Transmission Line Representations and Lossless Lines:

Assuming sinusoidal wave (so only spatial variance), we have a **representation**: $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$, $I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$

Complex Propagation Coefficient: $\gamma = \underbrace{\alpha}_{\text{loss}} + \underbrace{j\beta}_{\text{change of phase}}$

Characteristic Impedance: $Z_0 = \frac{V_0^+}{I_0^+} \overset{\text{lossless}}{=} \sqrt{\frac{L'}{C'}}$

Full Expression of Voltage Propagation in Space and Time: $V(z,t) = V_0^+ \underbrace{e^{-\alpha z}}_{\text{attenuation}} e^{j(\omega t - \beta z)} + V_0^- e^{-\alpha z} \underbrace{e^{j(\omega t + \beta z)}}_{\text{change in phase}}$

Phase Velocity: $v_p = \frac{\omega}{\beta} \overset{\text{lossless}}{=} \frac{1}{\sqrt{LC}}$

Reflection Coefficient: $\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$

Lossless Line: We ignore R and G in the distributed circuit

Propagation Constant: $\gamma = j\beta = \omega\sqrt{LC}$, $\alpha = 0$

In terms of phase velocity: $\beta = \frac{\omega}{v_p}$ **Wavenumber:** $k = \frac{2\pi}{\lambda} = \beta$

Wavenumber and Phase Constant in TE TM: $k = \beta$ only holds in TEM mode. In TE and TM we have - $k = \frac{\omega}{c}$, $\beta = k\sqrt{1 - \omega_c^2/\omega^2}$, $\beta \neq k$

Load Impedance	Γ_L
$Z_L = Z_0$	$\Gamma = 0$
$Z_L = 0$	$\Gamma = -1$
$Z_L = \infty$	$\Gamma = 1$
$Z_L =$ complex	$\Gamma =$ complex
$Z_L =$ imag.	$\Gamma_L =$ imag, $\lvert\Gamma\rvert = 1$

Load Reflection Coefficient: $\Gamma_L \overset{\text{def}}{=} \frac{V_1^-(l,t)}{V_1^+(l,t)} = \frac{V_1^-\left(t + \frac{l}{v}\right)}{V_1^+\left(t - \frac{l}{v}\right)} = \frac{R_L - Z_c}{R_L + Z_c}$

Source Reflection Coefficient: $\Gamma_G \overset{\text{def}}{=} \frac{V_2^+(0,t)}{V_1^-(0,t)} = \frac{R_G - Z_c}{R_G + Z_c}$

Wavelength: $\boldsymbol{\lambda = v_0/f}$ **Inductor Impedance:** $Z_{ind} = j\omega L$ **Phasor:** magnitude∠phase

Standing Waves - VSWR

Voltage Standing Wave Ratio (VSWR): $S = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$ **VSWR In Matched Line ($\boldsymbol{Z_L = Z_0}$):** $S = 1$

VSWR In an Open/Shorted Line ($\boldsymbol{Z_L = \infty, 0}$): $S = \infty$ **In General:** $1 \le S \le \infty$. Bigger S =worse match

Transmission Line Input Impedance

Input Impedance: $Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^+ e^{-\beta z} + V_0^- e^{\beta z}}{\frac{V_0^+}{Z_0}e^{-\beta z} - \frac{V_0^-}{Z_0}e^{+\beta z}} \overset{\text{simplifies algebaically}}{=} Z(d) = Z_0 \frac{1 + \Gamma e^{-j2\beta d}}{1 - \Gamma e^{-j2\beta d}}$, d =position on line

Direct Impedance Tracking: $Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma e^{+j\beta z}}{e^{-j\beta z} - \Gamma e^{+j\beta z}}$

Power at the Load (and other Power formulas)

Instantaneous: $P_{inst}(z,t) = V(z,t)I(z,t)$ **Incident:** $P_{inc}(z) = P_L + P_{ref} = \frac{1}{2}V(z)I^*(z) = \frac{|V_0(z)|^2}{2Z_0}$

Load: $P_L(z) = P_{inc} - P_{ref} = \frac{|V_0(z)|^2}{2Z_0}(1 - |\Gamma|^2)$ **Reflected:** $P_{ref}(z) = |\Gamma^2| \frac{|V_0(z)|^2}{2Z_0}$ Average: $P_{AV} = \frac{1}{2}VI^*$

Smith Charts

Normalized Impedance: $\overline{Z_L} = \frac{Z_L}{Z_0}$. Plot point on SC. Draw circle from center to it.

Reflection Coefficient (Normalized): $\Gamma(z) = \frac{\bar{Z}(z) - 1}{\bar{Z}(z) + 1}$

Line Length in Terms of Wavelength: l/λ

De-normalization: $\boldsymbol{Y_{in} = \overline{Y_{in}} \frac{1}{Z_c}}$

Input Impedance: proj. line from $\overline{Z_L}$ through center and further. Rotate line distance $c\lambda$ ie. By the length to get to your new point of interest in terms of wavelength. Project from the moved line back to original impedance circle. The first intersection is $\overline{Z_{in}}$, second is $\overline{Y_{in}}$

Relation between Impedance and Reflection Coefficient at Each Point: $\bar{Z}(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$

Double Stub Matching: =Y_{in}===/l2/Y_{in1}===Y_{in2}/l1/=|Z_L

Admittances: We work with admittances, since stubs are parallel to the line. We normalize the admittances since all lines have the same characteristic impedance

Matching Condition: $\overline{Y_{in}} = 1$

Input Admittance $\boldsymbol{\overline{Y_{in}}}$: Given by a parallel combination of the admittance of stub 2 ($\overline{Y_{l2}} = jx_2$), and the admittance $\overline{Y_{in_1}}$ or the remaining system.

Admittance $\boldsymbol{\overline{Y_{in_1}}}$: $\overline{Y_{l2}} = jx_2$, so $\overline{Y_{in_1}} = 1 - jx_2$. On the SC, this is located on the unit circle $r = 1$. This is the admittance past stub l2.

Admittance $\boldsymbol{Y_{in_2}}$: Given by parallel combination of $\overline{Y_{l1}} = jx_1$ and $\overline{Y_L}$, i.e. $\overline{Y_{in_2}} = Y_L + jx_1$. Also, $\overline{Y_{in}} = 1 - jx_2$

Stub Imaginary Admittance: Since $\overline{Y_{l1}}$ is purely imaginary, $Re\{\overline{Y_{in_2}}\} = Re\{\overline{Y_L}\}$, and $Im\{\overline{Y_{in_2}}\} = Im\{\overline{Y_L}\} + x_1$

Loci on the SC: Obtained from stub imaginary admittance section. $Re\{\overline{Y_{in1}}\} = 1$, $Re\{\overline{Y_{in_2}}\} = Re\{\overline{Y_L}\}$ =?. Plot both. They are circles with diameter from rightmost side to the value.

Intersecting Between Conditions, Moving Circles: Note that the conditions $Re\{\overline{Y_{in1}}\} = 1$ and $Re\{\overline{Y_{in_2}}\} = Re\{\overline{Y_L}\}$ might occur at different physical points on the line. We need to translate $Re\{\overline{Y_{in1}}\} = 1$ toward the end of the line. Rotate $Re\{\overline{Y_{in1}}\} = 1$ circle toward the load until you reach the end of the line. The new circle represents $\overline{Y_{in2}}$,, and so does the other $Re\{\overline{Y_{in_2}}\} = Re\{\overline{Y_L}\}$ circle.

Admittance $\boldsymbol{Y_{in2}}$ on SC: Intersection between the rotated circle & $Re\{\overline{Y_{in_2}}\} = Re\{\overline{Y_L}\}$ is admittance $\overline{Y_{in2}}$. 2 such points - A,B.

Impedance of a Stub: $x_{1,2}$ - by stub imaginary admittance, 2 знач. А,Б

Translating Admittances over The Line: $\overline{Y_{in_1}}$ через rotating solutions for $\overline{Y_{in_2}}$ toward $\overline{Y_{in_1}}$ position.

Length Of Stubs: On the SC, start from short circuit admittance ($Y_L \to \infty$), aka rightmost point. Move to generator until intersection with solutions for x_1, x_2. Length is how much you had to travel.

Forbidden Zone: All loads with $Re\{Y_L\} > 2$ cannot be matched.

Quarter-Wave Transformer Matching: $\boldsymbol{V_g, [R_g]}$: $\boldsymbol{Z_{in}}$==$\boldsymbol{l1}$==::($\boldsymbol{Z_{in2}, \Gamma_{L3}}$)==$\boldsymbol{\frac{1}{4}\lambda}$==::($\boldsymbol{Z_{in3}, \Gamma_{L2}}$)==$\boldsymbol{l3}$==|$\boldsymbol{Z_L}$

Application: match a load to a line to eliminate reflection at end of lin1, by a transformer $l_2 = 0.25\lambda$

Matching Requirement: $Z_{in2} = Z_{01}$. Using quarter-wave formula: $Z_{01} = Z_{02}^2/Z_{in3}$. Since Z_{01}, Z_{02} are both real, we can require that Z_{in3} also be real.

Length s.t. $\boldsymbol{Z_{in3}} \in \mathbb{R}$: We move the normalized load $\overline{Z_L} = Z_L/Z_{03}$ toward generator until we intersect the real axis. The intersection of the $\overline{Z_L}$ circle and the real axis is $\overline{Z_{in_3}}$, and the length is how much we moved.

Transmission Lines as Passive Components

Open Stub: $\Gamma = 1$ **Input Impedance (Open Stub):** $Z(z) = Z_0 \frac{2\cos(\beta z)}{j2\sin(\beta z)} = -jZ_0 \cot(\beta z)$

Shorted Stub: $\Gamma = -1$ **Input Impedance (Shorted Stub) :** $Z(z) = Z_0 \frac{j2\sin(\beta z)}{2\cos(\beta z)} = jZ_0 \tan(\beta z)$

Depending on the length of the stub, it can act as an inductor or a capacitor

Quarter Wavelength Matching on Transmission Lines

Propagation Constant in Quarter Wave Transformer: If we plug in $z = \frac{\lambda}{4}$ into βz, get: $\frac{\lambda}{4}\beta = \frac{\pi}{2}$

Input Impedance: $Z_{in}\left(\frac{\lambda}{4}\right) = \frac{Z_1^2}{Z_L}$ **Characteristic Impedance for Matching:** $Z_1 = \sqrt{Z_0 Z_L}$, $Z_0 = \frac{Z_1^2}{Z_L}$

Infinite Transmission Line

General Form: $V(z,t) = V^+\left(t - \frac{z}{v}\right) + V^-\left(t + \frac{z}{v}\right) + V_0$ **Possible Assumption:** No initial voltage V_0 or current I_0 on the line.

Significance: Since only a forward wave is generated, we can remove the negative propagation term, and have solution of the form $V(z,t) = V^+\left(t - \frac{z}{v}\right)$

Voltage Continuity: Voltage is continuous at $z = 0$ (and other junctions), ergo $V_g(t) = V(0,t) = V^+(0,t)$

General Solution: If $V_0^+(0,t)$ is function just time, we drop the 0 and $V^+(t) = S(t)$. $V(z,t) = V^+\left(t - \frac{z}{v}\right)$

Relation Between Current and Voltage: $I(z,t) = Y_c V^+\left(t - \frac{z}{v}\right) - Y_c V^-\left(t + \frac{z}{v}\right)$

Voltage over the Line: Given by adding a delay of $\frac{z}{v}$ to $V(t)$ to make it $V\left(t \pm \frac{z}{v}\right)$

Loaded Infinite Transmission Line

Behavior: Due to change at $z = 0$ (point where line meets load), we will have + and – traveling waves.

Ohm's Law: $\frac{V_L}{I_L} = R_L$ **Voltage and Current Continuity:** Suppose we have a junction at $z = 0$. Then: $V_L = V(z = 0, t) = V^+(t) + V^-(t)$, $I_L = I(z = 0, t) = Y_c(V^+(t) + V^-(t))$

V and I over the Line: $V^-(z,t) = V^+\left(t - \frac{z}{v}\right) + \Gamma_L V^+\left(t + \frac{z}{v}\right)$, $I(z,t) = Y_c\left[V^+\left(t - \frac{z}{v}\right) - \Gamma_L V^+\left(t + \frac{z}{v}\right)\right]$

Line with Initial Conditions

Open Ended Line: no load, or load far away

Common Init. Cond: $V_0 \neq 0, I_0 = 0$

Travel Time: travel time along the line $T = vl$

Types of time-dependent modifiers: switches closing at various times, such as $t = 0$, or other circuit changes

V and I for $t < 0$: from I.C $V(z,t) = V_0, I(z,t) = 0$

When switch closed we have $I(0,t)$ flowing right.

BC at Junction with Time-Dependent Modifier: $I(0,t)R_G + V(0,t) = 0$. Реш. В 3 частях (интервалах)

General Solution for Source Terminal: $V(z,t) = V_0 + V_1^+\left(t - \frac{z}{v}\right)$, $I(z,t) = I_1^+\left(t - \frac{z}{v}\right)$ — 0<t<2T

Continuity: $V(0,t) = V_0 + V_1^+(0,t) = -V_G$, $I(0,t) = \frac{1}{Z_c}V_1^+(0,t) = \frac{V_G}{R_G}$; $V_1^+(0,t) = -V_0\frac{Z_c}{Z_c+R_G}U(t)$

V,I over Line: $V(z,t) = V_0 + V_1^+(z,t) = V_0 - V_0\frac{Z_c}{Z_c+R_G}U\left(t - \frac{z}{v}\right)$, $I(z,t) = I_1^+(z,t) = -V_0\frac{Z_c}{Z_c+R_G}U\left(t - \frac{z}{v}\right)$; $\frac{z}{v} < t < 2T - \frac{z}{v}$

T<t<3T — **BC At Junction with a Load:** $I(l,t) = 0$, at t=T fwd wave arrives at load

BC Equation as Propagating Waves: $I(l,t) = I_1^+\left(t - \frac{l}{v}\right) + I_1^-\left(t + \frac{l}{v}\right) = \frac{1}{Z_c}V_1^+\left(t - \frac{l}{v}\right) - \frac{1}{Z_c}V_1^-\left(t + \frac{l}{v}\right) = 0$

Note: If the BC at load is to be satisfied, it is required that $V_1^-\left(t + \frac{l}{v}\right) = V_1^+\left(t - \frac{l}{v}\right)$. Тут Коф. Отраж. =1

$V(z,t) = V_0 + V_1^+(z,t) + V_1^-(z,t) = V_0 - V_0\frac{Z_c}{Z_c+R_G}U\left(t - \frac{z}{v}\right) - \Gamma_L V_0\frac{Z_c}{Z_c+R_G}U\left(t - 2T + \frac{z}{v}\right)$

$I(z,t) = I_1^+(z,t) + I_1^-(z,t) = -V_0\frac{1}{Z_c+R_G}U\left(t - \frac{z}{v}\right) + \Gamma_L V_0\frac{Z_c}{Z_c+R_G}U\left(t - 2T + \frac{z}{v}\right)$

$2T - \frac{z}{v} < t < 2T + \frac{z}{v}$

Generation of New Waves: $V_0, V_1^+(0,t), I_1^+(0,t)$ already satisfy BC at junction w. time dependent modifier (say z=0), but $V_1^-(0,t), I_1^-(0,t)$ disturb the BC, creating V_2^+ , I_2^+ — 2T<t<4T

Total Voltage and Current in a Line: $V(z,t) = V_0 + V_1^+\left(t - \frac{z}{v}\right) + V_2^+\left(t - \frac{z}{v}\right)$, $I(z,t) = I_1^+\left(t - \frac{z}{v}\right) + I_1^-\left(t + \frac{z}{v}\right) + I_2^+\left(t - \frac{z}{v}\right)$. Вставь в биси на з-ноль, тупо вычеркни в1+,в0,и1+

Reflections past the 1st: $\boldsymbol{V_2^+(0,t) = -\Gamma_g\Gamma_L V_0}\frac{Z_c}{Z_c+R_G}U\left(t - 2T - \frac{z}{v}\right)$, $2T + \frac{z}{v} < t < 4T - \frac{z}{v}$

Recitation 2:

Boundary Conditions: For a standard "generator-line-load" topology, we have two BC's:

(1) At $z = 0$ (start of line): $V_G(t) = I(0,t)R_G + V(0,t)$

(2) At $z = l$ (end of line): $V(l,t) = I(l,t)R_L$

General Transmission Line Solution Substitution Into Standard Boundary Conditions:

$$V_G(t) = \frac{1}{Z_c}\left[\sum_{n=1}^{\infty} V_n^+(t) - \sum_{n=1}^{\infty} V_n^-(t)\right]R_G + I_0R_G + \sum_{n=1}^{\infty} V_n^+(t) + \sum_{n=1}^{\infty} V_n^-(t) + V_0$$

$$\left[\sum_{n=1}^{\infty} V_n^+\left(t - \frac{l}{v_p}\right) + \sum_{n=1}^{\infty} V_n^-\left(t + \frac{l}{v_p}\right) + V_0\right] = \frac{1}{Z_c}\left[\sum_{n=1}^{\infty} V_n^+\left(t - \frac{l}{v_p}\right) - \sum_{n=1}^{\infty} V_n^-\left(t + \frac{l}{v_p}\right)\right]R_L + I_0R_L$$

Special Cases of Lossless Lines:

Line Length	Result
Half Wavelength	In the special case where $\beta l = n\pi$, where n is an integer (i.e. length of line is a multiple of half-wavelength), we have: $Z_{in} = Z_L$ for all n, including $n = 0$
Quarter Wavelength	In case where length of line is $\frac{1}{4}\lambda$ or an odd multiple of it: $Z_{in} = Z_0^2/Z_L$
Matched Load	If load impedance=char. impedance of the line (i.e. line=matched): $Z_{in} = Z_L = Z_0$
Shorted Line	If $Z_L = 0$ (short) : $Z_{in}(l) = jZ_0\tan(\beta l)$
Open Line	If $Z_L = \infty$ (open): $Z_{in}(l) = -jZ_0\cot(\beta l)$

Dispersive Transmission Line

Dispersive Line: A line in which $\beta(\omega)$ is not linear with ω. Since $\beta(\omega) = \frac{\omega}{v}$, this means that the frequency components propagate at different velocities.

Wave Spectrum: We define the spectrum of a wave according to Fourier transform. $V(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\tilde{V}(\omega)e^{j\omega t}$, $\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt V(t)e^{-j\omega t}$.

Since $V(t)$ is real valued, it follows that $\tilde{V}(-\omega) = \tilde{V}^*(\omega)$

Propagation in a Dispersive Line: Let the pulse at $z = 0$ be $V(0,t)$, then the pulse at a distance z is:

$V(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\,\tilde{V}(0,\omega)e^{-j\beta(\omega)z} = Re\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\,\tilde{V}(0,\omega)e^{-j\beta(\omega)z}\right\}$. We see that propagation by distance z is translated to multiplication of each spectral component $\tilde{V}(\omega)$ by $e^{-j\beta(\omega)z}$

Applicability of Second Expression of Propagation: The second expression applies because $V(z,t)$ is a real signal, so $\tilde{V}(\omega) = \tilde{V}^*(-\omega)$. This also implies $\beta(\omega) = -\beta(-\omega)$

Narrow Band Pulse: A modulated signal is given by $V(t) = f(t)\cos(\omega_0 t)$. $f(t)$ is the pulse, modulated by frequency ω_0 and the pulse is narrow relative to ω_0

Condition on Narrow Pulse: $\boldsymbol{\Delta\omega \ll \omega_0}$

Propagation Constant: $\beta(\omega) \approx \beta_0 + \beta_1(\omega - \omega_0)$. This is true if we assume narrow band pulse, which allows us to neglect the higher-order terms of the Taylor Expansion of $\beta(\omega)$ — $\beta_0 = \beta(\omega_0)$, $\beta_1 = \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}$

Voltage of a Narrow Pulse: $V(z,t) = f\left(t - \frac{z}{v_g}\right)\cos\left(\omega_0\left(t - \frac{z}{v_p}\right)\right)$

Gaussian Pulse: $f(t) = e^{-t^2/2T^2} \leftrightarrow \tilde{f}(\omega) = \sqrt{2\pi}e^{-\omega^2T^2/2}$. (T=pulse width)

Spectrum of a Gaussian Pulse: For a modulated signal $V(0,t) = f(t)\cos(\omega_0 t)$, the spectrum is: $\tilde{V}(0,\omega) = \frac{1}{2}\left[\tilde{f}(\omega - \omega_0) + \tilde{f}(\omega + \omega_0)\right]$

Z-Dependent Spectrum: $\tilde{V}(z,\omega) = \tilde{V}(0,\omega)e^{-j\beta(\omega)z}$

Propagating Signal (narrow pulse): $\cos\left(\omega_0\left(t - \frac{z}{v_p}\right)\right)f\left(t - \frac{z}{v_g}\right) = \cos\left(\omega_0\left(t - \frac{z}{v_p}\right)\right)e^{-\left(t - \frac{z}{v_g}\right)^2/2T^2}$

Phase Velocity: $\left.v_p\right|_{\omega_0} = \left.\frac{\omega}{\beta}\right|_{\omega_0} = \frac{\omega_0}{\beta_0}$

Group Velocity: $\left.v_g\right|_{\omega_0} = \left.\left(\frac{\partial\beta}{\partial\omega}\right)^{-1}\right|_{\omega_0} = \frac{1}{\beta_1}$

Gaussian Pulse Maximum: Has a max at $t = 0, z = 0$

Tracking the Max of a Propagating Gaussian Pulse: $V(z,t) = 0$ if the cosine term is 0. Find z s.t. its true.

Recitation 8 – Coaxial Stuff

[Coax] Inductance Per Unit Length: $L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$

[Coax] Capacitance Per Unit Length: $C = \frac{2\pi\epsilon}{\ln(b/a)}$

[Coax} Char.Impedance: $Z_c = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right)$

[Coax] Wavespeed: $v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c_0}{\sqrt{\mu_r\epsilon_r}}$, $c_0 = 3\cdot 10^8\left[\frac{m}{s}\right]$

EM Fields over a line (assuming $\boldsymbol{z = 0}$ is the junction between 2 coaxes):

$$E(\vec{r}) = \vec{e}(x,y)\begin{cases} V_1^+e^{-j\beta_1 z} + V_1^-e^{+j\beta_1 z} ; z < 0 \\ V_T^+e^{-j\beta_T z} + V_T^-e^{+j\beta_T z} ; z > 0\end{cases}, \vec{H}(\vec{r}) = \vec{h}(x,y)\begin{cases} Y_{C_1}V_1^+e^{-j\beta_1 z} - Y_{C_1}V_1^-e^{+j\beta_1 z}; z < 0 \\ Y_{C_T}V_T^+e^{-j\beta_T z} - Y_{C_T}V_T^-e^{+j\beta_T z}\end{cases}$$

Where: β_1, β_T =propagation constants; Y_{C_1}, Y_{C_T} = characteristic admittances, $V_1^{\pm}, V_T^{\pm}$=fwd/bckwd wave amplitudes over the lines.

[Coax] Mode Functions: $\vec{e}(x,y) = \frac{1}{r\ln\left(\frac{b}{a}\right)}\hat{r}$, $\vec{h}(x,y) = \frac{1}{2\pi r}\hat{\varphi}$

These functions depend only on the geometry of the cross section, and not on the medium properties. These functions are the same over all sections.

Continuity: Transversal fields must be continuous @ interfaces between lines. Ergo- continuity of V, I.

Wavelength: $\lambda = \frac{v_1}{f} = \frac{c_0}{f\sqrt{\epsilon_r}}$

Equivalent Circuit to some TL System (At Source Terminal): V_g , R_g, Z_{c1} all in series.

Input voltage to line 1: $V_{in} = V_1(0) = V_g(t)\frac{Z_{c1}}{Z_{c1}+R_g}$. In a matched line: $V_1(0) = V_1^+(0)$.

Recitation 9 + Book: Plane Waves

Wave Vector: $\underline{k} = (k_x, k_y, k_z) = k\underbrace{(\cos\theta_x, \cos\theta_y, \cos\theta_z)}_{\hat{\underline{k}}\text{ unit vector}} = k\hat{\underline{k}}$

Wave Number: $k = |\underline{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \omega\sqrt{\mu\epsilon} = \omega/c$

Wave Vector Angles: $\theta_{x,y,z}$ are angles from the x,y,z axes. They obey $\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1$

Plane Wave: $\underline{E}(\underline{r}) = \underline{E}_0e^{-j\underline{k}\cdot\underline{r}} = (E_{0x}, E_{0y}, E_{0z})e^{-j(k_xx+k_yy+k_zz)}$. Since planes of constant phase satisfy $\underline{k}\cdot\underline{r}$ = const., we call this a plane wave. These planes are perpendicular to $\underline{k}$.

Wavelength (Plane Waves): Distance between 2 planes with phase difference 2π: $k\lambda = 2\pi \rightarrow \lambda = \frac{2\pi}{k} = \frac{c}{f}$

Equation from Pranav: $\hat{\underline{k}}\cdot\hat{\underline{e}}_{TE} = \hat{\underline{k}}\cdot\hat{\underline{e}}_{TM} = \hat{\underline{e}}_{TE}\cdot\hat{\underline{e}}_{TM} = 0$

Plane Wave Structure: $\underline{k}\cdot\underline{E}_0 = \underline{k}\cdot\underline{H}_0 = \underline{E}_0\cdot\underline{H}_0 = 0$, thus $\underline{E}, \underline{H}, \underline{k}$ are perpendicular.

$\underline{H}$ Field in a Plane Wave: $\underline{H} = \underline{H}_0 e^{-j\underline{k}\cdot\underline{r}}$, $\underline{H}_0 = \eta^{-1}(\hat{\underline{k}} \times \underline{E}_0)$

Wave Impedance: $\eta = \sqrt{\mu/\epsilon}$ $[\Omega]$, $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 120\pi$

Plane Wave Interface

General Problem: 3 mediums, $1(\epsilon_1, \mu_0), 2(\epsilon_2, \mu_0$, length $L), 3(\epsilon_3, \mu_0)$, some incident field $\underline{E}^i = E_0 e^{-jk_1 z}\hat{\underline{y}}$

Incident Magnetic Field: In PW illumination: $\underline{H}^i = \eta^{-1}\hat{\underline{k}} \times \underline{E}^i = \eta_1^{-1}\hat{\underline{z}} \times E_0 e^{-jk_1 z}\hat{\underline{y}} = \eta_1^{-1}\hat{\underline{x}}E - 0$

What Happens? Due to dielectric discontinuity between the layers, forward and backward waves are generated at layer 2

EM Fields:
$E_y = E_y^+ + E_y^- = V_i^+(z) + V_i^-(z) = V_i^+ e^{-jk_i z} + V_i^- e^{+jk_i z}$
$H_x = H_x^+ + H_x^- = -(I_i^+ e^{-jk_i z} + I_i^- e^{+jk_i z}) = -\eta_i^{-1}(V_i^+ e^{-jk_i z} - V_i^- e^{jk_i z})$

Where $V_i^{\pm}$ are thefwd/bckwd propagating waves, $V^{\pm}$ are amplitudes.

Transmission Line Model: The line parameters are $\rightarrow$	$E_y = V(z)$	$-H_x = I(z)$	$\eta_i = Z_{c_i}$	$k_i = \beta_i$

We define $I(z) = -H_x$. For this choice, incident field satisfies $\frac{E_y^+}{-H_x^+} = \eta_i \rightarrow \frac{V_i^+(z)}{I_i^+(z)} = Z_{c_i} = \eta_i$ in the TL model

Wave Impedance & Phase Constants for Each Medium: $\eta_1 = \sqrt{\mu/\epsilon_1} = Z_{c_1}$ $\beta_1 = k_1$. Same for all others.

Transmission Line Model Circuit: Medium 1 is line 1, Medium 2 is line 2, medium 3 is line 3 (could be load). Γ_{in} is at entrance to medium 2, Γ_L is at exit of medium 2. Medium 1 is a TL that comes from $-\infty$

Voltages and Currents in the TL Model:
Line 1: $V_1(z) = V_1^+(z) + V_1^-(z) = V_1^+(0)e^{-jk_1 z} + V_1^+(0)e^{jk_1 z}\Gamma_{in}$
Line 2: $V_2(z) = V_2^+(z) + V_2^-(z) = V_2^+(0)e^{-jk_2 z}(1 + \Gamma_L e^{-2jk_2 s})$
Line 3: $V_3(z) = V_3^+(z) = V_3(z = L)e^{-jk_3(z-L)}$

Relevant Reflection Coefficients: $\Gamma_L = \Gamma_2(L)$, $\Gamma_{in} = \Gamma_1(0)$. So: $\Gamma_L = \frac{Z_{c_3} - Z_{c_2}}{Z_{c_3} + Z_{c_2}} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$

Forward Wave at First Junction ($\boldsymbol{V_1^+(0)}$): The incident field is related to $V_1^+(z)$, so $V_1^+(0) = E_0$

Voltage Continuity: lines 1 and 2: $V_1^+(0) = V_2^+(0)$, thus $V_1^+(0)(1 + \Gamma_{in}) = V_2^+(0)(1 + \Gamma_L e^{-2jk_2 L})$

Polarizations:

Wave Format: We work on wave propagating in z dir.	$\underline{E} = E_0 e^{-jkz}$	$\underline{E}_0 = \hat{\underline{x}}E_1 + \hat{\underline{y}}E_2$	$E_j = \lvert E_j\rvert e^{j\phi_j}$

Such that (whole line): $E_x(z,t) = |E_1|\cos(\omega t - kz + \phi_1)$ $\quad E_y(z,t) = |E_2|\cos(\omega t - kz + \phi_2)$

What Determines Polarization? (1) Relative Magnitude $|E_2|/|E_1|$, (2) Relative Phase $\delta = \phi_1 - \phi_2$

Linear Polarization:

Here, phases of E_x, E_y are the same, so $\delta = 0$ or π. We have the following

$$\begin{cases} E_x(z,t) = |E_1|\cos(\omega t - kz) \\ E_y(z,t) = \pm|E_2|\cos(\omega t - kz) \end{cases} \Rightarrow \frac{E_x(z,t)}{E_y(z,t)} = \pm\frac{|E_1|}{|E_2|} = \text{const.}; \text{for } \delta = \begin{cases} 0 \\ \pi \end{cases}$$

Thus, If we are looking at a given $z =$ const. plane, the signal oscillates as a function of t along the line.

Forward Wave at First Junction ($\boldsymbol{V_1^+(0)}$): The incident field is related to $V_1^+(z)$, so $V_1^+(0) = E_0$

Voltage Continuity: lines 1 and 2: $V_1^+(0) = V_2^+(0)$, thus $V_1^+(0)(1 + \Gamma_{in}) = V_2^+(0)(1 + \Gamma_L e^{-2jk_2 L})$

Circular Polarization:

In this case the phase difference between E_x, E_y is $\delta = \pm\frac{\pi}{2}$

If we have $\boldsymbol{|E_1| = |E_2| = A}$, then: $E_x(z,t) = A\cos(\omega t - kz)$, $E_y = A\cos\left(\omega t - kz \mp \frac{\pi}{2}\right)$

E_x, E_y oscillate in their axes s.t. their sum in the (x,y) plane (at a z=const.) traces out a circle with radius A.

Clockwise Circular Polarization (Right Hand): For $\delta = +\frac{\pi}{2}$, E_y is lagging $\frac{\pi}{2}$ after E_x.

Counter-Clockwise Circular Polarization (Left Hand): For $\delta = -\frac{\pi}{2}$, E_y is leading by $\frac{\pi}{2}$ b E_x.

Normal Incidence on a Dielectric Interface:

Consider 2 media. Medium 1 at $z < 0$ with μ_1, ϵ_1. Medium 2 at $z > 0$, with $\mu_2\epsilon_2$. The incident wave propagates in +z from $z < 0$

Normal Incidence: Propagation axis = Stratification Axis

A single Interface (half-space problem):
$$\left.\begin{aligned} \underline{E} &= \hat{\underline{x}}(E_0 e^{-jk_1 z} + \Gamma E_0 e^{+jk_1 z}) \\ \underline{H}^i &= \hat{\underline{y}}\left(\frac{E_0}{\eta_1}e^{-jk_1 z} - \Gamma\frac{E_0}{\eta_1}e^{+jk_1 z}\right) \end{aligned}\right\} z < 0 \qquad \left.\begin{aligned} \underline{E}^t &= \hat{\underline{x}}\tau E_0 e^{-jk_2 z} \\ \underline{H}^t &= \hat{\underline{y}}\tau\frac{E_0}{\eta_2}e^{-jk_2 z} \end{aligned}\right\} z > 0 \qquad \begin{aligned} k_i &= \omega\sqrt{\mu_i\epsilon_i} \\ \eta_i &= \sqrt{\mu_i/\epsilon_i} \end{aligned}$$

Reflection/Transmission coefficients (from continuity of tangential components): $1 + \Gamma = \tau$ $\quad \frac{1}{\eta_1}(1 - \Gamma) = \frac{1}{\eta_2}\tau \rightarrow$ $\quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ $\quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1}$

PEC In $\boldsymbol{z = 0}$ plane:
BC: $\underline{E}_{\tan} = 0 \rightarrow \Gamma = -1, \tau = 0$
Surface Current: $J_s = \hat{\underline{n}} \times \underline{H}|_{z=0} = -\hat{\underline{z}} \times \hat{\underline{y}}H_y\big|_{z=0} = \hat{\underline{x}}\frac{2E_0}{\eta_1} = 2\hat{\underline{\eta}} \times \underline{H}^i\big|_{z=0}$

Normal Incidence on a Multilayer Medium: conventional TL Model, in each layer:	$E_x = V(z)$	$H_y = I(z)$	$\eta = Z_c$	$k = \beta$	$S_z = \frac{1}{2}E_x H_y^* = \frac{1}{2}VI^*$

Plane Wave Propagation in Multi-Layered Medium:

Configuration: Layered medium with z =stratification axis. Each layer has a uniform medium μ_i, ϵ_i and width l_i (aka length). Denote the medium $z < 0$ as $i = 1$. The PW arrives from the left at an angle θ_1 wrt z.

Denote: $k_i = \omega\sqrt{\mu_i\epsilon_i} = k_0 n_i$ **Refractive Index:** $n_i = \sqrt{\mu_{ri}\epsilon_{ri}}$ **Wave Imped.:** $\eta_i = \sqrt{\mu_i/\epsilon_i} = \eta_0\sqrt{\mu_{ri}/\epsilon_{ri}}$

Location of $\underline{\boldsymbol{k}}$: $\underline{k}$ vecs in each plane are in (x, z) plane.

TE Polarization: $E_y, H_x, H_z \neq 0$ i.e. $H_y, E_x, E_z = 0$

2 Types of Uncoupled Polarization: TE ($\perp$), & TM ($\parallel$)

TM Polarization: $H_y, E_x, E_z \neq 0$ i.e. $E_y, H_x, H_z = 0$

Snell's Law Kinematic Properties:

Field Continuity along the interfaces requires the same $\boldsymbol{e^{-jk_x x}}$ behavior in all layers, so:
$$k_x = k_1\sin(\theta_1) = k_2\sin(\theta_2 \ldots = k_i\sin(\theta_i) = const$$
$$\text{Or } n_1\sin(\theta_1) = k_2\sin(\theta_2) \ldots = n_1\sin(\theta_i)$$
This is the generalized form of Snell's law. Given θ_1 it determines θ_i $\forall$ i

This relation implies: $k_{z_i} = \sqrt{k_i - k_x^2} = k_i\cos(\theta_i)$

Another Convenient form (if we use $\boldsymbol{k_x = k_1 n_1 \sin(\theta_1)}$): $k_{z_i} = k_i\underbrace{\sqrt{1 - \left(\frac{n_1}{n_i}\right)^2\sin^2(\theta_1)}}_{\cos(\theta_i)}$

Total Reflection and Evanescent Waves:

Suppose that in a certain layer l we have $n_l < n_1$. In that case there is a:

Critical Angle of Incidence: $\theta_c = \sin^{-1}\left(\frac{n_l}{n_1}\right)$

Propagation Angle: For $\theta_1 < \theta_c$ all k_{z_i} are real & wave propagates in all layers

For $\boldsymbol{\theta_1 > \theta_c}$ we have in the $\boldsymbol{l}$ layer: $k_{z_i} = -j\sqrt{k_x^2 - k_l^2} = -jk_1\sqrt{n_1^2\sin^2(\theta_1) - n_l^2}$. The sign of j is chosen by demanding that the transmitted field in the l layer $e^{-jk_{z_l}z}$ decays as z increases away from interface

Transmission Line Model (TE Polarization):

$\stackrel{*}{=}$ implies that the wave propagates in the layer (i.e. not evanescent), such that $k_z = k\cos(\theta)$

$H_z = \frac{k_x}{\omega\mu}E_y = \frac{1}{\eta}\frac{k_x}{k}E_y$ $\quad k_z = \sqrt{k^2 - k_x^2}$ $\quad Z_{TE} = \frac{\omega\mu}{k_z} = \eta\frac{k}{k_z} \stackrel{*}{=} \frac{\eta}{\cos(\theta)}$ $\quad E_y(x,z) \stackrel{\text{def}}{=} V(z)e^{-jk_x x}$, $H_x \stackrel{\text{def}}{=} -I(z)e^{-jk_x x}$

Transmission Line Equations: $-\partial_z V(z) = jk_z Z_{TE}I(z)$, $-\partial_z I(z) = jk_z Y_{TE}V(z)$

Field in Some Layer: Form of Solution in given layer:
$E_y(x,z) = [V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}]e^{-jk_x x}$
$H_x(x,z) = \underbrace{-Y_{TE}}_{\eta^{-1}\frac{k_z}{k}\stackrel{*}{=}\eta^{-1}\cos(\theta)}[V_0^+ e^{-jk_z z} - V_0^- e^{+jk_z z}]e^{-jk_x x}$

Rewriting for $\stackrel{*}{=}$ case:
$E_y(x,z) \stackrel{*}{=} [V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}]e^{-jk_x x}$
$H_x(x,z) \stackrel{*}{=} \eta^{-1}\cos(\theta)\,[V_0^+ e^{-jk_z z} - V_0^- e^{+jk_z z}]e^{-jk_x x}$
$H_z(x,z) \stackrel{*}{=} \eta^{-1}\sin(\theta)\,[V_0^+ e^{-jk_z z} +$

$H_z(x,z) = \underbrace{\eta^{-1}\frac{k_x}{k}}_{\stackrel{*}{=}\eta^{-1}\sin(\theta)} \left[V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}\right] e^{-jk_x x}$ | $V_0^- e^{+jk_z z}\big] e^{-jk_x x}$

One can Observe that:

$\left(E_y, H_x, H_z\right) = \left(1, \frac{-\cos(\theta)}{\eta}, \frac{\sin(\theta)}{\eta}\right) E_0^+ e^{-j(k_x x + k_z z)}$ is a TE Wave propagating at an angle θ wrt $+z$ axis.

$\left(E_y, H_x, H_z\right) = \left(1, \frac{\cos(\theta)}{\eta}, \frac{\sin(\theta)}{\eta}\right) E_0^- e^{-j(k_x x - k_z z)}$ is a TE wave going at an angle θ wrt $-z$ axis.

Poyinting Vector (TE): $S_z = Re\left\{\frac{1}{2}\underline{E} \times \underline{H}^* \cdot \hat{\underline{z}}\right\} = -Re\left\{\frac{1}{2} E_y H_x^*\right\} = \frac{1}{2} V(z) I^*(z)$

Transmission Line Model (TM Polarization):

$\stackrel{*}{=}$ implies that the wave propagates in the layer (i.e. not evanescent), such that $k_z = k\cos(\theta)$

$E_z = \frac{-k_x}{\omega\epsilon} H_y = -\eta \frac{k_x}{k} H_y$ | $k_z = \sqrt{k^2 - k_x^2}$ | $Z_{TM} = \frac{k_z}{\omega\epsilon} = \eta\frac{k_z}{k} \stackrel{*}{=} \eta\cos(\theta)$ | $H_y(x,z) \stackrel{\text{def}}{=} I(z) e^{-jk_x x}$, $E_x \stackrel{\text{def}}{=} V(z) e^{-jk_x x}$

Transmission Line Equations: $-\partial_z V(z) = jk_z Z_{TM} I(z)$, $-\partial_z I(z) = jk_z Y_{TM} V(z)$

Field in Some Layer: Form of Solution in given layer:

$H_y(x,z) = \underbrace{Y_{TM}}_{\eta^{-1}\frac{k}{k_z} \stackrel{*}{=} \frac{1}{\eta\cos(\theta)}} \left[V_0^+ e^{-jk_z z} - V_0^- e^{+jk_z z}\right] e^{-jk_x x}$

$E_x(x,z) = \left[V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}\right] e^{-jk_x x}$

$H_z(x,z) = \underbrace{-\frac{k_x}{k_z}}_{\stackrel{*}{=}-\tan(\theta)} \left[V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}\right] e^{-jk_x x}$

Rewriting for $\stackrel{*}{=}$ case:

$H_y(x,z) \stackrel{*}{=} \frac{1}{\eta\cos(\theta)} \left[V_0^+ e^{-jk_z z} - V_0^- e^{+jk_z z}\right] e^{-jk_x x}$

$E_x(x,z) \stackrel{*}{=} \left[V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}\right] e^{-jk_x x}$

$H_z(x,z) \stackrel{*}{=} -\tan(\theta) \left[V_0^+ e^{-jk_z z} + V_0^- e^{+jk_z z}\right] e^{-jk_x x}$

One can Observe that:

$\left(H_y, E_x, E_z\right) = \left(1, \eta\cos(\theta), -\eta\sin(\theta)\right) H_0^+ e^{-j(k_x x + k_z z)}$ is a TM Wave propagating at an angle θ wrt $+z$ axis, where $H_0^+ = \frac{V_0^+}{\eta\cos(\theta)}$ = amplitude of H_y

$\left(H_y, E_x, E_z\right) = \left(1, -\eta\cos(\theta), -\eta\sin(\theta)\right) H_0^- e^{-j(k_x x - k_z z)}$ is a TE wave going at an angle θ wrt $-z$ axis, where we denote $H_0^- = -\frac{V_0^-}{\eta\cos(\theta)}$ = amplitude of H_y

Poyinting Vector (TM) in a Given Layer: $S_z = Re\left\{\frac{1}{2}\underline{E} \times \underline{H}^* \cdot \hat{\underline{z}}\right\} = -Re\left\{\frac{1}{2} E_x H_y^*\right\} = \frac{1}{2} V(z) I^*(z)$

Oblique Incidence on a PEC:

Consider a PEC @ $z > 0$ and a wave incidence from $z < 0$ @ angle θ.

TE Case:

Incident Field: $\underline{E}^i = \hat{\underline{y}} E_0 e^{-j(k_x x + k_z z)}$

$\underline{H}^i = \left(-\hat{\underline{x}}\cos(\theta) + \hat{\underline{z}}\sin(\theta)\right)\frac{E_0}{\eta} e^{-j(k_x x + k_z z)}$

Where $k_x = k\sin(\theta)$, $k_z = k\cos(\theta)$

Excitation to the TL Problem: $V^+ = E_0 e^{-jk_z z}$

Solution to the TL Problem:

$V(z) = E_0 \underbrace{\left[e^{-jk_z z} - e^{jk_z z}\right]}_{-2j\sin(k_z z)}$; $I(z) = Y_{TE} E_0 \underbrace{\left[e^{-jk_z z} + e^{jk_z z}\right]}_{2\cos(k_z z)}$

Field Solution:

$E_y = E_0\left[e^{-jk_z z} - e^{jk_z z}\right] e^{-jk_x x}$

$H_x = -\underbrace{Y_{TE}}_{\eta^{-1}\cos(\theta)} E_0\left[e^{-jk_z z} + e^{jk_z z}\right] e^{-jk_x x}$

$H_z = \underbrace{\eta^{-1}\frac{k_x}{k}}_{\eta^{-1}\sin(\theta)} E_0\left[e^{-jk_z z} - e^{jk_z z}\right] e^{-jk_x x}$

TM Case:

Incident Field: $\underline{H}^i = \hat{\underline{y}} H_0 e^{-j(k_x x + k_z z)}$

$\underline{E}^i = \underbrace{E_0}_{\eta H_0} \left(\hat{\underline{x}}\cos(\theta) - \hat{\underline{z}}\sin(\theta)\right) e^{-j(k_x x + k_z z)}$

Excitation to the TL Problem: $I^+ = H_0 e^{-jk_z z}$

Solution to the TL Problem:

$I(z) = H_0\left[e^{-jk_z z} + e^{jk_z z}\right]$

$V(z) = \underbrace{H_0 Z_{TM}}_{H_0\eta\cos(\theta)} \left[e^{-jk_z z} - e^{jk_z z}\right]$

Field Solution:

$H_y = H_0\left[e^{-jk_z z} + e^{jk_z z}\right] e^{-jk_x x}$

$E_x = H_0 \underbrace{\eta\cos(\theta)}_{Z_{TM}} \left[e^{-jk_z z} + e^{jk_z z}\right] e^{-jk_x x}$

$E_z = -H_0\eta\sin(\theta)\left[e^{-jk_z z} + e^{jk_z z}\right]$

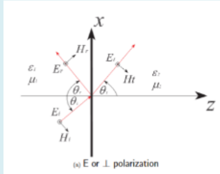
Surface Current: $\underline{J}_s(x)\big|_{z=0} = 2\hat{\underline{y}}\eta^{-1}\cos(\theta) E_0 e^{-jk_x x}$

Surf. Current: $\underline{J}_s(x)\big|_{z=0} = -\hat{\underline{z}} \times \underline{H}\big|_{z=0} = 2\hat{\underline{x}} e^{-jk_x x}$

P- Flow of Inc Wave: $\underline{S} = \left(\hat{\underline{z}}\cos(\theta) + \hat{\underline{x}}\sin(\theta)\right)\frac{|E_0^2|}{2\eta}$

Oblique Incidence on an Interface between 2 media

TE Polarization Case:



Incident Field: $\underline{E}^i = \hat{\underline{y}} E_0 e^{-j(k_x x + k_{z_1} z)}$

$\underline{H}^i = \eta_1^{-1} E_0\left(-\hat{\underline{x}}\cos(\theta_1) + \hat{\underline{z}}\sin(\theta_1)\right) e^{-j(k_x x + k_{z_1} z)}$

Where $k_x = k_1\sin(\theta_1)$, $k_{z_1} = k_1\cos(\theta_1)$, $k_1 = \omega\sqrt{\mu_1\epsilon_1}$, $\eta_1 = \sqrt{\mu_1/\epsilon_1}$

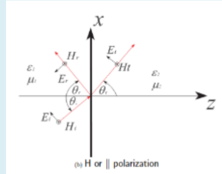
Excitation to the TL Problem: $E_0 e^{-jk_{z_1} z}$

Transmission Line Solution:

$$\left.\begin{aligned} V(z) &= E_0\left[e^{-jk_{z_1} z} + \Gamma_{TE} e^{jk_{z_1} z}\right] \\ I(z) &= Y_{TE} E_0\left[e^{-jk_{z_1} z} - \Gamma_{TE} e^{jk_{z_1} z}\right] \end{aligned}\right\} z < 0$$

$$\left.\begin{aligned} V(z) &= E_0\tau_{TE} e^{-jk_{z_1} z} \\ I(z) &= Y_{TE_2} E_0 \tau_{TE} e^{-jk_{z_2} z} \end{aligned}\right\} z > 0$$

TM Polarization Case:



Incident Field: $\underline{H}^i = \hat{\underline{y}} \underbrace{\eta_1^{-1} E_0}_{H_0} e^{-j(k_x x + k_{z_1} z)}$

$\underline{E}^i = \underbrace{E_0\left[\hat{\underline{x}}\cos(\theta_1)\right.}_{V_0^+} \left. - \hat{\underline{z}}\sin(\theta_1)\right] e^{-j(k_x x + k_{z_1} z)}$

Transmission Line Solution: Obtained by setting $V_0^+ = E_0\cos(\theta_1) = H_0 Z_{TM_1}$

$$\left.\begin{aligned} V(z) &= V_0^+\left[e^{-jk_{z_1} z} + \Gamma_{TM}^V e^{jk_{z_1} z}\right] \\ I(z) &= V_0^+ Y_{TM_1}\left[e^{-jk_{z_1} z} - \Gamma_{TM}^V e^{jk_{z_1} z}\right] \end{aligned}\right\} z < 0$$

$$\left.\begin{aligned} V(z) &= V_0^+ \tau_{TM}^V e^{-jk_{z_2} z} \\ I(z) &= V_0^+ \tau_{TM}^V Y_{TM_2} e^{-jk_{z_2} z} \end{aligned}\right\} z > 0$$

Note: The superscript V means that Γ_{TM}^V, τ_{TM}^V are reflection/transmission coefficients for $V = E_x$. They are NOT the same as those used in the literature, and are defined wrt total E field.

Snell's Law Kinematic Properties:

$k_1\sin(\theta_1) = k_2 sin(\theta_2) \stackrel{\text{def}}{=} k_x$ therefore $k_{z_2} = \sqrt{k_2^2 - k_x^2} = k_2\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2(\theta_1)}$. These expressions can be found in other forms, e.g.: $n_1\sin(\theta_1) = n_2\sin(\theta_2)$ or $\cos(\theta_2) = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2(\theta_1)}$

Total Reflection: If $n_2 < n_1$ then there is a "critical angle of incidence" θ_c such that for $\theta_1 < \theta_c$ there is no real solution for θ_2. θ_c is obtained by setting $\theta_2 = \frac{\pi}{2}$, giving: $\sin(\theta_c) = \frac{n_2}{n_1}$ if $n_2 < n_1$. There are 2 regions:

Refraction Zone: $\theta_1 < \theta_c$, where k_{z_2} is real and given by Snell's law formula from before

Total Reflection Zone: $\theta_1 > \theta_c$, where $k_{z\,2} = -j\sqrt{k_x^2 - k_2^2} = -jk_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2(\theta_1) - 1}$. The sign of j is determined by the demand that the transmitted field $e^{-jk_{z_t} z}$ will decay as z increases from the interface.

Transmission Line Impedance (TE): $Z_{TE_1} = \eta\frac{k_1}{k_{z_1}} = \frac{\eta_1}{\cos(\theta_1)}$, $Z_{TE_2} = \eta_2\frac{k_2}{k_{z_2}} \stackrel{*}{=} \frac{\eta_2}{\cos(\theta_2)}$. $\stackrel{*}{=}$ applies when $\theta_1 < \theta_c$ so that the wave in medium 2 propagates.

Transmission Line Impedance: $Z_{TM_1} = \eta_1\frac{k_{z_1}}{k_1} = \eta_1\cos(\theta_1)$, $Z_{TM_2} = \eta_2\frac{k_{z_2}}{k_2} \stackrel{*}{=} \eta_2\cos(\theta_2)$. $\stackrel{*}{=}$ applies when $\theta_1 < \theta_c$ so that the wave in medium 2 propagates. If $\theta_1 > \theta_c$ then k_{z_2} is imaginary so that Z_{TM_2} is also imaginary.

Reflection & Transmission Coefficients: These are known as Fresnel Coefficients:

Reflection & Transmission Coefficients:

$$\Gamma_{TE}=\frac{Z_{TE_2}-Z_{TE_1}}{Z_{TE_2}+Z_{TE_1}},\ \tau_{TE}=\frac{2Z_{TE_2}}{Z_{TE_2}+Z_{TE_1}}$$

$$\Gamma_{TM}^{V}=\frac{Z_{TM_2}-Z_{TM_1}}{Z_{TM_2}+Z_{TM_1}},\ \tau_{TM}^{V}=\frac{2Z_{TM_2}}{Z_{TM_2}+Z_{TM_1}}$$

As noted earlier, these are coefficients for E_x

Reflection & Transmission Special Case: In the case of a dielectric medium ($\mu_1=\mu_2=\mu_0$, $s.t\ n=\sqrt{\epsilon_r}$:

$$Z_{TE}=\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{n\cos(\theta)},\ \Gamma_{TE}=\frac{n_1\cos(\theta_1)-n_2\cos(\theta_2)}{n_1\cos(\theta_1)+n_2\cos(\theta_2)},$$

$$\tau_{TE}=\frac{2n_1\cos(\theta_1)}{n_1\cos(\theta_1)+n_2\cos(\theta_2)}.$$

Fresnel Coefficients: Noting that $E_x=E_{total}\frac{k_z}{k}\overset{*}{=}E_{total}\cos(\theta)$, we have:

$$\Gamma_{TM}\overset{\text{def}}{=}\frac{E_{tot}^{ref}}{E_{tot}^{inc}}=\frac{E_x^r}{E_x^i}=\Gamma_{TM}^V=\frac{Z_{TM_2}-Z_{TM_1}}{Z_{TM_2}+Z_{TM_1}}$$

$$\tau_{TM}\overset{\text{def}}{=}\frac{E_{tot}^{trans}}{E_{tot}^{inc}}=\frac{E_x^t}{E_x^i}\frac{k_2/k_{z_2}}{k_1/k_{z_1}}=\tau_{TM}^V\frac{k_2/k_{z_2}}{k_1/k_{z_1}}=\frac{2Z_{TM_2}(k_{z_1}/k_1)}{Z_{TM_2}+Z_{TM_1}}$$

Under Total Reflection Condition: If $\theta_1>\theta_c$ then k_{z_2} is imaginary and Z_{TE_2} is also imaginary, so $|\Gamma_{TE}|=1$

In the special case of a dielectric medium ($\mu_1=\mu_2=\mu_0$ such that $n=\sqrt{\epsilon_r}$):

$$Z_{TM}=\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\cos(\theta)}{n},\ \Gamma_{TM}=\frac{n_2^{-1}\cos(\theta_2)-n_1^{-1}\cos(\theta_1)}{n_2^{-1}\cos(\theta_2)+n_1^{-1}\cos(\theta_1)}$$

$$\tau_{TM}=\frac{2n_2^{-1}\cos(\theta_1)}{n_2^{-1}\cos(\theta_2)+n_1^{-1}\cos(\theta_1)}$$

Field Solution for $z<0$:

$$E_y=E_0\left[e^{-jk_{z_1}z}+\Gamma_{TE}e^{jk_{z_1}z}\right]e^{-jk_xx}$$

$$H_x=-\underbrace{\eta_1^{-1}\cos(\theta_1)}_{Y_{TE_1}}E_0\left[e^{-jk_{z_1}z}-\Gamma_{TE}e^{jk_{z_1}z}\right]e^{-jk_xx}$$

$$H_z=\underbrace{\eta_1^{-1}\sin(\theta_1)}_{\eta_1^{-1}\frac{k_x}{k_1}}E_0\left[e^{-jk_{z_1}z}-\Gamma_{TE}e^{jk_{z_1}z}\right]e^{-jk_xx}$$

Field Solution for $z<0$:

$$H_y=\underbrace{E_0\cos(\theta_1)\,Y_{TM}}_{E_0\eta_1^{-1}}\left[e^{-jk_{z_1}z}-\Gamma_{TM}^Ve^{jk_{z_1}z}\right]e^{-jk_xx}$$

$$E_x=E_0\cos(\theta_1)\left[e^{-jk_{z_1}z}+\Gamma_{TM}^Ve^{jk_{z_1}z}\right]e^{-jk_xx}$$

$$E_z=\underbrace{-E_0\cos(\theta_1)\,Y_{TM}\frac{k_x}{k_{z_1}}}_{-E_0\sin(\theta_1)}\left[e^{-jk_{z_1}z}-\Gamma_{TM}^Ve^{jk_{z_1}z}\right]e^{-jk_xx}$$

Field Solution for $z>0$:

$$E_y=\tau_{TE}E_0e^{-jk_{z_2}z}e^{-jk_xx}$$

$$H_z=\eta_2^{-1}\frac{k_x}{k_2}\tau_{TE}e^{-jk_{z_2}z}e^{-jk_xx}\overset{*}{=}\frac{\tau_{TE}E_0}{\eta_2}e^{-jk_{z_2}z}e^{-jk_xx}$$

$$H_x=-\underbrace{Y_{TE_2}}_{\eta_2^{-1}\frac{k_{z_2}}{k_2}\overset{*}{=}\eta_2^{-1}\cos(\theta_2)}\tau_{TE}E_0e^{-jk_{z_2}z}e^{-jk_xx}\overset{*}{=}-\frac{\tau_{TE}E_0}{\eta_2}\cos(\theta_2)\,e^{-jk_{z_2}z}e^{-jk_xx}$$

Field Solution for $z>0$:

$$H_y=E_0\cos(\theta_1)\,\tau_{TM}^V\underbrace{Y_{TM_2}}_{\frac{1}{\eta_2}=\frac{k_2}{k_{z_2}}\overset{*}{=}\frac{1}{\eta_2\cos(\theta_2)}}e^{-jk_{z_2}z}e^{-jk_xx}$$

$$E_x=E_0\cos(\theta_1)\,\tau_{TM}^Ve^{-jk_{z_2}z}e^{-jk_xx}$$

$$E_z=-E_0\cos(\theta_1)\,\tau_{TM}^V\underbrace{\frac{k_x}{k_{z_2}}}_{\overset{*}{=}\tan(\theta_2}e^{-jk_{z_2}z}e^{-jk_xx}$$

Brewster angle: The Brewster angle is defined by $\Gamma=0\to Z_1=Z_2$. Assuming dielectric medium ($\mu_1=\mu_2=\mu_0$) the wave impedances are: $Z_{TE}=\sqrt{\frac{\mu}{\epsilon}}\frac{1}{\cos(\theta)}=\eta_0\frac{1}{n\cos(\theta)}$, $Z_{TM}=\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\cos(\theta)}{n}=\frac{\eta_0\cos(\theta)}{n}$.

For TE Polarization: Brewster angle should satisfy $n_1\cos(\theta_1)=n_2\cos(\theta_2)$ but this contradicts Snell's law, so there is no solution.

For TM Polarization: Brewster angle is found to be: $coa(\theta_1)=\left(\frac{n_1}{n_2}\right)\cos(\theta_2)=\left(\frac{n_1}{n_2}\right)\sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2(\theta_1)}$

Formulation of the Radiation Problem:

We want to calculate the radiation into half space $z>0$ due to a given physical field $\underline{E}(x,y,0)$ in the $z=0$ plane. We assume initially that the medium at $z>0$ is uniform. Later we consider layered media.

Theorem of Determination of Radiated Field: The radiated field is fully determined by the transversal components of $\underline{E}(x,y,0)$, therefore we need to know: $\underline{E}_T(x,y,0)=\hat{\underline{x}}E_x(x,y,0)+\hat{\underline{y}}E_y(x,y,0)$

Initial Field: (Also referred to as "data") is typically generated by sources $\underline{J}(r)$ in the $z\le0$ domain.

Field for $z>0$: Is expressed as an angular spectrum of Plane Waves:

$$\underline{E}(\underline{r})=\frac{1}{(2\pi^2)}\iint dk_xdk_y\underline{\bar{E}}_0(k_x,k_y)e^{-j(k_xx+k_yy+k_zz)}\quad\Big|\quad\underline{H}(\underline{r})=\frac{1}{(2\pi^2)}\iint dk_xdk_y\underline{\bar{H}}_0(k_x,k_y)e^{-j(k_xx+k_yy+k_zz)}$$

For a given (k_x,k_y), $\underline{\bar{E}}_0(k_x,k_y)$ and $\underline{\bar{H}}_0(k_x,k_y)$ are constant vectors in space that define the PW amplitude. (k_x,k_y) defines direction. $\underline{\bar{E}}_0,\underline{\bar{H}}_0$ should destribe a valid plane wave, i.e. $\underline{\bar{E}}_0\perp\underline{\bar{H}}_0\perp\underline{k}$ and $|\underline{\bar{E}}_0|=\eta|\underline{\bar{H}}_0|$. The functions $\underline{\bar{E}}_0(k_x,k_y)$, $\underline{\bar{H}}_0(k_x,k_y)$ are the spectrum.

Propagating and Evanescent Spectra

k_z is determined from (k_x,k_y) via: $k_z=\sqrt{k^2-k_x^2-k_y^2}$, $k=\omega\sqrt{\mu\epsilon}=\omega/c$

Propagating Spectrum: $k_x^2+k_y^2<k^2$

Evanescent Spectrum: $k_x^2+k_y^2>k^2\to k_z=-j\sqrt{k_x^2+k_y^2-k^2}$

In the Propagating Regime: Plane Waves propagate in the spectral direction given by: $\hat{\underline{k}}=\left(\frac{k_x}{k},\frac{k_y}{k},\frac{k_z}{k}\right)=\left(\cos(\theta_x),\cos(\theta_y),\cos(\theta_z)\right)=(\sin(\theta)\cos(\varphi),\sin(\theta)\sin(\varphi),\cos(\theta))$. $\theta_{x,y,z}$ are the angles from the x,y,z axes satisfying $\cos^2\theta_x+\cos^2\theta_y+\cos^2\theta_z=1$. The (θ,φ) notation used is θ =polar, φ=azimuthal angles.

Resolution: Fine details in the data of order $\Delta<\frac{2\pi}{k}=\lambda$ are lost in the radiation, so it acts as a low pass filter for details smaller than λ.

Calculating the Spectral Amplitudes:

The **Spectral Amplitude** $\underline{\bar{E}}_0(k_x,k_y)$ is determined from the initial conditions:

$$\bar{E}_{0_x}(k_x,k_y)=\iint dxdy\ e^{j(k_xx+k_yy)}E_x(x,y,0)\quad\Big|\quad\bar{E}_{0_y}(k_x,k_y)=\iint dxdy\ e^{j(k_xx+k_yy)}E_x(x,y,0)$$

Since for each (k_x,k_y), $\underline{\bar{E}}_0(k_x,k_y)$ is a physical PW, it should satisfy the plane wave structure (the perpendicular shit). Thus $\bar{E}_{0_z}(k_x,k_y)=\frac{-k_x\bar{E}_{0_x}(k_x,k_y)+k_y\bar{E}_{0_y}(k_x,k_y)}{k_z}$.

The obtained **Spectral Field** $\underline{\bar{E}}_0(k_x,k_y)=\left(\bar{E}_{0_x}(k_x,k_y),\bar{E}_{0_y}(k_x,k_y),\bar{E}_{0_z}(k_x,k_y)\right)$ is then used in the angular spectrum integral.

Closed Form Expression for the Far Field:

For observation Points in the Far Zone: The field structure is a spherical wave, whose amplitude, the radiation pattern, is proportional to the spectral plane wave that propagates in the direction of $\underline{r}$.

$$\underline{E}(\underline{r})=-2jk\underbrace{\frac{e^{-jkr}}{4\pi r}}_{\text{Spherical Wave }g(r)}\underbrace{\cos(\theta)\,\underline{\bar{E}}_0\left(k_x=k\sin(\theta)\cos(\varphi),k_y=k\sin(\theta)\sin(\varphi)\right)}_{\text{Radiation Pattern}}$$

Far Zone (Fraunhofner Zone): $4\pi kL^2\cos(\theta)\ll r$, where L is a scale that defines the "spatial width" of the sources in the $z=0$ plane.

Far Field Structure:

Spherical Waves: Field dependence on the range r is governed by $g(r)$. Spherical **phase-fronts** (surfaces of constant phase) → field propagates outside at speed c.

$\frac{1}{r}$ **decay** → conservation of radiation power.

Note the difference between:

Spherical Phasefront: e^{-jkr} with $k=\omega/c$, always associated with $\frac{1}{r}$ decay.

Planar Phasefront: $e^{-j\underline{k}\cdot\underline{r}}$ with $|\underline{k}|=k=\omega/c$

Radiation Pattern: The radiation has a (θ,φ) pattern determined by the spectral plane wave in the observation direction

Local Plane Wave Structure: In any direction $\hat{\underline{r}}=(\theta,\varphi)$, the E field has the structure of a local plane wave $\underline{E}\perp\underline{H}\perp\hat{\underline{r}}$ and $\underline{H}=\eta^{-1}\hat{\underline{r}}\times\underline{E}$.

Propagation in Layered Media, TE/TM Decomposition:

The angular spectrum formulation from "formulation of the radiation pattern" only holds for $z > 0$ and a uniform medium. If the medium for $z > 0$ is layered, each PW has to be propagated in the medium.

This is done by decomposing $\bar{E}_0(k_x, k_y)$ into TE and TM polarizations wrt the "spectral plane of incidence", formed by the $\underline{k}$ vector associated with (k_x, k_y) and the stratification axis $\hat{\underline{z}}$. Thus:

For each $(\boldsymbol{k_x}, \boldsymbol{k_y})$ we identify 2 independent polarizations:

TE: $\bar{E}_{0_q}(k_x,k_y), \bar{H}_{0_p}(k_x,k_y)$, $\bar{H}_{0_z}(k_x,k_y)$	**TM:** $\bar{E}_{0_p}(k_x,k_y), \bar{E}_{0_z}(k_x,k_y)$, $\bar{H}_{0_q}(k_x,k_y)$

Define 2 Orthogonal Unit Vectors: $\hat{\underline{p}} = \frac{\hat{\underline{x}}k_x + \hat{\underline{y}}k_y}{\sqrt{k_x^2+k_y^2}}$, $\hat{\underline{q}} = \hat{\underline{z}} \times \hat{\underline{p}}$. The vectors $\hat{\underline{p}} = \parallel$, $\hat{\underline{q}} = \perp$ are in the plane of incidence and normal to it, respectively, such that $\hat{\underline{p}} \times \hat{\underline{q}} = \hat{\underline{z}}$

$\bar{E}_{0_p}, \bar{E}_{0_q}$ are found from Initial Conditions: $\bar{E}_{0_p}(k_x,k_y) = \hat{\underline{p}} \cdot \bar{E}_{0_T}(k_x,k_y), \bar{E}_{0_q}(k_x,k_y) = \hat{\underline{q}} \cdot \bar{E}_{0_T}(k_x,k_y)$

The other components are found by imposing the plane wave structure: $\bar{H}_0 = \eta^{-1}\hat{\underline{k}} \times \bar{E}_0$ and $\bar{E}_0 \cdot \bar{H}_0 \cdot \underline{k}$ (aka $\underline{k} \cdot \underline{E}_0 = \underline{k} \cdot \underline{H}_0 = \underline{E}_0 \cdot \underline{H}_0 = 0$), giving:

TE Polarization: $\bar{E}_{0_q}$, $\bar{H}_{0_p} = -\eta^{-1}\frac{k_z}{k}\bar{E}_{0_p}$, $\bar{H}_{0_z} = \eta^{-1}\frac{k_p}{k} \cdot \overline{E_{0_q}}$

TM Polarization: $\bar{E}_{0_p}$, $\bar{E}_{0_z} = -\frac{k_z}{k_p}\bar{E}_{0_p}$, $\bar{H}_{0_q} = \eta^{-1}\frac{k}{k_z}\bar{E}_{0_p}$

2D: Diffraction by Slits and Strips

Consider a 2D case, where a wave is impinging from $z < 0$ on an opaque screen in the $z = 0$ plane with a slit of width D, defined by $|x| < \frac{D}{2}$ (i.e. start of each slit is $\frac{D}{2}$ from the middle, vertically) . The wave propagates in the (x, z) plane, the problem is y —independent. There are 2 main Polarizations:

TE: $\underline{E} = (0, E_y, 0)$, $\underline{H} = (H_x, 0, H_z)$	**TM:** $\underline{E} = (E_x, 0, E_z)$, $\underline{H} = (0, H_y, 0)$

Since the problem is y-independent, we have $k_y \equiv 0$. Hence, instead of the standard "angular spectrum of plane waves" and "spectral amplitudes", we have:

[9.2] Spectrum: $\underline{E}(\underline{r}) = \frac{1}{2\pi}\int dk_x \underline{\bar{E}}_0(k_x)e^{-j(k_x x + k_z z)}$, $k_z = \sqrt{k^2 - k_x^2}$

Spectral Amplitudes:… where $\bar{E}_{0_T}$is calculated from the data via: $\bar{E}_{0_T}(k_x) = \int dx e^{jk_x x}\underline{E}_T(x, 0)$

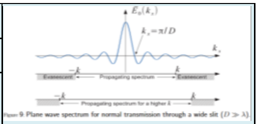
[9.4] Spectrum in TE Case: In TE, we have $\underline{E}_T(x, 0) = \hat{\underline{y}}E_y(x, 0)$ so that

$\bar{E}_{0y}(k_x) = \int dx e^{jk_x x}E_y(x,0) \rightarrow E_y(\underline{r}) = \frac{1}{2\pi}\int dk_x \underbrace{\left(\hat{\underline{x}} - \hat{\underline{z}}\frac{k_x}{k_z}\right)\bar{E}_{0x}(k_x)}_{\underline{\bar{E}}_0(k_x)} e^{-j(k_x x + k_z z)}$. For the accent, see "calculating the spectral amplitudes" – big fraction stuff, 3rd row cell.

Physical Optics (PO) Approximation for the field in the $\boldsymbol{z = 0^+}$ plane:
Field for $z > 0$ is given by [9.2], with $\underline{E}_T(x, 0^+) =$ field just after the screen. What is $\underline{E}_T(x, 0^+)$? In general it is found numerically, but we have an approximation:

For $D \gg \lambda$ (wide slits), we may approximate: $\underline{E}(x, 0^+) \simeq \begin{cases} \underline{E}^i(x,0) \,;\ |x| < D/2 \\ 0 \ ;\text{otherwise} \end{cases}$, as long as $D \gg \lambda$, where $\underline{E}^i$ is the incident field. This model is valid for $D \lesssim \lambda$

Normal Incidence on a Wide Slit (D>>λ):



Incident PW: For $z < 0$ can be separated into 2 polarizations:

TE: $\underline{E}^i = \hat{\underline{y}}Ae^{-jkz}$, $\underline{H}_i = -\hat{\underline{x}}\frac{A}{\eta}e^{-jkz}$	**TM:** $\underline{E}^i = \hat{\underline{x}}Ae^{-jkz}$, $\underline{H}^i = \hat{\underline{y}}\frac{A}{\eta}e^{-jkz}$

TE Case: For a wide screen, the field in the $z = 0^+$ plane is approximated by: $\underline{E}(x, 0^+) \simeq \begin{cases} \hat{\underline{y}}A \,;\ |x| < D/2 \\ 0 \ ;\text{otherwise} \end{cases}$

[9.9] The Spectral Field in [9.4] becomes:

$\bar{E}_{0y}(k_x) = \int dx E_y(x, 0^+)e^{jk_x x} \simeq \int_{-\frac{D}{2}}^{\frac{D}{2}} dx A e^{jk_x x} = A\frac{e^{jk_x\frac{D}{2}} - e^{-jk_x\frac{D}{2}}}{jk_x} = AD\text{sinc}(k_x\frac{D}{2})$

[9.10] And the **spectral integral** in [9.4] becomes:$\underline{E}(x,z) = \hat{\underline{y}}\frac{1}{2\pi}\int_{-k}^{k} dk_x\bar{E}_{0y}(k_x)\, e^{-j(k_x x + k_z z)}$, $k_z = \sqrt{k^2 - k_x^2}$, where we neglected the contribution of the evanescent spectrum, $|k_x| > k$.

TM Case: We use the PO Approximation for the field in the $z = 0^+$ plane: $\underline{E}(x, 0^+) \simeq \begin{cases} \hat{\underline{x}}A \,;\ |x| < D/2 \\ 0 \ ;\text{otherwise} \end{cases}$

Spectral Field: $\bar{E}_{0x}(k_x) = \int dx\ E_x(x, 0^+)e^{jk_x x} = AD\text{sinc}\left(k_x\frac{D}{2}\right)$

Spectral Integral for the Field at $\boldsymbol{z > 0}$: $\underline{E}(x,z) = \frac{1}{2\pi}\int_{-k}^{k} dk_x \left(\hat{\underline{x}} - \hat{\underline{z}}\frac{k_x}{k_z}\right)\underbrace{\bar{E}_{0x}(k_x)}_{AD\text{sinc}\left(k_x\frac{D}{2}\right)}\ e^{-j(k_x x + k_z z)}$

The Main Beam $\boldsymbol{\theta_D < \lambda/D}$:

[9.14] Main Lobe: The "main lobe" of the spectral function [9.9] is bounded in the spectral range: $-\pi < \frac{k_x D}{2} < \pi \rightarrow -\frac{2\pi}{D} < k_x < \frac{2\pi}{D}$.	Recall that radiation spectrum in [9.10] is bounded in $-\frac{2\pi}{\lambda} < k_x < \frac{2\pi}{\lambda}$. It follows that if $\lambda < D$ then all the main lobe is included in the radiation spectrum.

Direction of the Main Beam: Noting that $k_x = ksin(\theta)$, where θ is the PW angle from the z-axis, it follows from [9.14] that the main beam radiates in the directions: $-\frac{\lambda}{D} < \sin(\theta) < \frac{\lambda}{D} \overset{\text{"}if\ \lambda \ll D}{\Longrightarrow} -\frac{\lambda}{D} < \theta < \frac{\lambda}{D}$

Narrow Beam: Thus, if $\lambda \ll D$ then the spectrum is localized in a "narrow beam"

Diffraction Angle: The angle $\theta_D = \frac{\lambda}{D}$	If $D < \lambda$ then the radiated field spectrum radiates almost uniformly in all directions

Oblique Incidence on a Slit

Here, the incident PW propagates at an angle θ_0 relative to the z-axis, i.e. :

TE: $\underline{E}^i = \hat{\underline{y}}Ae^{-jk(x\sin(\theta_0) + z\cos(\theta_0))}$	**TM**: $\underline{H}^i = \hat{\underline{y}}Ae^{-jk(x\sin(\theta_0) + z\cos(\theta_0))}$

PO Approximation of the Field Beyond the Slit (TE): $\underline{E}(x, y, 0^+) = \begin{cases} \hat{\underline{y}}Ae^{-jkx\sin(\theta_0)} \,;\ |x| < D/2 \\ 0 \ ;\text{otherwise} \end{cases}$

Spectral Field in [9.4],[9.9] **Becomes**: $\underline{E}_{0y}(k_x) \simeq \int_{-\frac{D}{2}}^{\frac{D}{2}} dx\ Ae^{-jkx\sin(\theta_0)}e^{jk_x x} = ADsinc(\left(k_x - ksin(\theta)\frac{D}{2}\right)$

Radiated $\boldsymbol{E}$ field: Given, again, by [9.10]	**Direction of Max of Radiation:** As expected, max is in the direction $k_x = ksin(\theta_0) \rightarrow \theta = \theta_0$

Main Lobe: The main lobe of the "sinc" function is confined in the range $-\pi < \left(k_x - ksin(\theta_0)\right)\frac{D}{2}$ <\pi

The Plane Wave Angle Satisfies: $-\frac{\lambda}{D} < sin(\theta) - sin(\theta_0) < \frac{\lambda}{D}$, i.e a narrow spectral angle $\pm\frac{\lambda}{D}$ about θ_0

Oblique Incidence on a PEC Strip:

Strip is solid from $-\frac{D}{2}$ to $\frac{D}{2}$. We consider a PEC strip $|x| < \frac{D}{2}$ in $z = 0$, illuminated by the plane wave propagating at an angle θ_0 relative to the –z axis.

Incident Fields:

TE: $\underline{E}^i = \hat{\underline{y}}E_0 e^{-jk(x\sin(\theta_0) - z\cos(\theta_0))}$, $\underline{H}^i = (\hat{\underline{x}}\cos(\theta_0) + \hat{\underline{z}}\sin(\theta_0))\frac{E_0}{\eta}e^{-jk(x\sin(\theta_0) - z\cos(\theta_0))}$

TM: $\underline{H}^i = \hat{\underline{y}}\frac{E_0}{\eta}e^{-jk(x\sin(\theta_0) - z\cos(\theta_0))}$, $\underline{E}^i = (-\hat{\underline{x}}\cos(\theta_0) - \hat{\underline{z}}\sin(\theta_0))E_0 e^{-jk(x\sin(\theta_0) - z\cos(\theta_0))}$

Reflected fields at $\boldsymbol{z = 0^+}$ are given by (we only need the transversal E fields):

TE: $\underline{E}_T(x, y, 0^+) \simeq \begin{cases} \hat{\underline{y}}E_0 e^{-jkx\sin(\theta_0)} \,;\ |x| < D/2 \\ 0 \ ;\text{otherwise} \end{cases}$	**TM:** $\underline{E}_T(x, y, 0^+) \simeq \begin{cases} \hat{\underline{x}}E_0\cos(\theta_0)\, e^{-jkx\sin(\theta_0)} \,;\ |x| < D/2 \\ 0 \ ;\text{otherwise} \end{cases}$

Отсюдя продолжаен как в "oblique incidence on a slit".

Two Slits: Same problem of normal incidence on a slit, but …

Now there are 2 slits of width D, centered at $x = \pm\frac{L}{2}$, where $L > D$.

Spectral Field becomes: $\bar{E}_{0y}(k_x) = \int dx\, E_y(x, 0^+)e^{jk_x x} = AD\mathrm{sinc}\left(k_x\frac{D}{2}\right) \times 2\cos\left(k_x\frac{L}{2}\right)$

The resulting spherical field is obtained by superposing the 2 cos $\left(k_x\frac{L}{2}\right)$ function on the sinc function . If $L \gg D$ then this cos function is oscillating many times within the main lobe of the sinc function. Thus we obtain many maxima at angles, and nulls which are due to the positive/negative interference of the contributions of 2 slits. Directions of the maxima: $k_x\frac{L}{2} = n\pi \to \sin(\theta_n) = \frac{n\lambda}{L}$

A Multi-Layer Medium:

Consider a multilayer medim for $z > 0$. Как пример рассмотрим the TE problem with the PO approximation like in [9.8].[9.17],[9.21a], where the field is given by the plane wave integral (see [9.10]):

$$\underline{E}(x,z) = \underline{\hat{y}}\frac{1}{2\pi}\int_{-k}^{k} dk_x \bar{E}_y(z,k_x)e^{-jk_x x}\ , k_z = \sqrt{k^2 - k_x^2}$$

$\bar{E}_y(z,k_x)$ is a sol of TE spectral TL **whose parameters in the i[th] layer are:** $k_{z_i} = \sqrt{k_i^2 - k_x^2}$, $Z_{TE_i} = \frac{\omega\mu_i}{k_{z_i}} = \eta_i\frac{k_i}{k_{z_i}} \overset{*}{=} \frac{\eta}{\cos(\theta)}$, and excitation by a forward propagating wave: $V_0^+(z) = \bar{E}_{0_y}(k_x)e^{-jk_{z_0}z}$

RADIATION CHAPTER FROM HERE AND ON

Statement of the Problem: Find the EM field $(\underline{E}, \underline{H})$ due to a given current distribution $\underline{J}(\underline{r}, t)$ of finite support. The medium is homogenous & unbounded. For simplicity we consider the time harmonic formulation, i.e. $\underline{J}(\underline{r}, t) = \underline{\tilde{J}}(\underline{r})e^{j\omega t}$ and omit the tilde.

Lorenz Gauge:
[3.5]We choose: $\nabla\cdot\underline{A} = -j\omega\mu\epsilon\Phi$, leading to: $[\underline{\nabla}^2 + k^2]\underline{A}(\underline{r}) = -\mu\underline{J}(\underline{r})$, $k = \omega\sqrt{\mu\epsilon} = \omega/c$.
[3.6]In Cartesian, this vector equation describes 3 scalar eq-s: $(\nabla^2 + k^2)A_j(\underline{r}) = -\mu J_j(\underline{r}), j = x, y, z$
[3.7]Once $\underline{A}$ is solved via [3.5], we have: $\underline{H}(\underline{r}) = \frac{1}{\mu}\nabla\times\underline{A}$; $\underline{E}(\underline{r}) = -j\omega\underline{A} - \nabla\Phi = -j\omega\underline{A} + \frac{c^2}{j\omega}\nabla\nabla\cdot\underline{A} = -j\omega\left(1 + \frac{1}{k^2}\nabla\nabla\cdot\right)\underline{A}$

Retarded Potential Solution to the Radiation Problem:

The 3 solutions to [3.6] can be expressed in vector form: $\underline{A}(\underline{r}, t) = \mu\int_V dV'\underline{J}(\underline{r}')\frac{e^{-jkR}}{4\pi R}$, for $\underline{A}, \underline{J}$ Cartesian.

Once we know $\underline{A}$ we may calculate $\underline{E}, \underline{H}$ via [3.7]. $R = |\underline{r} - \underline{r}'|$

Green's Function:

$\nabla^2 G(\underline{r}, \underline{r}') = -\delta(\underline{r} - \underline{r}')$ where the 3D δ function is define via $\delta(\underline{r} - \underline{r}') = 0\ \forall\underline{r} \neq \underline{r}'$, and for any volume V: $\int_V dV\, \delta(\underline{r} - \underline{r}') = \begin{cases} 1\ ; \underline{r}' \in V \\ 0\ ; else \end{cases}$

$$G(\underline{r}, \underline{r}') = \frac{1}{4\pi R}$$

Time Dependent Case: $G(\underline{r}, \underline{r}') = \frac{e^{-jkR}}{4\pi R}$, $R = |\underline{r} - \underline{r}'|$

Anti-Causal Green's Func: $G^{\dagger}(\underline{r}, \underline{r}') = \frac{e^{+jkR}}{4\pi R}$, repr. an incoming wave that converges onto the sink at $\underline{r}'$

Radiation from a Short Electrical Dipole – Hertzian Dipole:

Hertzian Dipole: Consists of 2 equal and opposite charge reservoirs located a distance d from each other, connected through a wire.

[5.1]Consider a z –directed dipole at the origin. We have:
$\underline{r}' = (x', y', z') = \left(0, 0, \frac{d}{2}\right)$, $q(t) = q_0 e^{j\omega t}$ (if $(0, 0, -\frac{d}{2})$ then $-q(t)$)

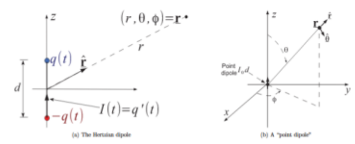


Figure 2: The Hertzian dipole

Current in the wire: $I(t) = q'(t) = \underbrace{j\omega q_0}_{I_0} e^{j\omega t}$ | **Dipole moment:** $I_0 d = j\omega q_0 d$ | Current Distribution: $\underline{J} = \underline{\hat{z}}(I_0 d)\delta(\underline{r} - \underline{r}')e$

If the wire is short such that $d \ll T$ (i.e. $d \ll \lambda$): The new may describe $\underline{J}$ as an effective current source at the origin, whose overall strength is: $\underline{J}(\underline{r}, t) = \underline{\hat{z}}(I_0 d)e^{j\omega t}\delta(\underline{r})$

The field:
From [4.5] we obtain: $\underline{A}(\underline{r}) = \underline{\hat{z}}\mu(I_0 d)g(r)$, $g(r) \overset{\text{def}}{=} G(\underline{r}, \underline{r}' = 0) = \frac{e^{-jkr}}{4\pi r}$
Polar Coordinate Replacement: The EM fields will be found in polar, so we replace the unit vectors as follows: $\underline{\hat{z}} = -\underline{\hat{\theta}}\sin(\theta) + \underline{\hat{r}}\cos(\theta) \to \underline{A}(\underline{r}) = \mu(I_0 d)[-\underline{\hat{\theta}}\sin(\theta) + \underline{\hat{r}}\cos(\theta)]g(r)$.We find $\nabla\times A$ using some long-ass algebra

EM Field is Given by [5.6],[5.7]:
$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \underline{\hat{\varphi}}(I_0 d)\left(jk + \frac{1}{r}\right)g(r)\sin(\theta)$, $\underline{E} = -j\omega\mu(I_0 d)\left[\underline{\hat{r}}\cos(\theta)\left(\frac{2j}{kr} + \frac{2}{(kr)^2}\right) - \underline{\hat{\theta}}\sin(\theta)\left(1 - \frac{j}{kr} - \frac{1}{(kr)^2}\right)\right]g(r)$

The "Near Field" ($r \ll \lambda$ or $kr \ll 1$): In this range, $g(r) = \frac{e^{-jkr}}{4\pi r} \approx \frac{1}{4\pi r}$. The dominant term has a r^{-3} singularity.
The following expression is the **quasi-static field of an electric dipole** with some time dependent moment $\underline{p}(r) = \underline{\hat{z}}\frac{(I_0 d)}{j\omega}e^{j\omega t} = \underline{\hat{z}}(q_0 d)e^{j\omega t}$: $\underline{E} \approx \underbrace{\frac{(I_0 d)}{j\omega}}_{p_0}\frac{1}{4\pi\epsilon r^3}\left(\underline{\hat{r}}2\cos(\theta) + \underline{\hat{\theta}}\sin(\theta)\right)$ **[5.8]**

[5.11]The "radiation field" ($kr \gg 1$): $\underline{E} \approx \underline{\hat{\theta}}j\underbrace{\omega\mu}_{k\eta}(I_0 d)g(r)\sin(\theta)$, $\underline{H} \approx \underline{\hat{\varphi}}jk(I_0 d)g(r)\sin(\theta) = \underline{\hat{\varphi}}\eta^{-1}E_\theta$

Observations:
Spherical Wave: The field dependence on the range r is governed by $g(r)$. Spherical phase fronts -> field propagates out at speed c. $\frac{1}{r}$ decay -> conservation of radiation power.
Radiation Pattern: Radiation is not uniform in all directions, but has a $\sin(\theta)$ pattern
Local Plane Wave: In any direction $\underline{\hat{r}} = (\theta, \varphi)$the E field is θ polarized and H field is φ polarized. The field has a structure of a "local plane wave", i.e. $\underline{E} \perp \underline{H} \perp \underline{\hat{r}}$, specifically $\underline{H}\eta^{-1}\underline{\hat{r}} \times \underline{E}$.
Poyinting Vector: Outgoing in $+\underline{\hat{r}}$ direction: $\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* \approx \frac{|E_0|^2}{2\eta} = (kI_0 d)^2\left(\frac{\sin(\theta)}{4\pi r}\right)^2\frac{\eta}{2}\underline{\hat{r}}$
Total Radiation Power: $P_{rad} \overset{\text{def}}{=} \oint\underline{S}\cdot d\underline{a} = \int_0^{\pi}\int_0^{\pi} S_r r^2\sin(\theta)\,d\theta d\varphi = \frac{(kI_0 d)^2\eta}{12\pi}$

Observations about the Near and Far Fields:
In the far zone, for $r \gg \lambda$, the E_r vanishes compared to E_θ (and H_φ). | In the far zone, we have an outgoing wave. |In the near zone, for $r \lesssim \lambda$, the main part of the field represents standing oscillations.

Characterization of the Far Radiation Fields: The most general structure of the radiation field:

$$\underline{E}(\underline{r}) = \underbrace{\frac{e^{-jkr}}{r}}_{\text{Spherical Wave}}\underbrace{\left[\underline{\hat{\theta}}E_\theta(\theta,\varphi) + \underline{\hat{\varphi}}E_\varphi(\theta,\varphi)\right]}_{\text{Radiation Pattern}}, \underline{H}(\underline{r}) = \underbrace{\frac{e^{-jkr}}{r}}_{\text{Spherical Wave}}\frac{1}{\eta}\underbrace{\left[\underline{\hat{\varphi}}E_\theta(\theta,\varphi) + \underline{\hat{\theta}}E_\varphi(\theta,\varphi)\right]}_{\text{Radiation Pattern}}$$

Parameter	Definition	For the Hertzian Dipole
Radiation Pattern:	$f(\theta,\varphi) = \frac{\lvert\underline{E}(\theta,\varphi)\rvert}{\lvert\underline{E}\rvert_{\max}}$	$= \sin(\theta)$
Power Density:	$S_r(\theta,\varphi) = \frac{\partial P}{\partial A} = \frac{1}{2\eta}\lvert\underline{E}\rvert^2$	$= (kI_0 d)^2\left(\frac{\sin(\theta)}{4\pi r}\right)^2\frac{\eta}{2}$
Radiation Intensity:	$U_r(\theta,\varphi) = \frac{\partial P}{\partial\Omega} = r^2 S_r$	$= (kI_0 d)^2\left(\frac{\sin(\theta)}{4\pi}\right)^2\frac{\eta}{2}$
Radiation Power:	$P_{rad} = \int_{4\pi} S_r(\theta,\varphi)r^2 d\Omega$	$= \frac{(kI_0 d)^2\eta}{12\pi}$
Gain:	$G(\theta,\varphi) = \frac{S_r}{P_{rad}/4\pi r^2}$	$= \frac{3}{2}\sin^2(\theta)$
Directivity:	$G_{\max}$	$= \frac{3}{2}$

Field of Several Dipoles: Interference Pattern:

Consider the field of several Hertzian dipoles located at points $\underline{r}_n$. All of them are $\hat{\underline{\xi}}$ polarized. The overall field is a simple vector summation. This yields an interference pattern with regions of minima and maxima.

Решаем так.. Field at n^{th} dipole is given in [5.6]-[5.7], in its own coordinate system $(r_n, \theta_n, \varphi_n)$

Since the unit vectors $(\underline{\hat{r}}_n, \underline{\hat{\theta}}_n, \underline{\hat{\varphi}}_n)$ are different for each dipole, the field of each dipole should be expressed as $\left(E_{n_x}, E_{n_y}, E_{n_z}\right)$ and then summed. This can be done by expressing the unit vectors $(\underline{\hat{r}}_n, \underline{\hat{\theta}}_n, \underline{\hat{\varphi}}_n)$ in terms of unit vectors $(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}})$, as shown in "two z-polarized dipoles along the z-axis"

Ex. Field of z-polarized Hertzian Dipoles:

The dipoles are located at $\underline{r}_n = (x_n, y_n, z_n)$, and we assume that all of them have the same vertizal z polarization.

Expressing the field of each dipole in its own coordinate system $(r_n, \theta_n, \varphi_n)$ as in [5.6-5.7], where:

$$r_n = \sqrt{(x - x_n)^2 + (y - y_n)^2 + (z - z_n)^2}\,, \cos(\theta_n) = \frac{z - z_n}{r_n}$$

Expanding Unit Vectors $(\underline{\hat{r}}_n, \underline{\hat{\theta}}_n, \underline{\hat{\varphi}}_n)$ in terms of $(\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}})$:

$$\begin{pmatrix} \underline{\hat{r}}_n \\ \underline{\hat{\theta}}_n \\ \underline{\hat{\varphi}}_n \end{pmatrix} = \begin{pmatrix} \underline{\hat{r}}_n \cdot \underline{\hat{x}} & \underline{\hat{r}}_n \cdot \underline{\hat{y}} & \underline{\hat{r}}_n \cdot \underline{\hat{z}} \\ \underline{\hat{\theta}}_n \cdot \underline{\hat{x}} & \underline{\hat{\theta}}_n \cdot \underline{\hat{y}} & \underline{\hat{\theta}}_n \cdot \underline{\hat{z}} \\ \underline{\hat{\varphi}}_n \cdot \underline{\hat{x}} & \underline{\hat{\varphi}}_n \cdot \underline{\hat{y}} & \underline{\hat{\varphi}}_n \cdot \underline{\hat{z}} \end{pmatrix} \begin{pmatrix} \underline{\hat{x}} \\ \underline{\hat{y}} \\ \underline{\hat{z}} \end{pmatrix} = \begin{pmatrix} \sin\theta_n \cos\varphi_n & \sin\theta_n \sin\varphi_n & \cos\theta_n \\ \cos\theta_n \cos\varphi_n & \cos\theta_n \sin\varphi_n & -\sin\theta_n \\ -\sin\varphi_n & \cos\varphi_n & 0 \end{pmatrix} \begin{pmatrix} \underline{\hat{x}} \\ \underline{\hat{y}} \\ \underline{\hat{z}} \end{pmatrix}$$

Ex. Two z-polarized dipoles along the y −axis:

Here $\underline{r}_1' = \left(0, -\frac{l}{2}, 0\right)$, $\underline{r}_2' = \left(0, \frac{l}{2}, 0\right)$	Excitations have same magnitude but a phase difference of α: $I_1 = A_0 e^{j\frac{\alpha}{2}}, I_2 = A_0 e^{-j\frac{\alpha}{2}}$

Observations:
(1)The field has many lobes. They represent interference of the individual contributions
(2)Number of lobes increases for large l/λ.
(3)For the case $\alpha = \pi$, the contributions from $\underline{r}_1', \underline{r}_2'$ interfere destructively in the $y = 0$ plane, and the field there vanishes. This resembles the case of a single dipole (say the one at $\underline{r}_2'$) located near a $y = 0$ PEC plane.

The "far field" (Fraunhofner) Approximation:

For $\underline{r}$ in the far zone such that $r \gg r'$ we have:

[7.1] Approximating $\boldsymbol{R} = |\underline{\boldsymbol{r}} - \underline{\boldsymbol{r}}'|$: $R \approx r - r'\cos(\psi) + \frac{r'^2}{2r}\sin^2(\psi) + \cdots$, where ψ is angle between $\underline{r}$,$\underline{r}'$.

The quadric term in [7.1] can be neglected in the phase since it vanishes as $r \to \infty$. The condition is: $\frac{k\left(\frac{L}{2}\right)^2}{2r} \ll \frac{\pi}{4} \to r \gg \frac{L^2}{\lambda}$ where L is the source dimension.

[7.4] In Fraunhofner Zone (Far Field) we Approximate: $G(\underline{r}, \underline{r}') = \frac{e^{-jk|\underline{r}-\underline{r}'|}}{4\pi \cdot |\underline{r}-\underline{r}'|} \approx g(r) e^{jk\underline{\hat{r}}\cdot\underline{r}'}$, $g(r) = \frac{e^{-jkr}}{4\pi r}$
$g(r)$ is the Green's function wrt origin. This equation is nice since it expresses G in terms of (r, θ, φ), where we use: $\underline{\hat{r}} = \underline{\hat{x}}\sin(\theta)\cos(\varphi) + \underline{\hat{y}}\sin(\theta)\sin(\varphi) + \underline{\hat{z}}\cos(\theta)$, where (θ, φ)= observation direction.

[7.6] Using this approximation we obtain: $\underline{A}(\underline{r}) = \mu g(\underline{r}) \int_V dV' \underline{J}(\underline{r}') e^{jk\underline{\hat{r}}\cdot\underline{r}'}$

Far Field Expressions for the $(\boldsymbol{E}, \boldsymbol{H})$ field:
From [7.6] we know $\underline{A}(\underline{r})$ so we can find $\underline{E}, \underline{H}$ via [3.7]
[12.3] $\underline{E} = -j\omega\left[\underline{A} + \frac{(-jk\underline{\hat{r}})(-jk\underline{\hat{r}}\cdot\underline{A})}{k^2}\right] = -j\omega\underbrace{[\underline{A} - \underline{\hat{r}}(\underline{\hat{r}}\cdot\underline{A})]}_{\underline{A}_\perp}$, where $\underline{A}_\perp$ is the component of $\underline{A}$ perpendicular to observation direction $\underline{\hat{r}}$.
[12.4]Likewise from [3.7]: $\underline{H} = \frac{1}{\mu}(-jk\underline{\hat{r}}) \times \underline{A} = -\frac{j\omega}{\eta}\underline{\hat{r}} \times \underline{A}_\perp = \frac{1}{\eta}\underline{\hat{r}} \times \underline{E}$, $\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* = \underline{\hat{r}}\frac{1}{2\eta}\ |\underline{E}|^2$

Thus the far field behaves like a spherical wave e^{-jkr}/r but in any direction $\underline{\hat{r}}$ it behaves like a local plane wave: $\underline{E} \perp \underline{H} \perp \underline{\hat{r}}$, $\frac{|E|}{|H|} = \eta$

The Far Field has the Form: $\underline{E} \sim \underline{F}_\perp(\theta, \varphi) g(r)$, $\underline{H} \sim \eta^{-1}\underline{\hat{r}} \times \underline{E}$, $\underline{S} = \frac{|\underline{F}_\perp(\theta,\varphi)|^2}{2(4\pi r)^2}\underline{\hat{r}}$, where $\underline{F}_\perp$ is a radiation pattern which depends on the direction $\underline{\hat{r}} = (\theta, \varphi)$ and has only transversal components

Field Calculation (Appendix B):
$\underline{A}$ in Cartesian coordinates is given via [7.6].
The transformation of $\underline{A}_\perp$ into polar representation is given by $\underline{A}_\perp = \underline{\hat{\theta}} A_\theta + \underline{\hat{\varphi}} A_\varphi$, where $A_\theta = -A_z\sin(\theta) + \left(A_x\cos(\varphi) + A_g\sin(\varphi)\right)\cos(\theta)$, and $A_\varphi = -A_x\sin(\varphi) + A_y\cos(\varphi)$.
Calculate $\underline{E}, \underline{H}$ via [12.3-12.4] ,i.e.: $\underline{E} = -j\omega(\underline{\hat{\theta}} A_\theta + \underline{\hat{\varphi}} A_\varphi)$, $\underline{H} = \eta^{-1}(\underline{\hat{\varphi}} E_\theta - \hat{\varphi}_\theta E\_\varphi)$
Finally, using K-domain representation it follows that: $\underline{E} \sim -jk\eta g(r) \underline{J}_\perp(k\underline{\hat{r}})$, where $\underline{J}_\perp$is the component of $\underline{J}$ that is perpendicular to $\underline{\hat{r}}$: $\underline{J}_\perp(k\underline{\hat{r}}) = \underline{J}(k\underline{\hat{r}}) - \underline{\hat{r}}\left(\underline{\hat{r}} \cdot \underline{J}(k\underline{\hat{r}})\right)$

Again, we see that the range dependence is governed by $g(r)$, so the field is spherical, the phase fronts are spherical, the field propagates outside, and the field decays with $1/r$.

Fourier (K-Space) Representation:

Define the 3D Spatial spectrum of the Currents:

[7.7a] $\underline{J}(\underline{K}) = \int_V dxdydz\, \underline{J}(\underline{r}) e^{j\underline{K}\cdot\underline{r}}$, $\underline{K} = \left(K_x, K_y, K_z\right)$ **[7.7b]** $\underline{J}(\underline{r}) = \left(\frac{1}{2\pi}\right)^2 \int dK_x dK_y dK_z \underline{J}(\underline{K}) e^{-j\underline{K}\cdot\underline{r}}$

The inverse transform expresses the current as a superposition of spectral components with planar phase $e^{j\underline{K}\cdot\underline{r}}$. These are NOT plane waves, since $|\underline{K}| \in [0, \infty)$ and it is $\neq \omega\sqrt{\mu\epsilon}$

Comparing [7.6] to [7.7]: Note that the far field in [7.6] can be expressed as $\underline{A}(\underline{r}) \sim \mu g(r) \underline{J}(k\underline{\hat{r}})$, i.e. $\underline{K} = k\underline{\hat{r}}$, where $\underline{\hat{r}}$ =observation direction.
In words: The ifeld $\underline{A}(\underline{r})$ in the far zone corresponds to spectral components of $\underline{J}$ that are located on a sphere in the K −space, with radius $k = \omega/c$, known as the Ewald Sphere.
Corollary: Measuring the far field is not sufficient to reconstruct $\underline{J}$

Arrays of Vertical Hertzian Dipoles:

Consider N Hertzian dipoles $I_n d_n, n = 1, \ldots, N$, located at points $\underline{r}_n'$. For simplicity we assume that all dipoles are co-polarized ($\underline{\hat{z}}$ polarization). We also consider time-harmonic source, so that (recalling [5.3])

Dipoles are defined by: $\underline{J}\ (\underline{r}, t) = \underline{\hat{z}}[I_n d_n]\delta(\underline{r} - \underline{r}'_n)e^{j\omega t}$. Henceforth Denote $[I_n d_n] = A_n$

Field is Given by: (from [5.4]) $\underline{A}(\underline{r}) = \sum_n \underline{\hat{z}}\mu A_n \frac{e^{-jkR_n}}{4\pi R_n}$, $R_n = |\underline{r} - \underline{r}_n'|$

For Observation Points in the Far Field: We may use $R_n \approx r - \underline{\hat{r}} \cdot \underline{r}_n'$, r =distance of $\underline{r}$ from the orgin. $\underline{\hat{r}}$ = observation direction (as in [7.5])

We obtain (see [7.6]) : $\underline{A}(\underline{r}) = \underline{\hat{z}}\mu g(r) \sum_n A_n e^{jk\underline{\hat{r}}\cdot\underline{r}_n'}$

Far Field Expression for (E,H), (as in [5.11]): $\underline{E} \approx \underbrace{\underline{\hat{\theta}} jk\eta g(r)\sin(\theta)}_{\text{Element Factor}} \underbrace{\sum_n A_n e^{jk\underline{\hat{r}}\cdot\underline{r}_n'}}_{\text{Array Factor}}$, $\underline{H} \approx \underline{\hat{\varphi}}\eta^{-1}E_\theta$, where we recall that $\underline{r}_n' = (x_n', y_n', z_n')$, so $\underline{\hat{r}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi\ , \cos\theta)$.

Element Factor: Depends only on the radiation properties of the individual elements, i.e. Hertzian Dipoles here.	**Array Factor:** Depends only on the relative position as well as the amplitudes of the elements. It describes the interference at a given direction between contributions of the individual dipoles.

Observations:
Field has **spherical wave structure**, spherical phase front, $\boldsymbol{\frac{1}{r}}$ **decay**.
Radiation Pattern is a function of (θ, φ) only. It consists of an element pattern and an array pattern.
Local Plane Wave: In any direction $\underline{\hat{r}} = (\theta, \varphi)$, $\underline{E} \perp \underline{H} \perp \underline{\hat{r}}$, $\underline{H} = \eta^{-1}\underline{\hat{r}} \times 3E$
Poyinting Vector: Outgoing in $+\underline{\hat{r}}$ direction, decays like $1/r^2$ so total radiation power independent of r.

Ex. Two Identical Elements along the y-axis:

$\underline{r}_1' = \left(0, -\frac{l}{2}, 0\right), \underline{r}_2' = \left(0, \frac{l}{2}, 0\right), A_1 = A_2 = A_0$

$\hat{\underline{r}} \cdot \underline{r}_1' = -\frac{l}{2}\sin(\theta)\sin(\varphi)$, $\hat{\underline{r}} \cdot \underline{r}_2' = \frac{l}{2}\sin(\theta)\sin(\varphi)$

For a given (θ, φ), ψ is the phase delay of the contribution of the source $\underline{r}_1'$ relative to the source $\underline{r}_2'$. Here: $\psi = k\hat{\underline{r}} \cdot \underline{r}_2' - k\hat{\underline{r}} \cdot \underline{r}_1'$

$$\underline{E} \approx \underbrace{\hat{\underline{\theta}} jk\eta g(r)\sin(\theta)}_{\text{Element Factor}} A_0 \underbrace{\left[e^{-j\frac{kl}{2}\sin(\theta)\sin(\varphi)} + e^{+j\frac{kl}{2}\sin(\theta)\sin(\varphi)}\right]}_{2\cos\left(\underbrace{\frac{kl}{2}\sin(\theta)\sin(\varphi)}_{\psi/2}\right)}$$

ψ is the phase delay of contribution from $\underline{r}_1'$ relative to contribution of $\underline{r}_2'$, hence we obtain:
Constructive Interference = Maxima: $kl\sin(\varphi) = 2m\pi$
Destructive Interference = Minima: $kl\sin(\varphi) = \pi + 2m\pi$
Principal Maxima: The Principal Maxima are $\varphi = 0, \pi$

Observations: Same as before

Two Elements with Opposite Phase along y-axis:

The element locations are $\underline{r}_1' = \left(0, -\frac{l}{2}, 0\right), \underline{r}_2' = \left(0, \frac{l}{2} 0\right)$. Unlike previous topics, here we have amplitudes $A_1 = -A_2 = A_0$, and the field: $\underline{E} \approx \hat{\underline{\theta}} jk\eta g(r)\sin(\theta) A_0 \underbrace{\left[e^{-j\frac{kl}{2}\sin(\theta)\sin(\varphi)} - e^{+j\frac{kl}{2}\sin(\theta)\sin(\varphi)}\right]}_{-2j\sin\left(\frac{kl}{2}\sin(\theta)\sin(\varphi)\right)}$. So now we have zeros in the principal directions $\varphi = 0, \pi$, representing destructive interference of the contributions arriving from the 2 dipoles. Etc.

Two Elements with Arbitrary Phase along the y axis:

The element locations are again $\underline{r}_1' = \left(0, -\frac{l}{2}, 0\right), \underline{r}_2' = \left(0, \frac{l}{2}, 0\right)$. But the amplitudes are $A_1 = A_0 e^{j\frac{\alpha}{2}}$, $A_2 = A_0 e^{-j\frac{\alpha}{2}}, -\pi < \alpha < \pi$.

The field: $\underline{E} \approx \hat{\underline{\theta}} jk\eta g(r)\sin(\theta) A_0 \underbrace{\left[e^{-j\frac{kl}{2}\sin(\theta)\sin(\varphi)} e^{j\frac{\alpha}{2}} - e^{+j\frac{kl}{2}\sin(\theta)\sin(\varphi)} e^{-j\frac{\alpha}{2}}\right]}_{-2\cos\left(\frac{kl}{2}\sin(\theta)\sin(\varphi) - \frac{\alpha}{2}\right)}$. The direction the principal maximum now depends on α. For example for $\theta = \frac{\pi}{2}$ the direction is $kl\sin(\varphi) = \alpha$

Observations:

α controls the main beam direction via $\sin(\varphi_{max}) = \frac{\alpha}{kl}$

Larger kl requires larger α

For $l = \frac{\lambda}{4}$ $(i.e.\, kl = \frac{\pi}{2})$ and $\alpha = \frac{\pi}{2}$ we obtain "**end-fire**" radiation wherein radiation along the +y axis is in phase, and radiation along the –y axis is out of phase

For $l = \frac{\lambda}{2}$ $(i, e, kl = \pi)$ and $\alpha = \pi$ we obtain "**end-fire**" radiation wherein the radiation of the 2 elements is in phase along +y and –y axes.

Two-Phased Elements Along the z-axis:

Here $\underline{r}_1' = \left(0,0, -\frac{l}{2}\right), \underline{r}_2' = \left(0,0, \frac{l}{2}\right)$. $A_1 = A_0 e^{j\frac{\alpha}{2}}, A_2 = A_0 e^{-j\frac{\alpha}{2}}, -\pi < \alpha < \pi$. Also: $\hat{\underline{r}} \cdot \underline{r}_1 = -\frac{l}{2}\cos(\theta)$, $\hat{\underline{r}} \cdot \underline{r}_2 = +\frac{l}{2}\cos(\theta)$. We get the field: $\underline{E} \approx \hat{\underline{\theta}} jk\eta g(r)\sin(\theta) A_0 \underbrace{\left[e^{-j\frac{kl}{2}\cos(\theta) + j\frac{\alpha}{2}} - A_0 e^{+j\frac{kl}{2}\cos(\theta) - j\frac{\alpha}{2}}\right]}_{2\cos\left(\frac{kl}{2}\cos(\theta) - \frac{\alpha}{2}\right)}$

The array has azimuthal symmetry (cylindrical around z-axis)

Maxima Occur At: $kl\cos(\theta) - \alpha = 2m\pi \rightarrow \frac{l}{\lambda}\cos(\theta_{\max}) = \frac{\alpha}{2\pi} + m, \quad m = 0, \pm 1, \pm 2 \ldots$

Principal Maximum Occurs at: $\theta_0 = \cos^{-1}\left(\frac{\alpha}{kl}\right)$ if $|\alpha| < kl$. Note: If $l < \frac{\lambda}{2}$ then a max occurs only if $|\alpha|$ is sufficiently smaller than π.

As before, the number of maxima and minima depends on kl

Phase Array of Hertzian Dipoles:

We consider an array of N identical Hertzian dipoles along the y-axis, with inter-element distance l, and inter-phase difference α. For simplicity, N is taken to be an odd number. The overall size of the array is $L = (N-1)l$

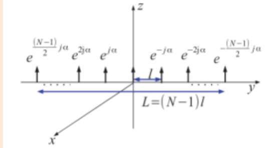


Figure 26 An array of N, z-polarized Hertzian dipoles along the y-axis.

Radiation Pattern: Recalling [8.7]we have: $\underline{r}_n{}' = (0, nl, 0)$, $A_n = A_0 e^{-jn\alpha}$, $n = -\frac{(N-1)}{2}, \ldots, \frac{N-1}{2}$:

$$\underline{E} \approx \hat{\underline{\theta}} jk\eta g(r)\sin(\theta) A_0 \sum_{n=-\frac{(N-1)}{2}}^{n=\frac{N-1}{2}} e^{jn(kl\sin(\theta)\sin(\varphi) - \alpha)} = \hat{\underline{\theta}} jk\eta g(r)\sin(\theta) A_0 \frac{\sin\left[\frac{N}{2}\overbrace{(kl\sin(\theta)\sin(\varphi) - \alpha)}^{\psi}\right]}{\sin\left[\frac{1}{2}(kl\sin(\theta)\sin(\varphi) - \alpha)\right]},$$

where we used the Dirichlet Function:

[9.3] Dirichlet Function (aka Periodic Sinc): $\sum_{n=-\frac{(N-1)}{2}}^{n=\frac{N-1}{2}} e^{j\omega\psi} = \frac{e^{-j\frac{N-1}{2}\psi} - e^{j\frac{N+1}{2}\psi}}{1 - e^{j\psi}} = \frac{\sin\left(N\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \stackrel{\text{def}}{=} \text{Psinc}(\psi\,; N)$

The **Major Maxima** are at the zeros of the denominator of the Psinc, i.e. $\psi_{\max} = 2m\pi$. Their magnitude is N. Between any two major maxima there are $N-2$ **minor maxima** whose magnitude is ~1, which are the maxima of the sine function in the numerator of [9.3].

[9.4] Physical Characteristics of the Radiation Pattern: For simplicity we consider the radiation pattern in the (x, y) plane. Here: $\underline{E} \approx -\hat{\underline{z}} jk\eta g(r) A_0 \frac{\sin\left[\frac{N}{2}(kl\sin(\theta)\sin(\varphi) - \alpha)\right]}{\sin\left[\frac{1}{2}(kl\sin(\theta)\sin(\varphi) - \alpha)\right]}$.

Major Maxima, the "Main Beam" and the "Grating Lobes":
[9.5] Major Maxima are zeros of denominator of the $Psinc$ function in [9.4]: $\psi = kl\sin(\varphi) - \alpha = 2m\pi$. Recalling that ψ is the phase-difference between the contributions of any 2 adjacent elements, it follows that in these directions the contributions of all the elements add up in phase (constructive interference), hence the **array factor** is $1 + 1 + \cdots = N$

Main Lobe: The $m = 0$ term in [9.5]. $kl\sin(\varphi) = \alpha \rightarrow \varphi_{\max_0} = \sin^{-1}\left(\frac{\alpha}{kl}\right) = \sin^{-1}\left(\frac{\lambda}{l} \cdot \frac{\alpha}{2\pi}\right)$

Steering Angle: α is the steering angle as it serves to steer the main beam.
Grating Lobes: The other maxima in [9.5] are called grating lobes. At the grating lobes, the phase difference between the contributions of any two adjacent dipoles is $2m\pi, m = \pm 1, \pm 2 \ldots$
[9.7]Grating Lobe Angles: $\sin\left(\varphi_{max_m}\right) = \underbrace{\frac{\alpha}{kl}}_{\sin(\varphi_{max_0})} + \frac{2\pi m}{kl}$

Eliminating the Grating Lobes:

Consider first that $\alpha = 0$: from [9.6-9.7]:

The main beam and grating lobes are at: $\varphi_{\max_0} = 0$, $\sin\left(\varphi_{max_m}\right) = \frac{2m\pi}{kl}$, thus if $kl < 2\pi$ i.e. if $l < \lambda$, there are no grating lobes.

Next we consider a general $-\pi < \alpha < \pi$. From [9.5], **there will be no grating lobes if**$|kl\sin(\varphi) - \alpha| = 2\pi$. This condition is satisfied for all $-\pi < \alpha < \pi$ is $kl < \pi$ i.e. $l < \frac{\lambda}{2}$

Alternatively we look at $|kl\sin(\varphi) - \alpha| = 2\pi \rightarrow -2\pi < kl\sin(\varphi) - \alpha < 2\pi \rightarrow -2\pi + \alpha < kl\sin(\varphi) < 2\pi + \alpha$, where $-\frac{\pi}{2\varphi} < \frac{\pi}{2}$ and then consider 2 limiting cases $\sin(\varphi) = \pm 1$

Minor Maxima (side lobes):

Around the main beam there are **small local maxima**, which are the maxima of the sine function in the numerator of the Psinc function in [9.2]. If kl is large s.t. there are **grating lobes** in addition to the main lobe, then there are $N-2$ side lobes between these lobes.

The nulls of the radiation pattern satisfy: $N\psi = k\underbrace{(Nl)}_{\approx L}\sin(\varphi) - N\alpha = 2n\pi, n = 1,2, \ldots N-1, etc.$

Thus the side lobes are generated when the phase difference between the contributions of the two farthest dipoles is $2m\pi$. The magnitude of the side lobes is N^{-1} relative to the main and grating lobes.

Beam Width and Angular Resolution

Assume first that $\alpha = 0$. Then the nulls on both sides of the main beam are described by $N\frac{kl}{2}\sin(\varphi) = \pm\pi \;\rightarrow\; \sin(\varphi_{null}) = \frac{\lambda}{Nl} = \frac{\lambda}{L}$, where L is the total size of the array. Assuming a large array with $L \gg \lambda$, then $\sin(\varphi_{null}) \approx \varphi_{null}$, and we obtain:

[9.11] Beamwidth$= \frac{2\lambda}{L}$ = angular resolution of the array.

Beam Steering

For a general α, the nulls on both sides of the main beam satisfy $\frac{N}{2}(kl\sin(\varphi_{null}) - \alpha) = \pm\pi$

Expressing $\varphi_{null} = \varphi_{\max_0} + \delta_\varphi$, where $\varphi_{\max_0}$ is the direction of the main beam and $\delta_\varphi \ll 1$, we obtain: $\frac{N}{2}\left(kl\left(\sin(\varphi_{\max_0} + \delta_\varphi\right) - \alpha\right) = \frac{N}{2}\left(kl\sin\left(\varphi_{max_0}\right) + \delta_\varphi\cos\left(\varphi_{max_0}\right) - \alpha\right) = \frac{N}{2}kl\delta_\varphi\cos(\varphi_{\max_0}$.

In the last expression we used $klsin\left(\varphi_{\max_0}\right) = \alpha$

We obtain a generalization of [9.11]: $\delta_\varphi \approx \pm\frac{\lambda}{Lcos(\varphi_{\max_0})} \rightarrow beamwidth = \frac{2\lambda}{Lcos(\varphi_{\max_0})} = \frac{2\lambda}{L_{eff}}$, were $Lcos\left(\varphi_{\max_0}\right)$ ="effective array size" = the array size L projected onto the main beam direction.

Notes:

s Axis: Define a new axis, s, such that at the load $s = l - z$.

Total Voltage and current in terms of Reflection over the line: $V(z) = V^+(z)\cdot[1+\Gamma(z)]$, $I(z) = Y_c V^+(z)\cdot[1-\Gamma(z)]$

Impedance at each point on the line: $Z(z) = \frac{V(z)}{I(z)} = Z_C\frac{1+\Gamma_L e^{-2j\beta s}}{1-\Gamma_L e^{-2j\beta s}} = Z_c\frac{Z_L + jZ_c\tan(\beta s)}{Z_c + jZ_L\tan(\beta s)}$

Direct Impedance Tracking: $\vec{Z}(z) = Z_c\frac{\vec{Z}(l)\cos\beta s + jZ_c\sin\beta s}{j\vec{Z}(l)\sin\beta s + Z_c\cos\beta s}$

Parallel Impedances: $Z_{L1} = Z_{in_2} \parallel Z_{in_3} = \frac{Z_{in_2}Z_{in_3}}{Z_{in_2}+Z_{in_3}}$. In particular this works for a ---< configuration, where we have parallel transmission Lines as a Load to some Primary Line (Equivalent Load):

Input Line Voltage: $V_{in} = V_g\frac{Z_{in}}{Z_{in}+Z_g}$, where g subscript indicates generator parameter.

Properties of the Standing Wave:

- The standing wave pattern has a $\frac{\lambda}{2}$ periodicity
- Maxima are obtain when $\vec{\Gamma}(z)$ is real and > 0
- Minima are obtained when $\vec{\Gamma}(z)$ is real and < 0, i.e.:
  - $\varphi_L - 2\beta s_{\min} = -(1+2n)\pi\; , n = 0,1,\ldots$
- The minima are sharper than the maxima. Therefore, in lab measurements the location of the minima can be measured more accurately than of the maxima. The maxima are $\frac{\lambda}{4}$ away from the minima.
- We have: $V_{Max} = |\tilde{V}^+|\cdot(1+|\Gamma_L|)$, $V_{Min} = |\tilde{V}^+|\cdot(1-|\Gamma_L|)$

Layer	1	…	*i*
$k = \frac{2\pi f}{c}\sqrt{\epsilon_r}, m^{-1}$	Value	…	Value
$k_z = kcos(\theta_i), m^{-1}$	Value	…	Value
$\lambda_z = \frac{2\pi}{k_z}, mm$	Value	…	Value

Magnitude of the Voltage: $|V(z)| = |V^+(z)|\cdot\left|1 + |\Gamma_L|e^{j\varphi_L}e^{-2j\beta s}\right|$

Maxima and Minima of the Voltage: $V_{\max} = |V^+|(1+|\Gamma_L|)$, $V_{\min} = |V^+|(1-|\Gamma_L|)$

$\sin(-\theta) = -\sin(\theta)$	$\sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta)$	$\sin(\pi-\theta) = \sin(\theta)$	$\cos(\pi-\theta) = -\cos(\theta)$	$\tan(\pi-\theta) = -\tan(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$	$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$	$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$	$\sin(\theta+2\pi) = \sin(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$tan\left(\frac{\pi}{2}-\theta\right) = \cot(\theta)$	$\sin^2\left(\frac{\theta}{2}\right) = \frac{(1-\cos(\theta))}{2}$	$\cos^2\left(\frac{\theta}{2}\right) = \frac{(1+\cos(\theta))}{2}$	$\cos(\theta+2\pi) = \cos(\theta)$
$\sin\left(\theta+\frac{\pi}{2}\right) = \cos(\theta)$	$\sin(\theta+\pi) = -\sin(\theta)$	$\cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}$		$\sin^2(\theta)+\cos^2(\theta) = 1$
$\cos\left(\theta+\frac{\pi}{2}\right) = -\sin(\theta)$	$\cos(\theta+\pi) = -\cos(\theta)$	$\sin(\theta) = \pm\sqrt{1-\cos^2(\theta)}$ Sign depends on quadrant of θ		
$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$		$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$		

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$	$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta) = 2\cos^2(\theta)-1 = 1-2\sin^2(\theta)$	
$\sin(3\theta) = -4\sin^3(\theta)+3\sin(\theta)$	$\cos(3\theta) = 4\cos^3(\theta)-3\cos(\theta)$	*Euler's formula*: $e^{ix} = \cos(x)+isin(x)$
$2\cos(\theta)\cos(\phi) = \cos(\theta-\phi)+\cos(\theta+\phi)$	$2\sin(\theta)\cos(\phi) = \sin(\theta+\phi)+\sin(\theta-\phi)$	
$2\sin(\theta)\sin(\phi) = \cos(\theta-\phi)-\cos(\theta+\phi)$	$2\cos(\theta)\sin(\phi) = \sin(\theta+\phi)-\sin(\theta-\phi)$	

INTEGRALS: Basic Forms

$\int x^n dx = \frac{1}{n+1}x^{n+1},\; n \neq -1$ | $\int \frac{1}{x}dx = \ln|x|$ | $\int udv = uv - \int vdu$ | $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$

Integrals of Rational Functions

$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$ | $\int \frac{1}{1+x^2}dx = \tan^{-1}x$ | $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ | $\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$

$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$ | $\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a}$ | $\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, a \neq b$

$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$ | $\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$ | $\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$

$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$ | $\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$

Integrals with Roots

$\int \sqrt{x-a}\;dx = \frac{2}{3}(x-a)^{3/2}$ | $\int \sqrt{x(ax+b)}\;dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|\right]$ | $\int \frac{x}{\sqrt{a^2-x^2}}\;dx = -\sqrt{a^2-x^2}$

$\int \frac{1}{\sqrt{x\pm a}}\;dx = 2\sqrt{x\pm a}$ | $\int \sqrt{x^3(ax+b)}\;dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|$

$\int \frac{1}{\sqrt{a-x}}\;dx = -2\sqrt{a-x}$ | $\int \sqrt{a^2-x^2}\;dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$ | $\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$ | $\int \sqrt{ax+b}\;dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$

$\int x\sqrt{x^2\pm a^2}\;dx = \frac{1}{3}(x^2\pm a^2)^{3/2}$ | $\int \frac{1}{\sqrt{x^2\pm a^2}}\;dx = \ln\left|x+\sqrt{x^2\pm a^2}\right|$ | $\int \frac{x}{\sqrt{x\pm a}}\;dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$ | $\int \frac{x}{\sqrt{x^2\pm a^2}}\;dx = \sqrt{x^2\pm a^2}$

Integrals with Exponentials

$\int e^{ax}\;dx = \frac{1}{a}e^{ax}$ | $\int xe^x\;dx = (x-1)e^x$ | $\int xe^{ax}\;dx = \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$ | $\int x^2e^{ax}\;dx = \left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)e^{ax}$ | $\int xe^{-ax^2}\;dx = -\frac{1}{2a}e^{-ax^2}$

$\int x^ne^{ax}\;dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}dx$ | $\int x^2e^x\;dx = (x^2-2x+2)e^x$ | $\int x^3e^x\;dx = (x^3-3x^2+6x-6)e^x$

Integrals with Trigonometric Functions

$\int \sin ax\;dx = -\frac{1}{a}\cos ax$ | $\int \sin^2ax\;dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$ | $\int \sin^3ax\;dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$

$\int \cos ax\;dx = \frac{1}{a}\sin ax$ | $\int \cos^2ax\;dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$ | $\int \cos^3ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$

$\int \sin^2x\cos x\;dx = \frac{1}{3}\sin^3x$ | $\int \cos^2ax\sin ax\;dx = -\frac{1}{3a}\cos^3ax$ | $\int \sin^2ax\cos^2ax\;dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$

$\int \tan ax\;dx = -\frac{1}{a}\ln\cos ax$ | $\int \tan^2ax\;dx = -x + \frac{1}{a}\tan ax$ | $\int \tan^3ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2ax$

$\int \cos x\sin x\;dx = \frac{1}{2}\sin^2x + c_1 = -\frac{1}{2}\cos^2x + c_2 = -\frac{1}{4}\cos 2x + c_3$

$\int \sin^2ax\cos bx\;dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$ | $\int \cos^2ax\sin bx\;dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$

$\int \cos ax\sin bx\;dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b$ | $\int \sin^2ax\cos^2bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)}$

Products of Trigonometric Functions and Exponentials

$\int e^x\sin x\;dx = \frac{1}{2}e^x(\sin x - \cos x)$ | $\int e^{bx}\sin ax\;dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$ | $\int e^x\cos x\;dx = \frac{1}{2}e^x(\sin x + \cos x)$

$\int e^{bx}\cos ax\;dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$ | $\int xe^x\sin x\;dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$ | $\int xe^x\cos x\;dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$

Products of Trigonometric Functions and Monomials

$\int x\cos x\;dx = \cos x + x\sin x$ | $\int x\cos ax\;dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$ | $\int x^2\cos x\;dx = 2x\cos x + (x^2-2)\sin x$

$\int x\sin x\;dx = -x\cos x + \sin x$ | $\int x^2\cos ax\;dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$ | $\int x\sin ax\;dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$

$\int x^2\sin x\;dx = (2-x^2)\cos x + 2x\sin x$ | $\int x^2\sin ax\;dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$ | $\int x\cos^2x\;dx = \frac{x^2}{4} + \frac{1}{8}\cos 2x + \frac{1}{4}x\sin 2x$

$\int x\sin^2x\;dx = \frac{x^2}{4} - \frac{1}{8}\cos 2x - \frac{1}{4}x\sin 2x$