MA573 Proofs

1. Proof or disprove: Suppose f is convex, and X is submartingale prove that $g(t) = E[f(X_t)]$ is increasing.

Proof: Since f is convex, this follows that its derivative is non-decreasing.

Since X is submartingale, this follows that $E[X_{t+1} | X_1, \dots, X_t] \geq X_t$.

This then follows that, $E[f(X_t)] \geq f(X_{t-1})$

So, $g'(t) = (E[f(X_t)])' > 0$

Hence, $g(t)$ is increasing.

2. Let $t \mapsto e^{-rt} S_t$ be a martingale, prove $C(t) = E[e^{-rt}(S_t - K)^+]$ is increasing.

Proof:
$$\begin{aligned}
\frac{\partial}{\partial t} C(t) &= \frac{\partial}{\partial t} E[e^{-rt}(S_t - K)^+] \\
&= E\left[\frac{\partial}{\partial t}(e^{-rt}(S_t - K)^+)\right] \\
&= E\left[(S_t - K)^+ \frac{\partial}{\partial t}(e^{-rt})\right] \\
&= E[(S_t - K)^+(-r)e^{-rt}] \\
&= -r(S_t - K)^+ E[e^{-rt}] < 0 \quad \text{decreasing?}
\end{aligned}$$

3. Suppose $r = 0$ and S is a martingale, Prove that $P(t) = E[(S_t - K)^-]$ is increasing.

Proof: S is a martingale, then $E[S_{t+1} | S_1, \dots, S_t] = S_t$

$$P'(t) = \frac{\partial}{\partial t} E[(K - S_t)^+]$$
$$= E[\tfrac{\partial}{\partial t}(K - S_t)^+]$$
$$= E[(K - S_t)^+]$$
$$= (K - S_t)^+ \geq 0$$

So, for all S, $P(t)$ is increasing.