Linear Space: The set V is called a linear space over the calar field F (either $\mathbb{R}$ or $\mathbb{C}$), if it has addition and scalar multiplication defined on it and pertains to the following conditions:

$u, v \in V$ then $u + v \in V$	$a \in F$ and $u \in V$ then $a \cdot u \in V$
$u, v, w \in V$ then $(u+v)+w = u+(v+w)$	There exists $\vec{0} \in V$ s.t. $u + \vec{0} = u$ for all $u \in V$
For each $u \in V$ there exist $-u \in V$ s,t, $u + (-u) = \vec{0}$	$u, v \in V$ then $u + v = v + u$
$a \in F,\ u, v \in V$ then $(a+b)u = au + av$	$a, b \in F,\ u \in V$ then $(a+b)u = au + bu$ and $a(bu) = ab(u)$
Let $1 \in F$ then for each $u \in V$ we have $1 \cdot u = u$	

Inner Product: Let V be a linear space over F, then: $\langle \cdot,\cdot \rangle : V \times V \to F$ is called an inner product if $\forall u, v, \omega \in V$:

$\langle u, v \rangle = \overline{\langle v, u \rangle}$	$\langle \alpha u, v \rangle = \alpha \langle u, v \rangle\ \forall \alpha \in F$
$\langle u + v, \omega \rangle = \langle u, \omega \rangle + \langle v, \omega \rangle$	$\langle u, u \rangle \geq 0;\ u = \vec{0} \Leftrightarrow \langle u, u \rangle = 0$

A linear space with an inner product defined on it is called an inner product space.

Inner Product Properties:

$\langle u, u \rangle = \overline{\langle u, u \rangle} \Rightarrow \langle u, u \rangle \in \mathbb{R}$	$\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$
$\langle \omega, u + v \rangle = \langle \omega, u \rangle + \langle \omega, v \rangle$	$\langle u, \vec{0} \rangle = \langle \vec{0}, u \rangle = 0$
$\langle u, v \rangle = \overline{\langle v, u \rangle}$	$\langle u, v \rangle = \sum_{i=1}^{n} u_i \bar{v}_i$
$\langle f, g \rangle = \int_{-L}^{L} f(x) \cdot \overline{g(x)}\, dx$	$\langle f, f \rangle = \int_{-L}^{L} f(x) \cdot \overline{f(x)}\, dx = \int_{-L}^{L} \lvert f(x) \rvert^2 dx = 0$
$f \equiv g$ iff $\langle f - g, f - g \rangle = \int_{-L}^{L} \lvert f(x) \rvert^2 dx = 0$	u, v Orthogonal if $\langle u, v \rangle = 0$

NORMS:

Norm: $\|\cdot\| : V \to \mathbb{R}$ is a norm if: 1) $\forall \alpha \in F, v \in V, \|\alpha \cdot v\| = |\alpha| \cdot \|v\|$. 2) $\|u + v\| \leq \|u\| + \|v\|\ u, v \in V$. 3) $\forall v \in V, \|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = \vec{0}$. Norm on $[-\pi, \pi]$: $\|f\| \triangleq \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx$

Inner product norm: $\|u\| \triangleq \sqrt{\langle u, u \rangle}$

Norm: $\langle f, f \rangle \triangleq \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx$, $\|f\| \triangleq \sqrt{\langle f, f \rangle} = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 dx}$

Norms Properties:

$\|u\| = \sqrt{\langle u, u \rangle}$	$\lvert\langle u, v \rangle\rvert < \|u\| \|v\|$
$\|\alpha v\| = \|\alpha\| \cdot \|v\|$	$\|v\| \geq 0$, $\|v\| = 0$ iff $v = \vec{0}$
$\|u + v\| \leq \|u\| + \|v\|$	If u, v orthogonal, $\|u + v\|^2 = \|u\|^2 + \|v\|^2$
$\text{dist.}(u, v) = \|v - u\|$	$\|v\|^2 = \langle v, v \rangle = \sum_{i=1}^{\infty} \lvert v_i \rvert^2$

ORTHOGONALITY/NORMALITY, NORMS & SPACES:

Orthogonality: Two vectors $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$. A set $A \subseteq V$ is an orthogonal system if each pair of vectors $u_1, u_2 \in A$ are orthogonal. A set $A \subseteq V$ is an orthonormal system if it is orthogonal and the norm of each vector $u \in A$ is $\|u\| = 1$. In other words $A = \{u_1, u_2, \dots, u_n\}$ then $\langle u_i, u_j \rangle = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. An orthogonal system is linearly independent.

Orthogonality Alternative: 1) u, v are orthogonal if $\langle u, v \rangle = 0$. 2) A set of vectors is an orthogonal system if each pair of vectors u_1, u_2 in the system maintains $\langle u_1, u_2 \rangle = 0$. 3) A set of vectors is an orthonormal system if $\langle u_n, u_m \rangle = 1$ (if $m = n$), 0 otherwise.

Notice: $\{e^{inx}\}_{n=-\infty}^{\infty}$ is orthogonal and $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n=-\infty}^{\infty}$ is orthonormal in both PC and L_2

Orthogonal projection: u "onto span" $\{v\}$: $z \to$"orthogonal projection" $z = \frac{\langle u, v \rangle}{\langle v, v \rangle} \cdot v$

Pythagoras Theorem: $u, v \in V$ an inner product space. $\langle u, v \rangle = 0$, Then: $\|u + v\|^2 = \|u\|^2 + \|v\|^2$. Notice that $\|u - v\|^2 = \|u\|^2 + \|v\|^2$ if $u \perp v, \omega = u + v \Rightarrow \|\omega\|^2 - \|v\|^2 = \|u\|^2$. The theorem will hold for $\|u - v\|^2$ as well.

Cauchy-Schwarz inequality: $u, v \in V$ an inner product space. Then: $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$

Closed System: $\{e_n\}_{n=1}^{\infty}$ an orthonormal system in an inner product space V. We can say that $\{e_n\}_{n=1}^{\infty}$ is a closed system if $\forall u \in V$: $\lim_{N \to \infty} \|u - \sum_{n=1}^{N} \langle u, e_n \rangle \cdot e_n\| = 0$ or: $u = \sum_{n=1}^{\infty} \langle u, e_n \rangle \cdot e_n$ in the norm sense.

Complete System: $\{u_k\}_{k=1}^{\infty}$ is called a complete orthonormal system if the only vector $u \in V$ that gives $\forall k\ \langle u, u_k \rangle = 0$ is $u = \vec{0}$. In other words we can't find another axis that will be orthogonal to the rest. A closed orthonormal system is complete

Hilbert space: A normed linear space that is complete and the norm is defined by an inner product.

The Central Theorem on Complete sets: Let $(V, <\cdot,\cdot>)$ be a Hilbert space over F and let $\{\Phi\}_{n=1}^{\infty}$ be an orthonormal set in V then the following conditions are equivalent:

1. For all $f \in V$ if $\langle f, \Phi_n \rangle = 0$ for all n, then $f \equiv 0$
2. For all $f \in V$ we have $\lim_{k - \infty} \|f - \sum_{n=1}^{k} \langle f, \Phi_n \rangle * \Phi_n\| = 0$
3. For all $f \in V$ we have $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \Phi_n \rangle|^2$

Perceval's Equality: the orthonormal system $\{u_n\}_{n=1}^{\infty}$ is closed in V iff $\sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2 = \|v\|^2$. Perceval $\Leftrightarrow$ closed orthonormal system $\Rightarrow$ complete orthonormal system $\Rightarrow$ closed orthonormal system only for Hilbert space.

Perceval's Equality alternative: $\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 = 2 \cdot \sum_{n=-\infty}^{\infty} |C_n|^2$ when $f(x)$ is piecewise continious and f uniformly converges to its Fourier.

BEST APPROXIMATION THEOREM:

Best Approximation Theorem: Let $A = \{u_1, u_2, \dots, u_n\}$ be an orthonormal system in an inner product space V, then for each choice of coefficents from the field $\{c_1, c_2, \dots, c_n\} \subset F$ we have: $\|v - \sum_{k=1}^{n} c_k u_k\|^2 \geq \|v\|^2 - \sum_{k=1}^{n} |\langle v, u_k \rangle|^2$

Best Approximation Theorem Alternative: Let $\{u_n\}_{n=1}^{N}$ be a finite orthonormal system in an inner product space, $v \in V$. Then for each choice of coefficients $\{c_n\}_{n=1}^{N}$ we have: $\|v - \sum_{n=1}^{N} c_n u_n\|^2 \geq \|v - \sum_{n=1}^{N} \langle v, u_n \rangle \cdot u_n\|^2$. Means: smallest distance between $\text{span}\{u_n\}$ and v is the orthogonal projection of v onto $\text{span}\{u_n\}$.Under Conditions of Best Approximation Theorem we Have: $\|v - \sum_{n=1}^{N} \langle v, u_n \rangle \cdot u_n\|^2 = \|v\|^2 - \sum_{n=1}^{N} |v, u_n|^2$

BESSEL'S INEQUALITY:

Bessel's Inequality: Let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal system in an inner product space V, then for all $v \in V$ we have: $\sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2 \leq \|v\|^2$

Bessel's Inequality alternative: let $f(x)$ be piecewise continuous defined on [-L,L], then: $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^{L} |f(x)|^2 dx$ where a_0, a_n, b_n are Fourier coefficients of $f(x$

GENERALIZED FOURIER:

Generalized Fourier Series: For a basis $\{u_n\}_{n=1}^{\infty}$. Orthonormal case: $f = \sum_{n=1}^{\infty} \langle f, u_n \rangle \cdot u_n$. Orthogonal case: $f = \sum_{n=1}^{\infty} \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n$

Generalized Fourier Coefficients: Let V be an inner product space, $u \in V$ and let $\{e_k\}_{k=1}^{n}$ (n can be ∞) an orthonormal system, then the coefficents $\langle u, e_k \rangle$ are called the generalized Fourier coefficents of the vector u with respect to the given orthonormal system.

FOURIER SERIES:

Fourier Series: If $f(x)$ is an even function: $f(x) \sim \frac{a_0}{2} + \sum^{n} a_n \cos\left(\frac{n\pi x}{L}\right)$. If $f(x)$is an odd function: $f(x) \sim \sum^{n} b_n \sin\left(\frac{n\pi x}{L}\right)$ (no a_0 because odd functions go through origin)

Complex Fourier Series: $f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}$ and: $C_n = \frac{a_n - ib_n}{2} (n > 0)$, $C_n = \frac{a_{-n} - ib_{-n}}{2} (n < 0)$, $C_0 = \frac{a_0}{2}$, $C_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx$

Theorem: let $f(x): \mathbb{R} \to \mathbb{R}$ be a 2π periodic continuous function, and let $f'(x)$ we piecewise continuous, then if $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \Rightarrow f'(x) \sim \sum_{-\infty}^{\infty} inc_n e^{inx}$

Theorem: let $f(x)$ be a 2π periodic function in C^{k-1}, and $f^{(k)}$ is p.w.c. then: $\lim_{n\to\infty}|n^k c_n| = 0$, where $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ (i.e. the coefficients decay faster than n^k

Fourier coefficients on general interval: Developing Fourier series of $f(x)$ on interval $[a,b]$: $\boldsymbol{a_0} = \frac{2}{b-a}\int_a^b f(x)dx$. $\boldsymbol{a_n} = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2\pi nx}{b-a}\right)dx$. $\boldsymbol{b_n} = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2\pi nx}{b-a}\right)dx$. $\boldsymbol{c_n} = \frac{1}{b-a}\int_a^b f(x)e^{-\frac{2i\pi nx}{b-a}}dx$

Term by term differentiation Fourier series: Let $f(x): [-\pi,\pi] \to \mathbb{R}$ be continuous and $f(-\pi) = f(\pi)$. Let $f'(x)$ be piecewise continuous. If: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ Then: $f'(x) \sim \sum_{n=1}^{\infty} nb_n \cos(nx) - na_n \sin(nx) \sim \sum_{n=-\infty}^{\infty} inC_n e^{inx}$. Notice that it's the same conditions as the conditions for uniform convergence. See "uniform convergence theorem"

Term by term differentiability Fourier series: Let $f(x)$ be continuous, with $f'(x)$ piecewise continuous on $[-\pi,\pi]$ with a Fourier series which converges uniformly on $[-\pi,\pi]$ to f(x), then the Fourier series is term by term differentiable.

Term by Term Integration Fourier: Let $f(x): [-\pi,\pi] \to \mathbb{R}$ be piecewise continuous and $\int_{-\pi}^{\pi} f(t)dt = 0$ $(a_{0(f)} = 0$, then its antiderivative: $F(x) = \int_0^x f(t)dt = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$, where $A_n = -\frac{b_n}{n}, B_n = \frac{a_n}{n}, A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)dt$

Term by Term Integration Without $\boldsymbol{a_0 = 0}$: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ then for all $x \in [-\pi,\pi]$: $F(x) = \int_0^x f(t)dt = \frac{a_0}{2}x + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(nx) - \frac{b_n}{n}(\cos(nx) - 1)$. Series on the right converges uniformly to the integral on the left. But series on the right is not a Fourier series because of the x term.

Odd/Even Continuation/Extension: If $f(x)$ is piecewise continuous on $[0,L]$, we can develop a Fourier series for $f(x)$ on $[-L,L]$. Even continuation: $g(x) = f(-x)$. Odd continuation: $g(x) = -f(-x)$

Uniqueness of Fourier Expansion Theorem: $f: [-\pi,\pi] \to \mathbb{R}$. $f(x) =^{pw} \sum_{n=-\infty}^{\infty} d_n e^{inx}$. If $\sum_{n=-\infty}^{\infty}|d_n| < \infty$ then: 1) $\sum_{n=-\infty}^{\infty} d_n e^{inx} \to^u f(x)$, 2) $d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$

Fourier is Cauchy Theorem: let $\{\Phi\}_{n=0}^{\infty}$ be an orthonormal set and let $S_k = \sum_{n=1}^{k}\langle f, \Phi_n\rangle \cdot \Phi_n$ then S_k is a Cauchy Sequence

Riemann-Lebesgue's Lemma: Let $f(x)$ be piecewise continuous defined on [-L,L], then:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx = 0$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx = 0$

The Fourier coefficients are tending to zero (this is a necessary condition for convergence but not sufficient)

FOURIER TRANSFORM:

Fourier Theorem: If $f(x)$ in $L_1(R)$is continuously differentiable, then: $\frac{1}{2\pi}\lim_{M\to\infty}\int_{-M}^{M}\hat{f}(\omega)e^{i\omega x}d\omega = \frac{f(x^+)+f(x^-)}{2} = \text{(for cont. functions)} = f(x)$

Fourier Transform (Harmonic): $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$. Inverse Transform: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$. $f, \hat{f}$ both must be in L_1

CONVERGENCE:

Convergence definition: A series of numbers $\sum_{n=0}^{N} a_n$ converges if the sequence of partial sums $S_n = \sum_{n=0}^{N} a_n$ tends to a finite limit i.e. $\lim_{n\to\infty} S_n = S < \infty$.

Uniform Convergence: A sequence of functions $\{f_n(x)\}$ converges uniformly to a limit function $f(x)$ if $\forall\epsilon > 0, \exists N\ \forall x$ s.t. $\forall n > N$, $f_n(x) - f(x) < \epsilon$. Find N for all x, not just for a specific x_0

Uniform Convergence Theorem: Let $f(x): [-\pi,\pi] \to \mathbb{R}$ be continuous which maintains $f(-\pi) = f(\pi)$ and $f'(x)$ is piecewise continuous. Then: $S_m(f,x) \longrightarrow f(x)$ (Fourier converges uniformly to $f(x)$). $S_m(f,x)$ is the finite Fourier sum (see Dirichlet's kernel). Uniform convergence implies norm convergence and point-wise convergence. Other way around does not work.

Convergence of Vectors: $\lim_{n\to\infty}\|u_n - u\| = 0$

Convergence proposition: If $\{f_n(x)\}$ is a sequence of odd/even functions and $f_n(x)$ converges pointwise to $f(x)$ then $f(x)$ is an odd/even function.

Normwise Convergence Theorem: Let $f(x)$ be piecewise continious on $[-\pi,\pi]$ then: $\lim_{m\to\infty}\|f - S_m\| = \lim_{m\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x) - S_m(f,x)|^2 dx = 0$, where $S_m(f,x) = \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos(nx) + b_n \sin(nx)$.

Normwise Convergence Alternative: $\{u_n\} \subset V$, a sequence in the inner product space is said to be Normwise convergent if $\exists u \in V$ s.t. $\|u_n - u\| \to 0$

FUNCTIONS:

Functions in $\boldsymbol{L_1}$: $f \in L_1$ if $\int_{-\infty}^{\infty}|f(x)|dx < \infty$

Smoothness of $\boldsymbol{f}$: If $|C_n| \le c \cdot \frac{1}{|n|^{k+\alpha}}$. c Is a constant, $\alpha > 1$, f is 2π periodic, then: $f \in C^k$ for all $\mathbb{R}$

Smoothness of $\boldsymbol{f}$ and Rate of Coefficient Decay: Let f be 2π periodic and $f \in C^{k-1}$ on all $\mathbb{R}$ and $f^{(k)}$ is piecewise continuous. Then: $\lim_{n\to\infty}|n^k a_n| = \lim_{n\to\infty}|n^k b_n| = 0$, $\lim_{n\to\infty}|n^k c_n| = 0$

Integrability: Let $f_n: [a,b] \to \mathbb{R}$ be a sequence of integrable functions that converge uniformly to $f(x)$. Then $f(x)$ is also integrable: $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$

Continuous differentiability: A real function f is piecewise continuously differentiable if it is piecewise continuous and $\lim_{h\to 0^+}\left|\frac{f(x+h)-f(x^+)}{h}\right| < \infty$, $\lim_{h\to 0^-}\left|\frac{f(x+h)-f(x^-)}{h}\right| < \infty$, $f(x_0^{\pm}) = \lim_{x\to x_0^{\pm}} f(x_0)$. $f \in C^k$ if f is continuously differentiable k times.

Gibbs Phenomenon: Let $f(x)$ be piecewise continuously differentiable, and let $x = d$ be a discontinuity point (jump) of $f(x)$. $d \in [-\pi,\pi]$. Then there exists a sequence $\{x_m\}_{m=1}^{\infty}$ $x_m > d$ $x_m \to d$ and $\lim_{m\to\infty}\frac{S_m(x_m) - f(x_m)}{f(d^-) - f(d^+)} = 0.089$

Uniqueness of Function Theorem: Let $f, g: [-\pi,\pi] \to \mathbb{R}$ be piecewise continuous and $f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{inx}$, $g(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{inx}$, then $f = g$ almost everywhere, up to a finite number of points because discontinuities at points don't affect integral. If f and g are both continious : $f = g$ everywhere.

Infinite Geometric series: If $|r| < 1$ then $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

DIRICHLET'S KERNEL:

Dirichlet's Kernel $\boldsymbol{D_m(t)}$ $= \frac{1}{2} + \sum_{n=1}^{m}\cos(nt)$, $\frac{1}{2}\sum_{n=-m}^{m} e^{int}$, $\frac{\sin\left(\left(m+\frac{1}{2}\right)t\right)}{2\sin\left(\frac{t}{2}\right)}$ $t \ne 2\pi k, k \in \mathbb{Z}$

Dirichlet Kernel: $S_m(f,x) = \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos(nx) + b_n \sin(nx)$ then: $S_m(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_m(t)dt$

Dirichlet's kernel facts: $\frac{1}{\pi}\int_{-\pi}^{\pi} D_m(t)dt = 1$. $\frac{1}{\pi}\int_{-\pi}^{0} D_m(t)dt = \frac{1}{\pi}\int_0^{\pi} D_m(t)dt = 1/2$

Dirichlet's Theorem: $f: [-\pi,\pi] \to \mathbb{R}$ be piecewise continuously differentiable $\Rightarrow$ for each $x \in (-\pi,\pi)$ the following equality holds: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x^+)+f(x^-)}{2}$, and for $x = \pi, -\pi$: $\frac{f(\pi^-)+f(-\pi^+)}{2}$. Equality breaks for discontinuity points. Also for discontinuous functions convergence cannot be uniform. The importance of the Dirichlet kernel comes from its relation to Fourier series. The convolution of $D_n(x)$ with any function f of period 2π is the nth-degree Fourier series approximation to f, i.e., we have

Partial Sums Reminder: If $\{F_n(x)\}$ is the sequence of partial sums of $\sum_{n=0}^{\infty} f_n(x)$, then: $\sum_{n=0}^{\infty} f_n(x) = \lim_{n\to\infty}\sum_{n=0}^{N} f_n(x) = \lim_{N\to\infty} F_N$

TRIGONOMETRIC IDENTITIES:

In terms of	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\sin(\theta)$	$\sin(\theta)$	$\pm\sqrt{1-\cos^2(\theta)}$	$\pm\dfrac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}}$
$\cos(\theta)$	$=\pm\sqrt{1-\sin^2(\theta)}$	$\cos(\theta)$	$\pm\dfrac{1}{\sqrt{1+\tan^2(\theta)}}$
$\tan(\theta)$	$\pm\dfrac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}}$	$\pm\dfrac{\sqrt{1-\cos^2(\theta)}}{\cos(\theta)}$	$\tan(\theta)$

$\sin(-\theta) = -\sin(\theta)$	$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$	$\sin(\pi-\theta) = \sin(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$	$\cos(\pi-\theta) = -\cos(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\tan\left(\frac{\pi}{2}-\theta\right) = \cot(\theta)$	$\tan(\pi-\theta) = -\tan(\theta)$
$\sin\left(\theta+\frac{\pi}{2}\right) = \cos(\theta)$	$\sin(\theta+\pi) = -\sin(\theta)$	$\sin(\theta+2\pi) = \sin(\theta)$
$\cos\left(\theta+\frac{\pi}{2}\right) = -\sin(\theta)$	$\cos(\theta+\pi) = -\cos(\theta)$	$\cos(\theta+2\pi) = \cos(\theta)$

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$	$\sin(\theta) = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$
$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$	$\cos(\theta) = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$
$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$	$\sin^2(\theta)+\cos^2(\theta) = 1$
$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta)$ $= 2\cos^2(\theta)-1$ $= 1-2\sin^2(\theta)$	$\sin(\theta) =$ $\pm\sqrt{1-\cos^2(\theta)}$ Sign depends on quadrant of θ
$\sin(3\theta) = -4\sin^3(\theta)+3\sin(\theta)$	$\cos(\theta)$ $= \pm\sqrt{1-\sin^2(\theta)}$
$\cos(3\theta) = 4\cos^3(\theta)-3\cos(\theta)$	
$\sin^2\left(\frac{\theta}{2}\right) = \dfrac{(1-\cos(\theta))}{2}$	
$\dfrac{\cos^2\left(\frac{\theta}{2}\right) = (1+\cos(\theta))}{2}$	
$2\cos(\theta)\cos(\phi) = \cos(\theta-\phi)+\cos(\theta+\phi)$	
$2\sin(\theta)\sin(\phi) = \cos(\theta-\phi)-\cos(\theta+\phi)$	
$2\sin(\theta)\cos(\phi) = \sin(\theta+\phi)+\sin(\theta-\phi)$	
$2\cos(\theta)\sin(\phi) = \sin(\theta+\phi)-\sin(\theta-\phi)$	

INTEGRALS:

Exponential on even interval: $\int_{-L}^{L} e^{\frac{im\pi}{L}x}\cdot e^{\frac{in\pi}{L}x}dx = 0(m\neq n); 2L(m=n)$

- Integral of piecewise continuous function is continuous
- If $f(x)$ is a complex function, we can write it as: $f(x) = u(x)+iv(x)$, where $u(x), v(x)$ are real functions. So: $\int_{-L}^{L} f(x)dx = \int_{-L}^{L}\big(u(x)+iv(x)\big)dx = \int_{-L}^{L} u(x)dx + i\int_{-L}^{L} v(x)dx$

$(odd)' = even$	$(even)' = odd$
$\int(odd) = even$	$\int(even) = odd$

- $\left|\int_a^b (f\cdot\bar{g})dx\right|^2 \leq \int_a^b |f|^2dx\cdot\int_a^b |g|^2dx$
- $\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$
- $\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0(m\neq n), L(m=n\neq 0), 2L(m,n=0)$
- $\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0(m\neq n), L(m=n),\ \ L>0; m,n\neq 0$

INTEGRAL TABLES:

Basic Forms

$\int x^n dx = \frac{1}{n+1}x^{n+1}$, $n\neq -1$	$\int u dv = uv - \int v du$
$\int \frac{1}{x}dx = \ln\lvert x\rvert$	$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln\lvert ax+b\rvert$

Integrals of Rational Functions

$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$	$\int \frac{1}{1+x^2}dx = \tan^{-1}x$
$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n\neq -1$	$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$	$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln\lvert a^2+x^2\rvert$
$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2-\frac{1}{2}a^2\ln\lvert a^2+x^2\rvert$	$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$
$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$	$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, a\neq b$
$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\lvert ax^2+bx+c\rvert - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$	$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x}+\ln\lvert a+x\rvert$

Integrals with Roots

$\int \sqrt{x-a}\ dx = \frac{2}{3}(x-a)^{3/2}$	$\int \sqrt{\frac{x}{a+x}}\ dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x}+\sqrt{x+a}\right]$
$\int \frac{1}{\sqrt{x\pm a}}\ dx = 2\sqrt{x\pm a}$	$\int x\sqrt{ax+b}\ dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b}$
$\int \frac{1}{\sqrt{a-x}}\ dx = -2\sqrt{a-x}$	$\int \sqrt{x(ax+b)}\ dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left\lvert a\sqrt{x}+\sqrt{a(ax+b)}\right\rvert\right]$
$\int \sqrt{ax+b}\ dx = \left(\frac{2b}{3a}+\frac{2x}{3}\right)\sqrt{ax+b}$	$\int \sqrt{x^3(ax+b)}\ dx = \left[\frac{b}{12a}-\frac{b^2}{8a^2x}+\frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left\lvert a\sqrt{x}+\sqrt{a(ax+b)}\right\rvert$
$\int (ax+b)^{3/2}\ dx = \frac{2}{5a}(ax+b)^{5/2}$	$\int \sqrt{x^2\pm a^2}\ dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$
$\int \frac{x}{\sqrt{x\pm a}}\ dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$	$\int \sqrt{a^2-x^2}\ dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$
$\int \sqrt{\frac{x}{a-x}}\ dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$	$\int x\sqrt{x^2\pm a^2}\ dx = \frac{1}{3}(x^2\pm a^2)^{3/2}$
$\int \frac{1}{\sqrt{x^2\pm a^2}}\ dx = \ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$	$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\frac{x}{a}$
$\int \frac{x}{\sqrt{x^2\pm a^2}}\ dx = \sqrt{x^2\pm a^2}$	$\int \frac{x}{\sqrt{a^2-x^2}}\ dx = -\sqrt{a^2-x^2}$
$\int \frac{x^2}{\sqrt{x^2\pm a^2}}\ dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \mp \frac{1}{2}a^2\ln\left\lvert x+\sqrt{x^2\pm a^2}\right\rvert$	$\int \sqrt{ax^2+bx+c}\ dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\lvert 2ax + b + 2\sqrt{a(ax^2+bx^+c)}\rvert$
$\int \frac{1}{\sqrt{ax^2+bx+c}}\ dx = \frac{1}{\sqrt{a}}\ln\lvert 2ax+b+2\sqrt{a(ax^2+bx+c)}\rvert$	$\int \frac{x}{\sqrt{ax^2+bx+c}}\ dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\lvert 2ax+b+2\sqrt{a(ax^2+bx+c)}\rvert$
$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$	

Integrals with Exponentials

$\int e^{ax}\ dx = \frac{1}{a}e^{ax}$	$\int xe^{x}\ dx = (x-1)e^{x}$
$\int xe^{ax}\ dx = \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$	$\int x^2e^{x}\ dx = (x^2-2x+2)e^{x}$
$\int x^2e^{ax}\ dx = \left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)e^{ax}$	$\int x^3e^{x}\ dx = (x^3-3x^2+6x-6)e^{x}$
$\int x^ne^{ax}\ dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\ \ dx$	$\int x^ne^{ax}\ dx = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n,-ax]$, where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\ \ dt$
$\int e^{ax^2}\ dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\big(ix\sqrt{a}\big)$	$\int e^{-ax^2}\ dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\big(x\sqrt{a}\big)$
$\int xe^{-ax^2}\ dx = -\frac{1}{2a}e^{-ax^2}$	$\int x^2e^{-ax^2}\ dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2}$
$\int \sqrt{x}e^{ax}\ dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}(i\sqrt{ax})$, whereerf$(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$	

Integrals with Trigonometric Functions

$\int \sin ax\ dx = -\frac{1}{a}\cos ax$	$\int \sin^2 ax\ dx = \frac{x}{2}-\frac{\sin 2ax}{4a}$
$\int \sin^3 ax\ dx = -\frac{3\cos ax}{4a}+\frac{\cos 3ax}{12a}$	$\int \cos ax\ dx = \frac{1}{a}\sin ax$
$\int \cos^2 ax\ dx = \frac{x}{2}+\frac{\sin 2ax}{4a}$	$\int \cos^3 ax dx = \frac{3\sin ax}{4a}+\frac{\sin 3ax}{12a}$
$\int \sin^2 x\cos x\ dx = \frac{1}{3}\sin^3 x$	$\int \cos^2 ax\sin ax\ dx = -\frac{1}{3a}\cos^3 ax$
$\int \sin^2 ax\cos^2 ax\ dx = \frac{x}{8}-\frac{\sin 4ax}{32a}$	$\int \tan ax\ dx = -\frac{1}{a}\ln\cos ax$
$\int \tan^2 ax\ dx = -x+\frac{1}{a}\tan ax$	$\int \tan^3 ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax$
$\int \cos x\sin x\ dx = \frac{1}{2}\sin^2 x + c_1 = -\frac{1}{2}\cos^2 x + c_2 = -\frac{1}{4}\cos 2x + c_3$	
$\int \sin^2 ax\cos bx\ dx = -\frac{\sin[(2a-b)x]}{4(2a-b)}+\frac{\sin bx}{2b}-\frac{\sin[(2a+b)x]}{4(2a+b)}$	
$\int \cos^2 ax\sin bx\ dx = \frac{\cos[(2a-b)x]}{4(2a-b)}-\frac{\cos bx}{2b}-\frac{\cos[(2a+b)x]}{4(2a+b)}$	
$\int \cos ax\sin bx\ dx = \frac{\cos[(a-b)x]}{2(a-b)}-\frac{\cos[(a+b)x]}{2(a+b)}, a\neq b$	
$\int \sin^2 ax\cos^2 bx dx = \frac{x}{4}-\frac{\sin 2ax}{8a}-\frac{\sin[2(a-b)x]}{16(a-b)}+\frac{\sin 2bx}{8b}-\frac{\sin[2(a+b)x]}{16(a+b)}$	

Products of Trigonometric Functions and Exponentials

$\int e^x\sin x\ dx = \frac{1}{2}e^x(\sin x - \cos x)$	$\int e^{bx}\sin ax\ dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$
$\int e^x\cos x\ dx = \frac{1}{2}e^x(\sin x + \cos x)$	$\int e^{bx}\cos ax\ dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$
$\int xe^x\sin x\ dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$	$\int xe^x\cos x\ dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$

Products of Trigonometric Functions and Monomials

$\int x\cos x\ dx = \cos x + x\sin x$	$\int x\cos ax\ dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$
$\int x^2\cos x\ dx = 2x\cos x + (x^2-2)\sin x$	$\int x^2\cos ax\ dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$
$\int x\sin x\ dx = -x\cos x + \sin x$	$\int x\sin ax\ dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$
$\int x^2\sin x\ dx = (2-x^2)\cos x + 2x\sin x$	$\int x^2\sin ax\ dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$
$\int x\cos^2 x\ dx = \frac{x^2}{4} + \frac{1}{8}\cos 2x + \frac{1}{4}x\sin 2x$	$\int x\sin^2 x\ dx = \frac{x^2}{4} - \frac{1}{8}\cos 2x - \frac{1}{4}x\sin 2x$

Integrals with Logarithms

$\int \ln ax\ dx = x\ln ax - x$	$\int x\ln x\ dx = \frac{1}{2}x^2\ln x - \frac{x^2}{4}$
$\int x^2\ln x\ dx = \frac{1}{3}x^3\ln x - \frac{x^3}{9}$	$\int x^n\ln x\ dx = x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right),\quad n \neq -1$
$\int \frac{\ln ax}{x}\ dx = \frac{1}{2}(\ln ax)^2$	$\int \frac{\ln x}{x^2}\ dx = -\frac{1}{x} - \frac{\ln x}{x}$
$\int \ln(ax+b)\ dx = \left(x+\frac{b}{a}\right)\ln(ax+b) - x, a \neq 0$	$\int \ln(x^2+a^2)\ \ dx = x\ln(x^2+a^2) + 2a\tan^{-1}\frac{x}{a} - 2x$
$\int \ln(x^2-a^2)\ \ dx = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x$	$\int x\ln(ax+b)\ dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right)\ln(ax+b)$
$\int (\ln x)^2\ dx = 2x - 2x\ln x + x(\ln x)^2$	$\int (\ln x)^3\ dx = -6x + x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x$
$\int x(\ln x)^2\ dx = \frac{x^2}{4} + \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x$	$\int x^2(\ln x)^2\ dx = \frac{2x^3}{27} + \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x$

SPAN:

Given a vector space *V* over a field *K*, the span of a set *S* of vectors (not necessarily infinite) is defined to be the intersection *W* of all subspaces of *V* that contain *S*. *W* is referred to as the subspace *spanned by S*, or by the vectors in *S*. Conversely, *S* is called a *spanning set* of *W*, and we say that *S spans W*.
Alternatively, the span of *S* may be defined as the set of all finite linear combinations of elements of *S*, which follows from the above definition.

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \,\middle|\, k \in \mathbb{N}, v_i \in S, \lambda_i \in \mathbf{K} \right\}.$$

In particular, if *S* is a finite subset of *V*, then the span of *S* is the set of all linear combinations of the elements of *S*. In the case of infinite *S*, infinite linear combinations (i.e. where a combination may involve an infinite sum) are excluded by the definition; a generalization that allows these is not equivalent

LINEAR SUBSET:

Let *K* be a field (such as the real numbers), *V* be a vector space over *K*, and let *W* be a subset of *V*. Then *W* is a **subspace** if:

1. The zero vector, **0**, is in *W*.
2. If **u** and **v** are elements of *W*, then the sum **u** + **v** is an element of *W*.
3. If **u** is an element of *W* and *c* is a scalar from *K*, then the scalar product *c***u** is an element of *W*.

A way to characterize subspaces is that they are closed under linear combinations. That is, a nonempty set *W* is a subspace if and only if every linear combination of (finitely many) elements of *W* also belongs to *W*. Conditions 2 and 3 for a subspace are simply the most basic kinds of linear combinations.

PROJECTIONS:

When the vector space *W* has an inner product and is complete (is a Hilbert space) the concept of orthogonality can be used. An **orthogonal projection** is a projection for which the range *U* and the null space *V* are orthogonal subspaces. Thus, for every *x* and *y* in *W*, $\langle Px, (y-Py)\rangle = \langle (x-Px), Py\rangle$, equivalently

$$\langle x, Py\rangle = \langle Px, Py\rangle = \langle Px, y\rangle$$

A projection is orthogonal if and only if it is self-adjoint.
Using the self-adjoint and idempotent properties of *P*, for any *x* and *y* in *W* we have $Px \in U, y - Py \in V$, and

$$\langle Px, y-Py\rangle = \langle P^2x, y-Py\rangle = \langle Px, P(I-P)y\rangle = \langle Px, (P-P^2)y\rangle = 0$$

Where $\langle\cdot,\cdot\rangle$ is the inner product associated with *W*.
Therefore, *Px* and *y* − *Py* are orthogonal. The other direction, namely that if *P* is orthogonal then it is self-adjoint, follows from

$$\langle x, Py\rangle = \langle Px, y\rangle = \langle x, P^*y\rangle$$

An orthogonal projection is a bounded operator. This is because for every *v* in the vector space we have, by Cauchy–Schwarz inequality:

$$\|Pv\|^2 = \langle Pv, Pv\rangle = \langle Pv, v\rangle \leqslant \|Pv\| \cdot \|v\|$$, thus $$\|Pv\| \leqslant \|v\|$$

For finite dimensional complex or real vector spaces, the standard inner product can be substituted for $\langle\cdot,\cdot\rangle$

The Riemann Localization Theorem: Let $f \in L([0, 2\pi])$ be a 2π –periodic function. Then the Fourier series generated by *f* will converge at a point *x* if and only if there exists a $b \in R$ with $0 < b \leq \pi$ such that

$$\lim_{n\to\infty} \frac{2}{\pi}\int_0^b \frac{f(x+t)+f(x-t)}{2} \cdot \sin(\frac{\left(n+\frac{1}{2}\right)t}{t} dt$$

exists in which case the Fourier series generated by *f* will converge at *x* to this limit.

Cauchy Sequences:

Cauchy sequence, is a sequence whose elements become *arbitrarily close to each other* as the sequence progresses. More precisely, given any small positive distance, all but a finite number of elements of the sequence are less than that given distance from each other.
It is not sufficient for each term to become arbitrarily close to the *preceding* term. For instance, in the harmonic series $\sum\frac{1}{n}$ the difference between consecutive terms in the sequence of partial sums decreases as $\frac{1}{n}$, however the series does not converge. Rather, it is required that *all* terms get arbitrarily close to *each other*, starting from some point. More formally, for any given $\epsilon > 0$(which means: arbitrarily *small*) there exists an *N* such that for any pair *m*,*n* > *N*, we have$|a_m - a_n| < \epsilon$ (whereas$|a_{n+1} - a_n| < \epsilon$ is not sufficient).

Since the definition of a Cauchy sequence only involves metric concepts, it is straightforward to generalize it to any metric space *X (distance is defined in the space)*. To do so, the absolute value $|x_m - x_n|$ is replaced by the *distance* $d(x_m, x_n)$ (where *d* denotes a metric) between x_m and x_n.
Formally, given a metric space (*X*, *d*), a sequence
$x_1, x_2, x_3, \ldots$
is Cauchy, if for every positive real number *ε* > 0 there is a positive integer *N* such that for all positive integers m, n > *N*, the distance
$d(x_m, x_n) < \varepsilon$.
Roughly speaking, the terms of the sequence are getting closer and closer together in a way that suggests that the sequence ought to have a limit in *X*. Nonetheless, such a limit does not always exist within *X*.

Plancherel's Theorem:

In mathematics, the **Plancherel theorem** is a result in harmonic analysis, proven by Michel Plancherel in 1910. It states that the integral of a function's squared modulus is equal to the integral of the squared modulus of its frequency spectrum.

$$\int_{-\infty}^{\infty} \|f(x)\|^2\,dx = \int_{-\infty}^{\infty} \|\widehat{f}(\xi)\|^2\,d\xi$$

The unitarity of the Fourier transform is often called Parseval's theorem in science and engineering fields, based on an earlier (but less general) result that was used to prove the unitarity of the Fourier series.
In its common form:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi,$$

where

$$\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx = \widehat{f}(\xi).$$

Convolution:

$$f(t) * g(t) \stackrel{\text{def}}{=} \underbrace{\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau}_{(f*g)(t)},$$

which has to be interpreted carefully to avoid confusion. For instance, $f(t) * g(t-t_0)$ is equivalent to $(f*g)(t-t_0)$, but $f(t-t_0) * g(t-t_0)$ is in fact equivalent to $(f*g)(t-2t_0)$.