Fourier Series:

Let $x(t) = x(t+T)$ be a periodic signal with period T, with finite energy in a single period.
For such a signal define the Fourier Series:

$$x(t) = \sum_{k\in\mathbb{Z}} c_k e^{j\frac{2\pi}{T}kt}$$

Where $\{c_k\}_{k\in\mathbb{Z}}$ are the Fourier Coefficients, given by:

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt\,, \qquad \forall k \in Z$$

Meaning: A periodic signal can be expressed by a linear combination of harmonic functions.

Fourier Transform:

Let $x(t)$ be a signal that is not necessarily periodic. We define its Fourier Transform as:

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \omega\left[\frac{rad}{sec}\right] = ang.\,freq$$

$$\tilde{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt \quad ; \quad f\left[\frac{1}{sec} = \mathrm{Hz}\right] = frequency$$

The inverse Fourier Transform:

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\left(= \mathcal{F}^{-1}\{\tilde{X}(f)\} = \int_{-\infty}^{\infty} \tilde{X}(f)e^{j2\pi f t}df\right)$$

Note/Deduction: $|X(\omega)|^2$ is a density function describing energy per unit frequency in the signal at angular frequency ω.

Parseval's Theorem:

For Fourier Series: $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k\in\mathbb{Z}} |c_k|^2$

For Fourier Transform: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \left(= \int_{-\infty}^{\infty} \left|\tilde{X}(f)\right|^2 df\right)$

FT Of a Periodic Signal (R1E1.1):

For a periodic signal $x(t) = x(t+T)$ with finite energy in a single period, with Fourier Coefficents $\{c_k\}_{k\in\mathbb{Z}}$, the FT is:

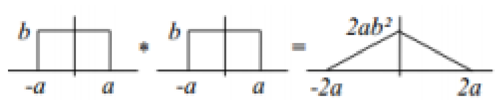
$$X(\omega) = 2\pi \sum_{k\in\mathbb{Z}} c_k \delta\left(\omega - \frac{2\pi}{T}k\right)$$

Triangular Signal (R1E1.2):

The signal:

$$p(t) = \begin{cases} 1 - \dfrac{|t|}{T_b} \; ; \; |t| \le T_b \\ 0 \,; \mathrm{else} \end{cases}$$

Is actually a convolution of two rectangles:



Fourier Transform: $P(\omega) = \mathcal{F}\{rect(t) * rect(r)\} = R(\omega)R(\omega) = T_b sinc^2\left(\frac{\omega T_b}{2\pi}\right)$

Random Processes:

A continuous-time random process $(\omega, t)^2$, where t=time and ω =an event from the sample space (where the sample space is usually denoted as Ω)

If t is fixed to t_0: We have $X(\omega, t_0)$ - a random variable
If ω is fixed to ω_0: We have $X(\omega_0, t)$ - a sample function

Expectation: $\eta_x(t) \triangleq E[X(t)]$, $\forall t$
Autocorrelation Function: $\widetilde{R_{xx}}(t_1, t_2) \triangleq E[X(t_1)X^*(t_2)]$, $\forall t_1, t_2$
Auto covariance Function: $\widetilde{C_{xx}}(t_1, t_2) \triangleq E\left[\left(X(t_1) - \eta_x(t_1)\right)\left(X(t_2) - \eta_x(t_2)^*\right)\right]$, $\forall t_1, t_2$

Autocorrelation - Auto covariance Relation: $\widetilde{C_{xx}}(t_1, t_2) \triangleq \widetilde{R_{xx}}(t_1, t_2) - \eta_x(t_1)\eta_x^*(t_2)$, $\forall t_1, t_2$

Wide Sense Stationary (WSS) Random Process:

A random process $X(t)$ is called WSS IFF the following hold:

i) $\eta_x(t) = \eta_x \,, \forall\, t$

ii) $\widetilde{R_{xx}}(t, t-\tau) = R_{xx}(\tau)$

Expectation is independent of time, and the autocorrelation is a function of the time difference (single argument) only.

Jointly WSS (JWSS) Random Processes:

$X(t), Y(t)$ are JWSS IFF they are both WSS, and following condition is satisfied: $\widetilde{R_{xy}}(t, t-\tau) \triangleq E[X(t)Y^*(t-\tau)] = R_{xy}(\tau)$

Power Spectral Density (PSD) Function:

Definition: For a WSS process $X(t)$, define $S_{xx}(\omega)$, the PSD function, as the FT of the autocorrelation function:

$$S_{xx}(\omega) \triangleq \mathcal{F}\{R_{xx}(\tau)\} = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau$$

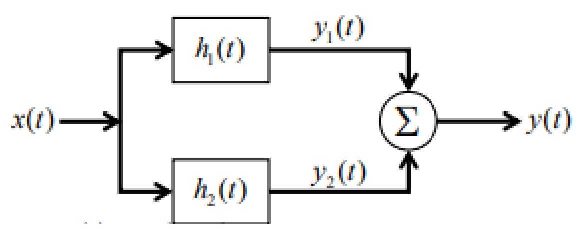
Properties: This is a deterministic, real and symmetric function: $\forall\omega\text{: } 0 \le S_{xx}(\omega) = S_{xx}(-\omega) \in \mathbb{R}$

In the context of an LTI System:

Let $H(\omega)$ =frequency response of the system, $Y(t)$ =output of the system for an input $X(t)$. Then:

1. $Y(t)$ is a WSS Process
2. ***PSD* of $\boldsymbol{Y(t)}$:** $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$
3. **Cross Spectrum of Y and X:** $S_{yx}(\omega) = S_{xx}(\omega)H(\omega)$
4. **Cross Spectrum definition is:** $S_{xy}(\omega) = \mathcal{F}\{R_{xy}(\tau)\} = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega\tau}d\tau$ for X,Y JWSS process
   a. From this definition: $S_{xy}(\omega) = S^*_{yx}(\omega) = S_{xx}(\omega)H^*(\omega)$

Two Parallel LTI Blocks summed at the output (R2E2.1):



Problem Definition: We have a WSS process $x(t)$, with autocorrelation $R_{xx}(\tau)$, PSD $S_{xx}(\omega)$. The output $y(t)$ is the sum of $y_1(t), y_2(t)$, obtained by passing $x(t)$ through two lti systems $h_1(t), = a\delta(t-t_1)\ ;\ h_2(t) = b\delta(t-t_2)$.

Spectra of the outputs:

$$y_1(t) \to R_{y_1y_1}(\tau) = |a|^2 R_{xx}(\tau) \to \text{and the spectra: } S_{y_1y_1} = |a|^2 S_{xx}(\omega)$$
$$y_2(t) \to \text{exactly as above} \to S_{y_2y_2} = |b|^2 S_{xx}(\omega)$$

Cross-correlation of the outputs (JWSS??):

$$R_{y_1y_2}(t+\tau.\tau) = ab^* R_{xx}(\tau + t_2 - t_1) \to JWSS$$

Spectrum of the final output:

$$S_{yy}(\omega) = |H(\omega|^2 S_{xx}(\omega)$$
$$|H(\omega)|^2 = |H_1(\omega)|^2 + |H_2(\omega\ )|^2 + 2\mathcal{R}e\{H_1(\omega)H_2^*(\omega)\}$$
$$|H_1(\omega)|^2 = |a|^2\ \ ;\ \ |H_2(\omega)|^2 = |b|^2\ \ ;\ 2\mathcal{R}e\{H_1(\omega)H_2^*(\omega)\} \neq 0$$

So the spectrum of the output is not equal to the sum of spectra of the sub-outputs.

In general: If $\mathcal{R}e\{H_1(\omega)H_2^*(\omega)\} = 0$ then the spectrum of $y(t)$ will equal to the sum of the spectra of $y_1(t), y_2(t)$

Wide Sense Cycle-Stationary Random Process:

A random process $x(t)$ is Wide Sense Cyclo-Stationary with period T IFF the following hold:

1. $\eta_x(t+T) = \eta_x(t) \quad ;\forall t$
2. $R_{xx}(t+T\,;\tau) = R_{xx}(t;\tau)\,;\forall t,\tau$

Fourier Series of the Autocorrelation: Due to the above, the autocorrelation is periodic in t, and can be expanded as an FS:
$R_{xx}(t,\tau) = \sum_{n=-\infty}^{\infty} R_{xx}^{n/T}(\tau)e^{j2\pi\frac{n}{T}t}$

Cyclic Autocorrelation Function: $R_{xx}^{n/T}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} R_{xx}(t,\tau)e^{-j2\pi\frac{n}{T}t}t$

Cyclic frequencies: The frequencies satisfying $\frac{n}{T}\ ,\ n \in \mathbb{Z}$

Cyclic Spectrum: The FT of the Cyclic Autocorrelation at some cyclic frequency α : $S_{xx}^{\alpha} = \int_{-\infty}^{\infty} R_{xx}^{\alpha}(\tau)e^{-j2\pi f\tau}d\tau$

Average Power Spectral Density (PSD): Cyclic Spectrum at zeroth cyclic frequency.

"WSS-ing" Algorithm:

We are given a WSS process $x(t)$, and the signal $y(t) = x(t)c(t)$, where $c(t)$ is a WS-Cyclo-Stationary periodic signal with period T. $y(t)$ is not necessarily WSS right now!

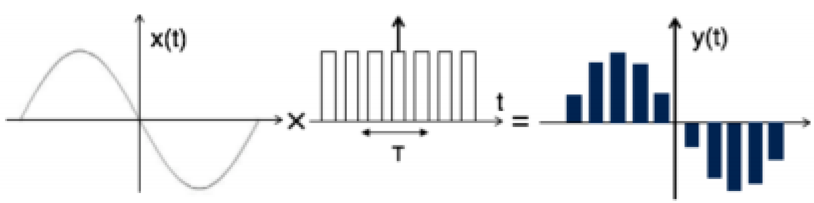
Adding a delay $\varphi \sim U(0,T)$, which is independent of $x(t)$ yields the signal $y(t) = x(t)c(t-\varphi)$, which is WSS!

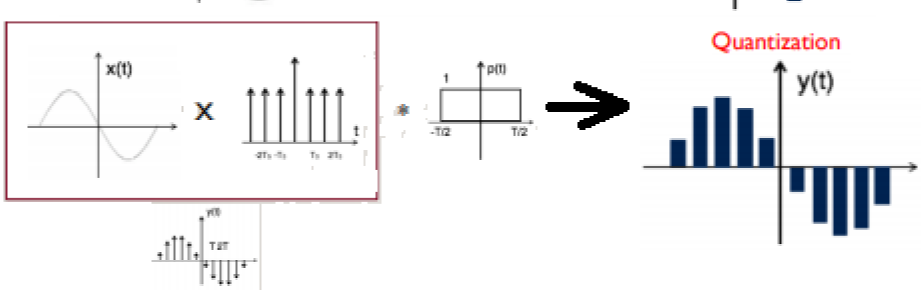
These conditions are SUFFICENT to ensure $y(t)$ stationary, but they are NOT NECCESARY.

Pulse Amplitude Modulation and Sampling:

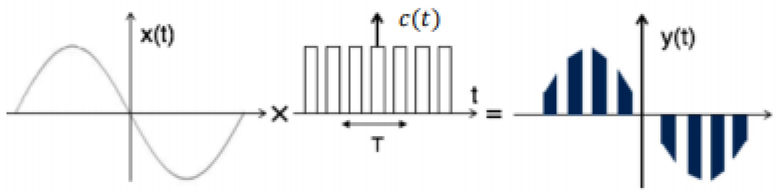
PAM Types Summary		
Sampling Method	Time Domain Signals	Frequency Domain Signals
Natural Sampling	$c(t) = \sum_{n\in\mathbb{Z}} p(t-nT_s)$ $y(t) = \sum_{n\in\mathbb{Z}} x(t)p(t-nT_s)$	$C(\omega) = \omega_s \sum_{k\in\mathbb{Z}} P(\omega_s k)\delta(\omega - k\omega_s)$ $Y(\omega) = \frac{1}{T_s}\sum_{k\in\mathbb{Z}} P(\omega_s k)X(\omega - k\omega_s)$
Flat Sampling	$x_s(t) = \sum_{n\in\mathbb{Z}} x(nT_s)\delta(t-nT_s)$ $y(t) = \sum_{n\in\mathbb{Z}} x(nT_s)p(t-nT_s)$	$X_s(\omega) = \frac{1}{T_s}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$ $Y(\omega) = \frac{1}{T_s}P(\omega)\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$ $\omega_s = 2\pi k/T_s$

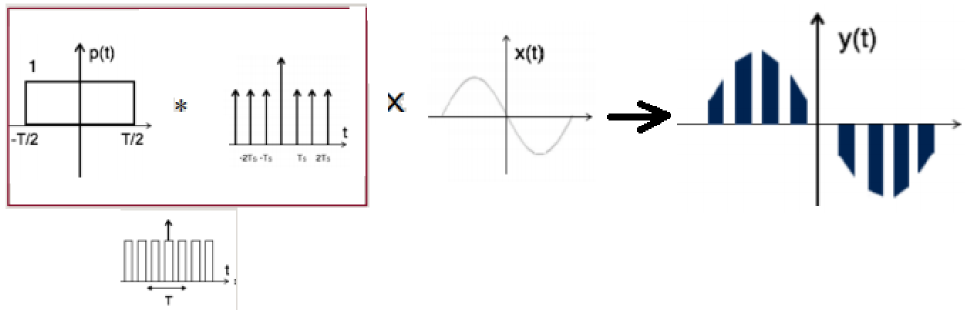
Flat Top Sampling: During transmission we introduce noise at the envelope of the transmission pulse. This noise is removed if the pulse is a flat top. $y(t) = \sum_{n\in\mathbb{Z}} x(nT_s)p(t-nT_s)$





Natural Sampling: Use pulse train instead of impulse train, i.e $y(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} P(t - nT)$





Fourier Transform of Pulse Train: $C(\omega) = \frac{2\pi}{T} \sum_{k \in \mathbb{Z}} P(\omega_s k)\delta(\omega - k\omega_s)$ $\left(c_k = \frac{1}{T} P(\omega)\right)$

Fourier Transform of the Output: $Y(\omega) = \frac{1}{T} \sum_{k \in \mathbb{Z}} P(\omega_s k) X(\omega - k\omega_s)$

Fourier Transform of the Input: $X(\omega) = 2\pi \sum_{k \in c} c_k \delta\left(\omega - \frac{2\pi}{T} k\right)$

Natural Sampling Analysis:

Let $x(t)$ be an information signal, and $c(t)$ be the "switching" signal, such that the **PAM signal is given by**:

$$y(t) = x(t)c(t) \ ; \ \text{where } c(t) = \sum_{n \in \mathbb{Z}} p(t - nT) \ ; p(t) = rect\left(\frac{t}{\tau}\right) \ , \qquad \tau \leq T$$

Fourier Transform of sampling signal $c(t)$:

$$C(\omega) = 2\pi \sum_{k \in \mathbb{Z}} c_k \delta\left(\omega - k\frac{2\pi}{T}\right) = \frac{2\pi\tau}{T} \sum_{k \in \mathbb{Z}} sinc\left(\frac{k\tau}{T}\right) \delta\left(\omega - \frac{2\pi k}{T}\right)$$

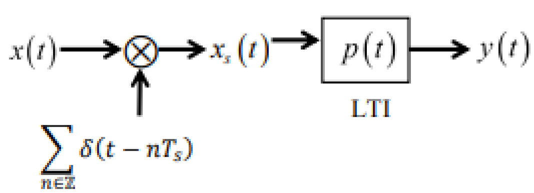
Where $\{c_k\}_{k \in \mathbb{Z}}$ are **Fourier Coefficients of $c(t)$, given by**:

$$c_k = \frac{\tau}{T} sinc\left(k\frac{\tau}{T}\right)$$

Fourier Transform of output signal $y(t)$:

$$Y(\omega) = \frac{1}{2\pi} C(\omega) * X(\omega) = \frac{\tau}{T} \sum_{k \in \mathbb{Z}} sinc\left(\frac{k\tau}{T}\right) X\left(\omega - \frac{2\pi k}{T}\right)$$

Flat Sampling Analysis:



Where $p(t) = sinc^2(t)$ is the impulse response of the LTI system. $x(t)$ is a band limited signal s.t $\forall |\omega| > B : X(\omega) = 0$

Necessary (but not sufficient) **condition for perfect reconstruction**:

$$P(\omega) = \frac{1}{2\pi}\left(rect\left(\frac{\omega}{2\pi}\right) * rect\left(\frac{\omega}{2\pi}\right)\right) = \frac{1}{2\pi} tri\left(\frac{\omega}{2\pi}\right) = \begin{cases} 1 - \frac{|\omega|}{2\pi} & ; |\omega| < 2\pi \\ 0 & ; \text{else} \end{cases}$$

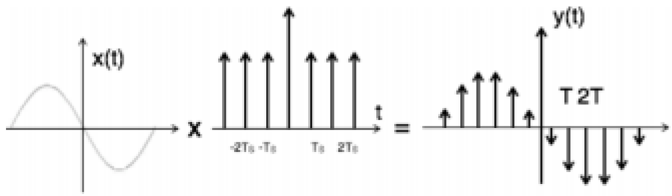
So, condition on B: $B < 2\pi$.

This is NOT SUFFICENT to reconstruct $x(t)$ from $y(t)$.

Niquist Sampling Condition: The sampling rate $\frac{1}{T_s}$ must satisfy $\frac{1}{T_s} > \frac{2B}{2\pi} = \frac{B}{\pi}$

Impulse Sampling:

Impulse sampling is the multiplication of an input signal $x(t)$ with an impulse train $\sum_{n=-\infty}^{\infty} \delta(t-nT)$, or period T.



Output of the sampler: $y(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)$

Fourier Transform of Impulse Train (Dirac Comb): $\mathcal{F}(\sum_{k\in\mathbb{Z}} \delta(t)) = 2\pi \sum_{k\in\epsilon} c_k \delta\left(\omega - \frac{2\pi}{T}k\right)$, $\omega_s = \frac{2\pi}{T}$, $c_k = \frac{1}{T}$

Fourier Transform of the Output: $Y(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$

Quantization:

The process of mapping a large set of input values to a countable small set

Quantization Error: $q[n] \triangleq x_q[n] - x[n]$, where $x[n] \triangleq x(nT_s)$, T_s =sampling interval

Optimal MMSE Quantizer:

Quantization error: $q \triangleq Q(X) - X$

Quantization level: Let y_i be the quantization level of the Quantizer for the cell $R_i = [t_{i-1}, t_i]$

Bias: the MMSE Quantizer is UNBIASED, i.e $E[Q(X)] = E[X]$

Quantization Error Property: the quantized signal is orthogonal to the quantization error, i.e $E[qQ(X)] = 0$

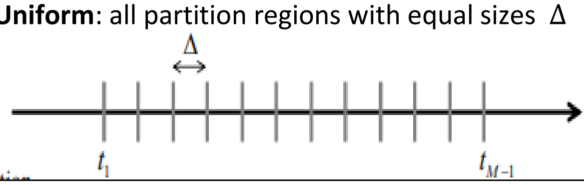
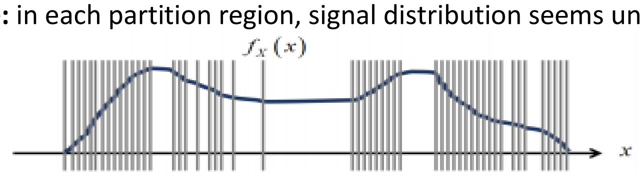
Optimal MMSE Quantization level:

MSE as a function of y_i: $E[q^2] = E[(Q(x)-x)^2] = \sum_{i=1}^{M} \Pr(X \in R_i)\left(y_i^2 - 2y_i E[X|X \in R_i] + E[X^2|X \in R_i]\right)$

Optimal Level: $y_i = E[X|X \in R_i]$ (derivative of MSE wrt y_i, equated to zero, solved for y_i)

Expression for the MSE: $\boldsymbol{MSE = E[q^2]} = E[(Q(X)-X)^2] = E[Q^2(X)] - 2E[XQ(X)] + E[X^2] = E[X^2] - E[Q^2(X)]$

Quantization Signal to Noise: $QSNR = \frac{E[x^2]}{E[q^2]}$

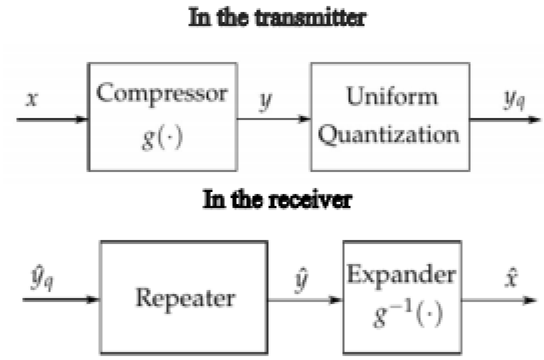
Quantization Types	
Uniform: all partition regions with equal sizes Δ 	**Dense:** in each partition region, signal distribution seems uniform: 

Companders:

Naïve Companding:

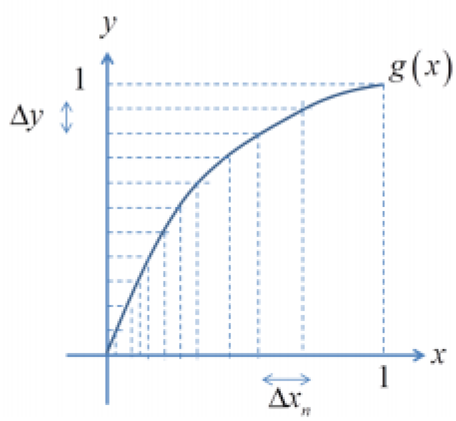
Motivation: It is easy to perform MMSE quantization for a uniformly distributed signal: We first choose partition regions with equal sizes Δ, thus performing dense linear quantization. Then we determine the representation values to be the middle of each section, which is the optimal representation choice in MMSE sense. We want to take advantage of this simplicity.

Result: We use the Compander, which contains a compressor and expander



The compressor applies $g(x)$ on the signal distribution, which transforms it into a uniform distribution. Then we perform uniform quantization, and transmit the signal. This way we almost ensure that we have MSE.

Compressor's $g(x)$ function:



If the signal distribution is known: $f_Y(y) = \frac{1}{g'(x)} f_X(x)|_{x=g^{-1}(y)}$. Since we want $f_Y(y) = 1$ uniforms distribution in [0,1], then we choose: $g'(x) = f_X(x) \rightarrow g(x) = F_X(x)$

Remarks:

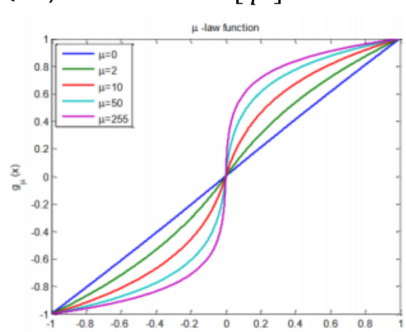
1. We can convert the distribution range to any range, not necessarily [0,1]
2. We cannot guarantee optimal quantization on the x signal, even if we choose $g(x) = F_X(x)$ and do optimal MMSE quantization on y_q
3. Since the signal distribution is not always known, we cannot use this kind of Compander.

μ −Law Compander:

Since we don't always know the signal distribution, we can't use the previously discussed Compander. To get around this, we use the $\mu - Law$ (or A-law) Compander.

$$g_\mu = sign(x)\frac{\ln(1+\mu|x|)}{\ln(1+\mu)} \quad , \qquad x \in [-1,1]$$

Where we select μ such that we maximize SQNR, i.e minimize $E[q^2]$



Algorithm:

1. If the signal x meets the requirement of [-1,1] probability distribution limits, then we should apply a linear function $z = h(x)$ on signal x. This would yield a signal z which has a distribution limited to the range of [-1,1]
2. The μ law compander is activated on z, and we get $y = g_\mu(z)$, distributed on [-1,1]
3. We apply uniform quantization with N levels on y, choosing $\Delta y = \frac{2}{N}$. We then find the optimal representative values according to the chosen distortion measure (usually MMSE, which means the optimal representation values are the middle of each section)
4. We extract $z = g_\mu^{-1}(y)$ by applying the inverse operation on y. We then extract x by applying $x = h^{-1}(z)$

Now, assuming N is sufficiently large, so that dense quantization is performed, and assuming $E[x] = 0$, we have:

$$SQNR_{dB} = 10\log\left(\frac{E[x^2]}{E[q^2]}\right) = 10\log\left(\frac{3\mu^2N^2}{\ln^2(1+\mu)}\right) - 10\log\left(\frac{1}{E[x^2]} + \frac{2\mu E[|x|]}{E[x^2]} + \mu^2\right)$$

$$MSE = E[q^2] = \frac{\ln^2(1+\mu)}{3\mu^2N^2}\left[1 + 2\mu E[|x|] + \mu^2 E[x^2]\right]$$

QSNR For an exponential distribution (R6E6.1):

Input: $X \sim \exp(\lambda)$

Variance of Exponential Distribution: $V(X) = \lambda^{-2}$

Quantizer Param: Dense, uniform quantization. Quantization cell width $\Delta = \frac{K\lambda^{-1}}{M}$, where $K\lambda^{-1}$ is highest threshold value.

QSNR Definition: For $E[X] \neq 0, E[\,q] = E[Q(X) - X] \neq 0$, we have $QSNR = \frac{V(X)}{V(q)}$

Quantization Error: $E[q^2] = E[q^2|X < K\lambda^{-1}]\,\Pr(X < K\lambda^{-1}) + E[q^2|X > K\lambda^{-1}\Pr(X > K\lambda^{-1}]$

$$E[q^2] = \frac{K^2\lambda^{-2}}{12M^2}\cdot(1 - e^{-K}) + \lambda^{-2}\left(\frac{K^2}{4M^2} + \frac{K}{M} + 2\right)\cdot e^{-K}$$

Quantization Noise Expectation:

$E[q] = [q|X < K\lambda^{-1}]\,\Pr(X < K\lambda^{-1}) + E[q|X > K\lambda^{-1}]\,\Pr(X > K\lambda^{-1}) = E[q|X > K\lambda^{-1}]e^{-K} = e^{-K}\lambda^{-1}\left(\frac{K}{2M} + 1\right)$

QSNR: $QSNR = \frac{V(X)}{V(q)} = \frac{\lambda^{-2}}{V(q)} \approx \frac{\lambda^{-2}}{E[q]} = \frac{\lambda^{-2}}{\frac{K^2\lambda^{-2}}{12M^2}(1-e^{-K}) + \lambda^{-2}\left(\frac{K^2}{4M^2}+\frac{K}{M}+2\right)e^{-K}} = \left(\frac{K^2}{12M^2}(1 - e^{-K}) + \left(\frac{K^2}{8M^2} + \frac{K}{2M} + 1\right)e^{-K}\right)^{-1}$

R6Q6.2 – Working with a given Quantization:

Problem Definition:

Input Variable*:* X is a continuous variable with a known Probability Density Function (PDF) $f_x(x)$, with finite support, such that $\forall|x| > 4: f_X(x) = 0$.

Quantization Function*:* $Q(x) = \begin{cases} -4\,; x \in (-\infty, -3) \\ -2\,; x \in (-3, -1) \\ 0\,; x \in (-1, 1) \\ +2\,; x \in (1, 3) \\ +4\,; x \in (3, \infty) \end{cases}$

A: Finding the PDF of the quantization error as a function of the input PDF:

Step 1: *Define quantization error*

$$q = X_q - X = Q(X) - X$$

Step 2: Write *a general expression for PDF of the error, using the law of total probability:*

$$f_q(q) = \sum_{x_q \in \{-4,-2,0,2,4\}} \Pr(X_q = x_q)\cdot f_{q|X_q}(q|x_q)$$

Step 3: *Expand the summation into individual terms*

$$f_q(q) = \Pr\left(X_q = -4\right) f_{q|X_q}(q|-4) + \Pr\left(X_1 = -2\right) f_{q|X_q}(q|-2) + \Pr\left(X_q = 0\right) f_{q|X_q}(q|0) + \Pr\left(X_q = 2\right) f_{q|X_q}(q|2) + \Pr\left(X_q = 4\right) f_{q|X_q}(q|4)$$

Step 4: *Characterize the unknown elements/variables of the expansion*

We ended up using a random variable $q|X_q = x_q$. This is essentially $x_q - X$, I.e the difference between the quantized X and the actual X for some specific quantization level.

Step 5: *Determine quantization error supports for each quantization level*

We know that our signal X is bounded by [-4,4], because that is the support of $f_X(x)$. In other words, $\forall |x| > 4 : f_X(x) = 0$ states that the probability of having an X outside [-4,4] is 0.

Given $x_q = -4$, we know that $Q(x) = -4$, which means $x \in (-\infty, -3)$. But we also know that $x \geq -4$, and thus $x \in [-4, -3]$. The error is $q = x_q - X$. Thus the range of is $q \in [-4 - (-3),\ -4 - (-4)] \to q \in D_{-4} = [-1,0]$
Given $x_q = -2$: $x \in (-3,-1) \to q \in D_{-2} = [-3 - (-2), -1 - (-2)] = [-1,1]$
Given $x_q = 0$: $x \in (-1,1) \to q \in D_0 = [-1 - 0, 1 - 0] = [-1,1]$
Given $x_q = 2$: $x \in (1,3) \to q \in D_2 = [1 - 2, 3 - 2] = [-1,1]$
Given $x_q = 4$: $x \in (3,\infty) \to x \in [3,4] \to q \in D_4 = [4 - 4, 4 - 3] = [0,1]$

Step 6: *Express the error pdf ($f_{q,X_q}(q, x_q)$) in terms of the input's pdf (f_X)*

$$f_{q,X_q}(q, x_q) = f_X(x_q - q) \quad \forall q \in D_{\hat{x}_q}$$

$$\text{Thus } f_{q|X_q}(q|x_q) \underbrace{=}_{\text{Bayes}} \frac{f_{q,X_q}(q, x_q)}{\Pr(X_q = x_q)} = \begin{cases} \dfrac{f_X(x_q - q)}{\Pr(X_q = x_q)} & ;\ q \in D_{x_q} \\ 0 ; \text{else} \end{cases}$$

Step 7: *Replace the previously unknown terms ($f_{q|X_q}$) from step 3, using the result of step 6*

$$f_q(q) = \begin{cases} \Pr(X_q = -4)\dfrac{f_X(-4-q)}{\Pr(X_q = -4)} + \Pr(X_q = -2)\dfrac{f_X(-2-q)}{\Pr(X_q = -2)} + \Pr(X_q = 0)\dfrac{f_X(0-q)}{\Pr(X_q = 0)} + \Pr(X_q = 2)\dfrac{f_X(2-q)}{\Pr(X_q = 2)} & ; q \in [- \\ \Pr(X_q = -2)\dfrac{f_X(-2-q)}{\Pr(X_q = -2)} + \Pr(X_q = 0)\dfrac{f_X(0-q)}{\Pr(X_q = 0)} + \Pr(X_q = 2)\dfrac{f_X(2-q)}{\Pr(X_q = 2)} + \Pr(X_q = 4)\dfrac{f_X(4-q)}{\Pr(X_q = 4)} & ; q \in [0 \\ 0 \ ; \text{else} \end{cases}$$

Note that the quantization error is not uniformly distributed, in general.

B: Finding the PDF of the quantization error as a function of some GIVEN PDF:

We are given that $f_X(x) = \begin{cases} \frac{1}{4} - \frac{|x|}{16} & ;\ |x| \leq 4 \\ 0 & ; \text{else} \end{cases}$

Step 1: Observe *any symmetry in the problem, to determine the ranges for which we must evaluate the PDF*

$f_X(x)$ is distributed symmetrically around zero. We also remember that the total quantization error supports, as found earlier, range in $[-1,1]$. Therefore we only need to cover the ranges $[-1,0]$, $[0,1]$

Step 2: *Express the PDF $f_q(q)$ in term of f_X, using the fact that $f_{q,X_q}(q, x_q) = f_X(x_q - q)$ (shown in step 6 of last section)*

$$f_q(q) = \begin{cases} f_X(-4-q) + f_X(-2-q) + f_X(-q) + f_X(2-q) & ; q \in [-1,0] \\ f_X(-2-q) + f_X(-q) + f_X(2-q) + f_X(4-q) & ; q \in [0,1] \\ 0 \ ; \ \text{else} \end{cases}$$

Step 3: *Evaluate all terms written in step 2. This is possible, since f_X is given. Algebra is not shown, only the final result is.*

$$f_q(q) = \begin{cases} \frac{1}{2} & ; q \in [-1,1] \\ 0 ; \text{ else} \end{cases}$$

Meaning that, for this particular given distribution of X, the quantization error is uniformly distributed: $q \sim U[-1,1]$

C: Compute the QSNR

Step 1: *Recall the QSNR Definition*

$$QSNR = \frac{E[X^2]}{E[q^2]}$$

Step 2: *Evaluate the denominator (for the specific PDF given to us before, the formula is known as q is uniform) :*

$$E[q^2] = \frac{\left(1-(-1)\right)^2}{12} = \frac{1}{3}$$

Step 3: *Evaluate the numerator*:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = 2\int_0^4 x^2 \left(\frac{1}{4} - \frac{x}{16}\right)dx = \frac{8}{3}$$

Step 4: *Plug results of steps 2,3 into step1, and evaluate:*

$$E[q^2] = \frac{8/3}{1/3} = 8 \approx 9dB$$

Grey Code(R6E6.3)

Setup:

Normal Binary Mapping: $0 \rightarrow' 00', 1 \rightarrow' 01', 2 \rightarrow' 10', 3 \rightarrow '11'$

Grey Code Mapping: $0 \rightarrow' 00', 1 \rightarrow' 01',\ 2 \rightarrow' 11',\ 3 \rightarrow '10'$

Receiver: Assume the received noisy symbols are denoted by $\widehat{s_n}$, and modeled by $\widehat{s_n} = \begin{cases} s_n & ; w.p\ 1-p \\ (s_n+1)_{mod4} & ; w.p\ p \end{cases}$.

This means each received symbol is estimated correctly with probability $1-p$, and estimated incorrectly with probability p.

Symbol probability: Assume all symbols transmitted with equal probability, i.e $\Pr(s_j) = \frac{1}{4}\ ; j \in \{1,2,34\}$

Define P_{b_i} = error probability in the ith bit., and $\Pr(b_i|s_j)$ =error probability in the i-th bit given that the symbol s_j was transmitted.

Analyze:

Bit Error Probability: $P_b = \frac{1}{K}\sum_{i=0}^{K-1} P_{b_i}$. This holds, assuming that symbols represent an equal number of bits, and K is the number of bits

Error probability in the i-th bit: $P_{b_i} = \sum_{j=0}^{M-1} \Pr(b_i|s_j)\Pr(s_j)$ (from law of total probability)

Apply to Specific Case:

We have $M = 4, \Pr(s_j) = \frac{1}{4}$

i-th Bit Error: $P_{b_i} = \frac{1}{4}\sum_{j=0}^{3} \Pr(b_i|s_j)$

Bit error in normal Binary Mapping:	**Bit Error in Gray Code Mapping:**
$$P_{b_0} = \frac{1}{4}\sum_{j=0}^{3} \Pr(b_0\|s_j) = \frac{1}{4}(p+p+p+p) = p$$	$$P_{b0} = \frac{1}{4}\sum_{j=0}^{3} \Pr(b_0\|s_j) = \frac{1}{4}(p+p) = \frac{p}{2}$$
$$P_{b_1} = \frac{1}{4}\sum_{j=0}^{3} \Pr(b_1\|s_j) = \frac{1}{4}(p+p) = \frac{p}{2}$$	$$P_{b_1} = \frac{1}{4}\sum_{j=0}^{3} \Pr(b_1\|s_j) = \frac{1}{4}(p+p) = \frac{p}{2}$$
$$P_b = \frac{1}{2}\sum_{i=0}^{1} P_{b_i} = \frac{1}{2}\left(p+\frac{p}{2}\right) = \frac{3p}{4}$$	$$P_b = \frac{1}{2}\sum_{i=0}^{1} P_{b_i} = \frac{1}{2}\left(\frac{p}{2}+\frac{p}{2}\right) = \frac{p}{2}$$

Inter Symbol Interference (ISI):

Raised Cosine: $p(t) = sinc\left(\frac{t}{T}\right) \cdot \frac{\cos\left(\frac{\pi\rho t}{T}\right)}{1-\frac{4\rho^2 t^2}{T^2}} \to P(f) = \begin{cases} T\,;\ 0 \le |f| \le \frac{1-\rho}{2T} \\ \frac{T}{2}\left(1+\cos\left[\frac{\pi T}{\rho}\left(|f| - \frac{1-\rho}{2T}\right)\right]\right);\ \frac{1-\rho}{2T} \le |f| \le \frac{1+\rho}{2T} \\ 0\,;\ |f| \ge \frac{1+\beta}{2T} \end{cases}$, where ρ is the roll-off factor, valued 0-1. $\frac{\rho}{2T}$ is called the excess bandwidth.

Transmitted Signal: Consider a signal $s(t) = \sum_{n=-\infty}^{\infty} s_n p(t-nT)$ where s_n represents the individual 'bits' of information and $p(t)$ is some kind of pulse.
Generation of the Transmitted Signal: We can generate $s(t)$ by: $\sum_{n=-\infty}^{\infty} s_n \delta(t-nT) \to \boxed{p(t)} \to s(t)$, where the $p(t)$ is an impulse response of some LTI system (it's still a pulse, though)

Peak ISI Distortion when using Niquist-I (Sinc) Pulses: $D_p = \frac{A}{T}\sum_{n=-\infty \neq 0}^{\infty} \left|sinc\left(\frac{t}{T} + k - n\right)\right|$ where k is the sampling period ($t = kT$), and

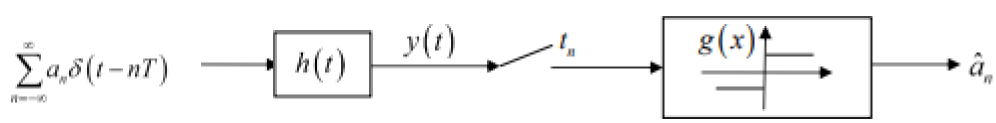
Peak ISI Distortion when using Niquist-II (Raised Cosine) Pulses: $D_p = \sum_{n=-\infty}^{\infty} \left|p_{raisedcos}\left(\frac{t}{T} + (n-k)\right)\right| \approx \frac{1}{n^3}$

No ISI (NISI) Condition: $p(nT) = 0,\ \ \forall n \neq 0$, in frequency domain: $\sum_{k=-\infty}^{\infty} P\left(\omega - \frac{2\pi}{T}k\right) = T \quad ;$ where $P(\omega) = \mathcal{F}\{p(t)\}$

Inter Symbol Interference NIQUIST Criterion (Google):
This condition must be satisfied by a communication channel (including responses of transmit and receive filters) in order to have no ISI. When consecutive symbols are transmitted over a channel by linear modulations , the impulse response of the channel causes a smearing of the transmitted signal in time domain. The criterion is: frequency shifted replicas of $H(f)$ must add up to a constant value. If we denote the channel impulse response as $h(t)$, then the NISI Condition is: $h(nT_s) = \begin{cases} 1\,;\ n = 0 \\ 0\,; n \neq 0 \end{cases}$ $\forall n \in \mathbb{Z}$. This is equivalent to $\frac{1}{T_s}\sum_{k=-\infty}^{\infty} H\left(f - \frac{k}{T_s}\right) = 1 \quad \forall f$.

Inter Symbol Interference Solving Algorithm (R7E7.1):

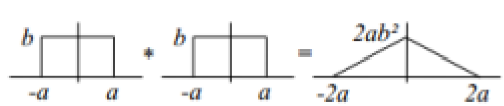
Setup:



The transmitted symbols are $a_n \in \{-1, +1\}$.
$h(t)$ is the impulse response of an LTI system.
$y(t)$ is sampled at times t_n and is passed through a slicer which operates according to $g(x) = sign(x)$
The frequency response of the LTI system $\widetilde{H}(f) = \begin{cases} 1 - T|f|\,; |f| \le \frac{1}{T} \\ 0\ \ ; |f| > \frac{1}{T} \end{cases}$

Finding the Impulse Response, and Checking $h(nT) = 0\ \forall n \neq 0$

Step 1: *Observe that $\widetilde{H}(f)$ is a triangular function. Recall that triangular function is a convolution of two boxes.*



Step 2: *Determine parameters of the rectangular functions which make up the convolution.*
We know that the triangular function will have zeroes at $\pm\frac{1}{T}$. Thus $2a = \frac{1}{T} \to \boxed{a = \frac{1}{2T}}$.

We also know the peak of the triangle is 1, thus: $2ab^2 = 1 \rightarrow b^2 = T \rightarrow \boxed{b = \sqrt{T}}$

Step 3: *Generate an expression for the rectangles in frequency and time:*
Denote $\widetilde{M}(f)$ as the rectangle. Using results of step 1:

$$\widetilde{M}(f) = \sqrt{T} \cdot rect(fT) \leftrightarrow m(t) = \frac{1}{\sqrt{T}} sinc\left(\frac{t}{T}\right)$$

Step 4: Take the inverse Fourier Transform of $\widetilde{H}(f)$ using the results of step 1 and 3.

$$h(t) = \mathcal{F}^{-1}\{\widetilde{H}(f)\} = \mathcal{F}^{-1}\{\widetilde{M}(f) * \widetilde{M}(f)\} = \mathcal{F}^{-1}\{\widetilde{M}(f)\}\mathcal{F}^{-1}\{\widetilde{M}(f)\} = m(t) \cdot m(t) = m^2(t) = \frac{1}{T} sinc^2\left(\frac{t}{T}\right)$$

Step 5: Check the condition (as requested by the question) $h(nT) = 0$:
We have that $h(t)|_{t=0} = h(0) = \frac{1}{T}$

$$h(t)|_{t=nT} = h(nT) = \frac{1}{T} sinc^2\left(\frac{nT}{T}\right) = \frac{1}{T} sinc^2(n) = 0 \quad \forall n \neq 0$$

For the rest of this question, it is not given that the sample time instances are $t_n = \frac{T}{4} + nT$

Checking the NISI Condition:

Step 1: *Find an expression for the LTI Impulse response given the sampling instances*:

$$h(t)|_{t=\frac{T}{4}=nT} = h\left(\frac{T}{4} + nT\right) = \frac{1}{T} sinc^2\left(\frac{1}{T}\left(\frac{T}{4} + nT\right)\right) = \frac{1}{T} sinc^2\left(\frac{1}{4} + n\right)$$

Step 2: *Evaluate the NISI Condition*:

$$\frac{1}{T} sinc^2\left(\frac{1}{4} + n\right) \neq 0 \ \forall n \neq 0 \rightarrow \text{NISI not satisfied}$$

Is it possible that an estimation error of $\widehat{a_n}$ will occur due to ISI (Hint: Use $\sum_{n\in\mathbb{Z}\neq 0}\left(\frac{1}{4n+1}\right)^2 = \frac{\pi^2}{8} - 1$) ?

Step 1: *Make assumptions that specify the problem: Without loss of generality, we assume that…*
Symbol to be estimated is $a_{n=0} = a_0$
The corresponding sample time is $t_{n=0} = t_0 = \frac{T}{4}$.
Therefore, the first thing we evaluate is $y\left(\frac{T}{4}\right)$

Step2: *Evaluate the "first transmission case" i.e the case where $t = \frac{T}{4}$*

$$y\left(\frac{T}{4}\right) = \sum_{n=-\infty}^{\infty} a_n h\left(\frac{T}{4} - nT\right) = a_0 h\left(\frac{T}{4}\right) + \sum_{n=-\infty}^{\infty} a_n h\left(\frac{T}{4} - nT\right)$$

Step 3: *Choose some specific symbol a_0 to be transmitted. If the system is symmetrical (like the slicer is here), then it doesn't matter if you choose $a_0 = -1$ or $+1$. Evaluate the error.*
Choose $a_0 = 1$. Then:
If $y\left(\frac{T}{4}\right) > 0$ then $\hat{a}_0 = g\left(y\left(\frac{T}{4}\right)\right) = 1 = a_0$ →no ISI error
If $y\left(\frac{T}{4}\right) < 0$ then $\hat{a}_0 = g\left(y\left(\frac{T}{4}\right)\right) = -1 \neq a_0$ →yes ISI error.

Step 4: *If an ISI is possible for some condition or case, verify that that case is actually possible.*
We have ISI Error if $y\left(\frac{T}{4}\right) < 0$

Step 4.0: *Establish the expression which would satisfy the condition:*
Note that the worst-case ISI error is obtained if all elements of the sum $y\left(\frac{T}{4}\right) = \sum_{n=-\infty}^{\infty} a_n h\left(\frac{T}{4} - nT\right)$ are negative, as this minimizes $y\left(\frac{T}{4}\right)$. This occurs at $a_n = -1$ since $h(t)$ is always positive.

In the case where $a_n = -1$ we have $y\left(\frac{T}{4}\right) = h\left(\frac{T}{4}\right) - \sum_{n=-\infty\neq 0}^{\infty} h\left(\frac{T}{4} - nT\right)$

Step 4.1 : *Evaluate the terms of the expression from 4.0:*

First, $n = 0$: $h\left(\frac{T}{4}\right) = \frac{1}{T}sinc^2\left(\frac{1}{T}\cdot\frac{T}{4}\right) = \frac{1}{T}\left(\frac{\sin\left(\frac{\pi}{4}\right)}{\pi/4}\right)^2 = \frac{1}{T}\left(\frac{\sqrt{2}/2}{\pi/4}\right)^2 = \frac{1}{T}\left(\frac{2\sqrt{2}}{\pi}\right)^2 = \frac{8}{\pi^2 T}$

And for $n > 0$: $h\left(\frac{T}{4} - nT\right) = \frac{1}{T}sinc^2\left(\frac{1}{T}\left(\frac{T}{4} - nT\right)\right) = \frac{1}{T}sinc^2\left(\frac{1}{4} - n\right) = \frac{1}{T}\cdot\left[\frac{\sin\left(\frac{\pi}{4} - \pi n\right)}{\frac{\pi}{4} - \pi n}\right]^2$

Step 4.2: *Simplify the result of step 4.1 into a workable form.*
We use $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$:

$$h\left(\frac{T}{4} - nT\right) = \frac{1}{T}\cdot\left[\frac{\sin\left(\frac{\pi}{4} - \pi n\right)}{\frac{\pi}{4} - \pi n}\right]^2 = \frac{1}{T}\left[\frac{\sin\left(\frac{\pi}{4}\right)\cos(\pi n) - \cos\left(\frac{\pi}{4}\right)\sin(\pi n)}{\frac{\pi}{4} - \pi n}\right]^2 = \cdots = \frac{8}{\pi^2 T}\cdot\frac{1}{(1-4n)^2}$$

Step 4.3: *Plug in the results of 4.2, 4.1 into the ISI condition found in 4.1:*

$$y\left(\frac{T}{4}\right) = h\left(\frac{T}{4}\right) - \sum_{n=-\infty\neq 0}^{\infty} h\left(\frac{T}{4} - hT\right) = \frac{8}{\pi^2 T} - \sum_{n=-\infty}^{\infty}\frac{8}{\pi^2 T}\cdot\frac{1}{(1-4n)^2} = \frac{8}{\pi^2 T}\left(1 - \sum_{n=-\infty}^{\infty}\frac{1}{(1-4n)^2}\right)$$
$$= \frac{8}{\pi^2 T}\left[1 - \left(\frac{\pi^2}{8} - 1\right)\right] = \frac{8}{\pi^2 T}\left[2 - \frac{\pi^2}{8}\right]$$

Step 4.4: *Evaluate the truthfulness of step 4.3*:
$y\left(\frac{T}{4}\right) = \frac{8}{\pi^2 T}\left[2 - \frac{\pi^2}{8}\right] > 0$, but we need < 0 to have ISI error. Therefore ISI error is not possible in this case.

Raised Cosine Exercise (R7E7.2):

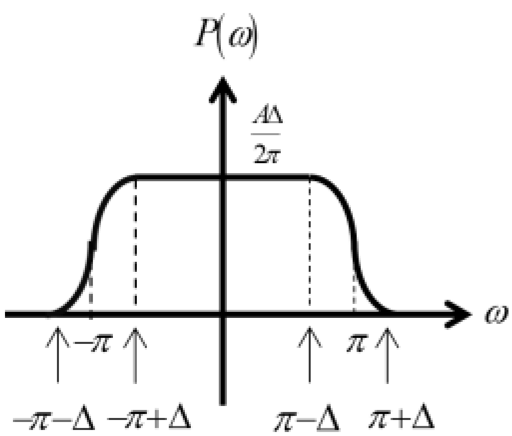
Setup:

We have a raised cosine pulse $p(t) = sinc\left(\frac{t}{T}\right)\cdot\frac{\cos\left(\frac{\pi\rho t}{T}\right)}{1 - \frac{4\rho^2 t^2}{T^2}}$. It is implemented as a multiplication of two signals, i.e $p(t) = c(t)\cdot r(r)$. We are given:

$$R(\omega) = \begin{cases}1\,; & |\omega| \le \pi\\ 0\,; & |\omega| > \pi\end{cases} = rect\left(\frac{\omega}{2\pi}\right) \quad ; \quad C(\omega) = \begin{cases}A\left(1 - \frac{|\omega|}{\Delta}\right) & ; |\omega| \le \Delta\\ 0\,; |\omega| > \Delta\end{cases} = A\cdot tri\left(\frac{\omega}{\Delta}\right)$$
$$R(\omega) = \mathcal{F}\{r(t)\} \quad ; \quad C(\omega) = \mathcal{F}\{c(t)\}$$

Assume $T = 1$, $\Delta < \pi$

Find the FT of the Raised Cosine Pulse:



$$P(\omega) = \begin{cases}\frac{A\Delta}{2\pi} & |\omega| \le \pi - \Delta\\ \frac{A\Delta}{4\pi}\cdot\left[2 - \left(1 + \frac{|\omega| - \pi}{\Delta}\right)^2\right] & \pi - \Delta < |\omega| \le \pi\\ \frac{A\Delta}{4\pi}\cdot\left(1 - \frac{|\omega| - \pi}{\Delta}\right)^2 & \pi < |\omega| \le \pi + \Delta\\ 0 & \pi + \Delta < |\omega|\end{cases}$$

Given $\int_{-\infty}^{\infty} c(t)dt = \frac{2\pi}{\Delta}$ find A and show that NISI holds for the pulse $p(t)$:

Step 1: *Identify where you can pull A out of.*

We observe that $A = C(0) = C(\omega)|_{\omega=0} = \int_{-\infty}^{\infty} c(t)e^{j\omega t}dt|_{\omega=0} = \int_{-\infty}^{\infty} c(t)dt = \frac{2\pi}{\Delta} = A$

Step 2: *Check the NISI Condition in time domain (because it is easy to take the inverse FT of c and r)*

$$r(t) = \mathcal{F}^{-1}\left\{rect\left(\frac{\omega}{2\pi}\right)\right\} = sinc(t) \quad ; \quad c(t) = \mathcal{F}^{-1}\left\{\frac{2\pi}{\Delta} \cdot tri\left(\frac{\omega}{\Delta}\right)\right\} = sinc^2\left(\frac{\Delta \cdot t}{2\pi}\right)$$

$$p(0) = c(0) \cdot r(0) = 1$$

$$p(nT)|_{T=1} = c(n) \cdot r(n) = 0, \qquad \forall n \neq 0$$

Determine the Raised Cosine Pulse $p(t)$ for the limiting case $\Delta \to 0, A \cdot \Delta = 2\pi$. Does the NISI Condition still hold?
As Δ goes to 0, we obtain $p(t) = r(t) = sinc(t)$, and the NISI still holds.

What is $p(t)$ as A=1 and $\Delta \to \infty$?

$$c(t) = sinc^2\left(\frac{\Delta \cdot t}{2\pi}\right) \to \delta(t) \to p(t) = c(t)r(t) = \delta(t)r(t) = \delta(t)r(0) = \delta(t)$$

Matched Filters (StackExchange):

Setup: Consider a finite energy signal $s(t)$. This signal goes into some filter with response $h(t)$. Thus, the output signal is

$$y(\tau) = \int_{-\infty}^{\infty} s(\tau - t)h(t)dt$$

The filter whose response maximizes $y(t_0)$ (t0 is a sampling instant), while subject to the fixed energy condition:

$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \mathbb{E} = \int_{-\infty}^{\infty} |h(t)|^2$$

Will be the matched filter.

Matched Filter: $h(t) = s(t_0 - t)$ produces the maximal response $y(t_0) = \mathbb{E}$ at the specified time t_0. In words: the filter is matched to $s(t)$ at time t_0.

Matched Filter Trivia:

1. **Unique Global Maximum:** The matched filter will have a unique global maximum value of $\mathbb{E}$ at t_0. For any other t we would have $y(t) < y(t_0) = \mathbb{E}$. Furthermore: $y(t) = \int_{-\infty}^{\infty} s(t-\tau)s(-\tau)d\tau = \int_{-\infty}^{\infty} s(\tau - t)s(\tau)d\tau = R_s(t)$, where $R_s(t)$ is the autocorrelation function of $s(t)$.
2. **Filter output at time t_0:** $R_s(t - t_0)$, i.e the autocorrelation function delayed to peak at time t_0.
3. **Time Reversal**: The impulse response $s(t_0 - t) = s\big(-(t - t_0)\big)$ of the matched filter for time t_0 is just $s(t)$ reversed in time and moved to the right by t_0.
4. **Non-Causality:** If $s(t)$ has a finite support (such as $[0, T]$), then the matched filter is noncausal if $t_0 < T$
5. **Matching to other times:** The filter matched to $s(t)$ at time $t_1 > t_0$ is simply the filter matched at time t_0, but with an additional delay of $t_1 - t_0$.
6. **Optimality:** No other filter for time t_0 can produce an output as large as $\mathbb{E}$ at t_0. However, it is possible to find filters which exceed $R_s(t_0)$ at t_0, though $R_s(t_0) < E$
7. **Filter Transfer Function**: $H(f) = S^*(f)$
8. **Output Transfer Function:** $Y(f) = \mathcal{F}\{y(t)\} = |S(f)|^2$

Matched Filter SNR & Statistics:

Assume the input signal is $s(t) + n(t)$ where $n(t)$ is additive white Gaussian noise, with two-sided PSD $\frac{N_0}{2}$. This is processed through the filter $h(t)$.

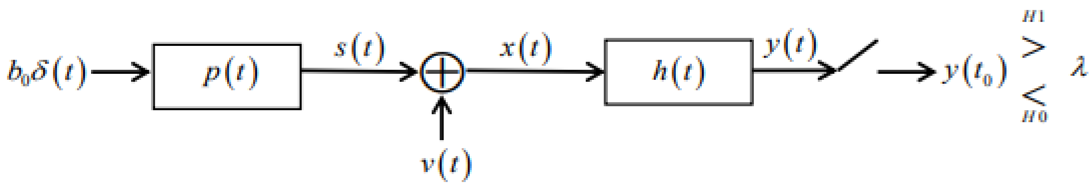
1. **Output Noise Process:** Zero mean stationary Gaussian Process with autocorrelation function $R_h(t) = \frac{N_0}{2} R_s(t)$
2. **Output Variance (for any filter under consideration):** $\sigma^2 = \frac{N_0}{2} R_s(0) = \frac{N_0}{2} \int_{-\infty}^{\infty} |h(t)|^2 dt$. This is the same regardless of when we sample the filter output
3. **Expression for SNR (regardless of what $h(t)$ is chosen):** $SNR = \frac{y(t_0)}{\sigma} \leq \sqrt{\frac{2\mathbb{E}}{N_0}}$

4. **SNR Optimality**: $SNR_{\max} = \sqrt{\frac{2E}{N_0}}$, which happens exactly when $h(t) = s(t_0 - t)$, i.e the filter matched to $s(t)$ at t_0.
5. **Variance for Matched Filter:** $\sigma^2 = \frac{\mathbb{E}N_0}{2}$

Where is SNR Maximized? SNR is only maximized at the sampling instant t_0. At other times, other filters could give a larger SNR than the matched filter is providing at t_1, but this is still smaller than the $SNR = \sqrt{\frac{2\mathbb{E}}{N_0}}$ that the matched filter gives at t_0.

Matched Filter Exercise (R8E8.1):

Setup:



Where $b_0 \in \{0,1\}$ and $S_{vv}(\omega) = \frac{N_0}{2}$. In addition it is given that $p(T) = \begin{cases} \frac{1}{2}(T-t) \ ; \ t \in [0,T] \\ 0 \ ; \ \text{else} \end{cases}$

Finding the Matched Filter for this System:

Step 1: *Recall the formula for MF*:
MF is given by $h_0(t) = p^*(t_0 - t)$, where t_0 is the "edge" of the pulse in time.

Step 2: *Apply the MF formula to your specific situation*:
Here we have $t_0 = T$, so: $h(t) = p^*(t_0 - t) = p^*(T-t) = \begin{cases} \frac{1}{2}\big(T-(T-t)\big) \ ; 0 \le T-t \le T \\ 0 \ ; \ \text{else} \end{cases} = \begin{cases} \frac{1}{2}t \ ; 0 \le t \le T \\ 0 \ ; \text{else} \end{cases}$

Step 3: *Compute* $\frac{|\mu|^2}{\sigma^2}$
Step 3.1: *Identify the distributions of $y(t_0)$ for the different cases $(H1, H0)$:*
Recall that:

$$y(t_0)|H0 \sim N(0, \sigma^2)$$
$$y(t_0)|H1 \sim N(\mu, \sigma^2)$$

Step 3.2: *Identify the formulas for μ and σ*

$$\mu = \int_{-\infty}^{\infty} h(\tau)p(t_0 - \tau)d\tau$$
$$\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |h(\tau)|^2 d\tau$$

Where the expression for σ^2 is only valid for AWGN

Step 3.3: *Evaluate the results of 3.2 by substituting explicit expressions for the filter and pulse*.

$$\mu = \int_{-\infty}^{\infty} h(\tau)p(T-\tau)d\tau = \int_{-\infty}^{\infty} |p(\tau)|^2 d\tau = \frac{1}{4}\int_0^T t^2 dt = \frac{T^3}{12}$$
$$\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |h(\tau)|^2 d\tau = \frac{N_0}{2}\int_{-\infty}^{\infty} |p(\tau)|^2 d\tau = \frac{N_0}{2} \cdot \frac{T^3}{12}$$

Step 3.4: *Finally compute* $\frac{|\mu|^2}{\sigma^2}$

$$\left(\frac{|\mu|^2}{\sigma^2}\right)_{MF} = \frac{T^2}{6N_0}$$

Now, the filter being used is $h(t) = p(t)$. Compute $\frac{|\mu|^2}{\sigma^2}$ for this filter and compare to previous result.

Step 1: *Re-calculate μ and σ^2 using the formulas from step 3.2 of past section, but applying the new change*

$$\mu = \int_{-\infty}^{\infty} h(t)p(T-t)dt = \int_{-\infty}^{\infty} p(t)p(T-t)dt = \int_0^T \frac{1}{2}(T-t)\cdot\frac{1}{2}tdt = \frac{1}{4}\int_0^T (Tt-t^2)dt = \frac{1}{4}\left[T\frac{t^2}{2}-\frac{t^3}{3}\right]_0^T = \frac{T^3}{24}$$

$$\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |h(\tau)|^2 d\tau = \frac{N_0}{2}\int_{-\infty}^{\infty} |p(\tau)|^2 d\tau = \frac{N_0 T^3}{24}$$

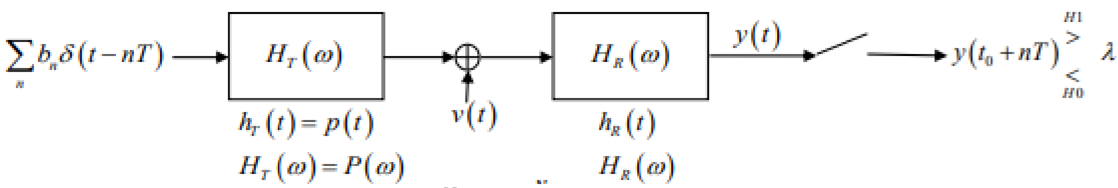
Step 2: Recalculate the ratio:

$$\frac{|\mu|^2}{\sigma^2} = \frac{1}{4}\cdot\left(\frac{|\mu|^2}{\sigma^2}\right)_{MF} < \left(\frac{|\mu|^2}{\sigma^2}\right)$$

The resulting SNR is lower, the error probability with the filter $h(t) = p(t)$ will be bigger than for the MF, as expected.

Matched Filter Exercise (R8E8.2):

Setup:



Where t_0 is an adjustable parameter and $S_{vv}(\omega) = \frac{N_0}{2} = 1$

We are given $p(t) = \delta(t-2) + \frac{1}{4}\big(\delta(t-4)+\delta(t)\big)$.

Minimized Error Probability Proof (for when $t_0 = 4$, $H_R(\omega) = H_T(\omega)$):

Step 1: *Realize that the minimum error probability is given by a Matched Filter. If* we find the MF, and it satisfies $H_R(\omega) = H_T(\omega)$, then we just proved the claim.

Step 2: *Find the Matched Filter:*

$$h_R(t) = p^*(t_0-t) = p^*(4-t) = \delta(4-t-2) + \frac{1}{4}\big(\delta(4-t-4)+\delta(4-t)\big) = \delta(t-2) + \frac{1}{4}\big(\delta(t)+\delta(t-4)\big)$$

$$h_R = p(t) = h_T(t) \leftrightarrow H_R(\omega) = H_T(\omega)$$

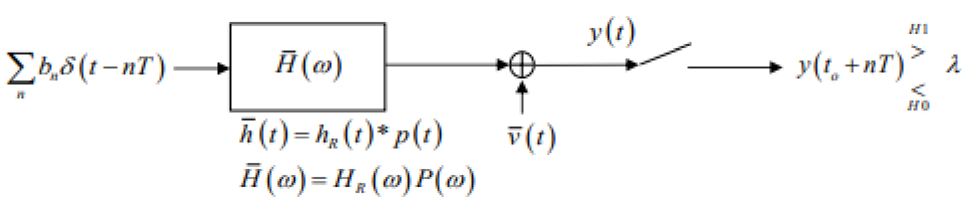
We know that $t_o = 4$, therefore $H_R(\omega) = H_T(\omega)$ is indeed the MF and the proof is complete.

Computing $P(\omega)$:

$$P(\omega) = \mathcal{F}\{p(t)\} = \mathcal{F}\left\{\delta(t-2)+\frac{1}{4}\big(\delta(t-4)+\delta(t)\big)\right\} = e^{-j2\omega} + \frac{1}{4}\big(e^{-j4\omega}+1\big) = e^{-j2\omega}\left(1+\frac{1}{4}\big(e^{-j2\omega}+e^{j2\omega}\big)\right)$$

$$= e^{-j2\omega}\left(1+\frac{1}{2}\cos(2\omega)\right) = e^{-j2\omega}\left(\frac{1}{2}+\cos^2(\omega)\right)$$

Equivalent System & Whitening Noise:



The above is a proposed equivalent system. Notice that $\bar{v}(t)$ is not white anymore. We want to find a whitening filter that is both causal and stable, with a FT $G(\omega)$. Can such a filter be found? It is even necessary to whiten with a whitening filter?

Step 1: *Recall the formula for the spectrum of equivalent noise $\bar{v}(t)$:*

$$S_{\bar{v}\bar{v}}(\omega) = \frac{N_0}{2}|H_R(\omega)|^2 = |H_T(\omega)|^2 = |P(\omega)|^2 = \left(\frac{1}{2}+\cos^2(\omega)\right)^2$$

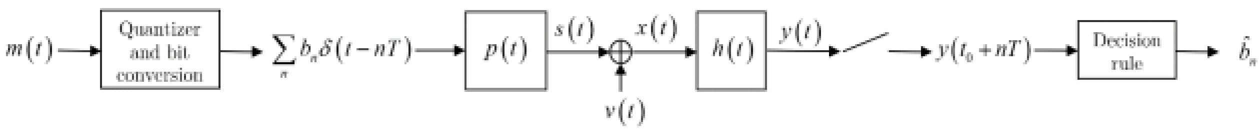
Step 2: *Use Paley-Wiener to determine the filter*:

We know that $0 < S_{\bar{v}\bar{v}}(\omega) < \infty$. From the Paley-Wiener causality condition & theorem: $P(\omega) = \sigma^2 G(\omega)G^*(\omega)$, where $G(\omega)$ is causal and stable.

Step 3: *Determine necessity of a whitening filter:*
We don't need a whitening filter because the noise coloration is caused by the matched filter, so an optimal decision (in the minimum error probability sense) may be taken according to the output signal, noised with $\tilde{v}(t)$

Matched Filter in Context of a Full System (with quantization) (R8E8.3):

Setup:



Where:
Input: $m(t) \sim U(-1,1)$
Quantizer: 4 level MMSE Quantization
Information Coding: Every symbols is mapped to bits by Grey code mapping, followed by 4-PAM mapping: $b_n \in \left\{\ell_0 = 0, \ell_1 = \frac{1}{3},\ \ell_2 = \frac{2}{3},\ \ell_3 = 1\right\}$
PSD: $S_{vv}(\omega) = \frac{N_0}{2}$
Period: $T = 1.5$
First system impulse response: $p(t) = u(t) - u(t-2)$

Finding $P(\omega)$:

Step 1: *Simple Fourier transform*

$$P(\omega) = \mathcal{F}\{p(t)\} = \mathcal{F}\{u(t) - u(t-2)\} = \mathcal{F}\left\{rect\left(\frac{t-1}{2}\right)\right\} = e^{-j\omega}\mathcal{F}\left\{rect\left(\frac{t}{2}\right)\right\} = 2e^{-j\omega}sinc(\omega)$$

Find the Impulse Response which will bring the Error Probability to Minimum:

Step 1: *Realize that the question is asking for a matched filter (Because it is guaranteed to give Pr(error)=min)*

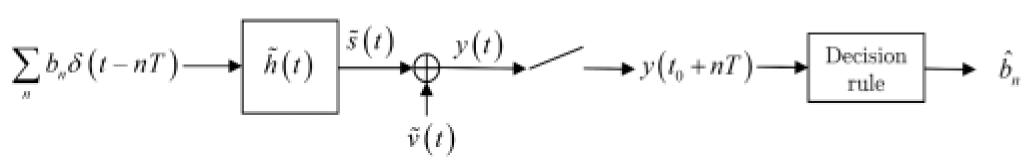
Step 2: *Determine t_0 of the matched filter, by looking at the edge of $p(t)$*
Since $\forall t > 2 : p(t) = 0$, we have $t_0 = 2$

Step 3: *Determine the matched filter response in time and frequency*

$$h(t) = p^*(t_0 - t) = p(2-t) = u(2-t) - u(2-t-2) = u(t) - u(t-2) = p(t)$$
$$\rightarrow\ H(\omega) = e^{-j\omega t_0}P^*(\omega) = P(\omega) = 2e^{-j\omega}sinc(\omega)$$

Finding Equivalent Representation of the System:

Step 1: *Determine the form of equivalent representation (usually given, like in this case):*



Step 2: *Find the response of the equivalent filter $\tilde{h}(t)$:*

$$\tilde{h}(t) = p(t) * h(t) = p(t) * p(t) = rect\left(\frac{t-1}{2}\right) * rect\left(\frac{t-1}{2}\right) = 2 \cdot tri\left(\frac{t-2}{2}\right)$$
$$\tilde{H}(\omega) = P(\omega)H(\omega) = P^2(\omega) = 4e^{-2j\omega}sinc^2(\omega)$$

Step 3: *Find the equivalent noise:*

$$\tilde{v}(t) = v(t) * h(t)$$
$$S_{\bar{v}\bar{v}}(\omega) = S_{vv}(\omega)|H(\omega)|^2 = \frac{N_0}{2}\left[2e^{-j\omega}sinc(\omega)\right]^2 = 2N_0 sinc^2(\omega)$$

Determining estimated symbols at the receiver (result: correct estimation):

Step 1: *Determine, what information is given and what we want to find*
We are given that $m(0) = 0.11$, $m(T) = 0.53$, $m(2T) = -0.41$, $m(3T) = -0.9$, and $\tilde{v}(t_0 + T) = 0.4$
We want to find the estimated symbol in the receiver for the value $m(T)$

Step 2: *Examine the quantization and bit conversion process:*

$$[-1, -0.5] \to y_0 = -0.75 \to '\ 00' \to \ell_0 = 0$$
$$[-0.5, 0] \to y_1 = -0.25 \to '\ 10' \to \ell_1 = \frac{1}{3}$$
$$[0, 0.5] \to y_2 = 0.25 \to '\ 11' \to \ell_2 = \frac{2}{3}$$
$$[0.5\ , 1] \to y_3 = 0.75 \to '\ 01' \to \ell_3 = 1$$

Where $\{y_i\}_{i=0}^{3}$ are quantization levels.

Step 3: *Plug in the given information about transmitted symbols (m(t)) into step 2, to obtain transmitted b values*

$$m(0) = 0.11 \in [0, 0.5] \to b_0 = \ell_2 = \frac{2}{3}$$
$$m(T) = 0.53 \in [0.5, 1] \to b_1 = \ell_3 = 1 \text{ (this is what we transmitted)}$$
$$m(2T) = -0.41 \in [-0.5, 0] \to b_2 = \ell_1 = \frac{1}{3}$$
$$m(3T) = -0.9 \in [-1, -0{,}5] \to b_3 = \ell_0 = 0$$

Step 4: *Determine which sample will decide the estimated symbol corresponding to the m of interest ($m(T)$)*

It will be decided by the sample $y(t_0 + T) = \tilde{s}(t_0 + T) + \tilde{v}(t_0 + T)$, where $\tilde{s}(t) = \sum_{n=-\infty}^{\infty} b_n \tilde{h}(t - nT)$

Step 5: *Notice that the sample from step 4 depending on $\tilde{s}(t)$, which is given in general. Determine what is for our sampling instance*.

At the sampling instance : $\tilde{s}(t_0 + T) = \sum_{n=-\infty}^{\infty} b_n \tilde{h}\,(t_0 + T - nT) = \cdots + b_{-1}\tilde{h}(5) + b_0\tilde{h}(3.5) + b_1\tilde{h}(2) + b_2\tilde{h}(0.5) + b_3\tilde{h}\,(-1) + \cdots = b_0\tilde{h}(3.5) + b_1\tilde{h}(2) + b_2\tilde{h}(0.5)$

$$s(t_0 + T) = \frac{1}{2}b_0 + 2b_1 + \frac{1}{2}b_2 = 2.5$$

Therefore: $y(t_0 + T) = \tilde{s}(t_0 + T) + \tilde{v}(t_0 + T) = 2.5 + 0.4 = 2.9$
Step 6: *Determine the "decision criteria" imposed on the symbol, and what symbol was estimated in the end*
We want to examine what symbol brings the metric $|\mu \cdot \ell - y(t_0 + T)|$ to its minimum.

Step 6.1: *To evaluate the above, we need to know μ:*

$$\mu = \int_{-\infty}^{\infty} h(\tau)p(t_0 - \tau)d\tau = h(t) * p(t)|_{t=t_0} = \tilde{h}(t_0) = \tilde{h}(2) = 2$$

Step 6.2: *Determine the ℓ (argument of expression in step 6) that minimizes the expression:*

$$\widehat{\ell_1} = \underset{\ell \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}}{argmin}(|\mu \cdot \ell - y(t_0 + T)|) = \underset{\ell \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}}{argmin}(|2\ell - 2.9|) = 1$$

The actual answer to $argmin(|2\ell - 2.9|)$ is 1.45, but that's an illegal ℓ value. The closest thing we can use is $\hat{\ell}_1 = 1$

Step 7: Compare estimated symbol with transmitted symbol
We know that we have $\ell = 1$, which corresponds to transmitting 0.75 (see step 3). This aligns with the $\hat{\ell}_1$ we found in step 6.2, so the symbol was estimated correctly.

Determining estimated symbols at the receiver (result: incorrect estimation):
Step 1: *Determine, what information is given and what we want to find*
We are given that $m(0) = 0.11$, $m(T) = 0.53$, $m(2T) = -0.41$, $m(3T) = -0.9$, just like in the previous section, but now we have: $\tilde{v}(t_0 + 2T) = 0.2$

We want to find the estimated symbol in the receiver for the value $m(2T)$

Step 2: *Examine the quantization and bit conversion process:*

$$[-1, -0.5] \to y_0 = -0.75 \to' 00' \to \ell_0 = 0$$
$$[-0.5, 0] \to y_1 = -0.25 \ \to' 10' \to \ell_1 = \frac{1}{3}$$
$$[0, 0.5] \to y_2 = 0.25 \to' 11' \to \ell_2 = \frac{2}{3}$$
$$[0.5\,, 1] \to y_3 = 0.75 \to' 01' \to \ell_3 = 1$$

Where $\{y_i\}_{i=0}^{3}$ are quantization levels.

Step 3: *Plug in the given information about transmitted symbols (m(t)) into step 2, to obtain transmitted b values*

$$m(0) = 0.11 \in [0, 0.5] \to b_0 = \ell_2 = \frac{2}{3}$$
$$m(T) = 0.53 \in [0.5, 1] \to b_1 = \ell_3 = 1$$
$$m(2T) = -0.41 \in [-0.5, 0] \to b_2 = \ell_1 = \frac{1}{3} \ \text{ (this is what we transmitted)}$$
$$m(3T) = -0.9 \in [-1, -0{,}5] \to b_3 = \ell_0 = 0$$

Step 4: *Determine which sample will decide the estimated symbol corresponding to the m of interest ($m(2T)$)*

It will be decided by the sample $y(t_0 + 2T) = \tilde{s}(t_0 + 2T) + \tilde{v}(t_0 + 2T)$
Were $\tilde{s}(t_0 + 2T) = \sum_{n=-\infty}^{\infty} b_n \tilde{h}(t_0 + 2T - nT)$

Step 5: *Notice that the sample from step 4 depending on $\tilde{s}(t)$, which is given in general. Determine what is for our sampling instance.*

At the sampling instance : $\tilde{s}(t_0 + 2T) == \cdots b_0\tilde{h}(t_0 + 2T) + b_1\tilde{h}(t_0 + T) + b_2\tilde{h}(t_0) + b_3\tilde{h}(t_0 - T) + b_4\tilde{h}(t_0 - 2T) + \cdots = b_0\tilde{h}(5) + b_1\tilde{h}(3.5) + b_2\tilde{h}(2) + b_3\tilde{h}(0.5) + b_4\tilde{h}(-1) + \cdots$

$$\tilde{s}(t_0 + 2T) = \frac{1}{2}b_1 + 2b_2 + \frac{1}{2}b_3 = 1.1667$$

Therefore: $y(t_0 + 2T) + \tilde{s}(t_0 + 2T) + \tilde{v}(t_0 + 2T) = 1.1667 + 0.2 = 1.3667$

Step 6: *Determine the "decision criteria" imposed on the symbol, and what symbol was estimated in the end*
We want to examine what symbol brings the metric $|\mu \cdot \ell - y(t_0 + T)|$ to its minimum.

Step 6.1: *To evaluate the above, we need to know μ:*

$$\mu = \int_{-\infty}^{\infty} h(\tau)p(t_0 - \tau)d\tau = h(t) * p(t)|_{t=t_0} = \tilde{h}(t_0) = \tilde{h}(2) = 2$$

Step 6.2: *Determine the ℓ (argument of expression in step 6) that minimizes the expression:*

$$\widehat{\ell_2} = \underset{\ell \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}}{argmin}(|\mu \cdot \ell - y(t_0 + 2T)|) \ = \underset{\ell \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}}{argmin}(|2\ell - 1.3667|) \ = \frac{2}{3}$$

Step 7: *Compare estimated symbol with transmitted symbol*
We know that we have $\hat{\ell}_2 = \frac{2}{3}$, which corresponds to estimating a symbol of 0.25. However, we transmitted $l_2 = \frac{1}{3}$ which means $b_2 = -0.25$, not +0.25, so the estimation is wrong. The error occurred to to ISI and the AWGN.

Communication model with random symbols (R8E8.4):

Setup:
We now consider a model where the symbols $\{b_n\}$ are random. This means that in this model our signal $s(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$ is *no longer deterministic*. Rather – it is a random process.

Assume that $\{b_n\}$ is a stationary discrete-time random process with $E[b_n] = \mu_b$, and $E[b_n b_{n-\ell}^*] = R_{bb}[\ell]$

Finding the Mean of our signal $s(t)$

$$E[s(t)] = E\left[\sum_{n=-\infty}^{\infty} b_n p(t-nT)\right] = \sum_{n=-\infty}^{\infty} E[b_n]p(t-nT) = \sum_{n=-\infty}^{\infty} \mu_b p(t-nT) = \mu_b \sum_{n=-\infty}^{\infty} p(t-nT)$$

Finding the Autocorrelation of our signal $s(t)$:

We know (given) that $R_{ss}(t;\tau) = E[s(t)s^*(t-\tau)]$

$$R_{ss}(t;\tau) = E[s(t)s^*(t-\tau)] = E\left[\sum_{n=-\infty}^{\infty} b_n p(t-nT) \sum_{m=-\infty}^{\infty} b_m^* p^*(t-\tau-mT)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[b_n b_m^*]p(t-nT)p^*(t-\tau=mT) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_{bb}[n-m]p(t-nT)p^*(t-\tau-mT)$$

$n-m=\ell$, therefore:

$$\sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_{bb}[1]p(t-(1+m)T)p^*(t-\tau-mT)$$

Determining if $s(t)$ is WSS and/or cyclo-stationary:

$s(t)$ is (generally) not stationary, since its mean and autocorrelation are time-dependent. Nevertheless, their dependence on time is periodic in T, and therefore $s(t)$ is Wide-Sense-Cyclo-Stationary.

Determining the mean ($\mu_s(t)$) for a rectangular input:

$$p(t) = ret\left(\frac{t}{T}\right) \rightarrow \mu_s(t) = E[s(t)] = \mu_b \sum_{n=-\infty}^{\infty} p(t-nT) = \mu_b$$

Matched Filter Exercise (2019B, Q1):

Setup:

A transmitter transmits a PCM signal given by $s(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT)$, where all $\{b_n\}$ take values in $\{0,1\}$, independently, and with equal probabilities. $p(t)$ is the pulse shape and $T > 0$ is a user selected constant.

The received signal is given by $x(t) = s(t) + n(t)$, where $n(t)$ is some zero-mean, stationary Gaussian noise, statistically independent of $\{b_n\}$

Part A:

Sub-Setup: $n(t)$ is white, with PSD 5, i.e $S_{nn}(\omega) = 5\ \forall\omega$. Assume first that only a single bit is transmitted, namely $s(t) = b_0 p(t)$.

The following decision rule (at the receiver) attains the smallest probability of error that can be achieved in deciding b_0 from $x(t)$: $\int_0^3 \tau \cdot x(3-\tau)d\tau \underset{\hat{b}_0=0}{\overset{\widehat{b_0}=1}{\gtrless}} \lambda$, where λ is a constant to be specified later.

A. Find and Sketch the Pulse shape $p(t)$ up to some (yet unknown) multiplicative constant c:

Step 1: *Realize, that since the decision rule is optimal, it must use a matched filter*

$$\int_0^3 \tau \cdot x(3-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t_0-\tau)d\tau$$

Where $h(\tau)$ must be the matched filter for white noise, namely $h(\tau) = c' \cdot p^*(t_0-\tau)$

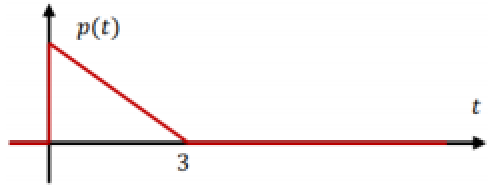
Step 2: *Determine the matched filter in τ, and extract $p(t)$ from this information*

t_0 must be 3 (argument of x in optimal decision),so $h(\tau) = c' \cdot p^*(t_0-\tau)$

In the integral $h(\tau) = \begin{cases} \tau & ;0 \le \tau \le 3 \\ 0\,; & \text{else} \end{cases}$

Therefore: $p(t) = c \cdot h^*(3-t) = c \cdot \begin{cases} 3-t\,; 0 \le t \le 3 \\ 0\ : \text{else} \end{cases}$, where $c = \frac{1}{c'} = \text{const}$

Step 3: *Sketch* $p(t)$



B. Given $\lambda = 18$, find the multiplicative constant c from earlier

Step 1: Find *the mean of the integral given* $b_0 = 1$

$$E\left[\int_0^3 \tau \cdot x(3-\tau)d\tau \mid b_0 = 1\right] = E\left[\int_0^3 \tau \cdot \big(1 \cdot p(3-\tau) + n(3-\tau)\big)d\tau\right] = \int_0^3 \tau \cdot p(3-\tau)d\tau + 0$$

Step 2: *Substitute* $p(t)$ *into the mean*

$$\int_0^3 \tau \cdot p(3-\tau)d\tau = \int_0^3 \tau \cdot c \cdot \tau d\tau = c \cdot \int_0^3 \tau^2 d\tau = c \cdot \left[\frac{\tau^3}{3}\right]_0^3 = 9c$$

Step 3: *Realize that the mean, given* $b_0 = 0$ *is zero, and find c:*

$$\lambda = \frac{1}{2} \cdot 9c = 18 \to c = 4$$

C. Finding the Probability of Error P_e in Terms of the Q Function:

Step 1: Find the standard deviation σ^2

$$\sigma^2 = N_0 \int_0^3 h^2(\tau)d\tau = 5\int_0^3 \tau^2 d\tau = 5\left[\frac{\tau^3}{3}\right]_0^3 = 5 \cdot 9 = 45$$

Step 2: Plug σ^2 into the Q function:

$$P_e = Q\left(\frac{\lambda}{\sigma}\right) = Q\left(\frac{18}{\sqrt{45}}\right) = Q\left(\frac{6}{\sqrt{5}}\right)$$

NOW, A SEQUENCE OF BITS $\{b_n\}$ IS TRANSMITTED

D. Find the value of T that would enable highest possible bit-rate transmission at the same probability of error P_e as found in the previous question, and without ISI at the receiver.

The double sided length of the convolution between $h(\tau)$ and $p(\tau)$ is 6, so the smallest possible T yielding highest bit rate would be $T = 3$

E. Formulate an expression for the Optimal Decision Rule (in the smallest probability of error sense) for deciding b_n from $x(t)$, assuming that T is set to the value you found in D.

The decision rule for b_n would be $\int_0^3 \tau \cdot x(3n+3-\tau)d\tau \underset{\hat{b}_0 = 0}{\overset{\widehat{b_0} = 1}{\gtrless}} 18$

Part B:

Sub-Setup:

Now assume that the noise is not white, but has a PSD $S_{nn}(\omega) = \frac{10}{2+\omega^2}$. Again, assume only a single bit is transmitted, i.e $s(t) = b_0 p(t)$

F. Does the decision rule described earlier still attain the smallest probability of error which can be attained in deciding b_0 from $x(t)$?

No, because the filter is no longer the matched filter

**G. Is the probability of error attained by the previous decision rule (in deciding b_0 from $x(t)$) larger/ smaller or equal to

P_e which we found in item C?

The error probability of the decision rule in (1) would be smaller than P_e from item c, because the spectral level of the new noise is weaker than the spectral level of the white noise in part A, at each ω (except $\omega = 0$, where they are equal).

$$S_{nn}(\omega) = \frac{10}{2+\omega^2} \leq 5 \ \ \forall\omega$$

Matched Filter Exercise (2018B, Q1):

Setup:

A system transmits a PCM signal given by $s(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT)$,
Where $\{b_n\}$ is a series of statistically independent symbols, each taking one of the values $\{0,1\}$ with equal probability. The pulse is given by $p(t) = u(t) - u(t-1)$, and $T > 0$ is a user-selected constant.
The received signal is given by $x(t) = s(t) + n(t)$, where $n(t)$ is some zero-mean, white Gaussian Noise, statistically independent of $\{b_n\}$, with PSD N_0, i.e $S_{nn}(\omega) = N_0 \ \ \forall\omega$
The receiver feed the received signal $x(t)$ to an LTI filter whose impulse response is $h(\tau) = e^{-\alpha t}u(t)$, and decides the values of $\{b_n\}$ based on the filter's output $y(t)$ as follows:

$$\hat{b}_n = \begin{cases} 1 \ ; \ y(nT+t_0) \geq \lambda \\ 0 \ ; \ y(nT+t_0) < \lambda \end{cases}$$

The parameters $\alpha > 0,\ t_0, \lambda$ are user-selected.
Assume first that only a single bit is transmitted, namely $s(t) = b_0 p(t)$.

A. Determine the mean of $y(t)$ given $b_0 = 0$, i.e determine $E[y(t)|b_0 = 0]$:

$$E[y(t)|b_0 = 0] = E[n(t)] = 0$$

B. Determine the variance of $y(t)$ given $b_0 = 0$, i.e determine $Var(y(t)|b_0 = 0)$:

$$\sigma^2 \triangleq Var(y(t)|b_0 = 0) = N_0 \int_{-\infty}^{\infty} h^2(\tau)d\tau = N_0 \int_0^{\infty} e^{-2\alpha\tau}d\tau = \frac{N_0}{2\alpha}$$

C. Determine the mean of $y(t)$ given $b_0 = 1$, i.e determine $E[y(t)|b_0 = 1]$. Sketch the result as a function of $t \in [-5.5]$, denoting important values on the axes.

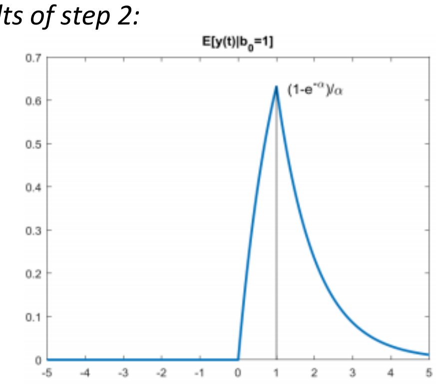
<u>Step 1:</u> *Find an expression for $\mu(t)$*

$$\mu(t) \triangleq E[y(t)|\ b_0 = 1] = \int_{-\infty}^{\infty} h(\tau)p(t-\tau)d\tau = \int_0^{\infty} e^{-\alpha\tau}p(t-\tau)d\tau$$

<u>Step 2:</u> *Determine $\mu(t)$ for each range of interest.*

For $t < 0$: $p(t-\tau) = 0$ throughout the integration range, so $\mu(t<0) = 0$
For $0 < t < 1$: $\mu(t) = \int_0^{\infty} e^{-\alpha\tau}p(t-\tau)d\tau = \int_0^t e^{-\alpha t}d\tau = \frac{1}{\alpha}(1-e^{-\alpha t})$
For $1 < t$: $\mu(t) = \int_0^{\infty} e^{-\alpha\tau}p(t-\tau)d\tau = \int_{t-1}^t e^{-\alpha\tau}d\tau = \frac{1}{\alpha}(e^{\alpha}-1)e^{-\alpha t}$

<u>Step 3:</u> *Sketch the mean according to results of step 2:*



D. What should be the optimal value of t_0 in order to enable attaining the minimal overall probability of error attainable by this receiver?

Since we want to maximize the ratio $\frac{\mu^2}{\sigma^2}$, and since σ^2 is independent of the sampling point, we should pick t_0 as the maximizer of $\mu(t)$, i.e $t_0 = 1$

E. Assuming that t_0 is set to the value you found in D, determine λ such that the probability of mis-detection (deciding $\hat{b}_0 = 0$ when $b_0 = 1$) equals the probability of false alarm (deciding $\hat{b}_0 = 1$ when $b_0 = 0$)

$$\mu = \frac{1}{\alpha}(1 - e^{-\alpha}), \quad thus \quad \lambda = \frac{\mu}{2} = \frac{1}{2\alpha}(1 - e^{-\alpha})$$

F. Express the resulting overall probability of error (using the values of t_0, λ as found earlier), in terms of the Q function.

$$P_{error} = Q\left(\frac{\lambda}{\sigma}\right) = Q\left(\frac{1 - e^{-\alpha}}{\sqrt{2\alpha N_0}}\right)$$

G. Can an optimal value of α be found for attaining the minimum overall probability of error attainable by this receiver?

Yes, by differentiating $\frac{1-e^{-\alpha}}{\sqrt{2\alpha N_0}}$ with respect to α, equating to zero, and verifying a maximum point.

H. Can an optimal value of α be found for attaining the minimum probability of error attainable in deciding b_0 from $x(t)$?

No, because the minimal attainable probability of error is only attainable by the matched filter, and no value of α can turn this filter into a matched filter.

I. Now assume a sequence of bits is transmitted. Are there any values of T that would guarantee no ISI in the decision process at the receiver? If so – find it.

No, because the convolution $p(t) * h(t)\ \ \forall\alpha$ is infinitely long and positive, and does not satisfy the NISI condition for any T.

Matched Filter Exercise (2017B, Q1):

Setup:

A system transmits a PCM Signal given by $s(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$

Where $\{b_n\}$ is a series of statistically independent symbols taking the values $\{0,2\}$ with equal probability.

The pulse is given by: $p(t) = \begin{cases} 1 - \frac{2t}{T} & ; 0 \le t \le T \\ 0 & : \text{else} \end{cases}$

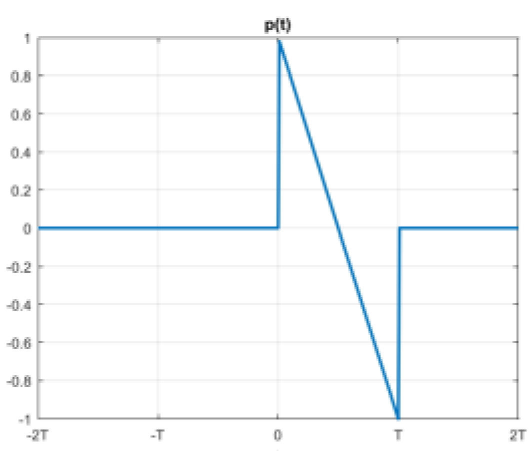
The received signal si given by $x(t) = s(t) + n(t)$, where $n(t)$ denotes white Gaussian noise with spectral level $S_{nn}(\omega) = N_0$.

The receiver applies an LTI filter with a user-selected impulse response $h(\tau)$ to the received signal $x(t)$, and samples the filter's output $y(t)$ at time instances $t_n = nT + 2T$ in order to decide the values of $\{b_n\}$ as follows:

$$\hat{b}_n = \begin{cases} 2 & ; \quad y(t_n) \ge \lambda \\ 0 & y(t_n) < \lambda \end{cases}$$

Where λ is a user selected constant.

A. Plot a sketch of the pulse $p(t)$ in the range $-2T \le t \le 2T$.



B. Find the impulse response $h(\tau)$ and the value of λ that would guarantee the smallest possible probability of error. Plot a sketch of this impulse response in $-2T \le \tau \le 2T$.

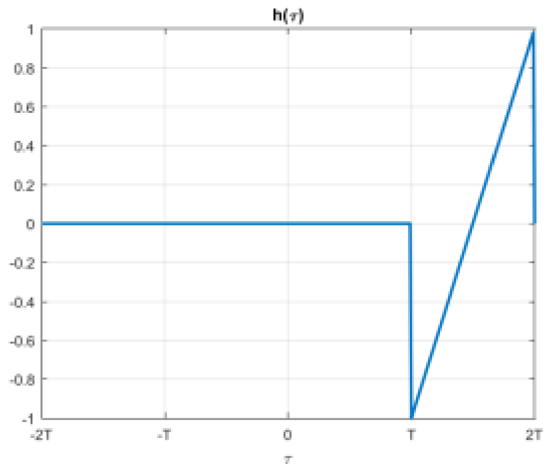
Step 1: *Determine* t_0

$$t_0 = 2T$$

Step 2: *Determine* $h(\tau) = p(t_0 - \tau)$

$$h(\tau) = p(2T-\tau) = \begin{cases} 1 - \dfrac{2(2T-\tau)}{T} & ; 0 \le 2T-\tau \le T \\ 0 \ ; \ \text{else} \end{cases} = \begin{cases} -3 + \dfrac{2\tau}{T} & ; T \le \tau \le 2T \\ 0 \ ; \text{else} \end{cases}$$

Step 3: *Sketch the result of step 2*



Step 4: *Find* $E[y(t_0)]$ *given that* $b_n = 0$ *or* $b_n = 2$

$$E[y(t_0)|b_0 = 0] = 0$$

$$E[y(t_0)|b_0 = 2] = \int_{-\infty}^{\infty} 2p(\tau)h(t_0-\tau)d\tau = 2\int_{-\infty}^{\infty} p^2(t)dt = 2\int_0^T \left(1 - \frac{2t}{T}\right)^2 dt = \frac{2\left(-\frac{T}{2}\right)}{3}\frac{1}{}\left[\left(1-\frac{2t}{T}\right)^3\right]_0^T = \frac{T}{3} + \frac{T}{3} = \frac{2T}{3}$$

Step 5: *Determine* λ

$$\lambda = \frac{1}{2}(E[y(t_0)|b_0 = 0] + E[y(t_0)|b_0 = 2]) = \frac{1}{2}\cdot\frac{2T}{3} = \frac{T}{3}$$

C. Would the system suffer from any ISI?

No, there will not be any ISI, because the convolution of $\tilde{p}(t) = p(t) * h(t)$ spans from T to 3T (centered around $t_0 = 2T$), and therefore satisfies $\tilde{p}(t + nT) = 0 \ \forall n \neq 0$, which is the NISI condition, but with a delay.

D. What is the resulting probability of error, in terms of the Q function?

In order to find the probability of error we must find the variance of $y(t_0)$ given each of the options, $b_0 = 0$, $b_0 = 2$. They are both the same,

$$\sigma^2 = Var(y(t_0)|b_0 = 2) = Var(y(t_0)|b_0 = 0) = N_0\int_{-\infty}^{\infty} h^2(\tau)d\tau = N_0\int_{-\infty}^{\infty} p^2(t)dt = \frac{N_0 T}{3}$$

Thus, $P_{error} = Q\left(\frac{\lambda}{\sigma}\right) = Q\left(\sqrt{\frac{T}{3N_0}}\right)$

Problem Update: Due to some malfunction in the transmitter, the transmitted pulse shapes are distorted, so that not

$p(t) = \begin{cases} 1 \ ; 0 \le t < 0.5T \\ -1 \ ; \ 0.5T \le t \le T \\ 0 \ ; \ \text{else} \end{cases}$. The receiver found in B remains unchanged.

E. Would the system suffer any ISI?

No, the delayed NISI condition is still satisfied, by the same reasoning as in C.

F. Would the resulting probability of error under these conditions be smaller/larger/equal to P_{error} found in D?
Since the system and the threshold value remain the same, $E[y(t_0)|b_0 = 0]$ remains zero. The false alarm probability remains the same. However, let's look at $E[y(t_0)|b_0 = 2]$:

$$E[y(t_0)|b_0 = 2] = \int_{-\infty}^{\infty} 2p(\tau)h(t_0 - \tau)d\tau = 2\left[\int_0^{0.5T}\left(1 - \frac{2t}{T}\right)dt - \int_{0.5T}^{T}\left(1 - \frac{2t}{T}\right)dt\right]$$
$$= 2\left(-\frac{T}{2}\right)\frac{1}{2}\left[\left(1 - \frac{2t}{T}\right)^2\right]_0^{0.5T} - 2\left(-\frac{T}{2}\right)\frac{1}{2}\left[\left(1 - \frac{2t}{T}\right)^2\right]_{0.5T}^{T} = 0 + \frac{T}{2} + \frac{T}{2} + 0 = T > \frac{2T}{3}$$

So the mis-detection probability is now smaller. Therefore, the overall probability of error is smaller.

Band-Pass Signals:

Band-Pass Signal Representations		
$v(t) = R(t)\cos\big(\omega_c t + \theta(t)\big)$	$v(t) = x(t)\cos(\omega_c t) - y(t)\sin(\omega_c t)$	$v(t) = \mathcal{R}e\{g(t)e^{j\omega_c t}\} = \frac{1}{2}g(t)e^{j\omega_c t} + \frac{1}{2}g^*(t)e^{-j\omega_c t}$

Where ω_C is the "carrier frequency"

Fourier Transform of Band pass Signal $v(t)$: $V(\omega) = \frac{1}{2}G(\omega - \omega_C) + \frac{1}{2}G^*(-\omega - \omega_c)$, where $G(\omega)$ is the FT of the complex envelope $g(t)$

Representation Function Relations	
Relation	Comment
$g(t) = x(t) + jy(t) = R(t)e^{j\theta(t)}$	Complex envelope of $v(t)$
$x(t) = \mathcal{R}e\{g(t)\} = R(t)\cos\big(\theta(t)\big)$	In phase component
$y(t) = \mathcal{I}m\{g(t)\} = R(t)\sin\big(\theta(t)\big)$	Quadrature Component
$R(t) = \lvert g(t)\rvert = \sqrt{x^2(t) + y^2(t)}$	Determines signal's power
$\theta(t) = \angle g(t) = \text{atan}(2\big(y(t), x(t)\big)$	

BandPass Signals Exercise (deterministic) (R10E10.1)

Setup: Consider the signal $v(t) = Asin(\omega_0 t)\cos\big(\omega_C t + m(t)\big)$, where $m(t) = 10\pi t$. We represent the signal $v(t)$ around the carrier frequency ω_c

A. Finding the Complex Envelope $g(t)$ of $v(t)$

Step 1: *Identify the representation of the BandPass Signal:*
We are given $v(t) = Asin(\omega_0 t)\cos\big(\omega_C t + m(t)\big)$
This is clearly in the format $v(t) = R(t)\cos\big(\omega_c t + \theta(t)\big)$, with $R(t) = Asin(\omega_0 t)$, and $\theta(t) = m(t) = 10\pi t$

Step2: *Write down the formula for $g(t)$, and plug in the known functions:*
$$g(t) = x(t) + jy(t) = R(t)e^{j\theta(t)} = Asin(\omega_0 t)e^{j\cdot 10\pi t}$$

B. Finding the In-Phase and Quadrature Components of $v(t)$:

Using the formulas, directly:
$$x(t) = \mathcal{R}e\{g(t)\} = Asin(\omega_0 t)\cos(10\pi t)$$
$$y(t) = \mathcal{I}m\{g(t)\} = Asin(\omega_0 t)\sin(10\pi t)$$

C. Finding the FT of the Complex Envelope $g(t)$ and the signal $v(t)$ $(G(\omega), V(\omega)$:
$$\mathcal{F}\{Asin(\omega_0 t)\} = \frac{A\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$
$$G(\omega) = \mathcal{F}\{Asin(\omega_0 t)e^{j10\pi t}\} = \mathcal{F}\{Asin(\omega_0 t)\}|_{\omega - 10\pi} = \frac{A\pi}{j}[\delta(\omega - 10\pi - \omega_0) - \delta(\omega - 10\pi + \omega_0)]$$
$$= jA\pi[\delta(\omega - 10\pi + \omega_0) - \delta(\omega - 10\pi - \omega_0)]$$

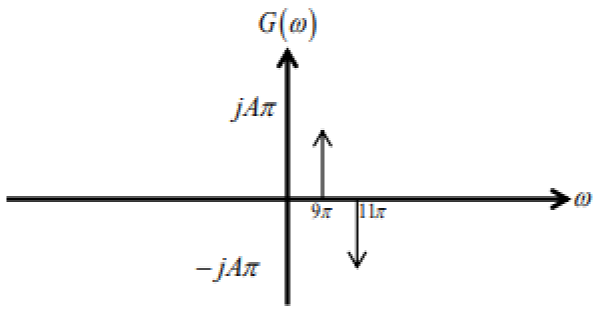
$$V(\omega) = \frac{1}{2}[G(\omega-\omega_c) + G^*(-\omega-\omega_c)]$$
$$= \frac{1}{2}(jA\pi[\delta(\omega-10\pi+\omega_0-\omega_C) - \delta(\omega-10\pi-\omega_0-\omega_c)] - jA\pi[\delta(-\omega-10\pi+\omega_0-\omega_c) - \delta(-\omega-10\pi-\omega_0-\omega_C)]) = \cdots$$
$$V(\omega) = \cdots = \frac{jA\pi}{2}(\delta(\omega-(10\pi-\omega_0+\omega_c)) - \delta\big(\omega-(10\pi+\omega_0+\omega_c)\big) - \delta\big(\omega-(-10\pi+\omega_0-\omega_c)\big) + \delta(\omega - (-10\pi-\omega_0-\omega_c)))$$

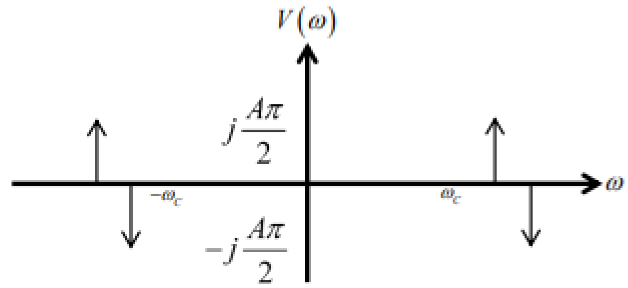
D. Drawing $G(\omega), V(\omega)$ for given parameters: $\omega_0 = \pi,\ A > 0,\ \omega_c \gg 10\pi$:

Step 1: *Plug in the now known values into the general expressions found earlier:*

$$G(\omega) = jA\pi[\delta(\omega-9\pi) - \delta(\omega-11\pi)]$$
$$V(\omega) = j\frac{A\pi}{2}\left[\delta\big(\omega-(9\pi+\omega_c)\big) - \delta\big(\omega-(11\pi+\omega_c)\big) - \delta\big(\omega-(-9\pi-\omega_c)\big) + \delta\big(\omega-(-11\pi-\omega_c)\big)\right]$$

Step 2: *Draw the graphs*





BandPass Exercise (from RANDOM SHITE) (REC10E10.2):

Setup: Consider a complex-valued random process $g(t)$ with mean $\tilde{\eta}_g(t)$, autocorrelation function $\tilde{R}_{gg}(t_1,t_2)$, and pseudo autocorrelation function $\tilde{R}_{gg^*}(t_1,t_2)$

Define the BP signal $v(t) = \mathcal{R}e\{g(t)e^{j\omega_c t}\}$

A. Compute the mean and autocorrelation function of the BP signal $v(t)$

Step 1: *Find the mean:*

The mean is given by $\tilde{\eta}_v(t) = E[v(t)] = E\left[\mathcal{R}e\{g(t)e^{j\omega_c t}\}\right] = E\left[\frac{1}{2}g(t)e^{j\omega_c t} + \frac{1}{2}g^*(t)e^{-j\omega_c t}\right] = \frac{1}{2}E[g(t)]e^{j\omega_c t} + \frac{1}{2}E[g^*(t)]e^{-j\omega_c t} = \frac{1}{2}\tilde{\eta}_g(t)e^{j\omega c t} + \frac{1}{2}\tilde{\eta}_g^*(t)e^{-j\omega_c t} = \mathcal{R}e\{\tilde{\eta}_g(t)e^{j\omega c t}\}$

Step 2: *Find the autocorrelation*

$$\tilde{R}_{vv}(t_1,t_2) = E[v(t_1)v^*(t_2)] = E\left[\frac{1}{2}\left(g(t_1)e^{j\omega_c t_1} + g^*(t_1)e^{-j\omega_c t_1}\right)\frac{1}{2}\left(g^*(t_2)e^{-j\omega_c t_2} + g(t_2)e^{j\omega_c t_2}\right)\right]$$
$$= \frac{1}{4}\left[\tilde{R}_{gg}(t_1,t_2)e^{j\omega_c(t_1-t_2)} + \tilde{R}_{gg^*}(t_1,t_2)e^{j\omega_c(t_1+t_2)} + \tilde{R}^*_{gg^*}(t_1,t_2)e^{-j\omega_c(t_1+t_2)} + \tilde{R}^*_{gg}(t_1,t_2)e^{-j\omega_c(t_1-t_2)}\right]$$
$$= \frac{1}{2}\mathcal{R}e\{\tilde{R}_{gg}(t_1,t_2)e^{j\omega_c(t_1-t_2)} + \tilde{R}_{gg^*}(t_1,t_2)e^{j\omega_c(t_1+t_2)}\}$$

B. Is $v(t)$ necessarily a WSS Process?

Step 1: *Check if the mean is time-independent*

$\tilde{\eta}_v(t) = \mathcal{R}e\{\tilde{\eta}_g(t)e^{j\omega_c t}\} = \mathcal{R}e\{0\cdot e^{j\omega_c t}\} = 0 = \eta_v$ – time independent, satisfies WSS

Step 2: *Check if the autocorrelation is time independent*

$$\tilde{R}_{vv}(t_1,t_2) = \frac{1}{2}\mathcal{R}e\{R_{gg}(t_1-t_2)e^{j\omega_c(t_1-t_2)} + \tilde{R}_{gg^*}(t_1,t_2)e^{j\omega_c(t_1+t_2)}\}$$

This is not necessarily time-independent, and therefore $v(t)$ is not necessarily a WSS process.

C. Determine whether $v(t)$ is WSS if $g(t)$ is proper

Since $g(t)$ is proper, this means that $\tilde{R}_{gg^*}(t_1, t_2) = 0$, and this means that

$$\tilde{R}_{vv}(t_1, t_2) = \frac{1}{2}\mathcal{R}e\left\{R_{gg}(t_1 - t_2)e^{j\omega_c(t_1-t_2)} + \underbrace{\tilde{R}_{gg^*}(t_1, t_2)e^{j\omega_c(t_1+t_2)}}_{0}\right\} = \frac{1}{2}\mathcal{R}e\{R_{gg}(t_1 - t_2)e^{j\omega_c(t_1-t_2)}\} = R_{vv}(t_1 - t_2)$$

Therefore, the BP signal $v(t)$ is WSS, because neither the mean nor the autocorrelation depend on time.

D. Compute the spectrum of $v(t)$, given that $g(t)$ is proper

We know that $R_{vv}(\tau) = \frac{1}{2}\mathcal{R}e\{R_{gg}(\tau)e^{j\omega_c\tau}\} = \frac{1}{2}R_{gg}(\tau)\cos(\omega_c\tau)$

And therefore $S_{vv}(\omega) = \frac{1}{4}\left(S_{gg}(\omega - \omega_C) + S_{gg}(-\omega - \omega_c)\right)$

Band Pass Signal with FT stuff (R10E10.4)

Setup: We have an input signal $x(t) = Asin(2\pi f_1 t)\cos(2\pi f_c t) + Bcos(2\pi f_2 t)\sin(2\pi f_c t)$

A. Find $\widetilde{G}_x(f)$, the FT of $g_x(t)$, the complex envelope of the signal x(t)

Step 1: *Identify the format of the input representation:*

The input is represented in the form: $v(t) = x(t)\cos(\omega_c t) - y(t)\sin(\omega_c t)$

The components are therefore:

$$x_{repr}(t) = Asin(2\pi f_1 t)\ \ ; y_{repr}(t) = Bcos(2\pi f_1 t)\ \ ; \omega_c = 2\pi f_c$$

Step 2: *Identify/construct the formula for the quantity of interest:*

$$g(t) = x(t) + jy(t)$$

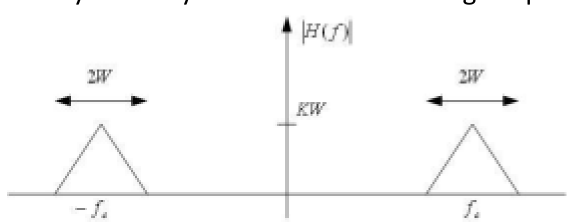
Step 3: *Evaluate the quantity of interest*:

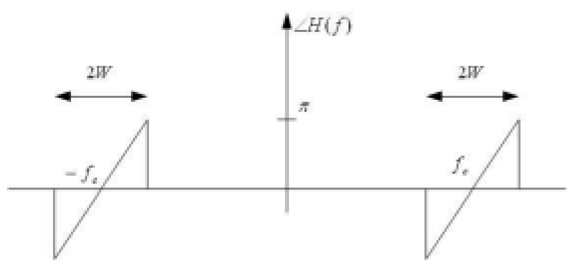
$$g_x(t) = x_{repr}(t) + jy_{repr}(t) = Asin(2\pi f_1 t) - jBcos(2\pi f_1 t)$$

Step 4: *Take the Fourier Transform:*

$$\tilde{G}_x(f) = \mathcal{F}\{g(t)\} = \frac{A}{2j}\big(\delta(f - f_1) - \delta(f + f_1)\big) - \frac{jB}{2}\big(\delta(f - f_2) + \delta(f + f_2)\big)$$

Sub-Setup: Now, the signal $x(t)$ is filtered by an LTI system with the following frequency response:





And it is given that $f_1, f_2 < W < f_c$

B. Find $\widetilde{G}_h(f)$, the FT of $g_h(t)$, the complex envelope of $h(t) = \mathcal{F}^{-1}\{\widetilde{H}(f)\}$:

$$g_h(t) = h_+(t)e^{-j2\pi f_c t}$$

$$\widetilde{G_h}(f) = 2KW\left(1 - \frac{|f|}{W}\right)e^{j\frac{\pi f}{W}}$$

C. Compute the output signal $y(t)$ and its complex envelope $g_y(t)$

$$\tilde{G}_y(f) = \frac{1}{2}\tilde{G}_x(f)\tilde{G}_h(f) = KW\left(1 - \frac{|f|}{W}\right)e^{j\frac{\pi f}{W}}\left[\frac{A}{2j}\big(\delta(f - f_1) - \delta(f + f_1)\big) - \frac{jB}{2}\big(\delta(f - f_2) + \delta(f + f_2)\big)\right]$$

$$g_y(t) = \mathcal{F}^{-1}\{\tilde{G}_y(f)\} = KA(W - f_1)\sin\left(2\pi f_1 t + \frac{\pi}{W}f_1\right) - jKB(W - f_2)\cos\left(2\pi f_2 t + \frac{\pi}{W}f_2\right)$$

$$\rightarrow y(t) = \mathcal{R}e\{g_y(t)e^{j2\pi f_c t}\} = KA(W - f_1)\sin\left(2\pi f_1 t + \frac{\pi}{W}f_1\right)\cos(2\pi f_c t) + KB(W - f_2)\sin\left(2\pi f_2 t + \frac{\pi}{W}f_2\right)\cos(2\pi f_c t)$$

Noise:

Noise Figure: $F \triangleq \frac{SNR_{in}}{SNR_{out}} = 1 + \frac{N_\ell}{N_{in}} > 1$, where $SNR_{in} = \frac{S_{in}}{N_{in}}$; $SNR_{out} = \frac{S_{out}}{N_{out}} = \frac{GS_{in}}{G(N_{in}+N_\ell)} = \frac{S_{in}}{N_{in}+N\_\ell}$

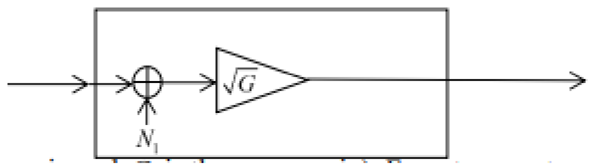
Usually, noise figure is given for the standard temperature of $T_0 = 290°K$, i.e $F = 1 + \frac{T_\ell}{T_0}$

For attenuating devices: $G = \frac{1}{L} < 1\,, \;\; F = L$

Cascaded Systems: For a system comprised of m cascaded devices, noise figure is $\bar{F} = F_1 + \frac{F_2-1}{G_1} + \frac{F_3-1}{G_1 G_2} + \cdots + \frac{F_m-1}{G_1 G_2 \cdot \ldots \cdot G_{m-1}}$

Noise Figure Exercise (R11E11.1):

Setup: Consider the following amplifier



Where $G = 10$ ($\sqrt{G} = Gain,\; G = PowerGain$) . For a temperature of T_0, the input noise power spectral level is N_0, and in this temperature $N_\ell = 3N_0$

A. Finding the Noise Figure of the Amplifier in dB:

Plug into the NF equations: $F_{amp} = 1 + \frac{N_\ell}{N_0} = 1 + \frac{3N_0}{N_0} = 1 + 3 = 4 = 2^2 \rightarrow F_{amp} = 6[dB]$

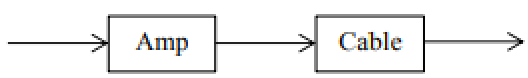
B. Finding the Noise Power at the Amplifier Output when there is no input signal (i.e only noise is input)

The noise power at the amplifier's output is given by $P_{out} = G(N_0 + N_\ell) = 4GN_0 = 40N_0$

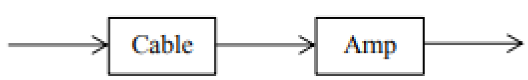
Sub-Setup: An engineer wants to minimize the NF of a system comprised of an amplifier and a cable. The cable is a passive device with attenuation of $16dB$.

C. Which of the following options should the engineer choose, given the cable defined in sub-setup?

Scheme A:



Scheme B:



1

<u>Step 1:</u> *Compute the gain and noise figure of the cable:*

$$L_{cable} = 16[dB] \rightarrow F_{cable} = 16[dB], \qquad G_{cable} = -16dB = -(10+3+3)dB \rightarrow G_{cable} = \frac{1}{10 \cdot 2^2} = \frac{1}{40}$$

<u>Step 2:</u> *Compute the equivalent noise figure for each option:*

Scheme A: Amp into cable: $\bar{F}_A = F_{amp} + \frac{F_{cable}-1}{G_{amp}} = 4 + \frac{40-1}{10} = 4 + 3.9 = 7.9 \rightarrow \bar{F}_A = 9dB$

Scheme B: Cable into amp: $\bar{F}_B = F_{Cable} + \frac{F_{amp}-1}{G_{cable}} = 40 + \frac{4-1}{\frac{1}{40}} = 40 + 3 \cdot 40 = 160 = 10 \cdot 2^4 \rightarrow \bar{F}_B \approx 10 + 4 \cdot 3 = 22dB$

<u>Step 3:</u> *Compare equivalent NFs and make a decision*:

$\bar{F}_A < \bar{F}_B$ →choose scheme A

D. In what temperature, the system described in scheme B will have an identical noise figure to the one in scheme A for a temperature of T_0?

Step 1: *Define the new temperature in terms of the old one:*

$$T_{new} = \alpha T_0$$

Step 2: *Find an equation for the equivalent temperature of scheme B:*

In general we have $F = 1 + \frac{T_\ell}{T_0} \to T_\ell = (F-1)T_0$, which in our case, for scheme B is: $\bar{T}_\ell = (\bar{F}_B - 1)T_0 = 159T_0$

Step 3: *Calculate the new equivalent noise figure for scheme B:*

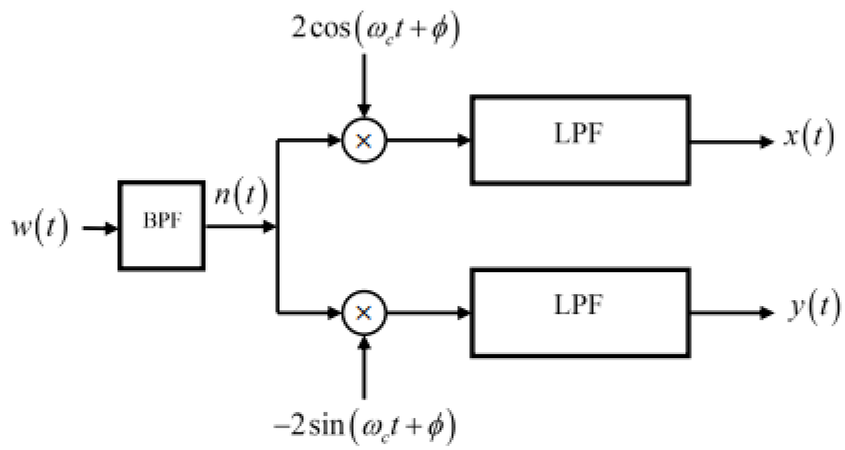
$$\bar{F}_B = 1 + \frac{\bar{T}_\ell}{T_{new}} = 1 + \frac{159T_0}{\alpha T_0} = 1 + \frac{159}{\alpha}$$

Step 4: *Enforce the demand $\bar{F}_B \equiv \bar{F}_A$, and find T_{new}*

$$7.9 = 1 + \frac{159}{\alpha} \to \alpha = \frac{159}{6.9} \approx 23.04 \to T_{new} = 23.04T_0$$

Noise Figure of Real Systems Exercise (R11E11.2):

Setup: Consider the following system



The spectra and cross-spectrum of $x(t), y(t)$ are given:

$$S_{xx}(\omega) = S_{yy}(\omega) = \begin{cases} 1+2|\omega| & ; |\omega| \le 0.5 \\ 3-2|\omega| & ; 0.5 \le |\omega| \le 1 \\ 0 & ; \text{else} \end{cases} \quad ; \quad S_{xy}(\omega) = j \cdot sign(\omega) \cdot \begin{cases} -1+2|\omega| & ; |\omega| \le 0.5 \\ +1-2|\omega| & ; \ 0.5 \le |\omega| \le 1 \\ 0 & ; \ \text{else} \end{cases}$$

A. Find $S_{nn}(\omega)$, the spectrum of $n(t)$

Step 1: *Recall a result from the lecture:*

$$S_{nn}(\omega) = P(\omega-\omega_c) + P(-\omega-\omega_C) = \{P(\omega) = \bar{P}(-\omega)\} = P(\omega-\omega_C) + \bar{P}(\omega+\omega_C)$$
$$S_{xx}(\omega) = S_{yy}(\omega) = P(\omega) + \bar{P}(\omega) = P(\omega) + P(-\omega)$$
$$S_{xy}(\omega) = j \cdot \left(P(\omega) - \bar{P}(\omega)\right) = j \cdot \left(P(\omega) - P(-\omega)\right)$$

Step 2: *Recognize that in order to find $S_{nn}(\omega)$, it is sufficient to find $P(\omega)$. Write down the equation for $P(\omega)$*

$$P(\omega) = \frac{1}{2}\left(S_{xx}(\omega) - j \cdot S_{xy}(\omega)\right) = \frac{1}{2}\left(P(\omega) + \bar{P}(\omega) + \left(P(\omega) - \bar{P}(\omega)\right)\right)$$

Step 3: *Compute $P(\omega)$*

We know $S_{xx}(\omega), S_{xy}(\omega)$. Plug them into $P(\omega)$ and evaluate. Not showing the math cause its trivial

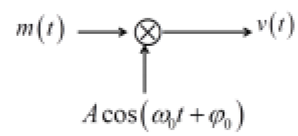
Step 4: *Obtain $S_{nn}(\omega)$ by substituting result of step 3 into $S_{nn}(\omega) = P(\omega-\omega_c) + P(-\omega-\omega_c)$*

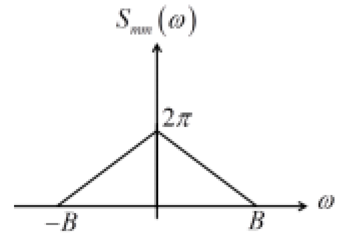
Double-Sideband Suppressed Carrier (DSB-SC)

DSB-SC: Double Sideband Suppressed Carrier Transmission, is transmission in which frequencies produce by amplitude modulation are symmetrically spaced above and below the carrier frequency and the carrier level is reduced to the lowest practical level, ideally being completely suppressed.

DSB-SC Exercise (R12E12.1):

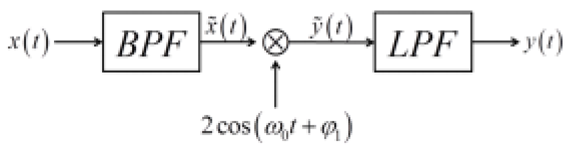
Setup: Consider a DSB-SC transmitter. The information signal $m(t)$ is a zero-mean WSS process, with spectrum $S_{mm}(\omega)$ and autocorrelation function $R_{mm}(\tau)$. The transmitter transmits the signal to the receiver, but the receiver receives two appearances of the signal – the first arrives directly while the second arrives attenuated with a delay (this can happen when the receiver sees the actual signal and its reflection from a longer path) .





The received signal is $x(t) = v(t) + \frac{1}{2}v(t-\Delta) + w(t)$, where $w(t)$ is additive noise with spectral density level $\frac{N_0}{2}$, independent of $v(t)$. Also assume $B \ll \omega_0$

The receiver is built as follows:



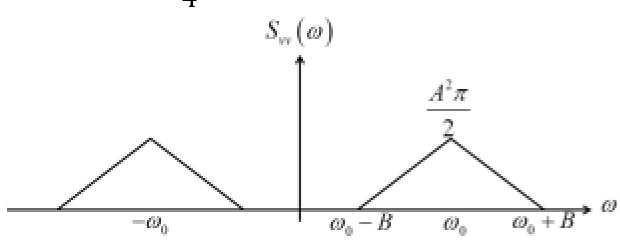
In addition, it is given that $H_{BPF}(\omega) = \begin{cases} 1 & ; |\omega| \in [\omega_0 - B,\ \omega_0 + B] \\ & 0\ ; \text{else} \end{cases}$; $H_{LPF}(\omega) = \begin{cases} 1 & ;\ |\omega| \le B \\ 0 & ;\ \text{else} \end{cases}$

A. Find the spectrum of $v(t)$:

$$(t) = Am(t) \cdot \cos(\omega_0 t + \varphi_0)$$
$$\rightarrow \tilde{R}_{vv}(t;\tau) = E[v(t+\tau)v^*(t)] = E\big[E[v(t+\tau)v(t)|\varphi_0]\big]$$
$$E[v(t+\tau)v(t)|\varphi_0] = E[Am(t+\tau)\cos(\omega_0 t + \omega_0\tau + \varphi_0)\, Am(t)\cos(\omega_0 t + \varphi_0)\,|\varphi_0]$$
$$= A^2\cos(\omega_0 t + \omega_0\tau + \varphi_0)\cos(\omega_0 t + \varphi_0)\,E[m(t+\tau)m(t)]$$
$$= \frac{A^2}{2}[\cos(2\omega_0 t + \omega_0\tau + 2\varphi_0) + \cos(\omega_0\tau)]R_{mm}(\tau)$$

$$\rightarrow \tilde{R}_{vv}(t;\tau) = E\big[E[v(t+\tau)v(t)|\varphi_0]\big] = E\left[\frac{A^2}{2}[\cos(2\omega_0 t + \omega_0\tau + 2\varphi_0) + \cos(\omega_0\tau)]R_{mm}(\tau)\right]$$
$$= \frac{A^2}{2}\cos(\omega_0\tau)\,R_{mm}(\tau) + \frac{A^2}{2}R_{mm}(\tau)E[\cos(2\omega_0 t + \omega_0\tau + 2\varphi_0)] = \frac{A^2}{2}\cos(\omega_0\tau)\,R_{mm}(\tau) \triangleq R_{vv}(\tau)$$

$$\rightarrow S_{vv}(\omega) = \frac{A^2}{4}[S_{mm}(\omega-\omega_0) + S_{mm}(\omega+\omega_0)]$$



B. Find the spectrum of $x(t)$

$$x(t) = v(t) + \frac{1}{2}v(t-\Delta) + w(t) = \tilde{v}(t) + w(t)$$

$$R_{xx}(\tau) = E[x(t+\tau)x(t)] = E\big[\big(\tilde{v}(t+\tau) + w(t+\tau)\big)\big(\tilde{v}(t) + w(t)\big)\big]$$
$$= R_{\tilde{v}\tilde{v}}(\tau) + \underbrace{E[\tilde{v}(t+\tau)w(t)]}_{0} + \underbrace{E[w(t+\tau)\tilde{v}(t)]}_{0} + R_{ww}(\tau) = R_{\tilde{v}\tilde{v}}(\tau) + R_{ww}(\tau)$$

$$\rightarrow S_{xx}(\omega) = S_{\tilde{v}\tilde{v}}(\omega) + S_{ww}(\omega)$$

$$R_{\tilde{v}\tilde{v}}(\tau) = E[\tilde{v}(t+\tau)\tilde{v}(t)] = E\left[\left(v(t+\tau) + \frac{1}{2}v(t+\tau-\Delta)\right)\left(v(t) + \frac{1}{2}v(t-\Delta)\right)\right]$$
$$= E\left[v(t+\tau)v(t) + \frac{1}{2}v(t+\tau)v(t-\Delta) + \frac{1}{2}v(t+\tau-\Delta)v(t) + \frac{1}{4}v(t+\tau-\Delta)v(t-\Delta)\right]$$
$$= R_{vv}(\tau) + \frac{1}{2}R_{vv}(\tau+\Delta) + \frac{1}{2}R_{vv}(\tau-\Delta) + \frac{1}{4}R_{vv}(\tau)$$

$$\rightarrow S_{\tilde{v}\tilde{v}}(\omega) = 1.25S_{vv}(\omega) + S_{vv}(\omega)\cos(\Delta\omega) = \frac{A^2}{4}(1.25 + \cos(\Delta\omega))[S_{mm}(\omega-\omega_0) + S_{mm}(\omega+\omega_0)]$$
$$\rightarrow S_{xx}(\omega) = \frac{N_0}{2} + \frac{A^2}{4}(1.25 + \cos(\Delta\omega))[S_{mm}(\omega-\omega_0) + S_{mm}(\omega+\omega_0)]$$

C. Find the spectrum of $\tilde{x}(t)$:

$$S_{\tilde{x}\tilde{x}}(\omega) = |H_{BPF}(\omega)|^2 S_{xx}(\omega) = \begin{cases} S_{xx}(\omega) & : \ |\omega| \in [\omega_0 - B, \omega_0 + B] \\ 0 & ; \text{ else} \end{cases}$$
$$= \begin{cases} \frac{N_0}{2} + \frac{A^2}{4}(1.25 + \cos(\Delta\omega))[S_{mm}(\omega-\omega_0) + S_{mm}(\omega+\omega_0)] & ; \ |\omega| \in [\omega_0 - B, \omega_0 + B] \\ 0 & ; \text{ else} \end{cases}$$

D. Find the SNR at the output of the BPF, given the following information:

Sub-Setup: It is given that $\Delta = 0.3\,[msec]$, $B = 5\pi\left[\frac{rad}{\text{sec}}\right]$, $\omega_0 = 1000\pi\left[\frac{rad}{\text{sec}}\right]$, $N_0 = 10^{-3}\left[\frac{W}{Hz\times rad}\right]$, $A = 2$

Step 1: *To find the SNR at the output of the BPF, we need to know P_{signal} and P_{noise}. Find them.*

$$P_{noise} = P_{\tilde{w}} = 2\cdot\frac{N_0}{2}\cdot 2B = 2N_0B = \pi\cdot 10^{-2}[W]$$
$$P_{signal} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{signal}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A^2}{4}(1.25 + \cos(\Delta\omega))[S_{mm}(\omega-\omega_0) + S_{mm}(\omega+\omega_0)]d\omega$$
$$= \frac{1}{2\pi}\frac{A^2}{4}\int_{-\infty}^{\infty}\big(1.25S_{mm}(\omega-\omega_0) + 1.25S_{mm}(\omega+\omega_0) + \cos(\Delta\omega)S_{mm}(\omega-\omega_0) + \cos(\Delta\omega)S_{mm}(\omega+\omega_0)\big)d\omega$$
$$= \frac{A^2}{8\pi}\left[\int_{-\infty}^{\infty} 1.25\left(\int_{-\infty}^{\infty}S_{mm}(\omega-\omega_0)d\omega + \int_{-\infty}^{\infty}S_{mm}(\omega+\omega_0)d\omega\right) + \int_{-\infty}^{\infty}\cos(\Delta\omega)S_{mm}(\omega-\omega_0)d\omega \right.$$
$$\left. + \int_{-\infty}^{\infty}\cos(\Delta\omega)S_{mm}(\omega+\omega_0)d\omega\right]$$
$$= \frac{A^2}{8\pi}\left[2.5\int_{-\infty}^{\infty}S_{mm}(\omega)d\omega + \int_{-\infty}^{\infty}\cos(\Delta\omega)S_{mm}(\omega-\omega_0)d\omega + \int_{-\infty}^{\infty}\cos(\Delta\omega)S_{mm}(\omega+\omega_0)d\omega\right]$$

Define $\alpha = \omega - \omega_0$; $\beta = \omega + \omega_0$

We solve each integral separately:

$$\int_{-\infty}^{\infty}S_{mm}(\omega)d\omega = \frac{2B\cdot 2\pi}{2} = 2\pi B = 10\pi^2$$
$$\int_{-\infty}^{\infty}\cos(\Delta\alpha + \Delta\omega_0)S_{mm}(\alpha)d\alpha = \cdots = 5.878$$
$$\int_{-\infty}^{\infty}\cos(\Delta\beta - \Delta\omega_0)S_{mm}(\beta)d\beta = \cdots = 5.878$$

$$\rightarrow P_{signal} = \frac{4}{8\pi}[2.5\cdot 10\pi^2 + 2\cdot 5.878] = 41.141$$

Step 2: *Compute the SNR:*

$$SNR = \frac{P_{Signal}}{P_{noise}} = \frac{41.141}{\pi\cdot 10^{-2}} = 1310 \approx 31.17[dB]$$

E. Find the required Phase Difference $\Delta\varphi = \varphi_0 - \varphi_1$, such that the dominant term stays (i.e $\neq 0$), and the term related to the reflection vanishes.

Step 1: *Examine the dominant term $v(t)$:*

We follow $v(t)$'s path through the receiver. The BPF doesn't change it. The multiplier does change it:
After the multiplier we have:
$\tilde{y}_v(t) = 2\cos(\omega_0 t + \varphi_1)\, v(t) = 2Am(t)\cos(\omega_0 t + \varphi_0)\cos(\omega_0 t + \varphi_1) = Am(t)[\cos(2\omega_0 t + \varphi_0 + \varphi_1) + \cos(\varphi_0 - \varphi_1)]$
The LPF also changes it. After the LPF:

$$y_v(t) = Acos(\varphi_0 - \varphi_1)m(t) = Acos(\Delta\varphi)m(t)$$

Step 2: *Identify a condition which would keep the dominant term nonzero:*
Demand $\Delta\varphi = \varphi_0 - \varphi \neq \frac{\pi}{2} + \pi k$. If this demand is met then the dominant term does not vanish

Step 3: *Examine the reflection-caused term :*
Again, we follow the signal path. The BPF doesn't change it, but the multiplier does.
After the multiplier:

$$\tilde{y}_{v^*}(t) = 2\cos(\omega_0 t + \varphi_1)\frac{1}{2}v(t - \Delta) = Am(t - \Delta)\cos(\omega_0(t - \Delta) + \varphi_0)\cos(\omega_0 t + \varphi_1)$$
$$= \frac{1}{2}Am(t - \Delta)[\cos(2\omega_0 t - \omega_0\Delta + \varphi_0 + \varphi_1) + \cos(-\omega_0\Delta + \varphi_0 - \varphi_1)]$$

And after the LPF:

$$y_{v^*}(t) = \frac{1}{2}Acos(-\omega_0\Delta + \varphi_0 - \varphi_1)m(t - \Delta) = \frac{1}{2}Acos(-\omega_0\Delta + \Delta\varphi)m(t - \Delta)$$

Step 4: *Find a condition that would make the reflection term vanish:*
We demand $-\omega_0\Delta + \Delta\varphi = \frac{\pi}{2} + \pi k \rightarrow \Delta\varphi = \omega_0\Delta + \frac{\pi}{2} + \pi k$

Step 5: *Reconcile the conditions found in Step 2 and Step 4:*
To have the reflection term vanish, we need $\Delta\varphi = \omega_0\Delta + \frac{\pi}{2} + \pi k$, but at the same time we want $\Delta\varphi = \varphi_0 - \varphi \neq \frac{\pi}{2} + \pi k$
Plugging in known numbers:

$$\Delta\varphi = \varphi_0 - \varphi_1 = \frac{\pi}{2} + \pi k + \omega_0\Delta = \frac{\pi}{2} + \pi k + 0.3\pi$$
$$\rightarrow \Delta\varphi = -0.2\pi(k = -1) \quad OR \quad \Delta\varphi = 0.8\pi(k = 0)$$

F. $\boldsymbol{\varphi_1}$ is now set as $\boldsymbol{\varphi_1 = \varphi_0 - \Delta\varphi}$, where $\boldsymbol{\Delta\varphi}$ is the (positive phase difference) value found in E. Notice that $\boldsymbol{\varphi_1}$ is an RV. Find the SNR at the output of the LPF.

Step 1: Define the signal at the output of the LPF:

$$y(t) = \underbrace{Acos(\varphi_0 - \varphi_1)m(t) +}_{y_s(t)} n(t)$$

Where
$n(t)$ – noise term
Further denote:
$w(t)$ – the channel's noise
$\tilde{w}(t)$ – the output of the BPF for the input $w(t)$ (just noise)
$\tilde{n}(t) -$ $\tilde{w}(t)$ after the multiplier
$n(t)$ – the output of the LPF for the input $\tilde{n}(t)$

Step 2: *Find the Signal's Power:*

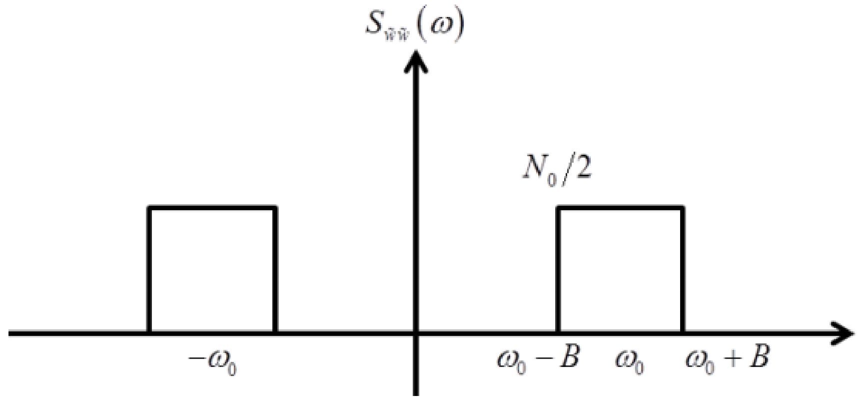
$$P_{signal} = R_{y_s y_s}(0) = E\left[\left(Acos(\varphi_0 - \varphi_1)m(t)\right)^2\right] = A^2\cos^2(\Delta\varphi)R_{mm}(0)$$
$$= A^2\cos^2(\Delta\varphi)\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{mm}(\omega)d\omega = A^2\cos^2(\Delta\varphi)\frac{1}{2\pi}\cdot\frac{2B\cdot 2\pi}{2}$$
$$= A^2Bcos^2(\Delta\varphi) = 2^2\cdot 5\pi\cos^2(0.8\pi) = 41.12$$

Step 3: *Find the Noise Power:*

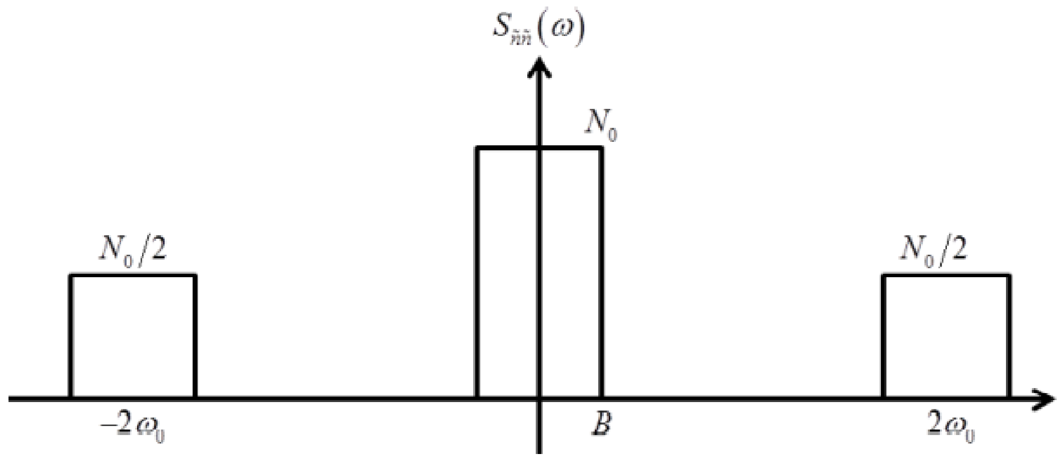
$$\tilde{n}(t) = 2\cos(\omega_0 t + \varphi_1)\,\tilde{w}(t)$$
$$R_{\tilde{n}\tilde{n}}(\tau) = E[\tilde{n}(t + \tau)\tilde{n}(t)] = E\left[\left(2\cos(\omega_0(t + \tau) + \varphi_1)\,\tilde{w}(t + \tau)\right)\left(2\cos(\omega_0 t + \varphi_1)\,\tilde{w}(t)\right)\right]$$
$$= 4E[\cos(\omega_0(t + \tau) + \varphi_1)\,cos(\omega_0 t + \varphi_1)]E[\tilde{w}(t + \tau)\tilde{w}(t)]$$
$$= 2E[\cos(2\omega_0 t + \omega_0\tau + 2\varphi_1) + \cos(\omega_0\tau)]R_{\tilde{w}\tilde{w}}(\tau) = \{E[\cos(2\omega_0 t + \omega_0\tau + 2\varphi_1)] = 0\}$$
$$= 2\cos(\omega_0\tau)\,R_{\tilde{w}\tilde{w}}(\tau)$$

$$\rightarrow S_{\tilde{n}\tilde{n}}(\omega) = S_{\widetilde{w}\widetilde{w}}(\omega - \omega_0) + S_{\widetilde{w}\widetilde{w}}(\omega + \omega_0)$$

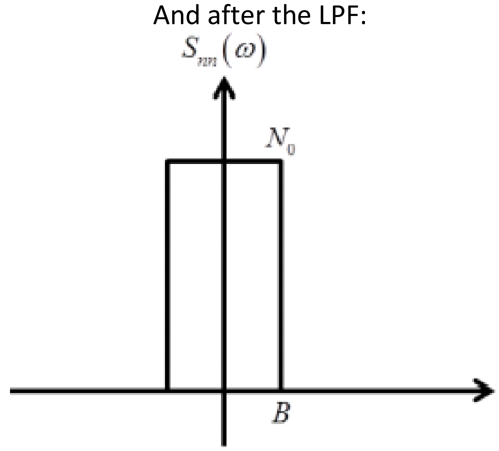
Recall that



Therefore:



And after the LPF:



And finally:

$$P_{noise} = R_{nn}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{nn}(\omega)d\omega = \frac{1}{2\pi}\cdot N_0 2B = \frac{N_0 B}{\pi} = 0.005\ [W]$$

<u>Step 4</u>: *Compute the SNR:*

$$SNR = \frac{P_{signal}}{P_{noise}} = \frac{41.12}{0.005} = 8224 \approx 39.14[dB]$$

<u>Single Side Band Transmission</u>

Hilbert Filter: $H_{Hilbert}(\omega) = -j \cdot sign(\omega) \underset{\mathcal{F}}{\leftrightarrow} h_{Hilbert}(t) = \frac{1}{\pi t}$

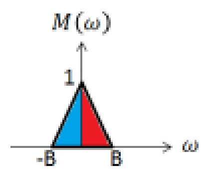
Side Band Transmission: A real-valued information signal $m(t) \in \mathbb{R}$ with an FT $M(\omega)$ may be transmitted by using only the positive frequency content of $M(\omega)$, by transmitting the Single-Side Band (SSB) Upper Side Band (USB) Signal:

$$v_{USB}(t) = A[m(t)\cos(\omega_0 t) - \widehat{m}(t)\sin(\omega_0 t)]$$

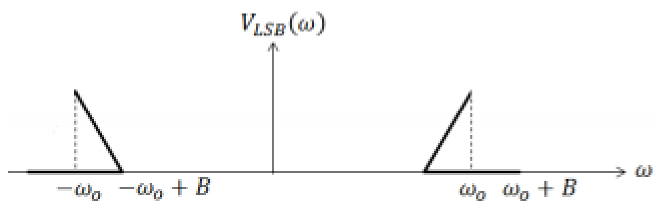
Where $\widehat{m}(t)$ denotes the milbert transform of $m(t)$, i.e $\widehat{m}(t) = \mathcal{F}^{-1}\{M(\omega)H_{Hilbert}(\omega)\}$, and ω_0 =carrier frequency.

SSB Lower Side Band (SSB-LSB)

We have a signal $M(\omega)$:



We want to transmit a signal $V_{LSB}(\omega)$ with a Fourier Transform:



Define:

Right side of $M(\omega)$: $M_U(\omega) \triangleq M(\omega) \cdot u(\omega)$

Left side of $M(\omega)$: $M_L(\omega) \triangleq M(\omega) \cdot u(-\omega)$

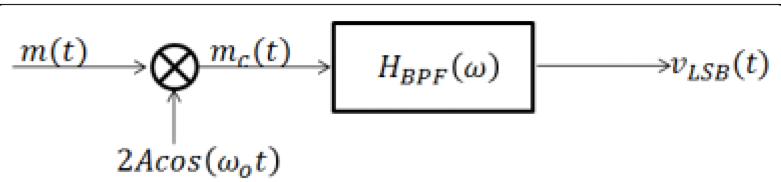
Where $u(\omega)$ is the standard step function

Thus, we want the **FT of the transmitted signal** to be $V_{LSB}(\omega) = A[M_L(\omega - \omega_0) + M_U(\omega + \omega_0)]$

Thus, the **SSB-LSB Signal is**: $v_{LSB}(t) = A[m(t)\cos(\omega_0 t) + \widehat{m}(t)\sin(\omega_0 t)]$

LSB-SSB Transmitters

Option A:



$$m_c(t) = 2Am(t)\cos(\omega_0 t)$$
$$= Am(t)\cos(\omega_0 t)\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$
$$\rightarrow M_c(\omega) = A(M(\omega - \omega_0) + M(\omega + \omega_0))$$
$$H_{BPF}(\omega) = \begin{cases} 1 \quad ; \quad |\omega| \in [\omega_0 - B, \omega_0] \\ 0 \quad ; \quad \text{else} \end{cases}$$

Option B:

Notice that $-j \cdot M_U(\omega) + j \cdot M_L(\omega) = -j \cdot sign(\omega)M(\omega) = H_{Hilbert}(\omega)M(\omega) = \widehat{M}(\omega)$

Since we also have $M(\omega) = M_U(\omega) + M_L(\omega)$, this means that we can represent $M_U(\omega), M_L(\omega)$ in terms of $M, \widehat{M}$:

$$M_U(\omega) = \frac{1}{2}\left(M(\omega) + j\widehat{M}(\omega)\right)$$
$$M_L(\omega) = \frac{1}{2}\left(M(\omega) - j\widehat{M}(\omega)\right)$$

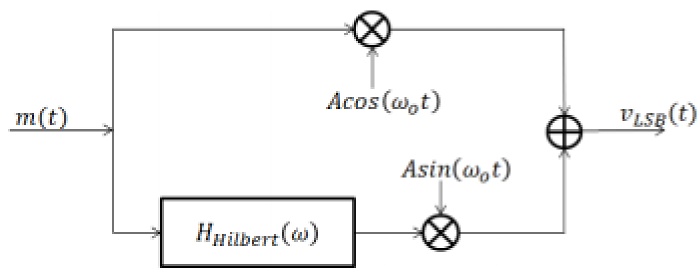
Substituting these definitions into $V_{LSB}(\omega) = A\big(M_L(\omega - \omega_0) + M_U(\omega - \omega_0)\big)$, we get:

$$
\begin{aligned}
V_{LSB}(\omega) &= A\big(M_L(\omega-\omega_0)+M_U(\omega-\omega_0)\big)\\
&= \frac{A}{2}\big[M(\omega-\omega_0)-j\widehat{M}(\omega-\omega_0)+M(\omega+\omega_0)+j\widehat{M}(\omega+\omega_0)\big]\\
&= \frac{A}{2}\left[M(\omega-\omega_0)+M(\omega+\omega_0)+\frac{\widehat{M}(\omega-\omega_0)j\widehat{M}(\omega+\omega_0)}{j}\right]\\
&= A\left[\frac{M(\omega-\omega_0)+M(\omega-\omega_0)}{2}+\frac{\widehat{M}(\omega-\omega_0)j\widehat{M}(\omega+\omega_0)}{2j}\right]
\end{aligned}
$$

Taking the Inverse FT of the above, we get:

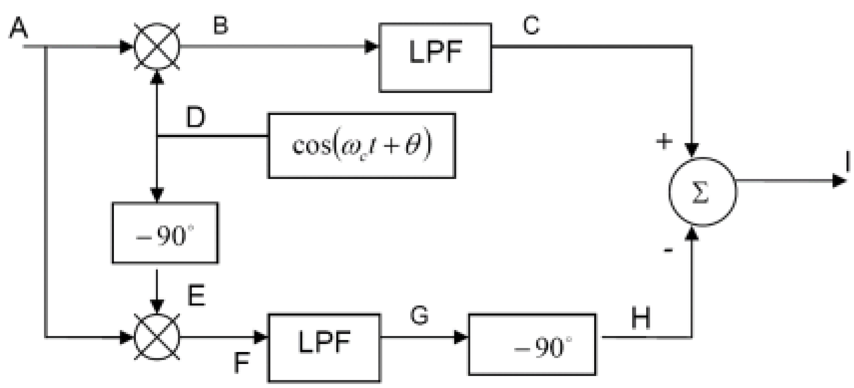
$$
\begin{aligned}
v_{LSB}(t) &= A\left[\frac{m(t)e^{j\omega_0 t}+m(t)e^{-j\omega_0 t}}{2}+\frac{\widehat{m}(t)e^{j\omega_0 t}-\widehat{m}(t)e^{-j\omega_0 t}}{2j}\right]\\
&= A[m(t)\cos(\omega_0 t)+\widehat{m}(t)\sin(\omega_0 t)]
\end{aligned}
$$

Therefore, in order to generate $v_{LSB}(t)$ from only $m(t)$, one may pass $m(t)$ through the following system:



SSB Execise (R13E13.1)

Setup: $x(t)$ is an SSB signal (we don't know if it is USB or LSB). The following *unsynchronized* receiver is used for discovering the SSB signal. Point A is the output of the receiver's BPF.



A. Find expressions for the signals at all the following points: B,C,D,E,F,G,H,I

Step 1: *Assume USB Signal:*

$$x_A(t) = m(t)\cos(\omega_0 t)-\widehat{m}(t)\sin(\omega_0 t)$$

Step 2: *Find the signal at points D,E (oscillators):*

$$x_D = \cos(\omega_0 t+\theta)$$
$$x_E(t) = \sin(\omega_0 t+\theta)$$

Step 3: *Find the signal after the multipliers (B, F):*

$$
\begin{aligned}
x_B(t) &= [m(t)\cos(\omega_0 t)-\widehat{m}(t)\sin(\omega_0 t)]\cos(\omega_0 t+\theta)\\
&= m(t)\cos(\omega_0 t)\cos(\omega_0 t+\theta)-\widehat{m}(t)\sin(\omega_0 t)\cos(\omega_0 t+\theta)\\
&= \frac{1}{2}[m(t)(\cos(\theta)+\cos(2\omega_0 t+\theta))-\widehat{m}(t)(\sin(-\theta)+\sin(2\omega_0 t+\theta))]
\end{aligned}
$$

$$
\begin{aligned}
x_F(t) &= [m(t)\cos(\omega_0 t)-\widehat{m}(t)\sin(\omega_0 t)]\sin(\omega_0 t+\theta)\\
&= \frac{1}{2}[m(t)(\sin(\theta)+\sin(2\omega_0 t+\theta))-\widehat{m}(t)(\cos(\theta)-\cos(2\omega_0 t+\theta))]
\end{aligned}
$$

Step 4: *Find the signals after the LPFs (C,G):*

After going through the LPFs, only the baseband frequencies components are left, so that

$$x_C(t) = \frac{1}{2}[m(t)\cos(\theta) + \hat{m}(t)\sin(\theta)]$$
$$x_G(t) = \frac{1}{2}[m(t)\sin(\theta) - \hat{m}(t)\cos(\theta)]$$

Step 5: *Find the signal after the Hilbert Filter (H):*

$$x_H(t) = \frac{1}{2}[\hat{m}(t)\sin(\theta) + m(t)\cos(\theta)]$$

Step 6: *Finally, the signal at the receiver's output:*

$$x_I = x_C(t) - x_H(t) = 0$$

Step 7: *Check result of step 6 against your assumption in step 1:*

Clearly, we don't want 0 output at the receiver, and therefore it must be meant for LSB.

Step 8: *IF Step 7 confirmed your assumption, you're done. IF not, continue:*

Step 9: *Assume LSB Signal:*

$$x_A(t) = m(t)\cos(\omega_0 t) + \hat{m}(t)\sin(\omega_0 t)$$

Step 10: *Correct steps 3-6, by changing the signs in the relevant places.*

$$x_D(t) = \cos(\omega_0 t + \theta) \quad ; \quad x_E(t) = \sin(\omega_0 t + \theta)$$
$$x_B(t) = \frac{1}{2}[m(t)(\cos(\theta) + \cos(2\omega_0 + \theta)) + \hat{m}(t)(\sin(-\theta) + \sin(2\omega_0 + \theta))]$$
$$x_F(t) = \frac{1}{2}[m(t)(\sin(\theta) + \sin(2\omega_0 t + \theta)) + \hat{m}(t)(\cos(\theta) - \cos(2\omega_0 t + \theta))]$$
$$x_C(t) = \frac{1}{2}[m(t)\cos(\theta) - \hat{m}(t)\sin(\theta)]$$
$$x_G(t) = \frac{1}{2}[m(t)\sin(\theta) + \hat{m}(t)\cos(\theta)]$$
$$x_H(t) = \frac{1}{2}[\hat{m}(t)\sin(\theta) - m(t)\cos(\theta)]$$

And the signal at the output:

$$x_I(t) = x_C(t) - x_H(t) = m(t)\cos(\theta) - \hat{m}(t)\sin(\theta)$$

B. Is the receiver designed for USB or LSB Signals? How should it be (simply) modified so that it would enable reception of the other type of SSB signal?

As established in part A, this is designed for LSB signals. When the receiver is synchronized, i.e $\theta = 0$, we have $x_I(t) = m(t)$, which means this is designed for LSB signals.

Note that θ represents synchronization importance. For example if the receiver is not synchronized, and $\theta = \frac{\pi}{2}$, then we have $x_I(t) = -\hat{m}(t) \neq m(t)$.

Also not that if we change $x_C(t) - x_H(t)$ to a summation instead, then the receiver becomes a USB receiver instead of LSB.

Mathematical Concepts:

Support: The support of a real-valued function f is the subset of the domain, which contains those elements which are not mapped to 0. In other words: $support(f) = \{x \in X \mid f(x) \neq 0\}$

Law of total probability: $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n)P(B_n)$ where A is an event of a sample space, and Bn is a finite partition of a sample space

Bayes' Theorem: $(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Paley-Wiener Theorem: A function $f \in L_2(-\infty, \infty)$ vanishes almost everywhere outside some interval $[-A, A]$ IFF its FT, defined as $F(y) = \int_{-\infty}^{\infty} f(x)e^{ixy}dx$, $y \in \boldsymbol{R}$, satisfied: $\int_{-\infty}^{\infty} |F(y)|^2 dy < \infty$.

Paley-Wiener Causality Criterion: A system is Causal if it satisfies: $\int_{-\infty}^{\infty} \frac{\ln(|H(f)|)}{1+f^2} df < \infty$

Proper Function: If $g(t)$ is proper, then $Var(\mathcal{Re}\{g(t)\}) = Var(\mathcal{Im}\{g(t)\})$, and pseudo autocorrelation $\tilde{R}_{gg^*}(t_1, t_2) = 0$