DOT/SCALAR PRODUCT

Dot Product: Let $\bar{a} = (a_1, a_2, a_3), \bar{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbb{R}^3$. The scalar product of $\bar{a}$ and $\bar{b}$ is defined as follows: $\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\bar{a}||\bar{b}|\cos(\theta)$. Behaves like normal multiplication: positive definite, commutative, and distributive.

Norm: $|\bar{a}| = \sqrt{\bar{a} \cdot \bar{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$. Notice that $|\bar{a}| = 0 \Leftrightarrow \bar{a} = 0$

Angle between two vectors: $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos(\theta)$, θ is the angle between$\bar{a}, \bar{b}$. Alternatively: $\cos(\theta) = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|\,|\bar{b}|}$

Orthogonal Vectors: Two vectors $\bar{a}, \bar{b} \in \mathbb{R}^3$ are orthogonal if $\bar{a} \cdot\ \bar{b} = 0$ i.e. $\theta = \frac{\pi}{2}$

Unit Vector: A unit vector in space is a vector of length 1. In general, every vector in space $\bar{a} \in \mathbb{R}^3$ has magnitude $|\bar{a}|$and direction which is given by the unit vector $\frac{\bar{a}}{|\bar{a}|}$

VECTOR/CROSS PRODUCT

Cross product: $\bar{a} \times \bar{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (\,a_3b_1 - a_1b_3)\hat{\jmath} + (\,a_1b_2 - a_2b_1)\hat{k} = \left(|\bar{a}||\bar{b}|\cos(\theta)\right)\hat{n}\ (n \perp \bar{a}, \bar{b})$

Properties:

1. $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$ for all $\bar{a}, \bar{b}, \in \mathbb{R}^3$, in particular, $\bar{a} \times \bar{a} = 0$
2. **Theorem**: $\bar{a} \times \bar{b}$ is a vector perpendicular to $\bar{a}$ and $\bar{b}$. Suppose we have chosen a unit vector $\bar{n}$ perpendicular to the plane by the right hand rule: this means we choose $\bar{n}$ to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from $\bar{a}$ to $\bar{b}$. Then: $\bar{a} \times \bar{b} = \left(|\bar{a}||\bar{b}|\sin(\theta)\right)\bar{n}$
3. **Parallel**: $\bar{a}, \bar{b} \neq 0$ are parallel iff $\bar{a} \times \bar{b} = 0$
4. **Corollary**: $|\bar{a} \times \bar{b}|$ is the area of the parallelogram determined by $\bar{a}$ and $\bar{b}$, $|\bar{a}|$ being the base, and $|\bar{b}|\sin(\theta)$ the height.

LINES AND PLANES IN SPACE:

Cartesian/canonical equation of a line: A point $(x, y, z) \in \mathbb{R}^3$ is on the line iff $\frac{x - x_0}{s_1} = \frac{y - y_0}{s_2} = \frac{z - z_0}{s_3}$, where $M_0 = (s_1, s_1, s_3)$ = a given point, and a=x0,y0,z0=direction vector.

Parametric equation of a line: $L = \{(x, y, z) \in \mathbb{R}^3 | (x, y, z) = (x_0 + \lambda s_1, y_0 + \lambda s_2, z_0 + \lambda s_3), \lambda \in \mathbb{R}\}$
$L = \{\vec{r_0} + \lambda\vec{s} | \lambda \in \mathbb{R}\} = \{(x_0, y_0, z_0) + \lambda(s_1, s_2, s_3) | \lambda \in \mathbb{R}\} = \{(x_0 + \lambda s_1, y_0 + \lambda s_2, z + 0 + \lambda s_3) | \lambda \in \mathbb{R}\}$.
Where the vector $\vec{r_0}$ represents the shift from the origin and the vector $\vec{s}$ is the direction or orientation.

Vector Equation of the plane: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \ \Rightarrow \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$

Scalar equation of the plane: (Point on plane $P_0 = (x_0, y_0, z_0)$ and normal vector $\vec{n} = (a, b, c)$) the equation is: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, often written as $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$

Angle between two planes: Defined to be the angle between the two normal $\vec{n_1}, \vec{n_2}$, i.e. $\cos(\theta) = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$

Tangent plane: Plane P is tangent to a surface S at point p_0 if $p_0 \in P$ and for any point $p \in S$, the angle between the vector $p \to p_0$ and the plane P tends to 0 as p approaches p_0. If the tangent plane P to $f(x, y)$ at (x_0, y_0) exists, then P is given by: $z = f(x_0, y_0) + \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0)$

Tangent plane alternative: Let $f(x, y)$ be a function, and let (x_0, y_0) be a point in the function's domain of definition. Suppose that the partial derivatives exist and are continuous. The graph of the function is a surface in space.
For $z_0 = (x_0, y_0)$ the tangent plane at (x_0, y_0, z_0) is given by $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The normal to the plane is given by the vector $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$

SETS AND REGIONS

Level Sets: A level set of a real function f of n variables is of the form:$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | f(x_1, x_2, \dots, x_n) = C\}$. When $n = 2$, the level sets are called level curves or contour lines, i.e. for some $z = f(x, y)$, the contour lines are all $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = C$ for some $C \in \mathbb{R}$. For $n = 3$, level sets are called level surfaces, i.e. for $\omega = f(x, y, z)$ the level surfaces are given by the equations $f(x, y, z) = a$ for some $a \in \mathbb{R}$. The level curves of the surface $f(x, y, z) = a$ are the curves we get for $z = k$

Closed and Open Regions: A region in $\mathbb{R}^2$ is closed if it includes its boundary. A region is called open if it doesn't include any of its boundary points.

Bounded Region: A region in $\mathbb{R}^2$ is called bounded if it can be completely contained in a disk. In other works, a region is bounded if it is finite.

Closed Path: A path C is called closed if its initial and final points are the same point. For example, a circle is a closed path.

Simple path: A path C is simple if it doesn't cross itself. A circle is a simple curve, while a figure 8 is not.

Open Region: A region D is open if it doesn't contain any of its boundary points.

Connected Region: A region D is connected if we can connect any two points in the region with a path that lies completely in D.

Simply-Connected Region: A region D is simply-connected if it is connected and contains no holes.

LIMITS:

Limit Definition: Let $f(x, y)$ defined on some neighborhood of a point $P_0 = (x_0, y_0)$, except maybe at P_0. We say that L is the limit of $f(x, y)$ as $P \to P_0$, and denote $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|P_0 - P| < \delta \Rightarrow |f(P) - L| < \varepsilon$.

Iterated limits: The limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ is the Double Limit. The limits $\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x, y)\right)$, $\lim_{y\to y_0}\left(\lim_{x\to x_0} f(x, y)\right)$ are called iterated limits.

Existence of Double and Iterated Limits: If the Double Limit exists, and the Iterated Limits exist, then they are all equal to each other. When Iterated limits exist but are different, the Double Limit does not exist. On the other hand, the existence and equality of the Iterated Limits does not imply the existence of the Double Limit.

Disproving existence of limits: By definition, the limit needs to be unique regardless of the path we take to approach the limit. For disproving the existence of a limit it is enough to find two different paths which give us two different limits.

Disproving existence of limits #2: We can make a polar change of variable: $x = rcos(\theta), y = rsin(\theta)$. If $(x, y)0 > (0,0), then\ r \to 0$. When the value of the limit depends on θ, this is equivalent to saying that the limit depends on the path, and thus does not exist.

Limit Properties:	
Suppose that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, and $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = K$, then:	**Sandwich Theorem:** Let I be an interval, and g, f, h be function on I. Suppose that for $\forall\ x \neq a \in I$, where a is the limit point, we have $g \le f \le h$. Also suppose that $\lim(g) = \lim(h) = L$. Then: $\lim_{x\to a} f = L$
$\lim_{(x,y)\to(x_0,y_0)} \alpha f(x, y) = \alpha L$	
$\lim_{(x,y)\to(x_0,y_0)} f(x, y)g(x, y) = LK$	
$\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K$	
$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \pm g(x, y) = L \pm K$	

Method to check Existence of limit: (1) Substitute the limit point into (x, y), if defined – limit exists. **(2)** Approach the limit from each axis by setting the other to 0. Compare the result to approaching from the other direction. If not equal, then limit does not exists. **(3)** Approach (x, y) from any nonvertical line by setting $y = mx$ and taking the limit as $x \to a$. If this limit depends on m, then it does not exists. **(4)** Rewrite $f(x, y)$ in cylindrical coordinates, and take limit $r \to a$. If this does not exists, limit of $f(x, y)$ does not exists.

LIMIT CALCULATION TECHNIQUES:

Sandwich Theorem: Let I be an interval, having the point a as a limit point. Let g, f, h be functions defined on I, except perhaps at a itself. Suppose that for every $x \neq a \in I$, we have $g(x) \le f(x) \le h(x)$. Also suppose that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} f(x) = L$. The functions g and h are said to be lower and upper bounds (respectively) of f. Also, a is *not* required to lie in the interior of I – if it is an endpoint, then the above limits are left or right-hand limits. This theorem is also valid for sequences: Let $(a_n), (c_n)$ be two sequences converging to l, and (b_n) a sequence. If $\forall n \ge N, N \in \mathbb{N}$, we have $a_n \le b_n \le c_n$, then (b_n) also converges to l.

Polar coordinates: Suppose that for every θ we have $|f(rcos(\theta), rsin(\theta) - A| \le \phi(r)$. If $\lim_{r\to 0} \phi(r) = 0$ then $\lim_{(x,y)\to(0,0)} f(x, y) = A$

DIFFERENTIABILITY AND CONTINUITY:

Continuity: $f(x, y)$ is continuous at the point (x_0, y_0) if $\lim_{(x,y)\to(0,0)} f(x, y)$ exists and equal to $f(x_0, y_0)$

Differentiability: A function $f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0) \cdot \Delta x + f_y(x_0, y_0) \cdot \Delta y + \varepsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2}$, where $\varepsilon(\Delta x, \Delta y) \to 0$ for $\Delta x, \Delta y \to 0$.

Existence of Partials: For $f(x, y)$ differentiable at (x_0, y_0), the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. On the other hand, the existence of partials doesn't imply differentiability of the function.

Partials & Differentiability: If the partials of $f(x, y)$are continuous at (x_0, y_0), then $f(x, y)$ is differentiable at (x_0, y_0).

Differentiability →Continuity: If a function $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

DIRECTIONAL DERIVATIVES:

Directional Derivative: The directional derivative of $f(x, y)$ at the point $a = (x_0, y_0)$ in the direction of the unit vector $\bar{v}$, $(|v| = 1)$ is defined in the following way: $D_v f(a) = \lim_{t\to 0} \frac{f(a+tv) - f(a)}{t}$. When $f(x, y)$is differentiable at the point a, we have $D_v f(a) = \nabla f(a) \cdot \bar{v}$. The maximum value occurs when $\bar{v}$ is parallel to $\nabla f(x, y, z)$, and is given by $|\nabla f(x, y, z)|$

Alt. Notation: Often written as $\frac{d}{ds} = \hat{s} \cdot \nabla = s_x \frac{\partial}{\partial x} + s_y \frac{\partial}{\partial y} + s_z \frac{\partial}{\partial z}$, where $\bar{s}$ denotes a unit vector in any given direction.

TAYLOR'S FORMULA:

$$f(a + h, b + k) = \sum_{i=0}^{n} \left(\frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(a, b) \right) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a + ch, b + ck), \qquad 0 < c < 1$$

PARTIAL DERIVATIVES:

Formal Definition: $f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$, $f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$

Implicit Differentiation: Start with a function in the form $F(x,y)=0$ (or bring it to this form), where $y=y(x)$. If you are looking for $\frac{dy}{dx}$, differentiate both sides with respect to x, to get $F_x + F_y\frac{dy}{dx} = 0$. Then isolate $\frac{dy}{dx}$, and you're done.

Implicit Differentiation 3D: Start in the form $F(x,y,z)=0$, assume that $z=f(x,y)$, and we want to find $\frac{\partial z}{\partial x}$. Start by trying to find $\frac{\partial z}{\partial x}$. Differentiate both sides with respect to x, treating y as a constant. Using the chain rule: $\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$. Now we have $\frac{\partial x}{\partial x} = 1$, $\frac{\partial y}{\partial x} = 0$. Plugging in and solving for $\frac{\partial z}{\partial x}$ gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

CRITICAL POINTS:

Critical point: The point (a,b) is a critical point of $f(x,y)$ provided one of the following is true: **(1)** $\nabla f(a,b) = \bar{0}$, **(2)** One or both partial derivate don't exist.

Extreme Value Theorem: If $f(x,y)$ is continuous in a closed, bounded set $D \in \mathbb{R}^2$, then there are points in D, (x_1,y_1) and (x_2,y_2) so that $f(x_1,y_1)$ is the absolute maximum, and $f(x_2,y_2)$ is the absolute minimum of the function in D

Local Extrema: If (x_0,y_0) is a local extrema of $f(x,y) \in C^1$, then$\nabla f(x_0,y_0) = 0$, and all partials are zero.

Classification: Denote $A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, $C = f_{yy}(x_0,y_0)$.

$AC-B^2>0$	$AC-B^2<0$	$AC-B^2=0$	$A>0$	$A<0$
(x_0,y_0) local extremum	Not a local extrema	Unknown. Maybe Inflection	Minimum	Maximum

Finding Absolute Extrema: (1) Find all the critical points of the function that lie in D, and determine the function's value at each of these points. **(2)** Find all extrema at the boundary. **(3)** The largest and smallest values found in (1), (2), are the absolute extrema of the function.

Lagrange Multipliers: We want to find the minimum and maximum of a function $f(x,y,z)$, under the constraint $g(x,y,z)=k$. The constraint may be the equation that describes the boundary of a region, or something else. **(1).** Solve the following system of equations: $\begin{cases}\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k\end{cases}$. **(2).** Plug in all the solutions (x_0,y_0,z_0) into $f(x,y,z)$ and identify the minimum and maximum values, provided they exist. The constant λ is called the Lagrange Multiplier.

Lagrange Multipliers explanation: System actually has four equations. By the definition of the gradient vector: $\langle f_x, f_y, f_z\rangle = \lambda\langle g_x, g_y, g_z\rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z\rangle$. For these to be equal, individual components must be equal. We have three equations: $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$. These, along with the constraint $g(x,y,z)=k$ give four equations with four unknowns, x,y,z,λ. In some cases minimums and maximums won't exist even though the method will imply that they do.

SINGLE INTEGRALS:

Surface Area: If rotation about x-axis, if $y=f(x), a\le x\le b$, then $S = \int 2\pi y ds$, where $ds = \sqrt{1+y''}dx$. If rotation about y-axis, if If $x=h(y), c\le y\le d$, then $S=\int 2\pi x ds$, where $ds = \sqrt{1+x''}dy$

Indefinite Integrals of Multivariate Functions: Suppose we have $\int x^3 - e^{-\frac{x}{y}}dx$. It is equal to $\frac{1}{4}x^4 + ye^{-\frac{x}{y}} + h(y)$. As seen , the "constants" of integration are now functions of the opposite variable.

Improper integrals: If $f(x)$ exists for every $t>a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$

Discontinuous Integrand: $\int_a^b f(x)dx =$

$f(x)$ is continuous on $[a,b)$, not continuous at $x=b$	$\lim_{t\to b^-}\int_a^t f(x)dx$
$f(x)$ is continuous on $(a,b]$, not continuous at $x=a$	$\lim_{t\to a^+}\int_t^b f(x)dx$
$f(x)$ Not continuous at $x=c$, $(a<c<b)$, And $\int_a^c f(x)dx$ and $\int_b^c f(x)dx$ are both convergent.	$\int_a^c f(x)dx + \int_c^b f(x)dx$
$f(x)$ Not continuous at $x=a, x=b$, And $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are both convergent	$\int_a^c f(x)dx + \int_c^b f(x)dx$

Integral Approximation: We want to approximate $\int_a^b f(x)dx$

Midpoint rule: Divide the interval $[a,b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$.	$\approx \Delta x[f(x_1^*+f(x_2^*)+\cdots+f(x_n^*)]$ $[x_0,x_1],[x_1,x_2]\ldots,[x_{n-1},x_n]$ where $x_0=a, x_n=b$.For each interval, let x_i^* be the midpoint of the interval.
Trapezoid rule: Divide the interval $[a,b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$	$\approx \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots 2f(x_{n-1})+f(x+n)$
Simpson's rule: Divide the interval $[a,b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$, we require that n is even.	$\approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)$

DOUBLE INTEGRALS:

Iterated Integrals (Fubini's Theorem): If $f(x,y)$ is continuous on a rectangular domain $R=[a,b]\times[c,d]$, then $\iint_R f(x,y)dA = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$. These are called iterated integrals.

Iterated integrals Separation of Variables: If $f(x,y) = X(x)Y(y)$, and we are integrating over the rectangle $R=[a,b]\times[c,d]$, then $\iint_R f(x,y) = \iint_R X(x)Y(y)dA = \left(\int_a^b X(x)dx\right)\left(\int_c^d Y(y)dy\right)$

Volume of a Solid: The volume of a solid that lies below the surface $z=f(x,y)$ and above the region D in the xy plane is given by $V = \iint_D f(x,y)dA$

Double Integrals in Polar Coordinates: $\iint_D f(x,y)dA = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))r\,drd\theta$. The dA, which is $dA=dxdy$ in Cartesian coordinates, becomes $dA = r\,drd\theta$ in polar coordinates. The bounds of integration are the polar boundaries of the domain, and <u>NOT</u> the Cartesian bounds.

Surface Area: We want to find the surface area of the surface given by $z=f(x,y)$, where (x,y) is a point from region D in the xy plane. In this case the surface area is given by: $S = \iint_D\sqrt{(f_x)^2+(f_y)^2+1}dA$

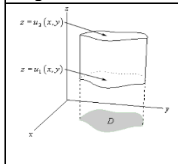
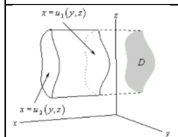
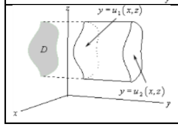
Properties of Double Integrals:

$\iint_D f(x,y)+g(x,y)dA = \iint_D f(x,y)DA + \iint_D g(x,y)dA$	$\iint_D cf(x,y)dA\, c\iint_D f(x,y)dA$, where c is any constant.

If D can be split into separate regions D_1 and D_2, can rewrite integral: $\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$

TRIPLE INTEGRALS:

Iterated Triple Integrals (Fubini's Theorem): If $f(x,y)$ is continuous on a box domain $B=[a,b]\times[c,d]\times[r,s]$, then$\iiint_B f(x,y,z)dV = \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz$. These are called iterated integrals.

Region	*Region Definition*	*Integral*
	$E=\{(x,y,z)\mid(x,y)\in D, u_1(x,y)\le x\le u_2(x,y)\}$ The region D is a region in the xy plane.	$\iiint_E f(x,y,z)dV = \iint_D\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right]dA$ Where we can use either iteration order for integrating D in the xz plane, or we can use polar coordinates if needed.
	$E=\{(x,y,z)\mid(y,z)\in D, u_1(y,z)\le x\le u_2(y,z)\}$ The region D is a region in the yz plane.	$\iiint_E f(x,y,z)dV = \iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right]dA$ Where we can use either iteration order for integrating D in the xz plane, or we can use polar coordinates if needed.
	$E=\{(x,y,z)\mid(x,z)\in D, u_1(x,z_{\le y}\le u_2(x,z)\}$ The region D is a region in the xz plane.	$\iiint_E f(x,y,z)dV = \iint_D\left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right]dA$ Where we can use either iteration order for integrating D in the xz plane, or we can use polar coordinates if needed.

Triple Integrals in Cylindrical Coordinates: To do the integral in cylindrical coordinates, we <u>convert</u> $dV = dxdydz$ to $dV = rdzdrd\theta$. The <u>region over which we are integrating</u> changes from $E=\{(x,y,z)\mid(x,y)\in D, u_1(x,y)\le x\le u_2(x,y)\}$ to $E=\{(r,\theta z)\mid \alpha\le\theta\le\beta, h_1(\theta)\le r\le h_2(\theta), u_1(r\cos(\theta, r\sin(\theta)\le z\le u_2(rcos(\theta, rsin(\theta)\}$. This specifically covers the case when D is in the xy plane, but can be modified if it is in some other plane. In terms of <u>cylindrical coordinates, a triple integral is</u>: $\iiint_E f(x,y,z)dV = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(rcos(\theta),rsin(\theta))}^{u_2(rcos(\theta),rsin(\theta))} rf(rcos\theta, rsin\theta, z)dzdrd\theta$.

Don't forget to add the r and make sure that all x's and y's also get converted to cylindrical coordinates.

Triple Integrals in Spherical Coordinates: After converting to Spherical coordinates, due to their nature we put the following restrictions on the coordinates: $\rho\ge 0, 0\le\varphi\le\pi$. For our integral, the variables will take the ranges $\le\rho\le b$, $\alpha\le\theta\le\beta, \delta\le\varphi\le\gamma$. Convert $dV = dxdydz$ to $dV = \rho^2\sin(\varphi)\,d\rho d\theta d\varphi$. Therefore the <u>integral becomes</u>: $\iiint_E f(x,y,z)dV = \int_\delta^\gamma\int_\alpha^\beta\int_a^b \rho^2\sin(\varphi)f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi))d\rho d\theta d\varphi$.

Volume and Area:	
Area of the region D	Area of D $=\iint_D dA$
Volume of the region E	Volume of E $=\iiint_E dV$
Volume of the region E ,if it can be defined as the region under the function $z=f(x,y)$ and above the region D in the xy plane	Volume of E$=\iint_D f(x,y)dA$

MULTIPLE INTEGRALS CHANGE OF VARIABLE:

Transformation: Equations that define a change of variables are called a transformation. Typically, we start out with a region, R, in xy coordinates, and transform it into a region in uv coordinates.

Jacobian of a Transformation (2D): Jacobian of transformation $\begin{matrix} x = g(u,v) \\ y = h(u,v) \end{matrix}$: $J_{\text{trans.}} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$.

Change of Variables (Double Integral): We want to integrate $f(x,y)$ over the region R. Under the transformation $\begin{matrix} x = g(u,v) \\ y = h(u,v) \end{matrix}$, the region becomes S and the integral becomes $\iint_R f(x,y)dA = \iint_S f\big(g(u,v),h(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$. If we look at the differentials in this formula, we can also say that $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$

Jacobian of a Transformation (3D): Jacobian of transformation $\begin{matrix} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{matrix}$: $J_{\text{trans.}} = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$.

Change of Variables (Triple Integral): The integral of $f(x,y,z)$ under the transformation given in [Jacobian of a Transformation (3D)] is: $\iiint_R f(x,y,z)dV = \iiint_S f\big(g(u,v,w),h(u,v,w),k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$. Looking at differentials, note that $dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$

LINE INTEGRALS:

Vector Field: A vector field on two (or three) dimensional space is a function $\bar{F}$ that assigns to each point (x,y) (or (x,y,z)) a two (or three) dimensional vector given by $\bar{F}(x,y)$ (or $\bar{F}(x,y,z)$. The standard notation for the 3D function $\bar{F}$ is $\bar{F}(x,y) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$. The functions P,Q,R are sometimes called scalar functions.

Gradient Vector: Given a function $f(x,y,z)$, the gradient vector is defined as $\nabla f = \langle f_x, f_y, f_z\rangle$. This is a vector field and if often called a gradient vector field.

Conservative Vector Field: Vector field $\bar{F}$ is called a conservative vector field if there exists a function f such that $\bar{F} = \nabla f$. If $\bar{F}$ is a conservative vector field, then the function f is called the potential function for $\bar{F}$. What this is saying, is that a vector field is conservative if it is also a gradient vector field for some function.

Parametric Equations: Instead of defining y in terms of x $\big(y = f(x)\big)$ or x in terms of y $\big(x = h(y)\big)$ we define both x and y in terms of a third variable called a parameter as follows: $x = f(t), y = g(t)$. Sometimes we will restrict the values of t, and sometimes we won't, This will depend on the problem.

Parameterization: Given a function/equation we want to write a set of parametric equations for it.

Curve		Parametric Equations	
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Counter Clockwise: $x = a\cos(t)$ $y = b\sin(t)$ $0 \le t \le 2\pi$	Clockwise: $x = a\cos(t)$ $y = -b\sin(t)$ $0 \le t \le 2\pi$
Circle	$x^2 + y^2 = r^2$	Counter Clockwise: $x = r\cos(t)$ $y = r\sin(t)$ $0 \le t \le 2\pi$	Clockwise: $x = r\cos(t)$ $y = -r\sin(t)$ $0 \le t \le 2\pi$
Line Segment	(x_0,y_0,z_0) to (x_1,y_1,z_1)	Form 1: $\bar{r}(t) = (1-t)\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $0 \le t \le 1$	Form 2: $x = (1-t)x_0 + tx_1$ $y = (1-t)y_0 + ty_1$ $z = (1-t)z_0 + tz_1$
$y = f(x)$		$x = t, y = f(t)$	
$x = g(y)$		$x = g(t), y = t$	

Smooth Curve: A curve is called smooth if $\bar{r}'(t)$ is continuous and $\bar{r}'(t) \neq 0$ for all t.

Piecewise Smooth Curve: A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves, $C_1, \dots, C_n$, where the end point of C_i is the starting point of C_{i+1}.

Line Integrals With Respect to Arc Length (ds):

Line integral: Let's start with the curve C, along which we want to integrate. It needs to be smooth. Suppose C is given by the parametric equations $\begin{matrix} x = h(t) \\ y = g(t) \end{matrix}$, $a \le t \le b$. We will often want to write the parameterization of the curve as a vector function, which in this case is $\bar{r}(t) = h(t)\hat{\imath} + g(t)\hat{\jmath}$, $a \le t \le b$. The line integral of $f(x,y)$ along C is denoted by $\int_C f(x,y)ds$.

Meaning of ds: We use a ds here to acknowledge that we are moving along a curve C, instead of the x −axis (denoted by dx) or the y −axis (denoted by dy). Because of the ds, this is sometimes called the line integral of f with respect to arc length. Recall from the formula for arc length of a parametric curve, $L = \int_a^b ds$, where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. This ds is the same for the line integral.

Computing Line Integrals: To compute a line integral we will convert everything over to parametric equations $\begin{pmatrix} x = h(t) \\ y = g(t) \end{pmatrix}, a \le t \le b$, so the line integral is then: $\int_C f(x,y)ds = \int_a^b f\big(h(t),g(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. Don't forget to plug the parametric equations into the function as well.

Computing Line Integrals (Vector notation): IF we use the vector form of the parameterization, we can simplify the notation by noticing that $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \|\bar{r}'(t)\|$, where $\|\bar{r}'(t)\|$ is the magnitude of the norm of $\bar{r}'(t)$. Using this notation, the integral becomes $\int_C f(x,y)ds = \int_a^b f\big(h(t),g(t)\big)\|\bar{r}'(t)\|dt$. Note that as long as the parameterization of the curve C is traced out exactly once as t increases from a to b, the value of the line integral will be independent of the parameterization of the curve.

Line Integrals over Piecewise Smooth Curves: All we do is evaluate the line integral over each of the pieces and then add them up. $\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \cdots + \int_{C_n} f(x,y)ds = \sum_{i=1}^{n}\left(\int_{C_i} f(x,y)ds\right)$

Line Integrals Direction of Curve: Suppose curve C has parameterization $\begin{matrix} x = h(t) \\ y = g(t) \end{matrix}$, with initial point on the curve A and final point B. The parameterization will determine the orientation of the curve, where the positive direction is the direction that is traced out as t increases. Let $-C$ be the curve with the same points as C, however this curve has B as the initial point and A as the final point. Again , t is increasing as we traverse this curve. In other words, given a curve C, the curve $-C$ is the same curve, except the direction is reversed. Then we have the following fact about line integrals w.r.t to arc length: $\int_C f(x,y)ds = \int_{-C} f(x,y)ds$. So, we can change the direction of the curve and not change the value of the integral.

Line Integrals in 3D: $\int_C f(x,y,z)ds = \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$. Note that often when dealing with 3D space, the parameterization will be given as a vector function $\bar{r}(t) = \langle x(t), y(t), z(t)\rangle$. Notice that $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \|\bar{r}'(t)\|dt$. Hence, alternatively: $\int_C f(x,y,z)ds = \int_a^b f\big(x(t),y(t),z(t)\big)\|\bar{r}'(t)\|dt$

Line Integrals With Respect to Variables (dx, dy…)

Line Integrals With Respect to Variables (Not Arc Length): Suppose we have a curve C with parameterization $\begin{matrix} x = h(t) \\ y = g(t) \end{matrix}$, $a \le t \le b$. The line integral of f with respect to x is: $\int_C f(x,y)dx = \int_a^b f\big(x(t),y(t)\big)x'(t)dt$. The Line integral of f with respect to y is: $\int_C f(x,y)dy = \int_a^b f\big(x(t),y(t)\big)y'(t)dt$. These two integrals often appear together, and have the shorthand notation: $\int_C Pdx + Qdy = \int_C P(x,y)dx + \int_C Q(x,y)dy$. Note that the only notational difference between these two and the line integral w.r.t arc length (from previous sections) is the differential. These have a dx or dy, as opposed to ds.

Relationship with Direction of Curve: If C is any curve, then, $\begin{matrix} \int_{-C} f(x,y)dx = -\int_C f(x,y)dx \\ \int_{-C} f(x,y)dy = -\int_C f(x,y)dy \end{matrix}$. With the combined form of these two integrals we get: $\int_{-C} Pdx + Qdy = -\int_C Pdx + Qdy$

Line Integrals W.R.T variables in 3D: Suppose curve $C$is parametrized by $\begin{matrix} x = x(t) \\ y = y(t), a \le t \le b \\ z = z(t) \end{matrix}$, then the line integral is: $\int_C f(x,y,z)dx = \int_a^b f\big(x(t),y(t),z(t)\big)x'(t)dt$. This also holds for y and z, just replace x with the variable in question. As with the two dimensional version, these three will also often occur together, and the shorthand we will be using is $\int_C Pdx + Qdy + Rdz = \int_C P(x,y,z)dx + \int_C Q(x,y,z)dy + \int_C R(x,y,z)dz$

Line Integrals of Vector Fields:

Line Integrals for Vector Fields: Suppose we have a vector field $\bar{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$, and the three dimensional, smooth curve is given by $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, $a \le t \le b$. Then, the line integral of $\bar{F}$

along C is $\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)dt$. Note the notation, "·" is a dot product, and the differential is a vector. Also, $\bar{F}(\bar{r}(t))$ is a shorthand for $\bar{F}(x(t), y(t), z(t))$.

Line Integral of Vector Fields w.r.t Arc Length: We can write line integrals of vector fields as line integrals with respect to arc length as follows: $\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{T}\, ds$, where $\bar{T}(t)$ is the unit tangent vector given by $\bar{T}(t) = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|}$. Using knowledge on how to compute integrals w.r.t arc length we see that this is equivalent to the original form: $\int_C \bar{F} \cdot \bar{T}\, ds = \int_a^b \bar{F}(\bar{r}(t)) \cdot \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|}\|\bar{r}'(t)\|dt = \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)dt$. In general, we use the first form for computation, as it's easier.

Relating Line Integrals of Vector Fields to Line Integrals w.r.t x, y, and z: Given the vector filed $\bar{F}(x, y, z) = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ and the curve C is parametrized by $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, $a \le t \le b$, then the line integral is $\int_C \bar{F} \cdot d\bar{r} = \int_a^b (P\hat{\imath} + Q\hat{\jmath} + R\hat{k}) \cdot (x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k})dt = \int_a^b Px' + Qy' + Rz'dt = \int_a^b Px'dt + \int_a^b Qy'dt + \int_a^b Rz'dt = \int_C Pdx + \int_C Qdy + \int_C Rdz = \int_C Pdx + Qdy + Rdz$.So we see that: $\int_C \bar{F} \cdot d\bar{r} = \int_C Pdx + Qdy + Rdz$. Note that this gives us another method for evaluating Line Integrals of Vector fields. This also allows us to state the following about the direction of the path of these integrals: $\int_{-C} \bar{F} \cdot d\bar{r} = -\int_C \bar{F} \cdot d\bar{r}$

Fundamental Theorem For Line Integrals:

Fundamental Theorem For Line Integrals: Suppose that C is a smooth curve given by $\bar{r}(t), a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then: $\int_C \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a))$. Note that $\bar{r}(a)$ represents the initial point on C, while $\bar{r}(b)$ represents the final point on C. Also, we didn't specify the number of variables for the function since the theorem holds regardless of the number of variables in the function.

Fundamental Theorem For Line Integrals Takeaways: The most important idea is that to evaluate these kinds of line integrals, we don't need to know the path to get the answer. In other words, we could use any path we want and we'll always get the same results. In earlier discussion of line integrals, we saw that often changing the path will change the value of the line integral. We now have a type of line integral for which changing the path will NOT change the value.

Line Integral Path Independence: $\int_C \bar{F} \cdot d\bar{r}$ is independent of path if $\int_{C_1} \bar{F} \cdot d\bar{r} = \int_{C_2} \bar{F} \cdot d\bar{r}$ for any two paths C_1 and C_2 in D, with the same initial and final points.

Line Integral Path Independence: By the Fundamental Theorem for Line Integrals, $\int_C \nabla f \cdot d\bar{r}$ is independent of path.

Conservative Field and Line Integral Path Independence: If $\bar{F}$ is a conservative vector field, then $\int_C \bar{F} \cdot d\bar{r}$ is independent of path. This is because if $\bar{F}$ is conservative, then it has a potential function f. Therefore, the line integral becomes $\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r}$, which, by the Fundamental Theorem for Line Integrals, is independent of path.

Line Integral Path Independence → Conservative Field: If $\bar{F}$ is a continuous vector field on an open connected region D, and if $\int_C \bar{F} \cdot d\bar{r}$ is independent of path (for any path in D), then $\bar{F}$ is a conservative vector field on D.

Path Independent Line Integral for Closed Curves: If $\int_C \bar{F} \cdot d\bar{r}$ is independent of path, then $\int_C \bar{F} \cdot d\bar{r} = 0$ for every closed path C. The reverse also holds: If $\int_C \bar{F} \cdot d\bar{r} = 0$ for every closed path C, then $\int_C \bar{F} \cdot d\bar{r}$ is independent of path.

Conservative Vector Fields:

Conservative Vector Field: Suppose that $\bar{F}$ is a continuous vector field in some domain D. $\bar{F}$ is a <u>conservative vector field</u> if there is a function f such that $\bar{F} = \nabla f$. The function f is called a <u>potential function</u> for the vector field.

Determining Conservatism in 2 Dimensions: Let $\bar{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field in an open and simply-connected region D. Then if P and Q have continuous first order partial derivatives in D, and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the vector field $\bar{F}$ is conservative.

Finding the Potential Function for a Vector Field (2D): First, assume that the vector field is conservative, so we know that a potential function $f(x, y)$ exists. We can then say that $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = P\hat{\imath} + Q\hat{\jmath} = \bar{F}$. By setting components equal, we have $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. By integrating each of these with respect to the appropriate variable we can arrive at the following two equations: $f(x, y) = \int P(x, y)dx$, $f(x, y) = \int Q(x, y)dy$. Note: If it is given that a 3D Vector field is conservative, we use the same process to arrive at $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k} = \bar{F}$

Green's Theorem:

Positive Orientation: We use the convention that the curve C has a positive orientation if it is trace out in a counter-clockwise direction (does not hold for regions with holes). Another way to think of a positive orientation is that as we traverse the path following the positive orientation, the region D enclosed by the path must always be on the left.

Green's Theorem: Let C be a positively oriented, piecewise smooth, simple, closed curve in the plane, and let D be the region enclosed by the it. If P and Q have continuous first order partials on D, then: $\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

Green's Theorem Notation: When working with a line integral in which the path satisfies the conditions of Green's Theorem, we will often denote the line integral as $\oint_C Pdx + Qdy$. Also, sometimes the curve C is not thought of as a separate curve, but instead as the boundary of some region D, and in these cases you may see C denoted as ∂D.

Green's Theorem for Domains With Holes: If Green's Theorem does not apply to a region D by default, you can slice the region into subregions $D = D_1 \cup D_2 \cup \ldots \cup D_n$ where each sub region on its own satisfies Green's Theorem. We can then break up line integrals into line integrals on each piece of the boundary. Of course, we also have to slice up the curves to fit our sliced domains. Boundaries that have the same curve, but in opposite directions will cancel.

Area Using Green's Theorem: Recall that we can determine the area of a region D with the double integral $A = \iint_D dA$. Let's think of this double integral as the result of Greens Theorem, i.e. assume that $Q_x - P_y = 1$. There are many functions that satisfy this, such as $\begin{matrix} P = 0 \\ Q = x \end{matrix}$, $\begin{matrix} P = -y \\ Q = 0 \end{matrix}$, $\begin{matrix} P = -y/2 \\ Q = x/2 \end{matrix}$. If we use Green's Theorem in reverse, we see that the area of region D can also be found by evaluating any of the following line integrals: $\oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx$. Where C is the boundary of region D.

Green's Theorem Curl Form: $\oint_C \bar{F} \cdot d\bar{r} = \iint_D \mathrm{curl}(\bar{F}) \cdot \hat{k}dA$, where $\hat{k}$ is standard unit vector in positive z direction.

Green's Theorem Divergence Form: $\oint_C \bar{F} \cdot \hat{n}ds = \iint_D \mathrm{div}(\bar{F})dA$

Outward Unit Normal to Curve C: Suppose curve C is parameterized by $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. Then, the outward unit normal is given by $\hat{n} = \frac{y'(t)}{\|\bar{r}'(t)\|}\hat{\imath} - \frac{x'(t)}{\|\bar{r}'(t)\|}\hat{\jmath}$

<u>CURL AND DIVERGENCE:</u>

Nabla (∇) Operator: $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$. Applied to a function : $\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$.

Laplace (∇^2) Operator: Take a look at $\mathrm{div}(\nabla f) = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{xx}$. The Laplace operator is defined as $\nabla^2 = \nabla \cdot \nabla$

Curl Definition: For $\bar{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$: $\mathrm{curl}(\bar{F}) = \nabla \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}$

Curl and Partial Derivatives: If $f(x, y, z)$ has continuous second order partial derivatives, then $\mathrm{curl}(\nabla f) = \bar{0}$

Curl, Given Conservative Fields: If $\bar{F}$ is a conservative vector field, then $\mathrm{curl}(\bar{F}) = \bar{0}$

Conservative Fields, given Curl: If $\bar{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous first order partial derivatives, and if $\mathrm{curl}\ (\bar{F}) = \bar{0}$, then $\bar{F}$ is a conservative vector field.

Divergence: Given vector field $\bar{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$. Divergence is defines as $\mathrm{div}(\bar{F}) = \nabla \cdot \bar{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

<u>SURFACES AND SURFACE INTEGRALS:</u>

Surfaces:

Closed Surface: A Surface S is closed if it is the boundary of some solid region E. A good example of a closed surface is the surface of a sphere.

Surface Orientation: The closed surface S has a <u>positive orientation</u> if we choose the set of unit normal vectors that point outward from the region E. <u>Negative orientation</u> will be the set of unit normal vectors that point in towards E. This convention is only used for closed surfaces.

Parametric Representation of A Surface: When we parameterized a curve, we took values of t from some interval $[a, b]$ and plugged them into $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$, and the resulting set of vectors was the position vectors for the points on the curve. With surfaces, we do something similar. We will take points (u, v) out of some 2-dimensional space D and plug them into $\bar{r}(u, v) = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}$. The resulting set of vectors will be the position vectors for the points on the surface S that we are trying to parameterize. This is called the parametric representation of surface$S$.

Parametric Equations for a Surface: These are the components of the parametric representation: $\begin{matrix} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{matrix}$.

Parameterizing a Surface: If we have a function $f(x, y, z)$, in basic form: $\begin{matrix} z = f(x, y) \Rightarrow \bar{r}(x, y) = x\hat{\imath} + y\hat{\jmath} + f(x, y)\hat{k} \\ x = f(y, z) \Rightarrow \bar{r}(y, z) = f(y, z)\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ y = f(x, z) \Rightarrow \bar{r}(x, z) = x\hat{\imath} + f(x, z)\hat{\jmath} + z\hat{k} \end{matrix}$

Tangent Plane to Parametric Surface: A parametric surface S is given by $\bar{r}(u, v) = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}$. First, define $\bar{r}_u(u, v) = \frac{\partial x}{\partial u}(u, v)\hat{\imath} + \frac{\partial y}{\partial u}(u, v)\hat{\jmath} + \frac{\partial z}{\partial u}(u, v)\hat{k}$, and $\bar{r}_v(u, v) = \frac{\partial x}{\partial v}(u, v)\hat{\imath} + \frac{\partial y}{\partial v}(u, v)\hat{\jmath} + \frac{\partial z}{\partial v}(u, v)\hat{k}$. Now, provided that $\bar{r}_u \times \bar{r}_v \ne \bar{0}$, it can be shown that the vector $\bar{r}_u \times \bar{r}_v$ will be orthogonal to the surface S. This means that it can be used as the normal vector that we need to write down the equation of a tangent plane.

Surface area of Parametric Surface: A parametric surface S is given by $\bar{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$. Provided that S is traced out exactly once as (u,v) ranges over the points in D, the surface area of S is given by $A = \iint_D \|\bar{r_u} \times \bar{r_v}\|$

Surface Integrals:

Surface Integral Notation: The surface integral will have a dS, while the standard double integral will have a dA.
Surface Integral Non-Parametric: Suppose a surface S is given by $z = g(x,y)$. In this case, the surface integral is:
$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA$.
Evaluating a Non-Parametric Surface Integral: We will first substitute the equation of the surface in for z, or a different variable if it is given in terms of a different variable. We will then add on the often messy square root. After that, the integral is a standard double integral, which we can deal with.
Surface Integral Parametric: Suppose a parameterization for surface S is $\bar{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$. In these cases, the surface integral is: $\iint_S f(x,y,z)dS = \iint_D f(\bar{r}(u,v))\|\bar{r_u} \times \bar{r_v}\|dA$, where D is the range of the parameters that trace out the surface S.
Connection Between Non-Parametric and Parametric Surface Integrals: Notice that we can parameterize a surface given by $z = g(x,y)$ as $\bar{r}(x,y) = x\hat{\imath} + y\hat{\jmath} + g(x,y)\hat{k}$. We can always use this form for Parametric Surfaces as well. In fact, it can be shown that $\|\bar{r_u} \times \bar{r_v}\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$ for these kinds of surfaces.
Splitting up Surfaces for Surface Integrals: In order to do integrals where a surface is given as a combination of other surfaces we let $S_1, \dots, S_n$ be a collection of surfaces that, when combined, make up the original surface over which you wish to integrate. (In our example, $S_1 =$portion of cylinder that goes from xy plane to the given plane, i.e. to the cap of the cylinder. $S_2 =$the cap of the cylinder, and S_3 is the disc in the xy plane. Just like in the standard double integral, if the surface is split up into pieces, we can also split up the surface integral $\iint_S \; = \iint_{S_1} \; + \iint_{S_2} \; \dots$ Evaluate each of the split integrals on their own, and then just combine.

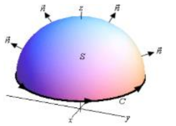
Surface Integrals of Vector Fields:

Unit Normal Vector for a Surface: In order to work with surface integrals of vector fields, we need the unit normal vector corresponding to the orientation. We have two ways of doing this, depending on how the surface was given to us.
Non-Parametric Unit Normal: Suppose the function is given by $z = g(x,y)$. In this case, we define a new function $f(x,y,z) = z - g(x,y)$. In terms of our new function, the surface is given by the equation $f(x,y,z) = 0$. Recall that ∇f will be orthogonal to the surface given by $f(x,y,z) = 0$. However, ∇f might not be a unit vector, so to rectify this, we define $\bar{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{-g_x\hat{\imath} - g_y\hat{\jmath} + \hat{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}$. *Note:* Notice that the z-direction component of the normal vector $(\hat{k})$ is always positive, so this normal vector will generally point upward (have an upwards component). When we are working with a closed surface, and want the positive orientation, we can look at the normal vector and determine if the "positive" orientation should point upwards or downwards. Remember that the "positive" orientation must point out of the region, and this could mean downwards in places. If we need the downward orientation, we can take the negative of this unit vector and we'll get the one that we need.
Parametric Unit Normal: Suppose a surface is given parametrically as $\bar{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$. Recall that the vector $\bar{r_u} \times \bar{r_v}$ will be normal to the tangent plane at a particular point. But if the vector is normal to the tangent plane at a point, then it will also be normal to the surface at that point. So, this is a normal vector. To make it into a unit vector, we define $\hat{n}$ as follows: $\hat{n} = \frac{\bar{r_u} \times \bar{r_v}}{\|\bar{r_u} \times \bar{r_v}\|}$. Note: As with the non-parametric case, we need to look at this vector and determine if it points in the correct direction or not. If it doesn't, we can take the negative of this vector, and it will point in the correct direction.
Surface Integrals of Vector Fields: Given a vector field $\bar{F}$ with a unit normal vector $\hat{n}$, then the surface integral of $\bar{F}$ over the surface S is given by: $\iint_S \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \hat{n}\, dS$, where the right hand integral is a standard surface integral. This is sometimes called the flux of $\bar{F}$ across S.
Unit Normal Vector Substitutions for Surface Integral: We can substitute in for the unit normal vector to get a *somewhat* easier formula to use. Be careful with each of the following formulas, as they assume a certain orientation, and the normal vector might require changing to match the given orientation. Let's start by assuming that the surface is given by $z = g(x,y)$. Let's also assume that the vector field is given by $\bar{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, and that the orientation we are after is "upwards". Under all of these assumptions, the surface integral of $\bar{F}$ over S is: $\iint_S \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \hat{n}dS = \iint_D (P\hat{\imath} + Q\hat{\jmath} + R\hat{k}) \cdot \left(\frac{-g_x\hat{\imath} - g_y\hat{\jmath} + \hat{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}\right)\sqrt{(g_x)^2 + (g_y)^2 + 1}\, dA = \iint_D (P\hat{\imath} + Q\hat{\jmath} + R\hat{k}) \cdot (-g_x\hat{\imath} - g_y\hat{\jmath} + \hat{k})dA = \iint_D -Pg_x - Qg_y + R\, dA$.

(Note 1) Now, remember that we assumed "upward" orientation. If we needed "downward" orientation, we would have to change the signs on the normal vector. Therefore, signs in the integrand would change as well. In general, it is best to rederive this formula as you need it. **(Note 2)** When we're given a non-parametric surface, there are 6 possible integrals here. Two for each form of the surface $z = g(x,y), y = g(x,z), x = g(y,z)$. For each form of the surface there are 2 possible normal vector, and we'll need to choose the correct one to match the given orientation of the surface. **(Note 3)** Notice as well that since we are using the unit normal vector, the messy square root will always drop out. This means that when we do need to derive the formula, we won't really need to put this in. All we'd need to work on is the numerator of the unit vector. **(Note 4)** It should also be noted that the square root is simply $\sqrt{(g_x)^2 + (g_y)^2 + 1} = \|\nabla f\|$
Surface Integral for Parametric Surfaces: Suppose the surface is given parametrically by $\bar{r}(u,v)$. In this case, the surface integral is $\iint_S \bar{F} \cdot d\bar{S} = \iint_S \bar{F} \cdot \hat{n}dS = \iint_D \bar{F} \cdot \left(\frac{\bar{r_u} \times \bar{r_v}}{\|\bar{r_u} \times \bar{r_v}\|}\right)\|\bar{r_u} \times \bar{r_v}\|dA = \iint_D \bar{F} \cdot (\bar{r_u} \times \bar{r_v})dA$. **(Note 1)** Again, note that we may have to change the sign on $\bar{r_u} \times \bar{r_v}$ to match the orientation of the surface, so once again there are really two formulas here. **(Note 2)** Also, again, the magnitude cancels in this case, so we don't need to worry about that in these problems either.

Stokes' Theorem (Relates Line Integral to Surface Integral):

Surface's Boundary Curve and Orientation: Suppose we have a surface with indicated orientation, for example the dome given to the left. Around the edge of this surface we have a curve C. This curve is called the boundary curve. The orientation of the surface S will induce the positive orientation of C. To get the positive orientation of C, think of yourself as walking along the curve. While you are walking along it, if your head is pointing in the same direction as the unit normal vectors, while the surface is on the left, then you are walking in the positive direction on C.



Stokes' Theorem: Let S be an oriented smooth surface, that is bounded by a simple, closed, smooth boundary curve C, with positive orientation. Also let $\bar{F}$ be a vector field. Then: $\int_C \bar{F} \cdot d\bar{r} = \iint_S \operatorname{curl}(\bar{F}) \cdot d\bar{S}$. In this theorem, note that the surface S can be any surface so long as its boundary curve is given by C. This is something that can be used to our advantage to simplify the surface integral on occasion.

Divergence/Gauss Theorem (Relates Surface Integrals to Triple Integrals):

Divergence Theorem: Let E be a simple solid region, and S is the boundary surface of E with positive orientation. Let $\bar{F}$ be a vector field whose components have continuous first order partial derivatives. Then, $\iint_S \bar{F} \cdot d\bar{S} = \iiint_E \operatorname{div}(\bar{F})dV$

COORDINATE SYSTEMS:

Cartesian Coordinates:

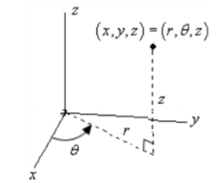
Distance Between Two Points: $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
General Equation for a Circle: For a circle with center (h,k) and radius r: $(x-h)^2 + (y-k)^2 = r^2$
General Equation for a Sphere: Center (h,k,l) and radius r: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$
Equations for coordinate planes: $z = 0 \rightarrow$ xy plane, $y = 0 \rightarrow$ xz plane, $x = 0 \rightarrow$ yz plane

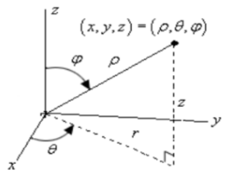
Spherical Coordinates:

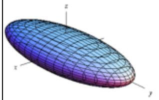
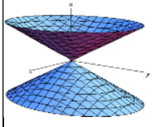
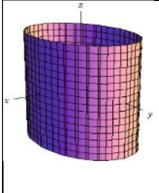
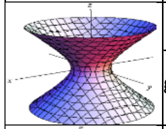
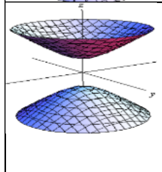
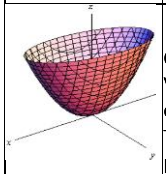
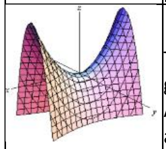
We require that $\rho \geq 0,\ 0 \leq \varphi \leq \pi$. Some sample surfaces: if $\rho = a$, the surface is a sphere of radius a centered at the origin. If $\varphi = a$, the surface is a cone that makes an angle a with the positive z-axis. If $\theta = \beta$, then the surface is a vertical plane that makes an angle of $\beta$with the positive x-axis.

COORDINATE SYSTEMS CONVERSIONS

		From		
		Cartesian	**Cylindrical**	**Spherical**
To	**Cartesian**	$x = x$ $y = y$ $z = z$	$x = rcos(\theta)$ $y = rsin(\theta)$ $z = z$	$x = \rho sin(\varphi)cos(\theta)$ $y = \rho sin(\varphi)sin(\theta)$ $z = \rho cos(\varphi)$
	Cylindrical	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $z = z$	$r = r$ $\theta = \theta$ $z = z$	$r = \rho sin(\varphi)$ $\theta = \theta$ $z = \rho cos(\theta)$
	Spherical	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\varphi = tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$	$\rho = \sqrt{r^2 + z^2}$ $\varphi = tan^{-1}\left(\frac{r}{z}\right)$ $\theta = \theta$	$\rho = \rho$ $\varphi = \varphi$ $\theta = \theta$





	**General Equation:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ If $a=b=c$, then we have a sphere. If $a=b\neq c$, then we have a spheroid (volume of revolution of an ellipse)	**Ellipsoid**
	**General Equation:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ Note that this equation is for a cone that opens along the z-axis. The variable on the RHS determines the orientation of the cone. E.g. $\frac{z^2}{c^2}+\frac{y^2}{b^2}=\frac{x^2}{a^2}$ is for a cone that opens along the x-axis.	**Cone**
	**General Equation:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Cross section is an ellipse, unless $a=b$, cross section is a circle, and the surface is given by $x^2+y^2=r^2$. The cylinder will be centered on the axis corresponding to the variable that doesn't appear in the equation. Be careful not to confuse this with a circle. In 2D – it is a circle, but in 3D it is a cylinder	**Cylinder**
	**General Equation:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The variable with the negative in front of it will give the axis along which the graph is centered.	**Hyperboloid of One Sheet**
	**General Equation:** $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (Or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$) The variable with the positive in front of it will give the axis along which the graph is centered. Notice that the only difference between one-sheet and two-sheet hyperboloids is the signs in front of the variables – they are exactly opposite signs.	**Hyperboloid of Two Sheets**
	**General Equation:** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ Cross section of an ellipse unless $a=b$, in which case circular. Variable that isn't squared determines the axis upon which the paraboloid opens up. The sign of c determines the direction in which the paraboloid opens. If c is positive, it opens up, and if c is negative, it opens down.	**Elliptic Paraboloid**
	**General Equation:** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$ The sign of c determines the direction in which the surface "opens up". The graph to the left is shows for c positive. As with the Elliptic Paraboloid, the surface can be moved up or down by adding/subtracting a constant from the LHS.	**Hyperbolic Paraboloid**

$\sin(-\theta)=-\sin(\theta)$	$\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$	$\sin(\pi-\theta)=\sin(\theta)$	$\cos(\pi-\theta)=-\cos(\theta)$	$\tan(\pi-\theta)=-\tan(\theta)$
$\cos(-\theta)=\cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$	$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}$	$\sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$	$\sin(\theta+2\pi)=\sin(\theta)$
$\tan(-\theta)=-\tan(\theta)$	$\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$	$\sin^2\left(\frac{\theta}{2}\right)=\frac{(1-\cos(\theta))}{2}$	$\cos^2\left(\frac{\theta}{2}\right)=\frac{(1+\cos(\theta))}{2}$	$\cos(\theta+2\pi)=\cos(\theta)$
$\sin\left(\theta+\frac{\pi}{2}\right)=\cos(\theta)$	$\sin(\theta+\pi)=-\sin(\theta)$	$\cos(\theta)=\pm\sqrt{1-\sin^2(\theta)}$	$\sin^2(\theta)+\cos^2(\theta)=1$	
$\cos\left(\theta+\frac{\pi}{2}\right)=-\sin(\theta)$	$\cos(\theta+\pi)=-\cos(\theta)$	$\sin(\theta)=\pm\sqrt{1-\cos^2(\theta)}$ Sign depends on quadrant of θ		
$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$		$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$		
$\sin(2\theta)=2\sin(\theta)\cos(\theta)=\frac{2\tan(\theta)}{1+\tan^2(\theta)}$		$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1=1-2\sin^2(\theta)$		
$\sin(3\theta)=-4\sin^3(\theta)+3\sin(\theta)$		$\cos(3\theta)=4\cos^3(\theta)-3\cos(\theta)$		*Euler's formula*:
$2\cos(\theta)\cos(\phi)=\cos(\theta-\phi)+\cos(\theta+\phi)$		$2\sin(\theta)\cos(\phi)=\sin(\theta+\phi)+\sin(\theta-\phi)$		$e^{ix}=\cos(x)+i\sin(x)$
$2\sin(\theta)\sin(\phi)=\cos(\theta-\phi)-\cos(\theta+\phi)$		$2\cos(\theta)\sin(\phi)=\sin(\theta+\phi)-\sin(\theta-\phi)$		

INTEGRALS: Basic Forms

$\int x^n dx=\frac{1}{n+1}x^{n+1},\ n\neq -1$	$\int \frac{1}{x}dx=\ln\lvert x\rvert$	$\int u\,dv=uv-\int v\,du$	$\int \frac{1}{ax+b}dx=\frac{1}{a}\ln\lvert ax+b\rvert$

Integrals of Rational Functions

$\int \frac{1}{(x+a)^2}dx=-\frac{1}{x+a}$	$\int \frac{1}{1+x^2}dx=\tan^{-1}x$	$\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int x(x+a)^n dx=\frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$		
$\int \frac{x}{a^2+x^2}dx=\frac{1}{2}\ln\lvert a^2+x^2\rvert$	$\int \frac{x^2}{a^2+x^2}dx=x-a\tan^{-1}\frac{x}{a}$	$\int \frac{1}{(x+a)(x+b)}dx=\frac{1}{b-a}\ln\frac{a+x}{b+x},a\neq b$
$\int \frac{x^3}{a^2+x^2}dx=\frac{1}{2}x^2-\frac{1}{2}a^2\ln\lvert a^2+x^2\rvert$	$\int \frac{x}{(x+a)^2}dx=\frac{a}{a+x}+\ln\lvert a+x\rvert$	$\int \frac{1}{ax^2+bx+c}dx=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$
$\int \frac{x}{ax^2+bx+c}dx=\frac{1}{2a}\ln\lvert ax^2+bx+c\rvert-\frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$	$\int (x+a)^n dx=\frac{(x+a)^{n+1}}{n+1},n\neq -1$	

Integrals with Roots

$\int \sqrt{x-a}\ dx=\frac{2}{3}(x-a)^{3/2}$	$\int \sqrt{x(ax+b)}\ dx=\frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)}-b^2\ln\left\lvert a\sqrt{x}+\sqrt{a(ax+b)}\right\rvert\right]$	$\int \frac{x}{\sqrt{a^2-x^2}}\ dx=-\sqrt{a^2-x^2}$
$\int \frac{1}{\sqrt{x\pm a}}\ dx=2\sqrt{x\pm a}$	$\int \sqrt{x^3(ax+b)}\ dx=\left[\frac{b}{12a}-\frac{b^2}{8a^2x}+\frac{x}{3}\right]\sqrt{x^3(ax+b)}+\frac{b^3}{8a^{5/2}}\ln\left\lvert a\sqrt{x}+\sqrt{a(ax+b)}\right\rvert$	
$\int \frac{1}{\sqrt{a-x}}\ dx=-2\sqrt{a-x}$	$\int \sqrt{a^2-x^2}\ dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$	$\int \frac{dx}{(a^2+x^2)^{3/2}}=\frac{x}{a^2\sqrt{a^2+x^2}}$
$\int \sqrt{ax+b}\ dx=\left(\frac{2b}{3a}+\frac{2x}{3}\right)\sqrt{ax+b}$		
$\int x\sqrt{x^2\pm a^2}\ dx=\frac{1}{3}(x^2\pm a^2)^{3/2}$	$\int \frac{1}{\sqrt{x^2\pm a^2}}\ dx=\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$	$\int \frac{x}{\sqrt{x\pm a}}\ dx=\frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$
$\int \frac{x}{\sqrt{x^2\pm a^2}}\ dx=\sqrt{x^2\pm a^2}$		

Integrals with Exponentials

$\int e^{ax}\ dx=\frac{1}{a}e^{ax}$	$\int xe^x\ dx=(x-1)e^x$	$\int xe^{ax}\ dx=\left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$
$\int x^2e^{ax}\ dx=\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)e^{ax}$	$\int xe^{-ax^2}\ dx=-\frac{1}{2a}e^{-ax^2}$	
$\int x^ne^{ax}\ dx=\frac{x^ne^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}dx$	$\int x^2e^x\ dx=(x^2-2x+2)e^x$	$\int x^3e^x\ dx=(x^3-3x^2+6x-6)e^x$

Integrals with Trigonometric Functions

$\int \sin ax\ dx=-\frac{1}{a}\cos ax$	$\int \sin^2 ax\ dx=\frac{x}{2}-\frac{\sin 2ax}{4a}$	$\int \sin^3 ax\ dx=-\frac{3\cos ax}{4a}+\frac{\cos 3ax}{12a}$
$\int \cos ax\ dx=\frac{1}{a}\sin ax$	$\int \cos^2 ax\ dx=\frac{x}{2}+\frac{\sin 2ax}{4a}$	$\int \cos^3 ax dx=\frac{3\sin ax}{4a}+\frac{\sin 3ax}{12a}$
$\int \sin^2 x\cos x\ dx=\frac{1}{3}\sin^3 x$	$\int \cos^2 ax\sin ax\ dx=-\frac{1}{3a}\cos^3 ax$	$\int \sin^2 ax\cos^2 ax\ dx=\frac{x}{8}-\frac{\sin 4ax}{32a}$
$\int \tan ax\ dx=-\frac{1}{a}\ln\cos ax$	$\int \tan^2 ax\ dx=-x+\frac{1}{a}\tan ax$	$\int \tan^3 ax dx=\frac{1}{a}\ln\cos ax+\frac{1}{2a}\sec^2 ax$
$\int \cos x\sin x\ dx=\frac{1}{2}\sin^2 x+c_1=-\frac{1}{2}\cos^2 x+c_2=-\frac{1}{4}\cos 2x+c_3$		
$\int \sin^2 ax\cos bx\ dx=-\frac{\sin[(2a-b)x]}{4(2a-b)}+\frac{\sin bx}{2b}-\frac{\sin[(2a+b)x]}{4(2a+b)}$	$\int \cos^2 ax\sin bx\ dx=\frac{\cos[(2a-b)x]}{4(2a-b)}-\frac{\cos bx}{2b}-\frac{\cos[(2a+b)x]}{4(2a+b)}$	
$\int \cos ax\sin bx\ dx=\frac{\cos[(a-b)x]}{2(a-b)}-\frac{\cos[(a+b)x]}{2(a+b)},a\neq b$	$\int \sin^2 ax\cos^2 bx dx=\frac{x}{4}-\frac{\sin 2ax}{8a}-\frac{\sin[2(a-b)x]}{16(a-b)}+\frac{\sin 2bx}{8b}-\frac{\sin[2(a+b)x]}{16(a+b)}$	

Products of Trigonometric Functions and Exponentials

$\int e^x\sin x\ dx=\frac{1}{2}e^x(\sin x-\cos x)$	$\int e^{bx}\sin ax\ dx=\frac{1}{a^2+b^2}e^{bx}(b\sin ax-a\cos ax)$	$\int e^x\cos x\ dx=\frac{1}{2}e^x(\sin x+\cos x)$
$\int e^{bx}\cos ax\ dx=\frac{1}{a^2+b^2}e^{bx}(a\sin ax+b\cos ax)$	$\int xe^x\sin x\ dx=\frac{1}{2}e^x(\cos x-x\cos x+x\sin x)$	$\int xe^x\cos x\ dx=\frac{1}{2}e^x(x\cos x-\sin x+x\sin x)$

Products of Trigonometric Functions and Monomials

$\int x\cos x\ dx=\cos x+x\sin x$	$\int x\cos ax\ dx=\frac{1}{a^2}\cos ax+\frac{x}{a}\sin ax$	$\int x^2\cos x\ dx=2x\cos x+(x^2-2)\sin x$
$\int x\sin x\ dx=-x\cos x+\sin x$	$\int x^2\cos ax\ dx=\frac{2x\cos ax}{a^2}+\frac{a^2x^2-2}{a^3}\sin ax$	$\int x\sin ax\ dx=-\frac{x\cos ax}{a}+\frac{\sin ax}{a^2}$
$\int x^2\sin x\ dx=(2-x^2)\cos x+2x\sin x$	$\int x^2\sin ax\ dx=\frac{2-a^2x^2}{a^3}\cos ax+\frac{2x\sin ax}{a^2}$	$\int x\cos^2 x\ dx=\frac{x^2}{4}+\frac{1}{8}\cos 2x+\frac{1}{4}x\sin 2x$
$\int x\sin^2 x\ dx=\frac{x^2}{4}-\frac{1}{8}\cos 2x-\frac{1}{4}x\sin 2x$		