MA 573 HW Correction for 3/3/2022

1. Let $t > s$, then,

$$g(t) = E[f(X_t)] = E[E[f(X_t)|X_s]], \text{ by independence.}$$

Since f is convex, we can apply Jensen's inequality, such that,

$$g(t) = E[E[f(X_t)|X_s]] \geq E[f(E[X_t|X_s])].$$

Since X is a martingale, we have, by definition,

$$g(t) \geq E[f(E[X_t|X_s])] = E[f(X_s)] = g(s).$$

Or simply saying, $g(t) \geq g(s)$. Hence, $g(t)$ is increasing.

2. Let $t > s$, then,

$$C(t) = E[e^{-rt}(S_t - K)^+] = E[(S_t e^{-rt} - Ke^{-rt})^+].$$

Let $X_t = S_t e^{-rt}$, then,

$$\begin{aligned} C(t) = E[(X_t - Ke^{-rt})^+] &\geq E[(X_t - Ke^{-rs})^+]; \\ &= E[f(X_t)]; \text{ where } f(x) = (x - Ke^{-rs})^+; \\ &\geq E[f(X_s)]; \\ &= E[(X_s - Ke^{-rs})^+] = C(s). \end{aligned}$$

Or simply saying, $C(t) \geq C(s)$. Hence, $C(t)$ is increasing.

3. Let $t > s$, then,

$$\begin{aligned} P(t) = E[(S_t - K)^-] &= E[(K - S_t)^+]; \\ &\geq E[(K - S_s)^+] = P(s). \end{aligned}$$

Or simply saying, $P(t) \geq P(s)$. Hence, $P(t)$ is increasing.

4. Let $t > s$, then,

$$P(t) = E[(S_t - K)^-] = E[(K - S_t)^+].$$

Since we let $r > 0$, then,

$$\begin{aligned} P(t) = E[(K - S_t)^+] &\leq E[(Ke^{-rt} - S_t)^+]; \\ &\leq E[(Ke^{-rs} - S_t)^+]; \\ &= E[f(S_t)]; \\ &\geq E[f(S_s)]; \\ &= E[(Ke^{-rs} - S_s)^+] \leq E[(K - S_s)^+] = P(s). \end{aligned}$$

The fact that $P(t) \leq P(s)$ challenges the claim that $P(t)$ increases.