MA 573 Note for Lecture Note 3

Definition of derivative by limit:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If h is small enough, then the definition of derivative will become:

$$f'(x) \simeq \frac{f(x+h)-f(x)}{h} = \delta_h f(x)$$

δ_h is finite difference operator (FD).

- If $h > 0$, then δ_h is forward finite difference operator (FFD).

$$\delta_h f(x) = \frac{f(x+h)-f(x)}{h}$$

- If $h < 0$, then δ_h is backward finite difference operator (BFD).

$$\delta_{-h} f(x) = \frac{f(x-h)-f(x)}{-h}$$

The central finite difference operator (CFD) is the average of FFD and BFD:

$$\delta_{\pm h} f(x) = \frac{1}{2}\big(\delta_h f(x) + \delta_{-h} f(x)\big) \simeq f'(x)$$

By Taylor's Theorem, which is a theorem gives an approximation of k-times differential function by a kth-order polynomial, both FFD and BFD have convergence order 1, such that, given $h \to 0$ or when h is small,

$$|\delta_h f(x) - f'(x)| = O(h) \to 0$$

And,

$$|\delta_{-h} f(x) - f'(x)| = O(h) \to 0$$

CFD has convergence order 2 because it yields a more accurate approximation by the 2nd-order polynomial or quadratic polynomial.

$$\left|\delta_{\pm h} f(x) - f'(x)\right| = O(h^2)$$

Proof:

$$\begin{aligned}
\left|\delta_{\pm h} f(x) - f'(x)\right| &= \left|\frac{1}{2}\big(\delta_h f(x) + \delta_{-h} f(x)\big) - f'(x)\right| \\
&= \left|\frac{1}{2}\delta_h f(x) + \frac{1}{2}\delta_{-h} f(x) - f'(x)\right| \\
&= \left|\frac{f(x+h)-f(x)}{2h} - \frac{f(x-h)-f(x)}{2h} - f'(x)\right| \\
&= \left|\frac{f(x+h)-f(x)-f(x-h)+f(x)}{2h} - f'(x)\right| \\
&= \left|\frac{f(x+h)-f(x-h)}{2h} - f'(x)\right|
\end{aligned}$$

Take the limit of the left term, and by *l'Hopitals* Rule, we have the relationship, such that

$$\frac{f(x+h)-f(x-h)}{2h}=f'(x)$$

This means that everything goes back to the original, such that

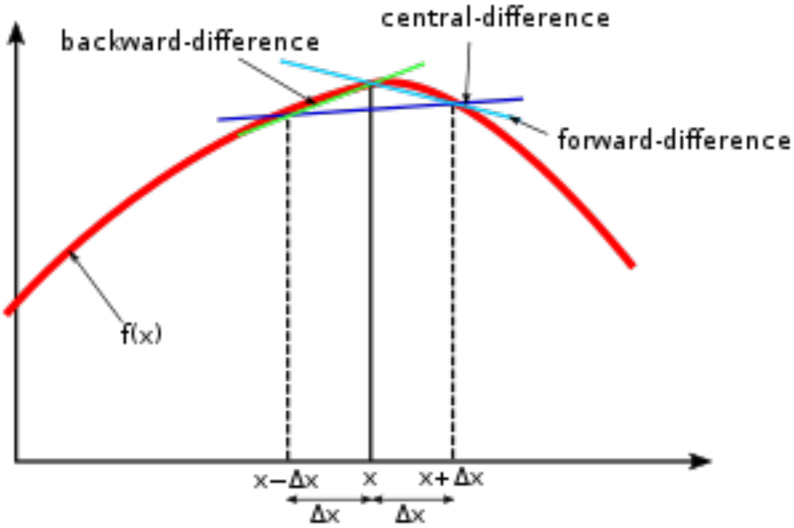
$$\left|\delta_{\pm h}f(x)-f'(x)\right|=O(h)$$

There is a proof of $g(x)=g(x^2)$, by continuity, we can use the similar idea, such that

$$O(h)=O(h^2)=O(h^4)=\cdots=O((h^2)^k)=O(h^{2k})$$

For $k=1,2,\ldots$

Since based on a graph that is about CFD,



The function is continuous, and by the continuity and the condition that h is small, we can also conclude that

$$O(h)=O(h^{2k})=O(0)=O(h^2)$$

Therefore,

$$\left|\delta_{\pm h}f(x)-f'(x)\right|=O(h^2)$$

Reference Information:

Since f is differentiable, we may use l'Hopitals rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}=\lim_{h\to 0}\frac{f'(x+h)-(-1)f'(x-h)}{2}=\lim_{h\to 0}\frac{f'(x+h)+f'(x-h)}{2}$$
$$=\frac{2f'(x)}{2}=f'(x).$$

https://en.wikipedia.org/wiki/Finite_difference
https://en.wikipedia.org/wiki/Taylor%27s_theorem
https://www.youtube.com/watch?v=JMttb5AsARE