the reparametrization of f, we have $\phi : I \to I$ such that $\phi(0) = 0, \phi(1) = 1$ and $g = f \circ \phi$. Now take a a map

$$F(s,t) = f(t\phi(s) + (1-t)s). \tag{126}$$

Then F is continuous, $F(s,0) = f(s)$ and $F(s,1) = f(\phi(s)) = g(s)$, and $F(0,t) = f(0), F(1,t) = f(1)$. Therefore F is homotopy or paths. □

Definition 45. The path $f : I \to X$ with basepoints x_0, x_0, i.e. $f(0) = f(1)$, is called a **loop**, and x_0 is called a **basepoint** of loop f. The set $\pi_1(X, x_0)$ is the set of all homotopy classes $[f]$ of loops $f : I \to X$ with basepoint x_0.

Proposition 46. *$\pi_1(X, x_0)$ is a group with the product $[f][g] = [f \cdot g]$, called **fundamental group**.*

Proof. Product is closed. Take $[f], [g] \in \pi_1(X, x_0)$. Since f, g are loops with basepoint x_0, $f \cdot g$ is a loop with basepoint x_0 also, thus $[f \cdot g] \in \pi_1(X, x_0)$. Hence the product is closed.

Associativity. For any paths $f, g, h : I \to X$ with $f(1) = g(0)$ and $g(1) = h(0)$, consider $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$. Take

$$\phi(s) = \begin{cases} \frac{s}{2}, & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4}, & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases} \tag{127}$$

Then $((f \cdot g) \cdot h) \circ \phi = f \cdot (g \cdot h)$, thus $(f \cdot g) \cdot h$ is reparametrization of $f \cdot (g \cdot h)$. Since the loop and its reparametrization is homotopy equivalent, we get $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$, Thus, restricting f, g, h as loops with basepoint x_0, $([f][g])[h] = [f]([g][h])$.

Identity. For a path $f : I \to X$ with $f(1) = x_1$, consider a constant path $c_{x_1} : I \to X$ such that $c_{x_1}(s) = x_1$. Take

$$\phi(s) = \begin{cases} 2s, & s \in [0, \frac{1}{2}] \\ 1, & s \in [\frac{1}{2}, 1] \end{cases} \tag{128}$$

Then $f \circ \phi = f \cdot c$, i.e. $f \cdot c$ is a reparametrization of f. Samely, for a path $f : I \to X$ with $f(0) = x_0$, consider a constant path $c_{x_0} : I \to X$ and take

$$\phi(s) = \begin{cases} 0, & s \in [0, \frac{1}{2}] \\ 2s - 1, & s \in [\frac{1}{2}, 1] \end{cases} \tag{129}$$

Then $f \circ \phi = c \cdot f$, i.e. $c \cdot f$ is a reparametrization of f. Thus, restricting f as a loop with basepoint x_0, $[f][c_{x_0}] = [c_{x_0}][f] = [f]$.

Inverse. For a path $f : I \to X$, define $\overline{f} : I \to X$ as $\overline{f}(s) = f(1-s)$. Now take the continuous map $H : I \times I \to X$ as

$$H(s,t) = \begin{cases} f(2ts), & s \in [0, \frac{1}{2}] \\ f(2t(1-s)), & s \in [\frac{1}{2}, 1] \end{cases} \tag{130}$$

Then we get $H(0,t) = H(1,t) = f(0)$ and $H(s,0) = f(0), H(s,1) = f \cdot \overline{f}(s)$. Therefore restricting f as a loop with basepoint x_0 gives $[f][\overline{f}] = [c_{x_0}]$. Just exchanging f and $\overline{f}$ gives $[\overline{f}][f] = [c_{x_0}]$. □

2019- 04- 08

Example 47. Since every paths of convex subset $X \subset \mathbb{R}^n$ with same endpoints are path homotopic, $\pi_1(X, x_0) = 0$.

Definition 48. For $x_0, x_1 \in X$, suppose that there is a path $h : I \to X$ whose endpoints are x_0, x_1. Then a **change-of-basepoint map** $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is defined as

$$\beta_h([f]) = [h \cdot f \cdot \overline{h}]. \tag{131}$$

This map is well defined because the path product is associative under path homotopy, which is proven in the associativity of $\pi_1(X, x_0)$. Furthermore if f is a loop based on x_1, then $(h \cdot f) \cdot \bar{h}$ is a loop based on x_0.

Proposition 49. *For $x_0, x_1 \in X$ and a path $h : I \to X$ whose endpoints are x_0, x_1, the change-of-basepoint map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is isomorphism.*

Proof. β_h is homomorphism because $\beta_h([f][g]) = \beta_h([f \cdot g]) = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h}][h \cdot g \cdot \overline{h}] = \beta_h([f])\beta_h([g])$. Also β_h is isomorphism with inverse $\beta_{\overline{h}}$ since $\beta_h \circ \beta_{\overline{h}}([f]) = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f]$, and exchanging $h, \overline{h}$ gives $\beta_{\overline{h}} \circ \beta_h([f]) = [f]$. □

Corollary 50. *If X is path connected, then $\pi_1(X, x_0) \simeq \pi_1(X)$, for some group $\pi_1(X)$.*

Proof. Since X is path connected, for any $x_0, x_1 \in X$, there is a path connecting x_0 and x_1, and thus $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. □

Definition 51. A space X is **simply connected** if X is path connected and $\pi_1(X) = 0$.

Proposition 52. *A space X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

The existence of homotopy class implies the path connectivity of X.

Proof. If X is simply connected, then for any paths $f, g : I \to X$ with same endpoints x_0, x_1, $f \cdot \overline{g} \simeq c_{x_0}$ and $\overline{g} \cdot g \simeq c_{x_1}$. Therefore, $f \simeq f \cdot \overline{g} \cdot g \simeq g$. Conversely, if there is a unique homotopy class of paths connecting any two points in X, then taking the paths connecting x_0 to itself gives $\pi_1(X, x_0) = 0$. □

Definition 53. For the spaces X, Y, A and continuous maps $f : A \to X$ and $p : Y \to X$, we call a continuous map $\tilde{f} : A \to Y$ a **lift** of f if $p \circ \tilde{f} = f$.

Example 54. Not every function has its lift. Consider the identity map $f : S^1 \to S^1$ and $p : \mathbb{R} \to S^1$ which is defined as $p(\theta) = (\cos\theta, \sin\theta)$. Then there is no lift $\tilde{f} : S^1 \to \mathbb{R}$ satisfying $p \circ \tilde{f} = f$. Indeed, consider $S^1 - x_0$. for $s_0 = (1, 0) \in S^1$. Since $S^1 - s_0$ is connected, $\tilde{f}(S^1 - s_0)$ is connected, and since $p^{-1} \circ f(S^1 - s_0) = \mathbb{R} - 2\pi\mathbb{Z}$, $\tilde{f}(S^1 - s_0)$ must be in the interval, which can be taken as $(0, 2\pi)$ WLOG. Due to the surjectivity of f, $\tilde{f}(S^1 - s_0) = (0, 2\pi)$ exactly. Now since S^1 is compact, $[0, 1] \subset \tilde{f}(S^1)$, thus $\{0, 1\} \subset \tilde{f}(s_0)$, contradiction.

Definition 55. For a space X, a **covering space** of X is a space $\tilde{X}$ with a map $p : \tilde{X} \to X$, called a **covering map**, such that for each $x \in X$, there is an open neighborhood $U \subset X$ of x such that $p^{-1}(U)$ is a union of disjoint open sets, where each are homeomorphic to U by p. We call such U **evenly covered**.

Example 56. The map $p : \mathbb{R} \to S^1$ defined as $p(\theta) = (\cos\theta, \sin\theta)$ is a covering map. Thus $\mathbb{R}$ is a covering space of S^1.

Lemma 57. *For a covering space and map $p : \tilde{X} \to X$ of X, if U is evenly covered open set and $W \subset U$ is also an open set, then W is also evenly covered.*

Proof. By the definition of evenly covered open set, we have a collection of open sets $p^{-1}(U) = \cup_\alpha U_\alpha$ where $p|_{U_\alpha}$ is homeomorphism. Now we may write $p^{-1}(W) = \cup_\alpha \left(U_\alpha \cap p^{-1}(W)\right)$, using de Morgan's law. Writing $U_\alpha \cap p^{-1}(W) = W_\alpha$, we can show that $W_\alpha \cup W_\beta = \emptyset$ if $\alpha \neq \beta$. Now since p is homeomorphism, the restriction of p on W_α is homeomorphism, and $p(W_\alpha) = p(U_\alpha \cap p^{-1}(W)) = p(U_\alpha) \cap p(p^{-1}(W)) = p(U_\alpha) \cap W = U \cap W = W$. $\square$

For any map $f : X \to Y$, if f is surjective and $B \subset Y$, then $f(f^{-1}(B)) = B$; if f is injective and $U, V \subset X$, then $f(U \cap V) = f(U) \cap f(V)$.

Lemma 58. *Take a covering space and map $p : \tilde{X} \to X$ of X. For a map $F : Y \times I \to X$ and a map $\tilde{F}_0 : Y \times \{0\} \to \tilde{X}$ lifting $F|_{Y \times \{0\}}$, there is a unique map $\tilde{F} : Y \times I \to \tilde{X}$ lifting F and $\tilde{F}|_{Y \times \{0\}} = \tilde{F}_0$.*

Proof. Take a point $y_0 \in Y, t \in I$. Then since X has a covering space, $F(y_0, t)$ has an open neighborhood U_t of $F(y_0, t)$ which is evenly covered. Thus, taking the neighborhood $N_t \times (a_t, b_t) \subset F^{-1}(U_t)$ of (y_0, t), we get $F(N_t \times (a_t, b_t)) \in U$. Now since $\{N_t \times (a_t, b_t) : t \in I\}$ is an open cover of $\{y_0\} \times I$, which is compact set, we may choose a finite subcover, $\{N_i \times (a_i, b_i) : i \in \{0, \cdots, m\}\}$, which also gives a finite partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\{N \times [t_i, t_{i+1}] : t \in \{0, \cdots, m\}\}$ is an open cover of $\{y_0\} \times I$, and $F(N \times [t_i, t_{i+1}]) \subset U_i$, taking $N = \cap_{i=0}^m N_i$.

Now we use induction. First, we already have a lifting $\tilde{F}_0|_N$ of $F|_{N\times\{0\}}$. Now assume that we already have a lifting $\tilde{F}$ on $N \times [0, t_i]$. For $F(N \times [t_i, t_{i+1}]) \subset U_i$, since U_i is evenly covered there exists $\tilde{U}_i \subset \tilde{X}$ so that $p(\tilde{U}_i) = U_i$ and $\tilde{F}(y_0, t_i) \in \tilde{U}_i$. If $\tilde{F}(N \times \{t_i\})$ is not contained in $\tilde{U}_i$, then we may take smaller open $N' \subset N$ so that $\tilde{F}(N \times \{t_i\}) \subset \tilde{U}_i$, which is defined as $N' \times \{t_i\} = N \times \{t_i\} \cap \tilde{F|_{N\times\{t_i\}}}^{-1}(\tilde{U}_i)$. Thus we may think that $\tilde{F}(N \times \{t_i\}) \subset \tilde{U}_i$. Now we may define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ as $p^{-1}|_{U_i} \circ F|_{N\times[t_i,t_{i+1}]}$, which is continuous due to the pasting lemma. Repeating this step finitely many times gives $\tilde{F} : N \times I \to \tilde{X}$.

For the uniqueness, first we show the uniqueness of the lift if $Y = \{y_0\}$ is a point: suppose that $\tilde{F}, \tilde{F}'$ are two lifts of $F : \{y_0\} \times I \to X$. such that $\tilde{F}(y_0, 0) = \tilde{F}'(y_0, 0)$. We can do the same procedure above, and so take a finite partition $0 = t_0 < t_1 < \cdots < t_m = 1$ so that $F(y_0, [t_i, t_{i+1}]) \subset U_i$ for some evenly covered U_i. Now again use induction, and consider $\tilde{F}|_{\{y_0\}\times[0,t_i]} = \tilde{F}'|_{\{y_0\}\times[0,t_i]}$. Since $[t_i, t_{i+1}]$ is connected, $\tilde{F}(y_0, [t_i, t_{i+1}]$ is connected, and thus must be connected in one of the disjoint open sets $\tilde{U}_i$ satisfying $p(\tilde{U}_i)$. Since $\tilde{F}(t_i) = \tilde{F}'(t_i)$, $\tilde{F}'([t_i, t_{i+1}]) \subset \tilde{U}_i$. Since p is injective on $\tilde{U}_i$ and $p \circ \tilde{F} = p \circ \tilde{F}' = F$, $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$, which shows that $\tilde{F} = \tilde{F}'$ by induction.

Finally, if $N \times I$ and $M \times I$ overlaps, then since the lifting on $\{y_0\} \times I$ is unique, the lifting on $N \times I \cap M \times I$ is uniquely determined. Thus, using all the neighbors of $y \in Y$, we get the lifting $\tilde{F} : Y \times I \to \tilde{X}$. This is continuous since this is continuous on each $N \times I$, and this is unique since it is unique on each $\{y_0\} \times I$. □

Lemma 59. *For each path $f : I \to X$ starting at a point x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x}_0$. Also, for each path homotopy $f_t : I \to X$ starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted path homotopy $\tilde{f}_t : I \to \tilde{X}$ starting at $\tilde{x}_0$.*

Proof. For the first statement, take $Y = \{y_0\}$ for some point y_0 and use Lemma above. For the second statement, take $F(s, t) = f_t(s)$, then by the first statement we get a unique lift $\tilde{F}_0 : I \times \{0\} \to \tilde{X}$, and by the Lemma above we get a unique lift $\tilde{F} : I \times I \to \tilde{X}$. Also, $\tilde{F}|_{\{0\}\times I}, \tilde{F}|_{\{1\}\times I}$ are the lifts of constant maps $F|_{\{0\}\times I}, F|_{\{1\}\times I}$ respectively, hence we may check that for each case the constant map is a lift, and since the uniqueness of lifting, the constant map is the lift. Thus $\tilde{F}$ is path homotopy. □

Theorem 60. *$\pi_1(S^1) \simeq \mathbb{Z}$, whose generator is the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based on $(1, 0)$.*

Proof. Let $f : I \to S^1$ is a loop with basepoint $x_0 = (1, 0)$. Then by the Lemma above for the path, we have a lifting of the path, $\tilde{f} : I \to \mathbb{R}$, starting at 0. Since $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$, this lifted path ends at $n \in \mathbb{Z}$. Now notice that

$$[\omega]^n = [\underbrace{\omega \cdot \omega \cdot \cdots \cdot \omega}_{n \text{ times}}] = [\omega_n] \tag{132}$$

where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$. Also, the lifting of $\omega_n(s)$ starting at 0 ends at n, by directly checking the path $\tilde{\omega}_n(s) = ns$ is the lifted path. Since $\mathbb{R}$ has a trivial fundamental group, $\tilde{\omega}_n \simeq \tilde{f}$ by some homotopy H, and taking $p \circ H$ gives the homotopy between ω_n and f. Therefore $[f] = [\omega_n]$.

To show that the fundamental group of S^1 is $\mathbb{Z}$, we need to show that $[\omega_n] = [\omega_m]$ then $n = m$. Choose the homotopy f_t between $f_0 = \omega_n$ and $f_1 = \omega_m$. By the Lemma above for the homotopy, we have a lifting of the homotopy, $\tilde{f}_t$, whose path starting at 0, and by the uniqueness of path lifting, $\tilde{f}_0 = \tilde{\omega}_n$ and $\tilde{f}_1 = \tilde{\omega}_m$. Finally, since $\tilde{f}_t$ is a path homotopy, $\tilde{f}_t(1)$ is constant function, thus $n = \tilde{\omega}_n(1) = \tilde{\omega}_m(1) = m$. □

2019- 04- 13

Theorem 61 (Fundamental theorem of algebra.). *Every nonconstant polynomials with coefficient in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Take the polynomial $p(x) = \sum_{i=0}^{n} a_i z^i$ where $a_i \in \mathbb{C}$ and $a_n \neq 0$. Dividing $p(x)$ by a_n, we may assume that $a_n = 1$. Now suppose that $p(z)$ has no roots in $\mathbb{C}$. Define a set of functions $f_r : I \to S^1 \subset \mathbb{C}$ for $r \in \mathbb{R}_{\geq 0}$ as

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|} \tag{133}$$

which is also a homotopy of loops based at 1, since $f_r(0) = f_r(1) = 1$. Since $f_0(s) = 1$, we conclude that $[f_r] \in \pi_1(S^1)$ for all $r \in \mathbb{R}_{\geq 0}$. Now, fix $r > \max(|a_0| + \cdots + |a_{n-1}|, 1)$. Then for $|z| = r$,

Indeed the exact loop homotopy must be given as $f_{rt}(s)$ which connects f_0 and f_r.

$$\begin{aligned} |z^n| &> (|a_0| + \cdots |a_{n-1}|)|z^{n-1}| \\ &> |a_0| + \cdots + |a_{n-1}z^{n-1}| \\ &> |a_{n-1}z^{n-1} + \cdots + a_0| \end{aligned} \tag{134}$$

Thus the polynomial

$$p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0) \tag{135}$$

has no roots for $|z| = r$ when $t \in [0,1]$. Also defining

$$f_{r;t}(s) = \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|} \tag{136}$$

gives the path homotopy from $f_{r;0}(s) = e^{2\pi ins} = \omega_n(s)$ to $f_{r;1}(s) = f_r(s)$: notice that $f_{r;t}(0) = f_{r;t}(1) = 1$ again. Since $[\omega_n] = [f_r] = 0$, $n = 0$, and thus the only polynomials without roots in $\mathbb{C}$ are constants. $\square$

Theorem 62 (2-dimensional Brouwer fixed point theorem.). *Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ such that $h(x) = x$.*

Proof. Suppose not. Then we may define a map $r : D^2 \to S^1$ as

$$r(x) = h(x)t + (1-t)x, \quad t \geq 0, \quad |h(x)t + (1-t)x| = 1. \tag{137}$$

This is just a restriction of continuous function, hence continuous. Furthermore, if $x \in S^1$ then $r(x) = x$. Therefore r is a retraction of D^2 onto S^1.

Now let f_0 is a loop in S^1. Because D^2 is convex, there is a path homotopy, f_t, of f_0 to a constant loop on the basepoint of f_0. Then the composition rf_t is a homotopy in S^1 from $rf_0 = f_0$ to the constant loop at x_0, hence $\pi_1(S^1) = 0$, contradiction. $\square$

This theorem holds for any dimension, which will be proven later.

Theorem 63 (2-dimensional Borsuk-Ulam theorem.). *For every continuous map $f : S^2 \to \mathbb{R}^2$, there is a pair of antipodal points $x, -x$ in S^2 with $f(x) = f(-x)$.*

This is best: consider the stereographic projection of S^2 on $\mathbb{R}^2$.

Proof. Suppose not. Then there is a map $g : S^2 \to S^1$ defined as

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}. \tag{138}$$

Now consider a loop $\eta(s) : I \to S^2$ as

$$\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0). \tag{139}$$

Consider a loop $h : I \to S^1$ as $h = g \circ \eta$. Since $g(-x) = -g(x)$ and $\eta(s) = -\eta(s + \frac{1}{2})$ for $s \in [0, \frac{1}{2}]$, we get $h(s + \frac{1}{2}) = -h(s)$ for all $s \in [0, \frac{1}{2}]$.

Now we can lift the loop h into a path $\tilde{h} : I \to \mathbb{R}$. Since $h(s + \frac{1}{2}) = -h(s)$, considering the covering map, we can see that

$$\tilde{h}\left(s + \frac{1}{2}\right) = \tilde{h}(s) + \frac{2n(s) + 1}{2} \tag{140}$$

for $n(s) \in \mathbb{Z}$ for each $s \in [0, \frac{1}{2}]$. But since we get

$$n(s) = \left(\tilde{h}\left(s + \frac{1}{2}\right) - \tilde{h}(s)\right) - \frac{1}{2}, \tag{141}$$

which is continuous function, $n(s)$ is a constant function, $n(s) = n$. Thus

$$\tilde{h}(1) = \tilde{h}\left(\frac{1}{2}\right) + \frac{2n(s) + 1}{2} = \tilde{h}(0)(2n + 1). \tag{142}$$

This implies that $[h] = [\omega_{2n+1}]$, and thus h is not nullhomotopic. Finally, considering the bijection between $S^2 - \{N\}$ and $\mathbb{R}^2$, which is given by the stereographic projection, we can see that η is loop homotopic with constant loop in S^2, by path homotopy η_t, which can be composed with g and give a loop homotopy $g \circ \eta_t$ between constant loop and $g \circ \eta = h$, contradiction. □

Corollary 64. *When S^2 is expressed as the union of three closed sets A_1, A_2, A_3, then at least one of them contain a pair of antipodal points $\{x, -x\}$.*

This is best: consider the projection of the four faces of the tetrahedron onto the inscribing sphere.

This can be extended in n dimensional case: S^n cannot be covered by $n+1$ closed sets where all of them does not contain a pair of antipodal points, but can by $n+2$ closed sets. The proof uses n-dimensional Borsuk-Ulam theorem and the counterexample uses n-dimensional tetrahedron.

Proof. For each A_i, let $d_i : S^2 \to \mathbb{R}$ defined as

$$d_i(x) = \inf_{y \in A_i} |x - y|. \tag{143}$$

Since this is distance function, it is continuous, thus we may use the Borsuk-Ulam theorem to the map $f : S^2 \to \mathbb{R}^2$ defined as

$$f(x) = (d_1(x), d_2(x)), \tag{144}$$

getting $x_0 \in S^2$ such that $d_1(x_0) = d_1(-x_0)$ and $d_2(x_0) = d_2(-x_0)$. If one of them are zero, then $x_0, -x_0$ both are included in A_1 or A_2, since they are closed sets. If not, then $x_0, -1x_0$ both are included in A_3. □

Proposition 65. *Suppose that X, Y are path-connected. Then $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.*

Proof. We know that $f : Z \to X \times Y$ is continuous if and only if the maps $g : Z \to X, h : Z \to Y$ defined by $f(z) = (g(z), h(z))$ are both continuous, due to the product topology. Now define a map

$$\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0) \tag{145}$$

defined as $\phi([f]) = ([g], [h])$, if $f = (g, h)$. This is well defined: suppose that $[f] = [f']$ and $f = (g, h), f' = (g', h')$. Then we have a loop homotopy $f_t : I \times I \to X \times Y$ with $f_0 = f, f_1 = f'$, and we can write $f_t = (g_t, h_t)$, where g_t, h_t are continuous and $g_0 = g, g_1 = g', h_0 = h, h_1 = h'$. This is bijection: For any $([g], [h])$ we have $f = (g, h)$ such that $\phi([f]) = ([g], [h])$, and for any $\phi([f]) = \phi([f'])$, we may take $\phi([f]) = ([g], [h])$ with $f = (g, h)$ and $\phi([f']) = ([g'], [h'])$ with $f' = (g', h')$, then we can find a loop homotopy g_t, h_t where $g_0 = g, g_1 = g', h_0 = h, h_1 = h'$, because $[g] = [g']$ and $[h] = [h']$. Now define $f_t = (g_t, h_t)$, which gives a loop homotopy with $f_0 = f, f_1 = f'$, and so $[f] = [f']$. This is finally homomorphism: $\phi([f][f']) = \phi([f \cdot f']) = \phi([(g \cdot g', h \cdot h')]) = ([g \cdot g'], [h \cdot h']) = ([g][g'], [h][h']) = ([g], [h])([g'], [h'])$. □

Example 66. A torus $T \simeq S^1 \times S^1$ has $\pi_1(T) = \pi_1(S^1)^2 = \mathbb{Z}^2$. A n-torus, $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$, has $\pi_1(T^n) = \mathbb{Z}^n$.

Definition 67. Suppose $\phi : X \to Y$ is a continuous map with $\phi(x_0) = y_0$ for some $x_0 \in X, y_0 \in Y$. The map $\phi_* : \phi_1(X, x_0) \to \phi_1(Y, y_0)$ defined as $\phi_*[f] = [\phi \circ f]$ is called a **induced homomorphism**.

Proposition 68. *For a continuous map $\phi : X \to Y$ with $\phi(x_0) = y_0$ for some $x_0 \in X, y_0 \in Y$, the induced homomorphism ϕ_* is indeed well defined and homomorphism.*

Proof. Suppose that $[f] = [f'] \in \pi_1(X, x_0)$. Then we have a loop homotopy f_t such that $f_0 = f, f_1 = f'$. Now taking $\phi \circ f_t$ gives a loop homotopy between $\phi \circ f_0 = \phi \circ f$ and $\phi \circ f_1 = \phi \circ f'$, which gives $[\phi \circ f] = [\phi \circ f']$. Also, $\phi([f][f']) = \phi([f \cdot f']) = [\phi \circ (f \cdot f')] = [(\phi \circ f) \cdot (\phi \circ f')] = \phi([f])\phi([f'])$. □

Proposition 69. *For a maps $\psi : (X, x_0) \to (Y, y_0)$ and $\phi : (Y, y_0) \to (Z, z_0)$, $(\phi \circ \psi)_* = \phi_* \circ \psi_*$. Also, $1_{X*} = 1_{\pi_1(X, x_0)}$.*

These properties shows that π_1 is a *functor*, which is a categorical concept, and will be defined exactly later.

Proof. Since $(\phi \circ \psi) \circ f = \phi \circ (\psi \circ f)$, $(\phi \circ \psi)_*([f]) = [\phi \circ (\psi \circ f)] = \phi_*[\psi \circ f] = \phi_* \circ \psi([f])$. Also, $1_{X*}([f]) = [1_X \circ f] = [f]$. □

Corollary 70. *If $\phi : (X, x_0) \to (Y, y_0)$ is a homeomorphism with inverse $\psi : (Y, y_0) \to (X, x_0)$, then ϕ_* is an isomorphism with inverse ψ_*.*

Proof. $\psi_* \circ \phi_* = (\psi \circ \phi)_* = 1_{X*}$ and $\phi_* \circ \psi_* = (\phi \circ \psi)_* = 1_{Y*}$. □

Lemma 71. *If a space X is a union of a collection of path connected open sets A_α each containing $x_0 \in X$ and all $A_\alpha \cap A_\beta$ is path connected, then every loop in X at x_0 is path homotopic to a product of loops each of wich is contained in a single A_α.*

Proof. Consider a loop $f : I \to X$ with basepoint x_0. For each point $f(s)$, we have an open neighborhood U_s which is contained in some A_{α_s}. Taking $f^{-1}(U_s \cap f(I))$ gives an open neighborhood $V_s \subset I$ satisfying $f(V_s) \subset A_{\alpha_s}$ makes possible to take the open interval $I_s \subset V_s$ containing s where $f(\mathrm{cl}(I)) \subset A_{\alpha_s}$. Since I is compact, we can take only finite number of $s \in I$ so that the collection of I_s cover I. Taking the endpoints of these intervals gives a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that each subinterval $[s_{i-1}, s_i]$ satisfies $f([s_{i-1}, s_i]) \subset A_{\alpha_i}$. Define paths $f_i : I \to X$ as

$$f_i(s) = f((1-s)s_{i-1} + ss_i). \tag{146}$$

Then, by taking appropriate reparametrization, f is path homotopic to $f_1 \cdot \cdots \cdot f_m$. Since $A_{\alpha_i} \cap A_{\alpha_{i+1}}$ is connected and contains x_0, we may choose a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i) \in A_i \cap A_{i+1}$. Then we may construct a loop

$$(f_1 \cdot \bar{g}_1) \cdot (g_1 \cdot f_2 \cdot \bar{g}_2) \cdot \cdots \cdot (g_{m-1} \cdot f_m) \tag{147}$$

which is path homotopic to f. Furthermore, $f_1 \cdot \bar{g}_1$ is a loop contained in A_{α_1}, $g_{m-1} \cdot f_m$ is a loop contained in A_{α_m}, and $g_i \cdot f_{i+1} \cdot \bar{g}_{i+1}$ is a loop contained in $A_{\alpha_{i+1}}$, showing the statement. □

Proposition 72. $\pi_1(S^n) = 0$ *if* $n \geq 2$.

Proof. Take a point $x_0 \in S^n$, and consider two open sets $A_1 = S^n - \{x_0\}$ and $A_2 = S^n - \{-x_0\}$. Notice that A_1, A_2 are homeomorphic to $\mathbb{R}^n$ and $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, hence path connected. Choose $x \in A_1 \cap A_2$. By the Lemma above, every loop in S^n based on x is homotopic to a product of loops in A_1 or A_2. Since $\pi_1(A_1) \simeq \pi_1(\mathbb{R}^n) \simeq \pi_1(A_2) = 0$, all those loops are nullhomotopic, hence every loop in S^n is nullhomotopic. □

Corollary 73. $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *for* $n \neq 2$.

This is true for any $\mathbb{R}^n$ and $\mathbb{R}^m$ with $n \neq m$, which can be shown using higher homotopy groups or homology groups.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. If $n = 1$, then $\mathbb{R}^2 - \{0\}$ is path connected but $\mathbb{R} - \{f(0)\}$ is not, thus there is no such homeomorphism. If $n > 2$, then since $\mathbb{R}^n - \{f(0)\} \simeq S^{n-1} \times \mathbb{R}$ by, for example, taking $f(0)$ WLOG and giving homeomorphism $\phi : \mathbb{R}^n - \{0\} \to S^{n-2} \times \mathbb{R}$ as

$$\phi(x) = (\frac{x}{|x|}, |x|), \tag{148}$$

$\pi_1(\mathbb{R}^n - \{x\}) \simeq \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$, which is trivial if $n > 2$ but $\mathbb{Z}$ if $n = 2$, contradiction. □

Proposition 74. *X retracts onto a subspace A, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism.*

Proof. If $r : X \to A$ is a retraction, then $r \circ i = 1_A$, thus $r_i \circ i_* = 1_{\pi_1(A,x_0)}$, which implies i_* is injective. If $r_t : X \to X$ is a deformation retraction of X onto A so that $r_0 = 1_X$, $r_t|_A = 1_A$, and $r_1(X) \subset A$, then for any loop $f : I \to X$ based on x_0, the composition $r_t \circ f$ is a loop homotopy between f and $r_1 \circ f$, a loop in A, which shows that

$$i_*([r_1 \circ f]) = [i \circ r_1 \circ f] = [r_1 \circ f] = [f] \tag{149}$$

thus i_* is surjective. □

Example 75. S^1 is not a retract of D^2.

This is proved in the proof of Brower fixed point theorem in different way.

Proof. If D^2 retracts onto S^1, then we must have a injective homomorphism $\phi : \pi_1(S^1) \to \pi_1(D^2)$, but since this must be an injective homomorphism $\phi : \mathbb{Z} \to 0$, which is impossible, there is no such retraction. □

Definition 76. A **homomorphism retraction** is a homomorphism $\rho : G \to H \leq G$ satisfying $\rho|_H = 1_H$.

If $H \trianglelefteq G$, then $G = H \times \ker(\rho)$. If $H \ntrianglelefteq G$, then G is the semi-direct product of H and $\ker(\rho)$. For detailed information see *Abstract Algebra, third edition, D. Dummit and R. Foote, Wiley*, section 5.5.

Proposition 77. *For the retraction $r : X \to A$, r_* is a homomorphism retraction.*

Proof. If f is a loop in A, then $r \circ f = f$, thus $r_*([f]) = [f]$. □

2019- 04- 15

Definition 78. Let $x_0 \in X$ and $y_0 \in Y$. If $\phi_t : X \to Y$ is a homotopy with $\phi_t(x_0) = y_0$ for all t, then we call ϕ a **basepoint-preserving homotopy**. If two maps $f, g : (X, x_0) \to (Y, y_0)$ are basepoint-preserving homotopic, then we write $f \simeq_0 g$. If two spaces with basepoints $(X, x_0), (Y, y_0)$ has maps $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (X, x_0)$ such that $\phi \circ \psi \simeq_0 1_Y$ and $\psi \circ \phi \simeq_0 1_X$, then we write $(X, x_0) \simeq (Y, y_0)$.

Proposition 79. *If $\phi_t : (X, x_0) \to (Y, y_0)$ is a basepoint-preserving homotopy, then $\phi_{0*} = \phi_{1*}$.*

Proof. Since ϕ is a basepoint-preserving homotopy, for a loop f in X with basepoint x_0, $\phi_t \circ f$ is a loop homotopy between $\phi_0 \circ f$ and $\phi_1 \circ f$. Therefore, $\phi_{0*}([f]) = [\phi_0 \circ f] = [\phi_1 \circ f] = \phi_{1*}([f])$. □

Corollary 80. *If $(X, x_0) \simeq (Y, y_0)$, then $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$.*

Proof. Since $(X, x_0) \simeq (Y, y_0)$, we have $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (X, x_0)$ such that $\phi \circ \psi \simeq_0 1_Y$ and $\psi \circ \phi \simeq_0 1_X$. By the proposition above, we get $\phi_*\psi_* = 1_{\pi_1(Y,y_0)}$ and $\psi_*\phi_* = 1_{\pi_1(X,x_0)}$. □

Lemma 81. *If $\phi_t : X \to Y$ is a homotopy and h is the path $\phi_t(x_0)$ for some $x_0 \in X$, then the three maps, induced homomorphisms $\phi_{0*} : \pi_1(X, x_0) \to \pi_1(Y, \phi_0(x_0)), \phi_{1*} : \pi_1(X, x_0) \to \pi_1(Y, \phi_1(x_0))$ and change-of-basepoint map $\beta_h : \pi_1(Y, \phi_1(X_0)) \to \pi_1(Y, \phi_0(x_0))$, we have $\phi_{0*} = \beta_h \phi_{1*}$.*

Proof. Let $h_t(s) = h(ts)$. Notice that $h_0(s) = h(0)$ and $h_1(s) = h(s)$. If f is a loop in X with basepoint x_0, then the map $h_t \cdot (\phi_t \circ f) \cdot \bar{h}_t$ is a loop homotopy with basepoint $\phi_0(x_0)$. Taking $t = 0, 1$ gives $\phi_0 \circ f$ and $h \cdot (\phi_1 \circ f) \cdot \bar{h}$, and since $\beta_h(\phi_{1*}([f])) = \beta_h([\phi_1 \circ f]) = [h \cdot (\phi_1 \circ f) \cdot \bar{h}] = [\phi_0 \circ f] = \phi_{0*}([f])$, we get the desired result. □

Proposition 82. *If $\phi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.*

Proof. Take a homotopy inverse $\psi : Y \to X$. Now consider

$$\pi_1(X, x_0) \underset{\phi_*}{\longrightarrow} \pi_1(Y, \phi(x_0)) \underset{\psi_*}{\longrightarrow} \pi_1(X, \psi \circ \phi(x_0)) \underset{\phi_*}{\longrightarrow} \pi_1(Y, \phi \circ \psi \circ \phi(x_0)) \tag{150}$$

Since $\psi \circ \phi \simeq 1_X$, $\psi_* \circ \phi_* = \beta_h$ for some path h, by the lemma. Since β_h is isomorphism, ϕ_* is injective and ψ_* is surjective. Using same arguement to $\phi_* \circ \psi_*$ gives ψ_* is injective and ϕ_* is surjective, and thus they are isomorphisms. □

Definition 83. Take a collection of groups G_α. The **word** is a finite or empty sequence of nonidentity elements $g_i \in G_{\alpha_i}$, which is written as $g_1 g_2 \cdots g_m$. If so, then we call m a **length** of word. If $m = 0$ then we write it e. For a word $g = g_1 g_2 \cdots g_m$, if $g_i \in G_{\alpha_i}, g_{i+1} \in G_{\alpha_{i+1}}$ then $\alpha_i \neq \alpha_{i+1}$ for all i, we call g a **reduced word**. For a word $g = g_1 g_2 \cdots g_m$, if $g_i, g_{i+1} \in G$, then replacing $g_i g_{i+1}$ in the sequence by the multiplicated result, and if it is identity then removing it, is the **reducing procedure**. Repeating reducing procedure, we get a reduced word $[g_1 \cdots g_m]$ of $g_1 \cdots g_m$. The set of reduced word is written as $*_\alpha G_\alpha$, called the **free product of groups**.

Proposition 84. *Consider a collection of groups G_α and their free product $*_\alpha G_\alpha$. For $g = g_1 \cdots g_m, h = h_1 \cdots h_n \in *_\alpha G_\alpha$, define their product as the word which is obtained by repeating reducing procedures to $g_1 \cdots g_m h_1 \cdots h_n$ until we get reduced word. Then the set $*_\alpha G_\alpha$ with the multiplication is a group.*

Proof. This product is closed since G_α are groups.

Identity. If we attach empty word to the left or right of some reduced word g, then we still get a reduced word g. Hence $eg = ge$.

Inverse. Consider a reduced word $g = g_1 \cdots g_m$. Consider $g^{-1} = g_m^{-1} \cdots g_1^{-1}$. Then $gg^{-1} = g_1 \cdots g_m g_m^{-1} \cdots g_1^{-1} = g_1 \cdots g_{m-1} g_{m-1}^{-1} \cdots g_1^{-1} = \cdots = g_1 g_1^{-1} = e$. Samely, $g^{-1} g = e$.

Associativity. For each $g \in G_\alpha$ define $L_g : *_\alpha G_\alpha \to *_\alpha G_\alpha$ as $L_g(g_1 g_2 \cdots g_m) = [g g_1 g_2 \cdots g_m]$. Now consider $L_g \circ L_{g'}(g_1 \cdots g_m) = [g(g' g_1 \cdots g_m)]$. The reducing procedure happens only when $g, g' \in G_\alpha$ or $g', g_1, \cdots, g_k \in G_\alpha$, or both. Those elements will be reduced into one element, and due to the associativity of G_α, this result is same with $[(gg')g_1 \cdots g_m]$. Therefore $L_g \circ L_{g'} = L_{gg'}$. Furthermore, $L_e \circ L_g = L_g \circ L_e = L_g$ and $L_g \circ L_{g^{-1}} = L_{g^{-1}} \circ L_g = L_e$. Using this data, the map $L : *_\alpha G_\alpha \to \mathrm{Hom}(*_\alpha G_\alpha)$ defined as $L(g_1 \cdots g_m) = L_{g_1 \cdots g_m}$ is well defined. Since $L_g \in \mathrm{Hom}(*_\alpha G_\alpha)$, L_g has associative structure, thus $*_\alpha G_\alpha$ also. □

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Definition 85. Let S be the set. The **free group** F_S **generated by** S is the free product of groups $*_{s\in S}s_{\mathbb{Z}}$. Here, $s_{\mathbb{Z}}$ is a group with elements $\{s^n | n \in \mathbb{Z}\}$.

Example 86. An integer group $\mathbb{Z}$ is a free group generated by $\{1\}$.

Proposition 87. *Let S be the set with size n. Then* $F_S \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$.

Proof. This is true since for each objects $s \in S$, $s_{\mathbb{Z}} \simeq \mathbb{Z}$. □

Theorem 88 (The universal property of free product.)**.** *Take the collection of groups G_α. For any group H and any collection of homomorphisms $\phi_\alpha : G_\alpha \to H$, there is a unique extension $\phi : *_\alpha G_\alpha \to H$.*

Proof. For the existence, we take $\phi(g_1 \cdots g_n) = \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_n}(g_n)$, where $g_i \in G_{\alpha_i}$. If the word $g_1 \cdots g_n$ is not reduced, then for each reducing step, i.e. $g_i g_{i+1}$, multiply $\phi_{\alpha_i}(g_i)\phi_{\alpha_{i+1}}(g_{i+1})$. Do the some reducing procedure, and then multiply all the leftings. Due to the associativity of group, the result is always same, independent to the number or sequence of reducing procedure. Therefore this is well defined, and therefore by definition it is homomorphism. Finally, to show the uniqueness, suppose that there is another extension $\psi : *_\alpha G_\alpha \to H$. Then $\psi(g_1 \cdots g_n) = \psi(g_1) \cdots \psi(g_n)$ since ψ is homomorphism. Since ψ is extension, $\psi(g_\alpha) = \phi(g_\alpha)$ for $g_\alpha \in G_\alpha$, which gives $\psi(g_1 \cdots g_n) = \phi(g_1 \cdots g_n)$. □

Theorem 89 (Van Kampen's Theorem.)**.** *If X is the union of path connected open sets A_α where each contains the basepoint $x_0 \in X$, and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. Furthermore if $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\operatorname{Ker}\Phi \simeq N$, where N is a group generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ with $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and thus Φ induces an isomorphism $\pi_1(X) \simeq *_\alpha \pi_1(A_\alpha)/N$. Here $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.*

Proof. We have already shown the first part: for the given condition, we have shown that all the loop in X at x_0 is path homotopic to a product of loops each of which is contained in a single A_α. We write those loops as $f_1, \cdots, f_n$ and call $[f_1][f_2] \cdots [f_n]$ a **factorization** of $[f]$. Thus, for the loops $f_i \subset A_{\alpha_i}$ and $f \in X$, if $i_{\alpha_1}([f_1]) \cdots i_{\alpha_n}([f_n]) = [f]$, then $[f_1] \cdots [f_n]$ is a factorization of $[f]$. Each factorization is a word in $*_\alpha \pi_1(A_\alpha)$ after reducing it.

Now notice that $i_\alpha(i_{\alpha\beta}(\omega)) i_\beta(i_{\beta\alpha}(\omega)^{-1}) = i_{\alpha\cap\beta}(\omega) i_{\alpha\cap\beta}(\omega)^{-1} = e$ where $i_{\alpha\cap\beta}$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$. This implies that $N \leq \operatorname{Ker}\Phi$. Now consider a trivial loop

f in X, and its factorization $[f_1]\cdots[f_n]$. If we show that $[f_1]\cdots[f_n] \in N$, then $\operatorname{Ker}\Phi \subset N$, and we have proven the theorem.

Since $f_1 \cdot \cdots \cdot f_n$ is homotopic to f, which is homotopic to constant loop, c_{x_0}, we can take a homotopy $F : I \times I \to X$ from $f_1 \cdot \cdots \cdot f_n$ to c_{x_0}. Now for each points $F(s,t)$, we have an open neighborhood $U_{s,t}$ which is contained in some A_{α_s}. Taking $F^{-1}(U_{s,t})$ gives an open neighborhood $V_{s,t} \subset I \times I$ of (s,t) satisfying $F(V_{s,t}) \subset A_{\alpha_{s,t}}$, which makes possible to take the open rectangle $(a_{s,t}, b_{s,t}) \times (c_{s,t}, d_{s,t})$ where $F([a_{s,t} - \epsilon_{s,t}, b_{s,t} + \epsilon_{s,t}] \times [c_{s,t} - \epsilon_{s,t}, d_{s,t} + \epsilon_{s,t}]) \subset A_{\alpha_{s,t}}$, for some $\epsilon_{s,t} > 0$. Since $I \times I$ is compact, we may choose a finitely many open rectangles which covers $I \times I$, where its slightly larger closure is fully contained in some A_α. Now by choosing all the vertices, and drawing vertical and horizontal lines on them, we can take a partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into a single A_{ij}, and a little bit of perturbation of vertical sides of the rectangles R_{ij} so that each point of $I \times I$ lies in at most three R_{ij}'s does not changes the result: still $F(R_{ij}) \subset A_{ij}$. Number the rectangles from left to right, down to right: R_1 for lowest, leftest rectangle, R_2 for the right one, and so on.

Notice that $F(0,t) = F(1,t) = x_0$. Now consider γ_r be the path separating the first r rectangles, $R_1, \cdots, R_r$, from the remaining rectangles. Then γ_0 is the bottom edge, where $F|_{\gamma_0}$ is a constant loop, and γ_{mn} is the top edge, where $F|_{\gamma_{mn}}$ is $f_1 \cdot \cdots \cdot f_n$. Furthermore, all γ_r are the loops with basepoint x_0.

Now call the corners of the R_r's as vertices. For each vertex v, if $F(v) \neq x_0$, then we may choose a path g_v from x_0 to $F(v)$ which lies in the intersection of the two or three A_{ij}'s corresponding to the R_r's containing v, because of the path connectivity of three intersections of A_α. This gives a factorization of $[F|_{\gamma_r}]$, by inserting the paths $\bar{g}_v g_v$ where the vertex v exists on γ_r. Indeed, for the upper edge, we need a bit more trick: choose the path g_v not only included in two A_α's corresponding to the R_s's, but also the one A_α, which contains f_i, which contains v in its domain. If v is the common endpoint of the domains of two consecutive f_i, then $F(v) = x_0$, so we do not need to choose such path.

Now consider the sliding-up of the L-shaped path to ⌝-shaped path on $I \times I$,

$$\gamma_t(s) = \begin{cases} (0, 1-3st), & s \in [0, \frac{1}{3}] \\ (3s-1, 1-t), & s \in [\frac{1}{3}, \frac{2}{3}] \\ (1, 3(1-t)(1-s)), & s \in [\frac{2}{3}, 1] \end{cases} \tag{151}$$

By this pushing-up, we may change γ_r to γ_{r+1} continuously, thus we may change $F|_{\gamma_r}$ to $F|_{\gamma_{r+1}}$ by a homotopy within A_r. This is just an ordinary loop homotopy, but if we consider the

Now we are done. The sliding-up process does not changes the representation on fundamental group, since the sliding-up process can be represented as a loop homotopy on each components. The nontrivial change only happens when we change the inclusion of one edge: for example, if the one edge represents a loop f_i contained in both A_α, A_β, and initially we have $[f_i]_\alpha \in \pi_1(A_\alpha)$, then we need to multiply $i_{\beta\alpha}([f_i]_{\alpha\beta}) i_{\alpha\beta}([f_i]_{\alpha\beta})^{-1}$, where $[f_i]_{\alpha\beta} \in \pi_1(A_\alpha \cap A_\beta)$. Thus the whole procedure to changing constant loop to $[f_1] \cdots [f_n]$ is the successive procedure of multiplying above elements between the loops, which gives that $[f_1] \cdots [f_n] \in N$. This proves the theorem. □

2019- 04- 23

Lemma 90. *Let A be the deformation retract of X. If A is path connected, then X is path connected.*

This can be more easily proven if we use the homology, or especially, H_0.

Proof. Take the deformation retract $F : X \times I \to X$. For any two points $x_0, x_1 \in X$, define $f_0(s) = F(x_0, s)$ and $f_1(s) = F(x_1, s)$. Then f_0 is a path from x_0 and $a \in A$, and f_1 is a path from x_1 and $b \in A$. Finally, since A is path connected, there is a path g connecting a, b. Then the path $(f_0 \cdot g) \cdot \bar{f}_1$ is a path connecting x_0 and x_1. □

Corollary 91 (Wedge sum.). *Let the for each space X_α there is a basepoint x_α and its open neighborhood U_α which deformation retracts to x_α. Then the wedge sum $\vee_\alpha X_\alpha$ identifying their basepoints has the fundamental group $\pi_1(\vee_\alpha X_\alpha) \simeq *_\alpha \pi_1(X_\alpha, x_\alpha)$.*

Proof. Take $A_\alpha = X_\alpha \vee \left(\vee_{\beta \neq \alpha} U_\beta\right)$. Then the intersection of two or more distinct A_α is $\vee_\alpha U_\alpha$. Since U_α has a deformation retract F_α to x_0, $\vee_\alpha U_\alpha$ has a deformation retract to x_0, which is defined from F_α, which is well-defined since $F_\alpha|_{x_0} = 1_{x_0}$ and continuous by pasting lemma. Since one-point set is path connected, $\vee_\alpha U_\alpha$ is path connected, thus we can use the van Kampen's theorem. Since $\vee_\alpha U_\alpha$ is simply connected, $\pi_1(X_\alpha) \simeq \pi_1(A_\alpha)$ and $i_{\alpha\beta}$ is a trivial map sending trivial loop to trivial loop, hence N is a trivial group. Therefore $*_\alpha \pi_1(X_\alpha) \simeq \pi_1(X)$. □

Example 92 (Loop deleted from $\mathbb{R}^3$.). Consider $X = \mathbb{R}^3 - S^1$ where S^1 lies on the xy plane. We can split this into right and left side with a little intersection, A_R and A_L, then since $A_{R,L}$ can deformation retract to S^1, $\pi_1(A_{R,L}) \simeq \mathbb{Z}$, and since $A_R \cap A_L$ can deformation retract to $S^1 \vee S^1$ we get, due to the van Kampen's theorem or due to the corollary above, $\pi_1(A_R \cap A_L) \simeq \mathbb{Z} * \mathbb{Z}$. Now write the generator of $\pi_1(A_{R,L})$ as r, l respectively, and the generator of $\pi_1(A_R \cap A_L)$ as a and b. Then the inclusion map $i_{RL} : \pi_1(A_R \cap A_L) \to \pi_1(A_R)$ is the map with $i_{RL}(a) = i_{RL}(b) = r$, and samely $i_{LR} : \pi_1(A_R \cap A_L) \to \pi_1(A_L)$ is the map with $i_{LR}(a) = i_{LR}(b) = l$. Then the elements of $\operatorname{Ker}\Phi$ is generated by rl^{-1}, which means that we need to quotient $\pi_1(A_R) * \pi_1(A_L)$ with $r = l$. Then we get $\pi_1(A_R \cup A_L) \simeq \pi_1(X) \simeq \mathbb{Z}$.

Example 93 (Two non-linked loops deleted from $\mathbb{R}^3$.)**.** Now consider deleting two S^1 rings from $\mathbb{R}^3$, which are not linked. Putting S^1 on the xy plane and splitting into right and left side as above gives A_R and A_L, which can deformation retract to $S^1 \vee S^1$, thus $\pi_1(A_R) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle r_1, r_2 \rangle$ and $\pi_2(A_R) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle l_1, l_2 \rangle$. Now the intersection $A_R \cap A_L$ can deformation retract to $S^1 \vee S^1 \vee S^1 \vee S^1$, thus $\pi_1(A_R \cap A_L) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \simeq \langle a, b, c, d \rangle$. Now the inclusion map $i_{RL} : \pi_1(A_R \cap A_L) \to \pi_1(A_R)$ is the map with $i_{RL}(a) = i_{RL}(b) = r_1, i_{RL}(c) = i_{RL}(d) = r_2$, and $i_{LR} : \pi_1(A_R \cap A_L) \to \pi_1(A_L)$ is the map with $i_{LR}(a) = i_{LR}(b) = l_1, i_{LR}(c) = i_{LR}(d) = l_2$. Thus the elements of $\operatorname{Ker}\Phi$ is generated by $r_1 l_1^{-1}$ and $r_2 l_2^{-1}$, which means that we need to quotient $\pi_1(A_R) * \pi_1(A_L)$ with $r_1 = l_1, r_2 = l_2$. Then we get $\pi_1(A_R \cup A_L) \simeq \pi_1(X) \simeq \mathbb{Z} * \mathbb{Z}$.

Indeed we can split this space into two spaces where each space contains one loop. Then the intersection is homeomorphic to $\mathbb{R}^3$, which is simply connected, hence we get $\mathbb{Z} * \mathbb{Z}$ again.

Example 94 (Loop and line deleted from $\mathbb{R}^3$.)**.** Before deleting two linked S^1 rings from $\mathbb{R}^3$, first consider deleting S^1 and $\mathbb{R}$ piercing through S^1 from $\mathbb{R}^3$. Putting S^1 on the xy plane and $\mathbb{R}$ at the center vertically, and splitting into right and left side gives A_R and A_L, which can deformation retract to $S^1 \vee S^1$, thus $\pi_1(A_R) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle r, r_{\mathbb{R}} \rangle$ and $\pi_2(A_R) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle l, l_{\mathbb{R}} \rangle$. Here the subscript $_{\mathbb{R}}$ implies the generator is a loop around vertical $\mathbb{R}$. Now the intersection $A_R \cap A_L$ can deformation retract to $S^1 \vee S^1 \vee S^1$, thus $\pi_1(A_R \cap A_L) \simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \simeq \langle a, b, c \rangle$. Here c is the generator of loop around $\mathbb{R}$. Calculating the inclusion map here, however, needs some care. Since at the center of this space we have vertical $\mathbb{R}$ bar, we need to take the basepoint not at the center, but near toward one of subspaces. Take the basepoint closer to A_R. Then the inclusion map $i_{RL} : \pi_1(A_R \cap A_L) \to \pi_1(A_R)$ is the map with $i_{RL}(a) = i_{RL}(b) = r$ and $i_{RL}(c) = r_{\mathbb{R}}$. For the inclusion map $i_{LR} : \pi_1(A_R \cap A_L) \to \pi_1(A_L)$, the result is different, which is because $i_{RL}(a) \neq i_{RL}(b)$. Indeed, $i_{LR}(c) = l_{\mathbb{R}}$ and $l_{\mathbb{R}}^{-1} i_{LR}(a) l_{\mathbb{R}} = i_{LR}(b)$. We now may take $i_{LR}(a) = l$. This gives $\pi_1(X) \simeq \langle r, r_{\mathbb{R}}, l, l_{\mathbb{R}} \rangle / \langle r_{\mathbb{R}} l_{\mathbb{R}}^{-1}, rl^{-1}, l_{\mathbb{R}}^{-1} l l_{\mathbb{R}} r^{-1} \rangle \simeq \langle r, R | R^{-1} r R r^{-1} \rangle$. Now notice that $R^{-1} r R r^{-1} = e$ implies $rR = Rr$, which means that we abelianize $\langle r, R \rangle \simeq \mathbb{Z} * \mathbb{Z}$. This is $\mathbb{Z} \times \mathbb{Z}$.

Indeed this space can be deformation retracted into the torus, T, which has a fundamental group $\pi_1(T) \simeq \mathbb{Z}$, and this confirms the above result.

Example 95 (Two linked loops deleted from $\mathbb{R}^3$.)**.** Finally we delete two linked S^1 rings from $\mathbb{R}^3$. Set one ring horizontally and split the space into left and right side, A_R and A_L, as we have done in one ring case. Consider A_R totally contains one another ring. Then we get $\pi_1(A_R) \simeq \mathbb{Z} \times \mathbb{Z} = \langle r, R \rangle / \langle rRr^{-1}R^{-1} \rangle$ and $\pi_1(A_L) \simeq \mathbb{Z} = \langle l \rangle$, and $\pi_1(A_R \cap A_L) \simeq \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$. Now $i_{RL} : \pi_1(A_R \cap A_L) \to \pi_1(A_R)$ is the map with $i_{RL}(a) = r, i_{RL}(b) = Rr^{-1}R^{-1}$, and $i_{LR}(a) = i_{LR}(b) = l$. Thus we get $\langle l, r, R \rangle / \langle rRr^{-1}R^{-1}, rl^{1-} \rangle \simeq \langle r, R \rangle / \langle rRr^{-1}R^{-1} \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ again.

Indeed this space also can be deformation retracted into the torus, T, which confirms the above result.

Therefore, we have proven that the space with two linked loops and the space with two non-linked loops are not homeomorphic, because $\mathbb{Z} \times \mathbb{Z} \not\simeq \mathbb{Z} * \mathbb{Z}$.

Lemma 96. *For a bounded subspace A of $\mathbb{R}^n$ with $n \geq 3$, $\pi_1(\mathbb{R}^n - A) \simeq \pi_1(S^n - A)$.*

Proof. Notice that S^n can be thought as the one-point compactification of $\mathbb{R}^n$. Now we write $S^n - A$ as the union of $\mathbb{R}^n - A$ and an open ball B, where $B = \{\bullet\} \cup (\mathbb{R}^n - B_A)$, where B_A is a closed ball containing A, which is possible to take since A is bounded. Then B is simply connected, and $B \cap (\mathbb{R}^n - A) \simeq S^{n-1} \times \mathbb{R}$ is also simply connected if $n \geq 3$, therefore by van Kampen's theorem we get the desired result. $\square$

Example 97 (Torus knot.)**.** Take a relative prime positive primes m, n, we call the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$ defined as $f(z) = (z^m, z^n)$ a **torus knot** and write $K_{m,n}$. Now we want to calculate $\pi_1(\mathbb{R}^3 - K_{m,n})$. Due to the lemma above, $\pi_1(\mathbb{R}^3 - K_{m,n}) \simeq \pi_1(S^3 - K_{m,n})$.

Now notice that $S^3 \simeq \partial D^4 \simeq \partial(D^2 \times D^2) \simeq \partial D^2 \times D^2 \cup D^2 \times \partial D^2 \simeq S^1 \times D^2 \cup D^2 \times S^1$. Thus we can think S^3 as a union of two solid torus, one can be thought as the ordinary torus mapped into the $\mathbb{R}^3$ and the other can be thought as the closure of lefting which is one-point compactificated. Notice that the meridian circle of $S^1 \times S^1$ bounds disk of first solid torus, and the longitudinal circle bounds disk of second solid torus. Denote the first solid torus as T_i and second solid torus as T_o.

Now delete $K_{m,n}$ from S^3. This gives two spaces $T_i - K_{m,n}$ and $T_o - K_{m,n}$, whose union is $S^3 - K_{m,n}$ and intersection is $S^1 \times S^1 - K_{m,n}$. Notice that $S^1 \times S^1 - K_{m,n}$ is path connected; indeed, if we shift $K_{m,n}$ a bit in $S^1 \times S^1$, then we can deformation retract $S^1 \times S^1 - K_{m,n}$ into the shifted knot, which is homeomorphic to S^1. Therefore $\pi_1(S^1 \times S^1 - K_{m,n}) \simeq \mathbb{Z}$. Also, since $T_{i,o} - K_{m,n}$ can be deformation retracted into the smaller torus, which also can be deformation retracted into a

Lemma 98. $\partial(X \times Y) \simeq (\partial X \times \bar{Y}) \cup (\bar{X} \times \partial Y)$.

Proof. Take $(x, y) \in \partial X \times \bar{Y}$, and let N be the neighbor of (x, y). By the definition of the product topology, we have open neighbor U of x and V of y such that $U \times V \subset N$. Since $x \in \partial X$, U intersects with both X and X^c. Also, since $y \in \bar{Y}$, the neighbor V must contain the element in Y. Thus $U \times V$ contains the element in $X \times Y$ and $(X \times Y)^c$, and so $(x, y) \in \partial(X \times Y)$. Samely, if $(x, y) \in \bar{X} \times \partial Y$, then $(x, y) \in \partial(X \times Y)$.

Now take $(x, y) \in \partial(X \times Y)$. Suppose that $(x, y) \in (\partial X \times \bar{Y})^c \cap (\bar{X} \times \partial Y)^c$. If $x \notin \bar{X}$ then there is an open neighborhood of x which does not contains any point of X, and same for y, $(x, y) \in \bar{X} \times \bar{Y}$. Therefore $x \notin \partial X$ and $y \notin \partial Y$. But then there is an open neighborhood of x which does not contains any point of X^c, contradiction. $\square$

circle, $\pi_1(T_{i,o} - K_{m,n}) \simeq \mathbb{Z}$. Denote k be the generator of $\pi_1(S^1 \times S^1 - K_{m,n})$ and a, b be the generator of $\pi_1(T_{i,o} - K_{m,n})$.

Now we need to think $i_{io}(k)$ and $i_{oi}(k)$. Indeed, k can be represented as the loop $K_{m,n}$, therefore we need to calculate which represents the loop $K_{m,n}$ in $T_{i,o}$. For T_i, since meridian circle bounds disk, winding around meridian circle is meaningless in the sense of fundamental group of T_i. Therefore the only meaningful winding is winding around longitudinal circle, which means, $i_{io}(k) = a^m$. In T_o, the result is same except we exchange the role of meridian and longitudinal circle, which gives $i_{oi}(k) = b^n$. Therefore, $\pi_1(\mathbb{R}^3) \simeq \langle a, b | a^m = b^n \rangle \simeq \mathbb{Z}_m * \mathbb{Z}_n$ where $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Now since the abelianization of $\mathbb{Z}_m * \mathbb{Z}_n$ is $\mathbb{Z}_m \times \mathbb{Z}_n$ and $\mathbb{Z}_m \times \mathbb{Z}_n \not\simeq \mathbb{Z}_k \times \mathbb{Z}_l$ if $\{m, n\} \neq \{k, l\}$, if we count the all the torus knots $K_{m,n}$ with different index has different knot group.

Lemma 99. *If G, H are abelian group, then the abelianization of $G * H$ is $G \times H$.*

Proof. By the abelianization, all the words becomes the form of $a^m b^n$ with $m, n \in \mathbb{Z}$. Take the map $\phi : (G * H)_{\text{ab}} \to G \times H$ as $\phi(a^m b^n) = (a^m, b^n)$. Then this is well defined, homomorphic, surjective. For injectivity, if $(a^m, b^n) = e$, then $a^m = e_G$ and $b^n = e_H$, thus $a^m b^n = e \in G * H$. Thus ϕ is isomorphism. □

2019- 05- 15

Example 100 (The shrinking wedge of circles and countable wedge sum of circles.)**.** Consider the space $X = \cup_{n=1}^{\infty} C_n \subset \mathbb{R}^2$, where C_n is a circle with radius $1/n$ and center $(1/n, 0)$. Now consider the retractions $r_n : X \to C_n$ which is defined as $r_n|_{C_n} = id_{C_n}$ and $r_n|_{X - C_n} = (0, 0)$. Each retraction induces a homomorphism $\rho_n : \pi_1(X) \to \pi_1(C_n) \simeq \mathbb{Z}$, which is surjective since r_n is a retraction. Now we define the direct *product* of these maps and define $\rho : \pi_1(X) \to \prod_{n=1}^{\infty} \mathbb{Z}$. Choose a sequence of integers $\{k_n\}$. Define a map $f : I \to X$ which is a glued map of $f_n : [1 - 1/n, 1 - 1/(n+1)] \to C_n$, where f_n wraps C_n k_n times. Notice that $f(0) = f(1) = (0, 0)$ Due to the pasting lemma, this map is continuous on $[0, 1)$. At the point 1, the open neighborhood of $(0, 0)$ always contains all but finitely many of the circles C_n, thus its inverse is open. Therefore f is a loop on X, and by definition $\rho([f]) = (k_n)$. This shows that ρ is surjective, hence $\pi_1(X)$ is uncountable.

The best representation of $\pi_1(X)$ group needs shape theory, which will not going to be treated here.

However, the fundamental group of countable wedge sum of circles is the free group generated by $S = \{s_n\}_{n \in \mathbb{Z}}$, which is smaller then the set of words generated by $S' = S \cup \{s_n^{-1} | n \in \mathbb{Z}\}$, i.e. $\cup_{n \in \mathbb{N}} S'^n$. Notice that S' is again infinitely countable, hence we may take a bijection $i : S' \to \mathbb{N} \cup \{0\}$. Now take a bijection $b : \cup_{n \in \mathbb{N}} S'^n \to \mathbb{N}$ as $b(s_{i_1} \cdots s_{i_j}) = 2^{i(s_{i_1})} 3^{i(s_{i_2})} \cdots p_j^{i(s_{i_j})}$, where p_j is the j-th prime number. Therefore the fundamental group of countable wedge sum of circles is countable, which cannot be equal to $\pi_1(X)$.

Proposition 101. *Let X be a path connected space and choose $x_0 \in X$. Consider we have a collection of 2-cells $e_\alpha^2 \simeq D^2 - S^1$ and a collection of* *maps $\phi_\alpha : S^1 \to X$. Define $Y = X \bigcup \cup_\alpha cl(e^2_\alpha)/(\phi_\alpha(s) \sim s, \forall s \in \partial e^2_\alpha)$. Choose $s_0 \in S^1$ as a basepoint, and take a path γ_α in X from x_0 to $\phi_\alpha(s_0)$. Take N the normal subgroup of $\pi_1(X, x_0)$ which is generated by all the loops $\gamma_\alpha \phi_\alpha \bar{\gamma}_\alpha$.*

(a) $\pi_1(Y) \simeq \pi_1(X)/N$.

(b) If we attach n-cells with $n > 2$ rather then 2-cells, then $\pi_1(Y) \simeq \pi_1(X)$.

(c) If X is a cell complex and X^2 is the 2-skeleton of X, then $\pi_1(X^2) \simeq \pi_1(X)$.

Proof.(a) We define Z as a expansion of Y, which is, attaching regular strips $I \times I$ on each γ_α following $I \times \{0\}$, and attaching $\{1\} \times I$ line on $\text{cl}(e^2_\alpha)$. Reducing the height of the strip, it is possible to deformation retract Z to Y. Also, choose y_α from e^2_α which is not on the attached part of the strip.

Now take $A = Z - \cup_\alpha\{y_\alpha\}$ and $B = Z - X$. Then since $e^2_\alpha - y_\alpha$ can be deformation retract to its boundary, A deformation retracts to X, and B is contractible. Furthermore, the intersection can be deformation retracted into the wedge sum of S^1's, whose fundamental group is $*_\alpha\mathbb{Z}$ and generated by $[\gamma_\alpha\phi_\alpha\bar{\gamma}_\alpha]$'s. This is trivial in B, therefore $\pi_1(Y) \simeq \pi_1(X)/N$ where N is generated by $\gamma_\alpha\phi_\alpha\bar{\gamma}_\alpha$'s.

(b) All the argument is same except the intersection is a wedge sum of S^{n-1}'s, where $n > 2$, hence the intersection is contractible. Therefore $\pi_1(Y) \simeq \pi_1(X)$.

(c) Let $f : I \to X$ be a loop at the basepoint $x_0 \in X^2$. Since the image is compact, it is in X^n for some finite n. By (b), f is homotopic to a loop in X^2, hence $\pi_1(X^2) \to \pi_1(X)$ is surjective. Now choose $f : I \to X^2$ a loop which is nullhomotopic in X by a homotopy $F : I \times I \to X$. Since the image is compact, it is in X^n for some finite $n > 2$. Since $\pi_1(X^2) \to \pi_1(X^n)$ is bijective by (b), and f is nullhomotopic in X^n, f is nullhomotopic in X^2.

□

Example 103. Orientable surface with g genus, M_g, has a cell structure with one 0-cell, $2g$ 1-cells, and one 2-cell. Before attaching 2-cell, we have the wedge sum of $2g$ cells, which gives the fundamental group $\langle a_1, b_1, \cdots, a_g, b_g\rangle$, i.e. a free group with $2g$ generators. Attaching 2-cell, the boundary of 2-cell is represented by the product of commutators of the generators, $[a_1, b_1][a_2, b_2]\cdots[a_g, b_g]$. Thus, $\pi_1(M_g) \simeq \langle a_1, b_1, \cdots, a_g, b_g | [a_1, b_1]\cdots[a_g, b_g]\rangle$.

Proposition 102. *A compact subspace in CW complex is contained in a finite subcomplex.*

Proof. Suppose that a compact subspace C of CW complex X meets infinitely many cells in X, therefore we can choose an infinite set $S = \{x_1, x_2, \cdots\} \subset C$ where all x_i lies in different cells. Notice that $S \cap X^0$ is closed in X^0, since it is the set of discrete points. Now suppose that $S \cap X^{n-1}$ is closed in X^{n-1}. For each e^n_α in X, $\phi^{-1}_\alpha(S)$ is closed in ∂D^n_α for attaching map ϕ_α, and since there is at most one point of $\Phi^{-1}_\alpha(S)$ in $\text{cl}(e^n_\alpha)$ where Φ_α is a characteristic map, $\Phi^{-1}_\alpha(S)$ is closed in $\text{cl}(e^n_\alpha)$. Thus S is closed in X. Using same argument shows that any subspace of S is closed, hence S has discrete topology. Since C is compact, S is finite, contradiction. Thus C intersects with finitely many cells. Furthermore, since the closure of cell is compact, closure of cells also intersects with finitely many cells.

Now if we show that every cells are contained in finite subcomplex, then we can show that C is contained in finite subcomplex. Indeed e^1_α is in finite subcomplex which is line, and the boundary of e^n_α is in X^{n-1} and compact thus in finite subcomplex $A \subset X^{n-1}$, therefore e^n_α is in finite subcomplex

Corollary 104. *If $g \neq h$, $M_g \not\simeq M_h$.*

Proof. The abelianization of $\pi_1(M_g)$ is the product of $2g$ copies of $\mathbb{Z}$, so if $M_g \simeq M_h$ then $g = h$. ☐

Corollary 105. *For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \simeq G$.*

Proof. Since every group is a quotient of free group, choose a representation $G = \langle g_\alpha | r_\beta \rangle$. Now attach 2-cells e^2_β to $\vee_\alpha S^1_\alpha$ by the loops specified by the relations r_β. ☐

For G, take a free group generated by all the elements of G. Now put all the multiplication relations as the relation condition of free group, and take the quotient. We can write the result as $\langle G | g_1 g_2 g_3^{-1} = e, \forall g_1, g_2 \in G, g_1 g_2 = g_3 \rangle$.

Example 106. Take $G = \langle a | a^n \rangle$. Then X_G is a circle S^1 with a cell e^2 attached by the map $z \mapsto z^n$.

Proposition 107. *For a space X, take a covering space $\tilde{X}$ and covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$. Then the induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective, and the image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts in $\tilde{X}$ starting at $\tilde{x}_0$ are loops.*

Proof. Take a loop $\tilde{f}_0 : I \to \tilde{X}$ where $f_0 = p \circ \tilde{f}_0$ is path homotopic to trivial loop f_1 by loop homotopy $f_t : I \to X$. By the homotopy lifting property, we have a loop homotopy $\tilde{f}_t : I \to \tilde{X}$ which is lifting of f_t and homotopy between $\tilde{f}_0$ and $\tilde{f}_1$, but since f_1 is a trivial loop, $\tilde{f}_1$ is also a trivial loop, thus $\ker(p_*) = 0$ and p_* is injective. Now if $f : I \to X$ is a loop based on x_0 whose lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x}_0$ is loop, then $p_*([\tilde{f}]) = [f]$, thus $[f] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Conversely, if $f : I \to X$ is a loop based on x_0 where there exists a loop $\tilde{f}' : I \to \tilde{X}$ based on $\tilde{x}_0$ with $p_*([\tilde{f}']) = [f]$, then $[p \circ \tilde{f}'] = [f]$, therefore there is a lifting loop $\tilde{f}$ of f based on $\tilde{x}_0$ satisfying $p \circ \tilde{f} = f$. ☐

Recall(homotopy lifting property): Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, there is a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ which lifts f_t.

Proposition 108. *The number of sheets of a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, where $X, \tilde{X}$ are path connected, equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.*

Proof. Denote $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Define $\Phi : G/H \to p^{-1}(x_0)$ as $\Phi(H[g]) = \tilde{g}(1)$ for the loops g in X based on x_0. Since $h \in H$ then $\tilde{h}$ is also a loop, $\widetilde{h \cdot g} = \tilde{h} \cdot \tilde{g}$ has a same endpoint with $\tilde{g}$. Therefore the map is well defined. Since $\tilde{X}$ is path connected, $\tilde{x}_0$ can be joined to any point $y \in p^{-1}(x_0)$ by a path $\tilde{g}$, thus we can define $g = p \circ \tilde{g}$ such that $\Phi(H[g]) = y$, therefore Φ is surjective. Now suppose that $\Phi(H[g_1]) = \Phi(H[g_2])$. This implies that $g_1 \cdot \bar{g}_2$ lifts to a loop in $\tilde{X}$ based on $\tilde{x}_0$, therefore $[g_1][g_2]^{-1} \in H$, hence $H[g_1] = H[g_2]$ so Φ is injective. ☐

Proposition 109 (Lifting criterion)**.** *Take a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a continuous map $f : (Y, y_0) \to (X, x_0)$ where Y is path* *connected and locally path connected space. Then a lift* $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ *exists if and only if* $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

A space X is locally path connected if for all $x \in X$ and open neighborhood U of x, there is a path connected open neighborhood V of x such that $x \in V \subset U$.

Proof. Suppose that the lift exists. Then since $f = p \circ \tilde{f}$, $f_* = p_* \circ \tilde{f}_*$, thus $f_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Now suppose that $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Let $y \in Y$ and γ be a path from y_0 to y, which exists due to the path connectivity of Y. The path $f \circ \gamma$ in X with starting point x_0 has a unique lift $\widetilde{f \circ \gamma}$ starting at $\tilde{x}_0$. Now define $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$. Notice that $p \circ \tilde{f}(y) = p \circ \widetilde{f \circ \gamma}(1) = f \circ \gamma(1) = f(y)$. Now choose another path γ' from y_0 to y. Since $h_0 = (f \circ \gamma') \cdot (\overline{f \circ \gamma}) = f \circ (\gamma' \cdot \bar{\gamma})$ is a loop with basepoint x_0, $[h_0] \in f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. This shows that there is a path homotopic loop h_1 with h_0 by path homotopy h_t lifts to the loop $\tilde{h}_1$ in $\tilde{X}$ with basepoint $\tilde{x}_0$, and by homotopy lifting property, there is a lifting $\tilde{h}_t$. Thus we get a lifted loop $\tilde{h}_0$ of h_0. Due to the uniqueness of lifted paths, $\tilde{h}_0 = (\widetilde{f \circ \gamma'}) \cdot (\widetilde{\overline{f \circ \gamma}})$. thus $f \tilde{\circ} \gamma(1) = f \tilde{\circ} \gamma'(1)$, and so $\tilde{f}$ is well defined.

Now let $U \subset X$ be an open neighborhood of $f(y)$ with a lift $\tilde{U} \subset \tilde{X}$ containing $\tilde{f}(y)$ such that $p : \tilde{U} \to U$ is a homeomorphism. Since Y is locally path connected, we may choose a path connected open neighborhood $V \subset f^{-1}(U)$ of y with $f(V) \subset U$. Now take a path γ from y_0 to y and a path η from y to $y' \in V$. Then a path $(f \circ \gamma) \cdot (f \circ \eta)$ in X has a lift $(\widetilde{f \circ \gamma}) \cdot (\widetilde{f \circ \eta})$, where $\widetilde{f \circ \eta} = p|_{\tilde{U}}^{-1} \circ f \circ \eta$. Since the endpoint of the last path is in $\tilde{U}$, $\tilde{f}(y') \in \tilde{U}$, thus $\tilde{f}(V) \subset \tilde{U}$. Furthermore, $\tilde{f}(y') = \widetilde{f \circ \eta}(1) = p|_{\tilde{U}}^{-1} \circ f \circ \eta(1) = p|_{\tilde{U}}^{-1} \circ f(y')$, $\tilde{f}|_V = p|_{\tilde{U}}^{-1} \circ f$. Since f and $p|_{\tilde{U}}^{-1}$ is continuous, $\tilde{f}$ is continuous on V, hence $\tilde{f}$ is continuous. □

Example 110. The locally path connected condition is crucial. Consider the **extended topologist's sine curve**, defined as $S = \{(x, y) : y = \sin\left(\frac{\pi}{x}\right), x \in (0, 1]\} \cup (\{0\} \times [-1, 1]) \cup P$, where P is the path connecting $(0, 0)$ and $(1, 0)$ which does not intersects with previous parts except the endpoints, for example, $P = \{(x-1)^2 + (y+1)^2 = 1 : x \in [1, 2]\} \cup [0, 1] \times \{-2\} \cup \{x^2 + (y+1)^2 = 1 : x \in [-1, 0]\}$. This space is not locally path connected, since every open neighborhood of $(0, 0)$ with radius less then 1 is not path connected. Consider the map $p : \mathbb{R} \to S^1$ as $p(\theta) = (\cos\theta, \sin\theta)$ and a continuous map $f : S \to S^1$, which is defined as the composition of two maps, $s \circ q = f$, where $q : S \to \overline{S}$ is defined as

$$q(x, y) = \begin{cases} (x, y), & (x, y) \in P \\ (x, 0), & (x, y) \in S - P \end{cases} \tag{152}$$

and $s : \overline{S} \to S^1$ is defined by mapping the upper line, rightmost half circle, lower line, and leftmost half circle to first, second, third, fourth quadrant of S^1 respectively. Notice that $f_*(\pi_1(S)) = 0$ thus $f_*(\pi_1(S)) \subset p_*(\pi_1(\mathbb{R}))$.

Now write $\{0\} \times [-1,1] = L$. WLOG we may assume that $f(L) = 1$, by rotating S^1 if needed. Now suppose $\tilde{f} : S \to \mathbb{R}$ is a lift of f, i.e. $p \circ \tilde{f} = f$. Since $S - L$ is connected, $\tilde{f}(S-L)$ is connected in $\mathbb{R}$. Also since $p^{-1} \circ f(S-L) = \mathbb{R} - 2\pi\mathbb{Z}$, $\tilde{f}(S-L)$ must be included in the interval, which can be chosen as $(0, 2\pi)$, WLOG. Since f is surjective, $\tilde{f}(S-L) = (0,2\pi)$. Since S is compact, $[0,2\pi] \subset \tilde{f}(S)$, thus $\{0, 2\pi\} \subset \tilde{f}(L)$, and since $f(L) = 1 = p \circ \tilde{f}(L)$, $\{0, 2\pi\} = \tilde{f}(L)$. But since L is connected set, it is contradiction.

Proposition 111 (Unique lifting property)**.** *For a covering space $p : \tilde{X} \to X$ and a map $f : Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ of f agree at one point of Y and Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. For $y \in Y$, let U is an evenly covered open neighborhood of $f(y)$, i.e. $p^{-1}(U)$ is a disjoint union of open sets $\tilde{U}_\alpha$ which is homeomorphic to U by the inverse of p. Let $\tilde{f}_1(y) \in \tilde{U}_1$ and $\tilde{f}_2(y) \in \tilde{U}_2$. Since $\tilde{f}_1, \tilde{f}_2$ are continuous, we have an open neighborhood N of y such that $\tilde{f}_1(N) \subset \tilde{U}_1, \tilde{f}_2(N) \subset \tilde{U}_2$. If $\tilde{f}_1(y) = \tilde{f}_2(y)$ then $\tilde{U}_1$ and $\tilde{U}_2$ intersects, hence $\tilde{U}_1 = \tilde{U}_2 = \tilde{U}$, so $p|_{\tilde{U}} \circ \tilde{f}_1|_N = p|_{\tilde{U}} \circ \tilde{f}_2|_N$. Since $p|_{\tilde{U}}$ is homeomorphism, $\tilde{f}_1|_N = \tilde{f}_2|_N$. If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then $\tilde{U}_1 \neq \tilde{U}_2$, therefore $\tilde{f}_1|_N \neq \tilde{f}_2|_N$. Now we can divide Y into two disjoint sets, $A = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$ and $B = \{y \in Y : \tilde{f}_1(y) \neq \tilde{f}_2(y)\}$. From the argument right before, A, B are open. Since Y is connected, A or B is empty. Since A is not empty, B is empty, and so for all $y \in Y$, $\tilde{f}_1(y) = \tilde{f}_2(y)$. □

2019- 05- 20

Definition 112. A space X is **semilocally simply connected** if for every $x \in X$ there is an open neighborhood $x \in U$ such that the inclusion-induced map $\pi_1(U,x) \to \pi_1(X,x)$ is trivial map.

Proposition 113. *If X has a simply connected covering space, then X is semilocally simply connected.*

Proof. Suppose that $p : \tilde{X} \to X$ is a covering map with simply connected covering space $\tilde{X}$. Then every point $x \in X$ has an open neighborhood U which have a lift $\tilde{U} \subset \tilde{X}$ which is homeomorphic to U by p. Take a loop $f : I \to U \subset X$. This can be lifted to a loop $\tilde{f} : I \to \tilde{U} \subset \tilde{X}$. Now, since $i_{\tilde{U}*}([\tilde{f}]) = 0$ where $i_{\tilde{U}*} : \pi_1(\tilde{U}) \to \pi_1(\tilde{X})$ is induced homomorphism of inclusion $i_{\tilde{U}} : \tilde{U} \hookrightarrow \tilde{X}$, $p_*(i_{\tilde{U}*}([\tilde{f}]) = [p \circ i_{\tilde{U}} \tilde{f}] = [i_U f] = i_{U*}([f]) = 0$, where $i_{\tilde{U}*} : \pi_1(U) \to \pi_1(X)$ is induced homomorphism of inclusion $i_U : U \hookrightarrow \tilde{X}$. Therefore X is semilocally simply connected. □

Example 114. The shrinking wedge of circles space is not semilocally simply connected, since every open neighborhood of $(0,0)$ contains infinitely many circles, whose loop can be nontrivial in whole space.

Proposition 115. *If X is path connected, locally path connected, and semilocally simply connected, then X has a simply connected covering space.*

Proof. Take a basepoint $x_0 \in X$. Define $\tilde{X}$ as a set of homotopy classes of γ, where γ is a path in X starting at x_0. Define $p : \tilde{X} \to X$ as $p([\gamma]) = \gamma(1)$, which is well defined, and since X is path connected p is surjective.

Now we define $\mathcal{U}$ as a collection of the path connected open sets $U \subset X$. Since X is semilocally simply connected, every points $x \in X$ contains such an open set. Also, every path connected open subset V of U satisfies $V \in \mathcal{U}$ if $U \in \mathcal{U}$, because the map $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is trivial. Therefore for any $U, U' \in \mathcal{U}$, $U \cap U'$ has a path connected open set V since X is locally path connected, which is contained in $\mathcal{U}$. Finally, for every open neighborhood $x \in U$, there is a path connected open set V satisfying $x \in V \in U$ since X is locally path connected. Thus $\mathcal{U}$ is the basis of X.

Now for $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let $U_{[\gamma]}$ be a set of $[\gamma \cdot \eta]$ where η is a path in U with $\gamma(1) = \eta(0)$. Notice that $p|_{U_{[\gamma]}}$ is surjective since U is path connected, and injective since $\pi_1(U) \to \pi_1(X)$ is trivial. Furthermore, if $[\gamma'] \in U_{[\gamma]}$ then $\gamma' = \gamma \cdot \eta$ for some path η in U, and then the elements of $U_{[\gamma]'}$ can be written as the form $[(\gamma \cdot \eta) \cdot \mu] = [\gamma \cdot (\eta \cdot \mu)]$, hence lie in $U_{[\gamma]}$. Similarly, the elements of $U_{[\gamma]}$ can be written as $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \bar{\eta} \cdot \mu] = [\gamma' \cdot \bar{\eta} \cdot \mu]$, thus lie in $U_{[\gamma']}$, hence $U_{[\gamma]} = U_{[\gamma']}$.

Therefore, for two $U_{[\gamma]}, V_{[\gamma']}$ and $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we have $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Thus choose $W \in \mathcal{U}$ such that $W \subset U \cap V$ and contains $\gamma''(1)$, then $[\gamma''] \in W_{[\gamma'']} \subset U_{[\gamma]} \cap V_{[\gamma']}$. Thus the collection of $U_{[\gamma]}$ can be thought as a basis. We give a topology of $\tilde{X}$ by using this basis.

Now take the map $p|_{U_{[\gamma]}}$. For $V \in \mathcal{U}$ contained in U and $[\gamma'] \in U_{[\gamma]}$ with $\gamma'(0), \gamma'(1) \in V$, $p(V_{[\gamma']}) = V$ and $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']} \cap U_{[\gamma']} = V_{[\gamma']}$ since $V_{[\gamma']} \subset U_{[\gamma]}$. Therefore $p|_{U_{[\gamma]}}$ is homeomorphism. The inverse image part shows that p is also continuous. Furthermore, $p^{-1}(U)$ is the union of $U_{[\gamma]}$ for varying $[\gamma]$, which are disjoint because $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ implies $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$.

Finally we need to show that $\tilde{X}$ is simply connected. For a point $[\gamma] \in \tilde{X}$, define $\gamma_t : I \to X$ as

$$\gamma_t(s) = \begin{cases} \gamma(s), & s \in [0,t] \\ \gamma(t), & s \in [t,1] \end{cases} \tag{153}$$

Then the function $t \mapsto [\gamma_t]$ is a path in $\tilde{X}$ lifting γ starting at $[c_{\gamma(0)}]$

and ending at $[\gamma]$, where $c_{\gamma(0)}$ is the constant path at $\gamma(0)$. Therefore $\tilde{X}$ is path connected. Now since p_* is injective, it is enough to show that $p_*(\pi_1(\tilde{X}, [c_{\gamma(0)}])) = 0$. Now the elements of $p_*(\pi_1(\tilde{X}, [c_{\gamma(0)}]))$ can be represented by the loops γ based on $\gamma(0)$ and lift to loops in $\tilde{X}$ based on $[c_{\gamma(0)}]$. Since γ lifts to $[\gamma_t]$, and this must be a loop, $[\gamma_1] = [\gamma] = [\gamma_0] = [c_{\gamma(0)}]$. Therefore $[\gamma]$ is trivial and $\pi_1(\tilde{X})$ is trivial. □

2019- 05- 21

Proposition 116. *Let X be a path connected, locally path connected, and semilocally simply connected space. Then for every subgroup $H \leq \pi_1(X, x_0)$, there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for some basepoint $\tilde{x}_0 \in X_H$.*

Proof. For points $[\gamma], [\gamma'] \in \tilde{X}$ where $\tilde{X}$ is a simply connected covering space of X, define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma'}] \in H$. Since $[\gamma \cdot \bar{\gamma}] = e$, $[\gamma] \sim [\gamma]$; since $[\gamma \cdot \bar{\gamma'}]^{-1} = [\gamma' \cdot \bar{\gamma}]$ and H is a group, $[\gamma] \sim [\gamma'] \leftrightarrow [\gamma'] \sim [\gamma]$; if $[\gamma \cdot \bar{\gamma'}], [\gamma' \cdot \bar{\gamma''}] \in H$, then $[\gamma \cdot \bar{\gamma'}][\gamma' \cdot \bar{\gamma''}] = [\gamma \cdot \bar{\gamma''}] \in H$ since H is a subgroup. Therefore $\sim$ is an equivalence relation.

Now let $X_H = \tilde{X}/\sim$. Then, for any $[\gamma], [\gamma'] \in \tilde{X}$, if any two points of $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in X_H, then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$ for some η, and then $[\gamma] \sim [\gamma']$, thus $U_{[\gamma]} = U_{[\gamma']}$. Thus the projection $X_H \to X$ induced by $[\gamma] \mapsto \gamma(1)$ is a covering map.

Finally choose $\tilde{x}_0 \in X_H$ which is the equivalence class of constant path $[c_{x_0}]$. Take a loop γ based on x_0. Then its lift to $\tilde{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, thus the image of the lift under $p_* : \pi_1(X_H, \tilde{x}_0) \to \pi_1(X, x_0)$ is a loop if and only if $[\gamma] \sim [c_{x_0}]$, or equivalently, $\gamma \in H$. Therefore $p_*(\pi_1(X_H, \tilde{x}_0)) = H$, and since p_* is injective, $\pi_1(X_H, \tilde{x}_0) \simeq H$. □

2019- 05- 23

Definition 117. For two covering spaces $p_1 : \tilde{X}_1 \to X, p_2 : \tilde{X}_2 \to X$, the **isomorphism** between these covering spaces is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ satisfying $p_1 = p_2 \circ f$.

Corollary 118. *If $f : \tilde{X}_1 \to \tilde{X}_2$ is an isomorphism between two covering spaces $p_1 : \tilde{X}_1 \to X, p_2 : \tilde{X}_2 \to X$, then f^{-1} is also.*

Proof. Since f is homeomorphism, $f^{-1} : \tilde{X}_2 \to \tilde{X}_1$ is also homeomorphism, and since $p_1 = p_2 \circ f$, $p_2 = p_1 \circ f^{-1}$. □

Proposition 119. *If X is path connected and locally path connected, then two path connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are iso-*

morphic by an isomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ *taking a basepoint* $\tilde{x}_1 \in p_1^{-1}(x_0)$ *to* $\tilde{x}_2 \in p_2^{-1}(x_0)$ *if and only if* $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

Proof. If we have the isomorphism, then since $p_1 = p_2 \circ f$ we have $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*} \circ f_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) \subset p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ and since $p_2 = p_1 \circ f^{-1}$ we have $p_2 * (\pi_1(\tilde{X}_2, \tilde{x}_2)) \subset p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1))$, thus we get the desired result. Conversely suppose that $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$. Since $\tilde{X}_1$ is path connected and locally path connected, and by the given condition of fundamental groups, we can use the lifting criterion and take the lifting of p_1 to $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \circ \tilde{p}_1 = p_1$. Similarly we can take the lifting of p_2 to $\tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$ with $p_1 \circ \tilde{p}_2 = p_2$. Notice that $p_2 \circ (\tilde{p}_1 \circ \tilde{p}_2) = p_2$, thus $\tilde{p}_1 \circ \tilde{p}_2$ is the lift of p_2 by p_2. Since $1_{\tilde{X}_2}$ is also the lift of p_2 by p_2, by the unique lifting property, $\tilde{p}_1 \circ \tilde{p}_2 = 1_{\tilde{X}_2}$. Similarly $\tilde{p}_2 \circ \tilde{p}_1 = 1_{\tilde{X}_1}$, therefore $\tilde{p}_1$ and $\tilde{p}_2$ are inverse isomorphisms. □

Lemma 120. *If* $p : \tilde{X} \to X$ *is a covering map and* X *is locally path connected, then* $\tilde{X}$ *is locally path connected.*

Proof. Take $\tilde{x} \in \tilde{X}$ and its open neighborhood $\tilde{x} \in V$. Let U be an evenly connected open neighborhood of $x = p(\tilde{x})$. Denote the evenly covering open set of $p^{-1}(U)$ containing $\tilde{x}$ as U_α. Since U_α is homeomorphic to U, $V \cap U_\alpha$ is homeomorphic to $p(V \cap U_\alpha) = p(V) \cap U$, which is an open set containing x. Since X is locally path connected, we may take an path connected open neighborhood W of x included in $p(V) \cap U$. Taking inverse image, $p|_{U_\alpha}^{-1}(W)$, gives a path connected open neighborhood of $\tilde{x}$ in V. □

Theorem 121 (Covering space classification theorem.). *Let* X *be a path connected, locally path connected, and semilocally simply connected space. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path connected covering spaces* $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *and the set of subgroups of* $\pi_1(X, x_0)$*, which is obtained by associating the subgroup* $p_*(\tilde{X}, \tilde{x}_0)$ *to the covering space* $(\tilde{X}, \tilde{x}_0)$*. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path connected covering spaces* $p : \tilde{X} \to X$ *and conjugacy classes of subgroups of* $\pi_1(X, x_0)$.

Proof. The first statement is proven by the previous propositions. For the second statement, we prove that for a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, if we change the basepoint $\tilde{x}_0$ to some point of $p^{-1}(x_0)$, then this corresponds to taking the conjugate of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$. Choose $\tilde{x}_1 \in p^{-1}(x_0)$ and take $\tilde{\gamma}$ a path from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\gamma = p \circ \tilde{\gamma}$ is a loop in X with basepoint x_0. Define $H_i = p_*(\pi_1(\tilde{X}, \tilde{x}_i))$ for $i = 0, 1$. For a loop $\tilde{f} : I \to \tilde{X}$ with basepoint $\tilde{x}_0$, $\bar{\tilde{\gamma}} \cdot \tilde{f} \cdot \tilde{\gamma}$ is a loop at $\tilde{x}_1$. Thus $[\gamma]^{-1} H_0 [\gamma] \subset H_1$. Similarly we can show $[\gamma] H_1 [\gamma]^{-1} \subset H_0$, thus $[\gamma]^{-1} H_0 [\gamma] = H_1$. Conversely, take $H_0 = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ and let $H_1 = g^{-1} H_0 g$ be a conjugate subgroup. Choose a loop γ representing g, and lift it to $\tilde{\gamma}$ starting at $\tilde{x}_0$. Now, take a loop $\tilde{f} : I \to \tilde{X}$ with basepoint $\tilde{x}_0$, then $g^{-1} p_*([\tilde{f}]) g = [\gamma]^{-1} p_*([f])[\gamma] = p_*([\tilde{\gamma}]^{-1} [\tilde{f}][\tilde{\gamma}]) = p_*([\bar{\tilde{\gamma}} \cdot \tilde{f} \cdot \tilde{\gamma}]) \in p_*(\pi_1(\tilde{X}, \tilde{x}_1))$, thus $H_1 \subset p_*(\pi_1(\tilde{X}, \tilde{x}_1))$. Similarly, using $H_0 = g H_1 g^{-1}$, we get $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \subset H_1$, thus $H_1 = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$. Therefore the conjugation gives the basepoint change. □

Definition 122. A simply connected cover of X, if exists, is called the **universal cover**.

Due to the covering space classification theorem, the universal cover is the covering space of every path connected covering space. Since it is unique up to isomorphism, it is called *the* universal cover.

Definition 123. Let $p : \tilde{X} \to X$ be a covering map and $x_0 \in X$. For a loop $\gamma : I \to X$ with basepoint x_0, the **(right) action of** $\pi_1(X, x_0)$ **on the fiber** $p^{-1}(x_0)$ can be defined as, for $\tilde{x}_1 \in p^{-1}(x_0)$, $\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_1$.

Definition 124. For a group G and a set X, a **left group action** is a function $\psi : G \times X \to X$, where $\psi(g, x)$ is often written as $g \cdot x$, which satisfies (1) $e \cdot x = x$ for all $x \in X$ where e is an identity of G, and (2) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$. a **right group action** is a function $\psi : X \times G \to X$, where $\psi(x, g)$ is often written as $x \cdot g$, which satisfies (1) $x \cdot e = x$ for all $x \in X$ where e is an identity of G, and (2) $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and $x \in X$.

Proposition 125. *For a covering map $p : \tilde{X} \to X$ and $x_0 \in X$, the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ is actually an action.*

Proof. First we need to show that the action is well defined. Take $\gamma_1, \gamma_2 : I \to X$ with basepoint x_0, and $[\gamma_1] = [\gamma_2]$. Then since γ_1 and γ_2 are path homotopic, there is a path homotopy $F : I \times I \to X$ between γ_1 and γ_2. Also, there is a lifting of those paths and path homotopy, $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{F}$, where $\tilde{\gamma}_{1,2}$ are the lifting of $\gamma_{1,2}$ and $\tilde{F}$ is a lifted path homotopy between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Therefore $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$.

Next, take a constant loop c_{x_0}. The lifting of this loop starting at $\tilde{x}_1$ is $c_{\tilde{x}_1}$, therefore $\tilde{x}_1 \cdot [c_{x_0}] = \tilde{x}_1$. Finally, take two loops γ_1, γ_2 with basepoint x_0. Then there is a lifting $\tilde{\gamma}_1$ of γ_1 starting at $\tilde{x}_1$, and a lifting $\tilde{\gamma}_2$ of γ_2 starting at $\tilde{\gamma}_2(1)$. Furthermore, notice that $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2$ is a lifting of $\gamma_1 \cdot \gamma_2$. Therefore, $(\tilde{x}_1 \cdot [\gamma_1]) \cdot [\gamma_2] = \tilde{\gamma}(1) \cdot [\gamma_2] = \tilde{\gamma}_2(1)$ and $\tilde{x}_1 \cdot ([\gamma_1][\gamma_2]) = \tilde{x}_1[\gamma_1 \cdot \gamma_2] = (\tilde{\gamma}_1 \cdot \tilde{\gamma}_2)(1) = \tilde{\gamma}_2(1)$. Thus this is an action. □

2019- 05- 30

Definition 126. For a covering space $p : \tilde{X} \to X$, the covering space isomorphisms $\tilde{X} \to \tilde{X}$ are called **deck transformations** or **covering transformations**.

Proposition 127. *For a covering space $p : \tilde{X} \to X$, the set of deck transformations $G(\tilde{X})$ with function composition as a binary operation is a group.*

Proof. Since the elements of $G(\tilde{X})$ are functions and binary operation is function composition, the binary operation is associative. Since the identity map is in $G(\tilde{X})$, we have an identity, and since the inverse of covering space isomorphism is also covering space isomorphism, we have inverse for every elements. □

Example 128. For the covering space $p : \mathbb{R} \to S^1$, take a homeomorphism $f : \mathbb{R} \to \mathbb{R}$. To make f a covering space isomorphism, we have $p \circ f = p$, which means $e^{if(\theta)} = e^{i\theta}$ for all $\theta \in \mathbb{R}$. This implies that $f(\theta) = \theta + 2n(\theta)\pi$ where $n(\theta) : \mathbb{R} \to \mathbb{N}$, and to make f homeomorphism $n(\theta)$ is a constant function. Thus $f(\theta) = \theta + 2n\pi$, and so we can take an isomorphism $\phi : G(\tilde{X}) \to \mathbb{Z}$ as $\phi(f) = f(0)/2\pi$.

Proposition 129. *Take a covering space $p : \tilde{X} \to X$ with connected $\tilde{X}$. If two deck transformations $f, g : \tilde{X} \to \tilde{X}$ agree at one point, then $f = g$.*

Proof. Since $p = p \circ f$ and $p = p \circ g$, f and g are lift of p. Since f, g agree at one point of $\tilde{X}$ and $\tilde{X}$ is connected, $f = g$ by unique lifting property. $\square$

Definition 130. A covering space $p : \tilde{X} \to X$ is called **normal** or **regular** if for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}'$ of x, there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

Proposition 131. *Take a regular covering space $p : \tilde{X} \to X$. For a set $F = p^{-1}(x)$ for some $x \in X$, define a map $\phi : G(\tilde{X}) \times F \to F$ as $\phi(f, \tilde{x}) = f(\tilde{x})$. Then this is a regular group action.*

Definition 132. An action of a group G on a set X is **regular** if for all x_1, x_2, there is a unique $g \in G$ such that $g \cdot x_1 = x_2$.

This is why we call this covering space regular.

Proof. Since we have a regular covering space, for all $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, there is a deck transformation f satisfying $f(\tilde{x}) = \tilde{x}'$. If there are two such deck transformations f, g, then since they agrees on one point, $f = g$. $\square$

Proposition 133. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path connected covering space of the path connected and locally path connected space X. Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$. Then,*

1. *This covering space is normal if and only if $H \trianglelefteq \pi_1(X, x_0)$.*
2. *$G(\tilde{X}) \simeq N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

This is why we call this covering space normal.

Definition 134. For a subset S of a group G, a **normalizer** of S, $N(S)$, in the group G is defined as $N(S) = \{g \in G | gS = Sg\}$.

Thus $G(\tilde{X}) \simeq \pi_1(X, x_0)/H$ if $\tilde{X}$ is a normal covering, and $G(\tilde{X}) \simeq \pi_1(X, x_0)$ if $\tilde{X}$ is a universal cover.

Proof. 1. From the proof of the covering space classification theorem, changing the basepoint $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ corresponds to conjugating H by $[\gamma] \in \pi_1(X, x_0)$ where $\gamma = p \circ \tilde{\gamma}$ with $\tilde{\gamma}$ is a path from $\tilde{x}_0$ to $\tilde{x}_1$. Thus $[\gamma] \in N(H)$ if and only if $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$. Due to the lifting criterion, this is equivalent to the existence of a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$, considering the lifting of maps $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and $p : (\tilde{X}, \tilde{x}_1) \to (X, x_0)$ by each other. Thus the covering space is normal if and only if for all $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ we have a path $\tilde{\gamma}$ connecting $\tilde{x}_0$ and $\tilde{x}_1$ such that $[\gamma] \in N(H)$ where $\gamma = p \circ \tilde{\gamma}$, which is equivalent with $N(H) = \pi_1(X, x_0)$, i. e. $H \leq \pi_1(X, x_0)$.

2. Define $\phi : N(H) \to G(\tilde{X})$ where $\phi([\gamma])$ is a deck transformation τ taking $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$, where γ lifts to $\tilde{\gamma}$ which is a path from $\tilde{x}_0$ to $\tilde{x}_1$. Since the lifting of path homotopic paths have same endpoints, this map is well defined. From above, $[\gamma] \in N(H)$ is equivalent with the fact that there is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$, thus the map is surjective. Finally, take another $[\gamma']$ where $\phi([\gamma']) = \tau'$ is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1'$. The lifting of γ' starting at $\tilde{x}_1$ can be written as $\tau \circ \tilde{\gamma}'$, since

$p \circ \tau = p$, therefore the lifting of $\gamma \cdot \gamma'$ is $\tilde{\gamma} \cdot (\tau \circ \tilde{\gamma}')$, which is a path between $\tilde{x}_0$ and $\tau \circ \tau'(\tilde{x}_0)$, thus $\tau \circ \tau'$ is the deck transformation corresponding to $[\gamma][\gamma']$. Therefore ϕ is homomorphism. The kernel of ϕ is the classes $[\gamma]$ where γ lifts to a loop in $\tilde{X}$, which is $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$. □

2019- 05- 31

Definition 135. For an action of a group G on a space X, the **orbits** are the sets $G \cdot x = \{g \cdot x | g \in G\}$ for all $x \in X$. The **orbit space** X/G is a quotient space $X/[x \sim g \cdot x, \forall g \in G]$, i.e. the space where all the points are orbits.

Definition 136. For an action of a group G on a space X, if each $x \in X$ has an open neighborhood U such that all the images $g(U)$ for $g \in G$ are disjoint for different g, then we call this a **covering space action**.

Proposition 137. *For a covering space $p : \tilde{X} \to X$, the action of deck transformation group $G(\tilde{X})$ acting on $\tilde{X}$ is a covering space action.*

Proof. Since $\tilde{X}$ is a covering space, we can choose an open neighbor U of $x \in X$ such that we can choose $\tilde{U} \subset \tilde{X}$ homeomorphic to U by the restriction of p. If $g_1 \cdot \tilde{U} \cap g_2 \cdot \tilde{U} \neq \emptyset$ for some $g_1, g_2 \in G(\tilde{X})$, then $g_1(\tilde{x}_1) = g_2(\tilde{x}_2)$ for some $\tilde{x}_1, \tilde{x}_2 \in \tilde{U}$. Thus $p(\tilde{x}_1) = p(\tilde{x}_2)$, but since $p|_{\tilde{U}}$ is a homeomorphism, $\tilde{x}_1 = \tilde{x}_2$, and g_1 and g_2 are deck transformations which agree at one point, thus $g_1 = g_2$. □

Proposition 138. *If an action of a group G on a space X is a covering space action, then*

1. *The quotient map $p : X \to X/G$ with $p(x) = G \cdot x$ is a normal covering space.*
2. *G is a group of deck transformations of the covering space $p : X \to X/G$ if X is path connected.*
3. *$G \simeq \pi_1(X/G)/p_*(\pi_1(X))$ if X is path connected and locally path connected.*

Proof. 1. Take an open neighborhood U of $x \in X$ which is given in the definition of covering space action. Then all the disjoint homeomorphic sets $g \cdot U$ satisfies $p(g \cdot U) = U$. Take $g = e$, hence $g \cdot U = U$. Then $p|_U : U \to p(U)$ is continuous and surjective. Suppose that $x_0, x_1 \in U$ satisfies $p(x_0) = p(x_1)$. Then $G \cdot x_0 = G \cdot x_1$, which means $x_0 = g \cdot x_1$ for some $g \in G$. But

Definition 139. A topological property $\mathcal{P}$ is **cohereditary** if for the space X with property $\mathcal{P}$ its quotient space has the property $\mathcal{P}$.

Proposition 140. *Path connectivity is cohereditary.*

Proof. Let X be a path connected space. Take a quotient map $p : X \to Q$ and $q_1, q_2 \in Q$. Then we may choose a point $x_1, x_2 \in X$ such that $p(x_1) = q_1$ and $p(x_2) = q_2$. Since X is path connected, there is a path $\gamma : I \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Now $p \circ \gamma : I \to Q$ is continuous since p and γ are continuous, and $p \circ \gamma(0) = q_1$ and $p \circ \gamma(1) = q_2$, thus $p \circ \gamma$ is a path connecting q_1, q_2. □

since $g \cdot U$ is disjoint with U, $g = e$, hence $x_0 = x_1$. Therefore p is injective. Also, define a map $f_g : X \to X$ as $f_g(x) = g \cdot x$. Since the action is an action on a space, f_g is homeomorphism, and since $p \circ f_g = p$, each elements $g \in G$ represents different covering isomorphisms. Finally, for any $[x_0] \in X/G$ and its liftings $g_1 \cdot x_0$ and $g_2 \cdot x_0$, $f_{g_2 g_1^{-1}}(g_1 \cdot x_0) = g_2 \cdot x_0$, thus p is normal.

2. Choose f as a deck transformation of the covering space $p : X \to X/G$. For $x \in X$, there is $g \in G$ such that $g \cdot x = f(x)$. Now since if two deck transformations agree at one point then they are same, $f = f_g$.

3. Since X is path connected and locally path connected, X/G is also path connected and locally path connected. Due to the proposition above, $G \simeq N(p_*(\pi_1(X)))/p_*(\pi_1(X))$. Since p is normal, $N(p_*(\pi_1(X))) \simeq \pi_1(X/G)$. Therefore $G \simeq \pi_1(X/G)/p_*(\pi_1(X))$.

$\square$

2019- 06- 03

Definition 142. For an action of a group G on a space X, if each $x \in X$ has an open neighborhood U such that all but finitely many images $g(U)$ for $g \in G$ are disjoint for different g, then we call this a **properly discontinuous action**.

Proposition 143. *If a group G acts on a Hausdorff space X freely and properly discontinuously, then the action is a covering space action.*

Definition 144. Take a group G acting on a set X. A point $x \in X$ is a **fixed point** if $g \cdot x = x$ for some nontrivial $g \in G$. If X has no fixed point, then the action is called a **free action**.

Proof. Suppose that for some $x \in X$ and its open neighborhood U all but finitely many $g \in G$ satisfies $U \cap g \cdot U = \emptyset$. Then for any $g' \in G$, all but finitely many $g'' \in G$ satisfies $g' \cdot U \cap g'' \cdot U = \emptyset$ because $g' \cdot U \cap g'' \cdot U$ is homeomorphic to $U \cap g'^{-1} g'' \cdot U$. Therefore it is enough to show that $U \cap g \cdot U = \emptyset$ for all $g \in G$. We can choose a finite elements $g_1, \cdots, g_n \in G$ such that if $g' \neq g_i$ for all $i = 1, \cdots, n$ then $U \cap g' \cdot U = \emptyset$. Now since the action is free, $g_i \cdot x \neq x$ for all $i = 1, \cdots, n$, and since X is Hausdorff there is disjoint open neighborhoods U_i, V_i of $x, g_i \cdot x$ respectively. Now take $V = U \cap \bigcap_i U_i \cap \bigcap_i g_i^{-1} \cdot V_i$. This is intersection of finitely many open sets, hence open neighborhood of x, and $V \cap g' \cdot V = \emptyset$ for all $g' \neq g_i$. Furthermore, $V \cap g_i \cdot V \subset U_i \cap g_i \cdot (g_i^{-1} \cdot V_i) = U_i \cap V_i = \emptyset$. Therefore the action is a covering space action. $\square$

Definition 145. Let G be a group which has a representation $\langle g_\alpha | r_\beta \rangle$. The **Cayley graph of G with respect to the generators** g_α is a graph whose vertices are the elements of g and edges join gg_α for each generators g_α. For each relations r_β, which is represented as a loop in Cayley graph, attaching 2-cell gives a cell complex $\tilde{X}_G$, which is called the **Cayley complex of** G.

Proposition 146. *Let $G = \langle g_\alpha | r_\beta \rangle$ is a group. Then the Cayley graph of G with respect to the generators g_α is path connected, and the group G acts on the Cayley complex of G, $\tilde{X}_G$, by the multiplication on the left, which is a covering space action. Also $\tilde{X}_G$, is the universal cover of X_G, a 2-dimensional cell complex which has fundamental group G which is shown in the previous corollary.*

Proof. Since every elements can be represented as the finite multiplication of generators, each vertex and vertex $\{e\}$ are path connected, thus the Cayley graph of G is path connected. Now consider the action of G on $\tilde{X}_G$ as, for $g \in G$, g takes a vertex $g' \in G$ to gg', an edge connecting g', g'' to an edge connecting gg', gg'', and a 2-cell with boundary loop passing $g_1, \cdots, g_n$ to a 2-cell with boundary loop passing $gg_1, \cdots, gg_n$. Now take a point of Cayley graph. If the point is a vertex, then choose an open neighborhood as a point and $1/3$ of its neighboring edges. If the point is not on vertex but on edge, then choose an open neighborhood which is totally contained in edge but does not contains any vertex, which is possible since the edge without its endpoints is an open set. If the point is neither on vertex nor on edge but on 2-cell, then choose an open neighborhood which is totally contained in 2-cell but does not contains any edge or vertex, which is possible since deleting boundary from 2-cell gives an open set. By this procedure, we can find an open neighborhood of every points in Cayley graph whose image of action of all elements of G is disjoint. Thus this action is a covering space action. Finally, $\tilde{X}_G/G$ and X_G are homeomorphic, since for the map $p : \tilde{X}_G \to X_G$ taking all the vertices into one point, all the edges connecting g, g_α to the edges representing g_α, and all the 2-cells with boundary loop passing $g, gg_1, \cdots, gg_1 \cdots g_n$ to the 2-cells with boundary loop passing $g_1, \cdots, g_n$, then this map is quotient map where all the orbits of the action of G on $\tilde{X}_G$ are quotiented. Now due to the previous proposition, $G \simeq \pi_1(\tilde{X}_G/G)/p_*(\pi_1(\tilde{X}_G)) \simeq \pi_1(X_G)/p_*(\pi_1(\tilde{X}_G)) \simeq G/p_*(\pi_1(\tilde{X}_G))$, thus $p_*(\pi_1(\tilde{X}_G)) = 0$, and since p_* is injective, $\pi_1(\tilde{X}_G) = 0$. □

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Definition 147. The ***n*-simplex**, or **simplex** if n is well known or not important, is a smallest convex set in a Euclidean space $\mathbb{R}^m$ containing ordered $n+1$ points $v_0, \cdots, v_n$ which do not lie in a hyperplane of dimension less then n, and written as $[v_0, \cdots, v_n]$. We call v_i a **vertices** of the simplex. The **standard *n*-simplex** is a set

$\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1, t_i \geq 0, \forall i\}$. For the linear homeomorphism $\phi : \Delta^n \to [v_0, \cdots, v_n]$ defined as $(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i$, the coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i$ in $[v_0, \cdots, v_n]$.

Since every simplex is linearly homeomorphic to standard simplex, if the vertices are well known or not important, we simply write the simplex as Δ^n.

Definition 148. For an n-simplex $[v_0, \cdots, v_n]$, **faces** of the simplex are the $n-1$ simplices $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. Here, the vertex under $\hat{}$ symbol is ignored. The union of all the faces of simplex Δ^n is the **boundary** of the simplex, which is written as $\partial\Delta^n$. The **open simplex** is a set $\Delta^n - \partial\Delta^n$, and written as $\mathring{\Delta}^n$.

Definition 149. A **Δ-complex structure** of a space X is a collection of maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ where n_α depends on α and often written as just n, such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and $x \in X$ is in the image of exactly one $\sigma_\alpha|_{\mathring{\Delta}^n}$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$, where the face of Δ^n and Δ^{n-1} are identified by the linear homeomorphism where the order of vertices are preserved.
3. $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

This condition bans the triviality, for example, all the maps $\sigma_\alpha : \Delta^n \to X$ are just a point map.

Definition 150. Consider a Δ-complex X. The **n-chain** of X is the free abelian group with basis $\sigma_\alpha : \Delta^n \to X$ and written as $\Delta_n(X)$. The **boundary homomorphism** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined as

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \tag{154}$$

, where the range of σ_α is $[v_0, \cdots, v_n]$.

Proposition 151. $\partial_{n-1} \circ \partial_n = 0$.

Proof.

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \partial_{n-1} \sum_i (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \\
&= \sum_{i<j} (-1)^{i+j} \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n]} \\
&+ \sum_{i>j} (-1)^{i+j-1} \sigma|_{[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n]} \\
&= 0
\end{aligned}$$

$\square$

Definition 152. Consider an abelian groups C_n and homomorphisms ∂_n for $n \in \mathbb{N} \cup \{0\}$ with structure

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{155}$$

with $\partial_n \circ \partial_{n+1} = 0$ for all n. This sequence is called a **(abelian group) chain complex** and written as $C_\bullet$ with homomorphisms $\partial_\bullet$. Elements of $\operatorname{Ker}\partial_n$ are called **cycles** and elements of $\operatorname{Im}\partial_n$ are called **boundaries**.

The chain complex can be defined on the R-modules for ring R and their module homomorphisms, which is a bit general case since abelian groups are $\mathbb{Z}$-modules.

Proposition 153. *Take a chain complex $C_\bullet$ with homomorphisms $\partial_\bullet$. Then* $\operatorname{Im}\partial_{n+1} \trianglelefteq \operatorname{Ker}\partial_n$.

Proof. Since $\partial_n \circ \partial_{n+1} = 0$, if $x \in \operatorname{Im}\partial_{n+1}$ then there is $x' \in C_{n+1}$ such that $\partial_{n+1}(x') = x$. Now since $\partial_n \circ \partial_{n+1}(x') = 0$, $\partial_n(x) = 0$ and thus $x \in \operatorname{Ker}\partial_n$ and $\operatorname{Im}\partial_{n+1} \leq \operatorname{Ker}\partial_n$. Since C_n are abelian, $\operatorname{Im}\partial_{n+1} \trianglelefteq \operatorname{Ker}\partial_n$. □

Definition 154. Take a chain complex $C_\bullet$ with homomorphisms $\partial_\bullet$. Then an abelian group $H_n = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$ is called a n**th homology group** of the chain complex. Elements of H_n are cosets of ∂_{n+1}, which is called **homology classes**. Two cycles representing the same homology class are said to be **homologous**.

Definition 155. Let X be a Δ-complex. The nth Homology group of a chain complex $\Delta_\bullet(X)$ with homomorphisms $\partial_\bullet$ is called the n**th simplicial homology group** of X, and written as $H_n^\Delta(X)$.

Example 156. Consider a space S^1. We can give a Δ-complex structure by one vertex v and one edge e. then $\Delta_0(S^1) = \langle v \rangle$, $\Delta_1(S^1) = \langle e \rangle$, and $\partial_1(e) = v - v = 0$. Therefore, $H_n^\Delta(S^1) \simeq \mathbb{Z}$ for $n = 0, 1$, and $H_n^\Delta(S^1) \simeq 0$ for all the others.

Example 157. Consider a space T, a torus. We can give a Δ-complex structure by one vertex v, three edges a, b, c, and two 2-simplices U, L, with $\partial_1 = 0$, $\partial_2 U = a + b - c = \partial_2 L$. Thus $H_0^\Delta(T) \simeq \mathbb{Z}$ and $H_1^\Delta(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$ since $\{a, b, a + b - c\}$ is a basis of $\Delta_1(T) = \operatorname{Ker}\partial_1$. Finally, $H_2^\Delta(T) = \operatorname{Ker}\partial_2$ is generated by $U - L$, hence $H_2^\Delta(T) \simeq \mathbb{Z}$. For $n \geq 3$, $H_n^\Delta(T) \simeq 0$.

Example 158. Now consider a space $\mathbb{R}P^2$. We can give a Δ-complex structure by two vertex v, w, three edges a, b, c, and two 2-simplices U, L, with $\operatorname{Im}\partial_1 = \langle w - v \rangle$, $\partial_2 U = -a + b + c$ and $\partial_2 L = a - b + c$. Therefore $H_0^\Delta(\mathbb{R}P^2) \simeq \mathbb{Z}$. Furthermore, since ∂_2 is injective, $H_2^\Delta(\mathbb{R}P^2) \simeq 0$. Finally, $\operatorname{Ker}\partial_1 \simeq \langle a - b, c \rangle$ and $\Im\partial_2 \simeq \langle a - b + c, 2c \rangle$. Since $\langle a - b, c \rangle \simeq \langle a - b + c, c \rangle$, $H_1^\Delta(X) \simeq \mathbb{Z}_2$. For $n \geq 3$, $H_n^\Delta(T) \simeq 0$.

Example 159. Give S^n a Δ-complex structure by taking two Δ^n and identifying their boundaries. Taking these simplices as U, L, then $\operatorname{Ker}\partial_n = \langle U - L \rangle$ and $\Im\partial_{n+1} = 0$ since there is no $n + 1$ simplex. Therefore $H_n^\Delta(S^n) \simeq \mathbb{Z}$.

Definition 160. A **singular** n**-simplex** in a space X is a map $\sigma : \Delta^n \to X$.

Definition 161. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$ are called **singular n-chains**. A boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined as

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]} \tag{156}$$

, where the range of σ is $[v_0, \cdots, v_n]$.

This definition is same with the boundary homomorphism of Δ-complex, thus $\partial_{n-1} \circ \partial_n = 0$. Often we write the boundary homomorphisms ∂_n as ∂, and $\partial_{n-1} \circ \partial_n = 0$ as ∂^2.

Definition 162. Let X be a space and $C_n(X)$ is a set of n-chains. The nth Homology group of a chain complex $C_\bullet(X)$ with homomorphisms $\partial_\bullet$ is called the **nth singular homology group** of X, and written as $H_n(X)$.

Proposition 163. *If X and Y are homeomorphic, then $H_n(X) \simeq H_n(Y)$ for all n.*

Proof. Suppose that $\phi : X \to Y$ is a homeomorphism. Then there are isomorphisms $\xi_n : C_n(X) \to C_n(Y)$ defined as $\xi_n(\sigma_X) = \phi \circ \sigma_X$, where $\partial_n^Y \circ \xi_n = \xi_{n-1} \circ \partial_n^X$. Thus $\operatorname{Ker} \partial_n^X \simeq \operatorname{Ker} \partial_n^Y$ and $\operatorname{Im} \partial_n^X \simeq \operatorname{Im} \partial_n^Y$, thus $H_n(X) \simeq H_n(Y)$. □

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Proposition 164. *If $X = \cup_\alpha X_\alpha$ where X_α are the path components of X, then $H_n(X) \simeq \oplus_\alpha H_n(X_\alpha)$.*

Proof. Since an image of singular simplex is always path connected, there is no singular simplex whose image is on two or more X_α. Therefore $C_n(X) \simeq \oplus_\alpha C_n(X_\alpha)$. Furthermore, since $\partial_n|_{C_n(X_\alpha)}$ takes $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, if we write $\partial_n|_{C_n(X_\alpha)}$ as ∂_n^α, then $\operatorname{Ker} \partial_n \simeq \oplus_\alpha \operatorname{Ker} \partial_n^\alpha$ and $\operatorname{Im} \partial_n \simeq \oplus_\alpha \operatorname{Im} \partial_n^\alpha$. Thus $H_n \simeq \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} \simeq \oplus_\alpha \operatorname{Ker} \partial_n^\alpha / \oplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha \simeq \oplus_\alpha (\operatorname{Ker} \partial_n^\alpha / \operatorname{Im} \partial_{n+1}^\alpha) \simeq \oplus_\alpha H_n(X_\alpha)$. □

Proposition 165. *If X is nonempty path connected space, then $H_0(X) \simeq \mathbb{Z}$. If there is a bijection between path components of X and a set A, then $H_0(X) \simeq \oplus_{\alpha \in A} \mathbb{Z}$.*

Proof. Since $\partial_0 = 0$, $H_0(X) \simeq C_0(X) / \operatorname{Im} \partial_1$. Now define $\epsilon : C_0(X) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. Since X is nonempty, ϵ is surjective. Now suppose that X is path connected. For a singular 1-simplex $\sigma : \Delta^1 \to X$, we have $\epsilon \circ \partial_1(\sigma) = \epsilon(\sigma|_{[v_1]}) - \epsilon(\sigma|_{[v_0]}) = 1 - 1 = 0$, therefore $\operatorname{Im} \partial_1 \subset \operatorname{Ker} \epsilon$. Conversely, suppose that $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i = 0$. Since σ_i are singular 0-simplexes, they are points in X. Now take a basepoint $x_0 \in X$ and choose a path $\tau_i : I \to X$ from x_0 to $\sigma_i(v_0)$. Also let σ_0 is a singular 0-complex with image x_0. Now $\tau_i : [v_0, v_1] \to X$ are singular 1-simplexes, and $\partial \tau_i = \sigma_i - \sigma_0$. therefore $\partial(\sum_i n_i \tau_i) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i$, therefore $\operatorname{Ker} \epsilon \subset \operatorname{Im} \partial_1$. Thus $\operatorname{Ker} \epsilon \simeq$

$\operatorname{Im}\partial_1$, and so $\mathbb{Z} \simeq C_0(X)/\operatorname{Ker}\epsilon \simeq C_0(X)/\operatorname{Im}\partial_1 \simeq H_0(X)$. The second statement follows from the previous proposition. □

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Proposition 166. *If X is a point set, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \simeq \mathbb{Z}$.*

Proof. For each n, there is a unique singular n-simplex σ_n, and $\partial(\sigma_n) = \sum_{i=0}^{n}(-1)^i\sigma_{n-1}$, which is 0 for odd n and σ_{n-1} for even n with $n \neq 0$. Thus the chain complex becomes

$$\cdots \to \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \tag{157}$$

For $\mathbb{Z} \xrightarrow{\simeq} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$, both kernel and image is $\mathbb{Z}$. For $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}$, both kernel and image is 0. therefore $H_n(X) \simeq 0$ for all $n > 0$. For $n = 0$, due to the previous proposition, $H_0(X) \simeq \mathbb{Z}$. □

Definition 167. For a nonempty space X and its singular chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0 \tag{158}$$

the **reduced homology groups** $\tilde{H}_n(X)$ is the homology groups of a chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{159}$$

where $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$.

Proposition 168. *The chain complex of reduced homology groups is actually a chain complex, and for a nonempty space X, $H_n(X) \simeq \tilde{H}_n(X)$ for $n > 0$ and $H_0 \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$.*

Proof. Since $\epsilon(\partial_1([v_0, v_1])) = \epsilon(v_0 - v_1) = 1 - 1 = 0$, and ϵ is surjective, it is a chain complex. Now notice that $H_0(X) \simeq C_0/\operatorname{Im}\partial_1$. Since $\epsilon(\operatorname{Im}\partial_1) = 0$, we may define an induced map $\tilde{\epsilon} : H_0(X) \to \mathbb{Z}$. Since X is nonempty, we may choose $n[v_0]$ as an element of $H_0(X)$ then $\tilde{\epsilon}(n[v_0]) = n$ for any $n \in \mathbb{Z}$, thus $\tilde{\epsilon}$ is surjective. Also $\operatorname{Ker}\tilde{\epsilon}$ is the elements of $H_0(X)$ which can be represented as $\sum_i n_i[v]_i$ with $\sum_i n_i = 0$. But these are the elements of $\tilde{H}_0(X) \simeq \operatorname{Ker}\epsilon/\operatorname{Im}\partial_1$, hence $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$. For $n > 0$, since the kernel and image structure are all same, $H_n(X) \simeq \tilde{H}_n(X)$. □

Definition 169. For two chain complexes $C_\bullet, D_\bullet$, the **homomorphism between chain complexes** $f : C_\bullet \to D_\bullet$ is a collection of homomorphisms $f_n : C_n \to D_n$. If the index n is well known or not important, we write f rather then f_n. If a homomorphism between chain complexes $f : C_\bullet \to D_\bullet$ satisfies $f \circ \partial_C = \partial_D \circ f$, then f is called a **chain**

map. Since the index C, D is obvious when the function is given, we often drop them.

Definition 170. For a map $f : X \to Y$ between two space, an **induced homomorphism** $f_\# : C_\bullet(X) \to C_\bullet(Y)$ is defined as, for each singular n-simplex $\sigma : \Delta^n \to X$ let $f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y$, then extending $f_\#$ linearly: $f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f_\#(\sigma_i)$.

Proposition 171. *For a map $f : X \to Y$, the induced homomorphism $f_\# : C_\bullet(X) \to C_\bullet(Y)$ is a chain map.*

Proof. For n-simplex σ, $f_\# \circ \partial(\sigma) = f_\#(\sum_i (-1)^i \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]}) = \sum_i (-1)^i f \circ \sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_n]} = \partial f_\#(\sigma)$. □

Proposition 172. *A chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.*

Proof. Let $f : X \to Y$ and $\alpha \in C_\bullet(X)$. Define $f_* : H_n(X) \to H_n(Y)$ as $f_*([\alpha]) = [f_\#(\alpha)]$. For $[\alpha] \in H_n(X)$, $\partial\alpha = 0$, and $\partial(f_\#(\alpha)) = f_\# \circ \partial(\alpha) = 0$, thus the codomain of map is indeed $H_n(Y)$. If $[\alpha] = [\beta]$ then $\alpha - \beta = \partial\gamma$, thus $f_\#(\alpha) - f_\#(\beta) = f_\#(\alpha - \beta) = f_\# \circ \partial(\gamma) = \partial(f_\#(\gamma))$, therefore $[f_\#(\alpha)] = [f_\#(\beta)]$ and so f_* is well defined. Since $f_\#$ is defined linearly, $f_*([\alpha] + [\beta]) = f_*([\alpha + \beta]) = [f_\#(\alpha + \beta)] = [f_\#(\alpha)] + [f_\#(\beta)] = f_*([\alpha]) + f_*([\beta])$, and so $f_\#$ is a homomorphism. □

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Proposition 173. *Consider $f : X \to Y, g : Y \to Z$. Then $(g \circ f)_* = g_* \circ f_*$, where $f_* : H_\bullet(X) \to H_\bullet(Y), g_* : H_\bullet(Y) \to H_\bullet(Z)$ are the induced homomorphisms. Furthermore, $(1_X)_* = 1_{H_n(X)}$, where $1_X : X \to X$ is an identity map.*

Proof. Take $[\alpha] \in H_n(X)$. Then $g_*(f_*([\alpha])) = g_*([f \circ \alpha]) = [g \circ (f \circ \alpha)]$ and $(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha]$, which are same because of the associativity of function composition. Also, $1_*([\alpha]) = [\alpha]$. □

Definition 174. For the homomorphism between two chain complexes $f, g : C_\bullet \to D_\bullet$, a map $h : C_n \to D_{n+1}$ is a **chain homotopy** if $f - g = \partial \circ h + h \circ \partial$.

This diagram does *not* commute.

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] \arrow[dl, "h"'] \arrow[d, "f-g"] & C_{n-1} \arrow[r, "\partial"] \arrow[dl, "h"'] \arrow[d, "f-g"] & \cdots \arrow[dl, "h"'] \\ \cdots \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r, "\partial"] & \cdots \end{tikzcd} \tag{160}$$

Theorem 175. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_\bullet(X) \to H_\bullet(Y)$. Furthermore, the chain-homotopic chain maps induce the same homomorphism on homology.*

Proof. Consider $\Delta^n \times I$ where $\Delta^n \times \{0\} = [v_0, \cdots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \cdots, w_n]$, where v_i, w_i are in the same image under the projection $\Delta^n \times I \to \Delta^n$. Now consider the $(n+1)$-simplexes, $[v_0, \cdots, v_i, w_i, \cdots, w_n]$, where $i = 0, \cdots, n$. Indeed, these simplexes can be obtained by dividing $\Delta^n \times I$ by the $\mathbb{R}^n$-planes containing $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$.

Now take a homotopy $F : X \times I \to Y$ from f to g. Take a simplex $\sigma : \Delta^n \to X$. Then we may take the composition $F \circ (\sigma \times 1_I) : \Delta^n \times I \to Y$. Define a *prism operators* $P : C_n(X) \to C_{n+1}(Y)$ as

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]} \tag{161}$$

Now notice that

$$\begin{aligned}
\partial \circ P(\sigma) = &\sum_{j\le i} (-1)^{i+j} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]} \\
&+ \sum_{j\ge i} (-1)^{i+j+1} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]} \\
= &\sum_{j<i} (-1)^{i+j} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]} \\
&+ \sum_{j>i} (-1)^{i+j+1} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]} \\
&+ \sum_i F \circ (\sigma \times 1_I)|_{[v_0,\cdots,\hat{v}_i,w_i,\cdots,w_n]} \\
&- \sum_i F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,\hat{w}_i,\cdots,w_n]} \\
= &\sum_{j<i} (-1)^{i+j} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]} \\
&+ \sum_{j>i} (-1)^{i+j+1} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]} \\
&+ F \circ (\sigma \times 1_I)|_{[w_0,\cdots,w_n]} - F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_n]} \\
= &\sum_{j<i} (-1)^{i+j} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]} \\
&+ \sum_{j>i} (-1)^{i+j+1} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]} \\
&+ g \circ \sigma - f \circ \sigma
\end{aligned}$$

Finally,

$$\begin{aligned}
P \circ \partial(\sigma) = &\sum_{i<j} (-1)^{i+j} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]} \\
&+ \sum_{i<j} (-1)^{i+j-1} F \circ (\sigma \times 1_I)|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}
\end{aligned} \tag{162}$$

which gives

$$\partial \circ P(\sigma) + P \circ \partial(\sigma) = g \circ \sigma - f \circ \sigma = g_\#(\sigma) - f_\#(\sigma) \tag{163}$$

and so $\partial \circ P + P \circ \partial = g_\# - f_\#$ and so P is a chain homotopy between chain maps $f_\#$ and $g_\#$. Now take $[\alpha] \in H_n(X)$, then $\partial\alpha = 0$, so $g_\#(\alpha) - f_\#(\alpha) = \partial \circ P(\alpha)$, hence $[g_\#(\alpha) - f_\#(\alpha)] = 0$, which means $g_*([\alpha]) = f_*([\alpha])$. □

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Definition 176. The sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots \tag{164}$$

of groups A_n is **exact** if $\operatorname{Ker}\alpha_n = \operatorname{Im}\alpha_{n+1}$.

Exact sequence is thus a chain complex, obviously.

Proposition 177. *Let A, B, C are groups and α, β are homomorphisms.*

1. $0 \to A \xrightarrow{\alpha} B$ *is exact if and only if* $\operatorname{Ker}\alpha = 0$*, i.e., α is injective.*
2. $A \xrightarrow{\alpha} B \to 0$ *is exact if and only if* $\operatorname{Im}\alpha = B$*, i.e., α is surjective.*
3. $0 \to A \xrightarrow{\alpha} B \to 0$ *is exact if and only if α is bijective.*
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is exact if and only if α is injective, β is surjective, and* $\operatorname{Ker}\beta \simeq \operatorname{Im}\alpha$*. Then $C \simeq B/A$.*

Proof. 1. Since the image of $0 \to A$ is 0, the sequence is exact if and only if $\operatorname{Ker}\alpha = 0$.

2. Since the kernel of $B \to 0$ is B, the sequence is exact if and only if $\operatorname{Im}\alpha = 0$.
3. By above two, the sequence is exact if and only if α is bijective.
4. Let the sequence is exact. By above two, α is injective and β is surjective, and $\operatorname{Ker}\beta \simeq \operatorname{Im}\alpha$ by definition. Conversely, injective α, surjective β, and $\operatorname{Ker}\beta \simeq \operatorname{Im}\alpha$ are the definition of the exact sequence. Finally, by the first homomorphisms theorem, $C \simeq B/\operatorname{Ker}\beta B/\operatorname{Im}\alpha$, and since α is injective $\operatorname{Im}\alpha \simeq A$, thus $C \simeq B/A$.

□

Definition 178. An exact sequence $0 \to A \to B \to C \to 0$ is called a **short exact sequence**.

Definition 179. Let X be a space and $A \subset X$. Define $C_n(X, A)$ as $C_n(X)/C_n(A)$. Define a boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$ from the boundary map $\partial' : C_n(X) \to C_{n-1}(X)$. This gives the chain complex

$$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots \tag{165}$$

and its homology group $H_n(X, A)$, which is called a **relative homology groups**.

Proposition 180. *The sequence* $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{q} C_n(X, A)$ *is a short exact sequence.*

Proof. Since i is inclusion and q is quotient, i is injective and q is surjective. Now $\operatorname{Ker} q \simeq C_n(A) \simeq \operatorname{Im} i$, because q is the quotient homomorphism taking $C_n(A)$ to 0. □

Proposition 181. *The following diagram commutes.*

Definition 182. This kind of diagram is called the **short exact sequence of chain complexes**.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial"] & C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_{n-1}(A) \arrow[r, "\partial"] \arrow[d, "i"'] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"'] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "q"'] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d] & C_n(X,A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X,A) \arrow[r, "\partial"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{166}$$

Proof. Since i and q are the induced homomorphism of inclusion and quotient map, i and q are chain map, thus $i \circ \partial = \partial \circ i$ and $q \circ \partial = \partial \circ q$. □

Theorem 183. *consider the following short exact sequence of chain complexes.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial"] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"'] & A_n \arrow[r, "\partial"] \arrow[d, "i"'] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i"'] & \cdots \\
\cdots \arrow[r, "\partial"] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"'] & B_n \arrow[r, "\partial"] \arrow[d, "j"'] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "j"'] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{167}$$

then the sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots \tag{168}$$

is exact. Here, the map $\partial : H_n(C) \to H_{n-1}(A)$ *is defined as* $\partial(c) = i^{-1} \circ \partial \circ j^{-1}(c)$.

Proof. First we need to show that the map $\partial : H_n(C) \to H_{n-1}(A)$ is well defined homomorphism. Take $[c] \in H_n(C)$, then $\partial c = 0$. Since j is injective, there is $b \in B_n$ such that $c = j(b)$. Also since $j \circ \partial(b) = \partial \circ j(b) = \partial c = 0$, $\partial b \in B_{n-1}$ is in the kernel of j, thus there is $a \in A_{n-1}$ such that $\partial b = i(a)$ since $\operatorname{Ker} j = \operatorname{Im} i$. Now, since i is injective, a is uniquely determined by ∂b. If we have another b' satisfying above, then $j(b) = j(b')$ thus $b' - b \in \operatorname{Ker} j = \operatorname{Im} i$. Thus $b' - b = i(a')$ for some $a' \in A_{n-1}$. Now notice that $i(a + \partial a') = i(a) + \partial i(a') = \partial(b + i(a'))$, thus we get $a + \partial a'$ is obtained from $b + i(a') = b$. Now $[a] = [a + \partial a']$ thus we get the same result. Finally, we may choose a different representation $c + \partial c'$. Since $c' = j(b')$ for some $b' \in B_{n+1}$, $c + \partial c' = c + \partial j(b') = c + j(\partial b') = j(b + \partial b')$, thus taking $c + \partial c'$ gives $b + \partial b'$, whose boundary is ∂b and gives the same result.

If $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$, then we have intermediate $b_1, b_2 \in B_n$ for each. Then $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$, therefore $\partial([c_1] + [c_2]) = [a_1] + [a_2]$. Thus ∂ is homomorphism.

Now we need to show that the sequence is exact.

1. $\operatorname{Im} i_* \leq \operatorname{Ker} j_*$. Since $j \circ i = 0$, $j_* \circ i_* = 0$.

2. $\operatorname{Im} j_* \leq \operatorname{Ker} \partial$. Notice that $\partial \circ j_*$ takes $[b]$ to $[i^{-1} \circ \partial \circ j^{-1} \circ j(b)] = [i^{-1} \circ \partial(b)]$, and since $\partial b = 0$, $\partial \circ j_* = 0$.

3. $\operatorname{Im} \partial \leq \operatorname{Ker} i_*$. Notice that $i_* \circ \partial$ takes $[c]$ to $[i \circ i^{-1} \circ \partial \circ j^{-1}(c)] = [\partial \circ j^{-1}(c)] = 0$, thus $i_* \circ \partial = 0$.

4. $\operatorname{Ker} j_* \leq \operatorname{Im} i_*$. The element of $\operatorname{Ker} j_*$ is a cycle $b \in B_n$ with $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjective, $c' = j(b')$ for some $b' \in B_{n+1}$. Now, since $\partial \circ j(b') = \partial c' = j(b)$, $j(b - \partial b') = 0$, thus $b - \partial b' = i(a)$ for some $a \in A_n$. Since $i(\partial a) = \partial \circ i(a) = \partial(b - \partial b') = \partial b = 0$ and i is injective, a is a cycle, thus a representation of $H_n(A)$. Therefore $i_*[a] = [b - \partial b'] = [b']$.

5. $\operatorname{Ker} \partial \leq \operatorname{Im} j_*$. The element of $\operatorname{Ker} \partial$ can be represented as $c \in H_n(C)$. Then there is $a \in A_{n-1}$ corresponding to c by the definition of $\partial : H_n(C) \to H_{n-1}(A)$ such that $a = \partial a'$ for $a' \in A_n$. Now the corresponding $b \in B_n$ satisfies $\partial(b - i(a')) = \partial b - i(\partial a') = \partial b - i(a) = 0$, thus $b - i(a')$ is a cycle, and $j(b - i(a')) = j(b) - j \circ i(a') = j(b) = c$, thus $j_*[b - i(a')] = [c]$.

6. $\operatorname{Ker} i_* \leq \operatorname{Im} \partial$. For a cycle $a \in A_{n-1}$ such that $i(a) = \partial b$ for some $b \in B_n$, since $\partial \circ j(b) = j(\partial b) = j \circ i(a) = 0$ thus $j(b)$ is a cycle, and $\partial[j(b)] = [i^{-1} \circ \partial \circ j^{-1} \circ j(b)] = [i^{-1} \circ \partial b] = [i^{-1} \circ i(a)] = [a]$.

□

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Corollary 184. *The sequence of homology groups*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$
$$\cdots \to H_0(X,A) \to 0 \quad (169)$$

is exact. Here, the map $\partial : H_n(X,A) \to H_{n-1}(A)$ *is defined as* $\partial([\alpha]) = [i^{-1} \circ \partial \circ j^{-1}(\alpha)]$.

The meaning of this boundary map $\partial : H_n(X,A) \to H_{n-1}(A)$ can be considered as $\partial[\alpha]$ is the class of the cycle $\partial\alpha$ for the relative cycle $\alpha \in H_n(X,A)$.

Proof. From the previous proposition, we can substitute A, B, C to $A, X, (X,A)$ in the previous theorem, which gives the desired exact sequence. □

Corollary 185. *Let* $A \neq \emptyset$. *The sequence of reduced homology groups and homology groups*

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \cdots$$
$$\cdots \to H_0(X,A) \to 0 \quad (170)$$

is exact.

Proof. It is enough to fill in the -1 dimensional short exact sequence to the short exact sequence of chain complexes, which is $0 \to \mathbb{Z} \xrightarrow{1_{\mathbb{Z}}} \mathbb{Z} \to 0 \to 0$. □

Example 186. Since D^n is contractible, $\tilde{H}_i(D^n) = 0$ for all i, therefore the maps $H_i(D^n, \partial D^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ and $H_0(D^n, \partial D^n) \simeq 0$.

Example 187. Since $\tilde{H}_n(x_0) = 0$ for all n, $H_n(X, x_0) \simeq \tilde{H}_n(X)$ for all n.

Definition 188. For a map $f : (X,A) \to (Y,B)$, the chain map $f'_\# : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$, which gives the quotient map $f_\# : C_n(X,A) \to C_n(Y,B)$. Since $\partial \circ f'_\# = f'_\# \circ \partial$, $\partial \circ f_\# = f_\# \circ \partial$, which induces the **induced homomorphism for relative homology** $f_* : H_n(X,A) \to H_n(Y,B)$.

Proposition 189. *If two maps* $f, g : (X,A) \to (Y,B)$ *are homotopic through map of pairs* $(X,A) \to (Y,B)$, *then* $f_* = g_* : H_n(X,A) rightarrow H_n(Y,B)$.

Proof. The proof is exactly same with nonrelative case, except we need to define a relative prism operator $P : C_n(X,A) \to C_{n+1}(Y,B)$ from the prism operator $P' : C_n(X) \to C_{n+1}(Y)$, which is well defined since $P(C_n(A)) \subset C_{n+1}(B)$. Since $g_\#$ and $f_\#$ does not changes by quotienting, $\partial \circ P' + P' \circ \partial = g_\# - f_\#$ induces $\partial \circ P + P \circ \partial = g_\# - f_\#$, thus they are chain homotopic on relative chain groups, and thus the induced homomorphisms are same. □

Proposition 190. *Consider $B \subset A \subset X$, and write it (X, A, B). Then the sequence of homology groups*

$$\cdots \to H_n(A, B) \xrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \cdots \tag{171}$$

is exact.

Proof. Take the short exact sequence

$$0 \to C_n(A, B) \xrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X, A) \to 0 \tag{172}$$

This is possible since $C_n(X, B)/C_n(A, B) \simeq (C_n(X)/C_n(B))/(C_n(A)/C_n(B)) \simeq C_n(X)/C_n(A)$ by the third isomorphism theorem. □

Definition 191. The **barycenter** of the simplex $[v_0, \cdots, v_n]$ is the point $b = \sum_i v_i/(n+1)$. The **barycentric subdivision** of the simplex $[v_0, \cdots, v_n]$ is the decomposition of $[v_0, \cdots, v_n]$ into the n-simplexes $[b, w_0, \cdots, w_{n-1}]$ where $[w_0, \cdots, w_{n-1}]$ is an $n-1$ simplex in the barycentric subdivision of a face $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$, whose induction trigger is that the barycentric subdivision of $[v_0]$ is $[v_0]$.

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Definition 192. The **diameter** of a space $X \subset \mathbb{R}^n$ is the maximum distance between any two points of X. The diameter of X is written as $d(X)$.

Proposition 193. *For a simplex $[v_0, \cdots, v_n]$, $d([v_0, \cdots, v_n]) = \max_{i,j} |v_i - v_j|$.*

Proof. For any two points $v, w = \sum_i t_i v_i \in [v_0, \cdots, v_n]$,

$$|v - \sum_i t_i v_i| = |\sum_i t_i(v - v_i)| \le \sum_i t_i |v - v_i| \le \max_i |v - v_i| \tag{173}$$

Letting $v = \sum_i k_i v_i$ and using above relation again, we get $|v - w| \le \max_{i,j} |v_i - v_j|$. Thus $d([v_0, \cdots, v_n]) \le \max_{i,j} |v_i - v_j|$. Since $d([v_0, \cdots, v_n]) \ge |v_i - v_j|$ for all i, j, we get $d([v_0, \cdots, v_n]) = \max_{i,j} |v_i - v_j|$. □

Proposition 194. *For a simplex $[v_0, \cdots, v_n]$, the diameter of each simplex of its barycentric subdivision is at most $d([v_0, \cdots, v_n]) \cdot n/(n+1)$.*

Thus, r times repeated barycentric subdivision of a n-simplex with diameter 1 gives the simplexes with radius $\left(\frac{n}{n+1}\right)^r$, which approaches to 0 when $r \to \infty$.

Proof. Let $[w_0, \cdots, w_n]$ be a simplex of the barycentric subdivision of $[v_0, \cdots, v_n]$. If w_i, w_j are not the barycenter b of the simplex $[v_0, \cdots, v_n]$, then w_i, w_j are in the barycentric subdivision of a face of $[v_0, \cdots, v_n]$. By induction with induction trigger $n = 1$, $|w_i, w_j| \le$

$\frac{n-1}{n}\max_{i,j\neq k}|v_i - v_j| \leq \frac{n-1}{n}\max_{i,j}|v_i - v_j| \leq \frac{n}{n+1}d([v_0, \cdots, v_n])$. Now let $w_i = b$. From the proof of previous proposition, we may suppose w_j as v_k. Now if b_i is the barycenter of the face $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$, then $b = \frac{1}{n+1}v_i + \frac{n}{n+1}b_i$, thus b lies on the line segment $[v_i, b_i]$, and the distance between b to v_i is $\frac{n}{n+1}d([v_i, b_i]) \leq \frac{n}{n+1}d([v_0, \cdots, v_n])$. $\square$

HYUNJUN PARK

HOMOLOGICAL ALGEBRA

2019- 07- 22

Definition 1. A **chain complex** $C_\bullet = C$ of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules with R-module maps $d = d_n : C_n \to C_{n-1}$ such that each composite $d \circ d : C_n \to C_{n-2}$ is zero. The maps $d_n = d$ are called the **differentials** of C. The kernel $\operatorname{Ker} d_n$ is the module of n**-cycles** of C, and denoted $Z_n = Z_n(C)$. The image $\operatorname{Im} d_{n+1}$ is the module of n**-boundaries** of C, and denoted $B_n = B_n(C)$. Since $d \circ d = 0$, $0 \subset B_n \subset Z_n \subset C_n$ for all n. The quotient $H_n(C) = Z_n/B_n$ is called the n**-th homology module** of C. We call a chain complex is **exact** if $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$ for all $n \in \mathbb{Z}$.

Exercise 2. Set $C_n = \mathbb{Z}/8$ for $n \geq 0$ and $C_n = 0$ for $n < 0$; for $n > 0$ let d_n send $x(\mod 8)$ to $4x(\mod 8)$. Show that $C_\bullet$ is a chain complex of $\mathbb{Z}/8$-modules and compute its homology modules.

Solution. Since $d \circ d(x) = 16x(\mod 8) = 0(\mod 8)$ for all $x \in \mathbb{Z}/8$, $C_\bullet$ is a chain complex. For $n < 0$, since $d = 0$, $H_n(C) = 0$. For $n > 0$, since the kernel is $\{0, 2, 4, 8\}$ and the image is $\{0, 4\}$, $H_n(C) = \mathbb{Z}/2$. For $n = 0$, since the kernel is $\mathbb{Z}/8$ and the image is $\{0, 4\}$, $H_n(C) = \mathbb{Z}/4$. Thus we get

$$H_n(C) = \begin{cases} \mathbb{Z}/2, & n > 0 \\ \mathbb{Z}/4, & n = 0 \\ 0, & n < 0 \end{cases} \tag{174}$$

■

Definition 3. The category $\mathsf{Ch}(\mathsf{mod} - R)$ is a category whose objects are chain complexes, and morphism $u : C_\bullet \to D_\bullet$ is the **chain complex map**, which is a family of R-module homomorphisms $u_n : C_n \to D_n$, which satisfies $u_{n-1} \circ d_n = d_n \circ u_n$. That is, such that the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "u"] & C_n \arrow[r, "d"] \arrow[d, "u"] & C_{n-1} \arrow[r, "d"] \arrow[d, "u"] & \cdots \\ \cdots \arrow[r, "d"] & D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1} \arrow[r, "d"] & \cdots \end{tikzcd} \tag{175}$$

Exercise 4. Show that a morphism $u : C_\bullet \to D_\bullet$ of chain complexes sends boundaries to boundaries and cycles to cycles, hence maps $H_n(C_\bullet) \to H_n(D_\bullet)$. Prove that each H_n is a functor from $\mathsf{Ch}(\mathsf{mod}_R)$ to mod_R.

Solution. For boundaries $d(C_n)$, $u \circ d(C_n) = d \circ u(C_n) \subset d(D_n)$, thus $u \circ d(C_n)$ are boundaries of D_n. For cycles Z, $d(Z) = 0$, and $d(u(Z)) = u(d(Z)) = 0$, thus $u(Z)$ are boundaries of D_n. Thus u :

$C_\bullet \to D_\bullet$ can be quotiented and gives $u : H_n(C_\bullet) \to H_n(D_\bullet)$, which is R-module map. To show H_n is a functor, we need to show that it takes identity morphism to identity morphism, and preserves the composition. The identity morphism $1_{C_\bullet}$ defines identity R-module map $H_n(C_\bullet) \to H_n(C_\bullet)$ by definition, and for two morphisms $u : C_\bullet \to D_\bullet$ and $v : D_\bullet \to E_\bullet$, $v \circ u$ are quotiented and gives $v \circ u : H_n(C_\bullet) \xrightarrow{u} H_n(D_\bullet) \xrightarrow{v} H_n(E_\bullet)$. ∎

Exercise 5 (Split exact sequences of vector spaces)**.** Choose vector spaces $\{B_n, H_n\}_{n\in\mathbb{Z}}$ over a field, and set $C_n = B_n \oplus H_n \oplus B_{n-1}$. Show that the projection-inclusions $C_n \to B_{n-1} = C_{n-1}$ make $\{C_n\}$ into a chain complex, and that every chain complex of vector spaces is isomorphic to a complex of this form.

Solution. Take $(b_n, h_n, b_{n-1}) \in B_n \oplus H_n \oplus B_{n-1}$. Then $d \circ d(b_n, h_n, b_{n-1}) = d(b_{n-1}, 0, 0) = (0, 0, 0)$, thus $C_\bullet$ is a chain complex. Now consider a chain complex $V_\bullet$ of vector spaces. Take B_n, H_n as the boundaries and homology modules of $V_\bullet$. Now if we show that $V_n = B_n \oplus H_n \oplus B'_{n-1}$, then the statement is proven. Notice that $H_n = Z_n/B_n$ thus $Z_n = H_n \oplus B_n$. Now due to the first isomorphism theorem, $V_n/Z_n = B_{n-1}$. Therefore $V_n = Z_n \oplus B_{n-1} = B_n \oplus H_n \oplus B_{n-1}$. ∎

Exercise 6. Show that $\{\mathrm{Hom}_R(A, C_n)$ forms a chain complex of abelian groups for every R-module A and every R-module chain complex C. Taking $A = Z_n$, show that if $H_n(\mathrm{Hom}_R(Z_n, C)) = 0$, then $H_n(C) = 0$. Is the converse true?

Solution. Define $d : \mathrm{Hom}_R(A, C_n) \to \mathrm{Hom}_R(A, C_{n-1})$ as $d(f : A \to C_n) = d \circ f$, which is a group homomorphism because $d(f+g)(c) = d \circ (f+g)(c) = d(f(c) + g(c)) = d(f(c)) + d(g(c)) = d(f)(c) + d(g)(c)$. Then $d \circ d(f) = (d \circ d) \circ f = 0$, thus this is a chain complex.

For second question, choose the inclusion $i_n : Z_n \hookrightarrow C_n$. Then we can see that $d_n \circ i_n = 0$, thus there is $u : Z_n \to C_{n+1}$ such that $i_n = d_{n+1} \circ u$. Then $Z_n = i_n(C_n) = d_{n+1} \circ u(C_n) \subset d_{n+1}(C_{n+1}) = B_n$, thus $H_n(C) = 0$.

Now consider the chain complex C as $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ with $Z_n = \mathbb{Z}/2$. Notice that $H_n = 0$. Now $\mathrm{Hom}_R(Z_n, 2\mathbb{Z}) = \mathrm{Hom}_R(Z_n, \mathbb{Z}) = 0$ and $\mathrm{Hom}_R(Z_n, \mathbb{Z}/2) = \mathbb{Z}/2$, thus we get the chain complex $0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$, and so $H_n(\mathrm{Hom}_R(Z_n, C)) = \mathbb{Z}/2 \neq 0$, so the converse is not true. ∎

Definition 7. A morphism $C_\bullet \to D_\bullet$ of chain complexes is called a **quasi-isomorphism** if the maps $H_n(C) \to H_n(D)$ are all isomorphisms.

Exercise 8. Show that the following are equivalent for every $C_\bullet$:

1. $C_\bullet$ is exact.

2. $C_\bullet$ is **acyclic**, that is, $H_n(C) = 0$ for all n.

3. The map $0 \to C_\bullet$ is a quasi-isomorphism.

Solution. $(1 \to 2)$ Since $C_\bullet$ is exact, $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$, thus $H_n(C) = 0$.

$(2 \to 3)$ Since $H_n(C) = 0$, all the maps $0 \to H_n(C)$ are isomorphisms.

$(3 \to 1)$ Since the maps $0 \to H_n(C)$ are isomorphisms, $H_n(C) = 0$, thus $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$. ■

All these definitions can be obtained by reindexing the chain complex C_n by $C^n = C_{-n}$.

Definition 9. A **cochain complex** $C^\bullet$ of R-modules is a family $\{C^n\}$ of R-modules with maps $d = d^n : C^n \to C^{n+1}$ such that $d \circ d = 0$. The kernel $\operatorname{Ker} d_n$ is the module of **n-cocycles** of C, and denoted $Z^n = Z^n(C)$. The image $\operatorname{Im} d_{n-1}$ is the module of **n-coboundaries** of C, and denoted $B^n = B_n(C)$. Since $d \circ d = 0$, $0 \subset B^n \subset Z^n \subset C^n$ for all n. The quotient $H^n(C) = Z^n / B^n$ is called the **n-th cohomology module** of C. The morphism $u : C^\bullet \to D^\bullet$ is a family of R-module homomorphisms $u^n : C^n \to D^n$ which satisfies $u^{n+1} \circ d^n = d^n \circ u^n$. A morphism $C^\bullet \to D^\bullet$ of chain complexes is called a **quasi-isomorphism** if the maps $H^n(C) \to H^n(D)$ are all isomorphisms.

Definition 10. A chain complex C is **bounded** if all but finitely many C_n are zero. If $C_n = 0$ unless $a \leq n \leq b$, then we say the complex has **amplitude** in $[a, b]$. A complex $C_\bullet$ is **bounded above(below)** if there is a bound $b(a)$ such that $C_n = 0$ for all $n > b(n < a)$. The bounded, bounded above, bounded below chain complexes form full subcategories of $\mathsf{Ch} = \mathsf{Ch}(R - \mathsf{mod})$ that are denoted $\mathsf{Ch}_b, \mathsf{Ch}_-, \mathsf{Ch}_+$. If a chain complex is bounded below with bound 0, then we call it **non-negative complex**, and its category is denoted as $\mathsf{Ch}_{\geq 0}$. All these definitions works with cochain complex, where all the subscripts are changed into superscripts.

Exercise 11 (Homology of a graph)**.** Let Γ be a finite loopless graph with V vertices $(v_1, \cdots v_V)$ and E edges $(e_1, \cdots e_E)$. If we orient the edges, we can form the **incidence matrix** of the graph. This is a $V \times E$ matrix whose (ij) entry is $+1$ if the edge e_j starts at v_i, -1 if the edge e_j ends at v_j, and 0 otherwise. Let C_0 be the free R-module on the vertices, C_1 the free R-module on the edges, $C_n = 0$ if $n \neq 0, 1$, and $d : C_1 \to C_0$ be the incidence matrix. If Γ is connected, show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions 1 and $E - V + 1$, which is the number of **circuits** of the graph, respectively.

Solution. What we need to find is $\operatorname{Im} d$ and $\operatorname{Ker} d$. For $\operatorname{Im} d$, choose the basis $\{v_0, v_1 - v_0, \cdots, v_V - v_0\}$. Then considering the paths connecting v_i and v_0, and take the edges of the paths. This edges gives $v_i - v_0$ when passing through d, thus the only unachievable basis is v_0, thus $H_0(C)$ is the free R-module with dimension 1. For $\operatorname{Ker} d$, notice that the rank of Γ is $V-1$ by above, and by the rank-nullity theorem, $\operatorname{Ker} d$ is the free R-module with dimension $E-(V-1) = E-V+1$. ■

Example 12 (Simplicial homology). Let K be a geometric simplicial complex, and let K_k, where $0 \leq k \leq n$, are the set of k-dimensional simplices of K. Each k-simplex has $k+1$ faces, which are ordered if the set K_0 of vertices is ordered, so we obtain $k+1$ set maps $\partial_i : K_k \to K_{k-1}$. The **simplicial chain complex** of K with coefficients in R is the chain complex $C_\bullet$ formed as follows. The set C_k is a free R module on the set K_k if $0 \leq k \leq n$, and $C_n = 0$ otherwise. Define $\partial_i : C_k \to C_{k-1}$ using the set map $\partial_i : K_k \to K_{k-1}$, and then their alternating sum $d = \sum(-1)^i \partial_i$ is the map $C_k \to C_{k-1}$ in the chain complex C. Showing $d \circ d = 0$ is equivalent to the fact that each $(k-2)$-dimensional simplex in a fixed k-simplex σ of K lies on exactly two faces of σ. The homology obtained from the chain complex $C_\bullet$ is called the **simplicial homology** of K with coefficients on R.

Exercise 13 (Tetrahedron). The tetrahedron T is a surface with 4 vertices, 6 edges, and 4 2-dimensional faces. Thus its homology is the homology of a chain complex $0 \to R^4 \to R^6 \to R^4 \to 0$. Write down the matrices in this complex and verify computationally that $H_2(T) \simeq H_0(T) \simeq R$ and $H_1(T) = 0$.

Proof. First and last map are trivial. For the second map $R^4 \to R^6$, denoting 4 faces as A, B, C, D, and 6 edges as a, b, c, d, e, f, then we can write

$$A \mapsto a - b + d, B \mapsto b - c + e, C \mapsto c - a + f, D \mapsto -(d + e + f) \quad (176)$$

For the third map $R^6 \to R^4$, denoting 4 vertices as v, w, x, y, then we can write

$$a \mapsto v - w, b \mapsto v - x, c \mapsto v - y, d \mapsto w - x, e \mapsto x - y, f \mapsto y - w \quad (177)$$

Consider $0 \to R^4 \to R^6$. The image is 0, and the kernel is R-module with basis $A + B + C + D$, thus we get $H_2(T) = R$.

Consider $R^4 \to R^6 \to R^4$. The image is R-module with basis $\{a - b + d, b - c + e, c - a + f\}$, and the kernel is R-module with basis $\{a - b + d, b - c + e, c - a + f\}$, thus we get $H_1(T) = 0$.

Consider $R^6 \to R^4 \to 0$. The image is R-module with basis $\{v - w, v - x, v - y\}$, and the kernel is v, w, x, y, thus we get $H_1(T) = R$. □

Example 14 (Singular homology)**.** Let X be a topological space and $S_k = S_k(X)$ be the free R-module on the set of continuous maps from the k-simplex Δ_k to X if $k \geq 0$ and $S_k = 0$ if $k < 0$. Restricting $\Delta_k \to X$ to $\Delta_{k-1} \to X$ gives an R-module homomorphism $\partial_i : S_k \to S_{k-1}$, and the alternating sum $d = \sum(-1)^i \partial_i : S_k \to S_{k-1}$ gives a chain complex $S_\bullet$. The reason why $d \circ d = 0$ is similar with simplicial homology case. The homology obtained from the chain complex $S_\bullet$ is called the **singular homology** of X with coefficients in R, and written $H_n(X; R)$. If X is a geometric simplicial complex, then the inclusion $\mathsf{C}_\bullet(X) \to S_\bullet(X)$ is a quasi-isomorphism, and so the simplicial and singular homology modules of X are isomorphic. For more details, see Algebraic Topology by Allen Hatcher.

Definition 15. A cataegory A is called an Ab**-category** if $\mathsf{A}(a, b)$ is given the structure of abelian group in such a way that composition distributes over addition. That is, if $f : a \to b, g, g' : b \to c, h : c \to d$ are morphisms of A, then $h \circ (g + g') \circ f = h \circ g \circ f + h \circ g' \circ f$.

Consider two Ab-categories A, B. A functor $F : \mathsf{B} \to \mathsf{A}$ is an **additive functor** if $F : \mathsf{B}(b, b') \to \mathsf{A}(F(b), F(b'))$ is a group homomorphism.

Consider an Ab-category A. Then A is an **additive category** if A has an object which is both initial and terminal, which is called a **zero object**, and a product $a \times b$ for objects a, b of A.

In additive category, the finite products are same with the finite coproducts.

Example 16. The category Ch is an Ab-category, since we can add chain maps $\{f_n\}, \{g_n\} : \mathsf{C}_\bullet \to D_\bullet$ degreewise, that is, their sum is a family of maps $\{f_n + g_n\}$. The zero object of Ch is the complex 0 of zero modules and maps. For a family $\{A_\alpha\}$ of complexes of R-modules, the product $\prod A_\alpha$ and coproduct $\oplus A_\alpha$ exist in Ch, and defined degreewise, that is, the differentials are the maps

$$\prod d_\alpha = \prod A_{\alpha,n} \to \prod A_{\alpha,n-1}, \quad \oplus d_\alpha : \oplus A_{\alpha,n} \to \oplus A_{\alpha,n-1} \tag{178}$$

This shows that Ch is an additive category.

Exercise 17. Show that direct sum and direct product commute with homology, that is, $\oplus H_n(A_\alpha) \simeq H_n(\oplus A_\alpha)$ and $\prod H_n(A_\alpha) \simeq H_n(\prod A_\alpha)$ for all n.

Proof. Before showing this, we need to show a small lemma: in category $R - \mathsf{mod}$, the product of epimorphisms is epimorphic. Notice that the product of morphisms is morphism in $R - \mathsf{mod}$, and the product of surjective functions is surjective, this is true. Now, since the direct sum and direct product are in dual relation, we only need to prove it on the direct product. Now consider the following diagram.

$$\begin{tikzcd} B \arrow[r, dotted, "h"] \arrow[rr, bend left, "f"] \arrow[dr, "f_\alpha"'] & \mathrm{Ker}(d) \arrow[r, hook, "i"] \arrow[d, "\pi_\alpha|_{\mathrm{Ker}(d)}"] & \prod A_{\alpha,n} \arrow[r, "d"] \arrow[d, "\pi_\alpha"] & \prod A_{\alpha,n-1} \arrow[d, "\pi_\alpha"] \\ & \mathrm{Ker}(d_\alpha) \arrow[r, hook, "i_\alpha"] & A_{\alpha,n} \arrow[r, "d_\alpha"] & A_{\alpha,n-1} \end{tikzcd} \tag{179}$$

Here B is an R-module. Now due to the definition of the product, the functions $i_\alpha \circ f_\alpha$ and projections π_α defines a unique function $f : B \to \prod A_{\alpha,n}$. Now notice that $\pi_\alpha \circ d \circ f = d_\alpha \circ \pi_\alpha \circ f = d_\alpha \circ i_\alpha \circ f_\alpha = 0$, thus again by the definition of the product, $d \circ f = 0$. Due to the universal property of the kernel, there is a unique $h : B \to \mathrm{Ker}(d)$ which makes the diagram above commutes. Therefore we showed that $\mathrm{Ker}(d) = \prod \mathrm{Ker}(d_\alpha)$.

Now from the short exact sequences

$$0 \to \mathrm{Ker}(d_\alpha) \to A_{\alpha,n+1} \to \mathrm{Im}(d_\alpha) \to 0 \tag{180}$$

we can build a sequence

$$0 \to \prod \mathrm{Ker}(d_\alpha) \to \prod A_{\alpha,n+1} \to \prod \mathrm{Im}(d_\alpha) \to 0 \tag{181}$$

which is left exact due to the above argument. Now since in R-module the product of epimorphisms are epimorphic, the above sequence is right exact, hence exact, and

$$\prod \mathrm{Im}(d_\alpha) \simeq \prod A_{\alpha,n+1} / \prod \mathrm{Ker}(d_\alpha) \simeq \prod A_{\alpha,n+1} / \mathrm{Ker}(d) \simeq \mathrm{Im}(d) \tag{182}$$

Now take the product of following sequence

$$0 \to \mathrm{Ker}(d) \to \mathrm{Im}(d) \to H_n(A_{\alpha,n}) \to 0 \tag{183}$$

which gives the desired result. □

Definition 18. Let C is a category and $f : b \to c$ is a morphism in C. Then f is a **constant morphism** or **left zero morphism** if for any object a in C and any morphisms $g, h : a \to b$, $f \circ g = f \circ h$. Dually, f is a **coconstant morphism** or **right zero morphism** if for any object d in C and any morphisms $g, h : c \to d$, $g \circ f = h \circ f$. If $f : b \to c$ is both a constant and coconstant morphism, we call it **zero morphism**. We often write zero morphism from b to c as 0_{bc}, and if its domain and codomain are obvious, 0. A **category with zero morphisms** is a category C such that for all object pairs $a, b \in \mathsf{C}$ there is a morphisms 0_{ab} such that for all objects $a, b, c \in \mathsf{C}$ and morphisms $f : b \to c$, $g : x \to$, the following diagram commutes.

$$\begin{tikzcd} a \arrow[r, "0_{ab}"] \arrow[d, "g"] \arrow[dr, "0_{ac}"] & b \arrow[d, "f"] \\ b \arrow[r, "0_{bc}"] & c \end{tikzcd} \tag{184}$$

Then the morphisms 0_{ab} are zero morphisms.

Example 19. Let a category C has a zero object 0. Then for all objects $b, c \in \mathsf{C}$, there are unique morphisms $f : b \to 0, g : 0 \to c$. Now define $0_{bc} = g \circ f$. Then this is a zero morphism from b to c, due to the definition of the zero object.

Example 20. Let C be an Ab-category. Then every morphism set $\mathsf{C}(x, y)$ is an abelian group, thus have a zero element. Denote it 0_{xy}. Now choose the morphisms $f : y \to z, g : x \to y$. Then since $f \circ 0_{xy} + f \circ 0_{xy} = f \circ (0_{xy} + 0_{xy}) = f \circ 0_{xy}$, thus $f \circ 0_{xy} = 0_{xz}$, and same for $0_{yz} \circ g$. Therefore 0_{xy} are zero morphisms and make C a category with zero morphisms.

Definition 21. In an additive category C, a **kernel** of a morphism $f : b \to c$ is a map $i : a \to b$ such that $f \circ i = 0$ and, for any $i' : a' \to b$ such that $f \circ i' = 0$, there is a unique morphism $u : a' \to a$ such that $i \circ u = i'$.

$$\begin{tikzcd}
& b \arrow[dr, "f"] & \\
& a \arrow[u, "i"] \arrow[r, "0"] & c \\
a' \arrow[uur, "i'"] \arrow[ur, dotted, "u"] \arrow[urr, "0"'] & &
\end{tikzcd} \tag{185}$$

A **cokernel** is a dual of a kernel.

Proposition 22. *Take an additive category* $\mathsf{C} = R - \mathsf{mod}$ *and its morphism* $f : x \to y$*. Show that the followings are equivalent.*

1. f *is monic, that is, for any morphisms* $h, k : w \to x$*,* $f \circ h = f \circ k$ *implies* $h = k$*.*
2. *For every map* $j : w \to x$*,* $f \circ j = 0$ *implies* $j = 0$*.*
3. f *is a kernel of some morphism* $g : y \to z$*.*

Dually, the followings are equivalent.

1. f *is epic, that is, for any morphisms* $h, k : y \to z$*,* $h \circ f = k \circ f$ *implies* $h = k$*.*
2. *For every map* $j : y \to z$*,* $j \circ f = 0$ *implies* $j = 0$*.*
3. f *is a cokernel of some morphism* $g : w \to x$*.*

Proof. Let f be monic. Due to the definition of zero morphism, $f \circ 0 = 0$, thus $f \circ h = f \circ 0$, thus $h = 0$. Conversely, suppose that for every map $j : w \to x$, $f \circ j = 0$ implies $j = 0$. Choose $f \circ h = f \circ k$ for some $h, k : w \to x$. Then $f \circ h - f \circ k = f \circ (h - k) = 0$, thus $h - k = 0$ and $h = k$.

Now notice that f is a kernel of $g : y \to z \in \mathsf{C} = R - \mathsf{mod}$ if and only if f is the injective morphism $f : \mathrm{Ker}(g) \hookrightarrow y$. Furthermore, f is monic if and only if f is injective, thus if f is a kernel then f is monic, and if f is a monic function then f is a kernel of the function g which sends $\mathrm{Im}(f)$ to 0 and $y - \mathrm{Im}(f)$ to $y - \mathrm{Im}(f)$ as identity function. □

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Exercise 23. For a category Ch, and f be its morphism, show that the complex $\mathrm{Ker}(f)$ is a kernel of f and the complex $\mathrm{coKer}(f)$ is a cokernel of f.

Solution. Let $i_n : \mathrm{Ker}(f)_n \to C_n$ be the kernel of $f_n : C_n \to D_n$. Then we have the universal properties for each components: for any $g_n : B_n \to C_n$ such that $f_n \circ g_n = 0$, there is a unique morphism $u_n : B_n \to C_n$ such that $i_n \circ u_n = g_n$. Now suppose that $\{B_n\}$ is a chain complex and $\{g_n\}$ is a chain map. What we now need to show is $\{u_n\}$ is a chain map, that is, $d \circ u_n = u_{n-1} \circ d$. Now notice that $i_{n-1} \circ u_{n-1} \circ d = g_{n-1} \circ d = d \circ g_n = d \circ i_n \circ u_n = i_{n-1} \circ d \circ u_n$, and by previous proposition we know that the kernel i_{n-1} is monic, thus $u_{n-1} \circ d = d \circ u_n$. ■

Definition 24. An **abelian category** is an additive category A such that

1. every map in A has a kernel and cokernel;
2. every monic in A is the kernel of its cokernel;
3. every epi in A is the cokernel of its kernel.

Example 25. The category $R - \mathsf{mod}$ is an abelian category. Indeed, every morphism $f : c \to d$ has a kernel $\mathrm{Ker}(f) = \{x \in c : f(x) = 0\} \hookrightarrow c$ and a cokernel $\mathrm{coKer}(f) = d \to d/\mathrm{Im}(f)$. For monic f, $\mathrm{Im}(f) \simeq c$, thus $\mathrm{coKer}(f) = d \to d/c$ and its kernel is c. For epic f, the cokernel of $\mathrm{Ker}(f)$ is $c \hookrightarrow c/\mathrm{Ker}(f)$, which is surjective and has a same structure with $f : c \to d$, thus it is f.

Definition 26. For an abelian category C and its morphism f, the **image** of a map $f : b \to c$ is the subobject of c defined as $\mathrm{Ker}(\mathrm{coKer}(f))$.

This definition is same with our previous definition of Im in $R - \mathsf{mod}$, because $\mathrm{Ker}(\mathrm{coKer}(f)) = \mathrm{Ker}(c \to c/\mathrm{Im}(f)) = \mathrm{Im}(f)$.

Proposition 27. *For an abelian category* A, *every morphism* $f : b \to c$ *factors as*

$$b \xrightarrow{e} \mathrm{Im}(f) \xrightarrow{m} c \tag{186}$$

where $e = \mathrm{coKer}(\mathrm{Ker}(f))$ *is an epimorphism and* $m = \mathrm{Ker}(\mathrm{coKer}(f))$ *is a monomorphism.*

Proof. Take $m = \text{Ker}(\text{coKer}(f))$, which is monic since it is a kernel. Since $\text{coKer}(f) \circ f = 0$ by definition, f factors as $f = m \circ e$ for some unique e, which is epic. Now for any $g : a \to b$, $f \circ g = 0$ if and only if $e \circ g = 0$, since m is monic. Thus $\text{Ker}(f) = \text{Ker}(e)$. But since e is epic, $e = \text{coKer}(\text{Ker}(e)) = \text{coKer}(\text{Ker}(f))$. $\square$

The argument of this statement is in Categories for the working mathematician, S. MacLane, p189.

Definition 28. For an abelian category A, a sequence $a \xrightarrow{f} b \xrightarrow{g} c$ is **exact** if $\text{Ker}(g) = \text{Im}(f)$.

Definition 29. For an abelian category A, the category $\mathsf{Ch}(\mathsf{A})$ is a category whose objects are chain complexes in A and morphisms are chain maps in A.

Theorem 30. *For an abelian category* A, *the category* $\mathsf{Ch}(\mathsf{A})$ *is an abelian category.*

Proof. The argument for showing additive category is same with $R - \mathsf{mod}$ case. The argument for the first condition is just same with the case Ch. For the second and third condition, consider the components of the morphism f, which are all monic(epic) if and only if f is monic(epic). Since A is an abelian category, the components of f_n is the kernel of its cokernel(cokernel of its kernel), thus f also is. $\square$

Exercise 31. Show that a sequence $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ of chain complexes is exact in Ch just in case each sequence $0 \to A_n \xrightarrow{f} B_n \xrightarrow{g} C_n \to 0$ is exact in A.

Solution. What we need to show is $\text{Ker}(g_\bullet) = \text{Im}(f_\bullet)$, which is equivalent to $\text{Ker}(g_n) = \text{Im}(g_n)$ for all n. $\blacksquare$

Example 32. A **double complex** or **bicomplex** in A is a family $\{C_{p,q}\}$ of objects in A, together with maps $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v = C_{p,q} \to C_{p,q-1}$ such that $d^h \circ d^h = d^v \circ d^v = d^v \circ d^h + d^h \circ d^h \circ d^v = 0$. If there are finitely many nonzero $C_{p,q}$ along each diagonal line $p + q = n$, then we call C **bounded**.

Due to the anticommutivity, the maps d^v are not maps in Ch, but the chain maps $f_{\bullet,q} : C_{\bullet,q} \to C_{\bullet,q-1}$ can be defined by introducing

$$f_{p,q} = (-1)^p d^v_{p,q} : C_{p,q} \to C_{p,q-1} \tag{187}$$

Example 33 (Total complexes)**.** For a bicomplex C, we define the **total complexes** $\text{Tot}^{\prod}(C)$ and $\text{Tot}^{\oplus}(C)$ as

$$\text{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q}, \quad \text{Tot}^{\oplus}(C)_n = \oplus_{p+q=n} C_{p,q} \tag{188}$$

Then $d = d^h + d^v$ defines maps $d : \text{Tot}^{\prod,\oplus}(C)_{\to}\text{Tot}^{\prod,\oplus}(C)_{n-1}$

such that $d \circ d = 0$ since $d^h \circ d^v + d^v \circ d^h = 0$, thus they are chain complexes. Notice that the total complexes does not always exists, because the infinite (co)direct products could not exists. An abelian category is **(co)complete** if all (co)direct products exist. Both $R-\mathsf{mod}$ and $\mathsf{Ch}(R-\mathsf{mod})$ are complete and cocomplete.

Exercise 34. For a bounded double complex C with exact rows(columns), show that $\mathrm{Tot}^{\prod}(C)=\mathrm{Tot}^{\oplus}(C)=\mathrm{Tot}(C)$ is acyclic.

Solution. Since C is bounded, we can write the element of $\mathrm{Tot}(C)$ as $c=(\cdots,0,c_{0,0},c_{1,-1},\cdots,c_{k,-k},0,\cdots)$, by some shifting of indexes if needed. Suppose that $d(c)=0$, which means,

$$(\cdots,0,d^v(c_{0,0}),d^v(c_{1,-1})+d^h(c_{0,0}),\cdots,d^h(c_{k,-k}),0,\cdots)=0 \quad (189)$$

Now we want to find the element b of $\mathrm{Tot}(C)$ such that $d(b)=c$. Without loss of generality, we may let the columns are exact. Then since $d^v(c_{0,0})=0$, there is $b_{1,0}$ such that $d^v(b_{1,0})=c_{0,0}$. Now then we have

$$d^v(c_{1,-1})+d^h(d^v(b_{1,0}))=d^v(c_{1,-1})-d^v(d^h(b_{1,0}))=d^v(c_{1,-1}-d^h(b_{1,0}))=0 \quad (190)$$

and due to the exactness we have $b_{1,-1}$ such that $d^v(b_{1,-1})=c_{1,-1}-d^h(b_{1,0})$. By doing this inductively, which has finitely many steps because C is bounded. ∎

Exercise 35. Give examples of

1. a second quadrant double complex C with exact columns such that $\mathrm{Tot}^{\prod}(C)$ is acyclic but $\mathrm{Tot}^{\oplus}(C)$ is not;
2. a second quadrant double complex C with exact rows such that $\mathrm{Tot}^{\oplus}(C)$ is acyclic but $\mathrm{Tot}^{\prod}(C)$ is not;
3. a double complex in the entire plane for which every row and every column is exact, yet neither $\mathrm{Tot}^{\prod}(C)$ nor $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Solution.

1. Consider the following double complex.

$$\begin{tikzcd} \ddots \arrow[d, "1"] & & \\ \mathbb{Z} & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, "1"] & \\ & \mathbb{Z} & \mathbb{Z} \arrow[l, "\times 2"'] \end{tikzcd} \quad (191)$$

   Here all the non-represented objects are zero objects and morphisms are zero morphisms. Notice that the columns are exact. Now notice that this double complex takes

$$(\cdots,a_{-2},a_{-1},a_0)\mapsto(\cdots,a_{-2}+2a_{-1},a_{-1}+2a_0) \quad (192)$$

For $\mathrm{Tot}^{\prod}(C)$, take $(\cdots, 4, -2, 1)$. This is in the kernel of above map, but not in the image of zero map. For $\mathrm{Tot}^{\oplus}(C)$, we get the number n such that $a_{-k} = 0$ for all $k > n$. Now since

$$(a_{-n}, \cdots, a_0) \mapsto (2a_{-n}, \cdots, a_{-1} + 2a_0) \tag{193}$$

thus if $(a_{-n}, \cdots, a_0)$ is the kernel of above map then $a_{-n} = 0$, and inductively all $a_0 = 0$.

2. Consider the following double complex.

$$\begin{tikzcd}
\ddots & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "1"] & \\
 & \mathbb{Z} & \mathbb{Z} \arrow[l, "1"]
\end{tikzcd} \tag{194}$$

Here all the non-represented objects are zero objects and morphisms are zero morphisms. Notice that the rows are exact. Now notice that this double complex takes

$$(\cdots, a_{-2}, a_{-1}, a_0) \mapsto (\cdots, 2a_{-2} + a_{-1}, 2a_{-1} + a_0, a_0) \tag{195}$$

For $\mathrm{Tot}^{\oplus}(C)$, suppose that we have $(\cdots, 2a_{-2} + a_{-1}, 2a_{-1} + a_0, a_0) = (\cdots, 0, 0, 1)$. Then $a_0 = 1$, thus $a_{-1} = -2$, and $a_{-2} = -4$, and so on so we get $a_{-n} = 2^n$, which is not in $\mathrm{Tot}^{\oplus}(C)$, thus $(\cdots, 0, 0, 1)$ is not in the image of the above map, but in the kernel of zero map. For $\mathrm{Tot}^{\prod}(C)$, for $(\cdots, b_{-2}, b_{-1}, b_0)$, we take $a_0 = b_0$ and $a_{-n} = b_{-n} - 2a_{-(n-1)}$, which is well defined for all n.

3. Consider the following double complex.

$$\begin{tikzcd}
\ddots \arrow[d, "-1"] & & \\
\mathbb{Z} & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "-1"] & \\
 & \mathbb{Z} & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "-1"] \\
 & & \ddots
\end{tikzcd} \tag{196}$$

Here all the non-represented objects are zero objects and morphisms are zero morphisms. Notice that the rows and columns are exact. Now notice that this double complex takes

$$(\cdots, a_{-1}, a_0, a_{-1}, \cdots) \mapsto (\cdots, -a_{-1} + a_0, -a_0 + a_1, \cdots) \tag{197}$$

For $\mathrm{Tot}^{\prod}(C)$, $(\cdots, 1, 1, \cdots)$ is in the kernel of above map, but not in the image of zero map. For $\mathrm{Tot}^{\oplus}(C)$, if $(\cdots, -a_{-1} + a_0, -a_0 +$

$a_1, -a_1 + a_2, \cdots) = (\cdots, 0, 1, 0, \cdots)$ then we get $\cdots = a_{-1} + 1 = a_0 + 1 = a_1 = a_2 = \cdots$, which is not in $\mathrm{Tot}^{\oplus}(C)$, thus $(\cdots, 0, 1, 0, \cdots)$ is not in the image of the above map, but in the kernel of zero map.

■

Definition 36. Let C be a chain complex and n be an integer. The complex $\tau_{\geq n} C$ defined by

$$(\tau_{\geq n} C)_i = \begin{cases} 0, & i < n \\ Z_n, & i = n \\ C_i, & i > n \end{cases} \tag{198}$$

is called the **truncation of** C **below** n. Notice that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0, & i < n \\ H_i(C), & i \geq n \end{cases} \tag{199}$$

The quotient $\tau_{<n} C = C/(\tau_{\geq n} C)$ is called the **truncation of** C **above** n. Notice that

$$H_i(\tau_{<n} C) = \begin{cases} H_i(C), & i < n \\ 0, & i \geq n \end{cases} \tag{200}$$

The complex $\sigma_{<n} C$ defined by

$$(\sigma_{<n} C)_i = \begin{cases} C_i, & i < n \\ 0, & i \geq n \end{cases} \tag{201}$$

is called the **brutal truncation of** C **above** n. Notice that

$$H_i(\tau_{\geq n} C) = \begin{cases} H_i(C), & i < n \\ 0, & i > n \\ C_n/B_n, & i = n \end{cases} \tag{202}$$

The quotient $\sigma_{\geq n} C = C/(\sigma_{<n} C)$ is called the **brutal truncation of** C **below** n. Notice that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0, & i < n \\ H_i(C), & i > n \\ C_n/B_n, & i = n \end{cases} \tag{203}$$

Definition 37. If C is a chain complex and p is an integer, we take a new complex $C[p]$ defined as

$$C[p]_n = C_{n+p} \tag{204}$$

with differential $(-1)^p d$. If C is a cochain complex, we take

$$C[p]^n = C^{n-p} \tag{205}$$

with differential $(-1)^p d$. This job is called **shifting indices** or **translation**. We call $C[p]$ the **p-th translate of** C. Notice that

$$H_n(C[p]) = H_{n+p}(C), \quad H^n(C[p]) = H^{n-p}(C) \tag{206}$$

for chain and cochain complex respectively.

For a (co)chain map $f : C \to D$, we define $f[p] : C[p] \to D[p]$ as

$$f[p]_n = f_{n+p}, \quad f[p]^n = f^{n-p} \tag{207}$$

for chain and cochain map respectively. This makes translation a functor.

Exercise 38. If C is a complex, show that there are exact sequences of complexes:

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0 \tag{208}$$

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0 \tag{209}$$

Solution. We can expand the first sequence as

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r, hook] \arrow[d, "d_{n+1}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "d_{n+1}"] & B_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, hook] \arrow[d, "d_n"] & C_n \arrow[r, "d_n"] \arrow[d, "d_n"] & B_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, hook] \arrow[d, "d_{n-1}"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "d_{n-1}"] & B_{n-2} \arrow[r] \arrow[d, "d_{n-2}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{210}$$

which commutes, and all the rows are exact by first isomorphism theorem, thus the sequence is exact. Similarly, we can expand the second sequence as the sequence of

$$0 \to H_n \overset{i_n}{\hookrightarrow} C_n/B_n \xrightarrow{d_n} Z_{n-1} \xrightarrow{q_{n-1}} H_{n-1} \to 0 \tag{211}$$

which is exact since $\operatorname{Im} i_n = H_n = Z_n/B_n = \operatorname{Ker} d_n$ and $\operatorname{Im} d_n = B_{n-1} = \operatorname{Ker} q_{n-1}$. ■

2019- 07- 28

Exercise 39 (Mapping cone)**.** Let $f : B \to C$ be a morphism of chain complexes. Form a double chain complex D out of f by thinking of f as a chain complex in Ch and using the sign trick, putting $B[-1]$ in the row $q = 1$ and C in the row $q = 0$. Thinking of C and $B[-1]$ as double complexes in the obvious way, show that there is a short exact sequence of double complexes

$$0 \to C \to D \xrightarrow{\delta} B[-1] \to 0 \tag{212}$$

Solution. We can take D as the following double chain complex.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "-d_B"] & B_{n-1} \arrow[r, "-d_B"] \arrow[d, "f"] & B_n \arrow[r, "-d_B"] \arrow[d, "f"] & B_{n+1} \arrow[r, "-d_B"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r, "d_C"] & C_{n-1} \arrow[r, "d_C"] \arrow[d] & C_n \arrow[r, "d_C"] \arrow[d] & C_{n+1} \arrow[r, "d_C"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{213}$$

The image of $C \to D$ is C in D, which is also the kernel of δ, thus the sequence is exact. ■

Theorem 40. *Let* $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ *be a short exact sequence of chain complexes. Then there are natural maps* $\partial : H_n(C) \to H_{n-1}(A)$*, called* ***connecting homomorphisms****, such that*

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} \cdots \tag{214}$$

is exact. Similarly, if $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ *is a short exact sequence of chain complexes, then there are natural maps* $\partial : H^n(C) \to H^{n+1}(A)$ *such that*

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} \cdots \tag{215}$$

is exact.

Proof. We will come back to the proof of this theorem after we show some small but important lemmas. □

Exercise 41. Let $0 \to A \to B \to C \to 0$ be a short exact sequences of complexes. Show that if two of the three complexes A, B, C are exact, then so is the third.

Solution. Let B, C are exact. From the previous theorem, we have a long exact sequence $0 \to 0 \to H^n(A) \to 0 \to \cdots$. Thus $H^n(A) = 0$ for all n, and so A is exact. The proof is same for the B, C case. ■

Exercise 42. Suppose given a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{216}$$

in an abelian category, such that every column is exact. Show the following:

1. If the bottom two rows are exact, so is the top row.
2. If the top two rows are exact, so is the bottom row.
3. If the top and bottom rows are exact, and the composite $A \to C$ is zero, the middle row is also exact.

Solution. From the previous exercise, what we only need to prove is that the rows above diagram is actually chain complexes.

1. Since the above rectangle commutes and $A \to C$ is zero, $A' \to C' \to C$ is zero. Since $C' \to C$ is monic, $A' \to C'$ is zero.
2. Since the below rectangle commutes and $A \to C$ is zero, $A \to A'' \to C''$ is zero. Since $A \to A''$ is epic, $A'' \to C''$ is zero.
3. The additional condition itself shows the middle row is a chain complex.

■

Lemma 43 (Snake lemma)**.** *Consider a commutative diagram of R-modules of the form*

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C &
\end{tikzcd} \tag{217}$$

If the rows are exact, there is an exact sequence

$$\mathrm{Ker}(f) \to \mathrm{Ker}(g) \to \mathrm{Ker}(h) \xrightarrow{\partial} \mathrm{coKer}(f) \to \mathrm{coKer}(g) \to \mathrm{coKer}(h) \tag{218}$$

where ∂ is defined by the formula $\partial(c') = i^{-1} \circ g \circ p^{-1}(c')$ when $c' \in \mathrm{Ker}(h)$. Moreover, if $A' \to B'$ is monic, then so is $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$, and if $B \to C$ is onto, then so is $\mathrm{coKer}(g) \to \mathrm{coKer}(h)$.

There are bunch of notes for this lemma. First, the term 'snake' comes from the shape the line of exact sequence when we add the kernels above the diagram and cokernels below the diagram. Second, this lemma holds in an arbitrary abelian category. This is the corollary of the Freyd-Mitchell embedding theorem which gives an exact fully faithful embedding of small abelian category into $R-\mathrm{mod}$ for some ring

Proof. First we need to show that ∂ is well defined. Since p is surjective, $p^{-1}(c')$ is a nonempty set of elements in B'. Since $h(c') = 0$, $g(p^{-1}(c'))$ is in the kernel of $B \to C$, thus in the image of $A \to B$. Since i is injective, we can take $i^{-1}(g(p^{-1}(c'))$. Now picking $b, b' \in B'$ such that $p(b) = p(b') = c$, we have $p(b - b') = 0$, thus there is $a \in A'$ such that $A' \to B'$ maps a to $b - b'$. Due to the commutativity, $f(a) = i^{-1} \circ g(b - b')$, which is zero in coKer$(A)$. Thus ∂ is well defined.

Now we need to show that the sequence is exact on Ker(h) and coKer(f), since others are trivial by the exactness of rows. Notice that $c \in \mathrm{Ker}(\partial)$, then $i^{-1}(g(p^{-1}(c))) = 0$ implies $g(p^{-1}(c)) = 0$. Choose $b \in p^{-1}(c)$, then $g(b) = 0$ thus $b \in \ker(g)$, and $p(b) = c$. Therefore $c \in \mathrm{Im}(\mathrm{Ker}(g) \to \mathrm{Ker}(h))$. Conversely choose $c \in \mathrm{Im}(\mathrm{Ker}(g) \to \mathrm{Ker}(h))$, and take $b \in \mathrm{Ker}(g)$ such that $p(b) = c$. Then $i^{-1}(g(p^{-1}(c))) = i^{-1}(g(b)) = i^{-1}(0) = 0$. Now take $a \in \mathrm{Im}(\partial)$, and take $c \in \mathrm{Ker}(h)$ such that $\partial(c) = a$. Then $i^{-1}(g(p^{-1}(c))) = a$ implies $g(p^{-1}(c)) = i(a) = 0$ in coKer(g). Finally, take $a \in \mathrm{Ker}(\mathrm{coKer}(f) \to \mathrm{coKer}(g))$, then there is $b \in B'$ such that $i(a) = g(b)$, thus $a = i^{-1}(g(b)) = i^{-1}(g(p^{-1}(p(b))))$, thus $a \in \mathrm{Im}(\partial)$.

Suppose that $A' \to B'$ is monic, that is, if $a \mapsto 0$ then $a = 0$. Thus for all $a \in \mathrm{Ker}(f)$, $a \mapsto 0$ implies $a = 0 \in \mathrm{Ker}(f)$, thus $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$ is monic.

Suppose that $B \to C$ is epic, that is, for all $c \in C$ there is $b \in B$ which maps to c. Now choose $[c'] \in \mathrm{coKer}(h)$. Taking the representation $c' \in C$ of $[c']$, we have $b' \in B$ which maps to c'. Then $[b']$ maps to $[c']$. $\square$

Exercise 44 (5-lemma)**.** In any commutative diagram

$$\begin{tikzcd} A' \arrow[r, "g'"] \arrow[d, "f_a"] & B' \arrow[r, "h'"] \arrow[d, "f_b"] & C' \arrow[r, "i'"] \arrow[d, "f_c"] & D' \arrow[r, "j'"] \arrow[d, "f_d"] & E' \arrow[d, "f_e"] \\ A \arrow[r, "g"] & B \arrow[r, "h"] & C \arrow[r, "i"] & D \arrow[r, "j"] & E \end{tikzcd} \tag{219}$$

with exact rows in any abelian category, show that if f_a, f_b, f_d, and f_e are isomorphisms, then f_c is also an isomorphism. More precisely, show that if f_b and f_d are monic and f_a is epic, then f_c is monic. Dually, show that if f_b and f_d are epic and f_e is monic, then f_c is epic.

Solution. By the Freyd-Mitchell embedding theorem, it is enough to show this theorem in $R-$mod category. By duality, we only need to prove the second statement. Take $c \in C'$ such that $f_c(c) = 0$. Then by commutativity, $f_d(i'(c)) = 0$. Since f_d is monic, $i'(c) = 0$, thus $c \in \mathrm{Ker}(i')$. Since the rows are exact, $c \in \mathrm{Im}(h')$, that is, we have $b \in B'$ such that $h'(b) = c$. By commutativity again, $h(f_b(b)) = 0$, thus $f_b(b) \in \mathrm{Ker}(h)$. Again since the rows are exact, $f_b(b) \in \mathrm{Im}(g)$, that is, we have $a' \in A$ such that $g(a') = f_b(b)$. Since f_a is surjective, there is $a \in A'$ such that $g(f_a(a)) = f_b(g'(a)) = f_b(b)$, and since f_b is monic, $g'(a) = b$. Since $c = h'(b) = h'(g'(a)) = 0$, we get f_c is monic. ■

Proof of long exact sequence with connecting homomorphisms. From the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \end{tikzcd} \tag{220}$$

The snake lemma implies the following rows are exact.

$$\begin{tikzcd} & A_n/d(A_{n+1}) \arrow[r] \arrow[d, "d"] & B_n/d(B_{n+1}) \arrow[r] \arrow[d, "d"] & C_n/d(C_{n+1}) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) & \end{tikzcd} \tag{221}$$

Notice that the kernel of $d : A_n/d(A_{n+1}) \to Z_{n-1}(A)$ is $Z_n/B_n = H_n(A)$, and the cokernel is $Z_{n-1}/B_{n-1} = H_{n-1}(A)$. Therefore snake lemma implies the sequence

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} \cdots \tag{222}$$

is exact. □

Proposition 45. *The construction of long exact sequence from short exact sequence defined as above is a functor from the category with short exact sequences to long exact sequence. That is, for every short exact sequence there is a long exact sequence, and for every map of short exact sequences there is a corresponding map of long exact sequences.*

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r, "\partial"] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots \end{tikzcd} \tag{223}$$

Proof. To prove this, we only need to show that the diagram above commutes. Since H_n is a functor, the left two squares commute. Due to the Freyd-Mitchell embedding theorem, we only need to work on $R-\mathsf{mod}$ category. Take $z \in H_n(C)$ which is represented by $c \in C_n$. Then the image of z, $z' \in H_n(C')$, is represented by the image of c. Also if $b \in B_n$ maps to c, then its image $b' \in'_n$ maps to c'. Now we observe that the element $d(b) \in B_{n-1}$ belongs to the submodule $Z_{n-1}(A)$ and represents $\partial(z) \in H_{n-1}(A)$, which can be found in the construction of ∂. Thus $\partial(z')$ is represented by the image of $d(b)$, which is the image of a representative of $\partial(z)$, thus $\partial(z')$ is the image of $\partial(z)$. □

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Exercise 46. Consider the boundaries-cycles exact sequence $0 \to Z \to C \to B[-1] \to 0$ associated to a chain complex C. Show that the corresponding long exact sequence of homology breaks up into short exact sequences.

Solution. Notice that $d(B) = 0$ since $d \circ d = 0$. Therefore we get the long exact sequence

$$\cdots \to 0 \to H_n(Z) \to H_n(C) \to 0 \to H_{n-1}(Z) \to \cdots \tag{224}$$

This shows that $H_n(Z) \simeq H_n(C)$. Indeed, since $\mathrm{Im}(d) = B_n$ and $\mathrm{Ker}(d) = Z_n$ in Z, $H_n(Z) = Z_n/B_n = H_n(C)$. ■

Exercise 47. Let f be a morphism of chain complexes. Show that if $\mathrm{Ker}(f)$ and $\mathrm{coKer}(f)$ are acyclic, then f is a quasi-isomorphism. Is the converse true?

Solution. Take $f : B \to C$. Notice that the sequences

$$0 \to \mathrm{Ker}(f) \to B \to \mathrm{Im}(f) \to 0 \tag{225}$$

and

$$0 \to \mathrm{Im}(f) \to C \to \mathrm{coKer}(f) \to 0 \tag{226}$$

are exact. Since $\mathrm{Ker}(f)$ and $\mathrm{coKer}(f)$ are acyclic, the long exact sequence shows that

$$H_n(B) \simeq H_n(\mathrm{Im}(f)) \simeq H_n(C) \tag{227}$$

and thus f is a quasi-isomorphism.

Conversely, consider the following morphism.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "0"] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{228}$$

Both sequences are exact, and hence f is a quasi-isomorphism. But both kernel $0 \to \mathbb{Z} \to 0 \to 0 \to 0$ and cokernel $0 \to 0 \to 0 \to \mathbb{Z} \to 0$ are not acyclic. ■

Exercise 48. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of double complexes of modules. Show that there is a short exact sequence of total complexes, and conclude that if $\mathrm{Tot}(C)$ is acyclic, then $\mathrm{Tot}(A) \to \mathrm{Tot}(B)$ is a quasi-isomorphism.

Solution. The last statement can be proven by using the long exact sequence. Now to make the short exact sequence

$$0 \to \mathrm{Tot}(A) \to \mathrm{Tot}(B) \to \mathrm{Tot}(C) \to 0 \tag{229}$$

define the maps as

$$\prod_{p+q=n} A_{p,q} \to \prod_{p'+q'=n} B_{p',q'}, \quad (\cdots, a_{p,q}, \cdots) \mapsto (\cdots, f_{p,q}(a_{p,q}), \cdots) \tag{230}$$

This is a chain map since f, g are map between double complexes, and short exact because f, g gives short exact sequence. ∎

Definition 49. A complex C is called **split** if there are maps $s_n : C_n \to C_{n+1}$ such that $d = d \circ s \circ d$. The maps s_n are called the **splitting maps**. If in addition C an exact sequence, then we say C is **split exact**.

Example 50. Let $R = \mathbb{Z}$ or $\mathbb{Z}/4$, and let C be a complex

$$\cdots \xrightarrow{\times 2} R \xrightarrow{\times 2} R \xrightarrow{\times 2} \cdots \tag{231}$$

This complex is exact but not split exact.

Exercise 51.

1. Show that acyclic bounded below chain complexes of free R-modules are always split exact.

2. Show that an acyclic chain complex of finitely generated free abelian groups is always split exact, even when it is not bounded below.

Solution.

1. First we want to show that if C is a free module, then every exact sequence
$$0 \to A \to B \to C \to 0 \tag{232}$$
has a split $s : C \to B$. Since C is free, there is a basis E of C, and since $B \to C$ is surjective, for every $e_\alpha \in E \subset C$ there is $b_\alpha \in B$ such that $b_\alpha \mapsto e_\alpha$. Now define $s : C \to B$ as $s(e_\alpha) = b_\alpha$. Now consider $d \circ s \circ d(b)$ for some $b \in B$. Since $d(b) \in C$, we may write $d(b) = \sum_i r_i e_i$. Now $d \circ s(d(b)) = d(\sum_i r_i s(e_i)) = \sum_i r_i d(b_\alpha) = \sum_i r_i d(b_\alpha)$. Now denote the chain as
$$\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \tag{233}$$

Indeed this sequence is split exact, since $B = A \oplus C$ thus we may take a map $B \to A$ by taking A to A with identity, C to 0, and define homomorphically for others. To show this, take $b \in B$ which maps to $c \in C$. Now take $b - s(c)$. Since $d(b - s(c)) = d(b) - d(s(c)) = c - c = 0$, $b - s(c) \in A$, thus $B = A + C$. Furthermore, suppose that $b \in B$ is in both A, C. Then b maps to 0, but since $s(b) = 0$, $b = 0$. This statement is related to the fact that the free modules are projective.

Now we have a following exact sequence which is split exact.

$$0 \to \mathrm{Ker}(d_1) \hookrightarrow C_1 \xrightarrow{d_1} C_0 \to 0 \tag{234}$$

Thus we may choose $s_0 : C_0 \to C_1$ such that $d_1 \circ s_0 \circ d_1 = d_1$. Also the following chain is exact.

$$\cdots \xrightarrow{d_3} C_2 \xrightarrow{d_2} \mathrm{Im}(d_2) \xrightarrow{0} \tag{235}$$

Now use induction steps to achieve s_n.

2. Consider the map $f : A \to B$ where A, B are finitely generated free abelian groups. Since the subgroup of free group is free, we may choose the finite generators of $\mathrm{Im}(A)$, and we may choose the orthogonal subgroup B' of B. Now for each generators $b \in \mathrm{Im}(A)$ there is $a \in A$ such that $f(a) = b$. Define $s : B \to A$ as $s(b) = a$ if b is a generator of $\mathrm{Im}(A)$, and $s(b) = 0$ if b is a generator of B'. Then we get $d \circ s \circ d = d$.

■

Lemma 52 (Splitting lemma)**.** *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \tag{236}$$

be a short exact sequence in $R - \mathsf{mod}$ category. Then the followings are equivalent.

1. *The sequence $0 \to A \to B$ splits.*
2. *The sequence $B \to C \to 0$ splits.*
3. *$A \oplus C \simeq B$.*

Proof. ($3 \Rightarrow 1$) Since $A \oplus C \simeq B$, we may identify A with $i(A)$. Define $s : B \to A$ as the projection operator. Then $i \circ s \circ i(a) = i(a)$ for all $a \in A$.

($3 \Rightarrow 2$) Since $A \oplus C \simeq B$, we may define $s : C \to B$ as the inclusion by identifying C with $s(C)$. Suppose that $b = a + c$ for $b \in B, a \in A, c \in C$. Then $j \circ s \circ j(b) = j \circ s(c) = j(c) = j(b)$ for all $b \in B$.

($1 \Rightarrow 3$) First, since i is injective, $i \circ s \circ i = i$ implies $s \circ i = 1_A$. Consider $s : B \to A$ such that $i \circ s \circ s = i$. Choose $b \in B$. Now notice that $b = (b - i \circ s(b)) + i \circ s(b)$. Notice that $i \circ s(b) \in \mathrm{Im}(i)$, and $s(b - i \circ s(b)) = s(b) - s \circ i \circ s(b) = s(b) - s(b) = 0$ thus $b - i \circ s(b) \in \mathrm{Ker}(s)$. Now, suppose that $b \in \mathrm{Im}(i) \cap \mathrm{Ker}(s)$. Then $i(a) = b$ for some $a \in A$ and $s(b) = 0$, thus $s \circ i(a) = 0$. Since $s \circ i = 1_A$, $a = 0$, $b = 0$. Hence $B = \mathrm{Im}(i) \oplus \mathrm{Ker}(s)$. Now since i is injective, $\mathrm{Im}(i) \simeq A$. Finally, consider $j : \mathrm{Ker}(s) \to C$ be the restricted map of j. For any $c \in C$ we have $b \in B$ such that $j(b) = c$, and then $j(b - i(s(b)) = c$. Thus j is injective. If $j(b) = 0$, then $j \in \mathrm{Im}(i)$, and since $\mathrm{Im}(i) \cap \mathrm{Ker}(s) = 0$, $b = 0$. Thus $j : \mathrm{Ker}(s) \to C$ is an isomorphism, and $\mathrm{Ker}(s) \simeq C$.

($2 \Rightarrow 3$) First, since j is surjective, $j \circ s \circ j = j$ implies $j \circ s = 1_C$. Choose $b \in B$. Now notice that $b = (b - s \circ j(b)) + s \circ j(b)$. Notice that $s \circ j(b) \in \mathrm{Im}(s)$, and $j(b - s \circ j(b)) = j(b) - j \circ s \circ j(b) = j(b) - j(b) = 0$ thus $b - s \circ j(b) \in \mathrm{Ker}(j)$. Now, suppose that $b \in \mathrm{Im}(s) \cap \mathrm{Ker}(j)$. Then $s(c) = b$ for some $c \in C$ and $j(b) = 0$, thus $j \circ s(c) = 0$. Since $j \circ s = 1_C$, $c = 0$. Hence $B = \mathrm{Im}(s) \oplus \mathrm{Ker}(j)$. Now since $\mathrm{Im}(i) \simeq \mathrm{Ker}(j)$ and i is injective, $\mathrm{Ker}(j) \simeq A$. Finally, since $j \circ s$ is a bijection, s is an injection, and thus $\mathrm{Im}(s) \simeq C$. □

Exercise 53. Let C be a chain complex, with boundaries B_n and cycles Z_n in C_n. Show that C is split if and only if there are R-module decomposition $C_n \simeq Z_n \oplus B'_n$ and $Z_n \simeq B_n \oplus H'_n$. Show that C is exact if and only if $H'_n = 0$.

Solution. The first statement shows second statement directly.

Suppose that C is split with splitting map s. Consider the map $d : s \circ d(C_n) \to \mathrm{Im}(d) = B_{n-1}$. If $d(c) = 0$ for $c \in s \circ d(C_n)$ then we have $c' \in C_n$ such that $c = s \circ d(c')$, thus $d \circ s \circ d(c') = d(c') = 0$ so $c = 0$. Hence $\mathrm{Ker}(d) = 0$. Also for all $c \in \mathrm{Im}(d)$, i.e. $c = d(c')$, $d \circ s \circ d(c') = d(c') = c$. Thus this map is isomorphic, and $s \circ d(C_n) \simeq B_{n-1}$. Now consider the following short exact sequence.

$$0 \to Z_n \to C_n \to B_{n-1} \to 0 \tag{237}$$

We may take the right splitting map $B_{n-1} \to C_n$ as the inclusion map $s \circ d(C_n) \hookrightarrow C_n$. This shows that $C_n \simeq Z_n \oplus B_{n-1}$ where $B'n \simeq B_{n-1} \simeq s \circ d(C_n)$.

Now consider $c \in \mathrm{Im}(d_{n+1})$, i.e. $c = d(c')$, then $c = d \circ s \circ d(c')$ thus $c \in d \circ s(C_n)$. Conversely if $c \in d \circ s(C_n)$ then $c \in \mathrm{Im}(d_{n+1})$ obviously, therefore $d \circ s(C_n) = \mathrm{Im}(d_{n+1}) = B_n$. Now consider the following short exact sequence.

$$0 \to B_n \to Z_n \to Z_n/B_n \to 0 \tag{238}$$

We may take the left splitting map $Z_n \to B_n \simeq d \circ s(C_n)$ as the map $d \circ s$. This shows that $Z_n \simeq B_n \oplus Z_n/B_n \simeq B_n \oplus H'_n$.

Finally suppose that $C_n \simeq Z_n \oplus B'_n$ and $Z_n \simeq B_n \oplus H_n$. Define the splitting map $s : C_n \to C_{n+1}$ as $s|_{B_n} = 1_{B_n} : B_n \mapsto B'_{n+1} \simeq B_n$, $s|_{H_n} = 0$, and $s|_{B'_n} = 0$. Notice that $d|_{B_n} = 0, d|_{H_n} = 0$, and $d|_{B'_n} = 1_{B'_n} : B'_n \simeq$

$B_{n-1} \to B_{n-1}$. This shows that $d \circ s \circ d = d$.

$$
\begin{tikzcd}
C_{n+1} \arrow[d, "d"] & H_{n+1} \arrow[drr, "0"] & \oplus & 0 & \oplus & B_{n+1} \arrow[dll, "0"] & \oplus & B'_{n+1} \arrow[dll, "\simeq"] \\
C_n \arrow[d, "d"] & H_n \arrow[drr, "0"] & \oplus & 0 & \oplus & B_n \arrow[dll, "0"] & \oplus & B'_n \arrow[dll, "\simeq"] \\
C_{n-1} & H_{n-1} & \oplus & 0 & \oplus & B_{n-1} & \oplus & B'_{n-1}
\end{tikzcd}
\tag{239}
$$

$$
\begin{tikzcd}
C_{n+1} & H_{n+1} & \oplus & 0 & \oplus & B_{n+1} & \oplus & B'_{n+1} \\
C_n \arrow[u, "s"] & H_n \arrow[urr, "0"] & \oplus & 0 & \oplus & B_n \arrow[urr, "\simeq"] & \oplus & B'_n \arrow[ullll, "0"] \\
C_{n-1} \arrow[u, "s"] & H_{n-1} \arrow[urr, "0"] & \oplus & 0 & \oplus & B_{n-1} \arrow[urr, "\simeq"] & \oplus & B'_{n-1} \arrow[ullll, "0"]
\end{tikzcd}
\tag{240}
$$

■

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Proposition 54. *For a two chain complexes C, D and maps $s_n : C_n \to D_{n+1}$. Define $f_n : C_n \to D_n$ defined as $f_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n$.*

$$
\begin{tikzcd}
C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "s"] \arrow[d, "f"] & C_{n-1} \arrow[dl, "s"] \\
D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1}
\end{tikzcd}
\tag{241}
$$

Then f is a chain map from C to D.

Proof. Direct calculation shows $d \circ f = d \circ (d \circ s + s \circ d) = d \circ s \circ d = (d \circ s + s \circ d) \circ d = f \circ d$. □

Definition 55. A chain map $f : C \to D$ is **null homotopic** if there are maps $s_n : C_n \to D_{n+1}$ such that $f = d \circ s + s \circ d$. The maps $\{s_n\}$ are called a **chain construction** of f.

Exercise 56. Show that C is a split exact chain complex if and only if the identity map on C is null homotopic.

Solution. Suppose that C is a split exact chain complex. Choose $s_n : C_n \to C_{n+1}$ as the split maps. From previous exercise, we may decompose C_n into $Z_n \oplus B'_n \simeq B_n \oplus B'_n$. The structure of d and s map can be drawn as below.

$$\begin{tikzcd}
C_{n+1} \arrow[d, "d"] & 0 & \oplus & B_{n+1} \arrow[dll, "0"] & \oplus & B'_{n+1} \arrow[dll, "\simeq"] \\
C_n \arrow[d, "d"] & 0 & \oplus & B_n \arrow[dll, "0"] & \oplus & B'_n \arrow[dll, "\simeq"] \\
C_{n-1} & 0 & \oplus & B_{n-1} & \oplus & B'_{n-1}
\end{tikzcd} \tag{242}$$

$$\begin{tikzcd}
C_{n+1} & 0 & \oplus & B_{n+1} & \oplus & B'_{n+1} \\
C_n \arrow[u, "s"] & 0 & \oplus & B_n \arrow[urr, "\simeq"] & \oplus & B'_n \arrow[ullll, "0"] \\
C_{n-1} \arrow[u, "s"] & 0 & \oplus & B_{n-1} \arrow[urr, "\simeq"] & \oplus & B'_{n-1} \arrow[ullll, "0"]
\end{tikzcd} \tag{243}$$

Thus $d \circ s$ is the projection on B_n and $s \circ d$ is the projection on B'_n. This shows that $d \circ s + s \circ d$ is identity.

Now suppose that the identity map on C is null homotopic, that is, we have the maps $s_n : C_n \to C_{n+1}$ such that $d \circ s + s \circ d = 1$. Then $d = d \circ s \circ d + s \circ d \circ d = d \circ s \circ d$, thus s_n are splitting maps. ■

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Definition 57. Two chain maps $f, g : C \to D$ are **chain homotopic** if $f - g$ is null homotopic, that is, if there are maps $s_n : C_n \to D_{n+1}$ such that $f - g = d \circ s + s \circ d$. The maps s_n are called a **chain homotopy** from f to g.

Definition 58. For a chain map $f : C \to D$, f is called a **chain homotopy equivalence** if there is a chain map $g : D \to C$ such that $g \circ f$ is chain homotopic to 1_C and $f \circ g$ is chain homotopic to 1_D.

Lemma 59. *If a chain map $f : C \to D$ is null homotopic, then every map $f_* : H_n(C) \to H_n(D)$ is zero. If f and g are chain homotopic, then they induce the same maps $f_* = g_* : H_n(C) \to H_n(D)$.*

Proof. The first statement shows the second statement directly.

Suppose that $f = s \circ d + d \circ s$ for some $s : C_n \to D_{n+1}$. Consider an n-cycle $c \in C_n$. Then $f(c) = s \circ d(c) + d \circ s(c) = d \circ s(c)$, thus $f(c)$ is the boundary in D. Hence $f_*(c) = 0$. □

The term homotopy comes from the following fact. For a map $f : X \to Y$ between topological spaces, there is a induced chain map $f_* : S(X) \to S(Y)$ between the corresponding singular chain complexes. If f is topologically null homotopic, then f_* is null homotopic; if f is a homotopy equivalence, then f_* is a chain homotopy equivalence; if f and another map $g : X \to Y$ are topologically homotopic, then f_* and g_* are chain homotopic.

Exercise 60. Consider the homology $H_n(C)$ of C as a chain complex with zero differentials. Show that if the complex C is split, then there is a chain homotopy equivalence between $C_\bullet$ and $H_\bullet(C)$. Conversely, if $C_\bullet$ and $H_\bullet(C)$ are chain homotopy equivalent, show that C is split.

Solution. Suppose that C is split. Then by previous exercise, we can take the chain map $f : C_\bullet \to H_\bullet(C)$ as the projection map and $g : H_\bullet(C) \to C_\bullet$ as the inclusion map: more precisely, $f = 1_C - s \circ d - d \circ s$ for splitting map s. Then $f \circ g$ is identity map itself. Also, $f \circ g - 1_C = s \circ d + d \circ s$, thus f and g are chain homotopy equivalences.

Now suppose that $C_\bullet$ and $H_\bullet(C)$ are chain homotopy equivalent, that is, take $f : C_\bullet \to H_\bullet(C)$ and $g : H_\bullet(C) \to C_\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to identities. since g is a chain map, $d \circ g = 0$. Also we may take maps $s_n : C_n \to C_{n+1}$ such that $1 - g \circ f = s \circ d + d \circ s$. Then,

$$d \circ s \circ d = d(s \circ d + d \circ s) = d(1 - g \circ f) = d - d \circ g \circ f = d \tag{244}$$

therefore s is a splitting map. ∎

Exercise 61. In this exercise we shall show that the chain homotopy classes of maps form a quotient category K of the category Ch of all chain complexes. The homology functors H_n on Ch will factor through the quotient functor $\mathsf{Ch} \to \mathsf{K}$.

1. Show that chain homotopy equivalences is an equivalence relation on the set of all chain maps from C to D. Let $\mathrm{Hom}_{\mathsf{K}}(C, D)$ denote the equivalence classes of such maps. Show that $\mathrm{Hom}_{\mathsf{K}}(C, D)$ is an abelian group.

2. Let f and g be chain homotopic maps from C to D. If $u : B \to C$ and $v : D \to E$ are chain maps, show that $v \circ f \circ u$ and $v \circ g \circ u$ are chain homotopic. Deduce that there is a category K whose objects are chain complexes and whose morphisms are given in 1.

3. Let f_0, f_1, f_0 and g_1 be chain maps from C to D such that f_i is chain homotopic to g_i for $i = 1, 2$. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category, and that $\mathsf{Ch} \to \mathsf{K}$ is an additive functor.

4. Is K an abelian category? Explain.

Solution. 1. For two chain maps $f, g : C \to D$, we say $f \sim g$ if $f - g = d \circ s + s \circ d$ for some maps $s_n : C_n \to D_{n+1}$. To show that $\sim$ is an equivalent relation, notice that $f \sim f$ by zero maps $0_n : C_n \to D_{n+1}$, and $f \sim g$ by s_n implies $g \sim f$ by $-s_n$. Finally, suppose that $f \sim g$ by s_n and $g \sim h$ by t_n. Then $f \sim h$ by $s_n + t_n$, since $f - h = (f - g) + (g - h) = (d \circ s + s \circ d) + (d \circ t + t \circ d) = d \circ (s + t) + (s + t) \circ d$.

Now for the equivalence classes $[f], [g] \in \mathrm{Hom}_{\mathsf{K}}(C, D)$, define $[f] + [g] = [f + g]$. This definition is well defined, since if $f \sim f'$ by s_n and $g \sim g'$ by t_n then $f + g \sim f' + g'$ by $s_n + t_n$. Now we can see that this addition is associative, and the zero map $[0]$ is identity and $[-f] = -[f]$. This gives the result.

2. Suppose that f and g are chain homotopic with s_n. Then $f - g = d \circ s + s \circ d$. Now since $v \circ f \circ u - v \circ g \circ u = v \circ (f - g) \circ u = v \circ (d \circ s + s \circ d) \circ u = d \circ (v \circ s \circ u) + (v \circ s \circ u) \circ d$ since u, v are chain maps, $v \circ f \circ u$ and $v \circ g \circ u$ are chain homotopic by $v \circ s \circ u$. Now define $[g] \circ [f] = [g \circ f]$. This definition is well defined, since if $f \sim f' : C \to D$ by s_n and $g \sim g' : D \to E$ by t, then $g' \circ f' \sim g \circ f$ since $g' \circ f' - g \circ f = (g' \circ f' - g' \circ f) + (g' \circ f - g \circ f) = g' \circ (d \circ s + s \circ d) + (d \circ t + t \circ d) \circ f = d \circ (g' \circ s + t \circ f) + (g' \circ s + t \circ f) \circ d$. Therefore the identity map 1 gives $[1] \circ [f] = [f]$ and $([h] \circ [g]) \circ [f] = [h \circ g] \circ [f] = [(h \circ g) \circ f] = [h \circ (g \circ f)] = [h] \circ ([g \circ f])$. Thus K is a category.

3. This is shown in 1., and this shows K is an Ab-category. Since the objects of K and Ch are same, K is an additive category: the zero objects are zero object, and the product is contained. Also, $[f + g] = [f] + [g]$ implies that $[\bullet] : \mathsf{Ch} \to \mathsf{K}$ is an additive functor.

4. No. Consider the chain map f between two chain complexes $\cdots \to 0 \to \mathbb{Z}/4 \to 0 \to \cdots$ and $\cdots \to 0 \to \mathbb{Z}/2 \to 0 \to \cdots$ defined by natural map $\mathbb{Z}/4 \to \mathbb{Z}/2$. TBD

■

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Definition 62. Let $f : B \to C$ be a chain map. The **mapping cone** of f is a chain complex $\mathrm{cone}(f)$ whose degree n part is $B_{n-1} \oplus C_n$ and the differential is $d(b, c) = (-d(b), d(c) - f(b))$.

Exercise 63. Let $\mathrm{cone}(C)$ denote the mapping cone of the identity map 1_C of C. Show that $\mathrm{cone}(C)$ is split exact, with $s(b, c) = (-c, 0)$ defining the splitting map.

Solution. Notice that $d \circ s \circ d(b, c) = d \circ s(-d(b), d(c) - b) = d(-d(c) + b, 0) = (-d(-d(c) + b), d(c) - b) = (-d(b), d(c) - b) = d(b, c)$. ■

Exercise 64. Let $f : C \to D$ be a chain map. Show that f is null homotopic if and only if f extends to a map $(-s, f) : \mathrm{cone}(C) \to D$.

Solution. Suppose that f is null homotopic, that is, $f = s \circ d + d \circ s$ for some $s_n : C_n \to D_{n+1}$. Then $d \circ (-s, f)(b, c) = d(f(c) - s(b)) = d \circ s \circ d(c) + d \circ d \circ s(c) - d \circ s(b) = d \circ s \circ d(c) - d \circ s(b)$ and $(-s, f) \circ d(b, c) = (-s, f)(-d(b), d(c) - b) = s \circ d(b) + f \circ d(c) - f(b) = -d \circ s(b) + d \circ s \circ d(c)$, thus $(-s, f)$ is a chain map. Conversely, if $(-s, f)$ is a chain map for some s, then we have $d \circ f(c) - d \circ s(b) = s \circ d(b) + f \circ d(c) - f(b)$, which implies $s \circ d + d \circ s = f$, thus f is null homotopic. ∎

Lemma 65. *For a chain map $f : B \to C$, the short exact sequence*

$$0 \to C \to cone(f) \xrightarrow{\delta} B[-1] \to 0 \tag{245}$$

gives the homology long exact sequence

$$\cdots \to H_{n+1}(cone(f)) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(cone(f)) \xrightarrow{\delta_*} \cdots \tag{246}$$

Then $\partial = f_*$.

Proof. For a cycle $b \in B_n$, The element $(-b, 0)$ of cone(f) lifts f via δ, and taking differential gives $d(-b, 0) = (d(b), f(b)) = (0, f(b))$. Thus $\partial[b] = [f(b)]$, which is $f_*([b])$. □

Corollary 66. *A chain map $f : B \to C$ is a quasi-isomorphism if and only if the mapping cone complex* **cone**(f) *is exact.*

Proof. If f is a quasi-isomorphism then ∂ is isomorphism, thus $H_n(C) \to H_n(\text{cone}(f))$ and δ_* are zero maps. Thus $H_n(\text{cone}(f)) = 0$ and so cone(f) is exact. Conversely, if cone(f) is exact then $H_n(\text{cone}(f)) = 0$, thus $H_n(B) \xrightarrow{\partial = f_*} H_n(C)$ is an isomorphism. □

Definition 67. Let K be a simplicial complex. The **topological cone** CK of K is obtained by adding a new vertex s to K and making the cone of the simplicies to get a new $n + 1$ simplex for every old n-simplex of K. Notice that the simplicial chain complex $C_\bullet(s)$ of the one point space $\{s\}$ is R in degree 0 and zero elsewhere. Then $C_\bullet(s)$ is a subcomplex of the simplicial chain complex $C_\bullet(CK)$ of the topological cone CK, and the quotient $C_\bullet(CK)/C_\bullet(s)$ is the chain complex cone$(C_\bullet K)$ of the identity map of $C_\bullet(K)$. The fact that cone$(C_\bullet K)$ is null homotopic reflects the fact that the topological cone CK is contractible.

Samely, if $f : K \to L$ is a simplicial map, the **topological mapping cone** Cf of f is obtained by glueing CK and L together, identifying the subcomplex K of CK with its image in L. If f is an inclusion of simplicial complexes, Cf is a simplicial complex. The quotient chain complex $C_\bullet(Cf)/C_\bullet(s)$ is the mapping cone cone(f_*) of the chain map $f_* : C_\bullet(K) \to C_\bullet(L)$.

Definition 68. For a chain complex map $f : B \to C$, the **mapping cylinder** of f is a chain complex $\mathrm{cyl}(f)$ whose degree n part is $B_n \oplus B_{n-1} \oplus C_n$ and the differential is $d(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b'))$.

Exercise 69. Let $\mathrm{cyl}(C)$ denote the mapping cylinder of the identity map 1_C of C. Show that two chain maps $f, g : C \to D$ are chain homotopic if and only if they extend to a map $(f, s, g) : \mathrm{cyl}(C) \to D$.

Solution. Suppose that f and g are chain homotopic, that is, $f - g = s \circ d + d \circ s$. Then $(f, s, g) \circ d(a, b, c) = (f, s, g)(d(a) + b, -d(b), d(c) - b) = f \circ d(a) + f(b) - s \circ d(b) + g \circ d(c) - g(b) = d \circ f(a) + d \circ s(b) + d \circ g(c) = d \circ (f, s, g)(a, b, c)$. Conversely, if $(f, s, g) \circ d(a, b, c) = d \circ (f, s, g)(a, b, c)$ then $f \circ d(a) + f(b) - s \circ d(b) + g \circ d(c) - g(b) = d \circ f(a) + d \circ s(b) + d \circ g(c)$ implies $(f - g)(b) = (s \circ d + d \circ s)(b)$, thus $f - g = s \circ d + d \circ s$. ■

Exercise 70. If $f : B \to C$, $g : C \to D$, and $e : B \to D$ are chain maps, show that e and $g \circ f$ are chain homotopic if and only if there is a chain map $\gamma = (e, s, g) : \mathrm{cyl}(f) \to D$. Note that e and g factor through γ.

Solution. Suppose that e and $g \circ f$ are chain homotopic, that is, $e - g \circ f = s \circ d + d \circ s$. Then $(e, s, g) \circ d(a, b, c) = (e, s, g)(d(a) + b, -d(b), d(c) - f(b)) = e \circ d(a) + e(b) - s \circ d(b) + g \circ d(c) - g \circ f(b) = d \circ e(a) + d \circ s(b) + d \circ g(c) = d \circ (e, s, g)(a, b, c)$. Conversely, if $(e, s, g) \circ d(a, b, c) = d \circ (e, s, g)(a, b, c)$ then $e \circ d(a) + e(b) - s \circ d(b) + g \circ d(c) - g \circ f(b) = d \circ e(a) + d \circ s(b) + d \circ g(c)$ implies $(e - g \circ f)(b) = (s \circ d + d \circ s)(b)$, thus $e - g \circ f = s \circ d + d \circ s$. ■

Lemma 71. *The subcomplex of elements $(0, 0, c)$ is isomorphic to C, and the corresponding inclusion $\alpha : C \to \mathrm{cyl}(f)$ is a quasi-isomorphism.*

Proof. Notice that $\mathrm{cyl}(f)/\alpha(C) = \mathrm{cone}(-1_B)$, which is split exact. Now from the short exact sequence

$$0 \to C \xrightarrow{\alpha} \mathrm{cyl}(f) \to \mathrm{cone}(-1_B) \to 0 \tag{247}$$

we have a long exact sequence

$$\cdots \to H_{n+1}(\mathrm{cyl}(f)) \to H_{n+1}(\mathrm{cone}(-1_B)) \to H_n(C) \to H_n(\mathrm{cyl}(f)) \to \cdots \tag{248}$$

Now since $\mathrm{cone}(-1_B)$ is exact, $H_n(C) \to H_n(\mathrm{cyl}(f))$ is an isomorphism. □

Exercise 72. Show that $\beta(b, b', c) = f(b) + c$ defines a chain map from $\mathrm{cyl}(f)$ to C such that $\beta \circ \alpha = 1_C$. Then show that the formula $s(b, b', c) = (0, b, 0)$ defines a chain homotopy from the identity of $\mathrm{cyl}(f)$ to $\alpha \circ \beta$. Conclude that α is in fact a chain homotopy equivalence between C and $\mathrm{cyl}(f)$.

Solution. First $d \circ \beta(b, b', c) = d(f(b) + c) = d \circ f(b) + d(c)$ and $\beta \circ d(b, b', c) = \beta(d(b) + b', -d(b'), d(c) - f(b')) = f \circ d(b) + f(b') + d(c) - f(b') = f \circ d(b) + d(c)$, thus β is a chain map. Now $\beta \circ \alpha(c) = \beta(0, 0, c) = c$ thus $\beta \circ \alpha = 1_C$. Finally, $(s \circ d + d \circ s)(b, b', c) = s(d(b) + b', -d(b'), d(c) - f(b')) + d(0, b, 0) = (0, d(b) + b', 0) + (b, -d(b), -f(b)) = (b, b', -f(b)) = (b, b', c) - (0, 0, f(b) + c)$, and since $\alpha \circ \beta(b, b', c) = \alpha(f(b) + c) = (0, 0, f(b) + c)$, $1 - \alpha \circ \beta = s \circ d + d \circ s$. Therefore α is a chain homotopy equivalence between C and $\mathrm{cyl}(f)$. ■

Definition 73. Let X be a cellular complex and $I = [0, 1]$. The space $I \times X$ is the **topological cylinder** of X, which is also a cell complex. If $C_\bullet(X)$ is the cellular chain complex of X, then the cellular chain complex $C_\bullet(I \times X)$ of $I \times X$ can be identified with the mapping cylinder chain complex of the identity map on $C_\bullet(X)$, $\mathrm{cyl}(1_{C_\bullet(X)})$.

Samely, if $f : X \to Y$ is a cellular map, then the **topological mapping cylinder** $\mathrm{cyl}(f)$ is obtained by glueing $I \times X$ and Y together, identifying $0 \times X$ with the image of X under f, which is also a cell complex. Then the cellular chain complex $C_\bullet(\mathrm{cyl}(f))$ can be identified with the mapping cylinder of the chain map $C_\bullet(X) \to C_\bullet(Y)$.

Lemma 74. *The subcomplex of elements $(b, 0, 0)$ in $cyl(f)$ is isomorphic to B, and $cyl(f)/B$ is the mapping cone of f. The composite $B \to cyl(f) \xrightarrow{\beta} C$ is the map f, where $\beta(b, b', c) = f(b) + c$ is the chain homotopy equivalence. Thus the map $f_* : H(B) \to H(C)$ factors through $H(B) \to H(cyl(f))$. Thus we may construct a commutative diagram of chain complexes with exact rows as following:*

$$\begin{tikzcd}
 & & C & & & \\
0 \arrow[r] & B \arrow[r] \arrow[ur, "f"] & cyl(f) \arrow[r] \arrow[u, "\beta"] & cone(f) \arrow[r] & 0 & \\
 & 0 \arrow[r] & C \arrow[r] \arrow[u, "\alpha"] & cone(f) \arrow[r, "\delta"] \arrow[u, equal] & B[-1] \arrow[r] & 0
\end{tikzcd} \tag{249}$$

and the homology long exact sequences can be drawn as following:

$$\begin{tikzcd}
\cdots \arrow[r, "-\partial"] & H_n(B) \arrow[r] \arrow[d, equal] \arrow[dr, "f"] & H_n(cyl(f)) \arrow[r] \arrow[d, equal] & H_n(cone(f)) \arrow[r, "-\partial"] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B[-1]) \arrow[r, "\partial"] & H_n(C) \arrow[r] & H_n(cone(f)) \arrow[r, "\delta"] & H_n(B[-1]) \arrow[r, "\partial"] & \cdots
\end{tikzcd} \tag{250}$$

This diagram commutes.

Proof. It suffices to show that the right square commutes. Let (b, c) be an n-cycle in $\mathrm{cone}(f)$, thus $d(b, c) = (-d(b), d(c) - f(b)) = 0$ implies

$d(b) = 0$ and $f(b) = d(c)$. Lifting (b, c) to $(0, b, c)$ in $\mathrm{cyl}(f)$ and taking differential gives $d(0, b, c) = (b, -d(b), d(c) - f(b) = (b, 0, 0)$. thus ∂ maps the class of (b, c) to the class of $b = -\delta(b, c)$ in $H_{n-1}(B)$. Thus the right square commutes. □

Proposition 75. *For any short exact sequence of complexes*

$$0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0 \tag{251}$$

the following natural isomorphism of long exact sequences holds.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] \arrow[d, equal] & H_n(cyl(f)) \arrow[r] \arrow[d, "\simeq"] & H_n(cone(f)) \arrow[r, "\partial"] \arrow[d, "\simeq"] & H_{n-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] & H_n(C) \arrow[r] & H_n(D) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd} \tag{252}$$

Proof. Consider a chain map $\phi : \mathrm{cone}(f) \to D$ defined by $\phi(b, c) = g(c)$. Then the following diagram commutes with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & \mathrm{cone}(f) \arrow[r, "\delta"] \arrow[d, equal] & B[-1] \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{cyl}(f) \arrow[r] \arrow[d, "\beta"] & \mathrm{cone}(f) \arrow[r] \arrow[d, "\phi"] & 0 & \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & D \arrow[r] & 0 &
\end{tikzcd} \tag{253}$$

Now since β is a quasi-isomorphism, by 5-lemma and the functority of long exact sequence, ϕ is a quasi-isomorphism. Thus the naturality of ∂ gives the diagram above commutes. □

Exercise 76. Considering B and C as modules considered as chain complexes concentrated in degree zero, $\mathrm{cone}(f)$ is the complex $0 \to B \xrightarrow{-f} C \to 0$. Show that ϕ defined in above proposition is a chain homotopy equivalence if and only if $f : B \hookrightarrow C$ is a split injection.

Proof. Since B and C are modules, D is also a module. Thus, ϕ is a chain homotopy equivalence if and only if there is a map $\alpha : D \to C$ such that there is $r : C \to B$ with $\alpha \circ \phi = 1 - f \circ r$ and $\phi \circ \alpha = 1$. Now since $\phi \circ f = 0$, $f - f \circ r \circ f = \alpha \circ \phi \circ f = 0$, thus $f = f \circ r \circ f$ and thus f is a split injection. Conversely, if $f = f \circ r \circ f$ for some r, then since f is injective $1 = r \circ f$. Now define $\alpha = (1 - f \circ r) \circ \phi^{-1}$. Notice that ϕ is surjective. Now if $a, b \in \phi^{-1}(c)$, then $\phi(a) - \phi(b) = \phi(a - b) = 0$, thus $a - b \in \operatorname{Ker} \phi = \operatorname{Im} f$. Thus there is e such that $f(e) = a - b$, and $(1 - f \circ r)(a - b) = (a - b) - f \circ r \circ f(e) = (a - b) - f(e) = 0$, thus this map is well defined. Finally, since $\phi \circ f = 0$, $\phi \circ \alpha = \phi \circ \phi^{-1} = 1$. □

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Exercise 77. Show that the composite

$$H_n(D) \simeq H_n(\mathrm{cone}(f)) \xrightarrow{-\delta_*} H_n(B[-1]) \simeq H_{n-1}(B) \tag{254}$$

is the connecting homomorphism ∂ in the homology long exact sequence for

$$0 \to B \to C \to D \to 0 \tag{255}$$

Solution. The result is obvious due to the result of previous proposition and lemma. ■

Exercise 78. Show that there is a quasi-isomorphism $B[-1] \to \mathrm{cone}(g)$ dual to ϕ. Then dualize the previous exercise, by showing that the composite

$$H_n(D) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{\simeq} H_n(\mathrm{cone}(g)) \tag{256}$$

is the usual map induced by the inclusion of D in $\mathrm{cone}(g)$.

Solution. This is basically the dual version of previous statements. ■

Exercise 79. Given a map $f : B \to C$ of complexes, let v denote the inclusion of C into $\mathrm{cone}(f)$. Show that there is a chain homotopy equivalence $\mathrm{cone}(v) \to B[-1]$. This equivalence is the algebraic analogue of the topological fact that for any map $f : K \to L$ of (topological) cell complexes the cone of the inclusion $L \hookrightarrow Cf$ is homotopy equivalent to the suspension of K.

Solution. First notice that $\mathrm{cone}(v)_n = C_{n-1} \oplus \mathrm{cone}(f)_n = C_{n-1} \oplus B_{n-1} \oplus C_n$ with differential operator $d(c_{n-1}, b_{n-1}, c_n) = (-d(c_{n-1}), d(b_{n-1}, c_n) - v(c_{n-1})) = (-d(c_{n-1}), -d(b_{n-1}), d(c_n) - f(b_{n-1}) - c_{n-1})$. Now consider $\psi : \mathrm{cone}(v) \to B[-1]$ defined as $\psi(c_{n-1}, b_{n-1}, c_n) = (-1)^n b_{n-1}$ and $\phi : B[-1] \to \mathrm{cone}(v)$ defined as $\phi(b_{n-1}) = ((-1)^{n-1} f(b_{n-1}), (-1)^n b_{n-1}, 0)$. Then $\psi \circ \phi = 1$ and $\phi \circ \psi(c_{n-1}, b_{n-1}, c_n) = \phi((-1)^n b_{n-1}) = (-f(b_{n-1}), b_{n-1}, 0) = (d(c_n), 0, c_n) + (-d(c_n) + f(b_{n-1}) + c_{n-1}, 0, 0) = d(-c_n, 0, 0) + s(-d(c_{n-1}), -d(b_{n-1}), d(c_n) - f(b_{n-1}) - c_{n-1}) = (d \circ s + s \circ d)(c_{n-1}, b_{n-1}, c_n)$ with $s(c_{n-1}, b_{n-1}, c_n) = (-c_n, 0, 0)$. ■

Exercise 80. Let $f : B \to C$ be a morphism of chain complexes. Show that the natural maps $\mathrm{Ker}(f)[-1] \xrightarrow{\partial} \mathrm{cone}(f) \xrightarrow{\beta} \mathrm{coKer}(f)$ give rise to a long exact sequence:

$$\cdots \xrightarrow{\partial} H_{n-1}(\mathrm{Ker}(f)) \xrightarrow{\alpha} H_n(\mathrm{cone}(f)) \xrightarrow{\beta} H_n(\mathrm{coKer}(f)) \xrightarrow{\partial} H_{n-2}(\mathrm{Ker}(f)) \xrightarrow{\alpha} \cdots \tag{257}$$

Solution. ■

Exercise 81. Let C and C' be split complexes, with splitting maps s, s'. If $f : C \to C'$ is a morphism, show that $\sigma(c, c') = (-s(c), s'(c') - s' \circ f \circ s(c))$ defines a splitting of $\mathrm{cone}(f)$ if and only if the map $f_* : H_*(C) \to H_*(C')$ is zero.

Solution. ■

Lemma 82. *Let* $\mathsf{C} \subset \mathsf{A}$ *be a full subcategory of an abelian category* A*.*

1. C *is additive* $\Leftrightarrow 0 \in \mathsf{C}$ *and* C *is closed under* $\oplus$*.*
2. C *is abelian and* $\mathsf{C} \subset \mathsf{A}$ *is exact* $\Leftrightarrow \mathsf{C}$ *is additive, and* C *is closed under* Ker *and* coKer*.*

Proof. 1. One direction is obvious. Suppose that $0 \in \mathsf{C}$ and C is closed under $\oplus$. Since C is a full subcategory, the morphism set are same, thus we can give the exactly same abelian group structure to the morphism set. Thus C is an Ab-category, and so an additive category.

2. For one direction, since C is abelian, C is additive, ans since $\mathsf{C} \subset \mathsf{A}$ is additive, C is closed under Ker and coKer. For the other direction, since C is closed under Ker and coKer, an exact chain in C is still an exact chain in A, thus $\mathsf{C} \subset \mathsf{A}$ is exact and A is an abelian category.

□

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A functor $F : \mathsf{C} \to \mathsf{D}$ is **full** if for each objects $x, y \in \mathsf{C}$, the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is surjective; **faithful** if for each objects $x, y \in \mathsf{C}$ the map $\mathsf{C}(x, y) \to \mathsf{D}(F(x), F(y))$ is injective; **embedding** if it is faithful and the map $F : \mathsf{ob}(\mathsf{C}) \to \mathsf{ob}(\mathsf{D})$ is also injective; **fully embedding** of C into D if it is full and embedding. If $F : \mathsf{C} \to \mathsf{D}$ is a full embedding, we call C a **full subcategory** of D. A functor $F : \mathsf{D} \to \mathsf{E}$ from an

Let A be an abelian group category and C be a torsion-free abelian group category. Consider $f : \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$ in C. This map has a cokernel $\mathsf{Z} \to 0$, which is not a cokernel in A. Indeed, there is no map $0 \to \mathbb{Z}/n$ which makes $\mathbb{Z} \to 0 \to \mathbb{Z}/n = \mathbb{Z} \to \mathbb{Z}/n$.

Example 83. 1. Inside $R - \mathsf{mod}$ we have an additive subcategory consists of the finitely generated R-modules, which is abelian if and only if R is noetherian.

2. Inside Ab, the torsion-free groups form an additive category, the p-groups form an abelian category, finite p-groups form an abelian category, and $\mathbb{Z}/p - \mathsf{mod}$ of vector spaces over the field $\mathbb{Z}/p$ is a full subcategory of Ab.

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This is because R is noetherian if and only if all the ideals are finitely generated.

Definition 84. Let C be any category and A be an abelian category. The **functor category** A^{C} is the abelian category whose objects are functors $F : \mathsf{C} \to \mathsf{A}$, and the maps are natural transformations.

Example 85. 1. If C is the discrete category of integers, then Ab^{C} contains the abelian category of **graded abelian groups** as a full subcategory.

2. If C is the poset category of integers, then the abelian category $\mathsf{Ch}(\mathsf{A})$ of cochain complexes is a full subcategory of A^{C}.

3. If R is a ring considered as a one-object category, then $R-\mathsf{mod}$ is the full subcategory of all additive functors in Ab^R.

Definition 86. Let X be a topological space and U the poset of open subsets of X. A contravariant functor $F : \mathsf{U} \to \mathsf{A}$ such that $F(\emptyset) = \{0\}$ is called a **presheaf** on X with values in A, and the presheaves are the objects of the abelian category $\mathsf{A}^{\mathsf{U}^{\mathrm{op}}} = \text{Presheaves}(X)$.

Example 87. Consider $C^0(U) = \{\text{continuous functions } f : U \to \mathbb{R}\}$. If $U \subset V$, then the maps $C^0(V) \to C^0(U)$ are given by restricting the domain of a function from V to U. Thus a functor C^0 from U to the category with continuous functions $C^0(U)$ is a presheaf.

Definition 88. A **sheaf** on X with values in A is a presheaf F satisfying the **sheaf axiom**: Let $\{U_i\}$ be an open covering of an open subset U of X. If $\{f_i \in F(U_i)\}$ are such that each f_i and f_j agree in $F(U_i \cap U_j)$, then there is a unique $f \in F(U)$ that maps to every f_i under $F(U) \to F(U_i)$. That is, the following sequence is exact.

$$0 \to F(U) \to \prod F(U_i) \xrightarrow{\text{diff}} \prod_{i<j} F(U_i \cap U_j) \tag{258}$$

Exercise 89. Let M be a smooth manifold. For each open $U \subset M$, let $C^\infty(U)$ be the set of smooth functions from U to $\mathbb{R}$. Show that $C^\infty(U)$ is a sheaf in M.

Solution. For the collection of maps $\{f_i \in F(U_i)\}$ such that f_i, f_j agree in $F(U_i \cap U_j)$, define f on U as $f(x) = f_i(x)$ if $x \in U_i$. This definition is well defined, since if $x \in U_i \cap U_j$ then $f_i(x) = f_j(x)$. Now since continuity and differentiability on point is determined by its open neighborhood, f is smooth since f_i are smooth. Finally, if f, g are both such maps, then $f(x) - g(x) = 0$, thus $f = g$. ■

Exercise 90. Let A be any abelian group. For every open subset U of X, let $A(U)$ denote the set of continuous maps from U to the discrete topological space A. Show that A is a sheaf on X.

Solution. Define the map as above, and prove the continuity in same way. ■

Example 91. The category Sheaves(X) is an abelian category, but not an abelian subcategory of Presheaves(X). For any space X, let O be the sheaf such that $\mathsf{O}(U)$ is the group of continuous maps from U into $\mathbb{C}$. Define O^* with $\mathbb{C}^*$. Then there is a short exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathsf{O} \xrightarrow{\exp} \mathsf{O}^* \to 0 \tag{259}$$

But this map is not an exact sequence of presheaves if $X = \mathbb{C}^*$, because the map $\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*$ is not surjective; notice that there is no global log function, thus there is no preimage of $z : \mathbb{C}^* \to \mathbb{C}^*$. Indeed, $H^1(\mathbb{C}^*, \mathbb{Z}) \simeq \mathbb{Z}$, and the contour integral $\frac{1}{2\pi i} \oint f'(z)/f(z)dz$ gives the image of $f(z)$ in the cokernel.

Definition 92. Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories. F is called **left(right) exact** if for every short exact sequences $0 \to A \to B \to C \to 0$ in A, the sequence $0 \to F(A) \to F(B) \to F(C)(F(A) \to F(B) \to F(C) \to 0)$ is exact in B. F is called **exact** if it is both left and right exact. A covariant functor F is called (left, right) exact if the corresponding covariant functor $F' : \mathsf{A}^{\mathrm{op}} \to \mathsf{B}$ is (left, right) exact.

Example 93. The inclusion of Sheaves(X) into Presheaves(X) is a left exact functor. The **Sheafification** Presheaves$(X) \to$ Sheaves(X) is an exact functor. The proof will given later.

Exercise 94. Show that the above definitions are equivalent to the following, which are often given as the definitions. A (covariant) functor F is left(right) exact if exactness of the sequence $0 \to A \to B \to C(A \to B \to C \to 00)$ implies exactness of the sequence $0 \to F(A) \to F(B) \to F(C)(F(A) \to F(B) \to F(C) \to 0)$.

Solution. On direction is obvious. For the other direction, first consider a monic $i : A \to B$. Then $0 \to A \xrightarrow{i} B \to \mathrm{coKer}(i) \to 0$ is exact, thus $0 \to F(A) \xrightarrow{F(i)} F(B) \to F(\mathrm{coKer}(i))$ is exact. Therefore $F(i)$ is monic. Now consider $f : B \to C$ such that $0 \to A \xrightarrow{i} B \xrightarrow{f} C$ is exact. Then $0 \to A \xrightarrow{i} B \xrightarrow{f} \mathrm{Im}(f) \to 0$ is exact, therefore $0 \to F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(f)} F(\mathrm{Im}(f))$ is exact. Since $\mathrm{Im}(f) \to C$ is monic, $F(\mathrm{Im}(f)) \to F(C)$ is monic, thus $0 \to F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(f)} F(C)$ is also exact. ■

Proposition 95. *Let* A *be an abelian category. Then* $Hom_{\mathsf{A}}(M, -)$ *is a left exact functor from* A *to* Ab *for every* M *in* A*. Thus, given an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* A*, the following sequence of abelian groups is also exact:*

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \tag{260}$$

Proof. Suppose that $\alpha \in \mathrm{Hom}(M, A)$. First $g_* \circ f_*(\alpha) = g \circ f \circ \alpha = 0$, this is a chain. If $f_*(\alpha) = f \circ \alpha = 0$ then $\alpha = 0$ since f is monic, thus f_* is monic. Finally, suppose that $\beta \in \mathrm{Hom}(M, B)$ satisfies $g_*(\beta) = g \circ \beta = 0$. Then $\beta(M) \subset \mathrm{Im}(f)$ due to the exactness, thus considering $A \rightarrowtail \mathrm{Im}(f) \rightarrowtail B \to C$, we have a map $\alpha : M \to A$ satisfying $\beta = f \circ \alpha$. □

Corollary 96. $\mathrm{Hom}_{\mathsf{A}}(-, M)$ *is a left exact contravariant functor.*

Proof. $\mathrm{Hom}_{\mathsf{A}}(A, M) = \mathrm{Hom}_{\mathsf{A}^{\mathrm{op}}}(M, A)$. □

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Definition 97. A (co)homological δ-**functor** between A and B is a collection of of additive functors $T_n : \mathsf{A} \to \mathsf{B}$ ($T^n : \mathsf{A} \to \mathsf{B}$) for $n \geq 0$, together with morphisms $\delta_n : T_n(C) \to T_{n-1}(A)(\delta^n : T^n(C) \to T^{n+1}(A))$ defined for each short exact sequence $0 \to A \to B \to C \to 0$ in A, which satisfies:

1. For each short exact sequences $0 \to A \to B \to C \to 0$, there is a long exact sequence

$$\cdots \to T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} \cdots \tag{261}$$

$$(\cdots \to T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} \cdots) \tag{262}$$

2. For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the morphisms δ give a commutative diagram

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \end{array} \tag{263}$$

$$\left(\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \end{array} \right) \tag{264}$$

Example 98. Homology gives a homological δ-functor H_8 from $\mathsf{Ch}_{\geq 0}(\mathsf{A})$ to A; cohomology gives a cohomological δ-functor H^* from $\mathsf{Ch}^{\geq 0}(\mathsf{A})$ to A.

Exercise 99. Let S be a category of short exact sequences $0 \to A \to B \to C \to 0$ in A. Show that δ_i s a natural transformation from the functor sending the sequence to $T_i(C)$ to the functor sending the sequence to $T_{i-1}(A)$.

Solution. The final commutating square shows the desired property. ■

Example 100 (p-torsion)**.** If p is an integer, the functors $T_0(A) = A/pA$, $T_1(A) = {}_pA := \{a \in A : pa = 0\}$, and $T_n(A) = 0$ for all $n \geq 2$, fit together to form a homological δ functor; taking $T^0 = T_1$ and $T^1 = T_0$ gives a cohomological δ functor. To show this, apply the snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "p"] & B \arrow[r] \arrow[d, "p"] & C \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \tag{265}$$

then we get

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0 \tag{266}$$

The same proof shows that if $r \in R$ for a ring R, then $T_0(M) = M/rM$ and $T_1(M) = {}_rM := \{m \in M : rm = 0\}$ fit together to form a homological δ functor from $R - \mathsf{mod}$ to Ab; taking $T^0 = T_1$ and $T^1 = T_0$ gives a cohomological δ functor.

Definition 101. A **morphism** $S \to T$ of δ functors is a system of natural transformations $S_n \to T_n$ that commute with δ. A homological δ functor T is **universal** if, given another δ functor S and a natural transformation $S_0 \to T_0$, there exists a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors that extends f_0. The dual statement defines cohomological δ functor T.

Example 102. We will show later that homology $H_\bullet : \mathsf{Ch}_{\geq 0}(\mathsf{A}) \to \mathsf{A}$ and cohomology $H_\bullet : \mathsf{Ch}^{\geq 0}(\mathsf{A}) \to \mathsf{A}$ are universal δ functors.

Exercise 103. If $F : \mathsf{A} \to \mathsf{B}$ is an exact functor, show that $T_0 = F$ and $T_n = 0$ for $n \neq 0$ defines both homological and cohomological universal δ functor.

Solution. Since F is exact functor, T is a δ functor with $\delta = 0$. Thus defining $\alpha_n : S_n \to T_n = 0$ as a zero map makes the diagram commutes; defining $\beta^n : T^n = 0 \to S^n$ as a zero map makes the diagram commutes. ∎

Definition 104. If $F : \mathsf{A} \to \mathsf{B}$ is an additive functor, we call the functors T_n of (co)homological δ functor T as the **left(right) satellite functors** of F if $T_0 = F(T^0 = F)$.

Definition 105. Let A be an abelian category. An object $P \in \mathsf{A}$ is **projective** if it satisfies the following **universal lifting property**: given an epimorphism $g : B \to C$ and a morphism $\gamma : P \to C$, there is a morphism $\beta : P \to B$ such that $\gamma = g \circ \beta$.

$$\begin{tikzcd} & P \arrow[dl, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd} \tag{267}$$

Proposition 106. *An R-module is projective if and only if it is a direct summand of a free R-module.*

Proof. Let $F(A)$ be a free module based on the module A. Then we have a natural surjection $\pi : F(A) \to A$, thus the sequence

$$0 \to \mathrm{Ker}(\pi) \to F(A) \xrightarrow{\pi} A \to 0 \tag{268}$$

is exact. Now if A is projecttive, then there is a following lifting.

$$\begin{tikzcd} & A \arrow[dl, "\exists i"'] \arrow[d, "1_A"] & \\ F(A) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \tag{269}$$

This makes the short exact sequence splits, thus $F(A) = A \oplus \mathrm{Ker}(\pi)$. Conversely, if A is a direct summand of a free R-module, then by lifting the image of basis using the surjectivity of $g : B \to C$, we can show that P is projective. ☐

Example 107. 1. Consider $R = R_1 \times R_2 = R_1 \times 0 \oplus 0 \times R_2$. Then $P = R_1 \times 0$ is projective, but not free.

2. Considering $R = M_n(F)$ and $V = F^n$, V can be considered as a left R-module, and $R = \underbrace{V \oplus \cdots \oplus V}_{n}$. But for every free modules of R their dimension on F must be dn^2 for some cardinal d, V is not free over R.

3. The finite abelian group category A is an abelian category without projective objects, since there is no nontrivial free object due to the finiteness.

Definition 108. For an abelian category A, we say that A has **enough projectives** if for every object A there is an epimorphism $P \to A$ with projective P.

Lemma 109. *Let P be an object of abelian category A. P is projective if and only if $\mathrm{Hom}_{\mathsf{A}}(P, -)$ is an exact functor.*

Proof. Suppose that $\mathrm{Hom}(P, -)$ is exact. Choose epic $g : B \to C$ and $\gamma \in \mathrm{Hom}(P, C)$. Since g_* is epic, we can find $\beta \in \mathrm{Hom}(P, B)$ satisfying $g_*(\beta) = g \circ \beta = \gamma$. Conversely, let P be projective. Since $\mathrm{Hom}(P, -)$ is left exact in general, what we need to show is $g_* : \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C)$ is epic. But for $\gamma \in \mathrm{Hom}(P, C)$, the universal lifting property shows we have $\beta \in \mathrm{Hom}(P, B)$ such that $\gamma = g_*(\beta)$. Thus $\mathrm{Hom}(P, -)$ is exact. ☐

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Definition 110. A chain complex P in abelian category is called a **chain complex of projectives** if all P_n are projective.

Exercise 111. Show that a chain complex P is a projective object in Ch if and only if it is a split exact complex of projectives. Their brutal truncations $\sigma_{\geq 0}$ form the projective objects in $\mathsf{Ch}_{\geq 0}$.

Solution. Notice that $\mathrm{cone}(1_P)$ is split exact, and the following sequence is short exact.

$$0 \to P \to \mathrm{cone}(1_P) \to P[-1] \to 0 \tag{270}$$

Since $\mathrm{cone}(1_P)$ is exact, P is exact. ∎

Exercise 112. Show that if an abelian category A has enough projectives, then so does the category $\mathsf{Ch}(\mathsf{A})$ of chain complexes over A.

Solution. ∎

Definition 113. Let M be an object of abelian category A. A **left resolution** of M is a complex $P_\bullet$ with $P_i = 0$ for $i < 0$ with a map $\epsilon : P_0 \to M$ so that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0 \tag{271}$$

is exact. If each P_i is projective, we call it a **projective resolution**.

Lemma 114. *Every R-module M has a projective resolution. Generally, if an abelian category A has enough projectives, then every object M in A has a projective resolution.*

Since we may find the surjective map $F(A) \to A$ and $F(A)$ is free object, which is itself the summand of a free object, $F(A)$ is projective, thus $R-\mathsf{mod}$ has enough projectives.

Proof. Consider a surjection $\epsilon_0 : P_0 \to M$ with projective P_0. Define $M_0 = \mathrm{Ker}\,\epsilon_0$. Now inductively, for a module M_{n-1}, choose $\epsilon_n : P_n \to M_{n-1}$ and take $M_n = \mathrm{Ker}\,\epsilon_n$.

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & & & 0 \\
& M_1 \arrow[ur] \arrow[dr] & & & & M \arrow[ur] & \\
P_2 \arrow[ur] \arrow[rr, "d_2"] & & P_1 \arrow[rr, "d_1"] \arrow[dr] & & P_0 \arrow[ur, "\epsilon_0"] & & \\
& & & M_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{272}$$

Define d_n as the composition of maps $P_n \to M_n \to P_{n-1}$. Now since $d_n(P_n) \simeq M_{n-1}$ and $\mathrm{Ker}(d_{n-1}) = \mathrm{Ker}(P_{n-1} \to M_0) = \mathrm{Im}(M_{n-1} \to$

$P_1) \simeq M_{n-1}$, we get the sequence is exact, hence the sequence is a projective resolution of M. □

Exercise 115. Show that if $P_\bullet$ is a complex of projectives with $P_i = 0$ for $i < 0$, then a map $\epsilon : P_0 \to M$ giving a resolution for M is the same thing as a quasi-isomorphism $\epsilon : P_\bullet \to M$, where M is considered as a complex concentrated in degree zero.

Solution. Since M has zero homology groups except $H_0(M) = M$, P_i is exact for $n > 0$ and $P_0/\operatorname{Im}(d_1) \simeq M$. Now since ϵ must induce the isomorphism to M, ϵ must be surjective, and considering $0 \to \operatorname{Ker}(\epsilon) \to P_0 \to M \to 0$ and $0 \to \operatorname{Im}(d_1) \to P_0 \to M \to 0$, which are both exact and there is a chain map between them, by 5-lemma $\operatorname{Ker}(\epsilon) \simeq \operatorname{Im}(d_1)$. ■

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Theorem 116 (Comparison Theorem)**.** *Let $P_\bullet \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' : M \to N$ in abelian category A. Then for every resolution $Q_\bullet \to \eta N$ of N, there is a chain map $f : P_\bullet \to Q_\bullet$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. The chain map is unique up to chain homotopy equialence.*

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\exists"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "\exists"] & M \arrow[r] \arrow[d, "f'"] & 0 \\ \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0 \end{tikzcd} \tag{273}$$

Proof. Denoting $f' = f_{-1}$, we will use induction. We can cut the chain into two chains as following.

$$\begin{tikzcd} \cdots \arrow[r, "d"] & P_{n+1} \arrow[r, "d"] \arrow[d, "\exists f_{n+1}"] & Z_n(P) \arrow[r] \arrow[d, "f_n'"] & 0 & 0 \arrow[r] & Z_n(P) \arrow[r, "d"] \arrow[d, "f_n'"] & P_n \arrow[r, "d"] \arrow[d, "f_n"] & \cdots \\ \cdots \arrow[r, "d"] & Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0 & 0 \arrow[r] & Z_n(Q) \arrow[r, "d"] & Q_n \arrow[r, "d"] & \cdots \end{tikzcd} \tag{274}$$

Here all the rows are exact. Since P_{n+1} is projective, we can lift $f_n' \circ d = f_n \circ d$ to f_{n+1} satisfying $f_n \circ d = d \circ f_{n+1}$. Now consider another lift $g : P_\bullet \to Q_\bullet$ and let $h = f - g$. To construct $s_n : P_n \to Q_{n+1}$, for $n < 0$ define $s_n = 0$; for $n = 0$, since $\eta \circ h_0 = \epsilon(f' - f') = 0$, h_0 sends P_0 to $Z_0(Q) = d(Q_1)$, thus we may lift h_0 to $s_0 : P_0 \to Q_1$ such that $h_0 = d \circ s_0 = d \circ s_0 + s_{-1} \circ d$. Now for given h_{n-1} satisfying $h_{n-1} = d \circ s_{n-1} + s_{n-2} \circ d$, then $d(h_n - s_{n-1} \circ d) = d \circ h - h \circ d + s \circ d \circ d = 0$. Thus $h_n - s_{n-1} \circ d$ takes P_n to $Z_n(Q)$, so we can lift it to $s_n : P_n \to Q_{n+1}$ such that $d \circ s_n = h_n - s_{n-1} \circ d$. □

Lemma 117 (Horseshoe Lemma). *Suppose we have a diagram*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} \tag{275}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the P_n assemble to form a projective resolution P of A, and the column lifts to an exact sequence of complexes $0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$, where $i_n : P_n' \to P_n$ and $\pi_n : P_n \to P_n''$ are the natural inclusion and projection, respectively.

Proof. Lift ϵ'' to a map $P_0'' \to A$, then the direct sum of the lifted map and $i_A \circ \epsilon : P_0' \to A$ gives a map $\epsilon : P_0 \to A$. Now we have the following commuting diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker}(\epsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\epsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\epsilon"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\epsilon'') \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{276}$$

Since the right two columns are short exact sequences, the snake lemma shows that the left column is exact and $\mathrm{coKer}(\epsilon) = 0$. Thus P_0 maps onto A. Now this process finishes the initial step and gives the following diagram, which can be filled up by induction.

$$\begin{tikzcd} & & 0 \arrow[d] & \\ \cdots \arrow[r] & P_1' \arrow[r] & \mathrm{Ker}(\epsilon') \arrow[r] \arrow[d] & 0 \\ & & \mathrm{Ker}(\epsilon) \arrow[d] & \\ \cdots \arrow[r] & P_1'' \arrow[r] & \mathrm{Ker}(\epsilon'') \arrow[r] \arrow[d] & 0 \\ & & 0 & \end{tikzcd} \tag{277}$$

□

Exercise 118. Show that there are maps $\lambda_n : P_n'' \to P_{n-1}'$ so that $d(p', p'') = (d'(p') + \lambda(p''), d''(p''))$.

Solution. Suppose that $d(p', p'') = (f_1(p') + f_2(p''), g_1(p') + g_2(p''))$. To commute with d', we have $(d'(p'), 0) = d'(p', 0) = (f_1(p'), g_1(p')$, thus $g_1 = 0$ and $f_1 = d$. To commute with d'', we have $g_1(p') + g_2(p'') = g_2(p'') = d''(p'')$, thus $g_2 = 0$. Therefore we get $d(p', p'') = (d(p') + f_2(p''), d(p''))$. ■

Definition 119. Let A be an abelian category. An object $I \in A$ is **injective** if it satisfies the following **universal lifting property**: given a monomorphism $f : A \to B$ and a map $\alpha : A \to I$, there is a morphism $\beta : B \to I$ such that $\alpha = \beta \circ f$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, "\exists\beta"] \\ & I & \end{tikzcd} \tag{278}$$

We say that A has **enough injectives** if for every object $A \in$ A there is a monomorphism $A \to I$ with injective I.

Theorem 120 (Baer's Criterion)**.** *A right R-module E is injective if and only if for every right ideal J of R, every map $J \to E$ can be extended to a map $R \to E$.*

Proof. One direction is obvious. Consider an R-module B, its submodule A, and a map $\alpha : A \to E$. Let $\mathcal{E}$ be the poset of all extensions $\alpha' : A' \to E$ of α to an intermediate submodule $A \subset A' \subset B$. We give the partial order $\alpha' \leq \alpha''$ if α'' extends α'. Then the Zorn's lemma shows that there is a maximal extension $\alpha' : A' \to E$ in $\mathcal{E}$. Now suppose that $b \in B - A'$. The set $J = \{r \in R : br \in A'\}$ is a right ideal of R, and by assumption the map $J \xrightarrow{b} A' \xrightarrow{\alpha'} E$ extends to a map $f : R \to E$. Let A'' be the submodule $A' + bR$ of B, and define $\alpha'' : A'' \to E$ by $\alpha(a + br) = \alpha'(a) + f(r)$ for all $a \in A', r \in R$. Since $\alpha'(br) = f(r)$ for $br \in A' \cap bR$, and since α'' extends α', this contradicts the maximality of α'. Thus there is no such b, and so $A' = B$. □

Exercise 121. Let $R = \mathbb{Z}/m$. Use Baer's criterion to show that R is an injective R-module. Then show that $\mathbb{Z}/d$ is not an injective R-module when $d \mid m$ and some prime p divides both d and m/d. (The hypothesis ensures that $\mathbb{Z}/m \neq \mathbb{Z}/d \oplus \mathbb{Z}/e$.)

Solution. The solution can be given by the following corollary, since $R = \mathbb{Z}/m$ is a principal ideal domain and $\mathbb{Z}/d$ is not divisible: if so, then for all $r \in \mathbb{Z}/m$ and $a \in \mathbb{Z}/d$, there is $b \in \mathbb{Z}/d$ such that $a = br \mod d$, that is, $a = br + dn$. This implies $\gcd(r, d)$ divides a. Take $a = 1$ and $r = m/d$ gives p divides 1, which is contradiction. ■

Corollary 122. *Suppose that R is a principal ideal domain. An R-module is injective if and only if it is divisible, that is, for every $r \neq 0 \in R$ and every $a \in A$, $a = br$ for some $b \in A$.*

Proof. By Baer's criterion, A is injective if and only if for every right ideal J of R, every map $J \to A$ can be extended to a map $R \to A$. Since R is a principal ideal domain, all J can be represented by (r). Each maps can be uniquely determined by the pair (r, a), where $r \in R$ and $a \in A$. Thus, if A is divisible, then there is $b \in A$ such that $a = br$, thus we can define $R \to A$ as $r \mapsto br$. Conversely, if A is injective, then there is an extension of the map f determined by (r, a), and taking $f(1) = b$ gives $a = br$. □

Example 123. The divisible abelian groups $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty} = \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$ are injective. Indeed, every injective abelian group is a direct sum of these. In particular, the injective abelian group Q/Z is isomorphic to $\oplus_p \mathbb{Z}_{p^\infty}$.

$\mathbb{Z}\left[\frac{1}{p}\right]$ is the group of rational numbers of the form $a/p^n, n \in \mathbb{N}$.

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Exercise 124. For an abelian group A, denote $I(A)$ as the product of copies of the injective group $\mathbb{Q}/\mathbb{Z}$ indexed by the set $\mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) - 0$, where 0 is the zero map. Then $I(A)$ is injective since it is a product of injectives, and there is a canonical map $e_A : A \to I(A)$. Show that e_A is an injection, and thus, show that Ab has enough injectives.

Solution. Take $a \in A$. Notice that $f \in \mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}$-th component of $e_A(a)$ is $f(a)$. Define $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a)$ as some nonzero value in $\mathbb{Q}/\mathbb{Z}$, for example, $\left[\frac{1}{2}\right]$. Now since we have a map $a\mathbb{Z} \to A$ and $\mathbb{Q}/\mathbb{Z}$ is injective, we can extend f to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$, which has $f'(a) \neq 0$. Therefore e_A is injective. ■

Exercise 125. Show that an abelian group A is zero if and only if $\mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) = 0$.

Solution. If A is zero then there is one trivial homomorphism. Suppose that $\mathrm{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) = 0$. We have defined an injection $e_A : A \to I(A)$ from previous exercise, but in this case $I(A) = 0$, thus $A = 0$. ■

Lemma 126. *The following are equivalent for an object I in an abelian category A:*

1. *I is injective in A;*
2. *I is projective in A;*
3. *The contravariant functor $\mathrm{Hom}_{\mathsf{A}}(-, I)$ is exact.*

Proof. Since the opposite category of abelian category is abelian, and the dual of injective object is project object, the lemma is shown by duality. □

Definition 127. Let M be an object of A. A **right resolution** of M is a cochain complex $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} \cdots \tag{279}$$

is exact, which is same as a cochain map $M \to I^\bullet$, where M is considered as a complex concentrated in degree 0. If each I^i is injective, the right resolution is called an **injective resolution**.

Lemma 128. *If the abelian category A has enough injectives, then every object in A has an injective resolution.*

Proof. This lemma is the dual version of projective resolution. □

Theorem 129 (Comparison Theorem.)**.** *Let $N \to I^\bullet$ be an injective resolution of N and $f' : M \to N$ be a map in A. Then for every resolution $M \to E^\bullet$, there is a cochain map $f : E^\bullet \to I^\bullet$ lifting f'. The map f is unique up to cochain homotopy equivalence.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f'"] & E^0 \arrow[r] \arrow[d, "\exists"] & E^1 \arrow[r] \arrow[d, "\exists"] & \cdots \\
0 \arrow[r] & N \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd} \tag{280}$$

Proof. This theorem is the dual version of projective comparison theorem. □

Exercise 130. Show that I is an injective object in the category of chain complexes if and only if I is a split exact complex of injectives. Then show that if A has enough injectives, so does the category $\mathsf{Ch}(\mathsf{A})$.

Proof. Taking the dual of projective chain complexes, we get the first statement. Now since $\mathsf{Ch}(\mathsf{A})^{\mathrm{op}} \simeq \mathsf{Ch}(\mathsf{A}^{\mathrm{op}})$, we get the second statement. □

Definition 131. A pair of functors $L : \mathsf{A} \to \mathsf{B}$ and $R : \mathsf{B} \to \mathsf{A}$ are **adjoint functors** if for all $A \in \mathsf{A}$ and $B \in \mathsf{B}$, there is a natural bijection

$$\tau = \tau_{AB} : \mathrm{Hom}_{\mathsf{B}}(L(A), B) \to \mathrm{Hom}_{\mathsf{A}}(A, R(B)) \tag{281}$$

that is, for all $f : A \to A' \in \mathsf{A}$ and $g : B \to B' \in \mathsf{B}$, the following diagram commutes.

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathsf{B}}(L(A'), B) & \xrightarrow{L(f)^*} & \mathrm{Hom}_{\mathsf{B}}(L(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathsf{B}}(L(A), B') \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \mathrm{Hom}_{\mathsf{A}}(A', R(B)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathsf{A}}(A, R(B)) & \xrightarrow{R(g)_*} & \mathrm{Hom}_{\mathsf{A}}(A, R(B')) \end{array} \tag{282}$$

We call L the **left adjoint functor** and R the **right adjoint functor**.

Lemma 132. *For every right R-module M, the natural map*

$$\tau : \mathrm{Hom}_{\mathsf{Ab}}(M, A) \to \mathrm{Hom}_{\mathsf{mod}-R}(M, \mathrm{Hom}_{\mathsf{Ab}}(R, A)) \tag{283}$$

is an isomorphism, where $\tau(f)(m)$ is the map $r \mapsto f(mr)$. Thus, forgetful functor and $\mathrm{Hom}_{\mathsf{Ab}}(R, -)$ are left and right adjoint functor pair.

Proof. Take $g : M \to \mathrm{Hom}_{\mathsf{Ab}}(R, A)$. Define $\mu : \mathrm{Hom}_{\mathsf{mod}-R}(M, \mathrm{Hom}_{\mathsf{Ab}}(R, A)) \to \mathrm{Hom}_{\mathsf{Ab}}(M, A)$ as $\mu(g)(m) = g(m)(1)$. Now $(\tau \circ \mu(g))(m)(r) = \mu(g)(mr) = g(mr)(1) = g(m)(1)r = g(m)(r)$ and $(\mu \circ \tau(f))(m) = \tau(f)(m)(1) = f(m)$, thus τ is an isomorphism. □

Proposition 133. *If an additive functor $R : \mathsf{B} \to \mathsf{A}$ is right adjoint to an exact functor $L : \mathsf{A} \to \mathsf{B}$ and I is an injective object of B, then $R(I)$ is an injective object of A.*

Dually, if an additive functor $L : \mathsf{A} \to \mathsf{B}$ is left adjoint to an exact functor $R : \mathsf{B} \to \mathsf{A}$ and P is a projective object of A, then $L(P)$ is a projective object of B.

In other words, right adjoint functor of exact functor preserves injectives.

In other words, left adjoint functor of exact functor preserves projectives.

Proof. For an injection $f : A \to A'$ in A, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{B}}(L(A'), I) & \xrightarrow{L(f)^*} & \mathrm{Hom}_{\mathsf{B}}(L(A), I) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_{\mathsf{A}}(A', R(I)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathsf{A}}(A, R(I)) \end{array} \tag{284}$$

Since L is exact and I is injective, $L(f^*)$ is onto. Hence f^* is onto, and so $\mathrm{Hom}_{\mathsf{A}}(-, R(I))$ is exact. Therefore $R(I)$ is injective. □

Corollary 134. *If I is an injective abelian group, then* $\mathrm{Hom}_{\mathsf{Ab}}(R, I)$ *is an injective R-module.*

Proof. This can be proven directly by previous lemma and proposition. □

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Exercise 135. Let M be an R-module and $Z(M)$ be a product of copies $I_0 = \mathrm{Hom}_{\mathsf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$, indexed by the set $\mathrm{Hom}_R(M, I_0) - 0$. Then $Z(M)$ is injective since it is a product of injectives, and there is a canonical map $e_M : M \to Z(M)$. Show that e_M is an injection, and thus, show that $R - \mathsf{mod}$ has enough injectives.

Solution. Take $r \in R$. Notice that $f \in \mathrm{Hom}_{\mathsf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$-th component of $e_M(r)$ is $f(r)$. Define $f : r\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(r)$ as some nonzero value in $\mathbb{Q}/\mathbb{Z}$, for example, $\left[\frac{1}{2}\right]$. Now since we have a map $r\mathbb{Z} \to R$ and $\mathbb{Q}/\mathbb{Z}$ is injective, we can extend f to a map $f' : R \to \mathbb{Q}/\mathbb{Z}$, which has $f'(r) \neq 0$. Therefore e_M is injective. ■

Definition 136. A set I is a **directed set** if there is a relation $\leq$ such that:

1. $i \leq i$ for all $i \in I$;
2. $i \leq j, j \leq k$ then $i \leq k$ for all $i, j, k \in I$;
3. For any $i, j \in I$, there is $k \in I$ such that $i \leq k$ and $j \leq k$.

For a directed set I, let $\{A_i : i \in I\}$ be a family of objects of category A indexed by I. Let $f_{ij} : A_i \to A_j$ be a morphism for all $i \leq j$ with:

1. f_{ii} is the identity of A_i;
2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

Then the pair $\langle A_i, f_{ij} \rangle$ is called a **direct system over** I.

Let $\langle A_i, f_{ij} \rangle$ be a direct system in A. A **target** of $\langle A_i, f_{ij} \rangle$ is a pair $\langle A, \phi_i \rangle$, where A is an object in A and $\phi_i : A_i \to A$ are morphisms satisfying $\phi_i = \phi_j \circ f_{ij}$ for all $i \leq j$. A **direct limit** of $\langle A_i, f_{ij} \rangle$ is a target $\langle A, \phi_i \rangle$ such that for all target $\langle B, \psi_i \rangle$, there is a unique morphism $u : X \to Y$ such that $u \circ \phi_i = \psi_i$ for all i.

$$\begin{tikzcd}
A_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & A_j \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& A \arrow[d, dotted, "u"] & \\
& B &
\end{tikzcd} \tag{285}$$

We write $A = \lim_{\to} A_i$.

Definition 137. For a topological space X and a sheaf F, a **stalk** of a sheaf F at a point $x \in X$ is the abelian group $F_x : \lim_{\to}\{F(U) : x \in U\}$.

Definition 138. For an abelian group A and a topological space X, the **skyscraper sheaf** $x_* A$ at the point $x \in X$ is the presheaf

$$(x_* A)(U) = \begin{cases} A & x \in U \\ 0 & x \notin U \end{cases} \tag{286}$$

Exercise 139. Show that $x_* A$ is a sheaf and that

$$\mathrm{Hom}_{\mathsf{Ab}}(F_x, A) \simeq \mathrm{Hom}_{\mathsf{Sheaves(X)}}(F, x_* A) \tag{287}$$

for every sheaf F. Thus if A_x is an injective abelian group, then $x_* A_x$ is an injective object in Sheaves(X) for each x, and that $\prod_{x \in X} x_* A_x$ is also injective.

Solution. Restricting $x_* A(U)$ to U_i gives 0 if $x \notin U_i$ and A if $x \in U_i$, which shows that $x_* A$ is a sheaf. To show the isomorphism, take $\tau : \mathrm{Hom}_{\mathsf{Ab}}(F_x, A) \to \mathrm{Hom}_{\mathsf{Sheaves(X)}}(F, x_* A)$ as $\tau(f)(U)$ is a zero map if $x \notin U$ and $\tau(f)(U)$ is a composition map $F(U) \to F_x \xrightarrow{f} A$. Now take $\mu : \mathrm{Hom}_{\mathsf{Sheaves(X)}}(F, x_* A) \to \mathrm{Hom}_{\mathsf{Ab}}(F_x, A)$ as the map which is uniquely generated by the direct limit. Since each construction gives the same commuting diagram, $\tau \circ \mu$ and $\mu \circ \tau$ are identity maps. Thus the stalk functor and skyscraper sheaf functor are left and right adjoints respectively. Since the stalk functor is exact, the previous proposition shows the last statement. ■

Stalk functor is exact since the direct limit functor is exact.

Example 140. Sheaves(X) has enough injectives. Indeed, given a fixed sheaf F, choose an injection $F_x \to I_x$ with I_x injective in Ab for each $x \in X$. Combining the natural maps $F \to x_* F_x$ with $x_* F_x \to x_* I_x$ gives a map from F to the injective sheaf $I = \prod_{x \in X} x_* I_x$. The map $F \to I$ can be shown that it is an injection.

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Example 141. Let I be a small category and A be an abelian category. If the product of any set of objects exists in A, which is, if A is complete, and A has enough injectives, then the functor category A^I has enough injectives. Indeed, for each $k \in I$, the coordinate functor $\mathsf{A}^I \to \mathsf{A}$ mapping $A \mapsto A(k)$ is an exact functor. Now for an object $A \in \mathsf{A}$, define the functor $k_*A : I \to \mathsf{A}$ as $k_*A(i) = \prod_{\mathrm{Hom}_I(i,k)} A$. Now if $\eta : i \to j$ is a map in I, then the map $k_*A(i) \to k_*A(j)$ is determined by the index map $\eta^* : \mathrm{Hom}(j,k) \to \mathrm{Hom}(i,k)$, that is, $\phi \in \mathrm{Hom}(i,k)$-th component becomes $\eta^*(\phi) = \phi \circ \eta$-th component. Now for a morphism $f : A \to B$, there is a corresponding morphism $k_*f : k_*A \to k_*B$ which is defined slotwise. This shows that $k_* : \mathsf{A} \to \mathsf{A}^I$ is an additive functor. The following exercise then shows that A^I has enough injectives.

Exercise 142. From the previous example, show that k_* is right adjoint to the k-th coordinate functor, so that k_* preserves injectives. Now for each $F \in \mathsf{A}^I$, embed $F(k)$ in an injective object $A_k \in \mathsf{A}$, and so let $F \to k_*A_k$ be the corresponding adjoint map. Show that $E = \prod_{k\in I} k_*A_k$ exists in A^I, that E is an injective object, and that $F \to E$ is an injection.

Solution. Choose $A \in \mathsf{A}$ and $F \in \mathsf{A}^I$. Then we have to show the isomorphism between $\mathrm{Hom}(F, k_*(A))$ and $\mathrm{Hom}(F(k), A)$. Let $f \in \mathrm{Hom}(F, k_*(A))$, then [LATER] ■

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Definition 143. Let $F : \mathsf{A} \to \mathsf{B}$ be a right exact functor between two abelian categories. If A has enough projectives, then we define the **left derived functors** L_iF for $i \geq 0$ of F as, for an object $A \in \mathsf{A}$, choose a projective resolution $P \to A$ and define

Since F is right exact, $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, thus $L_0F(A) \simeq F(A)$.

$$L_iF(A) = H_i(F(P)) \tag{288}$$

Lemma 144. *The objects $L_iF(A)$ of B are well defined up to natural isomorphisms. That is, if $Q \to A$ is another projective resolution, then there is a canonical isomorphism*

$$L_iF(A) = H_i(F(P)) \simeq H_i(F(Q)) \tag{289}$$

In particular, a different choice of the projective resolutions would yield new functors $\hat{L}_iF$, which are naturally isomorphic to the functors L_iF.

Proof. Due to the comparison theorem, there is a chain map $f : P \to Q$ lifting the identity map 1_A, which gives a map $f_* : H_iF(P) \to$

$H_iF(Q)$. Due to the uniqueness of the map f up to chain homotopy equivalence, f_* is canonical. Samely, there is a chain map $g : Q \to P$ lifting 1_A, and corresponding map $g_* : H_iF(Q) \to H_iF(P)$. Since $g \circ f$ and 1_P are both chain maps $P \to P$ lifting 1_A, and due to the uniqueness, we have $g_* \circ f_* = 1_{H_iF(P)}$. Samely, $f_* \circ g_* = 1_{H_iF(Q)}$. Thus f_*, g_* are isomorphisms. □

Corollary 145. *If A is projective, the $L_iF(A) = 0$ for $i \neq 0$.*

Proof. Consider the projective resolution $\cdots \xrightarrow{1_A} A \xrightarrow{0} A \xrightarrow{1_A} A \to 0$. This gives $L_iF(A) = 0$ for $i \neq 0$. □

Definition 146. Let $F : \mathsf{A} \to \mathsf{B}$ be a right exact functor between abelian categories. Then an object $Q \in \mathsf{A}$ is F-**acyclic** if $L_iF(Q) = 0$ for all $i \neq 0$. For an object $A \in \mathsf{A}$, a left resolution $Q \to A$ for which each Q_i is F-acyclic is an F-**acyclic resolution**.

Lemma 147. *If $f : A' \to A$ is any map in A, then there is a natural map $L_iF(f) : L_iF(A') \to L_iF(A)$ for each i.*

Proof. Let $P' \to A'$ and $P \to A$ be the projective resolutions. Then the comparison theorem gives a lift of f to a chain map $\tilde{f} : P' \to P$, hence a map $L_iF(f) := \tilde{f}_* : H_iF(P') \to H_iF(P)$, which is independent of the choice of $\tilde{f}$. □

Exercise 148. Show that $L_0F(f) = F(f)$ under the identification $L_0F(A) \simeq F(A)$.

Proof. $L_0F(f) : L_0F(A') \to L_0F(A) \simeq F(A') \to F(A)$, since the lifting of f should satisfy the commutativity of the diagram, which takes kernel to kernel. □

Theorem 149. *Each $L_iF : \mathsf{A} \to \mathsf{B}$ is an additive functor.*

Proof. First the identity map on P lifts the identity on A, thus $L_iF(1_A)$ is the identity map. Now consider the maps $A' \xrightarrow{f} A \xrightarrow{g} A''$ and chain maps $\tilde{f}, \tilde{g}$ lifting f, g respectively. Then $\tilde{g} \circ \tilde{f}$ lifts $g \circ f$, thus $g_* \circ f_* = (g \circ f)_*$, and so L_iF is a functor. Finally, if $f_1, f_2 : A' \to A$ are two maps with lifts $\tilde{f}_1, \tilde{f}_2$ respectively, then the sum $\tilde{f}_1 + \tilde{f}_2$ lifts $f_1 + f_2$, thus $f_{1*} + f_{2*} = (f_1 + f_2)_*$, thus L_iF is additive. □

Exercise 150. If $U : \mathsf{B} \to \mathsf{C}$ is an exact functor, then show that $U(L_iF) \simeq L_i(U \circ F)$.

Solution. For an object A, choose a projective resolution P, then

$$L_i(U \circ F)(A) = H_i(U \circ F(P)), \quad U(L_iF)(A) = U(H_i(F(P))) \tag{290}$$

Thus what we need to show is that if there is a chain complex B, then there is an isomorphism between $H_i(U(B))$ and $U(H_i(B))$. Now, consider the exact sequence $0 \to B_i(B) \xrightarrow{r} Z_i(B) \xrightarrow{q} H_i(B) \to 0$. Taking the exact functor U gives an exact sequence $0 \to U(B_i(B)) \xrightarrow{U(r)} U(Z_i(B)) \xrightarrow{U(q)} U(H_i(B)) \to 0$, thus $U(H_i(B)) \simeq U(Z_i(B))/U(B_i(B))$ with $B_i(U(B)) \simeq U(B_i(B))$ and $Z_i(U(B)) \simeq U(Z_i(B))$. This shows the desired result. ■

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Theorem 151. *Let $F : \mathsf{A} \to \mathsf{B}$ be a right exact functor between two abelian categories. The derived functors L_*F form a homological δ-functor.*

Proof. For a short exact sequence $0 \to A' \to A \to A'' \to 0$, choose projective resolutions $P' \to A'$ and $P'' \to A''$. Then by the horseshoe lemma, there is a projective resolution $P \to A$ which fits into a short exact sequence $0 \to P' \to P \to P'' \to 0$. Since P''_n are projective, each $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. Now since F is additive, it preserves the addition and zero map, thus $0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$ is split exact. This shows that $0 \to F(P') \to F(P) \to F(P'') \to 0$ is a short exact sequence of chain complexes. From this, we can take the corresponding long exact homology sequence, which gives

$$\cdots \xrightarrow{\partial} L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial} L_{i-1}F(A') \to \cdots \tag{291}$$

To show the naturality of ∂, consider a commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "i_B"] & B \arrow[r, "\pi_B"] & B'' \arrow[r] & 0 \end{tikzcd} \tag{292}$$

Take projective resolutions of the corners as $\epsilon' : P' \to A', \epsilon'' : P'' \to A'', \eta' : Q' \to B', \eta'' : Q'' \to B''$. By the horseshoe lemma, we get the corresponding projective resolutions $\epsilon : P \to A, \eta : Q \to B$. Also by the comparison theorem, we have chain maps $F' : P' \to Q'$ and $F'' : P'' \to Q''$ lifting the maps f', f'' respectively. Our aim is to show that there is a chain map $F : P \to Q$ lifting f, and giving a following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"] & 0 \\ 0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0 \end{tikzcd} \tag{293}$$

Since H_* is a homological δ-functor, this gives the naturality of ∂. If we define the maps $\gamma_n : P''_n \to Q'_n$ such that F_n are defined as

$$F_n(p', p'') = (F'(p') + \gamma(p''), F''(p'')) \tag{294}$$

then, if F is a chain map over f, this gives the commutative diagram. Now to make F a lifting of f, the map $\eta \circ F_0 - f \circ \epsilon : P_0 = P'_0 \oplus P''_0 \to B$ must be vanish. This implies

$$i_B \circ \eta' \circ \gamma_0 = f \circ \lambda_P - \lambda_Q F''_0 : P''_0 \to B \tag{295}$$

where λ_P, λ_Q are the restrictions of ϵ and η to P''_0 and Q''_0, respectively. Now since

$$\pi_B(f \circ \lambda - \lambda \circ F''_0) = f'' \circ \pi_A \circ \lambda - \pi_B \circ \lambda \circ F''_0 = f'' \circ \epsilon'' - \eta'' \circ F''_0 = 0 \tag{296}$$

thus there is $\beta : P''_0 \to B'$ so that $i_B \circ \beta = f \circ \lambda - \lambda \circ F''_0$. Now using projectivity, define γ_0 to be any lift of β to Q'_0, satisfying $\beta = \eta' \circ \gamma_0$. To make F a chain map, we have

$$(d \circ F - F \circ d)(p', p'') = ((d' \circ F' - F' \circ d')(p') + (d' \circ \gamma - \gamma \circ d'' + \lambda \circ F'' - F' \circ \lambda)(p''), (d'' \circ F'' - F'' \circ d'')(p'')) = 0 \tag{297}$$

This means the map $d' \circ \gamma_n : P''_n \to Q'_{n-1}$ must equal

$$g_n = \gamma_{n-1} \circ d'' - \lambda_n F'_n + F''_{n-1} \circ \lambda_n \tag{298}$$

Now use induction. Suppose γ_i defined for $i < n$, so that g_n exists. Then $d' \circ g_n = 0$ due to the inductive definition. Since Q' is exact, the map g_n factors through a map $\beta : P''_n \to d(Q'_n)$, and we take γ_n any lift of β to Q'_n. This constructs the desired chain map F. $\square$

HYUNJUN PARK

FUNCTIONAL ANALYSIS

2019- 09- 03

In this text, we will think the real field $F = \mathbb{R}$ or the complex field $F = \mathbb{C}$.

Definition 1. Let F be a field. A **scalar** is a member of the **scalar field** F. A **vector space over** F is a set X, whose elements are called **vectors**, with two binary operations: **addition**, denoted by $+ : X \times X \to X$ and written as $+(x, y) = x + y : X \times X \to X$, and **scalar multiplication**, denoted by $\cdot : F \times X \to X$ and written as $\cdot(\alpha, x) = \alpha \cdot x = \alpha x$. The addition and scalar multiplication satisfies following algebraic properties:

1. For all $x, y, z \in X$, $x + y = y + x$(commutativity) and $x + (y + z) = x + (y + z)$(associativity).

2. X contains a unique vector 0, which is called the **zero vector** or **origin** of X, such that $x + 0 = x$ for all $x \in X$.

3. For all $x \in X$, there is a unique vector $-x \in X$ such that $x + (-x) = 0$.

These three properties can be said that the pair $(X, +)$ is an abelian group.

4. To every pair $(\alpha, x) \in F \times X$, $1x = x$ and $\alpha(\beta x) = (\alpha\beta)x$.

The symbol 1 represents the identity in field F.

5. For every $\alpha, beta \in F$ and $x, y \in X$, $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$(distributive laws).

If $F = \mathbb{R}$, then X is called a **real vector space**. If $F = \mathbb{C}$, then X is called a **complex vector space**.

From now, if the field is not mentioned, then the statement applies for both real and complex vector space.

Definition 2. For a vector space X, take $A, B \subset X$, $x \in X$, and $\lambda \in F$. Then we define the following notations.

Notice that it may happen that $2A \neq A + A$.

$$
\begin{aligned}
x + A &:= \{x + a : a \in A\} \\
x - A &:= \{x - a : a \in A\} \\
A + B &:= \{a + b : a \in A, b \in B\} \\
\lambda A &:= \{\lambda a : a \in A\}
\end{aligned}
$$

Definition 3. Let X be a vector space. A set $Y \subset X$ is a **subspace** of X if Y itself is a vector space with respect to the same operations.

Proposition 4. *Let X be a vector space. Then a set $Y \subset X$ is a subspace of X if and only if $0 \in Y$ and $\alpha Y + \beta Y \subset Y$ for all scalars α, β.*

Proof. One direction is obvious. For opposite direction, notice that the condition shows that the addition and scalar multiplication are closed on Y. All properties except the existence and uniqueness of zero vector follows from the closure and the operations on X. The existence is given in the condition, and consider $0' \in Y$ is another zero vector, then $0 + 0' = 0 = 0'$. $\square$

Definition 5. Let X be a vector space. A set $C \subset X$ is **convex** if

$$tC + (1-t)C \subset C, \quad 0 \le t \le 1 \tag{299}$$

In other words, C should contain $tx + (1-t)y$ for all $x, y \in C$ and $0 \le t \le 1$.

Definition 6. Let X be a vector space. A set $B \subset X$ is **balanced** if $\alpha B \subset B$ for all $\alpha \in F$ with $|\alpha| \le 1$.

Example 7. Let $X = \mathbb{C}$ be a vector space over $\mathbb{C}$. Then the balanced sets are $\mathbb{C}$, $\emptyset$, and every open or closed circular disc centered at 0. If $X = \mathbb{R}^2$ be a vector space over $\mathbb{R}$, then there are more balanced sets, for example, any line segment with midpoint at $(0,0)$.

Definition 8. Let X be a vector space and $U = \{u_1, \cdots, u_n\} \subset X$. Then U is a **basis** of X if every $x \in X$ has a unique representation of the form

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n, \quad \alpha_i \in F \tag{300}$$

Then the **dimension** of X is $\dim X = n$. If n is finite, then X is said to have **finite dimension**. If $X = \{0\}$, then we say $\dim X = 0$.

Definition 9. A **topological space** S is a set with a collection of subsets τ, whose elements are called **open sets**, has been specified, which satisfies:

1. $S, \emptyset \in \tau$;
2. If $T_1, T_2 \in \tau$ then $T_1 \cap T_2 \in \tau$;
3. If $\{T_\alpha\} \subset \tau$ then $\cup_\alpha T_\alpha \in \tau$.

Then τ is called a **topology on** S.

Definition 10. Let S be a topological space with topology τ. A set $E \subset S$ is **closed** if its complement is open. The **closure** $\bar{E}$ of a set E is the intersection of all closed sets that contains E. The **interior** $\mathring{E}$ is the union of all open sets that are subsets of E. A **neighborhood** of a point $p \in S$ is any open set that contains p. A set $K \subset S$ is **compact** if for every open cover U_α of K, there is a finite subcover $U_{1,\cdots,n}$.

Definition 11. Let S be a topological space with topology τ. Then S is a **Hausdorff space**, and τ is a **Hausdorff topology**, if for any $x, y \in S$, there are neighborhoods U_x, U_y of x, y respectively, so that $U_x \cap U_y = \emptyset$.

Definition 12. Let S be a topological space with topology τ. Take a subcollection $\tau' \subset \tau$. Then τ' is a **base** of τ if every $U \in \tau$ is a union of members of τ'. A collection γ of neighborhoods of a point $p \in S$ is a **local base** at p if every neighborhood of p contains a member of γ.

Proposition 13. *Let S be a topological space. Then the collection of sets τ' is a base of a topology $\tau = \{\cup_\alpha U_\alpha : \{U_\alpha\} \subset \tau'\}$ if and only if τ' satisfies the following properties:*

1. *τ' covers S;*
2. *For $B, B' \in \tau'$ and $x \in B, B'$, there is $B'' \in \tau'$ such that $x \in B'' \subset B \cap B'$.*

Proof. Suppose that τ' is a base of a topology τ. Then since τ includes S, τ' covers S. Also, for any $B, B' \in \tau'$, $B, B' \in \tau$, thus $B \cap B' \in \tau$. This implies that $B \cap B'$ is also a union of elements of τ', so if we choose $x \in B \cap B'$, we can choose $B'' \in \tau'$ such that $x \in B'' \subset B \cap B'$. Conversely, suppose that the two conditions are true, and construct $\tau = \{\cup_\alpha U_\alpha : \{U_\alpha\} \subset \tau'\}$. Since τ' covers S, τ contains S; considering zero union, τ' contains $\emptyset$. Also the union of $\cup_\alpha U_\alpha$ for all $\alpha \in I$ is $\cup_{\beta \in \cup_{\alpha \in I}} U_\beta$. Finally, consider $\cup_\alpha U_\alpha$ and $\cup_\beta U_\beta$. Then $\cup_\alpha U_\alpha \bigcap \cup_\beta U_\beta = \cup_{\alpha,\beta}(U_\alpha \cap U_\beta)$. Since $U_\alpha, U_\beta \in \tau'$, for all $x \in U_\alpha \cap U_\beta$ we may choose $U_{\alpha,\beta;x_{\alpha\beta}} \in \tau'$ such that $x \in U_{\alpha,\beta;x_{\alpha\beta}} \in U_\alpha \cap U_\beta$. Thus $\cup_\alpha U_\alpha \bigcap \cup_\beta U_\beta = \cup_{\alpha,\beta;x_{\alpha\beta}} U_{\alpha,\beta;x_{\alpha\beta}}$. □

Proposition 14. *If S is a topological set with topology τ, $E \subset S$, and if σ is the collection of all intersections $E \cap V$ where $V \in \tau$, then σ is a topology on E.*

We call this the topology that E **inherits** from S.

Proof. $\emptyset \cap E = \emptyset, S \cap E = E$, thus $\emptyset, S \in \sigma$. Also, $\cup_\alpha(V \cap U_\alpha) = V \bigcap \cup_\alpha U_\alpha$. Finally, $(V \cap U_1) \cap (V \cap U_2) = V \cap (U_1 \cap U_2)$. □

Definition 15. A sequence $\{x_n\}$ in a Hausdorff space X **converges** to a point $x \in X$, or $\lim_{n\to\infty} x_n = x$, if every neighborhood of x contains all but finitely many of the points x_n.

Definition 16. A function $f : X \to Y$ with topological sets X, Y is **continuous** if $f^{-1}(V)$ is open in X for all open $V \subset Y$.

Definition 17. A vector space X is called a **normed space** if for every $x \in X$ there is an associated nonnegative real number $\|x\|$, which is called the **norm** of x, satisfying

1. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and α is a scalar;
3. $\|x\| > 0$ if $x \neq 0$.

A vector space X is a **metric space** if there is a function $d : X \times X \to [0, \infty)$ satisfying

1. $d(x, y) = 0$ if and only if $x = y \in X$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Taking $d(x,y) = \|x-y\|$, we can see that the normed space is a metric space.

For a metric space X, the **open ball** with **center** at x and **radius** r is the set

$$B_r(x) := \{y : d(x,y) < r\} \tag{301}$$

If X is a normed space, the sets

$$B_1(0) = \{x : \|x\| < 1\}, \quad \bar{B}_1(0) = \{x : \|x\| \leq 1\} \tag{302}$$

are called the **open unit ball** and **closed unit ball** of X, respectively.

Proposition 18. *Let X be a metric space with metric d. Then the collection of open balls, $\mathscr{B}$, is a base of a topology.*

If a topology τ is induced by a metric d as this way, then we say that d and τ are **compatible** with each other.

Proof. Since there always is a ball $B_1(x)$, $\mathscr{B}$ covers X. Now consider $B_r(x) \cap B_{r'}(x')$, which is nonempty set. Choose $x'' \in B_r(x) \cap B_{r'}(x')$ and $r'' = \min(r - d(x,x''), r' - d(x',x''))$. Choose $y \in B_{r''}(x'')$. Then $d(x,y) \leq d(x,x'') + d(x'',y) \leq d(x,x'') + r'' \leq d(x,x'') + r - d(x,x'') = r$ and $d(x',y) \leq d(x',x'') + d(x'',y) \leq d(x',x'') + r'' \leq d(x',x'') + r' - d(x',x'') = r'$, thus $y \in B_r(x) \cap B_{r'}(x')$. This shows that $B_{r''}(x'') \subset B_r(x) \cap B_{r'}(x')$, and so $\mathscr{B}$ is a base. □

Definition 19. A metric d is **complete** if every Cauchy sequence converges; that is, if for every $\epsilon > 0$, there is a positive integer N such that for all natural numbers $n,m > N$, $d(x_n, x_m) < \epsilon$, then $\{x_n\}$ converges.

A normed space X which is complete in the metric defined by its norm is called a **Banach space**.

Definition 20. Consider a vector space X and its topology τ such that

1. Every point of X is a closed set;
2. The vector space operations, addition and scalar multiplication, are continuous with respect to τ.

Then τ is calld a **vector topology** on X, and X is a **topological vector space**.

Definition 21. Let X be a topological vector space. For each vector $a \in X$ and each scalar $\lambda \neq 0$, the **translation operator** $T_a : X \to X$ and the **multiplication operator** $M_\lambda : X \to X$ are defined as $T_a(x) = a + x, M_\lambda(x) = \lambda x$.

Proposition 22. *T_a and M_λ are homeomorphisms from X to X.*

This shows that every vector topology τ is **translation invariant**, which means, a set $E \subset X$ is open if and only if $T_a(E)$ is open. Thus τ is completely determined by any local base. If we are talking about the vector space, the **local base** always mean a local base at zero vector 0.

Proof. From the vector space axioms, we can see that the operators T_{-a} and $M_{1/\lambda}$ are the inverses of T_a, M_λ, respectively. Also since addition and scalar multiplications are continuous, all the four functions are continuous. □

Definition 23. A metric d on a vector space X is **invariant** if $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$.

Definition 24. Let X be a topological vector space. A subset $E \subset X$ is **bounded** if to every neighborhood V of 0 in X corresponds a number $s > 0$ such that $E \subset tV$ for all $t > s$.

Definition 25. Let X be a topological vector space with topology τ.

1. X is **locally convex** if there is a local base whose members are convex.
2. X is **locally bounded** if 0 has a bounded neighborhood.
3. X is **locally compact** if 0 has a neighborhood whose closure is compact.
4. X is **metrizable** if τ is compatible with some metric d.
5. X is an F**-space** if its topology τ is induced by a complete invariant metric d.
6. X is a **Fréchet space** if X is a locally convex F-space.
7. X is **normable** if a norm exists on X such that the metric induced by the norm is compatible with τ.
8. X has the **Heine-Borel property** if every closed and bounded subset of X is compact.

2019- 09- 10

Proposition 26. *Let X be a topological vector space and W be a neighborhood of 0. Then there is a neighborhood U of 0 which is symmetric, which means $U = -U$, and which satisfies $U + U \subset W$.*

Proof. Since $0 + 0 = 0$ and the addition is continuous, there are neighborhoods V_1, V_2 of 0 such that $V_1 + V_2 \subset W$. Now take $U = V_1 \cap V_2 \cap -V_1 \cap -V_2$. □

Repeating this proof, we can always take a symmetric neighborhood U of 0 such that $\underbrace{U + \cdots + U}_{2^n} \subset W$.

Theorem 27. *Let X be a topological vector space, K be a compact subset, and C be a closed subset with $K \cap C = \emptyset$. Then 0 has a neighborhood V satisfying $(K+V) \cap (C+V) = \emptyset$.*

Since $K + V$ and $C + V$ are unions of open sets, it is open. Thus this theorem gives disjoint open sets that contains K and C, respectively.

Proof. If $K = \emptyset$ then $K + V = \emptyset$. Thus consider $K \neq \emptyset$ and choose $x \in K$. Considering the translational □

2019- 09- 19

Definition 28. A **functional** is a function $f : E \to \mathbb{R}$, where E is a vector space over $\mathbb{R}$. A functional f is **linear** if $f(v+w) = f(v) + f(w)$ and $f(\alpha v) = \alpha f(v)$.

Definition 29. A functional $f : E \to \mathbb{R}$ is a **Minkowski functional** if

1. $f(\lambda x) = \lambda f(x)$, for all $x \in E$ and $\lambda > 0$;
2. $f(x+y) \leq f(x) + f(y)$, for all $x, y \in E$

Theorem 30 (Analytic form of Hahn-Banach Theorem)**.** *Let E be a vector space over $\mathbb{R}$. Let $p : E \to \mathbb{R}$ be a Minkowski functional. Let $G \subset E$ be a linear subspace and $g : G \to \mathbb{R}$ be a linear functional such that*

$$g(x) \leq p(x), \quad \forall x \in G \tag{303}$$

Then there is a linear functional $f : E \to \mathbb{R}$ which extends g, that is, $g(x) = f(x)$ for all $x \in G$, and such that

$$f(x) \leq p(x), \quad \forall x \in E \tag{304}$$

Proof. Consider the set

$$P = \left\{ h : D(h) \subset E \to \mathbb{R} \;\middle|\; \begin{array}{l} D(h) \text{ is a linear subspace of } E, \\ h \text{ is linear, } G \subset D(h), \\ h \text{ extends } g, \text{ and } h(x) \leq p(x), \forall x \in D(h) \end{array} \right\} \tag{305}$$

We define the order relation on P as $h_1 \leq h_2$ if and only if $D(h_1) \subset D(h_2)$ and h_2 extends h_1. Since $g \in P$, P is nonempty. Let $Q = \{h_i\}_{i \in I}$ be a totally ordered subset. Define $D(h) = \cup_{i \in I} D(h_i)$ and $h(x) = h_i(x)$ if $x \in D(h_i)$ for some $i \in I$. Since Q is totally ordered, h is well defined. Also $h \in P$, and h is an upper bound of Q. Thus P is inductive, and we can apply Zorn's lemma. Take a maximal element $f \in P$. If we show that $D(f) = E$, then the theorem is proven.

Lemma 31 (Zorn's lemma)**.** *Every nonempty ordered inductive set has a maximal element.*

Suppose not, and choose $x_0 \in E - D(f)$. Set $D(h) = D(f) + \mathbb{R}x_0$, and $h(x + tx_0) = f(x) + t\alpha$ for all $x \in D(f)$ and $t \in \mathbb{R}$. Here, we choose $\alpha \in \mathbb{R}$ to make $h \in P$. Then we must ensure that

$$f(x) + t\alpha \leq p(x + tx_0), \quad \forall x \in D(f), \forall t \in \mathbb{R} \tag{306}$$

But since p is Minkowski, we only need to check

$$\begin{cases} f(x) + \alpha \leq p(x + x_0), & \forall x \in D(f) \\ f(x) - \alpha \leq p(x - x_0), & \forall x \in D(f) \end{cases} \tag{307}$$

Thus we may find

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \alpha \leq \inf_{x \in D(f)} \{p(x - x_0) - f(x)\} \tag{308}$$

But since

$$f(x) + f(y) \leq p(x + y) \leq p(x + x_0) + p(y - x_0) \tag{309}$$

thus

$$f(y) - p(y - x_0) \leq p(x + x_0) - f(x), \quad \forall x \in D(f), \forall y \in D(f) \tag{310}$$

and so we can choose such α. Finally $f \leq h$, but since f is maximal and $f \neq h$, this is contradiction. □

Definition 32. For a vector space E on $\mathbb{R}$, the **dual space** E^* is the space of all continuous linear functionals on E. If E is a normed vector space, then the (dual) norm on E^* is defined as

$$\|f\|_{E^*} = \sup_{\|x\| \leq 1, x \in E} |f(x)| \tag{311}$$

Given $f \in E^*$ and $x \in E$, we write $\langle f, x \rangle = f(x)$, in the sense that $\langle , \rangle$ is the **scalar product** for the duality E^*, E.

It is well known that E^* is Banach space even if E is not, because $\mathbb{R}$ is complete.

Corollary 33. *Let G be a linear subspace of E. If $g : G \to \mathbb{R}$ is a continuous linear functional, then there is $f \in E^*$ that extends g and*

$$\|f\|_{E^*} = \|g\|_{G^*} \tag{312}$$

Proof. Use the Hahn-Banach theorem with $p(x) = \|g\|\|x\|$, then we get $f \in E^*$ such that $f(x) \leq \|g\|\|x\|$, thus $\|f\| \leq \|g\|$. However since $f|_G = g$, $\|g\| \leq \|f\|$, thus $\|f\| = \|g\|$. □

The continuity does not changes the result of Hahn-Banach theorem, as we can see in the proof.

Corollary 34. *For every $x_0 \in E$, there is $f_0 \in E^*$ such that $\|f_0\| = \|x_0\|$ and $\langle f_0, x_0 \rangle = \|x_0\|^2$.*

Proof. From the previous Corollary, take $G = \mathbb{R}x_0$ and $g(tx_0) = t\|x_0\|^2$. Then $\|g\|_{G^*} = \|x_0\|$. Now we can take $f_0 \in E^*$ such that $\|f_0\|_{E^*} = \|x_0\|$. Also, since f extends g, $\langle f_0, x_0 \rangle = \langle g, x_0 \rangle = \|x_0\|^2$. □

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Corollary 35. *For every $x \in E$, we have*

$$\|x\| = \sup_{f \in E^*, \|f\| \leq 1} |\langle f, x \rangle| = \max_{f \in E^*, \|f\| \leq 1} |\langle f, x \rangle| \tag{313}$$

Proof. $x = 0$ case is trivial. If $x \neq 0$, then $f \in E^*$ satisfying $\|f\| \leq 1$ means $\sup_{\|x\| \leq 1, x \in E} |f(x)| \leq 1$, thus $|f(x/\|x\|)| \leq 1$ and so $|f(x)| \leq \|x\|$. Thus we have shown $\sup_{\|x\| \leq 1, x \in E} |\langle f, x \rangle| \leq \|x\|$. For converse, from previous corollary, choose $f_0 \in E^*$ such that $\|f_0\| = \|x\|$ and $\langle f_0, x \rangle = \|x\|^2$. Now take $f_1 = f_0/\|x\|$, which gives $\langle f_1, x \rangle = \|x\|$ and $\|f_1\| = 1$. Therefore $\max_{f \in E^*, \|f\|} |\langle f, x \rangle| \geq \|x\|$, showing the desired result. □

Definition 36. Let E be a normed vector space. An **affine hyperplane** is a subset $H \subset E$ of the form

$$H = \{x \in E : f(x) = \alpha\} \tag{314}$$

where f is a linear functional which does not vanish identically, and $\alpha \in \mathbb{R}$. We write $H = [f = \alpha]$, and say $f = \alpha$ the **equation** of H.

Proposition 37. *The hyperplane $H = [f = \alpha]$ is closed if and only if f is continuous.*

Proof. If f is continuous then since α is closed, $H = f^{-1}(\alpha)$ is closed. Now suppose that H is closed. Then $E \setminus H$ is open, and since f does not vanishes identically, $E \setminus H$ is nonempty. Thus we may choose $x_0 \in E \setminus H$. Suppose that $f(x_0) < \alpha$. We may choose an open ball $B(x_0, r) = \{x \in E : \|x - x_0\| < r\} \subset E \setminus H$. Suppose that there is $x_1 \in B(x_0, r)$ such that $f(x_1) > \alpha$. Considering the line segment connecting x_0 and x_1, $x(t) = (1-t)x_0 + tx_1, t \in [0,1]$, since this segment is contained in $B(x_0, r)$, $f(x(t)) \neq \alpha$ for all $t \in [0,1]$. But taking $t = \frac{f(x_1)-\alpha}{f(x_1)-f(x_0)}$, we get $f(x(t)) = \alpha$, contradiction. Thus $f(B(x_0, r)) < \alpha$, thus $f(x_0 + rz) < \alpha$ for all $z \in B(0,1)$. This shows that $\|f\| \leq \frac{1}{r}(\alpha - f(x_0))$, and so f is bounded, thus f is continuous. $\square$

Definition 38. For a normal vector space E, let $A, B \subset E$. We say the hyperplane $H = [f = \alpha]$ **separates** A and B if

$$f(x) \leq \alpha, \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha, \quad \forall x \in B \tag{315}$$

We say H **strictly separates** A and B if there is some $\epsilon > 0$ such that

$$f(x) \leq \alpha - \epsilon, \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha + \epsilon, \quad \forall x \in B \tag{316}$$

Lemma 39. *Let $C \subset E$ be an open convex set with $0 \in C$. For every $x \in E$, let*

$$p(x) = \inf\{\alpha > 0 : \alpha^{-1}x \in C\} \tag{317}$$

This p is called the **gauge of** C or the **Minkowski functional of** C.

Then p is a Minkowski functional, and:

1. *there is a constant M such that $0 \leq p(x) \leq M\|x\|$ for all $x \in E$;*
2. $C = \{x \in E : p(x) < 1\}$.

Proof. First, $p(\lambda x) = \inf\{\alpha > 0 : \alpha^{-1}\lambda x \in C\} = \lambda \inf\{\alpha > 0 : \alpha^{-1}x \in C\} = \lambda p(x)$. Now since C is open, we may take $r > 0$ such that $B(0, r) \subset C$. Then since $rx/\|x\| \in C$, $p(x) \leq \frac{1}{r}\|x\|$.

Now choose $x \in C$. Since C is open, we can choose $\epsilon > 0$ such that $(1+\epsilon)x \in C$. Thus $p(x) \leq \frac{1}{1+x} \leq 1$. Conversely, let $p(x) < 1$. Then there is $\alpha \in (0,1)$ such that $\alpha^{-1}x \in C$, and thus $x = \alpha\alpha^{-1}x + (1-\alpha)0 \in C$, since C is convex.

Finally, let $x, y \in E$ and $\epsilon > 0$. The previous results gives $\frac{p(x)}{p(x)+\epsilon} \in C$, thus $\frac{tx}{p(x)+\epsilon} + \frac{(1-t)y}{p(y)+\epsilon} \in C$ for all $t \in [0,1]$. Choosing $t = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon}$ gives $\frac{x+y}{p(x)+p(y)+2\epsilon} \in C$. Thus $p(x+y) < p(x) + p(y) + 2\epsilon$, and since ϵ is arbitrary, $p(x+y) \leq p(x) + p(y)$. □

Lemma 40. *Let $C \subset E$ be a nonempty open convex set, and let $x_0 \in E \setminus C$. Then there is $f \in E^*$ such that $f(x) < f(x_0)$ for all $x \in C$. In particular, the hyperplane $[f(x) = f(x_0)]$ separates x_0 and C.*

Proof. By translation, we may assume $0 \in C$. By previous lemma, we may introduce the gauge p of C. Consider the map $g : \mathbb{R}x_0 \to \mathbb{R}$ as $g(tx_0) = t$. If $t \leq 0$ then $g(tx_0) \leq 0 \leq p(tx_0)$, and if $t > 0$ then since $t^{-1}tx_0 \notin C$, $g(tx_0) = t \leq p(tx_0)$. Thus, by Hahn-Banach Theorem, there is a linear functional f on E which extends g and $f(x) \leq p(x)$ for all $x \in E$. Since p is bounded, f is bounded, and thus f is continuous, and $f(x_0) = 1$. Finally since $p(x) < 1$ for all $x \in C$, $f(x) < 1 = f(x_0)$ for all $x \in C$. □

Theorem 41 (First geometric form of Hahn-Banach Theorem)**.** *Let $A, B \subset E$ be two nonempty convex subsets such that $A \cap B = \emptyset$. If one of them is open, then there is a closed hyperplane that separates A and B.*

Proof. Consider $C = A - B$. For any $a - b, a' - b' \in C$, since $ta + (1-t)a'$ and $tb + (1-t)b'$ are in A and B, respectively, for all $t \in [0,1]$, $t(a-b) + (1-t)(a'-b') \in C$ for all $t \in [0,1]$. Hence C is convex. Since $C = \cup_{y \in B}(A - y)$, C is open, and $0 \notin C$ because $A \cap B = \emptyset$. Now by previous lemma, there is $f \in E^*$ such that $f(z) < 0$ for all $z \in C$, that is, $f(a) < f(b)$ for all $a \in A, b \in B$. Now we fix α satisfying

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y) \tag{318}$$

Then the hyperplane $[f = \alpha]$ separates A and B. □

Theorem 42 (Second geometric form of Hahn-Banach Theorem)**.** *Let $A, B \subset E$ be two nonempty convex subsets such that $A \cap B = \emptyset$. If A is closed and B is compact, then there is a closed hyperplane that strictly separates A and B.*

Proof. Consider $C = A - B$. As shown in above theorem, C is convex. Consider the neighborhoods U_b of each points in B. Since B is compact, there is a finite open cover of B. Now since

$$C = A - B = \cup(A - U_b) = \cup(A \cap U_b^c) \tag{319}$$

which is finite union of closed sets, C is closed. Finally $0 \notin C$. Hence there is $r > 0$ such that $B(0,r) \cap C = \emptyset$. By the previous theorem, there is a closed hyperplane that separates $B(0,r)$ and C, that is, there is $f \in E^*$ which does not vanish identically, such that $f(a-b) \leq f(rz)$ for all $a \in A, b \in B$, and $z \in B(0,1)$. Thus $f(a-b) \leq -r\|f\|$. Letting $\epsilon = \frac{1}{2}r\|f\| > 0$, we get

$$f(x) + \epsilon \leq f(y) - \epsilon \tag{320}$$

Now choosing α such that

$$\sup_{x \in A} f(x) + \epsilon \leq \alpha \leq \inf_{y \in B} f(y) - \epsilon \tag{321}$$

we can see that the hyperplane $[f = \alpha]$ strictly separates A and B. □

Corollary 43. *Let $F \subset E$ be a linear space such that $\bar{F} \neq E$. Then there is some $f \in E^*$ which does not vanish identically, such that $\langle f, x \rangle = 0$, for all $x \in F$.*

This, using the contrapositive statement, gives the fact that the linear subspace F of E is dense if every continuous linear functional on E that vanishes on F must vanish everywhere on E.

Proof. Let $x_0 \in E \setminus \bar{F}$. Using previous theorem with $A = \bar{F}$ and $B = \{x_0\}$, we have a closed hyperplane $[f = \alpha]$ strictly separating $\bar{F}$ and $\{x_0\}$, that is:

$$\langle f, x \rangle < \alpha < \langle f, x_0 \rangle, \quad \forall x \in F \tag{322}$$

Since $\lambda \langle f, x \rangle < \alpha$ for every $\lambda \in \mathbb{R}$, $\langle f, x \rangle = 0$ for all $x \in F$. □

Definition 44. Let E be a normed vetor space, and E^* be the dual space. The **bidual** E^** is the dual of E^* with the norm

$$\|\xi\|_{E^**} = \sup_{f \in E^*, \|f\| \leq 1} |\langle \xi, f \rangle| \tag{323}$$

Proposition 45. *There is a canonical injection $J : E \to E^**$, which maps x to the map $Jx : f \mapsto \langle f, x \rangle$.*

Proof. First notice that $\langle Jx, f \rangle_{E^**, E^*} = \langle f, x \rangle_{E^*, E}$ for all $x \in E, f \in E^*$. By the linearity of f, J is linear. Also J is an isometry, that is, $\|Jx\|_{E^**} = \|x\|_E$, because

$$\|Jx\| = \sup_{f \in E^*, \|f\| \leq 1} |\langle Jx, f \rangle| = \sup f \in E^*, \|f\| \leq 1 |\langle f, x \rangle| = \|x\| \tag{324}$$

□

Definition 46. If $M \subset E$ is a linear subspace, we define

$$M^{\perp} = \{f \in E^* : \langle f, x \rangle = 0, \quad \forall x \in M\} \tag{325}$$

If $N \subset E^*$ is a linear subspace, we define

$$N^{\perp} = \{x \in E : \langle f, x \rangle = 0, \quad \forall f \in N\} \tag{326}$$

We say $M^{\perp}$ and $N^{\perp}$ is the space orthogonal to M and N, respectively.

Proposition 47. *Let $M \subset E$ be a linear subspace. Then $(M^{\perp})^{\perp} = barM$. Let $N \subset E^*$ be a linear subspace. Then $(N^{\perp})^{\perp} \supset \bar{N}$.*

Proof. Obviously $M \subset (M^{\perp})^{\perp}$, and since $(M^{\perp})^{\perp}$ is closed, $\bar{M} \subset (M^{\perp})^{\perp}$. Same can be done in $(N^{\perp})^{\perp}$. Suppose that we have $x_0 \in (M^{\perp})^{\perp} \setminus \bar{M}$. By the second geometric form of Hahn-Banach Theorem, there is a closed hyperplane strictly separating $\{x_0\}$ and $\bar{M}$. Thus, there is $f \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle f, x \rangle < \alpha < \langle f, x_0 \rangle, \quad \forall x \in M \tag{327}$$

Since M is linear space, $\lambda \langle f, x \rangle < \alpha$, thus $\langle f, x \rangle = 0$ for all $x \in M$. Also $\langle f, x_0 \rangle > 0$. Therefore $f \in M^{\perp}$, and consequently $\langle f, x_0 \rangle = 0$. This gives contradiction. $\square$

Definition 48. Let E be a set. The **epigraph** of the function $\phi : E \to (-\inf, \inf]$ is the set

$$\text{epi}\phi = \{[x, \lambda] \in E \times \mathbb{R} : \phi(x) \leq \lambda\} \tag{328}$$

Definition 49. Let E be a set. A function $\phi : E \to (-\infty, \infty]$ is **lower semicontinuous** if for every $\lambda \in \mathbb{R}$ the set

$$[\phi \leq \lambda] = \{x \in E : \phi(x) \leq \lambda\} \tag{329}$$

is closed.

Proposition 50 (Without proofs)**.** *Let E be a set.*

1. *ϕ is lower semicontinuous if and only if epiϕ is closed in $E \times \mathbb{R}$.*

2. *ϕ is lower semicontinuous if and only if for every $\epsilon > 0$ there is some neighborhood V of x such that*

$$\phi(y) \geq \phi(x) - \epsilon, \quad \forall y \in V \tag{330}$$

   In particular, if ϕ is lower semicontinuous, then for every sequence $x_n \to x$ in E, we have

$$\liminf_{n \to \infty} \phi(x_n) \geq \phi(x) \tag{331}$$

   and converse holds if E is a metric space.

3. *If ϕ_1, ϕ_2 are lower semicontinuous, then $\phi_1 + \phi_2$ is also.*

4. *If $(\phi_i)_{i \in I}$ is a family of lower semicontinuous functions, then their **superior envelope**, which is defined as*

$$\phi(x) = \sup_{i \in I} \phi_i(x) \tag{332}$$

   is lower semicontinuous.

5. *If E is compact and ϕ is locally semicontinuous, then* $\inf_E \phi$ *exists.*

Definition 51. Let E be a vector space. A function $\phi : E \to (-\infty, \infty]$ is **convex** if

$$\phi(tx + (1-t)y) \le t\phi(x) + (1-t)\phi(y), \quad \forall x, y \in E, \quad \forall t \in (0,1). \tag{333}$$

Proposition 52 (Without proofs)**.** *1. ϕ is a convex function if and only if epiϕ is a convex set in $E \times \mathbb{R}$.*

2. *If ϕ is a convex function then for every $\lambda \in \mathbb{R}$, the set $[\phi \le \lambda]$ is convex.*

3. *If ϕ_1, ϕ_2 are convex, then $\phi_1 + \phi_2$ is also.*

4. *If $(\phi_i)_{i \in I}$ is a family of convex functions, then the superior envelope is convex.*

Definition 53. Let E be normed vector space and $\phi : E \to (-\infty, \infty]$ with $\phi \not\equiv \infty$. The **conjugate function** or **Legendre transformaion** of ϕ, $\phi^* : E^* \to (-\infty, \infty]$ is defined as

$$\phi^*(f) = \sup_{x \in E} \{\langle f, x \rangle - \phi(x)\}, \quad \forall f \in E^* \tag{334}$$

Proposition 54. *The conjugate function of ϕ, ϕ^*, is a convex function.*

Proof. Fix $x \in E$. Then the function $f \mapsto \langle f, x \rangle - \phi(x)$ is continuous. Also, $tf + (1-t)g \mapsto \langle tf + (1-t)g, x \rangle - \phi(x) = t(\langle f, x \rangle - \phi(x)) + (1-t)(\langle g, x \rangle - \phi(x))$. This the functions are convex and lower semicontinuous, and thus the superior envelope is also convex and lower semicontinuous. □

Definition 55. The **domain** of $\phi : E \to (-\infty, \infty]$ is the set

$$D(\phi) = \{x \in E : \phi(x) < \infty\} \tag{335}$$

Proposition 56. *Assume that $\phi : E \to (-\infty, \infty]$ is convex lower semicontinuous function with $\phi \not\equiv \infty$. Then $\phi^* \not\equiv \infty$, and thus ϕ is bounded below by an affine continuous function.*

Proof. Let $x_0 \in D(\phi)$ and $\lambda_0 < \phi(x_0)$. Using the second geometric form of Hahn-Banach Theorem in the space $E \times \mathbb{R}$ with $A = \text{epi}\phi$ and $B = \{[x_0, \lambda_0]\}$, there is a closed hyperplane $H = [\Phi = \alpha]$ in $E \times \mathbb{R}$ that strictly separates A and B. Now since $x \in E \mapsto \Phi([x, 0])$ is a continuous linear functional on E, thus $\Phi([x, 0]) = \langle f, x \rangle$ for some $f \in E^*$. Letting $k = \Phi([0, 1])$, we can write

$$\Phi([x, \lambda]) = \langle f, x \rangle + k\lambda, \quad \forall [x, \lambda] \in E \times \mathbb{R} \tag{336}$$

Writing $\Phi > \alpha$ on A and $\Phi < \alpha$ on B, we get

$$\langle f, x \rangle + k\lambda > \alpha, \quad \forall [x, \lambda] \in \text{epi}\phi \tag{337}$$

and $\langle f, x_0 \rangle + k\lambda_0 < \alpha$. This implies

$$\langle f, x \rangle + k\phi(x) > \alpha, \quad \forall x \in D(\phi) \tag{338}$$

and thus

$$\langle f, x_0 \rangle + k\phi(x_0) > \alpha > \langle f, x_0 \rangle + k\lambda_0 \tag{339}$$

thus $k > 0$. Now then

$$\left\langle -\frac{1}{k} f, x \right\rangle - \phi(x) < -\frac{\alpha}{k}, \quad \forall x \in D(\phi) \tag{340}$$

and thus $\phi^*(-\frac{1}{k} f) < \infty$. □

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Definition 57. We define

$$\phi^{**}(x) = \sup_{f \in E^*} \{\langle f, x \rangle - \phi^*(f)\}, \quad x \in E \tag{341}$$

Theorem 58 (Fenchel-Moreau)**.** *Assume that $\phi : E \to (-\infty, \infty]$ is convex and lower semicontinuous, and $\phi \not\equiv \infty$. Then $\phi^* * = \phi$.*

Proof. First we assume that $\phi \geq 0$. Then since $\langle f, x \rangle - \phi^*(f) \leq \phi(x)$ for all $x \in E$ and $f \in E^*$, $\phi^* * \leq \phi$. Suppose that $\phi^{**}(x_0) < \phi(x_0)$ for some $x_0 \in E$. Then $\phi^{**}(x_0)$ is finite. Applying second geometric form of Hahn-Banach theorem in the space $E \times \mathbb{R}$ with $A = \text{epi}\phi$ and $B = [x_0, \phi^{**}(x_0)]$, there is $f \in E^*, k \in \mathbb{R}, \alpha \in \mathbb{R}$ such that

$$\langle f, x \rangle + k\lambda > \alpha, \quad \forall [x, \lambda] \in \text{epi}\phi \tag{342}$$

and

$$\langle f, x_0 \rangle + k\phi^{**}(x_0) < \alpha \tag{343}$$

Fixing $x \in D(\phi)$ and taking $\lambda \to \infty$ gives $k \geq 0$. Now take $\epsilon > 0$. Since $\phi \geq 0$, we have

This result is different from previous one, because x_0 can or cannot be in $D(\phi)$.

$$\langle f, x \rangle + (k + \epsilon)\phi(x) \geq \alpha, \quad \forall x \in D(\phi) \tag{344}$$

Thus

$$\phi^* \left(-\frac{f}{k + \epsilon} \right) \leq -\frac{\alpha}{k + \epsilon} \tag{345}$$

and by the definition of $\phi^{**}(x_0)$,

$$\phi^{**}(x_0) \geq \left\langle -\frac{f}{k + \epsilon}, x_0 \right\rangle - \phi^* \left(-\frac{f}{k + \epsilon} \right) \geq \left\langle -\frac{f}{k + \epsilon}, x_0 \right\rangle + \frac{\alpha}{k + \epsilon} \tag{346}$$

Thus

$$\langle f, x_0 \rangle + (k + \epsilon)\phi^{**}(x_0) \geq \alpha, \quad \forall \epsilon > 0 \tag{347}$$

This contradicts $\langle f, x_0 \rangle + k\phi^{**}(x_0) < \alpha$.

Now for general case, fix $f_0 \in D(\phi^*)$, and define

$D(\phi^*) \neq \emptyset$, because of the previous proposition.

$$\bar{\pi}(x) = \phi(x) - \langle f_0, x \rangle + \phi^*(f_0) \tag{348}$$

Then $\bar{\phi}$ is convex, lower semicontinuous, $\bar{\phi} \not\equiv \infty$, and $\bar{\phi} \geq 0$. Thus $(\bar{\phi})^{**} = \bar{\phi}$. Now,

$$(\bar{\phi})^*(f) = \phi^*(f + f_0) - \phi^*(f_0) \tag{349}$$

and

$$(\bar{\phi})^{**}(x) = \phi^{**}(x) - \langle f_0, x \rangle + \phi^*(f_0) = \bar{\phi} \tag{350}$$

This gives $\phi^{**} = \phi$. □

Example 59. Consider $\phi(x) = \|x\|$. Then

$$\phi^*(f) = \sup_{x \in E} \{\langle f, x \rangle - \|x\|\} \tag{351}$$

Since $\langle f, kx \rangle - \|kx\| = k(\langle f, x \rangle - \|x\|)$, the result must be 0 or ∞. If $\|f\| \leq 1$, then at $\|x\| = 1$ we have $\langle f, x \rangle - \|x\| \leq 0$, and thus for all $x \in E$ it also holds. Thus $\phi^*(f) = 0$. If $\|f\| > 1$, then we have some $x \in E$ such that $\|x\| = 1$ and $\langle f, x \rangle > 1$. Thus $\phi^*(f) = \infty$. In summary,

$$\phi^*(f) = \begin{cases} 0, & \|f\| \leq 1 \\ \infty, & \|f\| > 1 \end{cases} \tag{352}$$

Now thus

$$\phi^{**}(x) = \sup_{f \in E^*, \|f\| \leq 1} \langle f, x \rangle \tag{353}$$

Using $\phi^{**} = \phi$, we again obtain

$$\|x\| = \sup_{f \in E^*, \|f\| \leq 1} \langle f, x \rangle \tag{354}$$

Example 60. For a nonempty set $K \subset E$, take

$$I_K(x) = \begin{cases} 0, & x \in K \\ \infty, & x \notin K \end{cases} \tag{355}$$

which is called the **indicator function**. Notice that I_K is convex if and only if K is convex, and I_K is lower semicontinuous if and only if K is closed. The conjugate function $(I_K)^*$ is called the **supporting function** of K. Now if $K = M$ is a linear subspace, then

$$(I_M)^*(f) = \sup_{x \in E} \{\langle f, x \rangle - I_K(x)\} \tag{356}$$

If $f \in M^\perp$, then all $x \in M$ gives $\langle f, x \rangle - I_M(x) = 0$ and all $x \notin M$ gives $-\infty$, thus $(I_M)^*(f) = 0$. If $f \notin M^\perp$, then there is $x \in M$ such that $\langle f, x\rangle - I_M(x) = \langle f, x\rangle \neq 0$. Putting λx and taking $\lambda \to \infty$ gives $(I_M)^*(f) = \infty$, thus $(I_M)^* = I_{M^\perp}$. Using this again, $(I_M)^{**} = I_{(M^\perp)^\perp}$. Setting M be a closed linear space to use $(I_M)^{**} = I_M$, we get $(M^\perp)^\perp = M$ again.

HYUNJUN PARK

K-THEORY

2019- 10- 10

Definition 1. An n**-dimensional vector bundle** is a map $p : E \to B$ with a real vector space structure on $p^{-1}(b)$ for each $b \in B$, such that the following **local triviality condition** is satisfied: there is an open cover $\{U_\alpha\}$ of B for each of which there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, whose restriction $h_\alpha|_{p^{-1}(b)} : p^{-1}(b) \to \{b\} \times \mathbb{R}^n$ is a homeomorphism for all $b \in U_\alpha$. We call h_α a **local trivialization**, B the **base space**, E the **total space**, and $p^{-1}(b)$ the **fibers**.

If $\mathbb{R}$ is changed to $\mathbb{C}$, then we call $p : E \to B$ a **complex vector bundle**.

We often abbreviate terminology by just calling E as the vector bundle.

Example 2. 1. The space $E = B \times \mathbb{R}^n$ with $p : E \to B$ the projection onto the first factor is **product bundle** or **trivial bundle**. The local trivialization is p itself.

2. Let E be the quotient space of $I \times \mathbb{R}$ with identifications $(0, t) \approx (1, -t)$. Then the projection map $p : I \times \mathbb{R} \to I$ induces a map $p : E \to S^1$, which is a 1-dimensional vector bundle, in other words, **line bundle**. We call E the **Möbius bundle**. For the local trivializations, choose any open connected proper subset containing $i \in I$.

3. For the unit sphere S^n in $\mathbb{R}^{n+1}$, consider $E = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\}$. By translation, we can think v as a tangent vector of S^n. Then the **tangent bundle** of the unit sphere is $p : E \to S^n$ defined as $p(x, v) = x$. For the local trivializations, for $x \in S^n$ let U_x be the open hemisphere containing x and bounded by the hyperplane through the origin orthogonal to x. Now define $h_: p^{-1}(U_x) \to U_x \times p^{-1}(x) \simeq U_x \times \mathbb{R}^n$ as $h_x(y, v) = (y, \pi_x(v))$, where π_x is the projection onto the hyperplane $p^{-1}(x)$.

4. For the unit sphere S^n in $\mathbb{R}^{n+1}$, consider $E = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : v = tx, t \in \mathbb{R}\}$. By translation, we can think v as a normal vector of S^n. Then the **normal bundle** of the unit sphere is a line bundle $p : E \to S^n$ defined as $p(x, v) = x$. For the local trivializations, for $x \in S^n$ let U_x be the open hemisphere containing x and bounded by the hyperplane through the origin orthogonal to x. Now define $h : p^{-1}(U_x) \to U_x \times p^{-1}(x) \simeq U_x \times \mathbb{R}^n$ as $h_x(y, v) = (y, \pi_x(v))$, where π_x is the projection onto the line $p^{-1}(x)$.

5. For the real projective n-space $\mathbb{R}P^n$, as the space of lines in $\mathbb{R}^{n+1}$ through the origin, the **canonical line bundle** $p : E \to \mathbb{R}P^n$ is a bundle with $E = \{(l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in l\}$ and $p(l, v) = l$. Local trivializations can be defined as the projection maps, as above.

6. Consider the infinite-dimensional projective space $\mathbb{R}P^\infty$, which is the union of finite-dimensional projective spaces $\mathbb{R}P^n$ under the inclusions $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$ induced from the natural inclusions $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$, and its topology is the weak or direct limit topology: a set in $\mathbb{R}P^\infty$ is open if and only if it intersects each $\mathbb{R}P^n$ in an open set. The canonical line bundle over $\mathbb{R}P^\infty$ is the direct limit of the canonical line bundles over $\mathbb{R}P^n$, and local trivializations are the induced maps.

7. The space $E^\perp = \{(l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \perp l$ with projection $p : E^\perp \to \mathbb{R}P^n$ defined as $p(l, v)$ is a vector bundle, and local trivializations are obtained by projecton maps.

Definition 3. For two vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ over the same base space, the **isomorphism** between E_1 and E_2 is a homeomorphism $h : E_1 \to E_2$ taking each $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism. We write $E_1 \simeq E_2$ if they are isomorphic.

Example 4. 1. The normal bundle of S^n in $\mathbb{R}^{n+1}$ is isomorphic to the product bundle $S^n \times \mathbb{R}$, by the map $(x, tx) \mapsto (x, t)$.

2. The tangent bundle of S^1 is isomorphic to the trivial bundle $S^1 \times \mathbb{R}$, by $(e^{i\theta}, ite^{i\theta}) \mapsto (e^{i\theta}, t)$.

3. The Möbius bundle is isomorphic to the canonical line bundle over $\mathbb{R}P^1 \simeq S^1$. Indeed, the line on angle 0 and π are identified, with $(0, x) \approx (\pi, -x)$, on the canonical line bundle.

Definition 5. For a vector bundle $p : E \to B$, a **section** is a map $s : B \to E$ assigning to each $b \in B$ a vector $s(b) \in p^{-1}(b)$, or equivalently, $p \circ s = 1_B$. The **zero section** is a section whose value is zero vector in each fiber. The **nonvanishing section** is section whose value is nonzero vector in each fiber.

Proposition 6. *If two bundles are isomorphic, then the complements of the zero sections of the two bundles are isomorphic.*

Proof. This is obvious since any vector bundle isomorphism takes the zero section to the zero section. □

Example 7. Consider the Möbius bundle and product bundle $S^1 \times \mathbb{R}$. The complement of the zero section of the Möbius bundle is connected but the complement of the zero section of the product bundle is disconnected, thus they are not isomorphic.

Example 8. For the tangent bundle of S^n, there is a nonvanishing section if and only if n is odd. Thus, if n is nonzero even number,

The proof will be given later.

then the tangent bundle and trivial bundle $S^n \times \mathbb{R}^n$ is not isomorphic, because the trivial bundle has a nonvanishing section, and an isomorphism between vector bundles takes nonvanishing sections to nonvanishing sections.

Lemma 9. *A continuous map $h : E_1 \to E_2$ between vector bundles over the same base space B is an isomorphism if it takes each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism.*

Proof. The only thing we need to check is that h^{-1} is continuous. Since the continuity is a local property, we can reduce the problem on an open set $U \subset B$ over which E_1 and E_2 are trivial. Restricting on $U \times \mathbb{R}^n$, we can see that $h : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$ is a continuous map with $h(x, v) = (x, g_x(v))$, where g_x is an element of the group $GL_n(\mathbb{R})$, and g_x depends continuously on x: which means the elements of g_x as $n \times n$ matrix are continuous functions. Thus the inverse g_x^{-1} depends continuously on x also, thus $h^{-1}(x, v) = (x, g_x^{-1}(v))$ is continuous. □

Proposition 10. *An n-dimensional bundle $p : E \to B$ is isomorphic to the trivial bundle if and only if it has n sections $s_1, \cdots, s_n$ such that the vectors $s_1(b), \cdots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$.*

Proof. For one direction, choose the basis of $\mathbb{R}^n$ as $\{e_1, \cdots, e_n\}$, then the sections $s_i(b) = (b, e_i)$ gives such sections. Conversely, if one has n linearly independent sections s_i, then the map $h : B \times \mathbb{R}^n \to E$ given by $h(b, t_1, \cdots, t_n) = \sum_i t_i s_i(b)$ is a linear isomorphism in each fiber, and is continuous since it is linear combination of continuous functions. Hence by previous lemma, h is an isomorphism. □

Example 11. The tangent bundle to S^1 is trivial, because it has the section $(x_1, x_2) \mapsto (-x_2, x_1)$ for $(x_1, x_2) \in S^1$.

Example 12 (Quaternions)**.** The quaternion space $\mathbb{H}$ has the elements $z = x_1 + ix_2 + jx_3 + kx_4$ with $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and there are multiplication rules $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$. Identifying $\mathbb{H}$ with $\mathbb{R}^4$, the unit sphere is S^3, and there are three sections of its tangent bundle defined as $z \mapsto iz, jz, kz$, which can be represented as $(x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, -x_4, x_3), (-x_3, x_4, x_1, -x_2), (-x_4, -x_3, x_2, x_1)$ respectively. These three vectors and (x_1, x_2, x_3, x_4) are orhthogonal to each other, thus we have three linearly independent nonvanishing tangent vector fields on S^3.

Example 13 (Octonians and Generalization)**.** The above argument holds because the quaternion multiplication satisfies $|zw| = |z||w|$, where $|\cdot|$ is the usual norm of vectors in $\mathbb{R}^4$. Now considering the

Cayley octonions, the space $\mathbb{R}^8$ thought as $\mathbb{H} \times \mathbb{H}$ and the multiplication defined as $(z_1, z_2)(w_1, w_2) = (z_1 w_1 - \bar{w}_2 z_2, z_2 \bar{w}_1 + w_2 z_1)$, and satisfies $|zw| = |z||w|$. This leads to the construction of seven orthogonal tangent vector fields on the unit sphere S^7, and so the tangent bundle to S^7 is also trivial. Indeed, the only spheres with trivial tangent bundle are S^1, S^3, S^7, which will be shown later.

Corollary 14. *For a vector bundle E, if there is a continuous projection map $E \to \mathbb{R}^n$ which is a linear isomorphism on each fiber, then E is isomorphic to the trivial bundle.*

Proof. Taking such a continuous map with the bundle projection $E \to B$, we get the continuous map $E \to B \times \mathbb{R}^n$ which takes each fibers to fibers, thus by previous lemma, this is isomorphism. $\square$

Definition 15.

1. For a vector bundle $p : E \to B$ and a subspace $A \subset B$, the map $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a vector bundle, which is called the **restriction of** E **over** A.

2. For vector bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$, $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is a vector bundle, whose fibers are the products $p_1^{-1}(b_1) \times p_2^{-1}(b_2)$ and the local trivialization is the product of local trivializations. This is called the **product of vector bundles**.

Definition 16. For two vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, the **direct sum** of E_1 and E_2 is the vector bundle

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\} \tag{357}$$

with the projection $E_1 \oplus E_2 \to B$ sending (v_1, v_2) to the point $p_1(v_1) = p_2(v_2)$.

We can also write $E_1 \oplus E_2$ as the restriction of the product $E_1 \times E_2$ over the diagonal $B = \{(b, b) \in B \times B\}$.

Definition 17. The bundle E is **stably trivial** if it becomes trivial after taking the direct sum with a trivial bundle.

Example 18.

1. All trivial bundles are stably trivial.

2. The direct sum of two trivial bundles is again a trivial bundle.

3. Consider the direct sum of the tangent and normal bundles to S^n in $\mathbb{R}^{n+1}$. The elements can be written as $(x, v, tx) \in S^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, where $x \perp v$. The map $(x, v, tx) \mapsto (x, v + tx)$ gives an isomorphism of the direct sum bundle with $S^n \times \mathbb{R}^{n+1}$. Thus the tangent bundle to S^n is stably trivial.

4. For the canonical line bundle $E \to \mathbb{R}P^n$, the direct sum $E \oplus E^\perp$ is isomorphic to the trivial bundle $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ by the map $(l, v, w) \mapsto (l, v + w)$ for $v \in l$ and $w \perp l$. If $n = 1$, then since $E^\perp$ is isomorphic to E itself, by the map that rotates each vector in the plane by 90 degrees. Thus the Möbius bundle is stably trivial.

Example 19. Consider $S^n \times \mathbb{R}^{n+1} \simeq TS^n \oplus NS^n$, where TS^n is the tangent bundle and NS^n is the normal bundle. Suppose we factor out by the identifications $(x, v) \approx (-x, -v)$ on both sides. Applied to TS^n, this identification yields $T\mathbb{R}P^n$, the tangent bundle to $\mathbb{R}P^n$. Applied to NS^n, the identification $(x, v) \approx (-x, -v)$ can be written as $(x, tx) \approx (-x, t(-x))$. This gives the product bundle $\mathbb{R}P^n \times \mathbb{R}$, because the section $x \mapsto (-x, -x) = (x, x)$ is well-defined. Now consider the identification $(x, v) \approx (-x, -v)$ in $S^n \times \mathbb{R}^{n+1}$. Considering the coordinate factors, the quotient is the direct sum of $n + 1$ copies of the line bundle E over $\mathbb{R}P^n$, obtained by making the identifications $(x, t) \approx (-x, -t)$ in $S^n \times \mathbb{R}$. Now identifying $S^n \times \mathbb{R}$ with NS^n by the isomorphism $(x, t) \mapsto (x, tx)$, we get $(-x, -t) \mapsto (-x, (-t)(-x)) = (-x, tx)$. This quotient gives the canonical line bundle over $\mathbb{R}P^n$, since the first identification gives $\mathbb{R}P^n$ structure, and the second identification is meaningless. Thus, the direct sum of the tangent bundle $T\mathbb{R}P^n$ with a trivial line bundle is isomorphic to the direct sum of $n + 1$ copies of the canonical line bundle over $\mathbb{R}P^n$.

Since we already shown that the tangent space of S^3, S^7 are trivial, $T\mathbb{R}P^3$ and $T\mathbb{R}P^7$ are trivial, which are isomorphic to 4 and 8 copies of canonical line bundle, respectively.. Later we will show that k-copies of the canonical line bundle over $\mathbb{R}P^n$ is stably trivial if and only if k is a multiple of $2^{\phi(n)}$, where $\phi(n)$ is the number of integers i in the range $0 < i \leq n$, with i congruent to $0, 1, 2, 4 \mod 8$. For $n = 3$, $\phi(3) = 2$ thus $2^{\phi(n)} = 4$, thus we need $4n$ copies; for $n = 7$, $\phi(7) = 3$ thus $2^{\phi(n)} = 8$, thus we need $8n$ copies. These results fits on the previous argument.

2019- 10- 11

Definition 20. An **inner product** on a vector bundle $p : E \to B$ is a map $\langle , \rangle : E \oplus E \to \mathbb{R}$ which restricts in each fiber to an inner product.

The **inner product** on vector space is a positive definite symmetric bilinear form.

Proposition 21. *For a vector bundle $p : E \to B$, an inner product exists if B is compact Hausdorff, or more generally, paracompact.*

Proof. To pull back the standard inner product in $\mathbb{R}^n$ to the inner product $\langle , \rangle$ continuously, we need to use the local trivialization $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. Now set $\langle v, w \rangle = \sum_\beta \phi_\beta \circ p(v) \langle v, w \rangle_\beta$. $\square$

A space X is **paracompact** if it is Hausdorff and every open cover of X has a **subordinate partition of unity**: a collection of maps $\phi_\beta : X \to I = [0, 1]$, each supported in some set of the open cover, and with $\sum_\beta \phi_\beta = 1$, only finitely many of the ϕ_β is nonzero near each point of X. Since Hausdorff compact set is normal, using Urysohn's Lemma, we can construct those functions when X is compact Hausdorff.

Definition 22. A **vector subbundle** of a vector bundle $p : E \to B$ is a subspace $E_0 \subset E$ intersecting each fiber of E in a vector subspace, such that the restriction $p|_{E_0} : E_0 \to B$ is a vector bundle.

Proposition 23. *If $E \to B$ is a vector bundle over a paracompact space B, and $E_0 \subset E$ is a vector subbundle, then there is a vector subbundle $E_0^\perp \subset E$ such that $E_0 \oplus E_0^\perp \simeq E$.*

Proof. Let $E_0^\perp$ is the subspace of E where in each fiber consists of all vectors orthogonal to vectors in E_0. Now our claim is that $E_0^\perp \to B$ is a vector bundle, so that we can take a continuous map $E_0 \oplus E_0^\perp \to E$, taking fibers to fibers by linear isomorphism, as $(v, w) \mapsto (v + w)$. To show the local triviality, notice that we may consider $E = B \times \mathbb{R}^n$, the trivial bundle, since the question is local. If we let E_0 an m-dimensional vector bundle, then it has m independent local sections near each points $b_0 \in B$. Now on $p^{-1}(b_0)$, we may take vectors $s_{m+1}(b_0), \cdots, s_n(b_0)$ which makes the set of vectors $\{s_i(b_0)\}$ independent. Take sections $s_{m+1}, \cdots, s_n$ as the constant function still makes the set $\{s_1(b), \cdots, s_m(b)\}$ independent for all b near b_0, due to the continuity of determinant. Now using the Gram-Schmidt process, orthogonalize the sections to $\{s_i'\}$. This is possible since the inner product is well defined, and all those sections are continuous since the inner product is continuous. Also, first m sections of $\{s_i'\}$ are the basis of E_0 in each fiber. Now we define a local trivialization $h : p^{-1}(U) \to U \times \mathbb{R}^n$ with $h(b, s_i'(b))$ equal to the i-th standard basis vector of $\mathbb{R}^n$. Now this h carries E_0 to $U \times \mathbb{R}^m$ and $E_0^\perp$ to $U \times \mathbb{R}^{n-m}$, so $h|_{E_0^\perp}$ is a local trivialization of $E_0^\perp$. □

Corollary 24. *For all vector bundles with an inner product, we can always choose isometric local trivializations.*

Proof. From previous proposition, take $E_0 = E$. Then the last statement of the proof gives that the given local trivialization is isometric. □

Proposition 25. *For each vector bundle $p : E \to B$ where B a compact Hausdorff, there is a vector bundle $E' \to B$ such that $E \oplus E'$ is the trivial bundle.*

Proof. For each $x \in B$, there is a neighborhood U_x over which E is trivial. By Urysohn's Lemma, there is a map $\phi_x : B \to [0, 1]$ that is 0 outside U_x and nonzero at x. Letting x vary, the sets $\phi_x^{-1}(0, 1]$ form an open cover of B, and by compactness we can choose a finite subcover, labeled as U_i and ϕ_i. Now define $g_i : E \to \mathbb{R}^n$ by

$$g_i(v) = \phi_i(p(v))(\pi_i \circ h_i(v)) \tag{358}$$

where π_i is a projection of $U_i \times \mathbb{R}^n$ to $\mathbb{R}^n$ and h_i is a local trivialization $h_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$. Then g_i is a linear injection on each fiber over $\phi_i^{-1}(0, 1]$. We define $g : E \to \mathbb{R}^N$ as the product of maps $g_i : E \to \mathbb{R}^n$, then g is a linear injection on each fiber.

Now define $f : E \to B \times \mathbb{R}^N$ as (p, g). Then the image of f is a subbundle of the product $B \times \mathbb{R}^N$, since the projection on $\mathbb{R}^N$ onto the i-th $\mathbb{R}^n$ factor gives the second coordinate of a local trivialization over $\phi_i^{-1}(0, 1]$. Thus we have E isomorphic to a subbundle of $B \times \mathbb{R}^N$, so by the preceding proposition, there is a complementary subbundle E' with $E \oplus E'$ isomorphic to $B \times \mathbb{R}^N$. □

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Definition 26. For two vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, the **tensor product** of E_1 and E_2 is a vector bundle $E_1 \otimes E_2 \to B$ defined as following:

1. Setwisely, $E_1 \otimes E_2$ is the disjoint union of the vector spaces $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for $x \in B$;
2. For the topology, choose $h_i : p_i^{-1}(U) \to U \times \mathbb{R}^{n_i}$ be the trivializations for each open set $U \subset B$ over which E_1, E_2 are trivial. Then the topology on the set $p_1^{-1}(U) \otimes p_2^{-1}(U)$ is the topology which makes the fiberwise tensor product map $h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ be a homeomorphism.
3. For the local trivialization, the maps $h_1 \otimes h_2$ and the open sets U given as above are desired local trivializations.

Proposition 27. *The topology defined as above is well-defined, which is, it does not depends on the choice of h_i's and U's.*

Proof. For the local trivialization h_i on U, any other choices are obtained by composing with isomorphisms of $U \times \mathbb{R}^{n_i}$ of the form $(x, v) \mapsto (x, g_i(x)(v))$, where $g_i : U \to GL_{n_i}(\mathbb{R})$ are continuous maps. Then $h_1 \otimes h_2$ changes by composing wit analogous isomorphisms of $U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$, where the second coordinates $g_1 \otimes g_2$ are continuous maps $U \to GL_{n_1 n_2})(\mathbb{R})$. Indeed this map is continuous because each entries of the matrices $g_1(x) \otimes g_2(x)$ are the products of the entries of $g_1(x)$ and $g_2(x)$. Now when we replace U by an open subset V, the topology on $p_1^{-1}(V) \otimes p_2^{-1}(V)$ induced by U is same with the topology induced by V, since local trivializations over U restrict to local trivializations over V. □

Example 28. Using above argument, we can consider another construction of vector space. Consider a vector bundle $p : E \to B$ and an open cover $\{U_\alpha\}$ of B with local trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. Then we can reconstruct E as the quotient space of the disjoint union $\sqcup_\alpha (U_\alpha \times \mathbb{R}^n)$, identifying $(x, v) \in U_\alpha \times \mathbb{R}^n$ with $h_\beta \circ h_\alpha^{-1}(x, v) \in U_\beta \times \mathbb{R}^n$ whenever $x \in U_\alpha \cap U_\beta$. Then the functions $h_\beta \circ h_\alpha^{-1}$ can be viewed as maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$, satisfying the **cocycle condition**, $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Thus any collection of **gluing functions** $g_{\beta\alpha}$ satisfying the cocycle condition can be used to construct a vector bundle $E \to B$.

With this constructions, we can obtain the tensor product of two vector bundles $E_i \to B$ with gluing functions $g^i_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_{n_i}(\mathbb{R})$ can be constructed with gluing functions $g^1_{\beta\alpha} \otimes g^2_{\beta\alpha}$, assigning each $x \in U_\alpha \cap U_\beta$ the tensor product $g^1_{\beta\alpha}(x) \otimes g^2_{\beta\alpha}(x)$.

Proposition 29. *The tensor product operation for vector bundles over a fixed base space is commutative, associative, and has an identity element: the trivial line bundle. Furthermore, it is also distributive with respect to direct sum.*

Proof. The proof fully depends on the fact that the properties in the proposition are all true in $\mathbb{R}^n$ space. $\square$

2019- 10- 22

Example 30. Consider the set $\mathrm{Vect}^1(B)$, the isomorphism classes of one-dimensional vector bundles over B. Then it is an abelian group with respect to the tensor product operation. The inverse of a line bundle $E \to B$ can be constructed by replacing its gluing matrices $g_{\beta\alpha}(x) \in GL_1(\mathbb{R})$ with their inverses: the cocycle condition is satisfied since 1×1 matrices commute. Now if we give E an inner product, we may rescale local trivializations h_α to be isometries, taking vectors in fibers of E to vectors in $\mathbb{R}^1$ of the same length. Then all the values of the gluing functions $g_{\beta\alpha}$ are ± 1, being isometries of $\mathbb{R}$. The gluing functions for $E \otimes E$ are the squares of these $g_{\beta\alpha}$'s, which is 1, thus $E \otimes E$ is the trivial line bundle, and so each element of $\mathrm{Vect}^1(B)$ is its own inverse.

Considering the complex vector bundles, we can get exactly same results, except the values of $g_{\beta\alpha}$'s are the complex numbers of norm 1. Thus we cannot expect $E \otimes E$ to be trivial.

The group $\mathrm{Vect}^1(B)$ is isomorphic to the cohomology group $H^1(B; \mathbb{Z}_2)$ when B is homotopy equivalent to a CW complex, which we will show later.

The group $\mathrm{Vect}^1_{\mathbb{C}}(B)$ is isomorphic to the cohomology group $H^2(B; \mathbb{Z})$, when B is homotopy equivalent to a CW complex, which we will show later.

Definition 31. Consider a complex vector bundle $E \to B$ with gluing maps $g_{\beta\alpha}$. The **conjugate bundle** $\bar{E} \to B$ is a complex vector bundle defined by the conjugate of gluing maps.

Example 32. For a complex vector line bundle $E \to B$. Then the complex line bundle $E \otimes \bar{E}$ is isomorphic to the trivial line bundle, since its gluing maps have values $z\bar{z} = 1$ for z a unit complex number.

Definition 33. Consider a vector bundle $E \to B$. The **exterior power** $\wedge^k(E)$ is a vector bundle generated by the exterior product of gluing maps.

Definition 34. Let F be a space. A **fiber bundle with fiber** F is a map $p : E \to B$ such that there is a cover of B by open sets U_α for each of which there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ taking $p^{-1}(b)$ to $\{b\} \times F$ for each $b \in U_\alpha$.

Example 35.

1. For a vector bundle E with an inner product, we can define a subspace $S(E)$ consisting of the unit spheres in all the fibers. Then the natural projection $S(E) \to B$ is a fiber bundle with sphere fibers, which is called a **sphere bundle**.

2. Similarly, we can define a **disk bundle** $D(E)$ consisting of the unit disks in all the fibers.

3. Indeed, we can define $S(E)$ and $D(E)$ without an inner product. For $S(E)$, we take the quotient of the complement of the zero section in E obtained by identifying each nonzero vector with all positive scalar multiples of itself. For $D(E)$, consider the mapping cylinder of the projection $S(E) \to B$, the quotient space of $S(E) \times [0,1]$ obtained by identifying two points in $S(E) \times \{0\}$ if they have the same image in B.

4. The canonical line bundle $E \to \mathbb{R}P^n$ has its unit sphere bundle $S(E)$ as the space of unit vectors in lines through the origin in $\mathbb{R}^{n+1}$. Since each unit vector uniquely determines the line containing it, $S(E)$ is S^n.

5. For a vector bundle $E \to B$, the **projective bundle**, $P(E) \to B$, is the fiber bundle where $P(E)$ is the space of all lines through the origin in all the fibers of E. We topologize $P(E)$ as the quotient of the sphere bundle $S(E)$ obtained by factoring out scalar multiplication in each fiber. Then the fibers are homeomorphic to $\mathbb{R}P^{n-1}$.

6. For an n-dimensional vector bundle $E \to B$, the **flag bundle** $F(E) \to B$ is a vector bundle with $F(E)$ the subspace of the n-fold product of $P(E)$, consisting of n-tubles of orthogonal lines in fibers of E. For any $k \leq n$, one could also take k-tuples of orthogonal lines in fibers of E, and get a bundle $F_k(E) \to B$.

7. The **Stiefel bundle** $V_k(E) \to B$ is a refinement of the flag bundle, that is, the points of $V_k(E)$ are k-tuples of orthogonal unit vectors in fibers of E.

8. The **Grassmann bundle** $G_k(E) \to B$, generalization of projective bundle, is the quotient space of Stiefel bundle $V_k(E)$, obtained by identifying two k-frames in a fiber if they span the same subspace of the fiber.

2019- 10- 23

Proposition 36. *Given a map $f : A \to B$ and a vector bundle $p : E \to B$, there is a vector bundle $p' : E' \to A$ with a map $f' : E' \to E$ taking the fiber of E' over each point $a \in A$ isomorphically onto the fiber of E over $f(a)$. Such a vector bundle E' is unique up to isomorphism.*

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ A & B \end{tikzcd} \tag{359}$$

Definition 37. The functor $\mathrm{Vect}^n : \mathsf{Top} \to \mathsf{Set}$ takes a topological set B to a set of real vector bundles over B. The functor $\mathrm{Vect}^n_{\mathbb{C}} : \mathsf{Top} \to \mathsf{Set}$ takes a topological set B to a set of complex vector bundles over B. The morphisms on Top becomes morphisms on Set which is described as above.

HYUNJUN PARK

MISCELLANEOUS THINGS

Miscellaneous thing

2019- 03- 09

Today I was quite busy finishing my small paper about Nagaoka's theorem, which takes whole time today from 3 p.m.(It's 11:40 already). And I need to solve some problems on *Quantum Phase Transitions* by Subir Sachdev. And I need to skim the *Algebraic Topology* by Alan Hatcher to do the mathematical physics seminar. And I need to study category theory. And I need to solve the SSL problems and Luttinger's theorem and so on. It could go crazy if I did not enjoy all those things. (Actually it is going crazy)

Anyway. No more works today. Maybe some Autochess and sleep then.

For the work I've done today, see Nagaoka's Theorem file.

Chern-Simons-Witten Theory

2019- 07- 10

Definition 1. Let M be a manifold and TM is the tangent space of M. Then a **differential k-form** on M, is a smooth section of $\wedge^k T^* M$.

Definition 2. Let M be a manifold and ω is a differential 2-form on M. Then ω is

1. **non-degenerate** if $\omega(X, Y) = 0$ for all $Y \in T_p M$ then $X = 0$.
2. **closed** if $d\omega = 0$.

Proposition 3. *Let M be a manifold and ω is a nondegenerate differential 2-form on M. Then M is a manifold with even dimension.*

Proof. Suppose that the dimension of M, n, is odd. Then using the matrix representation under local coordinate system, ω is a skew-symmetric matrix with size n. Now $\det(\omega) = \det(-\omega^T) = \det(-\omega) = (-1)^n \det(\omega) = -\det(\omega)$, thus $\det(\omega) = 0$, but ω is nondegenrate, contradiction. □

Definition 4. Let M be a $2n$ dimensional manifold and ω is a closed nondegenerate differential 2-form on M. Then (M, ω) is called a **symplectic manifold**.

Definition 5 (Classical Mechanics)**.** Let (M, ω) be a symplectic manifold. Choose $H \in C^\infty(M, \mathbb{R})$. Then a vector field X_H on M satisfying

$$dH(\cdot) = \omega(X_H, \cdot) \tag{360}$$

is called a **Hamiltonian vector field**.

Definition 6 (Poisson Bracket)**.** Let (M, ω) be a symplectic manifold. The bilinear skew-symmetric map

$$\{\cdot, \cdot\} : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \tag{361}$$

defined as

$$\{f, g\} = i_{v_f} \circ i_{v_g}(\omega) = \omega(v_f, v_g) \tag{362}$$

is called a **poisson bracket**.

Chern-Simons Theory

2019- 07- 13

Definition 7. A **fiber bundle** is a structure (E, B, π, F) where E, B, F are topological spaces and $\pi : E \to B$ is a continuous surjective map satisfying for every $x \in E$ there is an open neighborhood $U \subset B$ of $\pi(x)$ such that there is a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that $\mathrm{proj}_1 \circ \phi = \pi|_{\pi^{-1}(U)}$.

Definition 8. A **topological group** G is a topological space which is also a group such that the product map $(x, y) \mapsto xy$ and inverse map $x \mapsto x^{-1}$ are continuous.

Definition 9. Let M be a manifold and G be a topological group. A **principal G-bundle** is a fiber bundle $\pi : E \to M$ with a continuous right action $E \times G \to E$ such that

1. G preserves the fibers of P,
2. G acts on E freely and transitively.

Definition 10. Let (E, M) and (E', M') are principal G-bundles. If a map $\phi : E \to E'$ is a C^∞ map which commutes with right actions, then we call ϕ a **bundle map**.

Proposition 11. *Let (E, M) and (E', M') are principal G-bundles and $\phi : E \to E'$ is a bundle map. Then there is a unique map $f : M \to M'$ satisfying $\pi' \circ \phi = f \circ \pi$.*

Proof. Later. □

K-Theory of C^*-algebras in solid state physics

2019- 10- 18

This is the part of the digitalized version of the Bellissard's article written on 1986.

A real sample used in experimental solid state physics is made of a finite assembly of atoms bound together by electrostatic interactions. However, even though finite and relatively small(few millimeters in size, sometimes smaller in crystallography), any sample contains so many atoms that it is better described via the use of the infinite volume limit. On the other hand, it appears homogeneous at large scale while at atomic scale the disorder breaks any translation invariance.

When studying electronic properties of such a crystal, it is commonly admitted that a one electron approximation is usually quite good. Collective properties of the electrons gas are described through the model of a perfect Fermi gas. The disorder appears simply as external forces coming from the random positions of the atoms or impurities. In this set up, the Schrödinger operator for an electron, acting on the space $L^2(\mathbb{R}^D)$(D being the dimension of the crystal), is given by an effective Hamiltonian of the form:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \sum_i V_i(x - x_i) \tag{363}$$

where V_i is the potential created by the i-th atom or impurity and x_i denotes its position.

The disorder may have several sources. One is given by the randomness of the position of the atoms. This randomness may come from the occurrence of many defects due to the way the sample has been prepared. It may as well come from structural reasons, namely the thermodynamical equilibrium favors the occurence of some disorder: this is the case for amorphous materials(glasses) or quasi-crystals. Another source of disorder is also given by the impurities which modify the atomic potential at random positions.

In any case, to describe such a system in full generality, one usually introduces a probability space (Ω, Σ, μ), the points of which labelling the Hamiltonian and describing in an implicit way the configuration of the material. In other words H becomes a function of $\omega \in \Omega$. For obvious mathematical reasons, one demands that H be a measurable function of ωl at least in the strong resolvent sense(we recall that Borel sets are same for the norm and the strong topology in the algebra of bounded operator in a separable Hilbert space). To describe the macroscopic homogeneity of the material we just remark that translating the electron in the sample is equivalent to translating the atoms backward and since the sample looks almost the same at any place, this is just changing the configuration ω in Ω. Therefore, there must be an action $\omega \to T_x\omega$ of the translation group $\mathbb{R}^D$ on Ω such that, if $U(x)$ is the unitary operator representing the translation by x in the Hilbert space $L^2(\mathbb{R}^D)$ then

$$U(x)H_\omega U(x)^\dagger = H_{T_x\omega} \tag{364}$$

This action will be at least measurable and will satisfy the group property $T_xT_y = T_{x+y}$. As we shall see in the end of this section, we may assume Ω to be actually a compact space and the $\mathbb{R}^D$ action to be given by a group of homeomorphisms. The probability measure will serve later on in dealing with "self averaging" quantities.