Second order circuits:
- If $\omega_0^2 < 0 \to$ system is not stable
- Over-d: $\zeta > 1$;Critically-d: $\zeta = 1$; Under-d: $0 < \zeta < 1$
- If capacitance→ ∞ - capacitor can be ignored (shortened)
- If current source = 0, it is actually disconnected

Singularity functions:
- $\int_a^b \varphi(\lambda)\delta(\lambda - t_0) = \int_a^b \varphi(t_0)\delta(\lambda - t_0) = \begin{cases}\varphi(t_0) & a < t_0 < b \\ 0 & else\end{cases}$
- $\delta_{-1}(t) = u(t)$; $\delta_{-2}(t) = r(t) = tu(t)$
- $P_\Delta(t) = (u(t) - (u(t-\Delta))\frac{1}{\Delta}$

Laplace Transform:
- Constant's behavior: Same behavior as for constants inside integrals.
- Laplace of 1 is same as Laplace of step function
- $L^{-1}[F(s)\cdot e^{\alpha s}] = L^{-1}[F(s)]$ time shifted by $\alpha = f(t+\alpha)$
- Definition (unilateral Laplace transform) : $F(s) = \int_0^\infty f(t)e^{-st}dt$
- $s = jw$ in the Laplace domain
- Initial Value Theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$
- Final Value Theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
- Integration: $L\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s)$
- Time shift: $L\{f(t-\tau)\} = e^{-s\tau}F(s)$
- Frequency shift: $L\{e^{-at}f(t)\} = F(s+a)$
- Time Scaling: $L\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$
- Frequency Scaling: $L\{af(at)\} = F(\frac{s}{a})$
- Differentiation in time:
  $L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f^{(1)}(0^-) - \dots - f^{(n-1)}(0^-)$
- Differentiation in frequency: $-tf(t) \leftrightarrow \frac{dF(s)}{ds}$

Solving ODE with Laplace transform:
- Transform of a differential equation: $Y(s) = U(s)\frac{P_2(s)}{P_1(s)} - \frac{Q_1(s)}{Q_2(s)}$
- The term $G(s) = \frac{P_2(s)}{P_1(s)} = \frac{\text{output}}{\text{input}}$ is called the transfer function.
- The response to the input is $Y(s) = \frac{P_2(s)}{P_1(s)}U(s) = G(s)U(s)$
- Transfer function: For continuous-time input signal $U(s)$ and output $Y(s)$, the transfer function $G(s)$ is the linear mapping of the laplace transform of the input $U(s) = \mathcal{L}\{u(t)\}$ to the Laplace transform of the output $Y(s) = \mathcal{L}\{y(t)\}$ such that $Y(s) = G(s)U(s)$
- Transfer function = Laplace transform of the impulse response of an LTI system when initial conditions = 0
- For a sinusoidal steady state source: $V_s(\text{or } I_s) = Asin(\omega t)$, the output would be $A\underbrace{|G(s)|}_{\text{gain}}\cdot \sin\left(\omega t + \underbrace{\angle G(s)}_{\text{phase}}\right)$

Convolution:
- Define "impulse response" : if $x(t) = \delta(t)$, then $y(t) = h(t)$ is the impulse response. General solution would be $y_{total} = ZIR + convolution$
- The convolution integral: $y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$
- $h(\tau)$ May be nonzero only at $\tau > 0$, $h(t-\tau)$ can be nonzero for $t-\tau > 0$. Updated lims: $y(t) = x(t)*h(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_0^{\infty} x(t-\tau)h(\tau)d\tau$
- Impulse starting at $t=0$, $y(t) = x(t)*h(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t x(t-\tau)h(\tau)d\tau$
- Properties:
  - $x*h = h*x$
  - $(g+h)*x = g*x + h*x$
  - $(h*x)' = h'*x = h*x'$ (assuming derivative exists)
  - $x(t)*\delta(t) = x(t)$
  - Time shift: if $y(t) = x(t)*h(t) \Longrightarrow x(t)*h(t-t_0) = y(t-t_0)$
  - Specific case: if $h(t) = \delta(t-t_0) \Longrightarrow x(t)*h(t) = x(t-t_0)$
  - $x(t)*\delta'(t) = x'(t)*\delta(t) = x'(t)$

Phasors & Sinusoidal Steady State:
- Sinusoidal Steady State: a state of a system with a sine input, "after a long time has passed", i.e. all Transient effects are gone.
- It will be more convenient to work with complex numbers:
  $$A_m\cos(\omega t + \varphi) = Re\left[A_m e^{j(\omega t+\varphi)}\right] = Re\left[A_m e^{j\varphi}e^{j\omega t}\right]$$
- Define the Phasor as: $\tilde{A} \equiv A_m e^{j\varphi} \Longrightarrow A_m\cos(\omega t+\varphi) = Re\left[\tilde{A}e^{j\omega t}\right]$
- Transforming differential equations into algebraic equations: $\frac{d^n}{dt^n}(A_m\cos(\omega t+\varphi)) = \frac{d^n}{dt^n}\left(Re\left[\tilde{A}e^{j\omega t}\right]\right) = \left(Re\left[(j\omega)^n\tilde{A}e^{j\omega t}\right]\right)$
- Kirchhoff's laws apply for Phasors as well
- Alternate notation: $\tilde{A}\angle\varphi$
- If $Re[s] < 0 \to$ system is not stable, otherwise $\to$ stable
- If **feedback**: $H_{new} = \frac{G(s)}{G(s)\pm 1}$, (where $\pm$ = sign of feedback signal??)

Impedance/Admittance:
- Impedance: the ratio of the voltage and the current. For sinusoidal signals: $Z_x = \frac{\tilde{V}_x}{\tilde{I}_x}$. Admittance: $Y_x = \frac{\tilde{I}_x}{\tilde{V}_x}$
- Since $Z_x\tilde{I}_x = \tilde{V}_x$, the voltage and current might have different phases
  - For an Inductor, current lags behind voltage by $\frac{\pi}{2}$, for capacitors it's the other way around.
- In *Sinusoidal steady state*, each component can be treated as a "resistor" with R = impedance. Follows ohm's law (same for Phasors). Norton & Thevinin are legal.

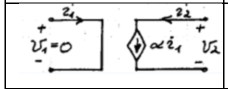
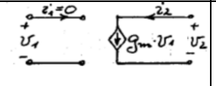
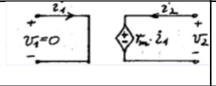
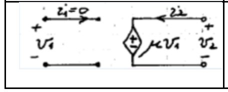
Power in Sinusoidal Steady State:
- For sinusoidal steady state we have:
  - $V(t) = V_m\cos(\omega t + \angle V)$; $I(t) = I_m\cos(\omega t + \angle I)$
  - $P(t) = V_m I_m\cos(\omega t + \angle V)\cos(\omega t + \angle I) = \underbrace{\frac{1}{2}V_m I_m\cos(\angle V - \angle I)}_{\text{average power}} + \underbrace{\frac{1}{2}V_m I_m\cos(2\omega t + \angle V + \angle I)}_{\text{time dependent power at double frequency}}$

Coupled Inductors:
- Coupled inductors are such that a change in the current through one inductor results in a change of the voltage across the other.
- The coupling equations are given in formulae , M=mutual inductance
- Coupling coefficient is defined as: $k = \frac{|M|}{\sqrt{L_1L_2}}, 0 \le k \le 1$
  - $k = 0 \Rightarrow$ no coupling ; $k = 1 \Rightarrow$ full coupling
- At Sinusoidal Steady State we have $\begin{cases}\tilde{V}_1 = j\omega L_1\tilde{I}_1 + j\omega M\tilde{I}_2 \\ \tilde{V}_2 = j\omega M\tilde{I}_1 + j\omega L_2\tilde{I}_2\end{cases}$

Ideal Transformers:
- Idealization assumptions: No energy dissipation, No flux leakage $\Rightarrow k = 1$ (full coupling), Infinite inductance: $L_1, L_2, M \to \infty$
- Ideal transformer is LTI.
- Since there is no power loss, any power entering on one side will emanate from the other: $V_1(t)I_1(t) + V_2(t)I_2(t) = 0$
- Ideal transformers different dots: $\frac{v_1}{v_2} = -\frac{n_1}{n_2}$ $\frac{i_1}{i_2} = \frac{n_2}{n_1}$

Reflection:
- Primary: the segment with a steady state input
- Secondary: the other inductor's segment
- Reflection process: Usually reflect secondary onto primary. If the opposite reflection is desired, n_1, n_2 switch places in all reflections. New circuit is w/o the coupled inductors and is connected at those inductors' terminals.
- If Circuit is in SSS, V=RI=ZI where Z is SSS condition (impedance)

Dependent sources:

Diagram	Name	Case Identifiers	Formula for the Unknown
	Current controlled Current source	$I_2 = \alpha I_1$ $V_1 = 0$	Current Ratio: $\alpha = \frac{I_2}{I_1}$
	Voltage controlled Current source	$I_1 = 0$ $I_2 = g_m V_1$	Transfer Conductance $g_m = \frac{I_2}{V_1}$
	Current controlled Voltage source	$V_1 = 0$ $V_2 = r_m I_1$	Transfer Resistance: $r_m = \frac{V_2}{I_1}$
	Voltage controlled Voltage source.	$I_1 = 0$ $V_2 = \mu V_1$	Voltage Ratio: $\mu = \frac{V_2}{V_1}$

- The sources are active. They may amplify the current/voltage/power
- Controlled sources may not be zeroed. While using superposition/ Norton/Thevinin equivalents, only independent sources are zeroed.

Translational Mechanical Systems:

	Mechanical	Electrical
Signals	V – velocity	V – voltage
	f – force	I – current
Components	Viscous friction or linear dashpot with constant B, that obeys $v = \frac{1}{B}f$	Resistor that obeys $V = RI$
	Spring with constant K $v = \frac{1}{k}\frac{df}{dt}$	Inductor that obeys $V = L\frac{dI}{dt}$
	Mass (M) that obeys $f = M\frac{dv}{dt}$	Capacitor that satisfies $I = C\frac{dV}{dt}$
voltage of ground=0 (you can apply any current to grnd. and voltage remains 0)	velocity of ground=0 (you can apply any force to ground and voltage remains 0)	

Rotational Mechanical Systems:

	Mechanical	Electrical
Signals	ω – velocity	V – voltage
	τ – torque	I – current
Components	Torsional friction (B), $\omega = \frac{1}{B}\tau$	Resistor that obeys $V = RI$
	Torsional spring (K) $\omega = \frac{1}{k}\frac{d\tau}{dt}$	Inductor that obeys $V = L\frac{dI}{dt}$
	Moment of inertia (J) $\tau = J\frac{d\omega}{dt}$	Capacitor that satisfies $I = C\frac{dV}{dt}$

- If connected through velocity – parallel. If connected through force – series.

Electrical	Mechanical
$\sum_{node} \text{current} = 0$	$\sum_{object} \text{forces} = 0$

State space representation:

- The following set is called a <u>state-space representation</u>:

$$\begin{cases} \bar{x}'(t) = A\bar{x}(t) + \bar{b}u(t) \\ y(t) = \bar{c}\bar{x}(t) + du(t) \\ \bar{x}(t=0) \equiv \bar{x}(0) \end{cases}$$

$\bar{x}(t)$	State variable
$u(t)$	Input
$y(t)$	Output
A	Dynamic matrix
$\bar{b}, \bar{c}, d$	---

Phase Variable

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \ddots & & \\ \vdots & & & \ddots & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \boldsymbol{\bar{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \boldsymbol{\bar{c}} = [b_0 \cdots b_{n-1}],$$

$$\boldsymbol{d} = 0$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{bmatrix} \bar{c} \\ \bar{c}A \\ \vdots \\ \bar{c}A^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y^{(1)}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix}$$

Where $a_0, a_1 \ldots$ are coefficients of y when y^n is ordered by ascending n.

- In general case, where the input appears with derivatives, the ODE can be written as

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) = \sum_{m=0}^{n-1} b_m u^{(m)}(t)$$

- Using linearity (or superposition), if we find the <u>result of a simple input</u> without derivatives $y_u(t)$, then the general solution is

$$y(t) = \sum_{m=0}^{n-1} b_m y_u^{(m)}(t)$$

- If the RHS of the ODE contains a $b_n u^{(n)}(t)$ term, then $d = 0$ changes to $d = b_n$, and $\bar{c}$ changes to

$$\bar{c} = [b_0, b_1, \cdots, b_{n-1}] + b_n[-a_0, -a_1, \cdots, -a_{n-1}]$$

Equivalent representations:

- Given a certain state variable $\bar{x}(t)$, we can use a transformation T to obtain a different representation. We are interested in representations where $\tilde{A}$ is diagonal (assuming A has distinct eigenvalues)
- <u>Steps to find a diagonal representation</u>:
  1. Find the eigenvalues of A by $\det(\lambda I - A) = 0$
  2. Find the eigenvectors of A by solving eq-n set $(\lambda_k I - A)\bar{u}_k = 0$
     a. If Eigenvector can't be found due to 2 unknowns and 1 eq. express $\underline{u_k}$ as the same component (eg. $\underline{u_k} = \begin{bmatrix} \alpha u_{k2} \\ u_{k2} \end{bmatrix}$) and factor out the component. The result is the basis of all eigenvectors of the eigenvalue.
     b. If a component of the eigenvector is independent, then it can be any value, but we select it to be 1 since that would be a basis for any other eigenvector of this eigenvalue.
  3. <u>Transformation matrix</u> - eigenvectors as columns $\boldsymbol{T} = [u_1 \, u_2 \ldots u_n]$
  4. Calculate the inverse matrix $\boldsymbol{T^{-1}}$.
     a. Matrix inverse 2x2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
     b. Matrix inverse 3x3:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{a(ei-fh)-b(di-fg)+c(dh-eg)} \begin{bmatrix} ei-fh & -(bi-ch) & bf-ce \\ -(di-fg) & ai-cg & -(af-cd) \\ dh-eg & -(ah-bg) & ae-bd \end{bmatrix}$$

  5. Find the new representation, eg. $\tilde{A} \equiv \Lambda = \mathrm{T}^{-1}A\mathrm{T} = \begin{bmatrix} \lambda_1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \lambda_n \end{bmatrix}$

State space Time Domain solution:

- Given a system $\begin{cases} \bar{x}'(t) = A\bar{x}(t) + \bar{b}u(t) \\ y(t) = \bar{c}\bar{x}(t) \\ \bar{x}(0) \end{cases}$, we would like to find an expression for $\bar{x}(t)$ and through that we find $y(t)$.
- (Input response?) $\bar{x}(t)$ consists of 2 parts: $\bar{x}(t) = \bar{x}_i(t) + \bar{x}_u(t)$, where:
  - $\bar{x}_i = \Phi(t)\bar{x}(0)$ is the homogenous solution (ZIR)
  - $\bar{x}_u = \int_0^t \Phi(t)\bar{b}u(t-\tau)d\tau$ is the response to the input (ZSR)
- The matrix $\Phi(t) = e^{At}$ is called the transition/transfer matrix
- We can express e^{At} by finding the diagonalizing transform:

$$\Lambda, T, T^{-1}, \boldsymbol{\Phi(t)} = e^{At} = Te^{\Lambda t}T^{-1}, \ \boldsymbol{e^{\Lambda t}} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & e^{\lambda_n t} \end{bmatrix}$$

- <u>Properties of matrix exponent:</u>
  - For $t = 0, e^{A0} = I$ (identity matrix)

$\frac{d\Phi(t)}{dt} = Ae^{At}$	$e^{ax}e^{bx} = e^{(a+b)x}$	$e^X e^{-X} = I$

  - If matrices obey XY=YX. Then $e^X e^Y = e^Y e^X = e^{Y+X}$

State space Laplace Domain solution:

- Use Laplace Transforms to solve state-space equations:

$$L\{\bar{x}(t) = A\bar{x}(t) + \bar{b}u(t)\} \Rightarrow s\bar{x}(s) - \bar{x}(0) = A\bar{x}(s) + \bar{b}U(s)$$
$$\Rightarrow (sI - A)\bar{x}(s) = \bar{x}(0) + \bar{b}U(s) \Rightarrow \bar{x}(s)$$
$$= (sI - A)^{-1}\bar{x}(0) + (sI - A)^{-1}\bar{b}U(s)$$

- Now, using $\bar{x}(s)$ we can find $\bar{Y}(s)$ using the row vector $\bar{c}$ and adding $dU(t)$ if $d \neq 0$:

$$\boldsymbol{Y(s)} = \bar{c}(sI - A)^{-1}\bar{x}(0) + \bar{c}(sI - A)^{-1}\bar{b}U(s) + dU(s)$$

OR:

$$\boldsymbol{Y(s)} = \bar{c}(sI - A)^{-1}\bar{x}(0) + G(s)U(s)$$

- The <u>transfer function</u> is $\left(\frac{\text{output}}{\text{input}}, 0 \text{ init. Cond.}\right)$

$$G(s) = \frac{Y(s)}{U(s)} = \bar{c}(sI - A)^{-1}\bar{b} + d$$

- $L\{e^{At}\} = (sI - A)^{-1}$

Diagonal representation:

- In the case that the transfer function $G(s)$ has distinct poles, we can expand it as: $G(s) = p_0 + \frac{p_1}{s-s_1} + \frac{p_2}{s-s_2} + \cdots + \frac{p_n}{s-s_n}$. Then a diagonal representation $\{A, \bar{b}, \bar{c}, d\}$, is given by: $\boldsymbol{A} = \begin{bmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n \end{bmatrix}$; $\boldsymbol{\bar{b}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$\boldsymbol{\bar{c}} = [c_1 \ldots c_n] \ ; \ \boldsymbol{d} = p_0 \ ; \ \boldsymbol{x_0} = \begin{bmatrix} \bar{c} \\ \bar{c}A \\ \vdots \\ \bar{c}A^{n-1} \end{bmatrix}^{-1} \bar{y}(0)$$

$\forall$ pairs c_i, b_i, s.t. $c_i \cdot d_i = p_i$ ascend.n.

- Two (of many) legitimate choices are:

$\boldsymbol{\bar{b}} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$; $\boldsymbol{\bar{c}} = [p_1 \ldots p_n]$	$\boldsymbol{\bar{b}} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$; $\boldsymbol{\bar{c}} = [1 \ldots 1] \leftarrow$ canonical form

Frequency response and resonance:

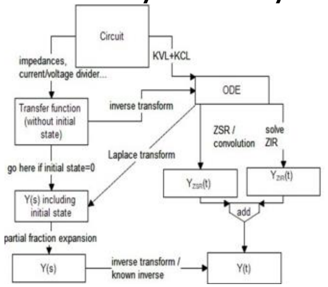
- For a transfer function $G(s)$ and an input $A\sin(\omega t)$, the output is : $y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \arg G(j\omega))$
- Transfer function is sometimes denoted as H instead of G.
- Since the Amplitude and phase depend on the frequency, we get a different response for each frequency
- Resonance Frequency: The frequency for which the voltage and current supplied to the circuit have the same phase. Denote as ω_0.

Bode Plots:

- dB=20 $\log(x)$
- DC Gain means 0 frequency: $\omega = 0$
- Real Gain: $= 20\log(|Q(s)|) - 20\log(|P(s)|$ where $G(s) = \frac{Q(s)}{P(s)}$
- Diff between real & asymptotic gain: $|G(s)|$
- Cutoff frequency: Where the gain is 0dB (when Bode plot intersects x-axis)
- Bode plot slope: $\frac{G(\omega_2) - G(\omega_1)}{\log(\omega_2) - \log(\omega_1)} \left[\frac{dB}{dec}\right]$
- If magnitude Bode plot doesn't start from the y-axis, unless stated otherwise assume that it comes from a very high/low value, then we have a zero/pole at the origin of order corresponding to the slope.
- Minimal phase: All poles & zeros are in the left side of the complex plane.
- Logarithm: $\log_b a = c \Leftrightarrow b^c = a$

	Difference between exact and asymptotic value @ the cutoff frequency
Simple Zero	+3[dB]
Simple Pole	-3[dB]
Conjugate pair of Zeroes	$-20\log_{10}\left(\frac{1}{2\|\zeta\|}\right)$
Conjugate pair of Poles	$+20\log_{10}\left(\frac{1}{2\|\zeta\|}\right)$

Circuits' analysis summary:



Filters: