Proof of the proposition that central finite difference operator (CFD) has convergence order 2, such that:

$$\left|\delta_{\pm h} f(x) - f'(x)\right| = O(h^2)$$

We know that the convergence order for forward finite difference operator (FFD) is 1, and it is expressed as:

$$|\delta_h f(x) - f'(x)| = O(h)$$

Where:

$$\delta_h f(x) = \frac{f(x+h) - f(x)}{h}$$

And the left-hand side of the equation or the absolute error term is expressed as or behaves as:

$$\epsilon(h) \simeq K h^{\alpha}$$

Where $\epsilon(h)$ is the absolute error term and α is its convergence order.
The convergence order of FFD being 1 can be shown by running a convergence analysis through plotting a range of data. If the graph shows a straight with slope α, then it is an indicator that the convergence analysis holds. The plotting process is based on an equation and the log_2 or ln of this equation, such that:

$$\epsilon_n(h) = \epsilon(2^{-n}) \simeq K(2^{-n})^{\alpha} = K 2^{-n\alpha}$$

Where h is defined as $h \in \{2^{-n}\text{: a range of } n\}$.
And the log_2 or ln of this equation is:

$$\log_2 \epsilon_n(h) \simeq \log_2 K - n\alpha$$

Or:

$$\ln \epsilon_n(h) \simeq \ln K - n\alpha$$

If consider the plotting process of the convergence analysis in terms of the CFD scenario, then we have the following equations:

$$\epsilon(h) = \left|\delta_{\pm h} f(x) - f'(x)\right| \simeq K h^{\alpha}$$

Where h is defined as $h \in \{2^{-n}\text{: a range of } n\}$.
And the log_2 or ln of this equation is:

$$\log_2 \epsilon_n(h) \simeq \log_2 K - n\alpha$$

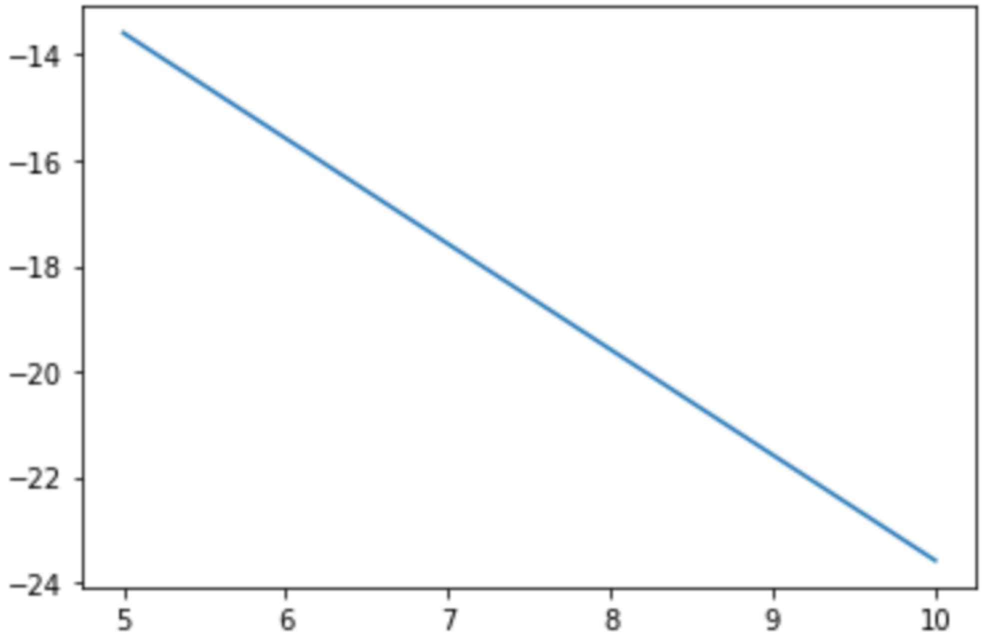
Or:

$$\ln \epsilon_n(h) \simeq \ln K - n\alpha$$

But in this scenario, the plotting graph must be a downward super-linear plot to show the convergence analysis is correct, and then the convergence order is 2 by the definition of quadratic convergence and super-linearity.
Now, to verify the convergence order of CFD, consider h, such that:

$$h \in \{e^{-n}: n = 5,6,\ldots,10\}$$

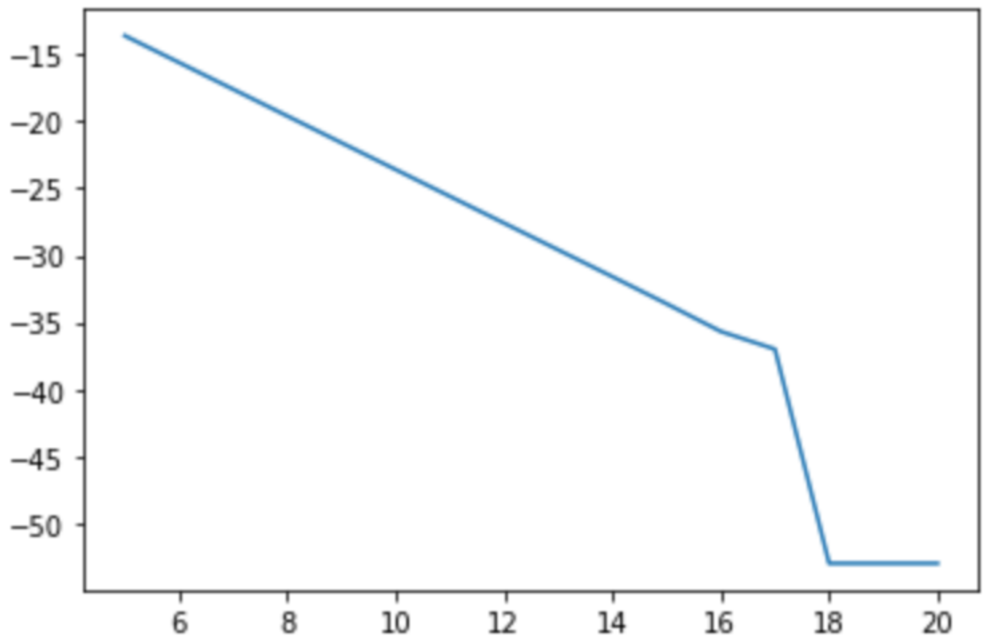
And consider the verification process with at $\pi/3$.
By running the code for convergence analysis, we get a graph of plotting, such that:



Although the graph looks like plotting line of linearity, the calculation shows that the convergence order is 1.99998839994406 which is close to 2, to be more careful we need to check whether the plotting is downward curving given a relatively larger range, such that:

$$h \in \{e^{-n}: n = 5,6,\ldots,20\}$$

After redo the whole procedure again, we have the following graph:



Based on this graph, the plotting is downward curving. And this fact is an indicator that the plotting satisfies the definition of quadratic convergence. The code for testing process is from the following link: https://github.com/YGuo00/MA-573-Independent-Study/blob/main/src/proof_of_convergence_order_2.ipynb.

Therefore, since the convergence analysis holds, CFD has convergence order 2.