MA 573 Reading for 3/17/2022

- Proof of proposition 1:

$$|\text{Bias}(\hat{\alpha})|^2 + \text{Var}(\hat{\alpha}) = |(E[\hat{\alpha}] - \alpha)|^2 + E[\hat{\alpha}^2] - (E[\hat{\alpha}])^2;$$
$$= \cancel{(E[\hat{\alpha}])^2} - 2\alpha E[\hat{\alpha}] + \alpha^2 + E[\hat{\alpha}^2] - \cancel{(E[\hat{\alpha}])^2};$$
$$= \alpha^2 - 2\alpha E[\hat{\alpha}] + E[\hat{\alpha}^2];$$
$$= E[\alpha^2] - E[2\alpha\hat{\alpha}] + E[\hat{\alpha}^2];$$
$$= E[\alpha^2 - 2\alpha\hat{\alpha} + \hat{\alpha}^2];$$
$$= E[(\alpha - \hat{\alpha})^2];$$
$$= E[(\hat{\alpha} - \alpha)^2].$$

- Why unbiased MC?

$$E[\hat{\alpha} - \alpha] = E[4I(X_i^2 + Y_i^2 < 1) - \tfrac{4n}{N}];$$
$$= 4E[I(X_i^2 + Y_i^2 < 1)] - 4E[\tfrac{n}{N}];$$

If both $E[I(X_i^2+Y_i^2<1)]$ and $E[\frac{n}{N}]$ represent the same thing, then Bias $= 0$, and by definition of unbiased, this shows why $\hat{\alpha}$ is unbiased.

Given Bias $= 0$, MSE $= 0$.

- Why unbiased β_N?

$$E[\beta_N - \alpha] = E[\tfrac{1}{N}\textstyle\sum_{i=1}^{N} \hat{\alpha}_i - \tfrac{1}{N} 4n];$$
$$= \tfrac{1}{N} E[\textstyle\sum_{i=1}^{N} \hat{\alpha}_i - 4n].$$

Since the probability of success is obtain by multiplying 4, and $E[\sum_{i=1}^{N} \hat{\alpha}_i]$ estimates the same thing,

Bias $= 0$.

- $\frac{1}{N}MSE(\hat{\alpha}) = \frac{1}{N}E[(\hat{\alpha}-\alpha)^2];$

$$= \frac{1}{N}E[\hat{\alpha}^2 - 2\alpha\hat{\alpha} + \alpha^2];$$

$$= E[\tfrac{1}{N}\hat{\alpha}^2 - \tfrac{2}{N}\alpha\hat{\alpha} + \tfrac{1}{N}\alpha^2].$$

- Why β_N a.s. constant?

β_N is an estimator and α is obtained in the same form of the estimator, plus β_N is unbiased.

- β_M biased?

$$E[\hat{\alpha}] - \alpha = E[\beta_M] - \bar{\alpha}_M;$$

$$= E[\tfrac{1}{M}\textstyle\sum_{i=1}^{M}(\alpha_i - \tfrac{1}{M}\sum_{i=1}^{M}\alpha_i)] - \tfrac{1}{M}\sum_{i=1}^{M}\alpha_i;$$

$$= E[\tfrac{1}{M}\textstyle\sum_{i=1}^{M}\alpha_i - \tfrac{1}{M^2}\sum_{i=1}^{M}\sum_{i=1}^{M}\alpha_i] - \tfrac{1}{M}\sum_{i=1}^{M}\alpha_i;$$

$$= \tfrac{1}{M}E[\textstyle\sum_{i=1}^{M}\alpha_i] - \tfrac{1}{M^2}\sum_{i=1}^{M}\alpha_i - \tfrac{1}{M}\sum_{i=1}^{M}\alpha_i \neq 0.$$