BASIC DEFINITIONS AND OPERATIONS:

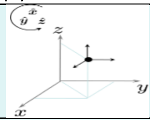
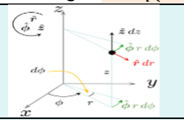
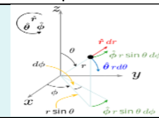
Dot Product:. Scalar product of $\bar{a}$ and $\bar{b}$: $\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\bar{a}||\bar{b}|\cos(\theta)$.

Angle between two vectors: $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos(\theta)$, θ is the angle between $\bar{a},\bar{b}$. Alternatively: $\cos(\theta) = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$

Orthogonal Vectors: Two vectors $\bar{a},\bar{b}\in\mathbb{R}^3$ are orthogonal if $\bar{a}\cdot\ \bar{b} = 0$ i.e. $\theta = \pi/2$

Cross product: $\bar{a}\times\bar{b} = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k} = \left(|\bar{a}||\bar{b}|\cos(\theta)\right)\hat{n}\ (n\perp\bar{a},\bar{b})$

Lorentz Law: Fields apply force on electric charge, according to $\bar{F} = q(\bar{E} + \mu_0\bar{V}\times\bar{H})$, where $\bar{V}$ =particle velocity

Cartesian, Cylindrical, Spherical				
Length element:	$d\bar{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$	$d\bar{l} = \hat{r}dr + \hat{\phi}rd\phi + \hat{z}dz$	$d\bar{l} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin(\theta)d\phi$	
Metric factors: $dl_i = h_idu_i$	$h_1 = h_2 = h_3 = 1$	$h_1 = 1, h_2 = r, h_3 = 1$	$h_1 = 1, h_2 = r, h_3 = r\sin(\theta)$	
Areas and volume:	$da_x = dydz$, $da_z = dxdy$, $da_y = dxdz$, $dv = dxdydz$	$da_r = rd\phi dz$, $da_z = rdrd\phi$, $da_\phi = drdz$, $dv = rdrd\phi dz$	$da_r = r^2\sin(\theta)d\theta d\phi$, $da_\theta = r\sin(\theta)drd\phi$	$da_\phi = rdrd\theta$, $dv = r^2\sin(\theta)drd\theta d\phi$

CHARGE, CURRENT, AND THEIR DENSITIES

Volume charge density: $\rho(r,t)$	Surface charge density: $\eta(r,t)$	Line charge density: λ

Total Charge inside volume: $Q = \iiint_V \rho dv = \iint_{S\in V}\eta da = \int_{l\in V}\lambda dl + \sum_{r_i\in V} q_i$

Representation of Plane charge q at r' via volume density: $\rho(\bar{r}) = q\delta(\bar{r}-\bar{r}') \equiv q\delta(x-x')\delta(y-y')\delta(z-z')$

Volume density: $J(r,t) = \rho\cdot v, dI = Jd\bar{a} = \hat{n}da$	Surface density: $k(r,t) - \eta\cdot v, dI = k\cdot\hat{n}\cdot dl, k = j\cdot\bar{n}$

Total Current Crossing Given Surface S: $I = \oiint_{S=\partial V}\bar{J}\cdot d\bar{a} + \int_{C\in S}\bar{K}\cdot\hat{n}dl + \sum_{r_i\in S} I_i$

Current that flows through surface element da: $\bar{J}\cdot d\bar{a}$

Conservation of charge: $\sum_{\text{currents leaving V}} I(r,t)da = -\frac{\partial}{\partial t}\sum_{\text{charges inside v}} Q$, da is pointing outward the volume V.

MAXWELL'S EQUATIONS

	Integral Form	Differential Form
Faraday's Law:	$\oint_{c=\partial S}\bar{E}(\bar{r},t)d\bar{l} = -\mu_0\frac{d}{dt}\iint_S\bar{H}(\bar{r},t)\cdot d\bar{a}$	$\nabla\times E(r,t) = -\mu\frac{\partial}{\partial t}H(r,t)$
Ampere's Law:	$\oint_{c=\partial S}\bar{H}(r,t)d\bar{l} = \epsilon_0\frac{d}{dt}\iint_S\bar{E}(r,t)d\bar{a} + \iint_S\bar{J}(r,t)\cdot d\bar{a}$	$\nabla\times H(r,t) = \epsilon\frac{\partial}{\partial t}E(r,t) + J(r,t)$
Gauss Law #1:	$\oiint_{S=\partial V}\epsilon_0 E(r,t)d\bar{a} = \sum_{\epsilon V} Q(t)\ \ (=\iiint_V\rho(r,t)dv)$,	$\nabla\cdot\epsilon_0\bar{E}(\bar{r},t) = \rho(\bar{r},t)$
Gauss Law #2:	$\oiint_{S=\partial V}\mu_0\bar{H}(\bar{r},t)d\bar{a} = 0$	$\nabla\cdot\mu_0\bar{H}(\bar{r},t) = 0$
Charge Conservation:	$\oiint_{S+\partial V}\bar{J}(\bar{r},t)\cdot d\bar{a} = -\frac{\partial}{\partial t}\iiint_V\rho(\bar{r},t)dv$	$\nabla\cdot\bar{J}(\bar{r},t) = -\frac{\partial}{\partial t}\rho(\bar{r},t)$

BOUNDAARY CONDITIONS:

Target	Interpretation	Condition
Electric Field:	The electric field normal component in the presence of a surface charge is discontinuous. Discontinuity is η/ϵ_0	$\hat{n}\cdot(E_1 - E_2) = \frac{\eta}{\epsilon_0}$
Magnetic Field	The magnetic field normal component is always continuous	$\hat{n}\cdot(H_1 - H_2) = 0$
Magnetic Field	The magnetic field tangential component in the presence of a surface current is discontinuous, discontinuity is $\bar{K}$.	$\hat{n}\times(H_1 - H_2) = K$
Electric Field	The electric field tangential component is always continuous	$\hat{n}\times(E_1 - E_2) = 0$
Charge Conservation	Final form:	$\hat{n}\cdot(\bar{J}_1 - \bar{J}_2) + \nabla_s\cdot\bar{K} = -\frac{\partial}{\partial t}\eta$
	$\nabla_s\cdot\bar{K}$ If the surface coincides with the (x,y) plane, then:	$\nabla_s\cdot\bar{K} = \frac{\partial}{\partial x}K_x + \frac{\partial}{\partial y}K_y$
	$\nabla_s\cdot\bar{K}$ In General:	$\nabla_s\cdot\bar{K} = \lim_{da\to 0}\frac{1}{da}\oint_{c=\partial da}\bar{K}\cdot\hat{n}_l dl$

PLANE WAVE SOLUTIONS:

Wave Equation: Assuming no sources in the medium $\bar{J} = 0, \rho = 0$, and $c^2 = (\mu_0\epsilon_0)^{-1}$, then $\nabla^2\bar{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{E} = 0$

Helmholtz Wave Equation: Assuming time-harmonic dependence $\bar{E} = \mathcal{R}\{\tilde{E}(\bar{r},\omega)e^{-i\omega t}\}$: $\nabla^2\tilde{E}(\bar{r},\omega) + \left(\frac{\omega}{c}\right)^2\tilde{E}(\bar{r},\omega) = 0$

Fields in the frequency domain: Assume that all physical quantities vary at a single frequency ω. Then: $\bar{E}(\bar{r},t) = \mathcal{R}\{\tilde{E}(\bar{r},\omega)e^{-i\omega t}\} = \frac{1}{2}[\tilde{E}(\bar{r},\omega)e^{-i\omega t} + \tilde{E}^*(\bar{r},\omega)e^{i\omega t}]$, $\bar{H}(\bar{r},t) = \mathcal{R}\{\tilde{H}(\bar{r},\omega)e^{-i\omega t}\}\frac{1}{2}[\tilde{H}(\bar{r},\omega)e^{-i\omega t} + \tilde{H}^*(\bar{r},\omega)e^{i\omega t}]$

Maxwell Equations in the frequency domain, with $e^{-i\omega t}$ time-dependence.		
Faraday's Law:	$\nabla\times\tilde{E}(\bar{r}) = i\omega\mu_0\tilde{H}(\bar{r})$	$\hat{n}\times(\tilde{E_1} - \tilde{E_2}) = 0$
Ampere's Law:	$\nabla\times\tilde{H}(\bar{r}) = -i\omega\epsilon_0\tilde{E}(\bar{r}) + \tilde{J}(\bar{r})$	$\hat{n}\times(\tilde{H_1} - \tilde{H_2}) = \tilde{K}$
Gauss Law #1:	$\nabla\cdot\epsilon_0\tilde{E}(\bar{r}) = \tilde{\rho}(\bar{r})$	$\hat{n}\cdot\epsilon_0(\tilde{E_1} - \tilde{E_2}) = \tilde{\eta}$
Gauss Law #2:	$\nabla\cdot\mu_0\tilde{H}(\bar{r}) = 0$	$\hat{n}\cdot\mu_0(\tilde{H_1} - \tilde{H_2}) = 0$
Charge Conservation:	$\nabla\cdot\tilde{J}(\bar{r}) = i\omega\tilde{\rho}(\bar{r})$	$\hat{n}\cdot\epsilon_0(\tilde{J_1} - \tilde{J_2}) + \nabla_s\cdot\tilde{K} = i\omega\eta$

Plane Wave solution: $\bar{E} = \mathcal{R}\{\tilde{E}(\bar{r},\omega)e^{-i\omega t}\} = \mathcal{R}\{\tilde{E}_0(\omega)e^{i(\bar{k}\cdot\bar{r}-\omega t)}\} = \hat{e}|\tilde{E}_0|\cos(\bar{k}\cdot\bar{r} - \omega t + \phi)$, $\bar{H} = \mathcal{R}\{\tilde{H}(\bar{r},\omega)e^{-i\omega t}\} = \mathcal{R}\{\tilde{H}_0(\omega)e^{i(\bar{k}\cdot\bar{r}-\omega t)}\} = \hat{h}\eta^{-1}|\tilde{E}_0|\cos(\bar{k}\cdot\bar{r} - \omega t + \phi)$, where $\tilde{E}_0 = \hat{e}\tilde{E}_0 = \hat{e}|\tilde{E}_0|e^{i\phi}$, $\tilde{H}_0 = \hat{h}\tilde{H}_0 = \hat{h}\frac{\tilde{E}_0}{\eta} = \hat{h}\frac{|\tilde{E}_0|}{\eta}e^{i\phi}$

Spectral Wave Equation: For a time dependent ($e^{-i\omega t}$) problem, where $\tilde{E}_0(x,y), \tilde{H}_0(x,y)$ are known, if we have some sources going through an aperture plane e.g. $z = 0$, and the half space $z > 0$ is homogenous and source free, E or H field at $z = 0$ is known, and the field at $z = 0$ only propagates into $z > 0$, then the field at $z > 0$ satisfies the following wave equation: $[\nabla^2 + k_0^2]\tilde{E} = 0,\ k_0 = \omega/c$.By applying 2D Fourier transform representations of the field to the wave equation, get the spectral wave equation: $\left[\frac{\partial^2}{\partial z^2} + k_0^2 - k_x^2 - k_y^2\right]\tilde{\tilde{E}}(k_x,k_y,z) = 0$, $k_z^2 = k_0^2 - \bar{k}_t\cdot\bar{k}_t$.

Spectral Wave Equation Solutions: $\tilde{\tilde{E}}(k_x,k_y,z) = \tilde{\tilde{E}}(k_x,k_y,0)e^{\pm ik_z z}$, where $\tilde{\tilde{E}}_0(k_x,k_y)$ =spectrum of the aperture field, and $k_z = \sqrt{k_0^2 - k_x^2 - k_y^2}$ ="Ewald Sphere" constraint

Spectral Plane Wave Representation: $\tilde{E}(x,y,z) = \frac{1}{4\pi^2}\iint dk_xdk_y\tilde{\tilde{E}}(k_x,k_y,z)e^{i(k_xx+k_yy)}$, $\tilde{\tilde{E}}(k_x,k_y,z) = \tilde{\tilde{E}}_0(k_x,k_y)e^{\pm ik_z z}$

ELECTRO-MAGNETIC ENERGY AND POWER:

Work done by EMF on moving charged particle: $dw = Fdr$

Power transmitted from EM field to moving charge: $p = \bar{F}\cdot\bar{V} = q(\bar{E} + \mu_0\bar{V}\times\bar{H})\cdot\bar{V} = q\bar{E}\cdot\bar{V}\ \ [W]$

Power transmitted to a unit volume: $P \equiv \frac{dp}{dv} = \frac{dq}{dv}\bar{E}\cdot\bar{V} = \rho\bar{E}\cdot\bar{V} = \bar{E}\cdot\bar{J}\,[\mathrm{W/m^3}]$

Poynting Theorem (Equations):

$-\nabla\cdot(\bar{E}\times\bar{H}) = \frac{\partial}{\partial t}\left[\frac{\epsilon_0}{2}\bar{E}\cdot\bar{E} + \frac{\mu_0}{2}\bar{H}\cdot\bar{H}\right] + \underbrace{\bar{E}\cdot\bar{J}}_{\text{Power trns-d to charges}}$ Or $-\nabla\cdot\bar{S} = \frac{\partial}{\partial t}[\varepsilon_E + \varepsilon_H] + P$

Where:		
	$\varepsilon_E = \frac{\epsilon_0}{2}\lvert\bar{E}^2\rvert$	Volume density of energy, stored in space, due to the presence of Electric field
	$\varepsilon_H = \frac{\mu_0}{2}\lvert\bar{H}^2\rvert$	Volume density of energy, stored in space, due to the presence of Magnetic Field
	$P \equiv \bar{E}\cdot\bar{J}$	Power transmitted from the fields to the charges

Poynting Vector: $\bar{S} = \bar{E}\times\bar{H}$. The vector physically means the energy transfer due to time-varying electric and magnetic fields is perpendicular to the fields.

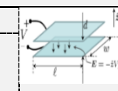
Power transmitted in presence of material with finite σ to charges and/or due to heat,: $\bar{J} = \sigma\bar{E} \to P = \bar{E}\cdot\bar{J} = \sigma\bar{E}^2$

Poynting Theorem (Integral version): $\underbrace{-\oint_{s=\partial v}\bar{S}\cdot d\bar{a}}_{\text{Net power flux entering V}} = \underbrace{\frac{d}{dt}\int_v[\mathcal{E}_E + \mathcal{E}_H]dv}_{\text{Total E and H energies inside V}} + \underbrace{\int_v Pdv}_{\text{Total P transferred to charges inside V}}$

Poynting Theorem Examples:

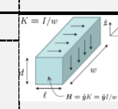
Parallel Plate Capacitor

Vol. dens. of stored energy: $\varepsilon_E = \frac{\epsilon_0}{2}\bar{E}\cdot\bar{E} = \frac{\epsilon_0}{2}\left(\frac{V}{d}\right)^2$. Tot. stored energy: $\int_{\text{v-cap}}\varepsilon_E dv = \frac{\epsilon_0}{2}\left(\frac{V}{d}\right)^2 lwv = \frac{1}{2}CV^2, C = \frac{\epsilon_0 lw}{d}$



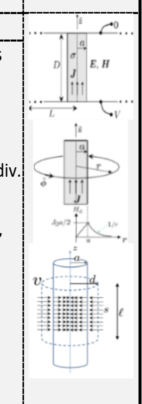
Square Cross-Section Inductor:

Given $K = \frac{I}{\omega}, \bar{H} = \hat{y}K = \hat{y}\frac{I}{\omega}$. Vol. dens. of stored energy: $\varepsilon_H = \frac{\mu_0}{2}\bar{H}\cdot\bar{H} = \frac{\mu_0}{2}\left(\frac{I}{\omega}\right)^2$. $\int_{V-ind}\varepsilon_H dv = \frac{\mu_0}{2}\left(\frac{I}{\omega}\right)^2 l\omega d = \frac{1}{2}LI^2$, $L = \frac{\mu_0 ld}{w}$



Long Cylindrical Resistor (resistor under static excitation)

A uniform current $\bar{J} = \hat{z}J_0$ is forced to flow in the resistor. Inside resistor, $\bar{E} = \hat{z}E_0 = \hat{z}J_0/\sigma$. Assume $L \gg D \gg a$. $\bar{E}$ is given by (both inside and outside the resistor) $\bar{E} = \hat{z}E_0, \forall\bar{r}, 0 \le z \le D$. $\bar{H}$ can be obtained by Ampere's law (see image): $\oint_{c=\partial s}\bar{H}\cdot dl = \int_s\bar{J}\cdot d\bar{a} = \begin{cases}J_0\pi r^2, r \le a\\ J_0\pi a^2, r > a\end{cases}$. But, due to symmetry $\bar{H} = \hat{\phi}H_\phi(r)$, thus: $\oint_{c=\partial s}\bar{H}\cdot dl = H_\phi 2\pi r$ and hence $\bar{H} = \hat{\phi}\frac{J_0}{2}\begin{cases}r, r \le a\\ \frac{a^2}{r}, r > a\end{cases}$. We have the fields, so we can get the Poynting vector, etc… The Poynting vector and its div. are $\bar{S} = \bar{E}\times\bar{H} = -\hat{r}\frac{J_0^2}{2\sigma}\begin{cases}r, r \le a\\ \frac{a^2}{r}, r > a'\end{cases}$ $-\nabla\cdot S = \begin{cases}\frac{J_0^2}{\sigma}, r \le a\\ 0 < r > a\end{cases}$. Note, in our problem $P = \bar{E}\cdot\bar{J} = \sigma\bar{E}^2 = \begin{cases}\frac{J_0^2}{\sigma}, r \le a\\ 0 < r > a\end{cases}$. So, we satisfy $-\nabla\cdot\bar{S} = P$, as by Poynting theorem $-\nabla\cdot\bar{S} = \frac{\partial}{\partial t}\underbrace{[\varepsilon_E + \varepsilon_H]}_{0} + P$. We check the total flux entering a finite volume: $\bar{S} = \bar{E}\times\bar{H} = -\frac{\hat{r}J_0^2}{2\sigma}\begin{cases}r,\ r \le a\\ \frac{a^2}{r}, r > a\end{cases} \to -\oint_{s=\partial v}\bar{S}\cdot d\bar{a} = \frac{J_0^2}{2\sigma}\begin{cases}2\pi d^2 l, d \le a\\ 2\pi a^2 l,\ d > a\end{cases}$. The total power entering a section of resistor of length l is: $\frac{J_0^2\pi a^2 l}{\sigma} = \frac{(J_0\pi a^2)^2}{\sigma\pi a^2}l = I_0^2R\ \ \left(I_0 = J_0\pi a^2, R = \frac{l}{\sigma\pi a^2}\right)$, as expected.



Time Dependent example: Plane wave fields are given in the "plane wave solution" section. $\hat{k}$ and the field polarizations $\hat{e},\hat{h}$ form an orthogonal triplet: $\hat{e}\perp\hat{k}, \hat{h}\perp\hat{k}, \hat{h}\perp\hat{e}, \hat{e}\times\hat{h} = \hat{k}$. Then, $\bar{S} = \bar{E}\times\bar{H} = \hat{k}\frac{|\tilde{E}_0|^2}{\eta}\cos^2(\bar{k}\cdot\bar{r} - \omega t + \phi)$. This is never negative, progagates in the direction of $\hat{k}$ =direction of propagation of plane wave, and is periodic. The divergence of this, using $k_x^2 + k_y^2 + k_z^2 = k_0^2 = \left(\frac{\omega}{c}\right)^2 = \omega^2\mu_0\epsilon_0$, $\eta = \sqrt{\mu_0/\epsilon_0}$: $-\nabla\cdot\bar{S} = \frac{k_0|\tilde{E}_0|^2}{\eta}\sin[2(\bar{k}\cdot\bar{r} - \omega t - \phi)] = \omega\epsilon_0|\tilde{E}_0|^2\sin[2(\bar{k}\cdot\bar{r} - \omega t - \phi)]$, and… (see stored electric and magnetic energy densities in t-dep plane wave.)

Stored electric and magnetic energy densities in T-Dep. Plane Wave: $\mathcal{E}_E = \frac{\epsilon_0}{2}\left|\tilde{E}_0\right|^2 \cos^2\left(\bar{k}\cdot\bar{r} - \omega t - \phi\right)$, $\mathcal{E}_H = \frac{\mu_0}{2\eta^2}\left|\tilde{E}_0\right|^2 \cos^2\left(\bar{k}\cdot\bar{r} - \omega t - \phi\right)$, which satisfies Pointing theorem. Note: Plane-wave solution is source free.

Complex Poynting Theorem for Time-Harmonic fields:

Complex Poynting Theorem: $-\nabla\cdot\left(\tilde{E}\times\tilde{H}^*\right) = i\omega\left(\epsilon_0\left|\tilde{E}\right|^2 - \mu_0\left|\tilde{H}\right|^2\right) + \tilde{E}\cdot\tilde{J}^*$

Where:		
	$\tilde{S} = \tilde{E}\times\tilde{H}^*$	The complex Poynting vector
	$\frac{1}{2}\mathcal{R}\{\tilde{S}\}$	The REAL AVERAGE power flux density carried by the fields
	$-\frac{1}{2}\mathcal{Im}\{\nabla\cdot\tilde{S}\} = 2\omega\left(\mathcal{E}_{E_a} - \mathcal{E}_{H_a}\right) + \frac{1}{2}\mathcal{Im}\{\tilde{E}\cdot\tilde{J}^*\}$	Reactive (imaginary) energy flow
	$-\frac{1}{2}\mathcal{R}\{\nabla\cdot\tilde{S}\} = \frac{1}{2}\mathcal{R}\{\tilde{E}\cdot\tilde{J}^*\}$	Avg. REAL power flux Div = Avg. REAL power density transmitted to charges.

QUASI STATICS

"Full Form" Of Maxwell's Equations (Including Time-Dependence):

Name	Differential Form	Boundary Condition
Faraday's Law:	$\nabla\times\bar{E}(\bar{r},t) = -\mu_0\frac{\partial}{\partial t}\bar{H}(\bar{r},t)$	$\hat{n}\times(E_1 - E_2) = 0$
Ampere's Law:	$\nabla\times\bar{H}(\bar{r},t) = \epsilon\frac{\partial}{\partial t}\bar{E}(\bar{r},t) + \bar{J}(\bar{r},t)$	$\hat{n}\times(H_1 - H_2) = K$
Gauss Law #1:	$\nabla\cdot\epsilon_0\bar{E}(\bar{r},t) = \rho(\bar{r},t)$	$\hat{n}\cdot(E_1 - E_2) = \frac{\eta}{\epsilon_0}$
Gauss Law #2:	$\nabla\cdot\mu_0\bar{H}(\bar{r},t) = 0$	$\hat{n}\cdot(H_1 - H_2) = 0$
Conservation Law:	$\nabla\cdot\bar{J}(\bar{r},t) = -\frac{\partial}{\partial t}\rho(\bar{r},t)$	$\hat{n}\cdot(\bar{J}_1 - \bar{J}_2) + \nabla_s\cdot\bar{K} = -\frac{\partial}{\partial t}\eta$

Static Solution: Solution at no time (e.g. if system was constant), usually given: $\bar{E}^{(0)}$ or $\bar{H}^{(0)}$

The Quasi-Static Algorithm:

Electro-Static Solution:		Push" Time as a Parameter:		Correct sources:
$\nabla\times\bar{E}^{(0)}(\bar{r}) = 0 \Rightarrow \bar{E}^{(0)}(\bar{r})$ $\bar{H}^{(0)}(\bar{r}) = 0, \quad \bar{J}(\bar{r}) = 0$	$\Rightarrow$	$\bar{E}^{(0)}(\bar{r}) \longmapsto \bar{E}^{(0)}(\bar{r},t)$ $\eta^{(0)}(\bar{r}) \longmapsto \eta^{(0)}(\bar{r},t)$ $\rho^{(0)}(\bar{r}) \longmapsto \rho^{(0)}(\bar{r},t)$	$\Rightarrow$	$\nabla\cdot\bar{J}^{(1)}(\bar{r},t) = -\frac{\partial}{\partial t}\rho^{(0)}(\bar{r},t)$ $\hat{n}\cdot\left(\bar{J}_1^{(1)} - \bar{J}_2^{(1)}\right) + \nabla_s\cdot\bar{K}^{(1)} = -\frac{\partial}{\partial t}\eta^{(0)}$ $\Rightarrow \bar{J}^{(1)}(\bar{r},t),\ \bar{K}^{(1)}(\bar{r},t)$

Find an estimation for $\bar{H}^{(1)}(\bar{r},t)$ (correction to the $\bar{H}$ field), via:		Find an estimation for $E^{(2)}(\bar{r},t)$ (correction to the $\bar{E}$ field) via:		Compare between Correction and Static:
$\nabla\times\bar{H}^{(1)}(\bar{r},t) = \epsilon_0\frac{\partial}{\partial t}E^{(0)}(\bar{r},t) + \bar{J}^{(1)}(\bar{r},t)$ Or via: $\hat{n}\times\left(\bar{H}_1^{(1)} - \bar{H}_2^{(1)}\right) = \bar{K}^{(1)}$	$\Rightarrow$	$\nabla\times\bar{E}^{(2)}(\bar{r},t) = -\mu_0\frac{\partial}{\partial t}\bar{H}^{(1)}(\bar{r},t)$	$\Rightarrow$	$\frac{\left\lvert\bar{E}^{(2)}\right\rvert}{\left\lvert\bar{E}^{(0)}\right\rvert} \ll 1\ ?$

	Electro Quasi-Statics (EQS)		Magneto Quasi-Statics (MQS)	
Fields	$\bar{\boldsymbol{E}}$	$\bar{\boldsymbol{H}}$	$\bar{\boldsymbol{E}}$	$\bar{\boldsymbol{H}}$
Static Solution	$\omega^0 = 1$	0	0	$\omega^0 = 1$
1st Correction	–	ω	ω	–
2nd Correction	ω^2	–	–	ω^2
3rd Correction	–	ω^3	ω^3	–

Quasi-Static Power Series decomposition		
ω^0	$\nabla\times\bar{E}^{(0)} = 0$	Static Problem
ω^1	$\nabla\times\bar{E}^{(1)} + \mu_0\frac{\partial}{\partial\tau}\bar{H}^{(0)} = 0$	1st correction
ω^2	$\nabla\times\bar{E}^{(2)} + \mu_0\frac{\partial}{\partial\tau}\bar{H}^{(1)} = 0$	2nd correction
ω^n	$\nabla\times\bar{E}^{(n)} + \mu_0\frac{\partial}{\partial\tau}\bar{H}^{(n-1)} = 0$	nth correction

Complete Quasi-Static Expansion Terms:

Zero order terms (leading terms):	Higher order terms (mth order "excites" the $m+1$ order):	At each stage, apply the Boundary Conditions:
$\nabla\times\bar{E}^{(0)} = 0$	$\nabla\times\bar{E}^{(m)} = -\frac{\mu_0\partial}{\partial t}\bar{H}^{(m-1)}$	$\hat{n}\times\left[\bar{E}_1^{(m)} - \bar{E}_2^{(m)}\right] = 0$
$\nabla\times\bar{H}^{(0)} = \bar{J}^{(0)}$	$\nabla\times\bar{H}^{(m)} = J^{(m)} + \frac{\epsilon_0\partial}{\partial t}\bar{E}^{(m-1)}$	$\hat{n}\times\left[\bar{H}_1^{(m)} - \bar{H}_2^{(m)}\right] = \bar{K}^{(m)}$
$\nabla\cdot\left[\epsilon_0\bar{E}^{(0)}\right] = \rho^{(0)}$	$\nabla\cdot\left[\epsilon_0\bar{E}^{(m)}\right] = \rho^{(m)}$	$\hat{n}\cdot\left[\bar{E}_1^{(m)} - \bar{E}_2^{(m)}\right] = \frac{\eta^{(m)}}{\epsilon_0}$
$\nabla\cdot\left[\mu_0\bar{H}^{(0)}\right] = 0$	$\nabla\cdot\left[\mu_0\bar{H}^{(m)}\right] = 0$	$\hat{n}\cdot\left[\bar{H}_1^{(m)} - \bar{H}_2^{(m)}\right] = 0$
$\nabla\cdot\bar{J}^{(0)} = 0$	$\nabla\cdot\bar{J}^{(m)} = -\frac{\partial}{\partial t}\rho^{(m-1)}$	$\hat{n}\cdot\left[\bar{J}_1^{(m)} - \bar{J}_2^{(m)}\right] = -\nabla_s\cdot\bar{K}^{(m)} - \frac{\partial}{\partial t}\eta^{(m-1)}$

Quasi-Static Power Series: Fields vary in time as $\cos(\omega t), \sin(\omega t), e^{i\omega t}$. Define dimensionless time variable $\tau = \omega t \rightarrow \frac{\partial}{\partial t} = \omega\frac{\partial}{\partial\tau}$. So time derivative transforms to derivative that doesn't change order. The power series is: $\nabla\times\bar{E}^{(0)} + \omega\left[\nabla\times\bar{E}^{(1)} + \mu_0\frac{\partial}{\partial\tau}\bar{H}^{(0)}\right] + \omega^2\left[\nabla\times\bar{E}^{(2)} + \mu_0\frac{\partial}{\partial\tau}\bar{H}^{(1)}\right] + \cdots + \omega^n\left[\nabla\times\bar{E}^{(n)} + \mu_0\frac{\partial}{\partial\tau}\bar{H}^{(n-1)}\right] + \cdots = 0$. Since entire sum must vanish for a continuum of frequencies, we obtain the Quasi Static Power Series Decomposition:

Energy in the Quasi-Static Power Series: Using Poynting & power series $\bar{E}(\bar{r},t) = \sum_{n=0}^{\infty}\omega^n E^{(n)}(\bar{r},\tau)$ (or same for $\bar{H}(\bar{r},t)$): $\bar{S} = \bar{E}\times\bar{H} = \sum_{m,n}\omega^{m+n}\bar{E}^{(m)}\times\bar{H}^{(n)} = \underbrace{\bar{E}^{(0)}\times\bar{H}^{(0)}}_{\bar{S}^{(0)}} + \omega\left[\underbrace{\bar{E}^{(1)}\times\bar{H}^{(0)} + \bar{E}^{(0)}\times\bar{H}^{(1)}}_{\bar{S}^{(1)}}\right] + \omega^2\left[\underbrace{\bar{E}^{(2)}\times\bar{H}^{(0)} + \bar{E}^{(1)}\times\bar{H}^{(1)} + \bar{E}^{(0)}\times\bar{H}^{(2)}}_{\bar{S}^{(2)}}\right] + \cdots = \sum_n\omega^n\bar{S}^{(n)}$

Power in the Quasi-Static Power Series: Similarly to energy: $\bar{E}\cdot\bar{J} = \sum_n\omega^n\bar{J}^{(n)} = \bar{E}\cdot\bar{J} = \bar{E}^{(0)}\cdot\bar{J}^{(0)} + \omega\left[\bar{E}^{(1)}\cdot\bar{J}^{(0)} + \bar{E}^{(0)}\cdot\bar{J}^{(1)}\right] + \omega^2\left[\bar{E}^{(2)}\cdot\bar{J}^{(0)} + \bar{E}^{(1)}\cdot\bar{J}^{(1)} + \bar{E}^{(0)}\cdot\bar{J}^{(2)}\right] + \cdots$

Stored Energy Densities in QS Series: $\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0\bar{E}\cdot\bar{E} + \mu_0\bar{H}\cdot\bar{H}\right) = \frac{\omega}{2}\frac{\partial}{\partial\tau}\left[\epsilon_0\sum_{m,n}\omega^{m+n}\bar{E}^{(m)}\cdot\bar{E}^{(n)} + \mu_0\sum_{m,n}\omega^{m+n}\bar{H}^{(m)}\cdot\bar{H}^{(n)}\right]$

Poynting theorem for Quasi-Statics: Substitute the Energy, Power, and Stored Energy Densities into the Poynting theorem, and equate powers of frequency.

Hierarchy of Power/Energy Balance (n^{th} order Power Energy Balance, where n is the exponent of ω)

ω^0: $-\nabla\cdot\left[\underbrace{\bar{E}^{(0)}\times\bar{H}^{(0)}}_{\bar{S}^{(0)}}\right] = \bar{E}^{(0)}\cdot\bar{J}^{(0)}$

ω^1: $-\nabla\cdot\left[\underbrace{\bar{E}^{(1)}\times\bar{H}^{(0)} + \bar{E}^{(0)}\times\bar{H}^{(1)}}_{\bar{S}^{(1)}}\right] = \frac{1}{2}\frac{\partial}{\partial\tau}\left[\epsilon_0\bar{E}^{(0)}\cdot\bar{E}^{(0)} + \mu_0\bar{H}^{(0)}\cdot\bar{H}^{(0)}\right] + \bar{E}^{(1)}\cdot\bar{J}^{(0)} + \bar{E}^{(0)}\cdot\bar{J}^{(1)}$

ω^2: $-\nabla\cdot\left[\underbrace{\bar{E}^{(2)}\times\bar{H}^{(0)} + \bar{E}^{(1)}\times\bar{H}^{(1)} + \bar{E}^{(0)}\times\bar{H}^{(2)}}_{\bar{S}^{(2)}}\right] = \frac{\partial}{\partial\tau}\left[\epsilon_0\bar{E}^{(1)}\cdot\bar{E}^{(0)} + \mu_0\bar{H}^{(1)}\cdot\bar{H}^{(0)}\right] + \bar{E}^{(2)}\cdot\bar{J}^{(0)} + \bar{E}^{(1)}\cdot\bar{J}^{(1)} + \bar{E}^{(0)}\cdot\bar{J}^{(2)}$

Electro Quasi Static Example

Simplest geometry: Parallel plate capacitor. Infinite geometry (finite, but ignore edge effects at static phase). Y-indep. excitation. 

Step 1: The Electro-Static solution:

Assume: No currents, invariant surface charge $\pm\eta_0\ \left(z = \pm\frac{d}{2}\right)$. Then: $\bar{H}^{(0)} = 0$, $\bar{E}^{(0)} = \begin{cases} -\frac{\hat{z}\eta_0}{\epsilon_0}, |z| \le d/2 \\ 0, |z| > d/2 \end{cases}$, $V = -\int_{z=-\frac{d}{2}}^{\frac{d}{2}}\bar{E}^{(0)}\cdot d\bar{l} = \epsilon_0^{-1}\eta_0 d$

Consider a section of dimensions $\ell\times w\ (x,y)$. For this section: $Q = \eta_0\ell w,\ C = \frac{Q}{V} = \epsilon_0\frac{\ell w}{d}$.

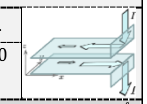
Step 2: Push time as a parameter

Let excitation change in time $\eta_0 \rightarrow \eta^{(0)}(t) = \eta_0\cos(\omega t) \Rightarrow \bar{E}^{(0)} = \begin{cases} -\hat{z}\epsilon_0^{-1}\eta_0\cos(\omega t), |z| \le d/2 \\ 0, |z| > d/2 \end{cases}$. $V = \epsilon_0^{-1}\eta^{(0)}d = \epsilon_0^{-1}d\eta_0\cos(\omega t)$, $Q = \ell w\eta_0\cos(\omega t)$, $C = \frac{Q}{V} = \epsilon_0\frac{\ell w}{d}$

Step 3: Correct Sources:

Since $\eta = \eta(t)$, we must have currents (charge conserv.): $\hat{\eta}\cdot\left(\bar{J}_1^{(1)} - \bar{J}_2^{(1)}\right) + \nabla_s\cdot\bar{K}^{(1)} = -\frac{\partial}{\partial t}\eta^{(0)},\ \eta^{(0)}(t) = \pm\eta_0\cos(\omega t)$.

Solve for currents under the assumptions: No volume currents, Y-Independent excitation, and B.C/I.C. : edge at $x = 0$ (no currents there!): $\frac{\partial}{\partial x}K_x^{(1)} = \pm\omega\eta_0\sin(\omega t) \Rightarrow K_x^{(1)} = \pm x\omega\eta_0\sin(\omega t), z = \pm d/2$.



Max I (@ electrodes) for $\ell\times w\ (x,y)$ section: $I = wK_x(x=\ell) = \ell w\eta_0\omega\sin(\omega t) \Rightarrow I = C\frac{d}{dt}V$ (Lmpd Cr-t. Theory). $V = \epsilon_0^{-1}d\eta_0\cos(\omega t), C = \epsilon_0\frac{\ell w}{d}$

Step 4: Find an Estimation for $\bar{H}^{(1)}$ (correction to the magnetic field $\bar{H}^{(0)} = 0$)

$\nabla\times\bar{H}^{(1)} = \epsilon_0\frac{\partial}{\partial t}\bar{E}^{(0)} + \bar{J}^{(1)}\quad \hat{n}\times\left(\bar{H}_1^{(1)} - \bar{H}_2^{(1)}\right) = \bar{K}^{(1)} \Rightarrow \bar{H}^{(1)} = \begin{cases}\hat{y}K_x^{(1)}\ \left(z = \frac{d}{2}\right), & |z| \le \frac{d}{2} \\ 0, & |z| > d/2\end{cases}$; $\oint_{\partial s}\bar{H}^{(1)}\cdot d\bar{l} = \epsilon_0\frac{d}{dt}\iint_s\bar{E}^{(0)}\cdot d\bar{a} + \iint_s\bar{J}^{(1)}\cdot d\bar{a}$; $\iint_s\bar{E}^{(0)}\cdot d\bar{a} = 0$

An estimation for $\bar{H}^{(1)}$: $K_x^{(1)}\left(z = \pm\frac{d}{2}\right) = \pm x\omega\eta_0\sin(\omega t)$: $\bar{H}^{(1)} = \begin{cases}\hat{y}K_x^{(1)}(z = d/2) = \hat{y}x\omega\eta_0\sin(\omega t), |z| \le d/2 \\ 0, |z| > d/2\end{cases}$

However, we obtained $E_z^{(0)}(|z| \le d/2) == -\epsilon_0^{-1}\eta_0\cos(\omega t)$, therefore: $H_y^{(1)}(|z| \le d/2) = \epsilon_0 x\frac{\partial}{\partial t}E_z^{(0)}(|z| \le d/2)$

Step 5: Find an estimation for $\bar{E}^{(2)}$ via:

$$\nabla\times\left(\underbrace{\bar{E}^{(0)}(\bar{r},t)}_{\nabla\times\bar{E}^{(0)}=0\ \text{(Static Solution)}} + \bar{E}^{(2)}(\bar{r},t)\right) = -\mu_0\frac{\partial}{\partial t}\left[\underbrace{\bar{H}^{(0)}(\bar{r},t)}_{\bar{H}^{(0)}=0} + \bar{H}^{(1)}(\bar{r},t)\right]$$

Hence: $\nabla\times\bar{E}^{(2)} = -\mu_0\frac{\partial}{\partial t}\bar{H}^{(1)} = -\hat{y}\epsilon_0\mu_0 x\frac{\partial^2}{\partial t^2}E_z^{(0)}\left(|z| \le \frac{d}{2}\right) = \hat{y}\epsilon_0\mu_0 x\omega^2 E_z^{(0)}$. The $\hat{y}$ component of $\nabla\times\bar{E}^{(2)}$ can only be $\frac{\partial}{\partial z}E_x^{(2)} - \frac{\partial}{\partial x}E_z^{(2)}$.

But, $E_x^{(2)} = 0$ on the metal bounds, and there is no z-dependece in the RHS, so we have: $\frac{\partial}{\partial x}E_z^{(2)} = -\epsilon_0\mu_0 x\omega^2 E_z^{(0)} \Rightarrow E_z^{(2)} = \frac{x^2\omega^2}{2c^2}E_z^{(0)}$

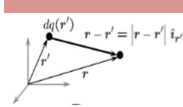
Step 6: Compare between correction and static

We have estimation: $E_z^{(2)} = \frac{x^2\omega^2}{2c^2}E_z^{(0)} \xrightarrow{\max} \frac{\ell^2\omega^2}{2c^2}E_z^{(0)}$, and compare between $E_z^{(2)}, E_z^{(0)}$: $\frac{\left|E_z^{(2)}\right|}{\left|E_z^{(0)}\right|} = \frac{\ell^2\omega^2}{2c^2}$. Therefore test for validity of the static approximation: $\frac{\ell^2\omega^2}{2c^2} \overset{?}{\ll} 1$, where $\frac{\ell^2\omega^2}{2c^2}$ is a typical physical dimension of the System. $\frac{\ell}{c} = \tau$ =time it takes for EM wave to cross the system, $\frac{2\pi}{\omega} = T$ =period of excitation, $\frac{2\pi c}{\omega} = \lambda$ =wavelength. So the test above can be rewritten as one of the following two tests:

$$\pi\frac{\tau}{T} = \pi\frac{\text{time of propagation through the system}}{\text{excitation period time}} \ll 1 \qquad \pi\frac{\ell}{\lambda} = \pi\frac{\text{physical dimensions of the system}}{\text{wavelength of the EM radiation}} \ll 1$$

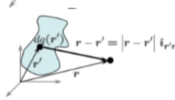
These conditions are equivalent. When one is satisfied, we are in the quasi-static regime. Lumped Circuit Theory approximately holds

ELECTROSTATICS:



Response field due to point source: In electrostatics context $d\bar{E}(\bar{r}) = \frac{dq(\bar{r}')}{4\pi\epsilon_0|\bar{r} - \bar{r}'|^2}\hat{\imath}_{\bar{r}'\bar{r}}$

Response field due to an arbitrary source distribution: $E(r) = \int\frac{\hat{\imath}_{\bar{r}'\bar{r}}\cdot dq(\bar{r}')}{4\pi\epsilon_0|\bar{r} - \bar{r}'|^2} + \sum_n\frac{\hat{\imath}_{\bar{r}_n\bar{r}}q_n}{4\pi\epsilon_0|\bar{r} - \bar{r}_n|^2}$, where the summation is only added if point charges are present. $\bar{r}_n$ =location of point charge # n. The presence of point charges can also be addressed via: $\rho(\bar{r}') = \sum_n q_n\delta(\bar{r}' - \bar{r}_n) = \sum_n q_n\delta(x' - x_n)\delta(y' - y_n)\delta(z' - z_n)$



Impulse Response POV: Point charge can be viewed as an impulse excitation in space. The space response (electric field) to an impulse of magnitude q at the origin is: $\bar{E}(\bar{r}) = \frac{q}{4\pi\epsilon_0|\bar{r}|^2}\hat{r}$. The convolution of this with a specific excitation $\rho(\bar{r})$ gives the response field due to this $\rho(\bar{r})$

ELECTRIC POTENTIAL

Time Independent Electric Fields: A time-independent electric field is conservative, $\nabla\times\bar{E} = 0 \Leftrightarrow \oint_{c=\partial s}\bar{E}\cdot d\bar{l} = 0$

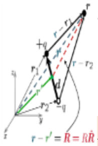
Scalar Potential (Electrostatic): Work needed to move a unit point charge from r_1 to r_2, given by $\bar{E}(r) = -\nabla\Phi(\bar{r})$, $\Phi(\bar{r}_2) - \Phi(\bar{r}_1) = \int_{r_1}^{r_2}\nabla\Phi(\bar{r})dl = -\int_{r_1}^{r_2}E(\bar{r})d\bar{l}$. Potential difference is independent of specific path. Electric potential in time independent fields is conservative.

Potential at a single point: Potential at a single point is defined up to an additive constant, i.e. at the reference $\bar{r}_1$. A common choice is $\bar{r}_1 = \infty$. Then: $\Phi(\bar{r}_p) = -\int_{r_1\to\infty}^{r_p} \bar{E}(\bar{r})\cdot d\bar{l}$.

Potential due to a Distribution of Charges: $\Phi(\bar{r}) = \int_{\text{all charges}} \frac{dq(\bar{r}')}{4\pi\epsilon_0|(\bar{r}-\bar{r}')|}$, where $dq(\bar{r}') = \eta(\bar{r}')da'$; surface charge, $\rho(\bar{r}')dv'$; volume charge, $\lambda(\bar{r}')dl'$; line charge

Potential at $\bar{r} = \bar{r}_p$ for a pt. charge @ origin: $\phi(\bar{r}_p) = -\int_{\bar{r}_1\to\infty}^{\bar{r}_p} \frac{q}{4\pi\epsilon_0 r^2}\hat{r}\cdot d\bar{l} = -\int_{\bar{r}_1}^{\bar{r}_p=|\bar{r}_p|} \frac{q}{4\pi\epsilon_0 r^2}\hat{r}\cdot d\bar{l} = \frac{q}{4\pi\epsilon_0 r_p}$

Potential due to point charge at arbitrary location: $\bar{\Phi}(\bar{r}) = \frac{q(\bar{r}')}{4\pi\epsilon_0|\bar{r}-\bar{r}'|}$



Simple Dipole: Set the center of the dipole as $\bar{r}'$. Dipole is not necessarily aligned with the z-axis. Define $d = (d_x, d_y, d_z), +q@\bar{r}_1 = \bar{r}' + \bar{d}/2, -q@\bar{r}_1 = \bar{r}' - \bar{d}/2$

Dipole moment (definition): $\bar{p} = q\bar{d}$

Potential for a Simple Dipole: $\Phi(\bar{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|(\bar{r}-\bar{r}_1)|} - \frac{1}{|(\bar{r}-\bar{r}_2)|}\right)$, Where r_1, r_2 are locations of $\pm q$, hence $\bar{r} - \bar{r}_1 = \bar{r} - \bar{r}' - \frac{\bar{d}}{2}$, $\bar{r} - \bar{r}_2 = \bar{r} - \bar{r}' + \frac{\bar{d}}{2}$, and $|\bar{r} - \bar{r}_{1,2}| = \sqrt{\left(\bar{r} - \bar{r}' \mp \frac{\bar{d}}{2}\right)\cdot\left(\bar{r} - \bar{r}' \mp \frac{\bar{d}}{2}\right)} = \sqrt{\left(x - x' \mp \frac{d_x}{2}\right)^2 + \left(y - y' \mp \frac{d_y}{2}\right)^2 + \left(z - z' \mp \frac{d_z}{2}\right)^2}$

Potential "far" from a Single Dipole: $\Phi(\bar{r}) \approx \frac{\bar{p}\cdot\hat{\imath}_{r',r}}{4\pi\epsilon_0|\bar{r}-\bar{r}'|^2}$, where $\hat{\imath}_{r',r} = \hat{R} = \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|}$

Dipole at Origin, along z −axis, $+q$ in $\hat{z}$ direction: $\bar{d} = \hat{z}d \Rightarrow \hat{r}\cdot\bar{p} = pcos\theta$, $\Phi(r) \equiv \frac{pcos\theta}{4\pi\epsilon_0 r^2}$,

$\bar{E} = -\nabla\Phi = \frac{pcos\theta}{4\pi\epsilon_0 r^3}(\hat{r}2cos\theta + \hat{\theta}sin\theta)$, in spherical coordinates. Note: no ϕ dependence.

Dipole with Arbitrary Location and Orientation: $\Phi(\bar{r}) \approx \frac{q\bar{d}\cdot\hat{R}}{4\pi\epsilon_0 R^2} = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot\hat{\imath}_{\bar{r}',\bar{r}}}{|\bar{r}-\bar{r}'|^2} = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$. Explicitly in Cartesian coordinates: $\Phi(\bar{r}) = \frac{1}{4\pi\epsilon_0}\frac{p_x(x-x')+p_y(y-y')+p_z(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$. Then : $\bar{E}(\bar{r}) = -\frac{1}{4\pi\epsilon_0|\bar{r}-\bar{r}'|^3}\cdot[\bar{p} - 3(\bar{p}\cdot\hat{\imath}_{r',r})\hat{\imath}_{r',r}]$

LAPLACE & POISSON EQUATIONS:

Poisson Equation: $\nabla^2\Phi(\bar{r}) = -\frac{\rho(\bar{r})}{\epsilon_0}$, $\rho \in V$, + B.C (on ∂V) **Laplace eq: (no charges):** $\nabla^2\Phi(r) = 0$

Derivation of Poisson Equation: We deal with static fields, thus $\nabla\times\bar{E}(\bar{r}) = 0 \to \bar{E}(\bar{r}) = -\nabla\Phi(\bar{r})$. Also, $\nabla\cdot\bar{E}(\bar{r}) = \rho/\epsilon_0$. Hence: $\nabla\cdot[\nabla\Phi(\bar{r})] = -\rho(\bar{r})/\epsilon_0$, and therefore $\nabla^2\Phi(\bar{r}) = -\rho(\bar{r})/\epsilon_0$

The Particular and Homogenous solution strategy: Let $\Phi_p(\bar{r})$ be a solution of the Poisson equation $\forall\bar{r}\in D$, where D is the domain of interest. Also let $\Phi_h(\bar{r})$ be a solution to the Laplace equation in the same domain. Then, by linearity of the Laplacian operator, superposition $\Phi(\bar{r}) = \Phi_p + \Phi_h$ is also a solution of the Poisson equation in the domain of interest: $\nabla^2\left(\Phi_p(\bar{r}) + \Phi_h(\bar{r})\right) = \nabla^2\Phi(\bar{r}) = -\frac{\rho(\bar{r})}{\epsilon_0}, \forall\bar{r}\in D$. How do we actually solve this? First, find Φ_p via the superposition/convolution approach: $\Phi_p(\bar{r}) = \int\frac{\rho(\bar{r}')dv'}{4\pi\epsilon_0|\bar{r}-\bar{r}'|}$. Add Φ_h so as to satisfy B.C. by $\Phi(\bar{r}) = \Phi_p + \Phi_h$

Minimum/Maximum Theorem: A solution of the Laplace equation $\Phi(\bar{r})$ inside a domain D ($\bar{r}\in D$) can't have extrema points inside D. Maximal and/or minimal values are attained only at the boundaries.

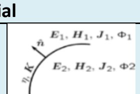
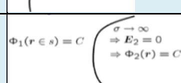
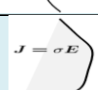
Consequences of Min/Max Theorem: If potential on closed surface of a source-free volume V is constant, it must also be constant inside V, i.e. the field inside must vanish: $\nabla^2\phi(r) = 0$. Another consequence is that the field inside a source free volume surrounded by a perfect conductor vanishes

Mean Value Theorem Laplace Equation: Let $\Phi(\bar{r})$ be a solution to the Laplace equation in domain D. Let $S(\bar{r}';a)$ be the surface of a sphere with radius a, centered at $\bar{r}'$, and completely contained in D. Then, we have: $\Phi(\bar{r}') = \frac{1}{4\pi a^2}\oint_{S(\bar{r}';a)}\Phi(\bar{r})ds$. This holds for any $S(\bar{r}';a)\in D$. Therefore: The value of Φ at any inner point $\bar{r}\in D$ is the average of its values on any spherical surface contained in D, and centered at $\bar{r}$.

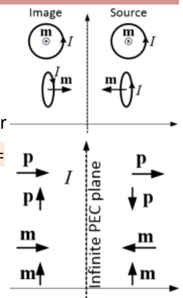
Uniqueness of Solution: Laplace – if the potential Boundary Condition $\Phi_{B.C}(\bar{r}_s)$ is given on the boundary of a volume V, then Laplace equation has a unique solution inside V. Poisson – If the potential $\Phi(\bar{r})$ or its normal derivative $\frac{\partial}{\partial n}\Phi(\bar{r})$ are known on the closed boundary of volume V, then up to an additive constant, the equation has a unique solution.

Uniqueness takeaway: Due to the uniqueness theorems, we know that once we found a solution to the problem (PDE+BC), then it is the one and only solution. If the potential boundary condition is given $\phi_{bc}(r_s)$ on the boundary of volume V, the Laplace equation possesses a unique solution inside V -> uniqueness of solution

Generalized Ohm's Law: $\bar{J} = \sigma\bar{E}$ (for a finite conductor)

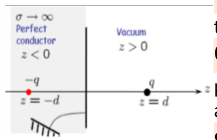
Typical Boundary Conditions for the Potential	
At an arbitrary surface: $\hat{n}\cdot(\bar{E}_1 - \bar{E}_2) = \frac{\eta}{\epsilon_0} \to \frac{\partial}{\partial n}\Phi_2 - \frac{\partial}{\partial n}\Phi_1 = \frac{\eta}{\epsilon_0}$	
Perfectly conducting boundary: Potential is constant along such boundary	
Inside a finite conductor: Current can't "jump" out from the surface. At the inner side of the surface, ON the surface: $\hat{n}\cdot\bar{E} = 0 \to \frac{\partial}{\partial n}\Phi = 0$	
Potential at infinity (this BC is always valid):	$\Phi(\bar{r}\to\infty) = 0$

METHOD OF IMAGES:

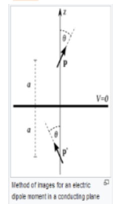


Volume Density due to Point Charge on Z-axis: $\rho(\bar{r}) = q\delta(x)\delta(y)\delta(z-d) = q\delta(\bar{r}-\bar{r}')$

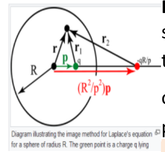
Point Charge near a Perfect Planar Conductor (With Derivations): $\nabla^2\Phi(\bar{r}) = -\rho(\bar{r})/\epsilon_0$. Charge density: $\rho(\bar{r}) = q\delta(x)\delta(y)\delta(z-d) = q\delta(\bar{r}-\bar{r}'), \bar{r}' = (0,0,d)$. We have B.Cs : $\Phi(x,y,z=0) = 0, \forall x,y$, $\Phi(\bar{r}\to\infty) = 0$. We express the solution as a sum of particular and homogenous solutions:



$\Phi(\bar{r}) = \Phi_p + \Phi_h$. For the particular: $\Phi_p(\bar{r}) = \int\frac{\rho(\bar{r}')dv\prime}{4\pi\epsilon_0|\bar{r}-\bar{r}'|} = \frac{q}{4\pi\epsilon_0|\bar{r}-\bar{r}'|}$. And for the homogenous: $\nabla^2\Phi_h(\bar{r}) = 0$, where the BC satisfied by Φ_h is $\Phi_h(x,y,z=0) = -\Phi_p(x,y,z=0) = -\frac{q}{4\pi\epsilon_0|(x,y,0)-(0,0,d)|} = \frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+d^2}}$. The only possible solution of the Laplace equation in the relevant domain ($z > 0$) (i.e. a solution that is homogenous in the domain) is: $\Phi_h(x,y,z) = -\frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z+d)^2}}$, as if a charge $-q$ is located at $\bar{r}' = (0,0,-d)$. The field at the entire half space $z > 0$ is given by $\Phi(\bar{r}) = \frac{q}{4\pi\epsilon_0|\bar{r}-\bar{r}'|} + \frac{-q}{4\pi\epsilon_0|\bar{r}-\bar{r}''|} = \Phi_p + \Phi_h$, $\bar{r}' = (0,0,d)$, $\bar{r}'' = (0,0,-d)$. The potential in cylindrical: $\Phi(r,\theta,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+(z-d)^2}} + \frac{-q}{\sqrt{r^2+(z+d)^2}}\right)$. The surface charge on the grounded PEC is: $\sigma = -\epsilon_0\frac{\partial\Phi}{\partial z}|_{z=0} = \frac{-qa}{2\pi(r^2+d^2)^{3/2}}$. Total charge induced on the PEC plane is the $-q$.

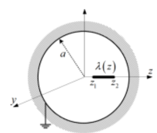


Reflection in a Conducting Plane, Electric Dipole Moments: Image of an electric dipole moment p at $(0,0,d)$ above an infinite grounded conducting plane in the xy plane is a dipole moment at $(0,0,-d)$ with equal magnitude and direction rotated azimuthally by π. That is, a dipole moment with Cartesian components $(psin(\theta)\cos(\phi), psin(\theta)\sin(\phi), pcos(\theta))$ will have an image dipole moment $(-psin(\theta)\cos(\phi), -psin(\theta)\sin(\phi), pcos(\theta))$. The dipole experiences a force in the z direction, given by $F = -\frac{1}{4\pi\epsilon_0}\frac{3p^2}{16d^4}(1+\cos^2(\theta))$, and a torque in the plane perpendicular to the dipole and the conducting plane: $\tau = \frac{1}{4\pi\epsilon_0}\frac{p^2}{16d^3}\sin(2\theta)$



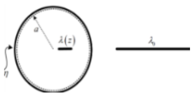
Reflection in a Conducting Sphere, Point Charges (Charge Inside): Find the potential inside a grounded sphere of radius R, centered at the origin, due to a point charge inside the sphere at position $\bar{P}$. Let q be the charge of this point. The image of this charge w.r.t the grounded sphere has charge $q' = \frac{qR}{d}$, and lies on a line connecting the center of the sphere and the inner charge at vector position $\left(\frac{R^2}{d^2}\right)\bar{P}$. The potential at a point specified by radius vector $\bar{r}$ due to both charges alone is given by: $\Phi(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+d^2-2\bar{r}\cdot\bar{P}}} - \frac{q}{\sqrt{\frac{r^2d^2}{R^2}+R^2-2\bar{r}\cdot\bar{P}}}\right)$. The potential vanishes on the surface of the sphere ($r = R$), and the potential inside is given by the above expression. It, however, is invalid outside the sphere. If the inner charge lies on the z −axis, then induced charge density: $\sigma(\theta) = \epsilon_0\frac{\partial\Phi}{\partial r}|_{r=R} = \frac{-q(R^2-d^2)}{r\pi R\left(R^2+d^2-2pRcos(\theta)\right)^{3/2}}$. Total charge is $-q$.

Reflection in a Conducting Sphere, Point Charges (Outside): In this case, the potential outside the sphere is given by the sum of the potentials of the charge and its image charge inside the sphere. Like in the first case, the image will have charge $-\frac{qR}{d}$ and located at vector position $\left(\frac{R^2}{d^2}\right)\bar{P}$. The potential inside the sphere depend only upon the true charge distribution inside the sphere. $\Phi_p(\bar{r}) = \frac{q}{4\pi\epsilon_0|\bar{r}-\bar{r}'|}$, and $\Phi_h(\bar{r}) = \frac{-Q}{4\pi\epsilon_0\left|r-(0,0,\frac{R^2}{d})\right|}$. The potential on the sphere surface is 0. The total charge in the sphere is given by the image charge: $-Q = -\frac{qR}{d}$. The True Charge is supplied by the ground.



Reflection in a Conducting Sphere, Line Charge (Inside): Inside a grounded, perfectly conducting sphere have a line charge distribution, given by $\lambda(z) = -\lambda_0\frac{a}{z}$, along the segment $[z_1, z_2] = \left[\frac{a}{4}, \frac{a}{2}\right]$. Outside the sphere $\Phi = 0$, since it is grounded. First, we divide the given segment into infinitesimal elements of length dz. Each element covers $[z, z+dz]$, and we consider such segments point charges of $dq' = \lambda(z)dz = -\lambda\frac{a}{z}dz$. Hence, $r' = \frac{R^2}{r} \to r = \frac{R^2}{r'}$. Also, $q' = -q\frac{R}{r} \to q = -q'\frac{r}{R} \to q = -q'\frac{R}{r'}$, where q' is the given charge inside, and q is the image charge outside (determined by who is inside and outside, not who is given and who is image). r is the distance of the outside charge to the sphere's center. Calculate the image of charge element dq' as $dq = -dq'\frac{a}{z} = \lambda_0\frac{a^2}{z^2}dz$. A general element dz on the given charge line is on the interval $[z, z+dz]$, and after the mapping we obtain an element on the interval $\left[\frac{a^2}{z}, \frac{a^2}{z+dz}\right]$, so its length is: $dl = \frac{a^2}{z} - \frac{a^2}{z+dz} = \frac{a^2}{z}\left(1 - \frac{1}{a+\frac{dz}{z}}\right)$. From Taylor Approximation $\left(\frac{1}{1+x} \approx 1 - x, x \ll 1\right)$ and get $dl = \frac{a^2}{z^2}dz$. Hence, linear charge density on the imaged charge line is $\lambda_{\text{imaged}} = \lambda_0\frac{\frac{a^2}{z^2}dz}{\frac{a^2}{z^2}dz} = \lambda_0$. The image lies between the points which are mapping of the edges of the given charge line: $z_{\text{left}} = \frac{a}{4} \to z_{\text{img,right}} = \frac{a^2}{z_{left}} = 4a$. $z_{\text{right}} = \frac{a}{2} \to z_{\text{img,left}} = 2a$. So – the image's charge is distributed on the segment $[2a, 4a]$ and has uniform distribution λ_0. Use superposition to find potential inside the sphere; $\Phi = \int_{a/4}^{a/2}\frac{-\lambda_0\frac{a}{z}dz\prime}{4\pi\epsilon_0 r} + \int_{2a}^{4a}\frac{\lambda_0 dz\prime}{4\pi\epsilon_0 r}$, where $r = \sqrt{x^2+y^2+(z-z')^2}$. The integrals can be found explicitly.

Imaging if Sphere is not Grounded: In this case, the boundary conditions on the sphere are a different. Potential does not have to be 0 on the sphere, but still has to be constant. The sphere is a PEC, so the potential on the outside takes the form of one generated by a point charge, equal to the total charge in the system: $Q = \int_{a/4}^{a/2} -\lambda_0 \frac{a}{z} dz = -\lambda_0 a ln(2)$. Hence, potential on the sphere is $\Phi(r = a) = -\frac{\lambda_0 a ln(2)}{4\pi\epsilon_0 a}$. For the inside, we know that the combination of line charges from before render the potential constant on the sphere, where the constant is zero. What image charges do we need to add to change this constant to $-\frac{\lambda_0 a ln(2)}{4\pi\epsilon_0 a}$? The only way to change a sphere's potential by a constant is to add a uniform surface charge distribution sphere, so that the total charge on it will equal Q. The charge density is found: $4\pi a^2 \eta = Q = -\lambda_0 a ln(2) \to \eta = \frac{-\lambda_0 a ln(2)}{4\pi a}$



Sphere potential: $\Phi_{sphere} = \frac{Q}{4\pi\epsilon_0 R}$

SEPARATION OF VARIABLES:

Separation of Variables – Cartesian Coordinates:

1	We seek a solution of the form $\Phi(\bar{r}) = X(x)Y(y)Z(z)$.
2	Substitute the solution into Laplace equation: $Y(y)Z(z)\frac{d^2}{dx^2}X(x) + X(x)Z(z)\frac{d^2}{dy^2}Y(y) + X(x)Y(y)\frac{d^2}{dz^2}Z(z) = 0$
3	Divide the equation by $\Phi(\bar{r}) = X(x)Y(y)Z(z)$. The result is $\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$,
4	From (3) we get $\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\frac{Z''(z)}{Z(z)} = \text{const.} = K_z^2$, and similarly: $\frac{X''(x)}{X(x)} = \text{const.} = -K_x^2$, $\frac{Y''(y)}{Y(y)} = \text{const.} = -K_y^2$, with the constraint (*needed to satisfy the original PDE*): $K_x^2 + K_y^2 + K_z^2 = 0$.
5	***The trivial solution*** $K_x = K_y = K_z = 0$: The functions must be first order polynomials: $X(x) = A + Bx$, $Y(y) = C + Dy$, $Z(z) = E + Fz$, where A,B,C,D,E,F are constants. $\Phi(\bar{r}) = (A + Bx)(C + Dy)(E + Fz)$.
6	***The non-trivial solution*** $K_x, K_y, K_z \neq 0$. Harmonic or Exponential Functions

$X(x) = Asin(K_x x) + Bcos(K_x x)$	$K_x^2 + K_y^2 + K_z^2 = 0$
$Y(y) = Csin(K_y y) + Dcos(K_y y)$	$\sin(i\alpha) = isinh(\alpha) \equiv \frac{i}{2}(e^{\alpha} - e^{-\alpha})$
$Z(z) = Esin(K_z z) + Fcos(K_z z)$	$\cos(i\alpha) = \cosh(\alpha) \equiv \frac{1}{2}(e^{\alpha} + e^{-\alpha})$
$\Phi(\bar{r}) = \left(Asin(K_x x) + Bcos(K_x x)\right)\left(Csin(K_y y) + Dcos(K_y y)\right)\left(Esin(K_z z) + Fcos(K_z z)\right)$	

Some examples of Possible Solutions:

The simplest: $K_x = K_y = K_z = 0 \Leftrightarrow \Phi(\bar{r}) = (A + Bx)(C + Dy)(E + Fz)$.

Non-trivial options: One possibility: $K_x \neq 0, K_y \neq 0, K_z' = \sqrt{K_x^2 + K_y^2}$, all Real.

It gives: $\Phi(\bar{r}) = \left(Asin(K_x x) + Bcos(K_x x)\right)\left(Csin(K_y y) + Dcos(K_y y)\right)\left(Esinh(K'_z z) + Fcosh(K'_z z)\right)$.

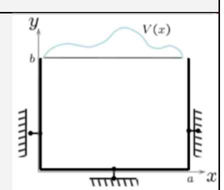
This is equivalent to $\Phi(\bar{r}) = \left(Asin(K_x x) + Bcos(K_x x)\right)\left(Csin(K_y y) + Dcos(K_y y)\right)\left(Ee^{K_z' z} + Fe^{-K_z' z}\right)$:.

Another possibility: $K_x \neq 0, K_y \neq 0, K_z = 0, \to K_y = iK_x, K_y' = K_x$, and the solution is: $\Phi(\bar{r}) = \left(Asin(K_x x) + Bcos(K_x x)\right)\left(Csinh(K' y) + Dcosh(K' y)\right)(E + FZ)$, etc… Many other options exist.

Properties of Solution: Any linear combination of homogenous solutions is also a solution. The constants $A, B, C, D, E, F, K_x, K_y, K_z$ are chose to satisfy B.C under the constraint $K_x^2 + K_y^2 + K_z^2 = 0$.

Example: Separation of Variables in Cartesian Coordinates

We have a ground plane with an infinite rectangular slit. It is covered by an electrode with potential distribution $V(x)$. The Boundary Conditions for this problem:



(1) $x = 0, 0 \le y \le b \Rightarrow \Phi = 0$	(2) $x = a, 0 \le y \le b \Rightarrow \Phi = 0$
(3) $y = 0, 0 \le x \le a \Rightarrow \Phi = 0$	(4) $y = b, 0 \le x \le a \Rightarrow \Phi = V(x)$

We try the non-trivial solution, with $K_z = 0$:

$$\Phi(\bar{r}) = [Asin(K_x x) + Bcos(K_x x)][Csin(K_y y) + Dcos(K_y y)], \quad K_x^2 + K_y^2 = 0$$

From BC (1): $B = 0 \Rightarrow \Phi = \sin(K_x x)\left[Csin(K_y y) + Dcos(K_y y)\right]$

From BC (2): $\sin(K_x a) = 0 \Rightarrow K_x = \frac{n\pi}{a}, n = 1,2,3 \ldots \Rightarrow \Phi = \Phi_n = \sin\left(\frac{n\pi}{a}x\right)\left[Csinh\left(\frac{n\pi}{a}y\right) + Dcosh\left(\frac{n\pi}{a}y\right)\right]$

From BC (3): $D = 0 \Rightarrow \Phi = \Phi_n = A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$

What about BC (4)? Try a Fourier Expansion (since each Φ_n satisfies homogenous B.C's, a sum of Φ_n conserves these B.C's). General expression of the solution: $\Phi(\bar{r}) = \sum_{n=1}^{\infty} A_n \Phi_n = \sum_{n=1}^{\infty} A'_n \sin\left(\frac{n\pi}{a}x\right) = V(x), 0 \le x \le a$

A Fourier expansion of B.C (4): $\Phi(x, y = b) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}x\right) = \sum_{n=1}^{\infty} A'_n \sin\left(\frac{n\pi}{a}x\right) = V(x), 0 \le x \le a$.

Multiply both sides by $\sin\left(\frac{m\pi x}{a}\right)$ and integrate $\int_0^a dx$. Use the identity $\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \begin{cases} 0, & n \neq m \\ \frac{a}{2}, & n = m \end{cases}$.

The result is: $\int_0^a V(x)\sin\left(\frac{m\pi x}{a}\right)dx = \frac{a}{2}A_m \sinh\left(\frac{m\pi b}{a}\right) \Rightarrow A_m = \frac{2}{a}\frac{\int_0^a V(x)\sin\left(\frac{m\pi x}{a}\right)dx}{\sinh\left(\frac{m\pi b}{a}\right)}$

The full exact solution is now given by: $\Phi(\bar{r}) = \sum_{m=1}^{\infty} A_m \sinh\left(\frac{m\pi}{a}y\right)\sin\left(\frac{m\pi}{a}x\right)$

Separation of Variables - Cylindrical coordinates:

Laplace Equation in Cylindrical Coordinates: $\nabla^2\Phi(\bar{r}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Phi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\Phi + \frac{\partial^2}{\partial z^2}\Phi = 0$, $\bar{r} = (r, \phi, z)$

1	Express the solution as $\Phi(\bar{r}) = R(r)\psi(\phi)f(z)$
2	Get the equation: $\psi(\phi)f(z)\frac{1}{r}\frac{d}{dr}[rR'(r)] + R(r)f(z)\frac{1}{r^2}\psi''(\phi) + R(r)\psi(\phi)f''(z) = 0$.
3	Divide by $R(r)\psi(\phi)f(z)$, and get: $\frac{1}{rR(r)}\frac{d}{dr}[rR'(r)] + \frac{1}{r^2}\frac{\psi''(\phi)}{\psi(\phi)} = -\frac{f''(z)}{f(z)}$. Hence RHS: $\frac{f''(z)}{f(z)} = -K_z^2 = \text{const.}$
4	Then, the LHS (after multiplication by r^2): $\frac{r}{R(r)}\frac{d}{dr}[rR'(r)] - K_z^2 r^2 + \frac{\psi''(\phi)}{\psi(\phi)} = 0$. Similarly: $\frac{\psi''(\phi)}{\psi(\phi)} = \text{const.}$
5	We end up with a set of three O.D.E's: $\frac{f''(z)}{f(z)} = -K_z^2$; $\frac{\psi''(\phi)}{\psi(\phi)} = -\nu_\phi^2$; $\frac{1}{r}\frac{d}{dr}[rR'(r)] - \left(\frac{\nu_\phi^2}{r^2} + K_z^2\right)R(r) = 0$ With two independent constants: K_z^2, ν_ϕ^2 (as in Cartesian coordinates: K_x^2, K_y^2 $\left(K_z^2 = -K_x^2 - K_y^2\right)$)
6	**Trivial solution:** $K_z = \nu_\phi = 0$ $f(z) = A + Bz$ $\psi(\phi) = C + D\phi \quad \Rightarrow \Phi(\bar{r}) = (A + Bz)(C + D\phi)(E + F\ln(r))$ $R(r) = E + F\ln(r)$
7	**Non Trivial Solutions Case (I):** $K_z = 0, \nu^2 > 0$ (ν real). Multiply the equation for R by r^2. Then: $r^2R''(r) + rR'(r) - \nu^2R(r) = 0 \to R(r) = Er^\nu + Fr^{-\nu}$ $f(z) = A + Bz$ $\psi(\phi) = C\sin(\nu\phi) + D\cos(\nu\phi) \quad \Rightarrow \Phi(\bar{r}) = (Ar^\nu + Br^{-\nu})[C\sin(\nu\phi) + D\cos(\nu\phi)](E + Fz)$ $R(r) = Er^\nu + Fr^{-\nu}$
8	**Non Trivial Solution Case (I) – modified:** $K_z = 0, \nu^2 < 0$ (ν imaginary: $\nu \mapsto i\nu'$) $f(z) = A + Bz$ $\psi(\phi) = C\sinh(\nu'\phi) + D\cosh(\nu'\phi)$ $R(r) = Er^{i\nu'} + Fr^{-i\nu'} = Ee^{\ln(r)i\nu'} + Fe^{-\ln(r)i\nu'} = E'\sin(\nu'\ln(r)) + F'\cos(\nu' \ln(r))$
9	**Non Trivial Solution Case (II):** $K_z^2 = -\gamma^2 < 0$ (γ real), $\nu^2 > 0$ (ν real) Multiply the equation for R by r^2. Then: $r^2R''(r) + rR'(r) + (\gamma^2r^2 - \nu^2)R(r) = 0$. Bessel functions of order ν: $R(r) = EJ_\nu(\gamma r) + FY_\nu(\gamma r)$ $f(z) = Ae^{\gamma z} + Be^{-\gamma z} = A'\sinh(\gamma z) + B'\cosh(\gamma z)$ $\psi(\phi) = C\sin(\nu\phi) + Dcos(\nu\phi) \quad \Rightarrow \Phi(\bar{r})$ $R(r) = EJ_\nu(\gamma r) + FY_\nu(\gamma r)$ $= (Ae^{\gamma z} + Be^{-\gamma z})[C\sin(\nu\phi) + Dcos(\nu\phi)][J_\nu(\gamma r) + FY_\nu(\gamma r)]$
10	**Non Trivial Solution Case (II) – modified:** $K_z^2 = -\gamma^2 > 0$ (K_z real, γ imaginary), $\nu^2 > 0$ (ν real) $f(z) = A\sin(K_z z) + B\cos(K_z z)$ $\psi(\phi) = C\sin(\nu\phi) + D\cos(\nu\phi)$ where $I_\nu(x), K_\nu(x)$ – Modified Bessel Functions of order ν $R(r) = EJ_\nu(\gamma r) + FY_\nu(\gamma r) = E'I_\nu(K_z r) + F'K_\nu(K_z r)$

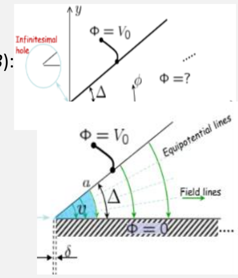
Bessel Functions	**Asymptotic Expressions for Integer order**
$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{n+2k}}{k!\,\Gamma(n+k+1)}$	$J_n(x) \approx \begin{cases} \frac{1}{2^n n!}x^n, \lvert x\rvert \ll 1 \\ \sqrt{\frac{2}{\pi x}}\cos\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right), \lvert x\rvert \gg 1 \end{cases}$
$Y_n(x) = \begin{cases} \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}, n \neq 0,1,2 \ldots \\ \lim_{p\to n}\frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}, n = 0,1,2 \ldots \end{cases}$	$Y_n(x) \approx \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right]\frac{x^n}{2^n n!} - \frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n}, \lvert x\rvert \ll 1 \\ \sqrt{\frac{2}{\pi x}}\sin\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right), \lvert x\rvert \gg 1 \end{cases}$ $\gamma = 0.57721 = \text{Euler's const}$
Modified Bessel Functions	**Asymptotic Expressions for Integer order**
$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(n+k+1)}$	$I_n(x) \approx \begin{cases} \left(\frac{x}{2}\right)^n\frac{1}{n!}, \lvert x\rvert \ll 1 \\ \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4n^2-1}{8x} + \cdots\right), \lvert x\rvert \gg 1 \end{cases}$
$K_n(x) = \begin{cases} \frac{\pi[I_{-n}(x) - I_n(x)]}{2\sin(n\pi)}, n \neq 0,1,2 \ldots \\ \lim_{p\to n}\frac{\pi[I_{-p}(x) - I_p(x)]}{2\sin(p\pi)}, n = 0,1,2 \ldots \end{cases}$	$K_n(x) \approx \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + \gamma\right]I_0(x),\ n = 0, \lvert x\rvert \ll 1 \\ \frac{1}{2}\left(\frac{x}{2}\right)^{-n}(n-1)!, n \neq 0, \lvert x\rvert \ll 1 \\ \frac{\sqrt{\pi}e^{-x}}{\sqrt{2x}}\left(1 + \frac{4n^2-1}{8x} + \cdots\right), \lvert x\rvert \gg 1 \end{cases}$

Orthogonality of Bessel Functions: Define $AJ_\nu(x) + BxJ'_\nu(x) = 0, \frac{A}{B} + \nu > 0, for\ all\ \frac{A}{B} + \nu > 0$ all roots are real. Then (orthogonality wrt weight r): $\int_0^a rJ_\nu\left(\gamma_{\nu,m}\frac{r}{a}\right)J_\nu\left(\gamma_{\nu,n}\frac{r}{a}\right)dr = \alpha_n\delta_{nm}$, where $\alpha_n = \frac{a^2}{2}\left[J_\nu'^2(\gamma_{\nu,n}) + \left(1 + \frac{\nu^2}{\gamma_{\nu,n}^2}\right)J_\nu^2(\gamma_{\nu,n})\right]$. Furthermore, the set is complete.

Fourier Series in terms of Bessel functions: If $f(r)$is of bounded variation in every subinterval $[r_1, r_2], 0 < r_1 < r_2 < a$, piecewise continuous, the integral $\int_0^a \sqrt{r}f(r)dr$ is finite, then $f(r)$ can be expanded in the Fourier series: $f(r) = \sum_{n=1}^{\infty} c_n J_\nu\left(\gamma_{\nu,n}\frac{r}{a}\right), 0 < r < a$, $c_n = \frac{1}{\alpha_n}\int_0^a rf(r)J_\nu\left(\gamma_{\nu,n}\frac{r}{a}\right)dr$. Note: the case $B = 0$ is included in this formula… then: $J_\nu(\gamma_{\nu,m}) = 0, J'_\nu(\gamma_{\nu,n}) = -J_{\nu+1}(\gamma_{\nu,n})$, and $\alpha_n = \frac{a^2}{2}J_{\nu+1}^2(\gamma_{\nu,n})$

Example: Separation of Variables in Cylindrical Coordinates #1

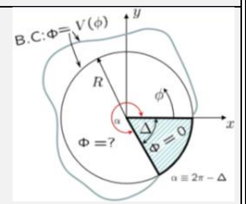
Infinite along $r, z -$ independent, has discontinuous potential at origin.



BC's: (1) $\phi = 0 \to \Phi = 0$ (2) $\phi = \Delta \to \Phi = V_0$ (3) $r \to 0$ Φ finite (Max/Min theorem).
We try the *trivial solution*, with no $z -$ dependence: $\Phi = (C + D\phi)(E + F\ln(r))$. *Due to BC (3)*: $F = 0 \to \Phi = C + D\phi$. *Due to BCs (1)+(2)*: $C = 0, D = V_0/\Delta$, and *hence* $\Phi = \frac{V_0\phi}{\Delta}$, $\bar{E} = -\nabla\Phi(\bar{r}) = -\hat{\phi}\frac{V_0}{r\Delta}$. The field goes to infinity near the origin (finite just of potential, over infinitely small distance) . So the field is singular. What about *energy density*? $\mathcal{E}_E = \frac{\epsilon_0}{2}\bar{E}\cdot\bar{E} = \frac{\epsilon_0}{2}\left(\frac{V_0}{r\Delta}\right)^2$ - singular as well. What about the *energy stored in a finite volume*?
$\int \mathcal{E}_E dv = \frac{\epsilon_0 V_0^2}{2\Delta^2}\int_\delta^a r dr\int_0^\Delta d\phi\int_0^\ell dz\frac{1}{r^2} = \frac{\epsilon_0 V_0^2\ell}{2\Delta}[\ln(r)]_\delta^a \to \infty$ as $\delta \to 0$, i.e. infinite, if our domain includes the origin, where the field is singular. It takes an infinite energy to sustain a potential discontinuity.

Example: Separation of Variables in Cylindrical Coordinates #2

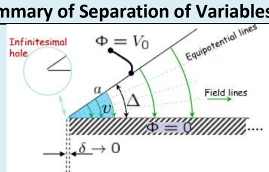
$z -$ independent structure, with the BC's:

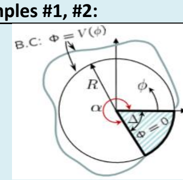


(1) $\phi = 0, 0 \le r \le R \Rightarrow \Phi = 0$	(2) $\phi = \alpha, 0 \le r \le R \Rightarrow \Phi = 0$
(3) $0 \le \phi \le \alpha,\ r = R \Rightarrow \Phi = V(\phi)$	(4) $0 \le \phi \le \alpha,\ r \le R \Rightarrow \Phi$ finite

Try non-trivial solution case (I) with no z- dependence. $\Phi(\bar{r}) = (Ar^\nu + Br^\nu)[C\sin(\nu\phi) + D\cos(\nu\phi)](E + Fz)$. The terms $(Ar^\nu + Br^\nu)$ and $(E + Fz)$ go to zero, by the minimum maximum theorem.
So: $\Phi(\bar{r}) = r^\nu[C\sin(\nu\phi) + D\cos(\nu\phi)]$. *Due to BC (1)*: $D = 0 \Rightarrow \Phi(\bar{r}) = Cr^\nu\sin(\nu\phi)$. *Due to BC (2)*: $\nu \mapsto \nu_n = n\frac{\pi}{\alpha} \Rightarrow \Phi \mapsto \Phi_n(\bar{r}) = C_n r^{\nu_n}\sin(\nu_n\phi)$. *We need to satisfy BC (3)*: *Use Fourier expansion*: $\Phi(\bar{r}) = \sum_n \Phi_n(\bar{r}) = \sum_n C_n r^{\nu_n}\sin(\nu_n\phi) = \sum_{n=1}^\infty A_n\sin(\nu_n\phi) = V(\phi), 0 \le \phi \le \alpha$. Multiply both sides by $\sin(\nu_m\phi)$ and integrate $\int_0^\alpha d\phi$: $\sum_{n=1}^\infty A_n\int_0^\alpha \sin\left(\frac{n\pi}{\alpha}\phi\right)\sin\left(\frac{m\pi}{\alpha}\phi\right)d\phi = \int_0^\alpha V(\phi)\sin\left(\frac{m\pi}{\alpha}\phi\right)d\phi \equiv V_m$. Now, use *orthogonality*: $\int_0^\alpha \sin\left(\frac{n\pi}{\alpha}\phi\right)\sin\left(\frac{m\pi}{\alpha}\phi\right)d\phi = \begin{cases}0, n \ne m\\ \frac{\alpha}{2}, n = m\end{cases}$. Then *we obtain*: $A_m = C_m R^{\nu_m} = \frac{2}{\alpha}V_m \Rightarrow C_n = \frac{2}{\alpha}V_n R^{-\nu_n}$. So we *end up with*: $\Phi(\bar{r}) = \frac{2}{\alpha}\sum_{n=1}^\infty V_n\left(\frac{r}{R}\right)^{\nu_n}\sin(\nu_n\phi)$, $\nu_n = \frac{n\pi}{\alpha}$. *The field*: $\bar{E} = -\nabla\Phi = -\hat{r}\frac{\partial\Phi}{\partial r} - \hat{\phi}\frac{1}{r}\frac{\partial\Phi}{\partial\phi}$. *Hence*:
$\bar{E} = -\frac{2}{\alpha}\sum_{n=1}^\infty V_n\frac{n\pi}{\alpha}\left(\frac{1}{R}\right)^{\frac{n\pi}{\alpha}}r^{\frac{n\pi}{\alpha}-1}\left[\hat{r}\sin\left(\frac{n\pi}{\alpha}\phi\right) + \hat{\phi}\cos\left(\frac{n\pi}{\alpha}\phi\right)\right]$. The power of $r^{\frac{n\pi}{\alpha}-1}$ is $> 0\ \forall n$ if $\alpha < \pi$, and the field is always regular. However, if $\alpha > \pi$, then the power of $r^{\frac{n\pi}{\alpha}-1}$ is < 0 for $n = 1$, thus field is singular at the origin. So the field may become singular. What about the energy density and the energy st[ored] $\frac{\epsilon_0}{2}\bar{E}\cdot\bar{E} =$
$\frac{\epsilon_0}{2}\frac{4\pi^2}{\alpha^4}\sum_{n=1}^\infty\sum_{m=1}^\infty V_n V_m nm\left(\frac{1}{R}\right)^{(n+m)\frac{\pi}{\alpha}}r^{(n+m)\frac{\pi}{\alpha}-2}\left[\sin\left(\frac{n\pi}{\alpha}\phi\right)\sin\left(\frac{m\pi}{\alpha}\phi\right) + \cos\left(\frac{n\pi}{\alpha}\phi\right)\cos\left(\frac{m\pi}{\alpha}\phi\right)\right]$. The energy density is singular at $r = 0$ if $\alpha > \pi$. Regarding the energy in a finite volume: note the orthogonality in ϕ. $\int\mathcal{E}_E dv = \int_0^a r dr\int_0^\alpha d\phi\int_0^\ell dz\mathcal{E}_E =$
$\frac{\epsilon_0}{2}\frac{4\pi^2\ell}{\alpha^2}\sum_{n=1}^\infty(nV_n)^2\left(\frac{1}{R}\right)^{\frac{2n\pi}{\alpha}}\int_0^a r^{\frac{2n\pi}{\alpha}-1}dr = \frac{\epsilon_0}{2}\frac{2\pi\ell}{\alpha^2}\sum_{n=1}^\infty nV_n^2\left(\frac{a}{R}\right)^{\frac{2n\pi}{\alpha}}$. Since $a \le R$, the total energy stored in a finite volume here is always finite.

Summary of Separation of Variables Cylindrical Examples #1, #2:





Potential BC	Discontinuous	Continuous, boundary w. edge	
		$\Delta < \pi$	$\Delta \ge \pi$
Field	Singular for any Δ	Singular at tip	Regular at tip
Energy Density	Singular at the tip for any Δ	Singular at tip	Regular at tip
Energy in a Finite Volume	Infinite for any Δ	Finite	Finite

Separation of Variables - Spherical coordinates:

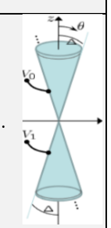
Laplace Equation in Spherical Coordinates: $\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi}{\partial\Phi^2} = 0$

Properties of Legendre functions: P_ℓ^m constitutes a multiple expansion. For $\ell = 0 \to P_0^0 = 1$, so we have point charge behaviour, no angular dependence, decays like r^{-1}. For $\ell = 1$ we have Dipole moment behavior, decays like r^{-2}. For $\ell = 2$ we have Quadrupole moment behavior, decays like r^{-3}

1	Express the solution as: $\Phi(\bar{r}) = R(r)T(\theta)\Psi(\phi)$
2	Substitute into the Laplace equation ($a' =$ derivative of a w.r.t to the argument): $\frac{\Psi T}{r^2}\frac{d}{dr}(r^2R') + \frac{R\Psi}{r^2\sin(\theta)}\frac{d}{d\theta}(\sin(\theta)T') + \frac{RT\Psi''}{r^2\sin^2(\theta)} = 0$
3	Multiply by $\frac{r^2\sin^2(\theta)}{RT\Psi}$ and obtain: $\frac{\sin^2(\theta)}{R}\frac{d}{dr}(r^2R') + \frac{\sin(\theta)}{T}\frac{d}{d\theta}(\sin(\theta)T') = -\frac{\Psi''}{\Psi}$. Notice that the RHS is a function of ϕ only, but the LHS is a function of r, θ only. Therefore $\frac{\Psi''}{\Psi} = \text{const.} = -m^2$
4	Use the constant in the equation above, and divide it by $\sin^2(\theta)$: $\frac{1}{R}\frac{d}{dr}(r^2R') + \frac{1}{T\sin(\theta)}\frac{d}{d\theta}(\sin(\theta)T') - \frac{m^2}{\sin^2(\theta)} = 0$. The first term is a function of r only, the other 2 terms are functions of θ only. Again, we must have $\frac{1}{R}\frac{d}{dr}(r^2R') = \text{const.} = \ell(\ell+1)$ (Euler type eq-n – polynom solution) , and $\frac{1}{T\sin(\theta)}\frac{d}{d\theta}(\sin(\theta)T') = \frac{m^2}{\sin^2(\theta)} - \ell(\ell+1)$
5	The chain rule: $\frac{d}{d\theta} = \frac{d}{d(\cos(\theta))}\frac{d(\cos(\theta))}{d\theta} = -\sin(\theta)\frac{d}{d(\cos(\theta))}$. Define $u = \cos(\theta)$. Then the equation for T becomes: $\frac{d}{du}\left[(1-u^2)\frac{dT}{du}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-u^2}\right]T(u) = 0$. In the most general case $(m, \ell \ne 0, \ell \ne -1)$: The associated Legendre equation. It may assume simpler forms otherwise.
6	We have: $\frac{\Psi''}{\Psi} = \text{const.} = -m^2, \frac{1}{R}\frac{d}{dr}(r^2R') = \text{const.} = \ell(\ell+1), \frac{d}{du}\left[(1-u^2)\frac{dT}{du}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-u^2}\right]T(u) = 0$
7	Trivial solution: Substitute $m = \ell = 0$, and obtain: $$\frac{\Psi''}{\Psi} = 0 \Rightarrow \Psi = A\phi + B$$ $$\frac{1}{R}\frac{d}{dr}(r^2R') = 0 \Rightarrow R' = \frac{c}{r^2} \Rightarrow R = \frac{C}{r} + D$$ $\frac{d}{du}\left[(1-u^2)\frac{dT}{du}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-u^2}\right]T(u) = 0 \Rightarrow T'(u) = \frac{C}{1-u^2} \Rightarrow T(u) = F + E\ln\left(\frac{1-u}{1+u}\right) = F + E\ln\left(tan\left(\frac{\theta}{2}\right)\right)$. To go from u to tan, we use $\frac{1-\cos(\theta)}{1+\cos(\theta)} = \tan\left(\frac{\theta}{2}\right)$. Therefore we obtain for the trivial solution: $\Phi(\bar{r}) = (A\phi + B)\left(\frac{c}{r} + D\right)\left(F + E\ln\left(tan\left(\frac{\theta}{2}\right)\right)\right)$
8	Non Trivial Solution (Equations for Ψ and R): substitute $m \ne 0, \ell \ne 0, -1$ and obtain: $$\Psi'' = -m^2\Psi \Rightarrow \Psi(\theta) = A\sin(m\phi) + B\cos(m\phi)$$ $\frac{d}{dr}(r^2R') - \ell(\ell+1)R = 0 \Rightarrow r^2R'' + 2rR' - \ell(\ell+1)R = 0$. We try $R = Ar^q$ and substitute: $A\frac{d}{dr}(r^2qr^{q-1}) - A\ell(\ell+1)r^q = 0 \Rightarrow A[q(q+1) - \ell(\ell+1)]r^q = 0$. Two solutions for q: $q = \ell,\ q = -\ell - 1$. Hence $R(r) = Ar^\ell + Br^{-\ell-1}$
9	Non Trivial Solution (Equation for T): $\frac{d}{du}\left[(1-u^2)\frac{dT}{du}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-u^2}\right]T(u) = 0$. The solutions are the "Associated Legendre Functions" of the 1st and 2nd kind. The solution: $T(u) = AP_\ell^m(u) + BQ_\ell^m(u), m = 0, \ldots, \ell$. IN OUR PROBLEMS $Q_\ell^m(u)$ IS NOT NEEDED SINCE IT IS SINGULAR AT $u = \pm 1\ (\theta = 0, \pi)$
10	Non Trivial Solution (General Solution)): The general solution $m \le \ell$: $$\Phi(\bar{r}) = (Ar^\ell + Br^{-\ell-1})\underbrace{(C\sin(m\phi) + D\cos(m\phi))P_\ell^m(\cos(\theta))}_{\text{Spherical harmonics } Y_\ell^m}$$ Note: $\int_{-1}^1 P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{m,n}$

Example: Separation of Variables in Spherical Coordinates #1 – Infinite Double Cone

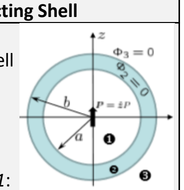
Infinite double cone. The structure is infinite in the z-direction. Find $\Phi, \bar{E}$ in the domain $\Delta \le \theta \le \pi - \Delta$.



(1) $\theta = \Delta \Rightarrow \Phi = V_0$	(2) $\theta = \pi - \Delta \Rightarrow \Phi = V_1$
(3) $\partial_\phi\Phi = 0$ (symmetry)	(4) $r \to 0, \Phi$ finite (min/max theorem)

Try the *trivial solution*: $\Phi(\bar{r}) = (A\phi + B)\left(\frac{C}{r} + D\right)\left[E\ln\left(tan\left(\frac{\theta}{2}\right)\right) + F\right]$. *From BC (3)*: $A = 0$. *From BC(4)*: $C = 0$.
Imposing BC's (1), (2) in $\Phi = E\ln\left(tan\left(\frac{\theta}{2}\right)\right) + F$: $V_0 = E\ln\left(tan\left(\frac{\Delta}{2}\right)\right) + F \Rightarrow F = \frac{V_0 + V_1}{2}$,
$V_1 = E\ln\left(tan\left(\frac{\pi-\Delta}{2}\right)\right) + F = -E\ln\left(tan\left(\frac{\Delta}{2}\right)\right) + F$. $E = \frac{V_0 - V_1}{2\ln\left(tan\left(\frac{\Delta}{2}\right)\right)}$. (by $\tan\left(\frac{\pi}{2} - \alpha\right) = \cot(\alpha) = 1/\tan(\alpha)$.
Finding the field: $\bar{E} = -\nabla\Phi = -\hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \hat{\theta}\frac{1}{r\sin(\theta)}\frac{V_1 - V_0}{\ln\left(tan\left(\frac{\Delta}{2}\right)\right)}$. *Note*: $\ln\left(tan\left(\frac{\Delta}{2}\right)\right)$ is always a negative number.

Example: Separation of Variables in Spherical Coordinates #2 – Point Dipole Inside Conducting Shell

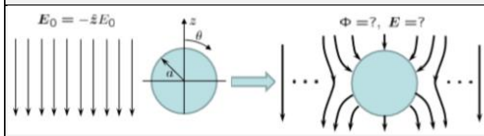
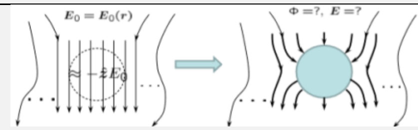


Point dipole inside a conducting spherical shell. Find the potential everywhere. The shell is a perfect conductor. The shell is not charged (i.e. total charge in the shell is zero). The dipole is located in the shell center. We have three regions:
$$0 \le r \le a \Leftrightarrow \Phi_1(\bar{r})$$
$$a \le r \le b \Leftrightarrow \Phi_2 = C = \text{const.}$$
$$b \le r \le \infty \Leftrightarrow \Phi_3(\bar{r})$$

Boundary Conditions:	
(1) $r = a \Rightarrow \Phi_1 = \Phi_2 = C$	(2) $r = b \Rightarrow \Phi_3 = \Phi_2 = C$
(3) $r \to \infty \Rightarrow \Phi_3 \to 0$	(4) $\oint_{S(r>b)}\bar{E}_3\cdot d\bar{a} = 0$

From BC(4) we know we must have $\Phi_3 = 0 \Rightarrow C = 0$. Symmetries: *Symmetry 1*: $\partial_\phi\Phi_1 = 0$. *Symmetry 2*: $\partial_\phi\Phi_3 = \partial_\theta\Phi_3 = 0 \Rightarrow \bar{E}_3 = \hat{r}E_{3_r}$ (since the BC for Φ_3 is const.). So – we only need to find Φ_1. Approach solution as $\Phi_1 = \Phi_p + \Phi_h$ a sum of particular and homogenous solution. We know that $\Phi_p(\bar{r})$ is for a dipole in free space, so $\Phi_p(\bar{r}) = \frac{P\cos(\theta)}{4\pi\epsilon_0 r^2}$, however, recall BC (1) for Φ_1: $r = a \Rightarrow \Phi_1 = \Phi_2 = C = 0$. Therefore we have BC for Φ_h: BC (1'): $\Phi_h(r=a) = -\Phi_p(r=a) = -\frac{P\cos(\theta)}{4\pi\epsilon_0 a^2}$. This condition cannot be satisfied by the trivial solution, so we try the *non-trivial*: $\Phi_h(\bar r) = (Ar^\ell + Br^{-\ell-1})(C\sin(m\phi) + D\cos(m\phi))P_\ell^m(\cos(\theta))$. Set $=0, \ell=1$ $(P_1^0(\cos(\theta) = \cos(\theta))$. Set $B=0$ (min/max theorem). Hence homogenous solution is: $\Phi_h(\bar r) = Ar\cos(\theta) \Rightarrow A = -\frac{P}{4\pi\epsilon_0 a^3}$, where A is found s.t. to satisfy BC (1'). Answer: $\Phi_1(\bar r) = \frac{P\cos(\theta)}{4\pi\epsilon_0 r^2} - \frac{Pr\cos(\theta)}{4\pi\epsilon_0 a^3}$

Example: Separation of Variables in Spherical Coordinates #3 – PEC Sphere Immersed in Uniform Electric Field

Scenario 1	Scenario 2
	 At the sphere location, excitation field $E_0(\bar r)$ is approx. const. and given by $-\hat z E_0$
$\bar E_0 = -\nabla\Phi_0, \nabla^2\Phi_0(\bar r) = 0, \Phi_0 = E_0 z = E_0 r\cos(\theta)$ $\Phi(\bar r) = ???$ We are interested in the potential outside the sphere. Equation to solve: $\nabla^2\Phi(\bar r) = 0$ BC's: (1): $\Phi(r\to\infty) = \Phi_0 = E_0 r\cos(\theta)$ (2): $\Phi(r=a) = \text{const.} = C$ We try the non-trivial solution with $m=0, \ell=1$: $\Phi(\bar r) = \left(Ar + \frac{B}{r^2}\right)\cos(\theta)$. Due to BC (1): $A = E_0$. Due to BC (2): $B = -E_0 a^3, C = 0$. Thereofre: $\Phi(\bar r) = E_0\left(r - \frac{a^3}{r^2}\right)\cos(\theta)$	$E_0 = -\nabla\Phi_0, \nabla^2\Phi_0(\bar r) = 0, \Phi = \Phi_0 + \Phi_{\text{dist.}}, \Phi_{\text{dist.}} = ???$ $\Phi_0(\bar r) \approx E_0 z = E_0 r\cos(\theta) r \le O(a)$ We are interested in the potential outside the sphere. Equation to solve: $\nabla^2\Phi_{\text{dist.}}(\bar r) = 0$ BC's: (1): $\Phi_{\text{dist}}(r\to\infty) \to 0$ (2): $\Phi_{\text{dist}}(r=a) = -\Phi_0(r=a) = -E_0 a\cos(\theta)$ We try the non-trivial solution with $m=0, \ell=1$: $\Phi(\bar r) = \left(Ar + \frac{B}{r^2}\right)\cos(\theta)$. Due to BC (1): $A=0$. Due to BC (2): $B = -E_0 a^3$. Therefore: $\Phi_{\text{dist}}(\bar r) = -E_0\frac{a^3}{r^2}\cos(\theta)$, and the answer: $\Phi(\bar r) = E_0\left(r - \frac{a^3}{r^2}\right)\cos(\theta)$

Discussion of Example #3: From the outside, the sphere looks like a dipole pointing in the $-z$ direction. The field: $\bar E(\bar r) = -\nabla\Phi = \bar E_0(\bar r) - E_0\frac{a^3}{r^3}(\hat r 2\cos(\theta) + \hat\theta\sin(\theta))$. At the sphere location, the excitation field $\bar E_0(\bar r)$ is approximately constant and given by $-\hat z E_0$. The field just outside the sphere: $\bar E(r=a) = -\hat z E_0 - E_0(\hat r 2\cos(\theta) + \hat\theta\sin(\theta)) = -3\hat r E_0\cos(\theta)$ (from the identity $\hat z = \hat r\cos(\theta) - \hat\theta\sin(\theta)$). The surface charge on the sphere: $\eta = \epsilon_0\hat n\cdot(\bar E_1 - \bar E_2) = -3\epsilon_0 E_0\cos(\theta)$. Note that the dipole is aligned with the local direction of the excitation field. The dipole strength is proportional to the local value of the excitation field. This calls for definition of particle polarizability α.

Particle Polarizability (scalar): The Polarizability α of an isotropic medium (isotropic=permittivity ϵ and permeability μ of the medium are uniform in all directions) is the ratio between the strength of the excited dipole $\bar p$ and the local value of the exciting field $\bar E^L$=the field at the location of the particle, in absence of that particle. $\bar p = \epsilon_0\alpha\bar E^L$. In the case of conducting sphere: $\alpha = 4\pi a^3 = 3V$

PARTICLE ARRAYS:

Polarizability matrix: For Anisotropic media, α is defined as a 3x3 matrix: $\begin{pmatrix}\alpha_{xx} & \alpha_{xy} & \alpha_{xz}\\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz}\\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz}\end{pmatrix}$. The elements along the diagonal describe the response parallel to the applied electric field. A large value of α_{yx} here means that an electric field applied in the x direction would strongly polarize the material in the y –direction.

Local Field of a Particle: $\bar E^L$ - field at the location of the particle, but with the absence of that particular particle.

Dipolar Potential: Dipolar potential at $\bar r$ due to point dipole $\bar p$ at $\bar r'$: $\Phi = \frac{\bar p\cdot\hat\imath_{r',r}}{4\pi\epsilon_0|\bar r - \bar r'|^2}$, $\hat\imath_{r',r}$ – unit vector from $\bar r'$ to $\bar r$.

Dipolar Field $\bar F(\bar r, \bar p)$: Taking the gradient of dipolar potential, we get: $\bar F(\bar r, \bar p) = -\frac{1}{4\pi\epsilon_0|\bar r - \bar r'|^3}\left[\bar p - 3(\bar p\cdot\hat\imath_{r',r})\hat\imath_{r',r}\right]$

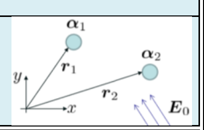
Riemann Zeta Function: $\zeta(s) = \sum_{m=1}^\infty \frac{1}{m^s}, \mathcal{R}(s) > 1$. Values: $\zeta(0) = -\frac{1}{2}, \zeta(2) = 1.6449, \zeta(3) = 1.2020569$

Edge Effects in Linear Array: Finite arrays response – solve numerically. Identical spherical particles. Define $\bar p_0 = \epsilon_0\alpha\bar E_0$ – the response of a single particle. Then the deviation of p/p_0 from 1 shows the effect of the array.

Two Particle System

The 2 particle system. Each of the particles has an induced dipole moment $\bar p_1, \bar p_2$. Define $\bar F(\bar r, \bar p)$- the Electric field at $\bar r$ due to dipole moment $\bar p$

$$\bar E_1^L = \bar E_0(\bar r_1) + \bar F(\bar r_1, \bar p_2) \to \bar p_1 = \epsilon_0\bar\alpha_1\bar E_1^L = \epsilon_0\bar\alpha_1[\bar E_0(\bar r_1) + \bar F(\bar r_1, \bar p_2)]$$
$$\bar E_2^L = \bar E_0(\bar r_2) + \bar F(\bar r_2, \bar p_1) \to \bar p_2 = \epsilon_0\bar\alpha_2\bar E_2^L = \epsilon_0\bar\alpha_2[\bar E_0(\bar r_2) + \bar F(\bar r_2, \bar p_1)]$$

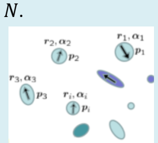


Many Particle System

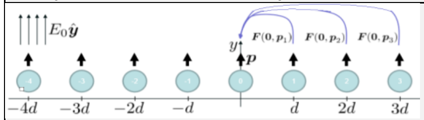
The previous example can be easily generalized to a system of many particles.

If we have N particles, the response of the $i'th$ particle: $\bar p_i = \epsilon_0\bar\alpha_i\left[\bar E_0(\bar r_i) + \sum_{j=1, j\ne i}^N \bar F(\bar r_i, \bar p_j)\right], i = 1, \dots, N$.

This formulation holds if the local field $\bar E^L$ of each particle can be considered constant over the location that will be occupied by the particle. Hence: the particles are small compared to the scale of variation of the excitation field (e.g. its wavelength). The particles are small compared to the inter-particle distances. If all the Polarizabilities are known, this will always yield a system of linear equations for all the dipole moment components.



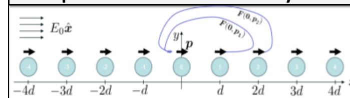
Example 1.1: Infinite Linear Array vertical field



Infinite array of identical particles is given, all with scalar Polarizability α. Particles are excited by a uniform electric field $E_0\hat y$. Find the dipolar moment induced in each particle.

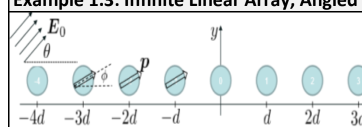
From symmetry and shift invariance – all dipoles are the same, and direction in $\hat y$. So: $\bar p_n = p_n\hat y = p\hat y$. Local field is made of the external field + all contributions from the rest of the particles. Define $\bar r_m = md\hat x$ – location of the m-th particle. **Local field at origin:** Contribution of m-th dipole to field at the origin is (by dipolar field), with $\hat\imath_{r',r} = -\text{sgn}(m)\hat x$: $\bar F(\bar 0, \bar p_m) = \frac{-1}{4\pi\epsilon_0|m|^3d^3}[p\hat y - 3(p\hat y\cdot\hat x)\hat x] = \frac{-p\hat y}{4\pi\epsilon_0|m|^3d^3}$. Hence: $\bar E^L(x=0) = E_0\hat y + \sum_{m=-\infty, m\ne 0}^{\infty}\left[\frac{-p\hat y}{4\pi\epsilon_0|m|^3d^3}\right]$ **Now we build our equation:** $\bar p = p\hat y = \epsilon_0\alpha\bar E^L(x=0) = \epsilon_0\alpha\left(E_0\hat y - \sum_{m=-\infty, m\ne 0}^{\infty}\left[\frac{-p\hat y}{4\pi\epsilon_0|m|^3d^3}\right]\right)$. We treat the infinite summation term $\sum_{m=-\infty, m\ne 0}^{\infty}\left[\frac{-p\hat y}{4\pi\epsilon_0|m|^3d^3}\right] = \frac{2p\hat y}{4\pi\epsilon_0 d^3}\sum_{m=1}^\infty\frac{1}{m^3}$. Recognize that the summation term is the Riemann Zeta function $\zeta(3) = 1.2020569$. We get: $p = \epsilon_0\alpha\left[E_0 - \frac{2p}{4\pi\epsilon_0 d^3}\zeta(3)\right] \to p = \frac{\epsilon_0\alpha E_0}{1 + \frac{\alpha\zeta(3)}{2\pi d^3}}$. The numerator $\epsilon_0\alpha E_0$= the response of a single particle if it was alone. Total response is weaker than that of each particle by itself – because field contributions from the neighbors are opposite to the external field.

Example 1.2: Infinite Linear Array horizontal field



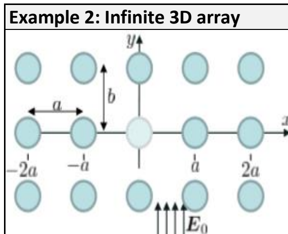
Exciting field is in the x-axis. From symmetry and shift invariance – all dipoles are same, directed in $\hat x$. Therefore $\bar p_n = p_n\hat x = p\hat x$. The local field is made of external field + contribution from the rest of the particles. Define $\bar r_m = md\hat x$- location of the m-th particle. **Local field at origin:** Contribution of the m-th dipole to the field at the origin: $\bar F(0, \bar p_m) = -\frac{1}{4\pi\epsilon_0|m|^3d^3}[p\hat x - 3(p\hat x\cdot\hat x)\hat x] = \frac{2p\hat x}{3\pi\epsilon_0|m|^3d^3}$. Hence: $\bar E^L(x=0) = E_0\hat x + \sum_{m=-\infty, m\ne 0}^{\infty}\left[\frac{2p\hat x}{4\pi\epsilon_0|m|^3d^3}\right] = E_0\hat x + \frac{p\hat x}{\pi\epsilon_0 d^3}\zeta(3)$. Again, we close the loop via $\bar p = p\hat x = \epsilon_0\alpha\bar E^L(x=0)$, substitute the $\bar E^L$ found above. Therefore: $p = \epsilon_0\alpha\left[E_0 + \frac{4p}{4\pi\epsilon_0 d^3}\zeta(3)\right] \to p = \frac{\epsilon_0\alpha E_0}{1 - \frac{\alpha\zeta(3)}{\pi d^3}}$. The response is stronger than that if each particle if it was alone – because the field contributions from the neighbors are directed in the same direction as the external field.

Example 1.3: Infinite Linear Array, Angled Field



If the field is not along the axes, e.g., $\bar E_0 = (\cos(\theta)\hat x + \sin(\theta)\hat y)E_0$. We simply superpose the results: $\bar p = \epsilon_0\alpha E_0\left[\frac{\cos(\theta)\hat x}{1 - \frac{\alpha\zeta(3)}{\pi d^3}} + \frac{\sin(\theta)\hat y}{1 + \frac{\alpha\zeta(3)}{2\pi d^3}}\right]$. In this case the dipole moment is NOT directed along $\bar E_0$. Rather: $\tan(\phi) = \frac{p_y}{p_x} = \frac{1 - \frac{\alpha\zeta(3)}{\pi d^3}}{1 + \frac{\alpha\zeta(3)}{2\pi d^3}}\tan(\theta)$

Example 2: Infinite 3D array



External field along y-axis: $\bar E_0 = E_0\hat y$. All particles are isotropic - α is a scalar. Define the location of the (m,n,k) particle: $\bar r_{m,n,k} = ma\hat x + nb\hat y + kc\hat z$. Symmetry and shift invariance imply that all particles will have same dipole moment, in the y direction: $\bar p_{m,n,k} = p\hat y$. The local field on each particle must also be in the same direction: $\bar E^L(\bar r_{m,n,k}) = E^L\hat y$. At the reference (origin): $\hat y E^L = \hat y E_0 + \sum_{m,n,k}^{(0)}\bar F(\bar 0, \bar p_{m,n,k})$. We express **field at origin due to the particle at location m,n,k** using Dipolar Field formula, with $\bar r = \bar 0$, $\bar r' = \bar r_{m,n,k} = ma\hat x + nb\hat y + kc\hat z$, $|\bar r - \bar r'| = \sqrt{(ma)^2 + (nb)^2 + (kc)^2}$, $\hat\imath_{r',r} = \frac{-\hat x ma - \hat y nb - \hat z kc}{\sqrt{(ma)^2 + (nb)^2 + (kc)^2}}$, $\bar p = \bar p_{m,n,k} = \epsilon_0\alpha\bar E^L = \epsilon_0\alpha\hat y E^L$. Then:

$$\bar F(0, \bar p_{m,n,k}) = -\frac{\alpha E^L}{4\pi[(ma)^2 + (nb)^2 + (kc)^2]^{3/2}}\left[\hat y - 3\left(\hat y\cdot\frac{-\hat x ma - \hat y nb - \hat z kc}{\sqrt{(ma)^2 + (nb)^2 + (kc)^2}}\right)\frac{-\hat x ma - \hat y nb - \hat z kc}{\sqrt{(ma)^2 + (nb)^2 + (kc)^2}}\right] = -\alpha E^L\frac{\hat y[(ma)^2 + (nb)^2 + (kc)^2] + \hat x mnab + \hat z knbc}{4\pi[(ma)^2 + (nb)^2 + (kc)^2]^{5/2}}$$

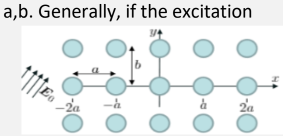
The $\hat x, \hat z$ components have odd dependence on the indices, and will not survive summation. Only the $\hat y$ will survive. So we have: $\hat y\cdot\bar F(0, \bar p_{m,n,k}) == -\alpha E^L\frac{(ma)^2 + (nb)^2 + (kc)^2}{4\pi[(ma)^2 + (nb)^2 + (kc)^2]^{5/2}}$, and $\hat y E^L = \hat y E_0 + \sum_{m,n,k}^{(0)}\bar F(\bar 0, \bar p_{m,n,k})$. These two equations lead to $E^L = E_0 + E^L\frac{\alpha}{4\pi b^3}S\left(\frac{a}{b}, \frac{c}{b}\right) \to E^L = \frac{E_0}{1 - \frac{\alpha}{4\pi b^3}S\left(\frac{a}{b}, \frac{c}{b}\right)}$, where $S(u,v)$ is the sum: $S(u,v) = \sum_{m,n,k}^{(0)}\frac{2n^2 - (mu)^2 - (kv)^2}{[(mu)^2 + n^2 + (kv)^2]^{5/2}} = S(v,u) \overset{u\approx v\approx 1}{\approx} \frac{\zeta(3)}{\pi} - 8\pi[K_0(2\pi u) + K_0(2\pi v)]$, Where K_0 is the Modified Bessel Function of the second kind. So, we finally got for the dipoles: $\bar p = \epsilon_0\alpha\bar E^L = \frac{\epsilon_0\alpha\hat y E_0}{1 - \frac{\alpha}{4\pi b^3}S\left(\frac{a}{b}, \frac{c}{b}\right)}$

Example 3: Infinite 3D Array with Angled Field

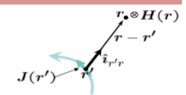
If the array was now excited by $\bar E_0 = \hat x E_0$, we would only need to interchange the role of a,b. Generally, if the excitation field is given by $\bar E_0 = \hat x E_{0_x} + \hat y E_{0_y} + \hat z E_{0_z}$, then $\bar p = \hat x p_x + \hat y p_y + \hat z p_z$:

$p_x = \frac{\epsilon_0\alpha E_{0_x}}{1 - \frac{\alpha}{4\pi a^3}S\left(\frac{b}{a}, \frac{c}{a}\right)} = C_{xx}E_{0_x}$ $\quad p_y = \frac{\epsilon_0\alpha E_{0_y}}{1 - \frac{\alpha}{4\pi a^3}S\left(\frac{b}{a}, \frac{c}{a}\right)} = C_{yy}E_{0_y}$ $\quad p_z = \frac{\epsilon_0\alpha E_{0_z}}{1 - \frac{\alpha}{4\pi a^3}S\left(\frac{b}{a}, \frac{c}{a}\right)} = C_{zz}E_{0_z}$

In matrix form: $\bar p = \bar C\bar E_0 \Leftrightarrow \begin{pmatrix}p_x\\ p_y\\ p_z\end{pmatrix} = \begin{pmatrix}C_{xx} & 0 & 0\\ 0 & C_{yy} & 0\\ 0 & 0 & C_{zz}\end{pmatrix}\begin{pmatrix}E_{0_x}\\ E_{0_y}\\ E_{0_z}\end{pmatrix}$



MAGNETOSTATICS



Governing Equations: $\nabla \times \bar{H} = \bar{J} \quad \nabla \cdot \mu_0 \bar{H} = 0$.

Vector Potential: $\mu_0 \bar{H} = \nabla \times \bar{A}$, where $\bar{A}$ =vector potential. **Coulomb Gauge Condition:** $\nabla \cdot \bar{A} = 0$

Poisson Equation for Vector Potential (derived from Ampere's law): $\nabla^2 \bar{A} = -\mu_0 \bar{J}$

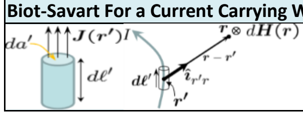
Solution for Vector Potential: $\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}dv'$. Consistent with the Gauge condition for static fields. In case of linear currents: $\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int \frac{Id\bar{l}}{|\bar{r}-\bar{r}'|}$, where dl is a length element in the direction of the current.

Magnetic Field from Vector Potential: $\bar{H}(\bar{r}) = \frac{1}{4\pi}\nabla \times \int \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}dv' = \frac{1}{4\pi}\int \nabla \times \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}dv'$. $\nabla \times$ operates on $\bar{r}$, not $\bar{r}'$.

Biot-Savart Law: $\bar{H}(\bar{r}) = \frac{1}{4\pi}\int \frac{\bar{J}(\bar{r}')\times \hat{i}_{r'r}}{|\bar{r}-\bar{r}'|^2}dv'$ (see diagram in top right corner) Provides a way to compute the magnetic field is all the currents are known. It can't handle BC's = cases where currents are "induced" on the other side of the boundary.

Vector Potential due to Magnetic Moment: $\bar{A} = \frac{\mu_0}{4\pi}\frac{\bar{m}\times\bar{r}}{|\bar{r}|^3}$ **Magnetic Moment due to Current Distribution:** $\bar{m} = \frac{1}{2}\int_V \bar{r}\times \bar{J}dV$

Biot-Savart For a Current Carrying Wire



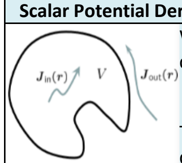
$d\bar{H}(\bar{r}) = \frac{\bar{J}(\bar{r}')dv'\times \hat{i}_{r'r}}{4\pi|\bar{r}-\bar{r}'|^2}$. From the diagram: $dv' = dl'da'_\perp \Rightarrow \bar{J}(\bar{r}')dv' = \bar{J}(\bar{r}')da'_\perp dl' = I'd\bar{l}'$. Hence: $d\bar{H}(\bar{r}) = \frac{I'd\bar{l}'\times\hat{i}_{r'r}}{|\bar{r}-\bar{r}'|^2}$. Summing it up: $\bar{H}(\bar{r}) = \frac{1}{4\pi}\int \frac{I'd\bar{l}'\times\hat{i}_{r'r}}{|\bar{r}-\bar{r}'|^2}$

Magnetic Scalar Potential: In a simply connected domain, where there is no free current $\nabla \times \bar{H} = 0$, we define a magnetic scalar potential Φ_m as $\bar{H} = -\nabla\Phi_m$. This also means that $\nabla^2\Phi_m = 0$. However, the domain V must be simply connected, as the passage from $\bar{H} = -\nabla\Phi$ to $\oint \bar{H}\cdot dl = 0\ \forall C \in V$ is guaranteed to hold only in simply connected domains.

Properties of Scalar Potential: Unlike the ES potential, Φ_m does not have the meaning of work or energy, and does not have to be continuous across boundaries.

Scalar Potential of Line Current: $\oint_C \bar{H}\cdot d\bar{s} = 2\pi r H_\phi = \int_S \bar{J}\cdot d\bar{a} = i$, so $H_\phi = \frac{i}{2\pi r}$, thus $\Phi_m = -\frac{i}{2\pi}\phi$. The potential is multivalued as the origin is encircled more than once. Because Φ_m is a solution to the Laplace equation, it must have an EQS analog. The ElectroQuasiStatic potential: $\Phi = -\frac{V}{2\pi}\psi, 0<\phi<2\pi$

Scalar Potential Derivation



We are interested in $\bar{H}(\bar{r})$ inside a given domain in space, $\bar{r}\in V$. The current $\bar{J}(\bar{r})$ is space is decomposed into two components: internal and external to V: $\bar{J}(\bar{r}) = \bar{J}_{in}(\bar{r}) + \bar{J}_{out}(\bar{r})$.

$$\bar{J}_{in}(\bar{r}) = \begin{cases}\bar{J}(\bar{r}), \bar{r}\in V\\ 0, \bar{r}\notin V\end{cases}, \bar{J}_{out}(\bar{r}) = \begin{cases}0, r\in V\\ \bar{J}(\bar{r}), \bar{r}\notin V\end{cases}.$$

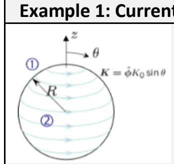
The governing equations are linear $\nabla\times\bar{H}(\bar{r}) = \bar{J}(\bar{r}) = \bar{J}_{in}(\bar{r}) + \bar{J}_{out}(\bar{r})$, $\nabla\cdot\mu_0\bar{H}(\bar{r}) = 0$, so we decompose $\bar{H}(\bar{r})$ in V into components, each associated with one of the currents: $\bar{H}(\bar{r}) = \bar{H}_{in}(\bar{r}) + \bar{H}_{out}(\bar{r})$, $\bar{r}\in V$. $\bar{J}_{in}(\bar{r}) \Leftrightarrow \bar{H}_{in}(\bar{r})$ = the field inside V due to current inside V. $\bar{J}_{out}(\bar{r}) \Leftrightarrow \bar{H}_{out}(\bar{r})$ = field inside V due to current outside V. Then, inside V: $\nabla\times\bar{H}_{in}(\bar{r}) = \bar{J}_{in}(\bar{r}), \bar{r}\in V$. $\bar{H}_{in}(\bar{r}) = \frac{1}{4\pi}\int\frac{\bar{J}_{in}(\bar{r}')\times\hat{i}_{r'r}}{|\bar{r}-\bar{r}'|^2}dv', \bar{r}\in V$.

$\nabla\times\bar{H}_{out}(\bar{r}) = \bar{J}_{out}(\bar{r}\in V) = 0$, $\bar{r}\in V$. Inside V, $\bar{H}_{out}(\bar{r})$ is a conservative field! $\Rightarrow \bar{H}_{out}(\bar{r}) = -\nabla\Phi_m(\bar{r}), \bar{r}\in V$.

He have expressions for $\bar{H}_{in}(\bar{r})$ and $\bar{H}_{out}(\bar{r})$, and $\bar{H}(\bar{r}) = \bar{H}_{in}(\bar{r}) + \bar{H}_{out}(\bar{r})$. $\bar{H}_{out}(\bar{r})$ is the homogenous solution, $\bar{H}_{in}(\bar{r})$ is the particular solution. Together, they satisfy B.C's on the boundary of V.Hence, the strategy above can be viewed as a particular+homogenous solutions construction (on the level of H): $\nabla\times\bar{H}(\bar{r}) = \bar{J}(\bar{r}) = \bar{J}_{in}(\bar{r}) + \bar{J}_{out}(\bar{r})$, $\nabla\cdot\mu_0\bar{H}(\bar{r}) = 0$.

Also, with the divergence-free equation, it follows that $\nabla^2\Phi_m(\bar{r}) = 0$, and we can use previously developed methods (e.g. SoV) to solve this equation. HOWEVER: V must be a simply connected domain.

Example 1: Current Carrying Spherical Shell



Current carrying spherical shell of radius R. Carries surface current $K = \hat{\phi}K_0 sin\theta \quad (@\, r = R)$. Find the magnetic field in the entire space. Sol: Divide space into two domains, both are simply connected: $r > R \to \phi_{m1}(r)$, $r < R \to \phi_{m2}(r)$, Both satisfying $\nabla^2\phi_{m1,2} = 0$ $\ H_{1,2}(r) = -\nabla_{m1,2}(r)$.

With the BC's:

$(\boldsymbol{I})\ r\to\infty, \phi_{m1}\to 0 \quad (\boldsymbol{II})\ r\to 0, \phi_{m2}\neq\infty \quad (\boldsymbol{III})\ r = R,\ H_{r,1} = H_{r,2}\ \ [\mu_0\hat{n}\cdot(H_1 - H_2) = 0]$

$(\boldsymbol{IV})\ r = R,\ \ H_{\theta,1} - H_{\theta,2} = K_0 sin\theta\ \ [\hat{n}\times(H_1 - H_2) = k]$

Also from symmetry considerations we must have $\frac{\partial}{\partial\phi} = 0$ hence $H_\phi = 0$. Chose the non-trivial solution in spherical coordinates with $m = 0$, $\ell = 1$ (bc #4). We are left with: $\phi_1 = \left(A_1 r + \frac{B_1}{r^2}\right)cos\theta$ (but $A_1 = 0$ due to bc#1), $\phi_2 = \left(A_2 r + \frac{B_2}{r^2}\right)\cos\theta$ (but $B_2 = 0$ due to bc#2). Now we find the fields H_1 H_2 in order to apply bc #3, #4: $\overline{H_1} = -\nabla\Phi_1$, $\overline{H_2} = -\nabla\Phi_2$. Solving for unknowns we get: $A_2 = -\frac{2B_1}{R^3} = -\left(\frac{2}{3}\right)K_0$ (BC III), $\frac{B_1}{R^3} - A_2 = K_0 \to B_1 = \left(\frac{K_0}{3}\right)R^3$ (BC IV). Note: The potential and field outside the shell have the same as from as those of an electric dipole. The field inside the shell is uniform, pointing in the +z direction

Example 2: Current Loop

Current is explicitly known in all space. $\Phi_m(r) = \frac{1}{4\pi}\frac{Ia\cdot\hat{i}_{r'r}}{|r-r'|^2} = \frac{1}{4\pi}\frac{\bar{m}\cdot\hat{i}_{r',r}}{|\bar{r}-\bar{r}'|}$ where $\bar{m} = I\bar{a} \to$ magnetic dipole moment. The far-field scalar magnetic potential has the same form as the electric dipole potential. CURRENT LOOP = "MAGNETIC DIPOLE"

Example 3: PEC Sphere Immersed in Uniform Magnetic Field

$\boldsymbol{H_0} = -\hat{z}H_0$ $\quad \phi_m(r) = zH_o = H_o rcos\theta$. The domain outside the sphere is simply connected and current free Hence scalar magnetic potential Φ_m can be used. Bc's:

$(\boldsymbol{I})r\to\infty \ \to\ \phi_m(r) = zH_o = H_o rcos\theta \quad (\boldsymbol{II})\ r = R \to \frac{\partial\phi_m(r)}{\partial r} = 0$ (the magnetic field normal to a perfect conductor must vanish). We try the non-trivial sol. Result: $\Phi_m(r) = H_0\left(r + \frac{R^3}{2r^2}\right)cos\theta = H_o rcos\theta + \frac{H_0 R^3}{2r^2}cos\theta$

Which is the initial potential + dipole term proportional to H_0. But dipole is directed towards +z while exiting field H_0 is toward –z, so we have negative Polarizability! Recall that magnetic potential due to a magnetic dipole is:

$\Phi_m(r) = \frac{1}{4\pi}\frac{Ia\cdot\hat{i}_{r'r}}{|r-r'|^2} = \frac{mcos\theta}{4\pi r^2}$ where $\boldsymbol{m} = \hat{z}m, r' = 0$, m=$Ia$ (dipole at origin directed in +z). We get the magnetic Polarizability of a perfectly conducting sphere: $\alpha_m = -2\pi R^3 = -\frac{3}{2}V$ where V is the volume. $\boldsymbol{m} = \alpha_m\boldsymbol{H}^L$. The magnetic field is given by: $H(r) = -\nabla\Phi_m(r) = -H_0(\hat{r}cos\theta - \hat{\theta}sin\theta) + H_0\left(\frac{R}{r}\right)^3\frac{1}{2}(\hat{r}2cos\theta + \hat{\theta}sin\theta)$ $\quad \hat{r}cos\theta - \hat{\theta}sin\theta = \hat{z}$

And the surface current:$\hat{n}\times(H_1 - H_2) = K \to \hat{r}\times\hat{\theta}H_0\frac{3}{2}sin\theta = \bar{K} \quad \bar{K} = \hat{\phi}\frac{3}{2}H_0 sin\theta$

TORQUES AND DIPOLES IN UNIFORM FIELDS:

Magnetic Dipole Moment: $\bar{m} = I\bar{a}$ **Torque on Dipole in Uniform Field**: $\bar{T} = \bar{r}\times\bar{F}$

Electric Dipole in Uniform Electric Field: Torque wrt center: $\bar{T} = \frac{\bar{d}}{2}\times\bar{E}q + \left(-\frac{\bar{d}}{2}\right)\times\bar{E}(-q) = q\bar{d}\times\bar{E} = \bar{p}\times\bar{E}$

Magnetic Dipole in Uniform Magnetic Field: Torque that a uniform $\bar{H}$ applies on a current loop: We know the Lorentz force $\bar{F} = q\bar{v}\times\mu_0\bar{H}$. Total torque $\bar{T} = 2\mu_0\bar{m}\times\bar{H} - \mu_0 I\pi R^2\hat{z}\times\bar{H} = \mu_0\bar{m}\times\bar{H}$

Electrostatic Torque & Potential due to non-uniform fields and dipole moments: $\bar{\tau} = \int_V d\bar{p}\times\bar{E}$, $U = \int_V d\bar{p}\cdot\bar{E}$

Magnetostatic Torque & Potential due to non-uniform fields and dipole moments: $\bar{\tau} = \int_V d\bar{m}\times H$, $U = \int_V dm\cdot\bar{H}$

MAGNETIZATION:

Magnetization Field: $\bar{M} = \frac{d\bar{m}}{dV}$, $\bar{m} = \iiint\bar{M}dV$, $\bar{m}$ is a magnetic moment, and the integral denotes integration over volume.

Magnetic Polarization $\bar{J}$: $\bar{J} = \mu_0\bar{M}$ **Magnetization Current**: $\bar{J}_m = \nabla\times\bar{M}$ **Bound Surface Current**: $\bar{K}_m = \bar{M}\times\hat{n}$

Total Current Density: $\bar{J} = \bar{J}_{free} + \bar{J}_m + \frac{\partial\bar{P}}{\partial t} = \bar{J}_{free} + \nabla\times M + \frac{\partial\bar{P}}{\partial t}$

Magnetostatic Equations: In the absence of free electric currents and time dependent fields: $\nabla\times\bar{H} = 0$, $\nabla\cdot\bar{H} = -\nabla\cdot\bar{M}$

MISCELLANEOUS:

Taylor Approximation: if $\sqrt{x} \ll 1$, then $\sqrt{1+x}\approx 1 + \frac{x}{2}$ **Logarithm definition:** $\log_b(x) = y \Leftrightarrow b^y = x$, Identitiy: $\ln(1+x)\approx x$ for small x. **Capacitor**: Field: $\bar{E} = -\frac{\eta}{\epsilon_0}\hat{z}$, Charge: $Q = \pi a^2\eta$, Capacitance: $C = \frac{Q}{V} = \epsilon_0\frac{\pi a^2}{d}$.

Capacitance of simple shapes: *Parallel-plate capacitor*: $C = \epsilon\frac{A}{d}$, *Coaxial cable*: $C = \frac{2\pi\epsilon\ell}{\ln(R_2/R_1)}$. *Pair of parallel wires*: $C = \frac{\pi\epsilon\ell}{\ln\left(\frac{d}{2a}+\sqrt{\frac{d^2}{4a^2}-1}\right)} = \frac{\pi\epsilon\ell}{arcosh\left(\frac{d}{2a}\right)}$. *Wire parallel to wall*: $C = \frac{2\pi\epsilon\ell}{arcosh\left(\frac{d}{a}\right)} = \frac{\pi\epsilon\ell}{\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)}$. *Concentric Spheres*: $C = \frac{4\pi\epsilon}{\frac{1}{R_1}-\frac{1}{R_2}}$.

Capacitor energy stored: $U = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2$ **Capacitor energy density**: $u = \frac{U}{Ad} = \frac{1}{2}\epsilon E^2$

Energy Density from Magnetic field: $u = \frac{B^2}{2A_0}$ **Dielectric Material**: *Material permittivity* $K_E = \frac{\epsilon_0}{\epsilon}$. *Permittivity*: $\epsilon = \epsilon_r\epsilon_0$

For a current loop (in spherical): $\bar{r}' = \text{acos}(\varphi')\,\hat{x} + \text{asin}(\varphi')\,\hat{y}$, $d\bar{l} = d\bar{r}' = (-\sin(\varphi')\,\hat{x} + \cos(\varphi')\,\hat{y})ad\varphi'$, observer position: $\bar{r} = rsin(\theta)\cos(\varphi)\,\hat{x} + rsin(\theta)\sin(\varphi)\,\hat{y} + rcos(\theta)\hat{z}$

Magnetic Field Infinite Current Carrying Wire: $\bar{H}_{\text{wire}} = \frac{\mu_0 I}{2\pi r}$ **Electrical Flux:** $\phi_E = \int_a^b\bar{E}\cdot\hat{n}d\bar{a} = \iint_A\bar{J}\cdot d\bar{a} = A\cdot Jcos(\theta)$

Electrodynamics: $I = \frac{dQ}{dt} = \int_S\bar{J}\cdot\hat{n}d\bar{a}$. $\bar{J} = \frac{I}{A}$. For finite resistance $\bar{J} = \frac{\bar{E}_{in}}{\rho} = \sigma\bar{E}$ **Magnetic Field for a toroid:** $\bar{B} = \frac{\mu_0 NI}{2\pi r}$

Displacement Field: In a dielectric material, an electric field $\bar{E}$ causes bound charges to separate, inducing a local electric dipole moment. Electric displacement field is defined as $\bar{D} = \epsilon_0\bar{E} + \bar{P}$, where $\bar{P}$ is the polarization density of the material.

Displacement Field Gauss Law: The field satisfies Gauss Law in a dielectric: $\nabla\cdot\bar{D} = \rho$

Current Density In Time-Invariant Fields: $\nabla\cdot\bar{J} = 0$

$\sin(-\theta) = -\sin(\theta)$	$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$	$\sin(\pi-\theta) = \sin(\theta)$	$\cos(\pi-\theta) = -\cos(\theta)$	$\tan(\pi-\theta) = -\tan(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$	$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$	$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$	$\sin(\theta+2\pi) = \sin(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$tan\left(\frac{\pi}{2}-\theta\right) = \cot(\theta)$	$\sin^2\left(\frac{\theta}{2}\right) = \frac{(1-\cos(\theta))}{2}$	$\cos^2\left(\frac{\theta}{2}\right) = \frac{(1+\cos(\theta))}{2}$	$\cos(\theta+2\pi) = \cos(\theta)$
$\sin\left(\theta+\frac{\pi}{2}\right) = \cos(\theta)$	$\sin(\theta+\pi) = -\sin(\theta)$	$\cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}$	$\sin^2(\theta)+\cos^2(\theta) = 1$	
$\cos\left(\theta+\frac{\pi}{2}\right) = -\sin(\theta)$	$\cos(\theta+\pi) = -\cos(\theta)$	$\sin(\theta) = \pm\sqrt{1-\cos^2(\theta)}$ Sign depends on quadrant of θ		
$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$		$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$		
$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$		$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta) = 2\cos^2(\theta)-1 = 1-2\sin^2(\theta)$		
$\sin(3\theta) = -4\sin^3(\theta)+3\sin(\theta)$		$\cos(3\theta) = 4\cos^3(\theta)-3\cos(\theta)$		*Euler's formula*: $e^{ix} = \cos(x)+isin(x)$
$2\cos(\theta)\cos(\phi) = \cos(\theta-\phi)+\cos(\theta+\phi)$		$2\sin(\theta)\cos(\phi) = \sin(\theta+\phi)+\sin(\theta-\phi)$		
$2\sin(\theta)\sin(\phi) = \cos(\theta-\phi)-\cos(\theta+\phi)$		$2\cos(\theta)\sin(\phi) = \sin(\theta+\phi)-\sin(\theta-\phi)$		

INTEGRALS: Basic Forms

$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$ $\int \frac{1}{x}dx = \ln|x|$ $\int u\,dv = uv - \int v\,du$ $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$

Integrals of Rational Functions

$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$ $\int \frac{1}{1+x^2}dx = \tan^{-1}x$ $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ $\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$

$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$ $\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$ $\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, a \neq b$

$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$ $\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$ $\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$

$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$ $\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$

Integrals with Roots

$\int \sqrt{x-a}\ dx = \frac{2}{3}(x-a)^{3/2}$ $\int \sqrt{x(ax+b)}\ dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\right]$ $\int \frac{x}{\sqrt{a^2-x^2}}\ dx = -\sqrt{a^2-x^2}$

$\int \frac{1}{\sqrt{x\pm a}}\ dx = 2\sqrt{x\pm a}$ $\int \sqrt{x^3(ax+b)}\ dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|$

$\int \frac{1}{\sqrt{a-x}}\ dx = -2\sqrt{a-x}$ $\int \sqrt{a^2-x^2}\ dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$ $\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$ $\int \sqrt{ax+b}\ dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$

$\int x\sqrt{x^2 \pm a^2}\ dx = \frac{1}{3}(x^2 \pm a^2)^{3/2}$ $\int \frac{1}{\sqrt{x^2\pm a^2}}\ dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$ $\int \frac{x}{\sqrt{x\pm a}}\ dx = \frac{2}{3}(x \mp 2a)\sqrt{x \pm a}$ $\int \frac{x}{\sqrt{x^2\pm a^2}}\ dx = \sqrt{x^2 \pm a^2}$

Integrals with Exponentials

$\int e^{ax}\ dx = \frac{1}{a}e^{ax}$ $\int xe^x\ dx = (x-1)e^x$ $\int xe^{ax}\ dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$ $\int x^2e^{ax}\ dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$ $\int xe^{-ax^2}\ dx = -\frac{1}{2a}e^{-ax^2}$

$\int x^ne^{ax}\ dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}dx$ $\int x^2e^x\ dx = (x^2-2x+2)e^x$ $\int x^3e^x\ dx = (x^3-3x^2+6x-6)e^x$

Integrals with Trigonometric Functions or their Products with Exponentials or Monomials

$\int \sin ax\ dx = -\frac{1}{a}\cos ax$ $\int \sin^2 ax\ dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$ $\int \sin^3 ax\ dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$

$\int \cos ax\ dx = \frac{1}{a}\sin ax$ $\int \cos^2 ax\ dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$ $\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$

$\int \sin^2 x\cos x\ dx = \frac{1}{3}\sin^3 x$ $\int \cos^2 ax\sin ax\ dx = -\frac{1}{3a}\cos^3 ax$ $\int \sin^2 ax\cos^2 ax\ dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$

$\int \tan ax\ dx = -\frac{1}{a}\ln\cos ax$ $\int \tan^2 ax\ dx = -x + \frac{1}{a}\tan ax$ $\int \tan^3 ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax$

$\int \cos x\sin x\ dx = \frac{1}{2}\sin^2 x + c_1 = -\frac{1}{2}\cos^2 x + c_2 = -\frac{1}{4}\cos 2x + c_3$ $\int x\sin^2 x\ dx = \frac{x^2}{4} - \frac{1}{8}\cos 2x - \frac{1}{4}x\sin 2x$

$\int \sin^2 ax\cos bx\ dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$ $\int \cos^2 ax\sin bx\ dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$

$\int \cos ax\sin bx\ dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b$ $\int \sin^2 ax\cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)}$

$\int e^x\sin x\ dx = \frac{1}{2}e^x(\sin x - \cos x)$ $\int e^{bx}\sin ax\ dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$ $\int e^x\cos x\ dx = \frac{1}{2}e^x(\sin x + \cos x)$

$\int e^{bx}\cos ax\ dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$ $\int xe^x\sin x\ dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$ $\int xe^x\cos x\ dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$

$\int x\cos x\ dx = \cos x + x\sin x$ $\int x\cos ax\ dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$ $\int x^2\cos x\ dx = 2x\cos x + (x^2-2)\sin x$

$\int x\sin x\ dx = -x\cos x + \sin x$ $\int x^2\cos ax\ dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$ $\int x\sin ax\ dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$

$\int x^2\sin x\ dx = (2-x^2)\cos x + 2x\sin x$ $\int x^2\sin ax\ dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$ $\int x\cos^2 x\ dx = \frac{x^2}{4} + \frac{1}{8}\cos 2x + \frac{1}{4}x\sin 2x$

Circular torus made from a conducting material of finite conductance σ. The geometry of toroidal ring is shown. Assume b<<a throughout the entire question. Torus is immersed in a magnetic field $\tilde{\bar{H}} = \hat{z}\tilde{H}_0 e^{-i\omega t}$.

a) *Use the integral form of Faraday's law and calculate $\int \bar{E}\cdot d\bar{l}$ along the toroidal ring. (note that under the assumption of quasi statics the first order of the electric field is considered).* **Sol:** Use of faraday's law: $\int_{c=\partial s}\bar{E}\cdot d\bar{l} = -\frac{d}{dt}\mu_0\iint_s \bar{H}\cdot d\bar{a} \Rightarrow \int_{c=\partial s}\bar{E}\cdot d\bar{l} = i\omega\mu_0\iint_s \bar{H}\cdot d\bar{a}$. So in our case (keep the right hand rule!) $2\pi a\tilde{E}_\phi = o\omega\mu_0\pi a^2\tilde{H}_z \Rightarrow \widetilde{E_\phi} = \frac{i}{2}a\omega\mu_0\tilde{H}_0$.

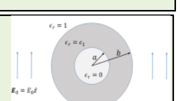
b) *Assume that the electric field is approximately uniform in the cross section of the torus. Calculate the electric field and deduce the current flowing along the toroidal ring.* **Sol:** We have $\tilde{J} = \sigma\tilde{E}$ therefore: $\tilde{J} = \hat{\phi}\widetilde{J_\phi} = \sigma\widetilde{E_\phi} = \frac{i}{2}\sigma a\omega\mu_0\widetilde{H_0}$. Since we consider the quantities to be uniform in the wire cross section, total current in the loop (in the ϕ direction) is: $\tilde{I} = \pi b^2\widetilde{J_\phi} = i\omega\mu_0\sigma\frac{\pi b^2 a}{2}\widetilde{H_0}$

c) *Calculate the magnetic dipole moment of the toroidal ring and use it to deduce it's magnetic Polarizability.* **Sol:** Magnetic dipole moment is $\tilde{m} = \tilde{I}A\hat{n}$ where $\tilde{I}$ is the current carried by the loop, A is its area, and $\hat{n}$ is the area normal unit vector, pointing in direction consistent with current direction and right hand rule. Thus: $\tilde{\bar{m}} = \hat{z}\widetilde{H_0}i\omega\mu_0\sigma\frac{\pi^2b^2a^3}{2} = \hat{z}\tilde{m}$. However, not that $\hat{z}\widetilde{H_0}$ is nothing but the Phasor of the local field. Therefore Polarizability is given by $\alpha_{zz} = i\omega\mu_0\sigma\frac{\pi^2b^2a^3}{2}$ (the double subscript z implies that it corresponds to the relation between z-directed local field and the resulting z-directed dipole).

d) *Find the range of parametric values under which the approximations used in your derivation are correct.* **Sol:** The calculation is valid as long as the loop diameter is much smaller than the wavelength, that is $\ll \lambda = \frac{2\pi c}{\omega}$

e) *The ring is placed at a distance d away from an infinite and perfectly conducting plane. The ring's axes is parallel to the surface. Assume d>>a. How would your answer to question 'c' change? Will the magnetic dipole be stronger? Weaker? Unchanged?* **Sol:** In order to satisfy our BC there must be an image magnetic dipole on the other side of the PEC wall. The imaged dipole is also pointing in the $+\hat{z}$ direction, and has the same magnitude as the original dipole. The magnetic field generated by the imaged dipole at the location of the original dipole is pointing in the $-\hat{z}$ direction, weakening the dipole.

A dielectric sphere of radius b consists of two layers as shown below. The sphere is placed in vacuum and its relative permittivity function is given by $\epsilon_r = \begin{cases} 0 & r<a \\ \epsilon_1 & a<r<b \\ 1 & r>b \end{cases}$. Assume that without the sphere there exists a uniform static electric field $\boldsymbol{E} = E_0\hat{z}$



a) *Write down the boundary conditions for the potential over all the boundaries.* **Sol**: (1) $\Phi_3 \to -E_0 r\cos(\theta)$ as $r \to \infty$. (2) On $r = b$: $\frac{d\Phi_3}{dr} = \epsilon_2\frac{d\Phi_2}{dr}$ and $\Phi_3 = \Phi_2$. (3) on $r = a$: $\frac{d\Phi_2}{dr} = 0$ and $\Phi_1 = \Phi_2$. (4) at $r = 0$: Φ_1 should be regular.

b) *Calculate the potential and use it to derive the electric field everywhere*. **Sol**: From BC1 we have to have a $\cos(\theta)$ dependence of the solution which implies the solution form: $\Phi_3 = \left(\frac{D}{r^2} - E_0 r\right)\cos(\theta)$, $\Phi_2 = \left(\frac{C}{r^2} + Br\right)\cos(\theta)$, $\Phi_1 = Ar\cos(\theta)$. From BC 3 on the normal derivative we conclude that solution for the three unknowns B, C, D can de decoupled from A. From these BCs we get the equations for the unknowns: $\frac{D}{b^2} - E_0 b = \frac{C}{b^2} + Bb$, $-\frac{2D}{b^3} - E_0 = -\frac{2C\epsilon_2}{b^3} + B\epsilon_2$, $-\frac{2C}{a^3} + B = 0$, $Aa = \frac{C}{a^2} + Ba$. By solving the first three equations, and then for the fourth we get: $A = \frac{9b^3}{2}\frac{E_0}{(\epsilon_2-1)a^3-(\epsilon_2+2)b^3}$, $B = \frac{3b^3E_0}{(\epsilon_2-1)a^3-(\epsilon_2+2)b^3}$, $C = \frac{3a^3b^3E_0}{(\epsilon_2-1)a^3-(\epsilon_2+2)b^3}$, $D = \frac{b^3(-2b^3(\epsilon_2-1)+a^3(\epsilon_2+2))E_0}{(\epsilon_2-1)a^3-(\epsilon_2+2)b^3}$. The E-field is given by $\bar{E} = -\nabla\Phi$, specifically in regions 1 this implies $E_1 = -\hat{z}A$

c) *Calculate the effective Polarizability of the sphere*. **Sol**: For the Polarizability of the core-shell structure we need to equate the secondary term of the potential in region 3 with that of an electric dipole of moment $\bar{p} = p\hat{z}$ at the origin. This reads: $\Phi_d = \frac{p}{4\pi\epsilon_0 r^2}\cos(\theta)$. Thus we conclude that $p = 4\pi\epsilon_0 D$ and therefore the electric Polarizability reads $\alpha = 4\pi D$

d) *Find a condition on ϵ_1, a and b (the inner and outer radii) so that the field outside the sphere will be the same as if there was no sphere at all. Explain the result.* **Sol**: If the induced electric dipole on the core-shell structure vanishes, then secondary potential outside the structure will be nullified and therefore we will see no effect of the core-shell on the uniform field outside. Thus the condition on problem parameters is such that Polarizability is zero, namely, $2b^3(\epsilon_2 - 1) = a^3(\epsilon_2 + 2)$. In this case the dipole on the shell and the dipole in the $\epsilon = 0$ regions will cancel out. Note that the electric Polarizability of a $\epsilon = 0$ sphere is negative.

e) *Assume all the parameters are chosen so that question 'd' is satisfied. What can be said about the field outside the sphere if we placed an object of arbitrary shape and electrical features inside the $r < a$ sphere?* **Sol**: Insertion of any object into region 1 will not affect potential and field problem in region 2 and 3 since at boundary of $r = a$, normal derivative of the potential Φ_2 is zero and indep. of the potential in the internal region. So, due to uniqueness theorem, solution we got is unique and unchanged.

The problem ahead is completely 2D, and there is no meaning for the z axis. Consider a material with constitutive relations $\bar{D} = \epsilon_{xx}E_x\hat{x} + \epsilon_{yy}E_y\hat{y}$, where $\epsilon_{xx}, \epsilon_{yy}$ are constants. **(a)** *Derive Laplace's Equation for this material.* **(b)** *Use separation of variables in order to find the general solution to the equation you have derived in the previous section.* **(c)** Consider the device in the figure below. The rectangular box has edges of lengths L_x, L_y. The upper edge is made of a series of electrodes that force the potential $V = V_0\sin(\pi x/L_x)$. The other 3 edges are made from a PEC and are grounded. The box is filled with the material from previous section. *Assume $L_x = L_y = L$ and find the electric potential inside the box.* **(d)** *Now we remove the material from the box, leaving vacuum. What should be the relations L_x/L and L_y/L in order for the potential to have the same mathematical form as before?*

a), b) Generally we still have $\bar{E} = -\nabla\Phi$ and $\nabla\cdot\bar{D} = 0$, but now $\bar{D} = \epsilon_{xx}\partial_x\Phi + \epsilon_{yy}\partial_y\Phi$, so the equations become: $\nabla\cdot\left(\epsilon_{xx}\partial_x\Phi\hat{x} + \epsilon_{yy}\partial_y\Phi\hat{y}\right) = 0$, $\epsilon_{xx}\partial_x^2\Phi + \epsilon_{yy}\partial_y^2\Phi = 0$, $\Phi(x,y) = X(x)Y(y)$, $\epsilon_x\frac{X''}{X} + \epsilon_y\frac{Y''}{Y} = 0$. Now the usual consideration of the separation of variables method lead to (for simplicity we omit one subscript of the epsilons, but still mean the same thing): $\epsilon_x X'' + k_x^2X = 0$, $\epsilon_y Y'' + k_y^2 Y = 0$, $k_x^2 + k_y^2 = 0$. Solutions: $\cos\left(\frac{k_x}{\sqrt{\epsilon_x}}x\right)$,$\sin\left(\frac{k_x}{\sqrt{\epsilon_x}}x\right)$,$\cosh\left(\frac{k_x}{\sqrt{\epsilon_x}}x\right)$,$\sinh\left(\frac{k_x}{\sqrt{\epsilon_x}}x\right)$. The same applies for 'Y'. Note, that as in the standard case, trigonometric function in 'X' should come with exponential in 'Y' and vice versa – in order to null the modified Laplace equation. **c)** $\Phi = \sin\left(\frac{k}{\sqrt{\epsilon_x}}x\right)\sinh\left(\frac{k}{\sqrt{\epsilon_x}}y\right)$ (already include B.C on x=0 and y=0). Boundary conditions: $\frac{k}{\sqrt{\epsilon_x}}L = \pi n$. The general solution: $\Phi = \sum_{n=1}^{\infty}A_n\sin\left(\frac{\pi}{L}n\cdot x\right)\sinh\left(\frac{\sqrt{\epsilon_x}}{\sqrt{\epsilon_y}}\frac{\pi}{L}n\cdot y\right)$, $\Phi = \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi\sqrt{\epsilon_x}}{L\sqrt{\epsilon_y}}y\right)$. The coefficients can be found from the Fourier series: $V_0\sin\left(\frac{\pi}{L}x\right) = \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi\sqrt{\epsilon_x}}{\sqrt{\epsilon_y}}\right) \to n = 1, A_1 = \frac{V_0}{\sinh\left(\frac{\pi\sqrt{\epsilon_x}}{\sqrt{\epsilon_y}}\right)}$ **d)** In the case of box with unequal sides, the solution is given by $\Phi = \sum_{n=1}^{\infty}A_n\sin\left(\frac{\pi}{L_x}n\cdot x\right)\sinh\left(\frac{\pi}{L_x}n\cdot y\right)$, and from the remaining BC: $n = 1, A_1 = V_0/\sinh\left(\frac{\pi L_y}{L_x}\right)$. Direct comparison shows that $L_x = L$, $\frac{L_y}{L_x} = \frac{\sqrt{\epsilon_x}}{\sqrt{\epsilon_y}}$

Magnetic charge model	Amperian currents model
Sources	
$\rho_m = -\nabla(\mu_0\mathbf{M})$ $\mathbf{J}_m = \frac{\partial}{\partial t}(\mu_0\mathbf{M})$ $\eta_m = -\hat{\mathbf{n}}\cdot(\mu_0\mathbf{M}_2 - \mu_0\mathbf{M}_1)$	$\mathbf{J}_a = \nabla\times\mathbf{M}$ $\hat{\mathbf{n}}\times(\mathbf{M}_2 - \mathbf{M}_1) = \mathbf{k}_a$
Maxwell's equations	
$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} - \mathbf{J}_m = -\frac{\partial\mathbf{B}}{\partial t}$ $\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$ $\nabla\cdot\mathbf{D} = \rho_f$ $\nabla\cdot\mu_0\mathbf{H} = \rho_m \iff \nabla\cdot\mathbf{B} = 0$ Magnetic charge conserved: $\nabla\cdot\mathbf{J}_m = -\frac{\partial\rho_m}{\partial t}$	$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}_a}{\partial t} = -\frac{\partial\mathbf{B}}{\partial t}$ $\nabla\times\mathbf{H} = \mathbf{J}_f + \mathbf{J}_a + \frac{\partial\mathbf{D}}{\partial t}$ $\nabla\cdot\mathbf{D} = \rho_f$ $\nabla\cdot\mu_0\mathbf{H}_a = 0 \iff \nabla\cdot\mathbf{B} = 0$
Boundary conditions	
$\hat{\mathbf{n}}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0$ $\hat{\mathbf{n}}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{k}_f$ $\hat{\mathbf{n}}\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \eta_f$ $\hat{\mathbf{n}}\cdot(\mu_0\mathbf{H}_2 - \mu_0\mathbf{H}_1) = \eta_m \iff \hat{\mathbf{n}}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0$ Magnetic charge conserved: $\hat{\mathbf{n}}\cdot(\mathbf{J}_2 - \mathbf{J}_1) = -\frac{\partial\eta_m}{\partial t}$	$\hat{\mathbf{n}}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0$ $\hat{\mathbf{n}}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{k}_f + \mathbf{k}_a$ $\hat{\mathbf{n}}\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \eta_f$ $\hat{\mathbf{n}}\cdot(\mu_0\mathbf{H}_{a,2} - \mu_0\mathbf{H}_{a,1}) = 0 \iff \hat{\mathbf{n}}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0$
Definition of magnetic flux	
$\mathbf{B} = \mu_0(\mathbf{H}_\mathbf{M})$	$\mathbf{B} = \mu_0\mathbf{H}_a$

Auxiliary fields, polarization, magnetization:

$$\bar{D}(\bar{r},t) = \epsilon_0\bar{E}(\bar{r},t) + \bar{P}(\bar{r},t)$$

$$\bar{H}(\bar{r},t) = \frac{1}{\mu_0}\bar{B}(\bar{r},t) - \bar{M}(\bar{r},t)$$

$\bar{P}$=Polarization field

$\bar{M}$= Magnetization field

Polarization Volume charge density: $\rho_p = -\nabla\cdot\bar{P}$

Polarization Surface Charge Density: $-\hat{n}\cdot(\bar{P}_1 - \bar{P}_2)$

Magnetic field generated by electric dipole rotating around the z axis with angular velocity ω, when the dipole is created by a small sphere with radius a in which a uniform polarization is present, choose solutions of the form $\Phi_{out} = \frac{B}{r^2}\cos(\theta)$, $\Phi_{in} = Ar\cos(\theta)$ for finding the Electric Field.

Current is time derivative of polarization density.