# POWER QUALITY:

**Phasors:** $a + jb = R\angle\gamma$

**Instantaneous Power:** $p(t) = v(t) \cdot i(t)\ [W]$

**Resistor:** $V_R = R \cdot i_R$

- Current and Voltage upon the resistor are in phase
- $Z_R = R$
- $V_R = Z_R \cdot \bar{I}_R$

**Inductor:** The current lags 90° behind the voltage

- **Voltage upon an inductor**: $V_L = L\frac{di_L}{dt}$
- **Inductor impedance**: $Z_L = j\omega L$

**Capacitor:** voltage lags 90° behind the current.

- **Current:** $i_C = C\frac{dV_c}{dt}$
- **Impedance:** $Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$

**RMS:** Defines a kind-of average, where the physical meaning is a representation of the power of the signal. $V_{RMS} = \sqrt{\frac{1}{T}\int_T V^2(t)dt}$

- **For** $\sin$ **or** $\cos$ **:** $V_{RMS} = \frac{V_m}{\sqrt{2}}$
- Phasor size is the RMS Value of the waveform.

**Average power:**
$P_{AV} = P = \frac{W_T}{T} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T \frac{V_m I_m}{2}(\cos(2\omega t - \theta) + \cos(\theta))dt = \frac{V_m I_m}{2}\cos(\theta) = V_{RMS}I_{RMS}\cos(\theta)$

**Apparent Power:** $|S| = |V \cdot I| = \sqrt{P^2 + Q^2}\ [VA]$, $S = V_{RMS} \cdot I_{RMS}\ [VA]$

**Reactive Power:** $Q = V_{RMS} \cdot I_{RMS} \cdot \sin(\theta) = I^2 \cdot \mathrm{X}\ [\mathrm{Var}]$

**Power Factor:** An indication of the effectiveness of power delivery between the source and load

- $PF = \frac{P}{S} = \frac{\text{real power}}{\text{apparent power}} = \frac{P}{V_{RMS} \cdot I_{RMS}}$
- **In a linear system:** $PF = \cos(\theta)$, where $\theta = \tan^{-1}\left(\frac{Q}{P}\right) = \tan^{-1}\left(\frac{X}{R}\right)$

**[NonLin] Harmonics:** When we have harmonics in the voltage and current waveforms, it is possible to express them as:
$v(t) = V_0 + \sum_{n=1}^{\infty} V_n\cos(n\omega t - \varphi_n)$
$i(t) = I_0 + \sum_{n=1}^{\infty} I_n\cos(n\omega t - \theta_n)$

- Harmonics are multiples of the fundamental frequency. For a fundamental of 60 [Hz], we have 2nd harmonic at 120 [Hz], 3rd at 180 [Hz]… etc.

**[NonLin] Average Power:** $P_{AV} = V_0 I_0 + \sum_{n=1}^{\infty}\frac{V_n I_n}{2}\cos(\varphi_n - \theta_n)$

- Only voltage and currents in the same harmonic deliver real power.

**[NonLin] RMS:** $V_{RMS} = \sqrt{V_0^2 + \sum_{n=1}^{\infty}\frac{V_n^2}{2}}$ ; $I_{RMS} = \sqrt{I_0^2 + \sum_{n=1}^{\infty}\frac{I_n^2}{2}}$

**[NonLin] Power Factor:** $PF = \frac{P_{AV}}{V_{RMS} \cdot I_{RMS}}$

- In the specific case that we have just the first harmonic in the voltage, but have harmonics in the current:

$$PF = \underbrace{\frac{\frac{I_1}{\sqrt{2}}}{\sqrt{I_0^2 + \sum_{n=1}^{\infty}\frac{I_n^2}{2}}}}_{\text{Distortion Factor}} \cdot \underbrace{\cos(\varphi_1 - \theta_1)}_{\text{Displacement Factor}}$$

**Distortion Factor:** It is associated with the harmonic voltages and currents in the system. It details the amount of harmonics in the circuit. $DF = \frac{I_1}{\sqrt{I_1 + I_2 + I_3 + \cdots}} = \frac{1}{\sqrt{1 + THD^2}}$

**Total Harmonic Distortion:** $THD = \sqrt{\frac{\sum_{n=2}^{\infty} I_n^2}{I_1^2}}$

# MAGNETIC CIRCUITS:

| DEFINITIONS: | | | |
|---|---|---|---|
| **Name** | **Symbol** | **Unit** | **Value** |
| Magnetic Field | $H$ | $\left[\frac{Amp \cdot Turns}{meter}\right] = \left[\frac{At}{m}\right]$ | N/A |
| Magnetic Flux | $\Phi$ | $[weber] = [wb]$ | N/A |
| Magnetic Flux Density | $B$ | $[Tesla] = [T]$ | N/A |
| Average Magnetic Length | $l$ | $[m]$ | N/A |
| Vacuum Permeability | $\mu_0$ | $\left[\frac{wb}{A \cdot t \cdot m}\right] = \left[\frac{Hy}{m}\right]$ | $4\pi \cdot 10^{-7}$ |
| Substance Permeability | $\mu$ | | N/A |
| Number of Turns | $N$ | N/A | N/A |

**Permeability:** Property of conduction of magnetic field, $\mu$

**Relative Permeability:** $\mu = \mu_r \cdot \mu_0$

**Magnetic Flux Density:** $B = \frac{\phi}{A}$, where $A$ =cross sectional area

**Magneto Motive Force (MMF):** Coil conducts current $i$ and has $N$ turns, we define $MMF = F = N \cdot i\ [At]$

**Ampere's Law:** $\sum H \cdot l = Ni$, where $Ni = MMF$ and is equal to all the "magnetic voltage drops"

**Flux Continuity Law:** $\sum \phi = 0$ i.e. fluxes entering and leaving a junction must sum to 0, similarly to Kirchhoff's law.

**Faraday's Law:** For a loop of area $A$, with a flux $\phi(t)$ crossing the loop, inducing $v(t)$ on the terminals, we have $v(t) = \frac{d}{dt}\phi(t)$

- **For a Unified Flux:** $\phi(t) = B(t) \cdot A_c$
- **Induced voltage:** $v(t) = A_c \cdot \frac{d}{dt}B(t)$

**Flux and Voltage in an inductor:** Induced voltage on the inductor: $e(t) = \frac{d}{dt}\lambda(t) = N\frac{d}{dt}\phi(t)$. The flux on the magnetic core is: $\phi(t) = \phi(0) + \frac{1}{N}\int_0^t e(\tau)d\tau$

**Relationship between $B$ and $H$:** $B = \mu H$

## ELECTRICAL EQUIV. MODEL OF A MAGNETIC CIRCUIT:

**Limitations of the Magnetic-Electric Analogy:**

- Electric currents represent a flow of particles and carry power, part of which is dissipated in resistances. Magnetic fields do not represent such flow, and no power is dissipated in reluctances.
- In an electric circuit, current is confined to the circuit. In a magnetic circuit, there may be significant flux leakage outside of the cores.
- Magnetic circuits are NONLINEAR; the reluctance in a magnetic circuit is not constant like Resistance, but varies depending on the magnetic field. At high magnetic fluxes the materials in the cores reach saturation, limiting the growth of flux through them.
- The ferromagnetic components of magnetic circuits suffer from hysteresis, so the flux in them depends not just on the instantaneous MMF, but also on its history.

**Power Source:** $MMF = Ni$

**Resistance:** Reluctance $\mathcal{R} = 1/\mu A$

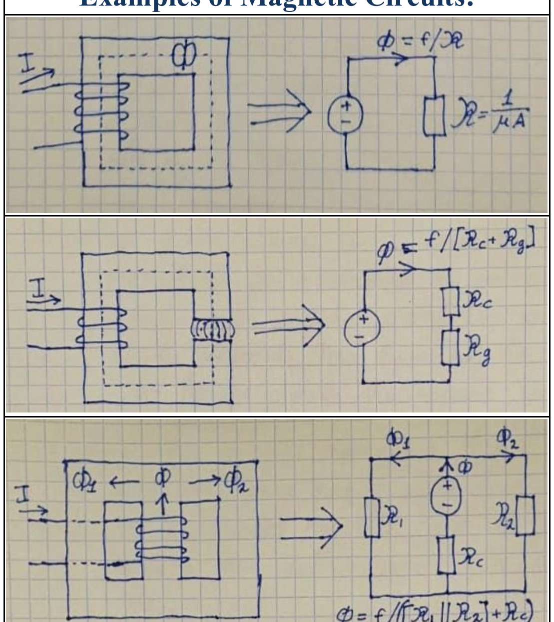
| TABLE OF MAGNETIC-ELECTRIC EQUIVALENCIES | | | | | |
|---|---|---|---|---|---|
| **MAGNETIC** | | | **ELECTRIC** | | |
| **Name** | **Symbol** | **Unit** | **Name** | **Symbol** | **Unit** |
| Magneto Motive Force: | $\mathcal{F} = \int H \cdot dl$ | $[At]$ | Electro Motive Force: | $\mathcal{E} = \int E \cdot dl$ | $[V]$ |
| Magnetic Field: | $H$ | $[A/m]$ | Electrical Field: | $E$ | $[V/m]$ |
| Magnetic Flux: | $\Phi$ | $[wb]$ | Electrical Current: | $I$ | $[A]$ |
| Hopkinson's Law: | $\mathcal{F} = \Phi R$ | $[At]$ | Ohm's Law: | $\mathcal{E} = IR$ | N/A |
| Reluctance: | $\mathcal{R}_m$ | $[1/Hy]$ | Electrical Resistance: | $R$ | $[\Omega]$ |
| Permeance: | $\mathcal{P} = \frac{1}{\mathcal{R}_m}$ | $[Hy]$ | Electrical Conductance: | $G = 1/R$ | $[1/\Omega]$ |
| **B** - **H** Relation: | $\vec{B} = \mu\vec{H}$ | N/A | Microscopic Ohm's Law: | $\vec{J} = \sigma\vec{E}$ | N/A |
| Magnetic Flux Density: | $B$ | $[T]$ | Current Density: | $\vec{J}$ | $[A/m^2]$ |
| Material Permeability: | $\mu$ | $[Hy/m]$ | Electrical Conductivity: | $\sigma$ | $\frac{1/\Omega}{m}$ |

## Magnetic Circuit Analysis Equations:

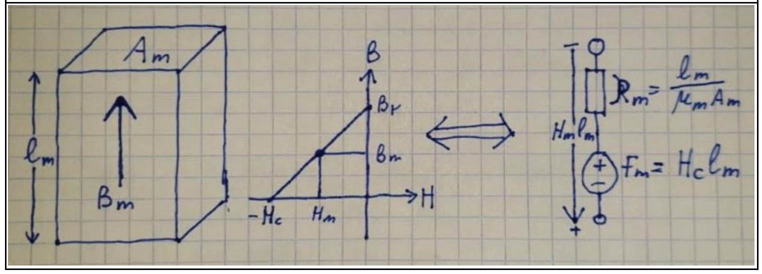
- **Reluctances in series:** $\mathcal{R}_{\text{total}} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \cdots$
- **Total Flux:** $\Phi = \frac{\text{MMF}}{\mathcal{R}_{\text{total}}} = \frac{Ni}{\mathcal{R}_{\text{total}}}$
- **Magnetic KCL:** Sum of magnetic fluxes into any node is $\sum_i \Phi_i = 0$
- **Magnetic KVL:** $\sum \mathcal{R}_k \phi_k = \sum F_k$

**Air Gaps:** Air gaps can be created in the cores of certain transformers to reduce the effects of saturation. This increases the reluctance of the magnetic circuit and enables it to store more energy before core saturation. We treat the gap as a reluctance $\mathcal{R}_g$, which we place in series with the other components.

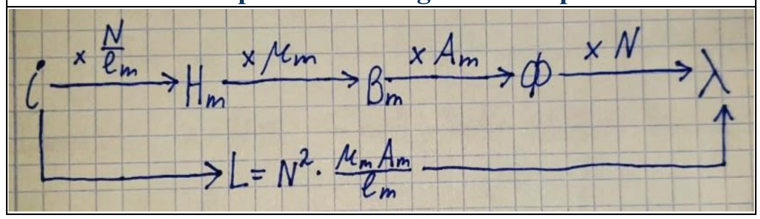
## Examples of Magnetic Circuits:




## Magnetic Circuit Model of a Magnet with Linear Demagnetization Curve:




## Relationship Between Magnetic Components:




**Magnetic "voltage drop" across a magnet:** $H_m l_m = \left[\frac{B_m}{\mu_m} - H_c\right] l_m = \mathcal{R}_m \phi_m - F_m$ (see linear demagnetization model)

**Magnet with a Non Linear Demagnetization Curve:** Linear magnet model is still valid, but permeability becomes $\mu_m = \frac{B_m}{H_m + H_c}$

**Induced Voltage:** $e = N\frac{d\phi_m}{dt}$

- For a single coil around a toroid, this is the voltage induced on inductor terminals
- We get an induced voltage if the flux changes in time. This can happen if the current itself changes with time, or the physical parameters change in time.

**Flux Linkage:** $= N\phi_m$ , i.e. the multiplication of the number of turns of a coil by its flux.

**Inductance:** $L = \frac{\lambda}{i}$

**Voltage generated on a coil's Terminals:** $e = \frac{d\lambda}{dt} = \frac{d}{dt}(Li) = L\frac{di}{dt} + i\frac{dL}{dt}$, where $i\frac{dL}{dt}$ is only present for geometric changes of the inductor.

**Inductor with Resistance:** If an inductor has resistance $r$, then: $V_L = r \cdot i + \frac{d\lambda}{dt} = r \cdot i + L\frac{di}{dt}$. From this we also get that $L = \frac{N^2/l_m}{l_m/\mu_m A_m} = N^2\frac{\mu_m A_m}{l_m}$

**Energy Stored in an Inductor:** $W = \frac{1}{2}Li^2$ [J]

**Self-Inductance:** $\lambda_{pp} = N_p\phi_{pp} = N_p \cdot \frac{N_p \cdot i_p}{\mathcal{R}_{N_p}} = \frac{N_p^2}{\mathcal{R}_{N_p}} \cdot i_p \rightarrow L_{pp} = \frac{N_p^2}{\mathcal{R}_{N_p}}$

**Mutual Inductance:** $L_{ps} = \frac{N_p \cdot N_s}{\mathcal{R}_{ps}}$

- **In matrix form:** $\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix} \cdot \begin{bmatrix}i_1\\ i_2\end{bmatrix}$
- **Inductor Voltages:** $V_1 = r_1 i_1 + \frac{d\lambda_1}{dt}$, $V_2 = r_2 i_2 + \frac{d\lambda_2}{dt}$

**Energy Density in a given magnetic field:** $w = \int H \cdot dB$ , $dw = H \cdot dB$, so total stored energy: $W = \int_v w dV$

**Variable Air Gap:** Suppose we have an air gap $\delta = \delta(t)$. The gap is modeled by a resistor with reluctance $\mathcal{R}_\delta(t)$. The flux in the circuit will always be $\Phi(t) = \frac{F}{\sum \mathcal{R}(t)}$

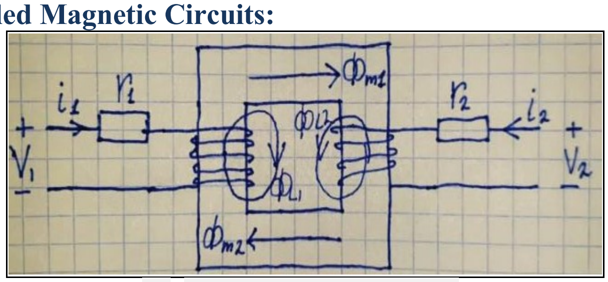
**Calculating current in an inductor:**

1. $Ni(t) = H(t) \cdot l$
2. $H = \frac{B(t)}{\mu}$
3. $B = \frac{\Phi(t)}{A}$
4. **From 1,2,3 we get:** $Ni(t) = \frac{\Phi(t) l}{\mu A}$ and $Ni_{\max} = \frac{\Phi_{\max}}{\mu A} \cdot l$
5. **Thus, max current:** $i_{\max} = \frac{\Phi_{\max}}{\mu A N} \cdot l$
6. $\Phi_{\max} = \frac{V_{inRMS}\sqrt{2}}{N\omega}$
7. $i_{\max} = \frac{V_{inRMS}\sqrt{2}}{N\omega} \cdot \frac{l}{\mu A N} = \frac{V_{inRMS} \cdot l\sqrt{2}}{N^2 \omega \mu A}$
8. **Effective current (in AC):** $i_{RMS} = \frac{i_{\max}}{\sqrt{2}}$
9. $i_{RMS} = \frac{V_{inRMS} \cdot l}{N^2 \omega \mu A}$
10. **Definition of induction:** $L = \frac{\lambda}{i} = \frac{N\Phi}{i} = \frac{N^2 \mu A}{l}$
11. **From 10,9 we get:** $I_{RMS} = \frac{V_{RMS}}{\omega L}$

**Flux Leakage:** Flux that "leaks" outside of the coil/core.

**Flux Leakage and Linkage:** $\Phi_{\text{total}} = \Phi_{\text{leakage}} + \Phi_{\text{linkage}}$

**Coupled Magnetic Circuits:**

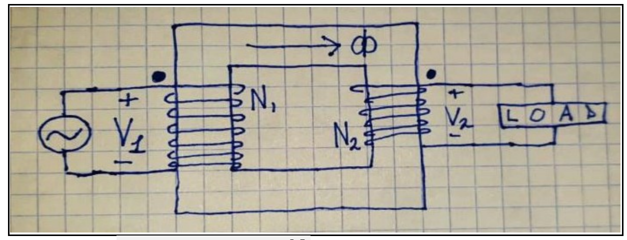



- **Flux through coil** $N_1$: $\Phi_1 = \Phi_{L1} + \Phi_{m1} + \Phi_{m2}$
- **Flux through coil** $N_2$: $\Phi_2 = \Phi_{L2} + \Phi_{m2} + \Phi_{m1}$
- **Voltage on the terminals of** $N_1$: $V_1 = r_1 i_1 + \frac{d\lambda_1}{dt}$
- **Voltage on the terminals of** $N_2$: $V_2 = r_2 i_2 + \frac{d\lambda_2}{dt}$
- **Flux passing through** $N_1$: $\Phi_1 = \frac{N_1 i_1}{\mathcal{R}_{L1}} + \frac{N_1 i_1}{\mathcal{R}_m} + \frac{N_2 i_2}{\mathcal{R}_m}$
- **Flux Linkage for** $N_1$: $\lambda_1 = \frac{N_1^2 i_1}{\mathcal{R}_{L1}} + \frac{N_1^2 i_1}{\mathcal{R}_m} + \frac{N_1 N_2 i_2}{\mathcal{R}_m}$
- **Flux Linkage for** $N_2$: $\lambda_2 = \frac{N_2^2 i_2}{\mathcal{R}_{L2}} + \frac{N_2^2 i_2}{\mathcal{R}_m} + \frac{N_2 N_1 i_1}{\mathcal{R}_m}$
- **Self-Inductance:** $L_{11} = \frac{N_1^2}{\mathcal{R}_{L1}} + \frac{N_1^2}{\mathcal{R}_m} = L_{L1} + L_{m1}$ ; $L_{22} = \frac{N_2^2}{\mathcal{R}_{L2}} + \frac{N_2^2}{\mathcal{R}_m} = L_{L2} + L_{m2}$
  - $\frac{L_{m2}}{N_2^2} = \frac{L_{m1}}{N_1^2}$
- **Mutual Inductance:** $L_{12} = \frac{N_1 N_2}{\mathcal{R}_m}$ , $L_{21} = \frac{N_2 N_1}{\mathcal{R}_m}$ $\rightarrow L_{12} = L_{21}$
  - ***Alternative Representation:*** $L_{12} = \frac{N_2}{N_1} L_{m1} = \frac{N_1}{N_2} L_{m2}$
- $\lambda_1 = L_{11} i_1 + L_{12} i_2$ ; $\lambda_2 = L_{21} i_1 + L_{22} i_2$

**Ideal Transformer Assumptions:**

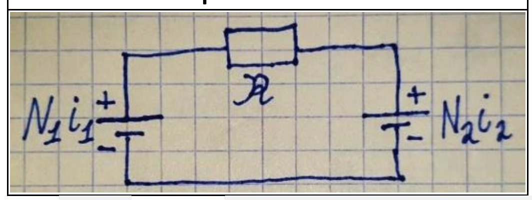
- Resistance of the turns is negligible
- All the flux goes through the core (no leakage)
- Losses in the core are negligible
- The permeability of the core is very large

**Ideal Transformer:**




- **Created EMF:** $V_1 = e_1 = N_1 \cdot \frac{d\Phi}{dt}$
- **Since all the flux goes through the core, the same flux also exits at the secondary, and an EMF is created:** $V_2 = e_2 = N_2 \frac{d\Phi}{dt}$
- **From the above:** $\frac{V_1}{V_2} = \frac{N_1}{N_2}$
- The 2 sources of emf are working in reverse directions
- **Flux:** The flux $\Phi$ is determined by the voltage $V_1$ and is independent of the current.
- Voltage on the terminal of $N_2$ is not influenced by current in $i_2$.
- The equivalent circuit is given and analyzed below

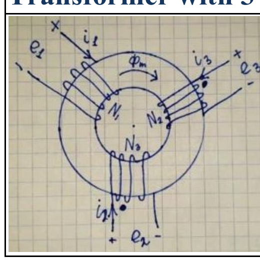
**Equivalent Circuit:**




- **Assuming** $\mathcal{R} \rightarrow 0$ **we get:** $N_1 i_1 = N_2 i_2 \rightarrow \sum Hl = \sum Ni$
- Due to the previous result, and since $\mu \rightarrow \infty$, $H$ is negligible, we conclude that all instantaneous power going into an ideal transformer is the same instantaneous power that goes to the load. The transformer also does not change phase.

- **An ideal transformer has:** $i_1 V_1 = i_2 V_2$

**Transformer with 3 sets of terminals:**



**Voltages at the terminals:** $e_J = N_J \frac{d\Phi_m}{dt}$

**From above:** $\frac{d\Phi_m}{dt} = \frac{e_1}{N_1} = \frac{e_2}{N_2} = \frac{e_3}{N_3}$

**Flux through the core:**

$\Phi_m = \frac{1}{N_1}\int e_1 dt \frac{1}{N_2}\int e_2 dt = \frac{1}{N_3}\int e_3 dt = \frac{N_1 i_1 + N_2 i_2 + N_3 i_3}{\mathcal{R}_m}$

## TRANSFORMERS:

**Operating Principle Transformers:** Faradays law - $e_{\text{induced}} = \frac{d\lambda}{dt}$

**Total Magnetic Flux in ALL the Turns:** $\lambda = \sum_{i=1}^{N} \Phi_i$

**Flux Linkage in a Transformer:** The previous formula "$\lambda = N \cdot \Phi$" does not hold. We define am average flux $\bar{\Phi}$ such that $\lambda = N \cdot \bar{\Phi}$.

- ***Faraday's law also becomes:*** $e_{\text{induced}} = N \cdot \frac{d\bar{\Phi}}{dt}$

**Average Flux:** If a voltage $V_P(t)$ is applied to the primary of the transformer, average flux is $\bar{\Phi} = \frac{1}{N_p}\int V_P(t)\, dt$

**Mutual Flux:** Denoted as $\Phi_M$, is the component of the flux generated by the primary, that goes through the core and arrives at the secondary.

## Fluxes in Primary and Secondary: $\bar{\Phi}_{\text{primary}} = \Phi_M + \Phi_{\text{leakage, primary}}$, $\bar{\Phi}_{\text{secondary}} = \Phi_M + \Phi_{\text{leakage, secondary}}$

## Voltage on the Primary and Secondary:

- **Voltage on primary is forced, so:** $V_P(t) = N_P \cdot \frac{d\bar{\Phi}}{dt} = N_P \cdot \frac{d}{dt}(\Phi_M + \Phi_{LP}) = N_P \cdot \frac{d\Phi_M}{dt} + N_p \frac{d\Phi_{LP}}{dt} = e_P(t) + e_{LP}(t)$
- **Similarly for secondary:** $V_S(t) = N_S \frac{d\Phi_M}{dt} + N_S \frac{d\Phi_{LS}}{dt} = e_S(t) + e_{LS}(t)$
- **Thus a condition exists:** $\begin{cases} e_P(t) = N_P \frac{d\bar{\Phi}_M}{dt} \\ e_S(t) = N_S \frac{d\bar{\Phi}_M}{dt} \end{cases} \to \frac{d\bar{\Phi}_M}{dt} = \frac{e_P(t)}{N_P} = \frac{e_S(t)}{N_S}$
  - This leads to: $\frac{e_P(t)}{e_S(t)} = \frac{N_P}{N_S} = a$

**Turn Ratio:** In a well-designed transformer where $\Phi_M \gg \Phi_{LP}, \Phi_{LS}$, we write the turn ratio as: $a = \frac{N_P}{N_S} = \frac{V_P(t)}{V_S(t)}$

**Magnetization Current:** Denoted by $i_m$, is the current required to create the flux in the magnetic core. It is modeled by an inductor $X_\mu$ parallel to the input power source; as the magnetization is proportional to the input voltage and Is lagging by $90°$ after said voltage.

**Core-Loss Current:** Denoted by $i_{h+e}$, is the current needed to overcome losses due to hysteresis and eddy currents.

**Excitation Current:** $i_{\text{ex.}} = i_m + i_{h+e}$, is the total current when the secondary is not loaded.

**Dots on Transformer Diagram:** The black dots indicate the direction of the coil with respect to the flux. I.e. the currents flowing into the dot create a positive MMF, and a current coming from the dot creates a negative EMF.

- **For a standard square-core loaded transformer:** The primary current makes a positive EMF $F_P = N_P \cdot i_P$, and the secondary current makes a negative EMF: $F_S = -N_S \cdot i_S$.
  - ***Their sum EMF:*** $\boldsymbol{F_{net} = N_P i_P - N_S i_S = \Phi\mathcal{R}}$

**Net MMF in a Transformer:** We have $F_{\text{net}} = N_P i_P - N_S i_S = \Phi\mathcal{R}$.

- **In a well-designed transformer:** The goal is to minimize the reluctance of the core, so that $F_{\text{net}} = N_P i_P - N_S i_S \approx 0$. This is true while the core is not at saturation.
- **Ratio between currents:** $N_P i_P \approx N_S i_S \to \frac{i_P}{i_S} \approx \frac{N_S}{N_P} = \frac{1}{a}$

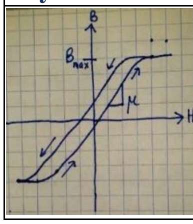
## Loss Mechanisms in a Transformer:

- **Copper Losses:** Loss due to resistance of the coils. Proportional to square of the current in the coils. Can be modeled by resistors $R_P$ and $R_S$.
- **Eddy Currents:** Losses in the magnetic core due to induced currents from Ampere's Law. Proportional to the square of the voltage on the transformer.
- **Hysteresis Losses:** Losses due to movements of magnetic dipoles inside the magnetic core.
- **Flux Leakage Losses:** caused by the leakage fluxes of the primary and the secondary, $\Phi_{LP}, \Phi_{LS}$, that create self-inductance in both coils, and are modeled by using inductors.

**Eddy Current Losses:** When an alternating magnetic field is applied to a magnetic material, an EMF is induced in the material itself according to Faraday's Law. Since the magnetic material is a conducting material, these EMFs circulate currents within the material. These are called eddy currents. They occur when the conductor feels a changing magnetic field.

- **How to reduce this loss?** Construct the magnetic core out of thin laminations parallel to the magnetic field.
- Magnetic flux $\propto$ current in the primary, so:
  - $\Phi(t) \propto i(t)$
  - $e_{\text{ind}} \propto \frac{d\Phi}{dt} \propto V(t)$
  - $P_e \propto \frac{V^2(t)}{R}$
  - $P_e \propto (f \cdot \Phi)^2$

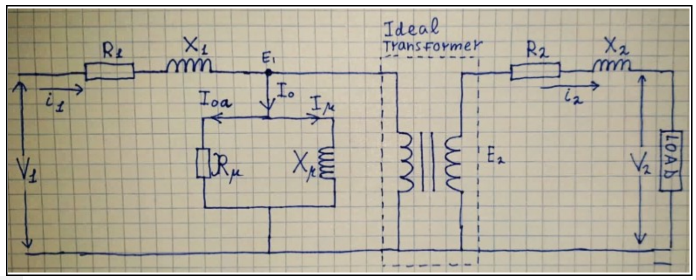
## Hysteresis Losses:




The Hysteresis loop occurs $f$ times/second so losses $\propto$ to the area of the loop and frequency.
**Area of the loop:** $S = B_{\max}^n \cdot k_n$, when $1.5 < n < 2.5$, and is determined by the material of the core
**Hysteresis Losses**: $P_h \propto f B_{\max}^n \propto f\left(\frac{E_{RMS}}{f}\right)^n = \frac{E_{RMS}}{f^{n-1}}$

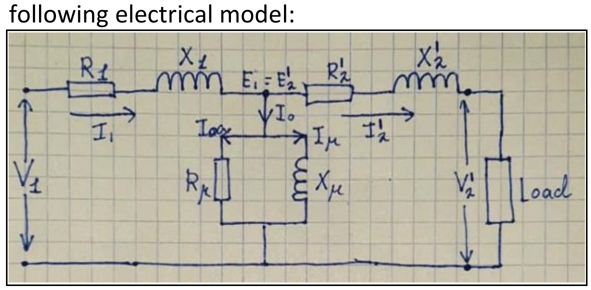
**Total Losses in the Magnetic Core:** $P_{\text{core}} = P_h + P_e = k_h \cdot E_{RMS}^2 + k_e E_{RMS}^2 = (k_n + k_e)E_{RMS}^2 = \frac{E_{RMS}^2}{R_\mu}$, where $R_\mu$ is the resistor in the electric model that represents core losses.

## Electrical Model of a Transformer:




- $R_1$ – Coil resistance of the primary
- $X_1$ – Leakage inductance of the primary
- $R_2$ – Coil resistance of the secondary
- $X_2$ – Leakage inductance of the secondary
- $R_\mu$ – Core Losses
- $X_\mu$ – Magnetization Current
- $a$ – Turn ratio between Primary and Secondary
- $V_2$ – Secondary Voltage
- $V_1$ – Primary Voltage
- $I_1$ – Primary Current
- $I_2$ – Secondary Current
- **Equivalent Resistance:** $R_2' = a^2 \cdot R_2$
- $I_0$ – Composed of two currents: $I_{0_a}, I_\mu$, that represent the currents of the core losses and the magnetization current. $I_0 = \sqrt{I_{0_a}^2 + I_\mu^2}$

**Reflected Electrical Model of a Transformer:** If we were to transform the secondary side to the primary side, we get the following electrical model:

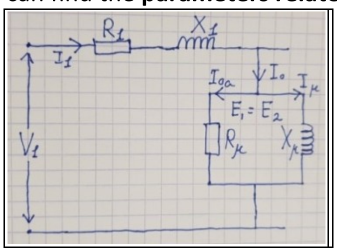



**With:**
- $R_2' = a^2 R_2$
- $X_2' = a^2 X_2$
- $a = \frac{E_1}{E_{2_0}}$

**Current $I_0$:** $I_0$ is composed of $I_{0_a}$ and $I_\mu$, that represent core loss and magnetization current. $I_0 = \sqrt{I_{0_a}^2 + I_\mu^2}$

**Nominal Values:** A value in which the transformer was designed to work. For example, $S = 1[kVA],\ 220\,[V], 110[V]$, the nominal load would be one that consumes a nominal current in the output, i.e. $I_{2n} = \frac{1\,[kVA]}{110\,[V]} = 9.1\,[A]$. The nominal current at the input of the transformer: $I_{1n} = \frac{1\,[kVA]}{220\,[V]} = 4.55\,[A]$

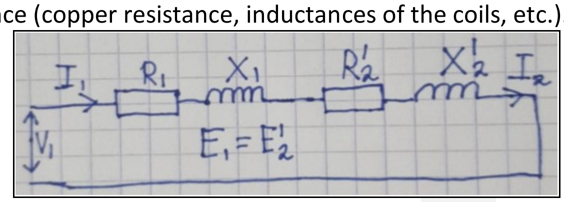
**Open Circuit Experiment:** There is no load across the secondary. We raise the voltage in the primary until reaching nominal voltage. Series Impedance is negligible compared to parallel impedance, so we can find the **parameters related to core losses:**




- $U_{O.C} \approx U_n$
- $P_{O.C} = U_{O.C} \cdot I_{O.C} \cdot \cos(\Phi_{O.C})$
- $I_{0_a} = I_{O.C} \cdot \cos(\Phi_{O.C})$
- $I_\mu = I_{O.C} \cdot \sin(\Phi_{O.C})$
- $R_\mu = \frac{U_{O.C}}{I_{0_a}}$
- $X_\mu = \frac{U_{O.C}}{I_\mu}$

- As there is no current to the load, all the currents will go to the parallel circuit.
  - $Q_0 = \sqrt{\underbrace{(U_0 I_0)^2}_{S_0} - P_0^2}$
  - $X_\mu = \frac{V_0^2}{Q_0} = \frac{V_0}{I_\mu}$
  - $P_0 \approx \Delta P_{fe}$
  - $P_0 = U_0 I_0 \cos(\Phi_0)$
  - $I_{0a} = I_0 \cos(\Phi_0)$
  - $I_{mu} = I_0 \sin(\Phi_0)$

**Short Circuit Experiment:** The secondary is shorted and we slowly raise the voltage on the primary, until we reach nominal current in the primary. The parallel circuit becomes negligible as the input voltage is low. We can find the parameters of the series impedance (copper resistance, inductances of the coils, etc.).




- **By dropping the parallel impedance we get:** $I_1 \approx I_2$, and also $P_k = I_1^2 R_1 + {I_2'}^2 \cdot R_2' = \Delta P_{cu2} = {I_1'}^2 (R_1 + R_2') = I_1^2 \cdot R_k$
- Obtain measurements of the short circuit current and the real power losses from the experiment. **Then:** $R_k = R_1 + R_2' = \frac{P_k}{I_k^2}$
- **We usually assume that:** $R_1 \approx R_2',\ X_1 \approx X_2'$ to get that
  - $Z_k = \frac{U_k}{T_k}$
  - $Z_k = \sqrt{R_k^2 + X_k^2}$
  - $X_k = X_1 + X_2' = \sqrt{Z_k^2 - R_k^2}$
- Some more formulas:
  - $R_k = R_1 + R_2'$
  - $X_k = X_1 + X_2'$
  - $Z_k = \sqrt{R_k^2 + X_k^2}$
- **Important ones:**
  - $Z_k = \frac{V_k}{I_k}$
  - $R_k = \frac{P_k}{I_k^2}$

**Copper Losses:** $\Delta P_{cu} = I^2 R_k \longleftarrow \begin{matrix} \Delta P_{cu1} = I_1^2 R_1 \\ \Delta P_{cu2} = I_2^2 R_2 = I_1^2 R_2' \end{matrix}$

**Transformer Efficiency:** Ratio between input and output power:
- $\eta = \frac{P_{out}}{P_{in} + P_{loss}}$, $P_{loss} = \Delta P_{fe_{(core)}} + \Delta P_{cu}$
  - $\Delta P_{FR} = \frac{E_T^2}{R_\mu} \approx \frac{U_1^2}{R_\mu}$
  - $\Delta P_{cu} = I_1^2 R_1 + I_2^2 R_2 \approx I_1^2 R_k = {I_2'}^2 R_k$
  - $\eta\% = \frac{V_2 I_2 \cos(\Phi_2)}{V_2 I_2 \cos(\Phi_2) + \Delta P_{cu} + \Delta P_{fe}}$
- **Maximum Efficiency:** $\eta_{\max}$ occurs when the variable losses equal the constant losses. Copper loss is load dependent, and thus it is variable. Core loss is constant.
- **Condition for maximum efficiency:** $x^2 P_{cufl} = P_{fe} \rightarrow x = \sqrt{\frac{P_{fe}}{P_{cufl}}}$, where $x =$ Load percentage of the rated load $S$ [VA] .
- **Efficiency in terms of load percentage:** $\eta = \frac{xS\cos(\theta)}{xS\cos(\theta) + x^2 P_{cufl} + P_{fe}}$

## THREE PHASE SYSTEMS:

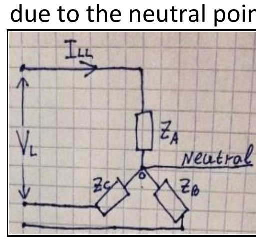
**Phasor representation of a 3-phase system:** A 3 phase system can be represented as Phasors if the system is linear and all voltages have the same frequency. The value of the Phasors will be the RMS value of the voltage. One of the voltages of the system will be the reference voltage with an angle 0°. Once chosen as reference it can't change and other voltage are denoted in reference to it.

**Line Voltage:** Voltage between two phases. $V_L = \sqrt{3} V_{\text{phase}},\ I_L = I_{\text{phase}}.\ V_{AB} = \sqrt{3} V_{RMS} \angle 30°,\ V_{BC} = \sqrt{3} V_{RMS} \angle -90°,\ V_{CA} = \sqrt{3} V_{RMS} \angle -210°$
- Phase voltage lags 30° behind Line voltage.

**Voltage of Each Phase to Neutral (in Israel):**
- $V_{AN} = \sqrt{2} V_{RMS} \cos(\omega t)$
- $V_{BN} = \sqrt{2} V_{RMS} \cos(\omega t - 120°)$
- $V_{CN} = \sqrt{2} V_{RMS} \cos(\omega t + 120°)$

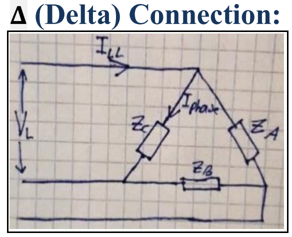
**Balanced three phase systems (balanced source and load):**
- **Instantaneous Power:** $p = V_{AN}(t) i_A(t) + V_{BN}(t) i_B(t) + V_{CN}(t) i_C(t)$. This power is constant and equal to three times the average phase power
- **In a YY connected system:** There is no current in the neutral.
- Three-phase reactive power is three times larger than the phase reactive power.

**Y (Star) Connection:** Best for long distance power transmission, due to the neutral point.




- $V_{LL} = \sqrt{3} V_{\text{phase}}$
- $I_{LL} = I_{\text{phase}}$
- $P = 3 V_{\text{phase}} I_{\text{phase}} \cos(\Phi) = \sqrt{3} V_{LL} I_{LL} \cos(\Phi)$
- No current in the neutral if current is balanced (same current in all three phases).

**Δ (Delta) Connection:**

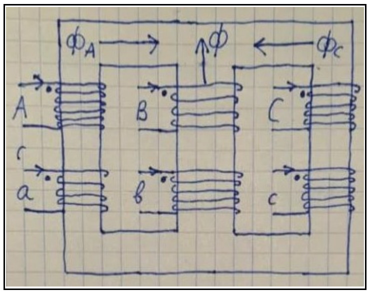



- $V_{LL} = V_{\text{phase}}$
- $I_{LL} = \sqrt{3} I_{\text{phase}}$
- $P = 3 V_{\text{phase}} I_{\text{phase}} \cos(\Phi) = \sqrt{3} V_{LL} I_{LL} \cos(\Phi)$
- Phase current lags 30° behind the line current.

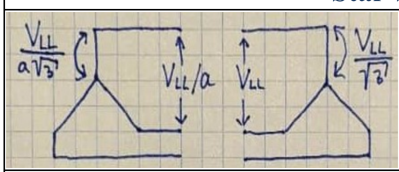
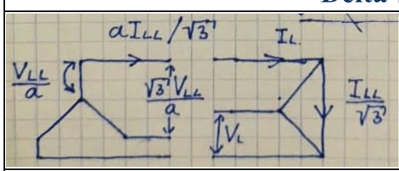
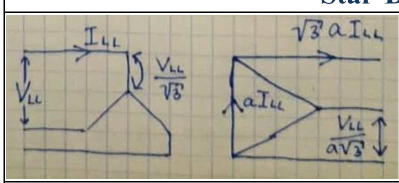
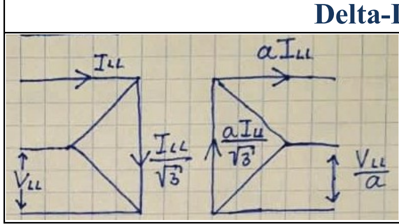
## Three Phase Power: three phase power is not affected by the connection options.

- $P = \sqrt{3}V_{LL}I_{LL}\cos(\Phi) = 3V_{ph}I_{ph}\cos(\Phi)$
- $Q = \sqrt{3}V_{LL}I_{LL}\sin(\Phi) = 3V_{ph}I_{ph}\sin(\Phi)$
- $S = \sqrt{3}V_{LL}I_{LL} = 3V_{ph}I_{ph}$
- $|S| = |P^2 + Q^2|$

## Three Phase Transformer:



- **Primary/Secondary Connection Options:** Both, primary and secondary can be connected in a star or delta connection
- **Transformation Ratio:** The transformation ratio between the primary and secondary in each of the single phase transformers is denoted as $a$.

| Three-Phase Connection Options | |
|---|---|
| **Star-Star** | |
|  | $S_{in} = \sqrt{3}I_{LL}V_{LL}$<br>$S_{out} = \sqrt{3}\frac{\sqrt{3}V_{LL}}{a}\cdot\frac{aI_{LL}}{\sqrt{3}} = \sqrt{3}V_{LL}I_{LL}$ |
| **Delta-Star** | |
|  | $S_{in} = \sqrt{3}I_{LL}V_{LL}$<br>$S_{out} = \sqrt{3}\frac{\sqrt{3}V_{LL}}{a}\cdot\frac{aI_{LL}}{\sqrt{3}} = \sqrt{3}V_{LL}I_{LL}$ |
| **Star-Delta** | |
|  | $S_{in} = \sqrt{3}I_{LL}V_{LL}$<br>$S_{out} = \sqrt{3}\frac{V_{LL}}{\sqrt{3}a}\cdot\sqrt{3}aI_{LL} = \sqrt{3}I_{LL}V_{LL}$ |
| **Delta-Delta** | |
|  | $S_{in} = \sqrt{3}I_{LL}V_{LL}$<br>$S_{out} = \sqrt{3}\frac{V_{LL}}{a}a\cdot I_{LL} = \sqrt{3}I_{LL}V_{LL}$ |

## Copper Losses in a Three Phase Transformer: $\Delta P_{cu} = 3I_1^2R_k$

## Efficiency of a Three Phase Transformer:

$$\eta = \frac{3V_{ph2}I_{ph2}\cos(\Phi_2)}{3V_{ph2}I_{ph2}\cos(\Phi_2)+3I_2^2R_k''+\Delta P_{fe}} = \frac{3V_{L2}I_{L2}\cos(\Phi_2)}{3V_{L2}I_{L2}\cos(\Phi_2)+3I_2^2R_k''+\Delta P_{fe}}$$

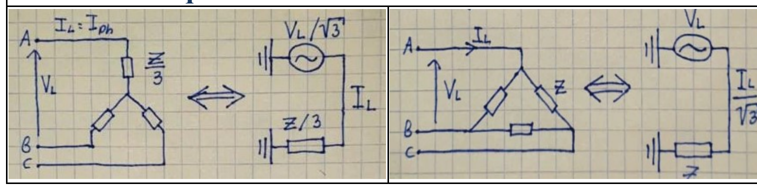
- **Maximum efficiency:** From $\frac{d\eta}{dI_{ph2}} = 0$ we get: $\Delta P_{cu} = \Delta P_{fe} = 3I_{ph2}^2R_k''$

## Comparison between Star and Delta Loads:

- **Star:**
  - $V_{ph} = Z_\Delta I_{ph}$
  - $V_{ph} = V_L$
  - $I_{ph} = \frac{1}{\sqrt{3}}I_L$
  - $V_L = \frac{1}{\sqrt{3}}Z_\Delta I_L$
- **Delta:**
  - $V_{ph} = Z_r I_{ph}$
  - $V_{ph} = \frac{1}{\sqrt{3}}V_L$
  - $I_L = I_{ph}$
  - $V_L = \sqrt{3}Z_Y I_L$

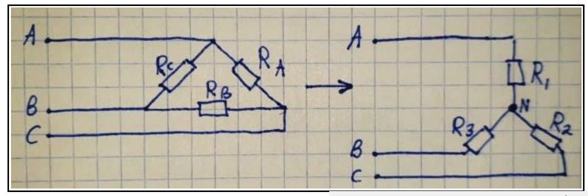
- **Delta in Terms of Star:** Assuming that same power is drawn, we wish to express the delta load as a function of the star load, and vice versa:
  - $\frac{1}{\sqrt{3}}Z_\Delta I_L = \sqrt{3}Z_Y I_L \rightarrow Z_\Delta = 3Z_Y$
  - $I_{Z_\Delta} = \frac{1}{\sqrt{3}}I_{Z_Y} = \frac{1}{\sqrt{3}}I_L,\ V_L = V_{Z_\Delta} = \sqrt{3}V_{Z_Y}$

## Electrical Equivalents of the Star and Delta Connections:



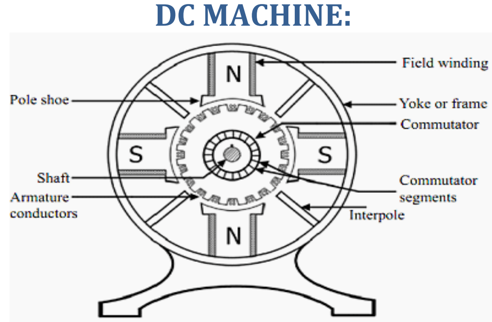
## Converting 3-Phase to Single Phase: A 3 phase system can be modeled as three single-phase systems, as long as it is connected in a Star-Star (YY) Configuration.

## Conversion of Delta into Star Connection:



- **Equivalent Resistance from A to B:** $R_A \parallel (R_B + R_C) = \frac{R_A(R_B+R_C)}{R_A+R_B+R_C}$
- **Similarly, from B to C:** $R_B \parallel (R_A + R_C) = \frac{R_B(R_C+R_A)}{R_B+R_C+R_A}$
- **And from C to A:** $R_C \parallel (R_A + R_B) = \frac{R_C(R_B+R_A)}{R_C+R_B+R_A}$

# DC MACHINE:




**Definition of Common Parameters:**

| Parameter | $E_a$ | $k_e$ | $k_m$ | $I_a$ | $\Phi$ | N | $R_f$ |
|---|---|---|---|---|---|---|---|
| **Meaning** | Induced Voltage (rotor voltage) | Electrical Constant of the machine | Mechanical constant of the machine | Armature Current | Magnetic Flux | RPM | Field Resistor |

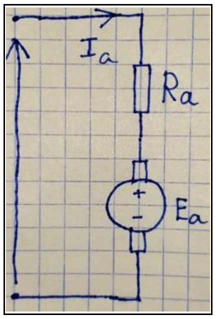
## Working Principle: Based on the movement of a current carrying wire inside a magnetic field, in accordance with the right hand rule of $f = BLi$, and if there is no current, $e = BLV$

## DC Machine Model:

- **Induced voltage in the Armature :** $E_a = k_e \cdot \Phi \cdot n$
- **Mechanical Torque:** $f = BLi \rightarrow M_m = k_m\Phi I_a$
- **Electromagnetic Power:** $P_{em} = E_a \cdot I_a$
- **Electromagnetic Torque:** $M_{em} = \frac{P_{em}}{\omega} = \frac{P_{em}}{\frac{2\pi}{60}n} = 9.55\frac{P_{em}}{n} = 9.55\frac{E_aI_a}{n}\ [N\cdot m]$. This is a connection between the electro-magnet power and the electro-magnet torque (once for RPM and once for frequency)
- **Counter EMF (Back EMF):** $E_b = k_b\Phi \cdot n$

- **Voltage Balance Equation:** The DC Motor's input voltage must overcome the counter EMF as well as voltage drop created by armature current across the motor resistance (combined resistance of the brushes, armature windings, and series field windings, if any): $V_m = E_b + R_m I_a$
- **Speed Equation:** Since $n = \frac{E_b}{k_b \Phi}$ and from the voltage balance equation, we have: $n = \frac{V_m - R_m I_a}{k_b \Phi} = k_n \frac{V_m - R_m I_a}{\Phi}$ , $k_n = \frac{1}{k_b}$

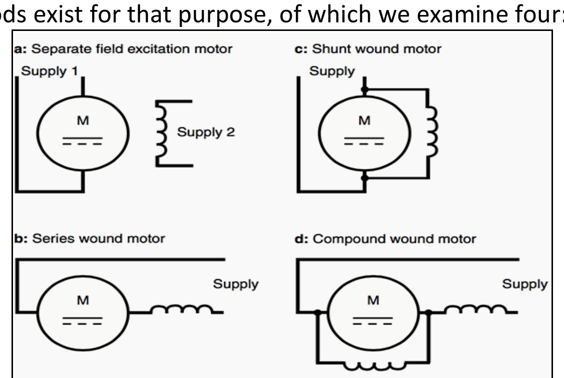
## Armature Model:

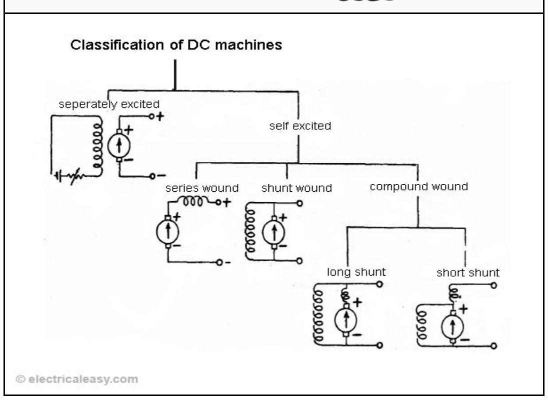



- **If the mechanical losses are insignificant:**
  - $P_{mech} = M \cdot \omega = k_m \cdot \Phi \cdot I_a \cdot \omega$
  - $P_{elec} = E_a \cdot I_a = k_e \cdot \Phi \cdot a \cdot I_a$
- **By comparing the powers obtain:**
  - $P_{mech} = P_{elec} \rightarrow k_e \cdot n = k_m \cdot \omega$
- **From Connection Between RPM & Freq:**
  - $n = 60f = 60\frac{\omega}{2\pi} \rightarrow n = 9.55\omega$
- **Mechanical and Electrical Constants:**
  - $k_m = 9.55 \cdot k_e$

## Activation Techniques:

In order for the rotor to be able to turn inside a magnetic field, such a field must first be generated. A number of methods exist for that purpose, of which we examine four:



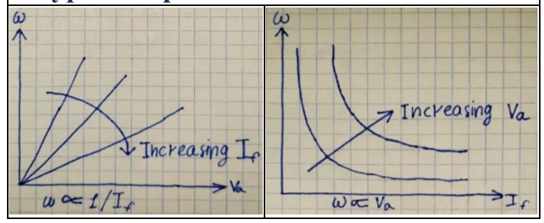




- **Separate Activation:** Source of activation is different to the source of armature
- **Parallel Activation (Separate or Shunt):** Source current is divided into the coil and armature currents. The coils, armature and field coil are connected in parallel or supplied via two sources with different voltages in order to adapt to the characteristics of the machine (e.g.: armature voltage 400 volts and field coil voltage 180 volts).
- **Series Activation:** Currents of the coil and armature are the same. The design of this motor is similar to that of the separate field excitation motor. The field coil is connected in series to the armature coil, hence its name.
- **Combined Activation:** Combination of series and parallel.

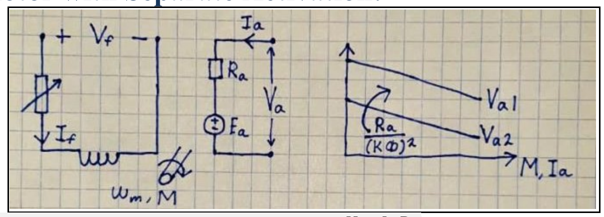
## Speed Control in DC Motors:

Speed Control is possible by altering the voltage of the armature $V_a$ or by controlling the current of the magnetic field $I_f$. It is possible to see that if the current of the field, $I_f$ is disconnected, the speed of the motor will increase and the motor might even break.

### Types of Speed Control in DC Motors:

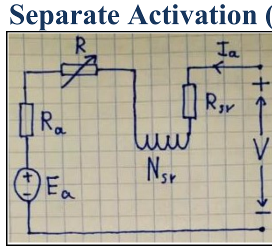



## DC Motor with Separate Activation:

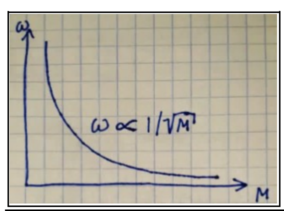



- $E_a = K \cdot \Phi \cdot \omega = V_a - I_a R_a \rightarrow \omega = \frac{V_a - I_a R_a}{k\Phi}$
- $M = K \cdot \phi \cdot I_a$
- $\omega = \frac{V_a}{k\Phi} - \frac{R_a}{(k\Phi)^2} M$
- The speed drops with increased torque
- Motor Speed: $n = \frac{E_a}{k_e \Phi}$
- **In a True Motor:** the flux is not constant but will drop due to the response of the armature when $I_a$ increases; thus dropping the speed.
- **We can regulate the speed by:**
  - Voltage of the Armature
  - The flux of the Coil
  - Resistance of the Armature

## Separate Activation (Motor):

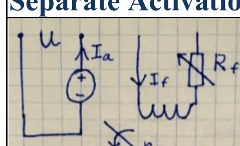
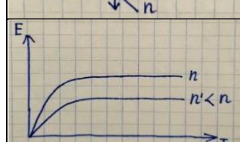
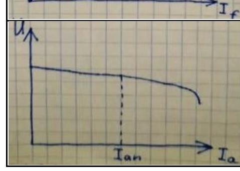



- **In the Linear area of operations:**
  - $k\Phi = k_{sr} I_a \leftarrow \Phi \propto I_a$
  - $M = k\Phi I_a = k_{sr} I_a \omega$
  - $M = k\Phi I_a = k_{sr} I_a^2$
  - $E_a = V - I_a(R_a + R + R_{sr})$
- $\omega = \frac{V}{k_{sr} I_a} - \frac{R_a + R + R_{sr}}{k_{sr}} = \frac{V}{\sqrt{k_{sr} M}} - \frac{R_a + R + R_{sr}}{k_{sr}}$

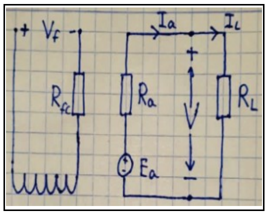



- The graph shows that we can get a high torque at a low velocity.
- Applications: Car starter, blenders, electric cranes, elevators, etc.

## Separate Activation (Generator):

| | |
|---|---|
|   | • **With no load:** $U = E$, as the armature current is 0. |
|   | • The resultant graph is only true for a specific speed $n$ |
|   | • **Under Load:** The voltage of the armature drops with an inclination to the armature resistance $R_a$, until the graph drops due to the response of the armature. See graph below. |

## DC Generator with Separate Activation:




- $M = k_m \Phi I_a$
- $V_L = E_a - I_a R_a$
- $I_a = I_L$
- $\begin{cases} V_L = K\Phi\omega - I_a R_a \\ \quad \Phi \propto I_f \end{cases}$
- Induced Armature Voltage: $E_a = U + I_a R_a$

## DC Motor with Parallel Excitation:

- **Excitation Coil Current:** $I_f = \frac{U}{R_f}$
- **Excitation Coil Losses:** $P_{R_f} = V_{in} \cdot I_f$
- $r_a = \frac{P_{r_a}}{I_{an}^2}$
- $I_{an} = I_{in_{nom}} - I_f$ [See 2017B, Q1.B for source]
- **Armature Current:** $I_a = I_{in} - I_f$
- $\Delta P = P_{in} - P_{out} = P_{in}(1 - \eta)$
- $\Delta p_{cu} = I_f^2 \cdot R_f + I_a^2 \cdot R_a$
- **Mechanical/ No-Load Losses:** $\Delta P_{fe} + \Delta P_{mech} = P_{loss} - \Delta P_{cu}$

## DC Generator with Parallel Excitation:

- **Armature Current:** $I_a = I_f + I_{Load}$
- **Parallel Excitation Losses:** $\Delta P_{fe} = V_t^2 / R_f$
- **Armature Losses:** $\Delta P_{cu} = I_a^2 R_a$
- **Output Power:** $P_{out} = V_t \cdot I_{Load}$
- **Input Power:** $P_{in} = E_a \cdot I_a$
- **Induced Voltage:** $E_a = V_t + +I_a R_a$

## DC Motor with Series Excitation:

- $\Phi = const. \cdot I_a$
- $M = k_m \Phi I_a = k_m \cdot const \cdot I_a \cdot I_a = k_m \cdot C \cdot I_a^2$
- **Connection between torque and armature (which is also the excitation current):** $M = k_m \cdot C I_a^2$

## DC Machine Starting Current:

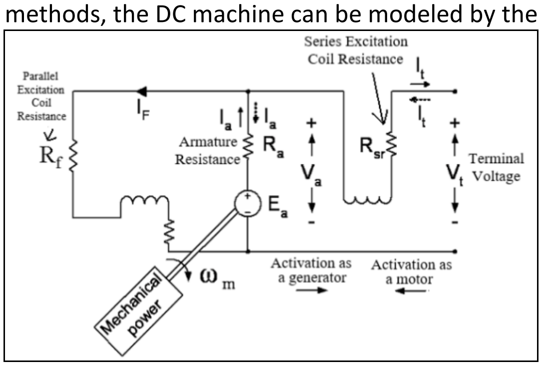
- **Armature Current:** $I_a = (V_a - E_a) R_a$, where $E_a$ is voltage induced in armature (back emf), and $V_a$ is the voltage supplied.
- When the motor starts, its speed remains 0, so $E_a = 0$, since back emf $\propto n$
- **So, Starting Armature Current:** $I_{a_{start}} = V_a R_a$
- $M = k_m \Phi I_a = k_m \cdot const \cdot I_a \cdot I_a = k_m \cdot C \cdot I_a^2$
- **Connection between torque and armature (which is also the excitation current):** $M = k_m \cdot C I_a^2$

## Finding the Constants $k_e\Phi,\ k_m\Phi$: [This was done for parallel connected DC Machine]

- $E_{an} = V_{in} - I_{an} r_a$
- $E_a = k_e \Phi \cdot n$ rearrange into $k_e \Phi = \frac{E_a}{n}$, substitute $E_{an}$ into $E_a$
- **At nominal condition:** $M_n = 9.55 \frac{P_n}{n_n} = k_m \Phi_n I_{an} \rightarrow k_m \Phi = \frac{M_n}{I_{an}}$

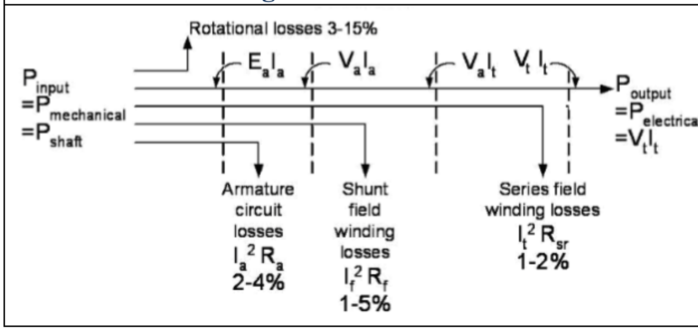
## DC Machine Losses:

For a general case, where we have mixed excitation methods, the DC machine can be modeled by the circuit:

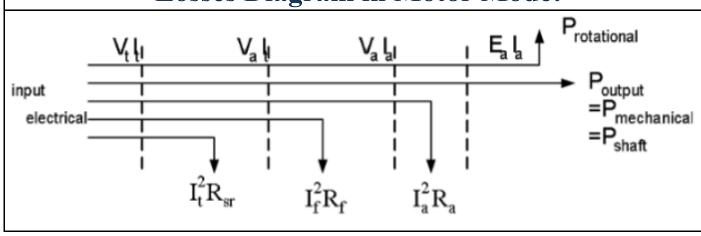



- **From Kirchhoff Equations we get the following on the motor:**
  - $I_t = I_a + I_f$
  - $V_t = V_a + I_t R_{sr}$
- **DC Machine Losses can be divided as such:**
  - ***Sliding rings voltage drop, (Arc Drop):*** $= 2\Delta V_d$. We can assume this to be $2\ [V]$. Therefore the losses will be $P_{\text{sliding ring}} = 2 \cdot I_a \cdot V_D$
  - ***No Load Losses or Rotational Losses:*** These are about 3-15% of the input power. These include bearings friction, wind windage, and core losses. They will be given if they exist, and cannot be neglected.
  - ***Armature (Rotor) Cupper Losses:*** About 3-4% of the input power, given by $I_a^2 \cdot R_a$
  - ***Parallel Excitation Field Winding Cupper Losses:*** About 1-5% of the input power, calculated using $I_f^2 \cdot R_f$
  - ***Series Excitation Field Winding Cupper Losses:*** Given by $I_t^2 \cdot R_{sr}$

### Losses Diagram in Generator Mode:




### Losses Diagram in Motor Mode:




## DC Machine Efficiency: $\eta = \frac{P_{out}}{P_{in}}$

## Meaning of "Load in terms of motor/generator":

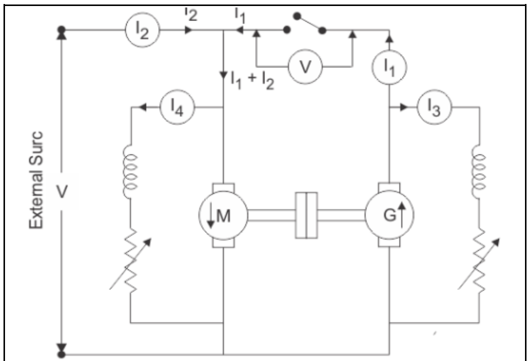
When we use the DC machine as a motor, and want to refer to the motor's load, we actually mean the amount of torque we get from the motor. For example, when saying "a specific motor preserves the same load" we mean "the motor preserves the same torque". If we say that the load was reduced by half, this means the torque was halved, and so on.

Recall that DC machine obeys the equation $M_m = k_m \Phi I_a$, which means that for a parallel excitation machine, where the excitation is constant with the input voltage, then the load remains the same, i.e. the torque remains the same, i.e. $I_a$ remains the same. And if our load was halved, that means $I_a$ was halved.

In generator mode, the meaning of the load is simply the output current of the generator. "Load remains the same" = "Output current remains same" etc.

## Hopkinson's Test (Regenerative Test):

Method to test efficiency of a DC Machine. It is a full load test, which requires two identical machines which are coupled to each other. One of the machines is working as a generator, supplying mechanical power to the other machine, which is working as a motor, which is driving the generator.

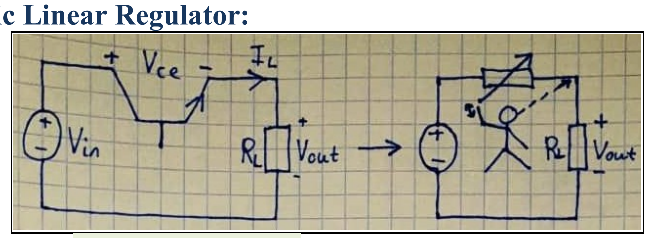


Hopkinson's Test of DC Machine

- If there are no losses in the machine, the no external power supply would be needed. But due to the drop in generator output voltage, we need an extra voltage source to supply the proper input voltage to the motor. Hence the power drawn from then external supply is used to overcome the internal losses of the motor-generator set.
- A motor and a generator, both identical, are coupled together. When the machine is started, it is started as a motor. The shunt field resistance of the machine is adjusted so that the motor can run at its rated speed.
- The generator is made equal to the supply voltage by adjusting the shunt field resistance across the generator.
- Calculating Efficiency:
  - Let $V =$ supply voltage of the machines, then $\text{Motor input} = V(I_1 + I_2)$, where $I_1 =$current from the generator, and $I_2 =$current from external source. And, $\text{Generator output} = VI_1$
  - Both machines are at same efficiency $\eta$.
  - Output of motor: $\eta \times input = \eta \times V(I_1 + I_2)$
  - Input to generator = output of motor = $\eta \times V(I_1 + I_2)$
  - Output of generator = $\eta \times input = \eta \times [\eta \times V(I_1 + I_2)] = \eta^2 V(I_1 + I_2)$
  - Thus: $\eta = \sqrt{\frac{I_1}{I_1+I_2}}$
  - Stray losses in both machines: $W = VI_2 - (I_1 + I_2 - I_4)^2 R_a + VI_4 + (I_1 + I_3)^2 R_a + VI_3$, and stray loss in each machine is $W/2$
  - Efficiency of Generator: $\eta_g = \frac{VI_1}{VI_1 + W_g}$, where $W_G = (I_1 + I_3)^2 R_a + VI_3 + \frac{W}{2}$ are the total losses in the generator.
  - Efficiency of Motor: $\eta_M = \frac{V(I_1+I_2) - W_M}{V(I_1+I_2}$, where $W_M = (I_1 + I_2 - I_4)^2 R_a + VI_4 + \frac{W}{2}$ are the total losses in the motor.

# DC-DC CONVERTERS:

## Basic Linear Regulator:




- Can only reduce voltage levels.
- Transistor is working in the linear operation area, so the load current is proportional to the base current.
- For each current we get a different voltage drop $V_{ce}$
- **Output voltage:** $V_{out} = V_{in} - V_{ce}$

**Switching Period:** $T_s = t_{on} + t_{off} = 1/f_s$ , where $f_s$ is the switching frequency.

**Duty Cycle:** $D = \frac{t_{on}}{t_{on}+t_{off}} = \frac{t_{on}}{T} = t_{on} \cdot f$

**Output Voltage Averaging:** The average of the square wave, $V_{out}$, generated by the switching action is: $V_{out} = \frac{1}{T}\int_0^T V_{out}(t)dt = \frac{1}{T}\int_0^{DT} V_{in}\,dt = V_{in}D$.

- It is possible to see that the output voltage is determined by the input voltage and the Duty Cycle, with no apparent losses.

**Steady State Converters:** We only analyze converters in steady state after the output is stabilized. In steady state, the following rules are used:

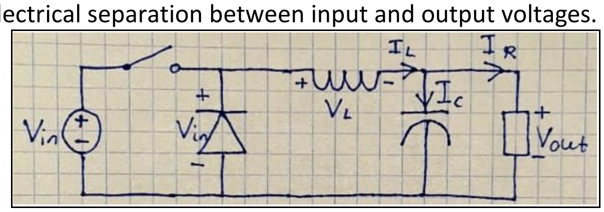
- $i_L(t + T) = i_L(t) \rightarrow$ the current of the inductor is periodic
- $V_L = \frac{1}{T}\int_t^{t+T} V_L(\lambda)d\lambda = 0 \rightarrow$average voltage on the inductor is 0
- $I_c = \frac{1}{T}\int_t^{t+T} i_c(\lambda)d\lambda = 0 \rightarrow$ average current of the capacitor is 0
- $P_{in} = P_{out} \rightarrow$ ideal components, no losses

## Work Assumptions for Converter Analysis:

- The circuit is working in steady state
- The current of the inductor is always above 0
- The output capacitor is very large
- During the period $T$, the switch is closed at $TD$, and open during $(1 - T)D$
- All components are ideal

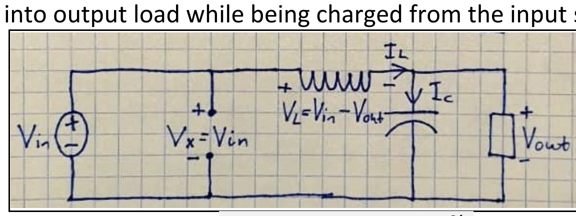
## BUCK CONVERTER:

This converter is used to lower the voltage. It is not isolated, i.e. there is no electrical separation between input and output voltages.



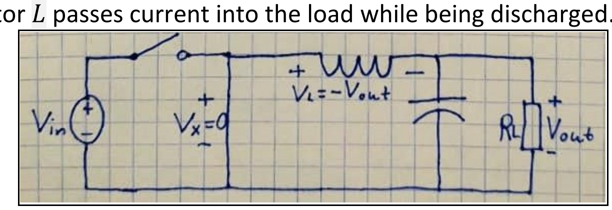

- In a practical circuit, the switch is usually a MOSFET.
- There are two distinct modes, closed switch and open switch.

**Closed Switch (Buck):** Switch is "on" (closed). Inductor $L$ passes current into output load while being charged from the input source.




- **Voltage on the inductor:** $V_L = V_{in} - V_{out} = L\frac{di_L}{dt}$
  - $\rightarrow \frac{di_L}{dt} = \frac{\Delta i_L}{\Delta t} = \frac{\Delta i_L}{DT} = \frac{V_{in} - V_{out}}{L}$
- **Current ripple of the inductor:** $(\Delta i_L)_{closed} = \left[\frac{V_{in} - V_{out}}{L}\right] DT$

**Open Switch (Buck):** When the switch is "off" (i.e. open), the inductor $L$ passes current into the load while being discharged.




- As the diode is forward biased, the inductor is parallel to the output, so the **voltage on the inductor:** $V_L = -V_{out} = L\frac{di_L}{dt}$
  - $\rightarrow \frac{\Delta i_L}{\Delta t} = \frac{\Delta i_L}{(1-D)T} = -\frac{V_{out}}{L}$
- **Inductor ripple:** $(\Delta i_L)_{open} = -\left[\frac{V_{out}}{L}\right](1 - D)T$

## Analysis of the Complete Buck Circuit:

- The change of current of the inductor over a single period should be zero, thus: $(\Delta i_L)_{closed} + (\Delta i_L)_{open} = 0$
- From previous analysis, obtain: $\left[\frac{V_{in}-V_{out}}{L}\right]DT - \left[\frac{V_{out}}{L}\right](1-D) = 0$
- **Finally, the Stead State Transfer Function:** $V_{out} = V_{in}D$
- **Inductor Voltage:** $V_L = L\frac{dI_L}{dt}$
- **Inductor Current:** $\Delta I = \frac{V_L}{L}\Delta t = \frac{(V_{in}-V_{out})}{L}t_{on} = \frac{V_{out}}{L}t_{off}$
- **When assigning:** $V_{out} = D_{on}V_{in},\ D_{on} = \frac{t_{on}}{T} = t_{on}f \to t_{on} = \frac{D_{on}}{f}$
- **Inductor Current Oscillations:** $\Delta I = \frac{V_{in}(1-D_{on})}{L}\frac{D_{on}}{f}$

**Output Voltage Ripple (Buck):** The output capacitor will charge when the inductor current is higher than its average current, and will discharge when its current is below the average. We get: $\frac{\Delta V_{out}}{V_{out}} = \frac{1-D}{8LCf^2}$, $\Delta V_{out} = \frac{\Delta IT}{8C_{out}} = \frac{(1-D)DV_{in}}{8LCf^2}$, and $C_{out} = \frac{1-D}{8L\left(\frac{\Delta V_{Cout}}{V_{out}}\right)f^2}$

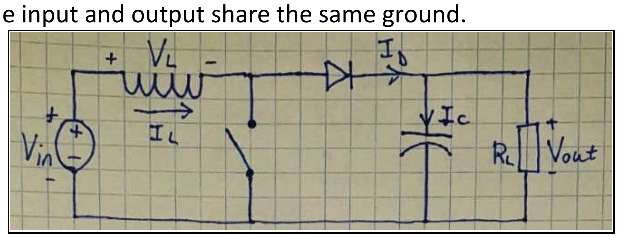
**Minimum Inductor Size for CCM (Buck):** Buck converter operates in continuous mode if the current through the inductor never falls to zero during the commutation cycle.

- **Minimum Inductor Current:** $I_{L_{\min}} = I_{L_{av}} - \frac{\Delta I_L}{2} = V_{out}\left[\frac{1}{R_L} - \frac{1-D}{2Lf}\right]$
- **Maximum inductor current:** $I_{L_{\max}} = I_{L_{av}} + \frac{\Delta I_L}{2} = V_{out}\left[\frac{1}{R_L} + \frac{1-D}{2Lf}\right]$
- When the inductor current touches 0, this will force the **minimum inductor size:** $0 = I_{L_{\min}} = V_{out}\left[\frac{1}{R_L} - \frac{1-D}{2Lf}\right]$, thus $L_{\min} = \frac{R(1-D)}{2f}$

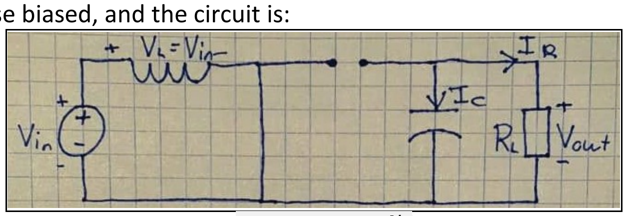
**Discontinuous Mode (Buck):** Energy required by the load is too small. In this case, the current through the inductor falls to zero during part of the period. The inductor current falling below zero results in the discharging of the output capacitor during each cycle and therefore higher switching losses. We get: $V_{out} = V_{in}\frac{1}{\frac{2LI_{out}}{D^2V_{in}T}+1}$

## BOOST CONVERTER (STEP UP):

Used to increase voltage levels. Similar to the Buck, it's not isolated, and the input and output share the same ground.

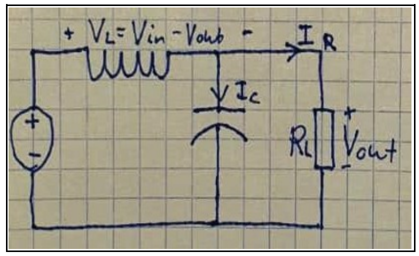



**Closed Switch (Boost):** When the switch is "on" (i.e. closed), the inductor $L$ is being charged by the input source. The diode is also reverse biased, and the circuit is:




- **Voltage on the inductor:** $V_L = V_{in} = L\frac{di_L}{dt}$
  - $\frac{di_L}{dt} = \frac{\Delta i_L}{\Delta t} = \frac{\Delta i_L}{DT} = \frac{V_{in}}{L}$
- **Ripple current in the inductor:** $(\Delta i_L)_{closed} = \frac{V_{in}DT}{L}$

**Open Switch (Boost):** When the switch is "off" (i.e. open), the inductor L passes current into the load while being discharged. The diode is also forward biased, so we treat it as a short:




- When the diode is shorted, **voltage on the inductor:** $V_L = V_{in} - V_{out} = L\frac{di_L}{dt}$
  - $\frac{\Delta i_L}{\Delta t} = \frac{\Delta i_L}{(1-D)T} = \frac{V_{in}-V_{out}}{L}$
- **Ripple Current:** $(\Delta i_L)_{open} = \left(\frac{V_{in}-V_{out}}{L}\right)(1-D)T$

## Analysis of the Complete Circuit (Boost):

- $(\Delta i_L)_{closed} + (\Delta i_L)_{open} = 0$
- $\frac{V_{in}DT}{L} + \left(\frac{V_{in}-V_{out}}{L}\right)(1-D)T = 0$
- $V_{in}(D+1-D) + V_{out}(1-D) = 0$
- **Transfer Function:** $V_{out} = \frac{V_{in}}{1-D}$
- **Inductor Voltage:** $V_L = L\frac{di_L}{dt}$
- **Inductor Current:** $\Delta I = \frac{V_{out}-V_{in}}{L}t_{off}$
- **When assigning:** $V_{out} = \frac{1}{1-D_{on}}V_{in}, D_{on} = \frac{t_{on}}{T} = t_{on}f \to t_{on} = \frac{D_{on}}{f}$
- **Inductor Current Oscillations:** $\Delta I = \frac{V_{out}(1-D_{on})}{L}\frac{D_{on}}{f}$

**Minimum Inductor Size for CCM (Boost):**

- **For an Ideal Converter:** we have power conservation between input and output power: $I_{in}V_{in} = I_{out}V_{out}$
- **Avg. Inductor Current:** $I_{L_{av}} = \frac{I_{out}V_{out}}{V_{in}} = \frac{I_{out}}{1-D_{on}} = \frac{1}{1-D_{on}}\cdot\frac{V_{out}}{R_L}$
- **Max. Inductor Current:** $I_{L_{\max}} = I_{L_{av}} + \frac{\Delta I_L}{2} = V_{out}\left[\frac{1}{1-D_{on}}\cdot\frac{1}{R_L} + \frac{D_{on}(1-D_{on})}{2Lf}\right]$
- **Min. Inductor Current:** $I_{L_{\max}} = I_{L_{av}} - \frac{\Delta I_L}{2} = V_{out}\left[\frac{1}{1-D_{on}}\cdot\frac{1}{R_L} - \frac{D_{on}(1-D_{on})}{2Lf}\right]$
- **Min. Inductor Size** is found from the equation of the minimum current when the current touches 0 Amperes: $0 = I_{L_{\min}} \to L_{\min} = \frac{R_LD_{on}(1-D_{on})^2}{2f}$

**Discontinuous Mode (Boost):** If the ripple amplitude of the current is too high, the inductor may be completely discharged before the end of a whole commutation cycle. In this case, the current through the inductor falls to zero during part of the period.

We get: $\frac{V_{out}}{V_{in}} = 1 + \frac{V_{in}D^2T}{2LI_{out}} = \frac{1+\sqrt{1+\frac{4D^2}{\frac{2L}{RT}}}}{2}$, where $R$ is the load.

**Output Voltage Ripple (Boost):** The output capacitor will be charged during $t_{off}$. During $t_{on}$, however, the output capacitor supplies the output energy by itself.

- **Assuming the capacitor current is equal to the output average current, we get:** $\Delta Q = C_{out}\Delta V_{C_{out}} = I_{av}t_{on} = \frac{V_{out}}{R_L}t_{on} = \frac{V_{out}}{R_L}D_{on}T = \frac{V_{out}}{R_L}\frac{D_{on}}{f}$
- **Extracting** $\Delta V_{C_{out}}$: $\Delta V_{C_{out}} = \frac{V_{out}D_{on}}{R_LfC} \to C = \frac{\frac{V_{out}}{\Delta V_{C_{out}}}D_{on}}{R_Lf}$
- **For resistor out (book output):** $\frac{\Delta V_{out}}{V_{out}} = \frac{D}{RCf}$, $\Delta V_{out} = \frac{D}{RCf(1-D)}V_{in}$

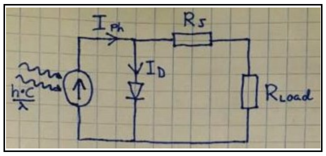
---

## SOLAR PANELS:

**Energy to Excite a Solar Cell:** Photons with energy $\frac{h\cdot c}{\lambda} > E_{gap}$ can create current flow.

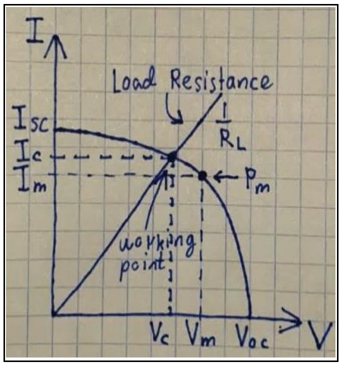
**Fluorescence Intensity:** Uses units of the Sun: $1\,[sun] = 1\left[\frac{kW}{m^2}\right]$

## Electrical Model of a Photovoltaic Cell:



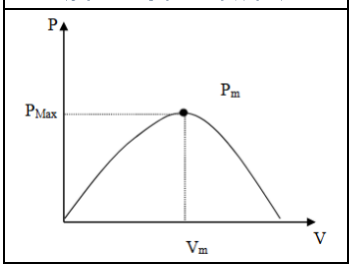

- $I_{ph}$ = photovoltaic current
- $I_D$ = dark current
- $R_S$ = self-resistance of the cell

## I-V Curve of the Photovoltaic Cell:




- Curve is specific for a given irradiation intensity
- **Straight Line:** load Resistance
- **Intersection of Straight Line and Curve:** Working point
- **Short Circuit Current of the Cell:** $I_{sc}$ This is the maximum current we can get from it
- **Open Circuit Voltage of the Cell:** $V_{oc}$, it is the maximum voltage we can get from the cell.
- **Maximum Power Point:** $I_M, V_M$ is the point where we get maximum power from the cell
  - $P_M = I_M \cdot V_M$

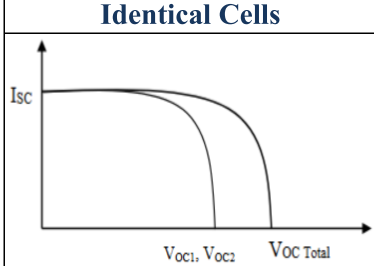
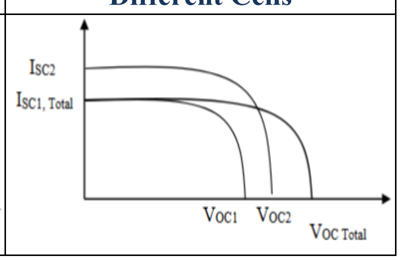
**Solar Cell Power:**




## PHOTOVOLTAIC CELL CONNECTIONS:

**Series:** If the cells are identical, their overall voltage will be equal to the sum of their voltages. The current will be one for a single solar cell. If the cells are different, the total voltage is still their sum, but the current will be the minimum among them. In a series connection, if one of the cells is damaged, the entire power of the chain is lost.

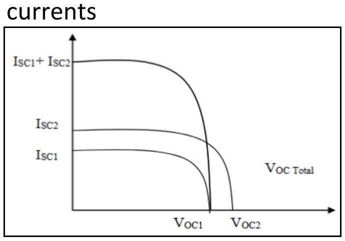
**Series Connection of Cells**

| Identical Cells | Different Cells |
|---|---|
|  |  |



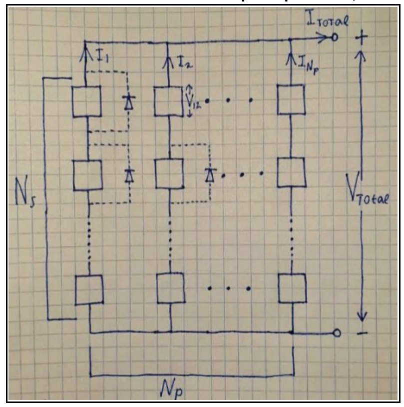

- $V_{oc_{total}} = V_{oc_1} + V_{oc_2}$
- $I_{sc_{total}} = \min(I_{sc_1}, I_{sc_2})$

**Parallel:** When the cells are parallel connected, the total voltage is the minimum between them, but this time the current is the sum of currents




- $I_{sc} = I_{sc_1} + I_{sc_2}$
- $V_{oc} = \min(V_{oc_1}, V_{oc_2})$

**Mixed Connection:** Mixed connection allows us to benefit from the pros of both connection types, and increase the overall available power from the solar array. We can build any solar array to fit our needs in terms of output power, current and voltage.

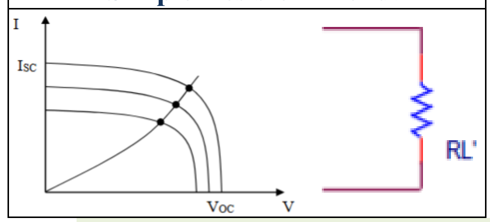



- $N_p$ Columns of cells, with $N_s$ series cells in each.
- **Voltage on a specific cell**: $V_{ij}$
- **Current on a specific cell:** $I_{ij}$
- **Total Current:** $I_{total} = N_p \cdot i_j$, where $i_j$ is the current of cells in column $j$.
- **Total Voltage:** $V_{total} = N_s \cdot V_{ij}$
- Diode in parallel to each cell in order to allow for current flow even if the cell is shaded or broken.

## POSSIBLE LOADS ON THE PHOTOVOLTAIC ARRAY/CELL:

### Simple Resistor:

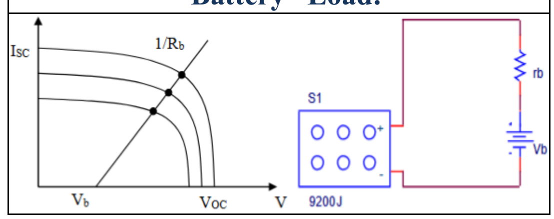
**"Simple Resistor" Load:**




- **On a Single Cell:** Connection between the voltage and current in a resistor $R_L'$ is given by $V = IR_L'$
- **Load Resistor into "Per Cell" Resistor:** We are given an Array with $N_s, N_p$ values, and the graph for the cells. We want to find the point of load on the graph. We can convert the resistor given for the array, $R_L$, into resistance which each sell "sees", $R_L'$, by : $R_L' = \frac{N_p}{N_s} = R_L$

### Battery:

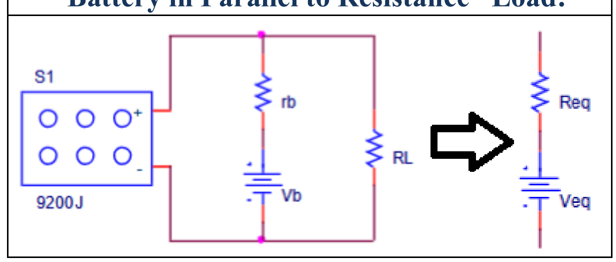
**"Battery" Load:**




- We have to account for the internal resistance of the battery, $r_a$.

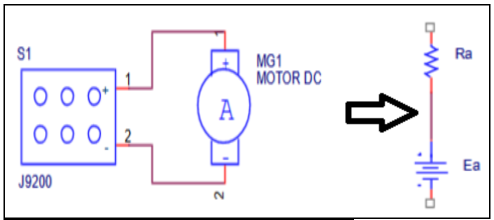
### Battery and Load Resistance in parallel:

**"Battery in Parallel to Resistance" Load:**

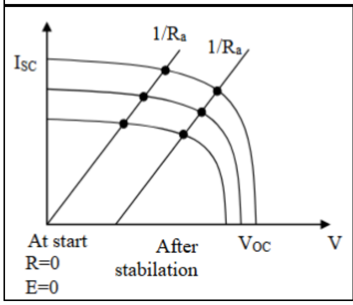



- **Equivalent circuit from Thevinin Theorem:**
  - $V_{eq} = V_b \frac{R_L}{R_L + r_b} = V_b \frac{1}{1 + \frac{r_b}{R_L}}, \; R_{eq} = R_L \parallel r_b$
- In order to be sure the battery is not discharged into the solar panels, we insert a diode in series with the panel (not shown)

## DC Motor:




- $E = C_e \cdot n$
- $V = E + I \cdot R_a$
- $M = C \cdot I$
- $V_a = E + I \cdot R_a = C_e \cdot n + I \cdot R_a$

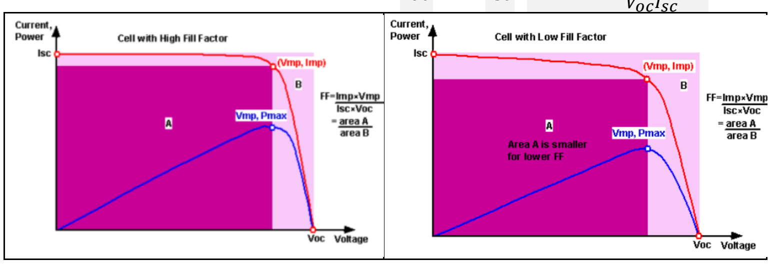



- **Difference From Other Loads:** Although this equivalent load looks similar to the previous loads (specifically – battery ∥ resistor), but this is fine, because this load has 2 operating points: During nominal condition, when it is rotating, and during startup, where $E = 0$

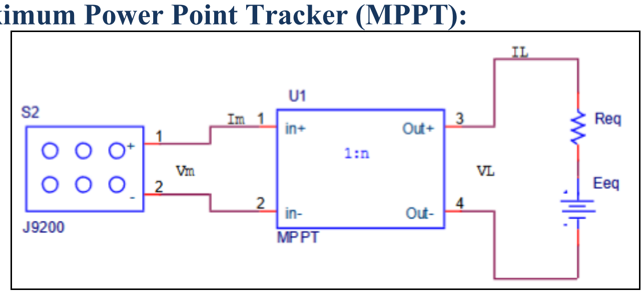
**Plotting a Resistance on the IV Curve:** The line $\frac{1}{R}$ becomes steeper as the resistance decreases. Find the numerical value of $\frac{1}{R}$, and you get $I = \frac{1}{R}V$.

**Efficiency of a Solar Cell:** Efficiency of the cell is the fraction of incident power that is converted to electricity, i.e. $\eta = \frac{P_{\max}}{P_{in}} = \frac{V_{OC}I_{SC}FF}{P_{in}}$, where FF is the fill factor.

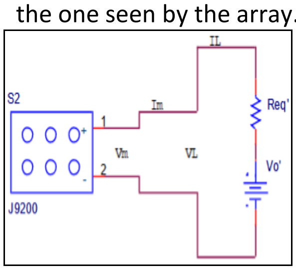
**Fill Factor:** The FF is defined as the ratio of the maximum power from the solar cell to the product of $V_{oc}$ and $I_{sc}$. $FF = \frac{V_{maxp}I_{maxp}}{V_{OC}I_{SC}}$




## Maximum Power Point Tracker (MPPT):




- The MPPT is like a transformer, but for DC
- By changing $n$ – the transformance ratio, we make sure the PV array produces maximum power, so the output voltage and current of the array stay at $V_M, I_M$
- This device is actually an SMPS (Switched Mode Power Supply), from the DC converters section
- We need to find $n$ because this is how we reflect the actual load to the one seen by the array.




- $R'_{eq} = \frac{R_{eq}}{n^2}$
- $V'_o = \frac{E_{eq}}{n}$
- $V_M = I_M R'_{eq} + V'_o \rightarrow V_m = I_m \frac{R_{eq}}{n^2} + \frac{E_{eq}}{n}$
- $n = \frac{E_{eq}}{2V_m} + \sqrt{\left(\frac{V_{eq}}{2V_M}\right)^2 + \left(\frac{I_M R_{eq}}{V_M}\right)}$
- $V_L = nV_M$
- $I_L = \frac{I_M}{n}$
- $P_M = I_M V_M = I_L V_L = I_L^2 R$
- For $E_{eq} = 0$, we get: $n = \sqrt{\frac{I_M}{V_M} R_{eq}}$
- For $R_{eq} \rightarrow 0$, we get: $n = \frac{E_{eq}}{V_m}$

When the MPPT is connected, it always finds the maximum power available for the current radiation settings.

The actual MPPT is not ideal, and has some power loss, and an efficiency that is around 90%.

This means that the use of MPPT is only warranted when the gain of power achieved by this system is worth its power loss due to its efficiency.

## GENERAL FORMULAS:

**Resistors in Series:** $R_{tot} = R_1 + R_2 + R_3 \ldots$

**Resistors in Parallel:** $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots$

**Inductors in Series:** $L_{total} = L_1 + L_2 + L_3 \ldots$

**Inductors in Parallel:** $L_{total} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \cdots \frac{1}{L_n}}$

**Capacitors in Series:** $C_{total} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \cdots \frac{1}{C_n}}$

**Capacitors in Parallel:** $C_{total} = C_1 + C_2 + \cdots C_n$

**Kirchhoff's Current Law:** The algebraic sum of currents in a network of conductors meeting at a point is zero, $\sum_{k=1}^{n} I_k = 0$

**Kirchhoff's Voltage Law:** The directed sum of the voltages around any closed loop is zero, $\sum_{k=1}^{n} V_k = 0$

| **Basic Forms** | |
|---|---|
| $\int x^n dx = \frac{1}{n+1}x^{n+1}$, $n \neq -1$ | $\int u dv = uv - \int v du$ |
| $\int \frac{1}{x} dx = \ln\|x\|$ | $\int \frac{1}{ax+b} dx = \frac{1}{a}\ln\|ax+b\|$ |
| **Integrals of Rational Functions** | |
| $\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a}$ | $\int \frac{1}{1+x^2} dx = \tan^{-1}x$ |
| $\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$ | $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ |
| $\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$ | $\int \frac{x}{a^2+x^2} dx = \frac{1}{2}\ln\|a^2+x^2\|$ |
| $\int \frac{x^3}{a^2+x^2} dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln\|a^2+x^2\|$ | $\int \frac{x^2}{a^2+x^2} dx = x - a\tan^{-1}\frac{x}{a}$ |
| $\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$ | $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, a \neq b$ |
| $\int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a}\ln\|ax^2+bx+c\| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$ | $\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln\|a+x\|$ |
| **Integrals with Roots** | |
| $\int \sqrt{x-a}\ dx = \frac{2}{3}(x-a)^{3/2}$ | $\int \sqrt{\frac{x}{a+x}}\ dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x} + \sqrt{x+a}\right]$ |
| $\int \frac{1}{\sqrt{x \pm a}}\ dx = 2\sqrt{x \pm a}$ | $\int x\sqrt{ax+b}\ dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b}$ |
| $\int \frac{1}{\sqrt{a-x}}\ dx = -2\sqrt{a-x}$ | $\int \sqrt{x(ax+b)}\ dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left\|a\sqrt{x} + \sqrt{a(ax+b)}\right\|\right]$ |
| $\int \sqrt{ax+b}\ dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$ | $\int \sqrt{x^3(ax+b)}\ dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left\|a\sqrt{x} + \sqrt{a(ax+b)}\right\|$ |
| $\int (ax+b)^{3/2}\ dx = \frac{2}{5a}(ax+b)^{5/2}$ | $\int \sqrt{x^2 \pm a^2}\ dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2\ln\left\|x + \sqrt{x^2 \pm a^2}\right\|$ |
| $\int \frac{x}{\sqrt{x \pm a}}\ dx = \frac{2}{3}(x \mp 2a)\sqrt{x \pm a}$ | $\int \sqrt{a^2-x^2}\ dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$ |
| $\int \sqrt{\frac{x}{a-x}}\ dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$ | $\int x\sqrt{x^2 \pm a^2}\ dx = \frac{1}{3}(x^2 \pm a^2)^{3/2}$ |

| | |
|---|---|
| $\int \frac{1}{\sqrt{x^2 \pm a^2}} \; dx = \ln\left\|x + \sqrt{x^2 \pm a^2}\right\|$ | $\int \frac{1}{\sqrt{a^2 - x^2}} \; dx = \sin^{-1}\frac{x}{a}$ |
| $\int \frac{x}{\sqrt{x^2 \pm a^2}} \; dx = \sqrt{x^2 \pm a^2}$ | $\int \frac{x}{\sqrt{a^2 - x^2}} \; dx = -\sqrt{a^2 - x^2}$ |
| $\int \frac{x^2}{\sqrt{x^2 \pm a^2}} \; dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln\left\|x + \sqrt{x^2 \pm a^2}\right\|$ | $\int \sqrt{ax^2 + bx + c} \; dx = \frac{b + 2ax}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a^{3/2}} \ln\left\|2ax + b + 2\sqrt{a(ax^2 + bx^{+}c)}\right\|$ |
| $\int \frac{1}{\sqrt{ax^2 + bx + c}} \; dx = \frac{1}{\sqrt{a}} \ln\left\|2ax + b + 2\sqrt{a(ax^2 + bx + c)}\right\|$ | $\int \frac{x}{\sqrt{ax^2 + bx + c}} \; dx = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a^{3/2}} \ln\left\|2ax + b + 2\sqrt{a(ax^2 + bx + c)}\right\|$ |
| $\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$ | |

## Integrals with Exponentials

| | |
|---|---|
| $\int e^{ax} \; dx = \frac{1}{a}e^{ax}$ | $\int xe^{x} \; dx = (x - 1)e^{x}$ |
| $\int xe^{ax} \; dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$ | $\int x^2e^{x} \; dx = (x^2 - 2x + 2)e^{x}$ |
| $\int x^2e^{ax} \; dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$ | $\int x^3e^{x} \; dx = (x^3 - 3x^2 + 6x - 6)e^{x}$ |
| $\int x^ne^{ax} \; dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax} \; dx$ | $\int x^ne^{ax} \; dx = \frac{(-1)^n}{a^{n+1}}\Gamma[1 + n, -ax]$, where $\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t} \; dt$ |
| $\int e^{ax^2} \; dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right)$ | $\int e^{-ax^2} \; dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(x\sqrt{a}\right)$ |
| $\int xe^{-ax^2} \; dx = -\frac{1}{2a}e^{-ax^2}$ | $\int x^2e^{-ax^2} \; dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}\left(x\sqrt{a}\right) - \frac{x}{2a}e^{-ax^2}$ |
| $\int \sqrt{x}e^{ax} \; dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}\left(i\sqrt{ax}\right)$, where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ | |

## Integrals with Trigonometric Functions

| | |
|---|---|
| $\int \sin ax \; dx = -\frac{1}{a}\cos ax$ | $\int \sin^2 ax \; dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$ |
| $\int \sin^3 ax \; dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$ | $\int \cos ax \; dx = \frac{1}{a}\sin ax$ |
| $\int \cos^2 ax \; dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$ | $\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$ |
| $\int \sin^2 x\cos x \; dx = \frac{1}{3}\sin^3 x$ | $\int \cos^2 ax\sin ax \; dx = -\frac{1}{3a}\cos^3 ax$ |
| $\int \sin^2 ax\cos^2 ax \; dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$ | $\int \tan ax \; dx = -\frac{1}{a}\ln\cos ax$ |
| $\int \tan^2 ax \; dx = -x + \frac{1}{a}\tan ax$ | $\int \tan^3 ax dx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax$ |
| $\int \cos x\sin x \; dx = \frac{1}{2}\sin^2 x + c_1 = -\frac{1}{2}\cos^2 x + c_2 = -\frac{1}{4}\cos 2x + c_3$ | |
| $\int \sin^2 ax\cos bx \; dx = -\frac{\sin[(2a - b)x]}{4(2a - b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a + b)x]}{4(2a + b)}$ | |
| $\int \cos^2 ax\sin bx \; dx = \frac{\cos[(2a - b)x]}{4(2a - b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a + b)x]}{4(2a + b)}$ | |
| $\int \cos ax\sin bx \; dx = \frac{\cos[(a - b)x]}{2(a - b)} - \frac{\cos[(a + b)x]}{2(a + b)}, a \neq b$ | |
| $\int \sin^2 ax\cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a - b)x]}{16(a - b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a + b)x]}{16(a + b)}$ | |

## Products of Trigonometric Functions and Exponentials

| | |
|---|---|
| $\int e^x\sin x \; dx = \frac{1}{2}e^x(\sin x - \cos x)$ | $\int e^{bx}\sin ax \; dx = \frac{1}{a^2 + b^2}e^{bx}(b\sin ax - a\cos ax)$ |
| $\int e^x\cos x \; dx = \frac{1}{2}e^x(\sin x + \cos x)$ | $\int e^{bx}\cos ax \; dx = \frac{1}{a^2 + b^2}e^{bx}(a\sin ax + b\cos ax)$ |
| $\int xe^x\sin x \; dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$ | $\int xe^x\cos x \; dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$ |

## Products of Trigonometric Functions and Monomials

| | |
|---|---|
| $\int x\cos x \; dx = \cos x + x\sin x$ | $\int x\cos ax \; dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$ |
| $\int x^2\cos x \; dx = 2x\cos x + (x^2 - 2)\sin x$ | $\int x^2\cos ax \; dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax$ |
| $\int x\sin x \; dx = -x\cos x + \sin x$ | $\int x\sin ax \; dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$ |
| $\int x^2\sin x \; dx = (2 - x^2)\cos x + 2x\sin x$ | $\int x^2\sin ax \; dx = \frac{2 - a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$ |
| $\int x\cos^2 x \; dx = \frac{x^2}{4} + \frac{1}{8}\cos 2x + \frac{1}{4}x\sin 2x$ | $\int x\sin^2 x \; dx = \frac{x^2}{4} - \frac{1}{8}\cos 2x - \frac{1}{4}x\sin 2x$ |