MA 573 HW for 3/3/2022

1. We know that, $p \in (f_{min}, f_{max})$ and f is continous, then, by the definition of continuity, we have;

   There exists a $\delta > 0$, such that, $|f(x) - f(\hat{\sigma})| < \varepsilon$, for $|x - \hat{\sigma}| < \delta$.

   This follows that, $f(x) - \varepsilon < f(\hat{\sigma}) < f(x) + \varepsilon$, for all $x \in (\hat{\sigma} - \delta, \hat{\sigma} + \delta)$.

   We know that, $\lim_{x \to \infty} f(x) = f_{max}$.

   By the concept of supremum, there exists some $a \in (\hat{\sigma} - \delta, \hat{\sigma}]$, such that,

   $$f(\hat{\sigma}) < f(a) + \varepsilon \leq p + \varepsilon. \quad (1)$$

   We know that, $f(0) = f_{min}$.

   By the concept of infimum, there exists some $b \in [\hat{\sigma}, \hat{\sigma} + \delta)$, such that,

   $$p - \varepsilon \leq f(b) - \varepsilon < f(\hat{\sigma}). \quad (2)$$

   Combine equation (1) and (2), we have,

   $$p - \varepsilon < f(\hat{\sigma}) < p + \varepsilon, \text{ for all } \varepsilon > 0.$$

   This follows that, $f(\hat{\sigma}) = p$ as unique. (Condition is similar to intermediate value theorem)

   We know that, $f(0) = f_{min}$. Then,

   $$\hat{\sigma} = \arg\min_{\sigma \in (0,\infty)} |f(\sigma) - p| = p.$$

2. We are given that, vol_ratio $= \sigma$, spot_price $= 100$, drift_ratio $= .0475$ strike $= 110$, and maturity $= 1$.

   $$\text{So, } d_1 = \frac{\ln\left(\frac{100}{110}\right) + \left(0.0475 + \frac{1}{2}(\sigma)^2 \cdot 1\right)}{\sigma\sqrt{1}} = \frac{-0.0478 + \frac{1}{2}\sigma^2}{\sigma}$$

   $$d_2 = d_1 - \sigma\sqrt{1} = \frac{-0.0478 + \frac{1}{2}\sigma^2}{\sigma} - \sigma$$

   $$\begin{aligned} f_{min} &= (110)e^{-(0.0475)}\Phi(-d_2) - (100)\Phi(-d_1); \\ &= 104.90\,\Phi(-d_2) - (100)\Phi(-d_1); \\ &= 10.60 \end{aligned}$$

   $$f_{max} = 104.90\,\Phi(-d_2) - (100)\Phi(-d_1) = 45.60$$

   If market put is 10, the implied volatility is:

   $$10 - 104.90\,\Phi(-d_2) + 100\,\Phi(-d_1).$$

If market call is 10, the implied volatility is:

$10 - 100\Phi(d_1) + 104.90\Phi(d_2)$.