Name	Form	Domain	Solution
Infinite Strings	$u_{tt}(x,t)=a^2u_{xx}(x,t)$ $u(x,0)=f(x)$ $u_t(x,0)=g(x)$	$x\in(-\infty,\infty)$ $t\geq 0$	$u(x,t)=\frac{f(x-at)+f(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}g(s)ds$ Where f is twice differentiable and g is differentiable
Half-Infinite Strings (*fixed boundary*)	$u_{tt}(x,t)=a^2u_{xx}(x,t)$ $u(x,0)=f(x)$ $u_t(x,0)=g(x)$ $u(0,t)=0$	$0\leq x<\infty$ $t\geq 0$	$\tilde{u}(x,t)=\frac{\tilde{f}(x-at)+\tilde{f}(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\tilde{g}(s)ds$ Where $\tilde{f}=\begin{cases}f(x);x\geq 0\\-f(-x);x<0\end{cases}$, $\tilde{g}=\begin{cases}g(x);x\geq 0\\-g(-x);x<0\end{cases}$ Where f is twice differentiable, $f(0)=0$, g is differentiable, and $g(0)=0$.
Finite Strings (*fixed boundary*)	$u_{tt}(x,t)=a^2u_{xx}(x,t)$ $u(x,0)=f(x)$ $u_t(x,0)=g(x)$ $u(0,t)=u(l,t)=0$	$0\leq x\leq l$ $t\geq 0$	$\tilde{u}(x,t)=\frac{\tilde{f}(x-at)+\tilde{f}(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\tilde{g}(s)ds$ Where $\tilde{f}$ and $\tilde{g}$ are the $2l$ periodic extensions of the odd extensions of f and g respectively, f is twice differentiable, $f'(0)=0$, g is differentiable, and $g'(0)=0$
Finite Strings (*fixed boundary*) (Fourier Method)	$u_{tt}(x,t)=a^2u_{xx}(x,t)$ $u(0,t)=0$ $u(l,t)=0$ $u(x,0)=f(x)$ $u_t(x,o)=g(x)$	$0\leq x\leq l$ $t\geq 0$	$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi a}{l}t\right)+B_n\sin\left(\frac{n\pi a}{l}t\right)\right)\sin\left(\frac{n\pi}{l}x\right)$ $A_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{k\pi}{l}x\right)dx$ $B_n=\frac{2}{n\pi a}\int_0^l g(x)\sin\left(\frac{n\pi}{l}x\right)dx$ In general, $u(x,t)=\sum_{n=1}^{\infty}T(t)X(x)$ | $A_n=\frac{2}{l}\int_0^l f(x)X_n(x)$ | $B_n=\frac{2}{a\lambda l}\int_0^l g(x)X_n(x)$
Laplace Equation (*Rectangular Domain*)	$\Delta u=0;(x,y)\in D$ $u(x,0)=\varphi_0(x)$ $u(x,b)=\varphi_1(x)$ $u(0,y)=\psi_0(y)$ $u(a,y)=\psi_1(y)$ In case of non-homo conditions: ⋆ $u_x(0,y)=\psi_0(y)$ $u_x(a,y)=\psi_1(y)$ (1) Do Sturm Liouville to find $X(x)$ and λ_n (consider $\lambda=0$). (2) $Y_n(y)=A_n\sinh(\lambda y)+B_n\sinh(\lambda(b-y)$ (consider $\lambda=0$)	$D=[0,a]\times[0,b]$ ⋆↓ (3) $u(x,t)=\sum X(x)Y(y)$, use boundary condition to find some coefficents. (4) For other coefficient use other B.C (non-homogeneous). Plug in, then integrate both sides for a coefficient. For the same equation you got from plugging in the non-homo condition, multiply both sides by $X_n(x)$, then integrate.	$u(x,y)=v(x,y)+w(x,y)$ Where (this is achieved using separation of variables, detailed at the end of the block) $v(x,y)=\sum_{n=1}^{\infty}\left(A_n\sinh\left(\frac{n\pi}{a}y\right)+B_n\sinh\left(\frac{n\pi}{a}(b-y)\right)\right)\sin\left(\frac{n\pi}{a}x\right)$ $w(x,y)=\sum_{n=1}^{\infty}\left(C_n\sinh\left(\frac{n\pi}{b}x\right)+D_n\sinh\left(\frac{n\pi}{b}(a-x)\right)\right)\sin\left(\frac{n\pi}{a}y\right)$ $A_n=\frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a\varphi_1(x)\sin\left(\frac{n\pi}{a}x\right)dx$ | $C_n=\frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)}\int_0^b\psi_1(y)\sin\left(\frac{n\pi}{b}y\right)dy$ $B_n=\frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a\varphi_0(x)\sin\left(\frac{n\pi}{a}x\right)dx$ | $D_n=\frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)}\int_0^b\psi_0(y)\sin\left(\frac{n\pi}{b}y\right)dy$ SEP: $\lambda_n=\frac{n\pi}{a}$, therefore $X_n(x)=\sin(\lambda x)$ and $Y_n(y)=A_n\sinh(\lambda y)+B_n\sinh(\lambda(b-y))$
Laplace Equation (*Disk Domain*)	$\Delta u=0;(x,y)\in D$ $u(x,y)=f(x,y)$ $(x,y)\in\partial D$	$D=\{(x,y)\mid x^2+y^2<R_0^2\}$	Inside the disk: $u(r,\theta)=A_0+\sum_{n=1}^{\infty}r^n(A_n\cos(n\theta)+B_n\sin(n\theta))$ Outside the disk: $u(r,\theta)=A_0+\sum_{n=1}^{\infty}r^{-n}(A_n\cos(n\theta)+B_n\sin(n\theta))$ For a Ring: $u(r,\theta)=A_0+B_0\ln(r)+\sum_{n=1}^{\infty}(A_nr^n\cos(n\theta)+A_{-n}r^{-n}\cos(n\theta)+B_nr^n\sin(n\theta)+B_{-n}r^{-n}\sin(n\theta)$ Where $A_n=\frac{1}{\pi R_0^n}\int_0^{2\pi}h(\theta)\cos(n\theta)\,d\theta$ $B_n=\frac{1}{\pi R_0^n}\int_0^{2\pi}h(\theta)\sin(n\theta)\,d\theta$ $A_0=\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta$
	From Sep.Var. $(u(r,\theta)=R(r)\Theta(\theta))$. Convert to polar, then separate. In this case $\lambda=n$, hence $\Theta_n(\theta)=A_n\cos(\lambda\theta)+B_n\sin(\lambda\theta)$ and 2π periodic. $R(r)$ is found from separation of variables: $r^2R''(r)+rR'(r)-\lambda^2R(r)=0$. Solve for cases where n=0 and $n\neq 0$. In our case $R_0(r)=C_0+D_0\ln(r)$ and $R_n(r)=C_nr^{\lambda_n}+D_nr^{-\lambda_n}$		
Heat Equation	$u_t=a^2u_{xx}+F(x,t)$ $u(x,0)=f(x)$ $u(0,t)=\mu(t)$ | $u(l,t)=\nu(t)$	$0\leq x\leq l$ $0\leq t\leq t_1$	For $0\leq t\leq t_1$ it is said to be a non-homogenous heat equation. The function $u(x,t)$ represents the temperature of a wire from 0 to l, at position x at time t. The solutions of this equation and its properties are presented later in the sheet.

Definition (Laplacian): $\nabla\cdot\nabla=\nabla^2=\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

Extensions: Even $f(-x)$. Odd $-f(-x)$. Periodic: extension with period 2l, $f(x)=f(x\pm 2l)$. Don't actually state the extended function with an even or odd extension. If $u(0,t)=0$, or if it's not given, odd extension. If $u_x(0,t)=0$, even

METHOD OF SEPARATION OF VARIABLES (FOURIER METHOD):

1	Let $u(x,t)=X(x)T(t)$ Substitute into the original problem
2	Separate $T(t)$ and $X(x)$ in the new equation. Make it to be in the form $\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda^2$ (other things can be used instead of $-\lambda^2$, such as $-\lambda$)
3	Convert the result of (2) into two ODEs for $X(x)$ and $T(t)$. For example, $\frac{X''(x)}{X(x)}=-\lambda^2$ becomes $X''(x)+\lambda^2X(x)=0$
4	Obtain the boundary conditions for $X(x)$ and $T(t)$ from the original conditions. Use the conditions that are equal to 0. For example, $u(0,t)=0\to X(0)=0$
5	You will end up with a particular case of the Sturm-Liouville problem Find the eigenvalues (values of λ) and eigenfunction s$X_n(x)$. Some common cases are in the **Eigen-table**
6	Having found $X_n(x)$, find $T(t)$ by using the same λ, and solving the ODE using the characteristic equation (assume solution is in the form e^r, go from there).You will end up with $T_n(t)=e^{i\alpha t}$, which, by Euler's formula is $A_n\cos(\alpha t)+B_n\sin(\alpha t)$.
7	Find $u_n(x,t)$ according to definition $u_n(x,t)=X_n(x)T_n(t)$
8	Take the infinite summation $u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)$
9	Find the Fourier Coefficients using:$A_n=\frac{2}{l}\int_0^l f(x)X_n(x)$; $B_n=\frac{2}{a\lambda l}\int_0^l g(x)X_n(x)$. If formulas don't apply, express original eq. in terms of your solution, do the same thing as last steps of Neumann Problem Laplace Equation on a disc.

Dirichlets Problem $X''+\lambda^2X=0$ $X(0)=X(l)=0$	$\lambda_n=\frac{n\pi}{l}, n=1,2,\ldots$ $X_n(x)=\sin\left(\frac{n\pi}{l}x\right)$	**Neumann Problem** $X''+\lambda^2X=0$ $X'(0)=X'(l)=0$	$\lambda_n=\frac{n\pi}{l}, n=0,1,2,\ldots$ $X_n(x)=\cos\left(\frac{n\pi}{l}x\right)$
Mixed Boundary Value Problem A $X''+\lambda^2X=0$ $X(0)=X'(l)=0$	$\lambda_k=\frac{(2k+1)\pi}{2l}, k=0,1,2,\ldots$ $X_n(x)=\sin\left(\frac{(2k+1)\pi}{2l}x\right)$	**Mixed boundary Value Problem B** $X''+\lambda^2X=0$ $X'(0)=X(l)=0$	$\lambda_k=\frac{(2k+1)\pi}{2l}, k=0,1,2,\ldots$ $X_n(x)=\cos\left(\frac{(2k+1)\pi}{2l}x\right)$
The Periodic BVP $X''+\lambda^2X=0$ $X(-l)=X(l)=0$ $X'(-l)=X'(l)=0$		$\lambda_n=\frac{n\pi}{l}, n=0,1,2,\ldots$ $X_n(x)\in\{1,\cos\left(\frac{n\pi}{l}x\right),\sin\left(\frac{n\pi}{l}x\right)\}$	

Finding Eigenvalues in Sturm Liouville:

1. Let $\lambda>0$ and assume that $\lambda=\omega^2$ where $\omega>0$.
2. Find the characteristic equation of the ODE, and solve it.

ENERGY METHOD TO PROVE UNIQUENESS:

Equation	Form	Energy
Wave Equation	$\rho u_{tt}-Tu_{xx}=0$	$E(t)=\frac{1}{2}\int_0^L(\rho u_t^2+Tu_x^2)dx$
Heat Equation	$u_t-ku_{xx}=0$	$E(t)=\frac{1}{2}\int_0^L u(x,t)^2dx$

1	Define $\omega=u_1-u_2$ where u_1,u_2 are two distinct solutions.
2	Rewrite the problem as a homogenous problem in terms of ω, with all conditions, including the original equation equaling 0.
3	Write the energy integral that corresponds to the type of equation, and take its derivative with respect to time$\left(\frac{d}{dt}\right)$. When taking the derivative, use Leibniz rule, and don't actually evaluate the integral yet.
4	Use integration by parts to rewrite the integral in terms of the homogenous conditions as much as possible.
5	Manipulate the resultant integral, using initial conditions to cancel terms, until you show that $E'(t)\leq 0$
6	Having shown that the energy is decreasing or 0, we say that the energy of the entire system at $t=0$ follows: $E(0)\geq E(t)\geq 0$
7	Prove that $E(0)=0$. Once proven, it follows from the sandwich theorem that $E(t)=0$, which means that the integrand is 0, and therefore $w=0$, which proves uniqueness.

UNIQUENESS USING GREEN'S FORMULA:

1	Consider the problem: $u_{xx}+u_{yy}=0$; $(x,y)\in D$ | $u(x,y)=g(x,y)$; $(x,y)\in D$
2	Let u_1,u_2 be two distinct solutions to the problem. Goal: show that they're equal.
3	Let $\omega=u_1-u_2$. We must show that $\omega=0$ on $D\cup\partial D$
4	Rewrite the PDE in terms of ω, thus forming a new BVP.
5	If we restrict x,y to ∂D, we get $w(x,y)=g(x,y)-g(x,y)=0$, so we get the problem: $\omega_{xx}+\omega_{yy}=0$; $(x,y)\in D$, $\omega(x,y)=0$; $(x,y)\in\partial D$
6	We will use greens first identity, but u and v are replaced with ω. $\oint_{\partial D}\omega\frac{\partial\omega}{\partial n}ds=\oint_{\partial D}\omega\Delta\omega\cdot\vec{n}ds=\iint_D(\omega\nabla^2\omega+\nabla\omega\cdot\nabla\omega)dA.$
7	We take the following part of the identity: $\oint_{\partial D}\omega\Delta\omega\cdot\vec{n}ds=\iint_D(\omega\nabla^2\omega+\nabla\omega\cdot\nabla\omega)dA$, and will try to simplify it. $\oint_{\partial D}\omega\Delta\omega\cdot\vec{n}ds$ is an integral along the boundary of ω. From our boundary condition, we know that $\omega(x,y)=0$ on the boundary. Therefore our integrand is 0, and we have $\iint_D(\omega\nabla^2\omega+\nabla\omega\cdot\nabla\omega)dA=0$
8	$\omega\nabla^2\omega=\omega(\omega_{xx}+\omega_{yy})=\omega\cdot 0$, so we are left with $\iint_D(\nabla\omega\cdot\nabla\omega)dA=\iint_D(\|\Delta\omega\|^2)dA=0$. Integral is 0 when its integrand is 0. $\|\Delta\omega\|^2=0$. Norm is 0 if its contents are $\vec{0}$, so $\Delta\omega=\vec{0}$
9	Show that all 1st order partials of ω are 0. This means that $\omega=c$. We know that $\omega=0$ on the boundary, and since it is constant and continuous, it must keep same value throughout, so $\omega=0$.

Mixed Partial Equality: If $\exists\ f_{xy}, f_{yx}$ and they are continuous, then $f_{xy} = f_{yx}$

CLASSIFICATION AND CANONICAL FORM:

1 Consider the equation: $Au_{xx} + 2Bu_{xy} + Cu_{yy} + F(x,y,u,u_x,u_y)$

2 Using the coefficients, construct ODE: $Ay'^2 - 2By' + C = 0$.The ODE has two roots: $k_1(x,y) = \frac{B+\sqrt{B^2-AC}}{A}$; $k_2(x,y) = \frac{B-\sqrt{B^2-AC}}{A}$. Also define $y_1' = k_1(x,y)$, $y_2' = k_2(x,y)$. Identify the type of PDE by using the following table:

Hyperbolic	Parabolic	Elliptical
$u_{\xi\eta} = \Phi(\xi,\eta,u,u_\xi,u_\eta)$	$u_{\eta\eta} = \Phi(\xi,\eta,u,u_\xi,u_\eta)$	$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha,\beta,u,u_\alpha,u_\beta)$ $\alpha = \mathcal{R}\{\xi\}$ and $\beta = \mathcal{Im}\{\xi\}$
$k_1, k_2 \in \mathbb{R}$, $k_1 \neq k_2$	$k_1, k_2 \in \mathbb{R}$, $k_1 = k_2$	$k_1, k_2 \in \mathbb{C}$
$B^2 - AC > 0$	$B^2 - AC = 0$	$B^2 - AC < 0$

3 There are two ways to find η and ξ. We can find y_1, y_2 from integrating y_1', y_2' the constant of integration for y_1 is ξ and for y_2 is η. Alternatively, We define η and ξ from k_1 and k_2.

Hyperbolic	Parabolic	Elliptic
$\xi = y_1 - \int k_1 dx$ $\eta = y_2 - \int k_2 dx$	$\xi = y_1 - \int k_1 dx$ $\eta = x$	$\xi = \varphi(x,y) = c, \eta = \overline{\varphi(x,y)} = c$ c = const. of integration from finding y_1, y_2

4 Convert all partials into partials with respect to ξ, η, and find the new $\tilde{A}, \tilde{B}, \tilde{C}$

u_x	$u_x = u_\xi \xi_x + u_\eta \eta_x$
u_y	$u_y = u_\xi \xi_y + u_\eta \eta_y$
u_{xx}	$u_{xx} = (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx}$
u_{xy}	$u_{xy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\xi \xi_{xy} + u_\eta \eta_{xy}$
u_{yy}	$u_{yy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\xi \xi_{yy} + u_\eta \eta_{yy}$
$\tilde{A}$	$\tilde{A} = A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2$
$\tilde{B}$	$\tilde{B} = A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y$
$\tilde{C}$	$\tilde{C} = A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2$

5 Rewrite the original PDE in terms of converted partials and new coefficents from step 4. (This is the canonical form), and simplify, using mixed partial equality.

6 We proceed to solve the canonical form using classical methods. The final form of the solution will be something like $u(x,y) = H(x,y) + F(\xi) + G(\eta)$. Express η and ξ in terms of x,y using their definitions (from step 3). $H(x,y)$ is some expression. Functions F, G don't matter – add a note saying they're arbitrary.

Laplace Equation: $\Delta u = 0$. A function that satisfies the Laplace equation is called a ***harmonic function***.

Laplace Equation in Polar: $\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

Dirichlet's Problem: Let D be an open and bounded domain in $\mathbb{R}^2$. Let ∂D be the boundary of the domain D. Then, the problem

$\Delta u = F(x,y)$
$u(x,y) = f(x,y)$

For all $(x,y) \in D$ is called the Dirichlet problem. The boundary condition $u(x,y) = f(x,y)$ is called the Dirichlet boundary condition.

Well-Posedness of Dirichlet Problem: If there exists a solution to the Dirichlet problem, where F is defined on D and f is defined on ∂D, then the problem is well posed.

Poisson equation: $\Delta u = F(x,y)$

Maximum/Minimum Principle Poisson: Let u be continuous on a bounded and closed domain $D \cup \partial D$, twice differentiable on the open domain D and satisfying the Poisson equation $\Delta u(x,y) = F(x,y)$.

$F \geq 0$ on D: Maximum value of u in $D \cup \partial D$ is on the boundary ∂D.
$F \leq 0$ on D: Minimum value of u in $D \cup \partial D$ is on the boundary ∂D.
$F = 0$ on D: Minimum and maximum value of u in $D \cup \partial D$ are on the boundary ∂D. If $\max(u) = M$, $\min(u) = m$, then $m \leq u(x,y) \leq M$

Poisson kernel: $I = \frac{R_0^2 - r^2}{R_0^2 - 2R_0 r cos(\theta-\psi) + r^2}$. **Green Function for the Dirichlet Problem of the Laplace Equation in a disk**: $G(r,\theta-\psi) = \frac{1}{2\pi}I$

Solution to Laplace Equation in Polar Coordinates: The solution to

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$
$$u(R_0,\theta) = h(\theta)$$

Where $0 \leq r \leq R_0$, $0 \leq \theta \leq 2\pi$

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta-\psi)h(\psi)d\psi = \frac{1}{2\pi}\int_0^{2\pi} \frac{R_0^2 - r^2}{R_0^2 - 2R_0 r cos(\theta-\psi) + r^2} h(\psi)d\psi$$

And the boundary condition is $h(\theta_0) = \lim_{(r,\theta)\to(R_0,\theta_0)} u(r,\theta)$

Green's First Formula: Let $\frac{\partial u}{\partial n}$ be the directional derivative of $u(x,y)$ w.r.t the outward unit normal $\hat{n}$ to ∂D. Then: $\iint_D u\Delta v dxdy = \int_{\partial D} u\frac{\partial v}{\partial n}ds - \iint_D \nabla u \cdot \nabla v dxdy$

Green's Second Formula: Let $\frac{\partial u}{\partial n}$ be the directional derivative of $u(x,y)$ w.r.t the outward unit normal $\hat{n}$ to ∂D. Then: $\iint_D (u\Delta v - v\Delta u)dxdy = \int_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$

Existence of Solution of Neumann Problem of the Poisson Equation: A necessary condition for existence of a solution of the equation $\Delta u = F(x,y)$ for $(x,y) \in D$, *with a boundary condition* $\frac{\partial u(x,y)}{\partial n} = g(x,y)$ for $(x,y) \in \partial D$ *is* $\iint_D F(x,y)dxdy = \int_{\partial D} g(x,y)ds$

Property of solutions to the Neumann problem of Poisson Equation: Let $u_1(x,y), u_2(x,y)$ be solutions to the Neumann problem of the Poisson equation: $\Delta u = F(x,y)$ for $(x,y) \in D$, with a boundary condition $\frac{\partial u(x,y)}{\partial n} = g(x,y)$ for $(x,y) \in \partial D$. Then: $u_1(x,y) - u_2(x,y) = c$, where c is a constant

Existence of Solution of Neumann Problem of the Laplace Equation: A conditions for existence of a solution to the equation $\Delta u = 0$ for $(x,y) \in D$, *with a boundary condition* $\frac{\partial u(x,y)}{\partial n} = f(x,y)$ for $(x,y) \in \partial D$ is $\int_D f(x(s),y(s))ds = 0$

Neumann Problem Laplace Equation on a disc:
Convert everything into polar coordinates. For the Neumann condition $\frac{\partial u(x,t)}{\partial n} = f(x,t)$ the conversion is $\frac{\partial u(R_0,\theta)}{\partial r} = f(R_0,\theta)$ where R_0 is the radius of the disk. You will end up with the following problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\ , 0 < r < R_0\ ; 0 \leq \theta \leq 2\pi$$
$$\frac{\partial u(R_0,\theta)}{\partial r} = f(R_0,\theta)\ \ ; 0 \leq \theta \leq 2\pi$$

Our boundary condition is a partial with respect to r, and in the general solution formula, the only r dependent term is r^n. Therefore we have

$$\frac{\partial u(R_0,\theta)}{\partial r} = f(R_0,\theta) = \sum_{n=1}^{\infty} nR_0^{n-1}(A_n\cos(n\theta) + B_n\sin(n\theta))$$

Simplify $f(R_0,\theta)$ such that it becomes a sum of 1[st] order trigonometric functions. From the processed $f(R_0,\theta)$, *find the non-zero coefficients*. For example, if you have a term $8\cos(2\theta)$ inside $f(R_0,\theta)$, then the coefficient A_n that gives that term is given by $n = 2$, $nA_nR_0^{n-1} = 8$. Find all the coefficients that you can using this method. *All the others are 0*. Write $u(r,\theta)$ using only the coefficients you found. *Convert back* to Cartesian coordinates, and you're done.

Polar Coordinates: x= $r\cos(\theta)$, $y = r\sin(\theta)$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

Maximum Principle for the Heat Equation:

Domain D: $0 \leq x \leq l, 0 \leq t \leq t_1$	
Equation Form $u_t = a^2u_{xx} + F(x,t)$	**Assumption** $u(x,t)$ and $F(x,t)$ continuous on D
Maximum Let $F(x,t) \leq 0$. Let $u(x,t) \leq M$ for $t = 0, x = 0$, or $x = l$	**Minimum** Let $F(x,t) \geq 0$. Let $u(x,t) \geq m$ for $t = 0, x = 0$, or $x = l$
$u(x,t) \leq M$ in the domain. The maximum of $u(x,t)$ is on the boundary of the domain, excluding the top boundary	$u(x,t) \geq m$ in the domain. The minimum of $u(x,t)$ is on the boundary of the domain, excluding HWthe top boundary
General: If $m \leq u(x,t) \leq M$, For $t = 0, x = 0$, or $x = l$	
Then: $m \leq u(x,t) \leq M$, For $0 \leq x \leq l, 0 \leq t \leq t_1$.	

Heat Equation Well Posedness: If there exists a solution to the heat equation, then the problem is well posed.

Heat Equation Separation of Variables: The solution to the heat equation, where $0 \leq x \leq l, t \geq 0$ is:

$u_t = a^2u_{xx} + F(x,t)$
$u(x,0) = f(x)$
$u(0,t) = u(l,t) = 0$

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx\right)e^{-\left(\frac{n\pi a}{l}\right)^2 t}\sin\left(\frac{n\pi}{l}x\right)$$

$X_n(t)$ is found via classical separation method. $T_n(t)$ is as follows:

$T_n(t) + \left(\frac{n\pi}{l}\right)^2 a^2T_n(t) = 0$, therefore $T_n(t) = A_ne^{\left(-\frac{n\pi a}{l}\right)^2 t}$. Therefore,

$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{\left(-\frac{n\pi a}{l}\right)^2 t}\sin(\frac{n\pi}{l}x)$, $A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$

Heat Equation Cauchy Problem Solution:

$u_t = a^2u_{xx}$ $u(x,0) = f(x)$	$u(x,t) = \int_{-\infty}^{\infty} G(x,y,t)f(y)dy$
$G(x,y,t) = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{(x-y)^2}{4a^2t}}$ Green function for the Cauchy problem of the heat equation on an infinite interval.	Useful identity: $\frac{2}{\sqrt{\pi}}\int_{-z}^0 e^{-\alpha^2}d\alpha = -\text{erf}(-z) = \text{erf}(z)$ In integration, use substitution: $\alpha = \frac{y-x}{2a\sqrt{t}}$

Gaussian Error function: $\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds.$

$\text{erf}(0) = 0$	$\text{erf}(\pm\infty) = \pm 1$	$\text{erf}(0) = 0$
$\text{erf}(-z) = -\text{erf}(z)$	$\frac{d\,\text{erf}(z)}{dz} = \frac{2}{\sqrt{\pi}}e^{-z^2}$	$\text{erf}(z) \approx \frac{2}{\sqrt{\pi}}e^{-z2}, z \ll 1$
$\frac{d^2\,\text{erf}(z)}{dz^2} = -\frac{4}{\sqrt{\pi}}ze^{-z^2}$	$erfc(z) = 1 - \text{erf}(z)$	$\int_0^\infty erfc(z)dz = \frac{1}{\pi}$
$erfc(z) \approx \frac{1}{\sqrt{\pi}}\frac{e^{-z^2}}{z}, z \gg 1$	$\int_0^z erfc(y)dy = z\cdot\text{erfc}\,(z) + \frac{1}{\sqrt{\pi}}\left(1 - e^{-z^2}\right)$	$\frac{2}{\sqrt{\pi}}\int_{-\infty}^0 e^{-\alpha^2}d\alpha = 1$
$\frac{2}{\sqrt{\pi}}\int_{-z}^0 e^{-\alpha^2}d\alpha = -\phi(-z) = \phi(z)$	$\frac{2}{\sqrt{\pi}}\int_0^{-z} e^{-\alpha^2}d\alpha = \phi(-z) = -\phi(z)$	$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\alpha^2}d\alpha = 1$

Solution of Poisson Equation on a Disk: For the problem

$u_{xx} + u_{yy} = F(x,y), x^2 + y^2 < R_0^2$ | $u(x,y) = 0, x^2 + y^2 = R_0^2$

1. Convert to polar coordinates

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = F(r,\theta), 0 < r < R_0, 0 \leq \theta \leq 2\pi$ | $u(R_0,\theta) = 0$

2. Express $u(r,\theta)$ as a Fourier series w.r.t θ, coefficients depend on r.

$$u(r,\theta) = \frac{a_0(r)}{2} + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n(r)\sin(n\theta)$$

3. Express the non-homogenous part $F(r,\theta)$ as a series too:

$$F(r,\theta) = \frac{A_0(r)}{2} + \sum_{n=1}^{\infty} A_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} B_n(r)\sin(n\theta)$$

$A_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(r,\varphi)\cos(n\varphi)d\varphi$ | $B_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(r,\varphi)\sin(n\varphi)d\varphi$

4. Calculate the derivatives of $u(r,\theta)$ that a relevant to the equation.

$$u_r(r,\theta) = \frac{a_0'(r)}{2} + \sum_{n=1}^{\infty} a_n'(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n'(r)\sin(n\theta)$$
$$u_{rr}(r,\theta) = \frac{a_0''(r)}{2} + \sum_{n=1}^{\infty} a_n''(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n''(r)\sin(n\theta)$$
$$u_{\theta\theta}(r,\theta) = -\sum_{n=1}^{\infty} n^2a_n(r)\cos(n\theta) - \sum_{n=1}^{\infty} n^2b_n\sin(n\theta)$$

5. *Substitute into the polar form of the equation*

$$\frac{a_0''(r)}{2}+\frac{1}{r}\frac{a_0'(r)}{2}+\sum_{n=1}^{\infty}\left(a_n''(r)+\frac{1}{r}a_n'(r)-\frac{n^2}{r^2}a_n(r)\right)\cos(n\theta)$$
$$+\sum_{n=1}^{\infty}\left(b_n''(r)+\frac{1}{r}b_n'(r)-\frac{n^2}{r^2}b_n(r)\right)\sin(n\theta)=$$
$$\frac{A_n(r)}{2}+\sum_{n=1}^{\infty}A_n(r)\cos(n\theta)+\sum_{n=1}^{\infty}B_n(r)\sin(n\theta)$$

6. *Get a system of equations*

$$\begin{cases}\frac{a_0''(r)}{2}+\frac{1}{r}\frac{a_0'(r)}{2}=\frac{A_n(r)}{2}\\ a_n''(r)+\frac{1}{r}a_n'(r)-\frac{n^2}{r^2}a_n(r)=A_n(r)\,,n=1,2\ldots\\ b_n''(r)+\frac{1}{r}b_n'(r)-\frac{n^2}{r^2}b_n(r)=B_n(r),n=1,2,3\ldots\end{cases}$$

7. *Substitute the polar boundary condition $u(R_0,\theta)=0$:*

$$0=u(R_0,\theta)=\frac{a_0(R_0)}{2}+\sum_{n=1}^{\infty}a_n(R_0)\cos(n\theta)+\sum_{n=1}^{\infty}b_n(R_0)\sin(n\theta)$$

Therefore $\begin{cases}a_n(R_0)=0\,,n=0,1,2\ldots\\ b_n(R_0)=0,\ n=1,2,3\ldots\end{cases}$

8. The system in (6) contains linear non-homogenous ODEs of the second order. Solve the system using boundary conditions from the (7), and using the fact that $a_n(0)$ & $b_n(0)$ are bounded (since $u(r,\theta)$ is continuous on the disc including the origin).

$\sin(-\theta)=-\sin(\theta)$	$\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$	$\sin(\pi-\theta)=\sin(\theta)$
$\cos(-\theta)=\cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$	$\cos(\pi-\theta)=-\cos(\theta)$
$\tan(-\theta)=-\tan(\theta)$	$tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$	$\tan(\pi-\theta)=-\tan(\theta)$
$\sin\left(\theta+\frac{\pi}{2}\right)=\cos(\theta)$	$\sin(\theta+\pi)=-\sin(\theta)$	$\sin(\theta+2\pi)=\sin(\theta)$
$\cos\left(\theta+\frac{\pi}{2}\right)=-\sin(\theta)$	$\cos(\theta+\pi)=-\cos(\theta)$	$\cos(\theta+2\pi)=\cos(\theta)$
$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$		$\sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$
$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$		$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}$
$\sin(2\theta)=2\sin(\theta)\cos(\theta)=\frac{2\tan(\theta)}{1+\tan^2(\theta)}$		$\sin^2(\theta)+\cos^2(\theta)=1$
$\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1=1-2\sin^2(\theta)$		$\sin(\theta)=\pm\sqrt{1-\cos^2(\theta)}$ Sign depends on quadrant of θ
$\sin(3\theta)=-4\sin^3(\theta)+3\sin(\theta)$		$\cos(\theta)=\pm\sqrt{1-\sin^2(\theta)}$
$\sin^2\left(\frac{\theta}{2}\right)=\frac{(1-\cos(\theta))}{2}$	$\cos^2\left(\frac{\theta}{2}\right)=\frac{(1+\cos(\theta))}{2}$	$\cos(3\theta)=4\cos^3(\theta)-3\cos(\theta)$
$2\cos(\theta)\cos(\phi)=\cos(\theta-\phi)+\cos(\theta+\phi)$		$2\sin(\theta)\sin(\phi)=\cos(\theta-\phi)-\cos(\theta+\phi)$
$2\sin(\theta)\cos(\phi)=\sin(\theta+\phi)+\sin(\theta-\phi)$		
$2\cos(\theta)\sin(\phi)=\sin(\theta+\phi)-\sin(\theta-\phi)$		
$\sinh(x)=\frac{e^x-e^{-x}}{2}=-i\sin(ix)$		$\cosh(x)=\frac{e^x+e^{-x}}{2}=\cos(ix)$
Euler's formula: $e^{ix}=\cos(x)+i\sin(ix)$		

INTEGRALS: Basic Forms

$\int x^n dx=\frac{1}{n+1}x^{n+1}$	$\int u dv=uv-\int v du$	$\int\frac{1}{x}dx=\ln\lvert x\rvert$	$\int\frac{1}{ax+b}dx=\frac{1}{a}\ln\lvert ax+b\rvert$

Integrals of Rational Functions

$\int\frac{1}{(x+a)^2}dx=-\frac{1}{x+a}$	$\int\frac{1}{1+x^2}dx=\tan^{-1}x$
$\int(x+a)^n dx=\frac{(x+a)^{n+1}}{n+1},n\neq-1$	$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int x(x+a)^n dx=\frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$	$\int\frac{x}{a^2+x^2}dx=\frac{1}{2}\ln\lvert a^2+x^2\rvert$
$\int\frac{x^3}{a^2+x^2}dx=\frac{1}{2}x^2-\frac{1}{2}a^2\ln\lvert a^2+x^2\rvert$	$\int\frac{x^2}{a^2+x^2}dx=x-a\tan^{-1}\frac{x}{a}$
$\int\frac{1}{ax^2+bx+c}dx=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$	$\int\frac{1}{(x+a)(x+b)}dx=\frac{1}{b-a}\ln\frac{a+x}{b+x},a\neq b$
$\int\frac{x}{ax^2+bx+c}dx=\frac{1}{2a}\ln\lvert ax^2+bx+c\rvert-\frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$	$\int\frac{x}{(x+a)^2}dx=\frac{a}{a+x}+\ln\lvert a+x\rvert$

Integrals with Roots

$\int\sqrt{x-a}\ dx=\frac{2}{3}(x-a)^{3/2}$	$\int\sqrt{\frac{x}{a+x}}\ dx=\sqrt{x(a+x)}-a\ln\left[\sqrt{x}+\sqrt{x+a}\right]$
$\int\frac{1}{\sqrt{x\pm a}}\ dx=2\sqrt{x\pm a}$	$\int x\sqrt{ax+b}\ dx=\frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b}$
$\int\frac{1}{\sqrt{a-x}}\ dx=-2\sqrt{a-x}$	$\int\sqrt{x(ax+b)}\ dx=\frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)}-b^2\ln\lvert a\sqrt{x}+\sqrt{a(ax+b)}\rvert\right]$
$\int\sqrt{ax+b}\ dx=\left(\frac{2b}{3a}+\frac{2x}{3}\right)\sqrt{ax+b}$	$\int\sqrt{x^3(ax+b)}\ dx=\left[\frac{b}{12a}-\frac{b^2}{8a^2x}+\frac{x}{3}\right]\sqrt{x^3(ax+b)}+\frac{b^3}{8a^{5/2}}\ln\lvert a\sqrt{x}+\sqrt{a(ax+b)}\rvert$
$\int\frac{x}{\sqrt{x\pm a}}\ dx=\frac{2}{3}(x\mp2a)\sqrt{x\pm a}$	$\int\sqrt{a^2-x^2}\ dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$
$\int\sqrt{\frac{x}{a-x}}\ dx=-\sqrt{x(a-x)}-a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$	$\int x\sqrt{x^2\pm a^2}\ dx=\frac{1}{3}(x^2\pm a^2)^{3/2}$
$\int\frac{1}{\sqrt{x^2\pm a^2}}\ dx=\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$	$\int\frac{1}{\sqrt{a^2-x^2}}\ dx=\sin^{-1}\frac{x}{a}$; $\int\frac{x}{\sqrt{a^2-x^2}}\ dx=-\sqrt{a^2-x^2}$
$\int\frac{x}{\sqrt{x^2\pm a^2}}\ dx=\sqrt{x^2\pm a^2}$	$\int\frac{x}{\sqrt{ax^2+bx+c}}\ dx=\frac{1}{a}\sqrt{ax^2+bx+c}-\frac{b}{2a^{3/2}}\ln\left\lvert 2ax+b+2\sqrt{a(ax^2+bx+c)}\right\rvert$
$\int\frac{1}{\sqrt{ax^2+bx+c}}\ dx=\frac{1}{\sqrt{a}}\ln\left\lvert 2ax+b+2\sqrt{a(ax^2+bx+c)}\right\rvert$	

Integrals with Exponentials

$\int e^{ax}\ dx=\frac{1}{a}e^{ax}$	$\int xe^{x}\ dx=(x-1)e^{x}$
$\int xe^{ax}\ dx=\left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$	$\int x^2e^{x}\ dx=(x^2-2x+2)e^{x}$
$\int x^2e^{ax}\ dx=\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)e^{ax}$	$\int x^3e^{x}\ dx=(x^3-3x^2+6x-6)e^{x}$
$\int e^{ax^2}\ dx=-\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}(ix\sqrt{a})$	$\int e^{-ax^2}\ dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}(x\sqrt{a})$
$\int xe^{-ax^2}\ dx=-\frac{1}{2a}e^{-ax^2}$	$\int x^2e^{-ax^2}\ dx=\frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}(x\sqrt{a})-\frac{x}{2a}e^{-ax^2}$
$\int\sqrt{x}e^{ax}\ dx=\frac{1}{a}\sqrt{x}e^{ax}+\frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}(i\sqrt{ax})$, where $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$	

Integrals with Trigonometric Functions

$\int\sin ax\ dx=-\frac{1}{a}\cos ax$	$\int\sin^2ax\ dx=\frac{x}{2}-\frac{\sin2ax}{4a}$
$\int\cos ax\ dx=\frac{1}{a}\sin ax$	$\int\cos^2ax\ dx=\frac{x}{2}+\frac{\sin2ax}{4a}$
$\int\sin^2x\cos x\ dx=\frac{1}{3}\sin^3x$	$\int\cos^2ax\sin ax\ dx=-\frac{1}{3a}\cos^3ax$
$\int\sin^2ax\cos^2ax\ dx=\frac{x}{8}-\frac{\sin4ax}{32a}$	$\int\tan ax\ dx=-\frac{1}{a}\ln\cos ax$
$\int\tan^2ax\ dx=-x+\frac{1}{a}\tan ax$	$\int\cos x\sin x\ dx=\frac{1}{2}\sin^2x+c_1=-\frac{1}{2}\cos^2x+c_2$
$\int\sin^2ax\cos bx\ dx=-\frac{\sin[(2a-b)x]}{4(2a-b)}+\frac{\sin bx}{2b}-\frac{\sin[(2a+b)x]}{4(2a+b)}$	
$\int\cos^2ax\sin bx\ dx=\frac{\cos[(2a-b)x]}{4(2a-b)}-\frac{\cos bx}{2b}-\frac{\cos[(2a+b)x]}{4(2a+b)}$	
$\int\cos ax\sin bx\ dx=\frac{\cos[(a-b)x]}{2(a-b)}-\frac{\cos[(a+b)x]}{2(a+b)},a\neq b$	

Products of Trigonometric Functions and Exponentials

$\int e^x\sin x\ dx=\frac{1}{2}e^x(\sin x-\cos x)$	$\int e^{bx}\sin ax\ dx=\frac{1}{a^2+b^2}e^{bx}(b\sin ax-a\cos ax)$
$\int e^x\cos x\ dx=\frac{1}{2}e^x(\sin x+\cos x)$	$\int e^{bx}\cos ax\ dx=\frac{1}{a^2+b^2}e^{bx}(a\sin ax+b\cos ax)$
$\int xe^x\sin x\ dx=\frac{1}{2}e^x(\cos x-x\cos x+x\sin x)$	$\int xe^x\cos x\ dx=\frac{1}{2}e^x(x\cos x-\sin x+x\sin x)$

Products of Trigonometric Functions and Monomials

$\int x\cos x\ dx=\cos x+x\sin x$	$\int x\cos ax\ dx=\frac{1}{a^2}\cos ax+\frac{x}{a}\sin ax$
$\int x^2\cos x\ dx=2x\cos x+(x^2-2)\sin x$	$\int x^2\cos ax\ dx=\frac{2x\cos ax}{a^2}+\frac{a^2x^2-2}{a^3}\sin ax$
$\int x\sin x\ dx=-x\cos x+\sin x$	$\int x\sin ax\ dx=-\frac{x\cos ax}{a}+\frac{\sin ax}{a^2}$
$\int x^2\sin x\ dx=(2-x^2)\cos x+2x\sin x$	$\int x^2\sin ax\ dx=\frac{2-a^2x^2}{a^3}\cos ax+\frac{2x\sin ax}{a^2}$
$\int x\cos^2x\ dx=\frac{x^2}{4}+\frac{1}{8}\cos2x+\frac{1}{4}x\sin2x$	$\int x\sin^2x\ dx=\frac{x^2}{4}-\frac{1}{8}\cos2x-\frac{1}{4}x\sin2x$

Energy Method Uniqueness for Neumann problem, Wave Equation (Just the Integral)

The problem:
$u_{tt}-4u_{xx}=0,0<x<1,t>0$
$u(x,0)=\cos^2(\pi x),0\leq x\leq1$
$u_t(x,0)=\sin^2(\pi x)\cos(\pi x)\,,0\leq x\leq1$
$u_x(0,t)=u_x(1,t)=0,t\geq0$

$E(t)=\frac{1}{2}\int_0^1(w_t^2+4w_x^2)dx$. $E'(t)=\int_0^1(w_tw_{tt}+4w_xw_{xt})dx=$ $\int_0^1 w_tw_{tt}dx+[4w_xw_t]_0^1-4\int_0^1 w_tw_{xx}dx=[4w_xw_t]_0^1=4w_x(1,t)w_t(1,t)-4w_x(0,t)w_t(0,t)\equiv0$

Energy Method Uniqueness for Dirichlet Problem of the Wave Equation

Prove uniqueness the solution to the following.
$u_{tt}-u_{xx}=xt,\ 0<x<1,t>0$
$u_x(0,t)=g(t),\ u(1,t)=h(t),t\geq0$
$u(x,0)=x^2-1,\ u_t(x,0)=x^{2016}-1,0\leq x\leq1$

Prove that $w=0$, where w solves the homog. problem with zero RHS of the problem (write it!). Define energy integral for w: $E(t)=\frac{1}{2}\int_0^1(w_t^2+w_x^2)dx$. Derivative of the energy integral: $E'(t)=\int_0^1(w_tw_{tt}+w_xw_{xt})dx=\int_0^1 w_tw_{tt}dx+[w_xw_t]_0^1-\int_0^1 w_tw_{xx}dx=[w_xw_t]_0^1=w_x(1,t)w_t(1,t)-w_x(0,t)w_t(0,t)\equiv0$. Here we used that $w_{tt}-w_{xx}=0$ and homogenous B.Cs ($w_x(0,t)=0$ and $w(1,t)=0\rightarrow w_t(1,t)=0$). Since $E'(t)\equiv0$, then $E(t)\equiv$ const., but $w(x,0)=0$, and therefore, $w_x(x,0)=0$, as well as $w_t(x,0)=0$, then $E(0)=0$. Hence, $E(t)=0\rightarrow w_t(x,t)=w_x(x,t)=0\rightarrow w(x,t)\equiv C$, but $w(x,0)=0$, then $w(x,t)=0$

Maximum/Minimum Principle, Domain Shenanigans

Let D be a bounded domain in the xy plane, and let C be the boundary of D. Define $\bar{D}=D\cup C$. The functions $u(x,y)$ and $v(x,y)$ are twice differentiable by x and y in the domain D and continuous in $\bar{D}$. Given $\Delta v\geq0$ and $\Delta u=0$ in the domain D, and $u=v$ over the boundary C. Prove that $v(x,y)\leq u(x,y),\forall(x,y)\in\bar{D}$. **Solution**: Define $w=v-u$ and get the new problem: $\Delta w=\Delta v-\Delta u\geq0,(x,y)\in D$,$w=v-u=0,(x,y)\in C$. By the maximum principle, w gets the maximum on the boundary C. Since $w=0$ on the boundary, then $w\leq0$ in $\bar{D}$, i.e., $w=v-u\leq0,\forall(x,y)\in\bar{D}$, or $v\leq u,\forall(x,y)\in\bar{D}$. QED.

Canonical Form

$u_{xx}+2u_{xy}+(\cos^2(x))u_{yy}-\cot\left(x(u_x+u_y)\right)=0$. **Solution**: The characteristic equation is: $(y')^2-2(y')+\cos^2(x)=0$. **Roots**: $(y')_{1,2}=1\pm\sin(x)$. **Solutions**: $y-x+\cos(x)=C$, $y-x-\cos(x)=C$. Variables: $\xi=y-x+\cos(x),\eta=y-x-\cos(x)$. Calculate **derivatives**: $u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi(-1-\sin(x))+u_\eta(-1+\sin(x))$. $u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi+u_\eta$.
$u_{xx}=-\cos(xu_\xi)+(-1-\sin(x))\left(u_{\xi\xi}(-1-\sin(x))+u_{\xi\eta}(-1+\sin(x))\right)+\cos(xu_\eta)+(-1+\sin(x))\left(u_{\eta\xi}(-1-\sin(x))+u_{\xi\xi}(-1+\sin(x))\right)=u_{\xi\xi}(1+2\sin(x)+\sin^2(x))+2u_{\xi\eta}\cos^2(x)+u_{\eta\eta}(1-2\sin(x)+\sin^2(x))+(u_\eta-u_\xi)\cos(x)$.
$u_{xy}=(-1-\sin(x))(u_{\xi\xi}+u_{\xi\eta})+(-1+\sin(x))(u_{\eta\xi}+u_{\eta\eta})=-u_{\xi\xi}(1+\sin(x))-2u_{\xi\eta}-u_{\eta\eta}(1-\sin(x))$
$u_{yy}=u_{\xi\xi}+u_{\xi\eta}+u_{\eta\xi}+u_{\eta\eta}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$
Substitute into the equation and get $-4(\sin^2(x))u_{\xi\eta}=0$. The canonical form is: $u_{\xi\eta}=0$. Integrate with respect to η: $u_\xi=f(\xi)$. Integrate with respect to ξ: $u(\xi,\eta)=F(\xi)+G(\eta)$, and finally, $u(x,y)=F(y-x+\cos(x))+G(y-x-\cos(x))$ where F and G are arbitrary functions of a single variable, twice differentiable.

Uniqueness Proof Using Green's First Identity:

Show solution uniqueness:
$\Delta u-e^xu=1,x^2+y^2<2$
$\frac{\partial u}{\partial n}=x^2,\ x^2+y^2=2$

Define $v=u_1-u_2$. Then: $\begin{cases}\Delta V-e^xv=0,\ \ x^2+y^2<2\\ \frac{\partial v}{\partial n}=0,\ \ x^2+y^2=2\end{cases}$.
Multiply both sides of the equation by v and integrate: $\iint_{x^2+y^2<2}v\Delta v dxdy-\iint_{x^2+y^2<2}e^xv^2dxdy=0$ **(1).** Using Green's First formula: $\iint_{x^2+y^2<2}v\Delta v dxdy=\oint_{x^2+y^2=2}v\frac{\partial v}{\partial n}ds-\iint_{x^2+y^2<2}\lvert\Delta v^2\rvert dxdy$. By the B.C $\frac{\partial v}{\partial n}=0$ we get: $\iint_{x^2+y^2<2}v\Delta v dxdy=-\iint_{x^2+y^2<2}\lvert\Delta v^2\rvert dxdy$. Plug this into **(1)**, get: $-\left(\iint_{x^2+y^2<2}\lvert\Delta v^2\rvert dxdy+\iint_{x^2+y^2<2}e^xv^2dxdy\right)=0$. This will only hold when $v(x,y)\equiv0$ on $x^2+y^2\leq2$, so the solution is unique.

Neumann Problem for the Laplace equation in a disc: Existence, Solution, Min/Max

Neumann Problem for Laplace in disc:
$$\Delta u(x,y)=0, x^2+y^2<4$$
$$\frac{\partial u}{\partial n}(x,y)=x^2-2y^2+K, x^2+y^2=4$$
And $u(0,0)=\frac{1}{2}$.

a)Find the constant K for which exists a necessary condition for solution existence. b)Solve the problem for K you found c)Find the maximum value of the solution in the given disc. **a)** The Neumann problem existence condition for the Laplace equation is: $\int_{\partial D} f(x(s),y(s))ds=0$, where $\left[\frac{\partial u}{\partial n}\right]_{\partial D}=f(x,y)=x^2-2y^2+K$.Here,$\partial D=\{(x,y)|x^2+y^2=4\}$,Hence: $\int_{\partial D} f(x(s),y(s))ds=R\int_0^{2\pi} f(R\cos(\phi),R\sin(\phi))d\phi=2\int_0^{2\pi}(4\cos^2(\phi)-8\sin^2(\phi)+K)d\phi=4\pi(K-2)=0\rightarrow K=2$. **b)** Now, substitute $K=2$. The normal is in the radial direction: $\frac{\partial}{\partial n}=\frac{\partial}{\partial r}$. Then, the boundary condition in polar coordinates is: $\frac{\partial u}{\partial r}(2,\phi)=4\cos^2(\phi)-8\sin^2(\phi)+2$. The general solution for the Laplace equation in a disc is: $u(r,\phi)=A_0+\sum_{n=1}^{\infty}(A_n\cos(n\phi)+B_n\sin(n\phi))r^n$. $\frac{\partial u}{\partial r}(r,\phi)=\sum_{n=1}^{\infty}(A_n\cos(n\phi)+B_n\sin(n\phi))nr^{n-1}$. $\frac{\partial u}{\partial r}(2,\phi)=\sum_{n=1}^{\infty}(A_n\cos(n\phi)+B_n\sin(n\phi))n2^{2-1}=4\cos^2(\phi)-8\sin^2(\phi)+2=6\cos(2\phi)$. $4A_2=6\rightarrow A_2=\frac{3}{2}$, $A_n=0,\forall n\neq 2$, $B_n=0,\forall n$. So: $u(r,\phi)=A_0+\frac{3}{2}r^2\cos(2\phi)$. As we know, the Neumann problem is unique up to a constant. The given point $u(0,0)=\frac{1}{2}$ lets us determine $u(0,0)=A_0=\frac{1}{2}$. The solution: $u(r,\phi)=\frac{1}{2}+\frac{3}{2}r^2\cos(2\phi)$. **c)** The Maximum is on the boundary, by the maximum principle. Hence: $\max_{\bar{D}} u(r,\phi)=\max_{\partial D} u(r,\phi)=\frac{13}{2}$, for $r=2,\phi=0$. Remember, by the way, that the minimum for the Laplace equation is also on the boundary, so $\min_{\bar{D}} u(r,\phi)=\min_{\partial D} u(r,\phi)=-\frac{11}{2}$ $\left(2,\frac{\pi}{2}\right)$

Maximum/Minimum For Heat Equation with Dirichlet Conditions, and an initial condition

$$u_t=u_{xx}$$
$$u(x,0)=\sin^2(x), 0\leq x\leq\pi$$
$$u(0,t)=u(\pi,t)=0, 0\leq t\leq T$$
$$Q=\{(x,t)|0<x<\pi, 0<t<T\}$$

Prove that $0\leq u(x,t)\leq 1$ in the rectangle Q. Solution: By the Max/Min principle for the heat equation, the maximum and minimum of the solution are on the boundary of the domain $Q=[0,\pi]\times[0,T]$, without the upper part. Denote the boundary by ∂Q. Then: $\min_{\partial Q} u(x,t)=0$, $\max_{\partial Q} u(x,t)=\max_{\partial Q}\sin^2(x)=1$. Hence, $0\leq u(x,t)\leq 1$

Wave Equation Separation of Variables:

Find a solution to
$$u_{tt}=a^2u_{xx}$$
$$u(x,0)=0$$
$$u_t(x,0)=\sin\left(\frac{\pi x}{l}\right)$$
$$u(0,t)=u_x(l,t)=0$$
$$t>0, 0<x<l$$

Separation of variables: $u(x,t)=X(x)T(t)\rightarrow\frac{X''}{X}=\frac{1}{a^2}\frac{Y''}{Y}=-\lambda$. First, we'll **solve for X**: $X''+\lambda X=0$, $X(0)=X'(l)=0$. **Case a:** $\lambda<0\rightarrow$trivial. **Case b:** $\lambda=0\rightarrow$trivial. **Case c:** $\lambda>0$: $X(x)=c_1\cos(\sqrt{\lambda}x)+c_2\sin(\sqrt{\lambda}x)$. By the B.Cs: $X(0)=0\rightarrow c_1=0\rightarrow X(x)=c_2\sin(\sqrt{\lambda}x)$. $X'(l)=c_2\sqrt{\lambda}\cos(\sqrt{\lambda}l)=0\rightarrow\sqrt{\lambda_k}l=\frac{\pi(2k+1)}{2}, k=0,1,2\ldots$.

The eigenvalues: $\sqrt{\lambda_k}=\frac{\pi(2k+1)}{2l}, k=0,1,2\ldots$, and **the eigenfunction**: $X_k(x)=c_k\sin(\sqrt{\lambda_k}x)$. **Solve for T:** $T''(t)+a^2\lambda_kT(t)=0$. General solution of form: $T_k(t)=a_k\cos(\sqrt{\lambda_k}at)+b_k\sin(\sqrt{\lambda_k}at)$. We get: $u(x,t)=\sum_{k=0}^{\infty}(\tilde{a_k}\cos(\sqrt{\lambda_k}at)+\tilde{b_k}\sin(\sqrt{\lambda_k}at))\sin(\sqrt{\lambda_k}x)$. $u(x,0)=0\rightarrow\tilde{a_k}=0$. $u_t(x,t)=\sum_{k=0}^{\infty}(\sqrt{\lambda_k}a\tilde{b_k}\cos(\sqrt{\lambda_k}at))\sin(\sqrt{\lambda_k}x)$. $u_t(x,0)=\sum_{k=0}^{\infty}(\sqrt{\lambda_k}a\tilde{b_k})\sin(\sqrt{\lambda_k}x)=g(x)=\frac{\sin(\pi x)}{l}$. To find series coefficents, multiply both sides by $\sin(\sqrt{\lambda_n}x)$, and integrate w.r.t to x over $[0,l]$. Using orthogonality of the system $\{\sin(\sqrt{\lambda_k}x)\}$, get $\tilde{b_n}=\frac{4}{\pi(2n+1)a}\int_0^l\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi(2n+1)}{2l}x\right)dx=\frac{2}{\pi(2n+1)a}\int_0^l\left[\cos\left(\frac{\pi x}{2l}(2n-1)\right)-\cos\left(\frac{\pi x}{2l}(2n+3)\right)\right]dx=\cdots=\frac{4l}{\pi^2(2n+1)a}\left[\frac{(-1)^{n+1}}{2n-1}-\frac{(-1)^{n+1}}{2n+3}\right]=\frac{(-1)^{n+1}16l}{\pi^2(2n-1)(2n+1)(2n+3)a}$. Solution: $u(x,t)=\frac{16l}{\pi^2a}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)}\sin\left(\frac{(2n+1)\pi at}{2l}\right)\sin\left(\frac{(2n+1)\pi x}{2l}\right)$

Separation of variables for non-standard PDEs (e.g. heat equation with extra u term)

$$u_t=au_{xx}-cu, 0<x<1, t>0$$
$$u_x(0,t)=u_x(1,t)=0$$
$$u(x,0)=f(x)$$

a) Find the solution of the problem, where $a>0$, c are constants. Hint: define the cases of the solution by separation of variables for $\tilde{\lambda}=\left(\lambda-\frac{c}{a}\right)$, b) find the solution for $f(x)=\cos^2(\pi x)$. **A):** Look for a solution of the form $u(x,t)=X(x)T(t)$. $XT'=aTX''-cXT\rightarrow\frac{T'}{aT}=\frac{X''}{X}-\frac{c}{a}=-\lambda$. We get the **Sturm-Liouville problem for X**: $X''+\left(\lambda-\frac{c}{a}\right)X=0$, $X'(0)=X'(1)=0$. **For $\lambda<\frac{c}{a}$**, we get a trivial solution. **For $\lambda=\frac{c}{a}$**, the general solution is $X(x)=ax+b$, and from the B.Cs we get $X_0(x)=b$. The corresponding $T_0(t)=a_0e^{-ct}$. **For $\lambda>\frac{c}{a}$**, general solution is $X(x)=c_1\cos\left(\sqrt{\lambda-\frac{c}{a}}x\right)+c_2\sin\left(\sqrt{\lambda-\frac{c}{a}}x\right)$. From the B.C $X'(0)=0\rightarrow c_2=0$ and $X'(1)=0\rightarrow\sin\left(\sqrt{\lambda-\frac{c}{a}}\right)=0$. Hence, the eigenvalues are $\lambda_n=n^2\pi^2+\frac{c}{a}$, $n=1,2,3\ldots$, and the eigenfunctions are $X_n(x)=\cos(n\pi x)$. For $T(t)$, we get the problem $T'(t)+a\lambda T(t)=0$, and the general solution is $T_n(t)=A_ne^{-a\lambda_nt}=A_ne^{-(an^2\pi^2+c)t}$. The general solution of the problem is: $u(x,t)=X_0(x)T_0(t)+\sum_{n=1}^{\infty}X_n(x)T_n(t)=A_0e^{-ct}+\sum_{n=1}^{\infty}A_ne^{-(an^2\pi^2+c)t}\cos(n\pi x)$. Using the initial conditions: $u(x,0)=A_0+\sum_{n=1}^{\infty}A_n\cos(n\pi x)=f(x)$. Then, $A_0=\int_0^1 f(x)dx$, $A_n=2\int_0^1 f(x)\cos(n\pi x)dx$. Substitute into the general solution to get unique $u(x,t)$.**B)** $f(x)=\cos^2(\pi x)=\frac{1}{2}(1+\cos(2\pi x))$. $A_0=A_2=\frac{1}{2}$, $A_n=0,\forall n\neq 0,2$. $u(x,t)=\frac{e^{-ct}}{2}+\frac{e^{-(4a\pi^2+c)t}}{2}\cos(2\pi x)$.

Heat Equation Cauchy Problem Solution Using Green's Function & ERF

Solve:
$$u_t=a^2u_{xx}, t>0, -\infty<x<\infty$$
$$u(x,0)=\begin{cases}T_1, x\geq 0\\ T_2, x<0\end{cases}$$

By formula (green funct.):$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4a^2t}}f(y)dy=\frac{T_1}{2a\sqrt{\pi t}}\int_0^{\infty}e^{-\frac{(x-y)^2}{4a^2t}}dy+\frac{T_2}{2a\sqrt{\pi t}}\int_{-\infty}^0 e^{-\frac{(x-y)^2}{4a^2t}}dy$. Change of variables: $\alpha=\frac{y-x}{2a\sqrt{t}}$: $y=-\infty\rightarrow\alpha=-\infty$; $y=\infty\rightarrow\alpha=\infty$; $y=0\rightarrow\alpha=\frac{-x}{2a\sqrt{t}}=-z$. Then, $u(x,t)=\frac{T_1}{\sqrt{\pi}}\int_{-z}^{\infty}e^{-\alpha^2}d\alpha+\frac{T_2}{\sqrt{\pi}}\int_{-\infty}^{-z}e^{-\alpha^2}d\alpha$. $u(x,t)=\frac{T_1}{2}\left(\frac{2}{\sqrt{\pi}}\int_{-z}^0 e^{-\alpha^2}d\alpha+\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-\alpha^2}d\alpha\right)+\frac{T_2}{2}\left(\frac{2}{\sqrt{\pi}}\int_{-\infty}^0 e^{-\alpha^2}d\alpha+\frac{2}{\sqrt{\pi}}\int_0^{-z}e^{-\alpha^2}d\alpha\right)$. Here, $\varphi(z)=\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$. Since: $\frac{2}{\sqrt{\pi}}\int_{-z}^0 e^{-\alpha^2}d\alpha=-\varphi(-z)=\varphi(z)$, $\frac{2}{\sqrt{\pi}}\int_0^{-z}e^{-\alpha^2}d\alpha=\varphi(-z)=-\varphi(z)$, $\frac{2}{\sqrt{\pi}}\int_{-\infty}^0 e^{-\alpha^2}d\alpha=\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-\alpha^2}d\alpha=1$, we get: $u(x,t)=\frac{T_1}{2}(\varphi(z)+1)+\frac{T_2}{2}(1-\varphi(z))=\frac{T_1+T_2}{2}+\frac{T_1-T_2}{2}\varphi\left(\frac{x}{2a\sqrt{t}}\right)$

Non Homogenous Heat Equation Neumann B.C.

Solve:
$$u_t=a^2u_{xx}, t>0, 0<x<L$$
$$u(x,0)=x$$
$$u_x(0,t)=1, u_x(L,t)=2$$

Look for the solution in the form $u(x,t)=v(x,t)+z(x,t)$. Since we are dealing with Neumann B.Cs, we take for z the function $z(x,t)=x+\frac{x^2}{2L}$. Put this into the PDE: $v_t=u_t=a^2u_{xx}=a^2v_{xx}+a^2z_{xx}=a^2v_{xx}+\frac{a^2}{L}\rightarrow v_t=a^2v_{xx}+\frac{a^2}{L}$, $v(x,0)=-\frac{x^2}{2L}$, $v_x(0,t)=v_x(L,t)=0$. We get the non-homog. heat equation with non-homog. Initial condition but homog. B.Cs. We separate the problem into 2 problems: $v(x,t)=w(x,t)+h(x,t)$, so that: $\begin{cases}w_t=a^2w_{xx}, 0<x<L, t>0\\ w(x,0)=-\frac{x^2}{2L}\\ w_x(0,t)=w_x(L,t)=0\end{cases}$; $\begin{cases}h_t=a^2h_{xx}+\frac{a^2}{L}, 0<x<L, t>0\\ h(x,0)=0\\ h_x(0,t)=h_x(L,t)=0\end{cases}$. We solved a similar problem in class. Using solution from class, we define $A=\frac{a^2}{L}$, $f(x)=-\frac{x^2}{2L}$. Denoting the new coefficients for $f(x)$ as $\tilde{a_0},\tilde{a_n}$: $a_0=-\frac{1}{2L}a_0=-\frac{L}{6}$. $\tilde{a_n}=\frac{1}{2L}a_n=-\frac{1}{2L}\left(\frac{4L^2}{\pi^2}\frac{(-1)^n}{n^2}\right)=\frac{2L(-1)^{n+1}}{\pi^2n^2}, n=1,2\ldots$. Therefore, $w(x,t)=-\frac{L}{6}+\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-a^2\left(\frac{n\pi}{L}\right)^2t}\cos\left(\frac{n\pi}{L}x\right)$, $h(x,t)=\frac{a^2}{L}t$, $u(x,t)=z(x,t)+h(x,t)+w(x,t)\rightarrow u(x,t)=x+\frac{x^2}{2L}+\frac{a^2}{L}t-\frac{L}{6}+\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-a^2\left(\frac{n\pi}{L}\right)^2t}\cos\left(\frac{n\pi}{L}x\right)$

Non-Homogenous Heat Equation, other B.C.

Solve:
$$u_t=u_{xx}+1, t>0, 0<x<1$$
$$u(x,0)=\frac{1-\cos(x\pi)}{4}$$
$$u(0,t)=1, u(1,t)=\frac{1}{2}$$

We look for the solution $u(x,t)=v(x,t)+z(x,t)$. Using the table, $\boldsymbol{z(x,t)}=1-\frac{x}{2}$. **New problem**: $v(x,t)=u(x,t)-z(x,t)$. By substituting $u(x,t)$ into the problem we get for $v(x,t)$: $\begin{cases}v_t=v_{xx}+1\\ v(x,0)=u(x,0)-1+\frac{x}{2}=\frac{1-\cos(x\pi)}{4}-1+\frac{x}{2}\\ v(0,t)=v(1,t)=0\end{cases}$. We get the non-homog. Heat equation with non-homog. I.C but homog. B.Cs. **Separate the problem** to 2 problems: $v(x,t)=w(x,t)+h(x,t)$, so that: $\begin{cases}w_t=w_{xx}, 0<x<1, t>0\\ w(x,0)=-\frac{3}{4}+\frac{x}{2}-\frac{\cos(\pi x)}{4}\\ w(0,t)=w(1,t)=0\end{cases}$; $\begin{cases}h_t=h_{xx}+1, 0<x<1, t>0\\ h(x,0)=0\\ h(0,t)=h(1,t)=0\end{cases}$.

Solving for $w(x,t)$ using separation of variables: $w(x,t)=X(x)T(t)\rightarrow\frac{X''}{X}=\frac{Y'}{Y}=-\lambda$. This is a Sturm-Liouville problem with non-trivial solution only for $\lambda>0$. The e.values and e.functions are: $X_n(x)=\sin(n\pi x)$, $\lambda_n=n^2\pi^2$, $n=1,2,3\ldots$. For T: $T_n'+n^2\pi^2T_n=0\rightarrow T_n=a_ne^{-n^2\pi^2t}$. Hence: $w(x,t)=\sum_{n=1}^{\infty}a_ne^{-n^2\pi^2t}\sin(n\pi x)$. Put it into the initial condition: $w(x,0)=\sum_{n=1}^{\infty}a_n\sin(n\pi x)=-\frac{3}{4}+\frac{x}{2}-\frac{\cos(\pi x)}{4}$. **Calculate the coefficients**: $a_n=2\int_0^1\left(-\frac{3}{4}+\frac{x}{2}-\frac{\cos(\pi x)}{4}\right)\sin(n\pi x)dx$. $\int_0^1\left(-\frac{3}{4}\right)\sin(n\pi x)dx=\left[\frac{3}{4}\frac{\cos(n\pi x)}{n\pi}\right]_0^1=\frac{3}{4n\pi}((-1)^n-1)$. $\int_0^1\left(\frac{x}{2}\right)\sin(n\pi x)dx=\frac{1}{2}\left(\left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^1+\frac{1}{n\pi}\int_0^1\cos(n\pi x)dx\right)=-\frac{1}{2}\frac{(-1)^n}{n\pi}$. $\int_0^1\left(-\frac{\cos(\pi x)}{4}\right)\sin(n\pi x)dx=-\frac{1}{8}\int_0^1[\sin(\pi(n+1)x)+\sin(\pi(n-1)x)]dx=\frac{1}{8}\left(\left[\frac{\cos(\pi(n+1)x)}{\pi(n+1)}\right]_0^1+\left[\frac{\cos(\pi(n-1)x)}{\pi(n-1)}\right]\right)=\frac{1}{8\pi}\left(\frac{(-1)^{n+1}-1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}\right)=\frac{(-1)^{n+1}-1}{8\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right)=\frac{n((-1)^{n+1}-1)}{4\pi(n^2-1)}$. $a_n=\frac{1}{2n\pi}((-1)^n-3)+\frac{n((-1)^{n+1}-1)}{4\pi(n^2-1)}$ for $n\neq 1$, and $a_1=-\frac{2}{\pi}$, $a_0=0$. **For $h(x,t)$ we get**: $h(x,t)=\sum_{n=1}^{\infty}h_n(t)\sin(n\pi x)$, $F(x,t)=\sum_{n=1}^{\infty}F_n(t)\sin(n\pi x)=1$, $F_n(t)=2\int_0^1\sin(n\pi x)dx=\frac{2}{n\pi}(1-(-1)^n)$. Put these h and F into the problem for h: $\begin{cases}h_n'(t)+n^2\pi^2h_n(t)=F_n(t)\\ h_n(0)=0\end{cases}$, and the solution: $h_n(t)=\int_0^t e^{-n^2\pi^2(t-\tau)}F_n(\tau)d\tau=\frac{2}{n^3\pi^3}(1-e^{-n^2\pi^2t})(1-(-1)^n)$. For an even n: $h_n(t)=0$, so we define $n=2k-1$, and so: $h(x,t)=\frac{4}{\pi^3}\sum_{k=1}^{\infty}\frac{1-e^{-(2k-1)^2\pi^2t}}{(2k-1)^3}\sin((2k-1)\pi x)$. The final solution: $u(x,t)=z(x,t)+w(x,t)+h(x,t)=1-\frac{x}{2}+\frac{1}{2\pi}\sum_{n=2}^{\infty}\left[\frac{1}{n}((-1)^n-3)+\frac{n((-1)^{n+1}-1)}{(n^2-1)}\right]e^{-n^2\pi^2t}\sin(n\pi x)+\left(-\frac{2}{\pi}\right)e^{-\pi^2t}\sin(\pi x)+\frac{4}{\pi^3}\sum_{k=1}^{\infty}\frac{1-e^{-(2k-1)^2\pi^2t}}{(2k-1)^3}\sin((2k-1)\pi x)$

Harmonic Function , Specific Solution, Min/Max

Let $u(r,\theta)$be a harmonic function in the disk $x^2+y^2<R^2$ and satisfies the following condition on the boundary of the disk: $u(R,\theta)=f(\theta)=\begin{cases}\sin^2(2\theta), |\theta|\leq\frac{\pi}{2}\\ 0, \frac{\pi}{2}<|\theta|\leq\pi\end{cases}$. A) Find $u(0,0)$ without solving the boundary problem. B) Prove that for any (r,θ) in the disk it is true that $0\leq u(r,\theta)\leq 1$

A) Use the Poisson formula (pay attention, $\int_0^{2\pi}=\int_{-\pi}^{\pi}$ for 2π periodic functions). $u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}f(\varphi)d\varphi$. For $(x,y)=(0,0)$ take $(r,\theta)=(0,0)$. It is given that $f(\varphi)=\begin{cases}\sin^2(2\varphi), |\varphi|\leq\frac{\pi}{2}\\ 0, \frac{\pi}{2}<|\varphi|\leq\pi\end{cases}$, so: $u(0,0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\varphi)d\varphi=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(2\varphi)d\varphi=\frac{1}{4}$, $u(0,0)=\frac{1}{4}$ **B)** By the maximum principle, the max/min of the harmonic function is on the boundary. $\max_{x^2+y^2=R^2}u(R,\theta)=\max_{-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}}\sin^2(2\theta)=1$, and $\min_{x^2+y^2=R^2}u(r,\theta)=\min_{-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2},\frac{\pi}{2}\leq|\theta|\leq\pi}\{\sin^2(2\theta), 0\}=0$, therefore, $0\leq u(r,\theta)\leq 1\ \forall(r,\theta)$ in the disk.

Finding a Harmonic Function under certain constraints/conditions

Find a harmonic function in the disk $x^2+y^2<6$, which satisfies $u(x,y)=y+y^2$ on the boundary of the disk. Write the final answer in x,y coordinates.

Rewrite the problem in polar coordinates: $\begin{cases}u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi}=0, 0<r<\sqrt{6}, 0\leq\varphi\leq 2\\ u(\sqrt{6},\varphi)=\sqrt{6}\sin(\varphi)+6\sin^2(\varphi)\end{cases}$ A general solution is $u(r,\varphi)=A_0+\sum_{n=1}^{\infty}(A_n\cos(n\varphi)+B_n\sin(n\varphi))r^n$. Then, $u(\sqrt{6},\varphi)=A_0+\sum_{n=1}^{\infty}(A_n\cos(n\varphi)+B_n\sin(n\varphi))\sqrt{6}^n=\sqrt{6}\sin(\varphi)+6\sin^2(\varphi)=\sqrt{6}\sin(\varphi)+\frac{6}{2}(1-\cos(2\varphi))\rightarrow A_0=3$, $6A_2=-3\rightarrow A_2=-\frac{1}{2}$, $A_n=0\ \forall n\neq 0,2$. $\sqrt{6}B=\sqrt{6}\rightarrow B_1=1$, $B_n=0,\forall n\neq 1$. $u(r,\varphi)=3-\frac{1}{2}\cos(2\varphi)r^2+\sin(\varphi)r$. Go back to (x,y): $r\sin(\varphi)=y$, $r^2\cos(2\varphi)=r^2(\cos^2(\varphi)-\sin^2(\varphi))=x^2-y^2\rightarrow u(x,y)=3+y-\frac{1}{2}(x^2-y^2)$