MA 573 HW Correction for 3/17/2022

- Correction of the second part of the first proof:

The proof of there exists $\hat{\sigma}$, such that $f(\hat{\sigma})=P$, is given by the proof of intermediate value theorem.

Given $f(\hat{\sigma})=P$, it is obviously saying that,

$$|f(\hat{\sigma})-P|=|P-P|=0.$$

Since zero is less than or equal to non-zero values, and $\sigma \geq 0$, for $\sigma \in (0,\infty)$, we have,

$$|f(\hat{\sigma})-P|=0 \leq \sigma \leq |f(\sigma)-P|.$$

By definition of arg min, we have,

$$\hat{\sigma}=\underset{\sigma\in(0,\infty)}{\arg\min}\,|f(\sigma)-P|.$$

- Justify f strictly increasing on $(0,\infty)$:

We know that, the Black-Scholes formula for put is,

$$f(\sigma)=Ke^{-rT}\Phi(-d_2)-S(0)\Phi(-d_1),$$

where $d_1=\dfrac{\ln\left(\frac{S(0)}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2=d_1-\sigma\sqrt{T}$.

We are given that, $S(0)=100$, $r=0.0475$, $K=110$, and $T=1$. Then,

$$f(\sigma)=110e^{-(0.0475)(1)}\Phi(-d_2)-100\Phi(-d_1);$$

$$=110e^{-0.0475}\Phi(-d_2)-100\Phi(-d_1),$$

where $d_1=\dfrac{\ln\left(\frac{100}{110}\right)+\left(0.0475+\frac{1}{2}\sigma^2\right)(1)}{\sigma\sqrt{1}};$

$$=\frac{-0.0953+0.0475+\frac{1}{2}\sigma^2}{\sigma}=\frac{-0.0478+\frac{1}{2}\sigma^2}{\sigma}.$$

$$d_2=d_1-\sigma\sqrt{1}=d_1-\sigma.$$

Now, $\text{Vega} = \frac{\partial f}{\partial \sigma} = 110 e^{-0.0475} \Phi'(-d_2)\left(-\frac{\partial d_2}{\partial \sigma}\right)$

$$-100\,\Phi'(-d_1)\left(-\frac{\partial d_1}{\partial \sigma}\right);$$

$$\frac{\partial d_1}{\partial \sigma} = \left(\frac{-0.0478 + \frac{1}{2}\sigma^2}{\sigma}\right)' = \left(-0.0478\,\sigma^{-1} + \frac{1}{2}\sigma\right)';$$

$$= 0.0478\,\sigma^{-2} + 0.5 = \frac{0.0478}{\sigma^2} + 0.5.$$

$$\frac{\partial d_2}{\partial \sigma} = (d_1 - \sigma)' = (0.0478\,\sigma^{-2} + 0.5 - \sigma)';$$

$$= -0.0956\,\sigma^{-3} - 1 = -\frac{0.0956}{\sigma^3} - 1.$$

So, $\text{Vega} = 110 e^{-0.0475} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{d_2^2}{2}} \left(\frac{0.0956}{\sigma^3} + 1\right)$

$$-100 \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{d_1^2}{2}} \left(-\frac{0.0478}{\sigma^2} - 0.5\right).$$

$$e^{\frac{d_2^2}{2}} = e^{\frac{(d_1-\sigma)^2}{2}} = e^{\frac{d_1^2 - 2d_1\sigma + \sigma^2}{2}} = e^{\frac{d_1^2}{2} - d_1\sigma + \frac{\sigma^2}{2}};$$

$$= e^{\frac{d_1^2}{2} - (-0.0478 + \frac{\sigma^2}{2}) + \frac{\sigma^2}{2}};$$

$$= e^{\frac{d_1^2}{2} + 0.0478 - \cancel{\frac{\sigma^2}{2}} + \cancel{\frac{\sigma^2}{2}}}$$

$$= e^{\frac{d_1^2}{2}} \cdot e^{0.0478}.$$

Then, $\text{Vega} = 110 e^{-0.0475} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{d_1^2}{2}} e^{0.0478} \left(\frac{0.0956}{\sigma^3} + 1\right)$

$$-100 \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{d_1^2}{2}} \left(-\frac{0.0478}{\sigma^2} - 0.5\right).$$

So, $\text{Vega} = \frac{1}{\sqrt{2\pi}} e^{\frac{d_1^2}{2}} \left(110 e^{0.0003} \left(\frac{0.0956}{\sigma^3} + 1\right)\right.$

$\left. - 100\left(-\frac{0.0478}{\sigma^2} - 0.5\right)\right) > 0.$

This follows that, $f(\sigma)$ is strictly increasing.