Cheese Model (A simple version)

1. Math Notation

$y_{ij} := \log Q_{ij}$, $x_{ij} := \log P_{ij}$, $\beta_{0i} := \log \alpha_i$

2. The Model

$$y_{ij} = \beta_{0i} + \beta_{1i} \cdot D_{ij} + \beta_{2i} \cdot x_{ij} + \beta_{3i} \cdot D_{ij} \cdot x_{ij} + \epsilon_{ij} \text{ , where}$$

- $\epsilon_{ij} \sim N(0, \sigma^2)$.
- $\beta_{0i} \sim N(\mu_0, V_0)$, $\beta_{1i} \sim N(\mu_1, V_1)$, $\beta_{2i} \sim N(\mu_2, V_2)$, $\beta_{3i} \sim N(\mu_3, V_3)$. $i = 1, 2, \cdots, n$

  denoted by n is the total number of shops, it's a large enough number — so we can obtain $\mu_{0:3}$ and $V_{0:3}$ directly from ols solutions.
- $\lambda := 1/\sigma^2 \sim Ga(\frac{1}{2}, \frac{1}{2})$.

3. Inference

3.1 Posteria

$$f(\underline{\beta_0}, \underline{\beta_1}, \underline{\beta_2}, \underline{\beta_3}, \lambda \mid \underline{Y})$$

$$\propto f(\underline{Y} \mid \underline{\beta_0}, \underline{\beta_1}, \underline{\beta_2}, \underline{\beta_3}, \lambda) \cdot f(\underline{\beta_0}) \cdot f(\underline{\beta_1}) \cdot f(\underline{\beta_2}) \cdot f(\underline{\beta_3}) \cdot f(\lambda)$$

$$\propto \lambda^{\frac{mn}{2}} \exp\left\{ -\frac{\lambda}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (y_{ij} - \beta_{0i} - \beta_{1i} D_{ij} - \beta_{2i} x_{ij} - \beta_{3i} D_{ij} x_{ij})^2 \right\}$$

$$\cdot \exp\left\{ -\frac{1}{2V_0^2} \sum_{i=1}^{n} (\beta_{0i} - \mu_0)^2 \right\} \cdot \exp\left\{ -\frac{1}{2V_1^2} \sum_{i=1}^{n} (\beta_{1i} - \mu_1)^2 \right\}$$

$$\cdot \exp\left\{ -\frac{1}{2V_2^2} \sum_{i=1}^{n} (\beta_{2i} - \mu_2)^2 \right\} \cdot \exp\left\{ -\frac{1}{2V_3^2} \sum_{i=1}^{n} (\beta_{3i} - \mu_3)^2 \right\} \cdot \lambda^{-1/2} \exp\left\{ -\frac{\lambda}{2} \right\}$$

3.2 Full conditionals.

- $f(\beta_{0i} \mid \cdots) = N(\beta_{0i} \mid \varphi_{0i}, \gamma_{0i})$ where

$$\gamma_{0i} = \left(\frac{1}{V_0^2} + m \cdot \lambda \right)^{-1}, \quad \varphi_{0i} = \gamma_{0i} \cdot \left[\lambda \sum_{j=1}^{m} (y_{ij} - \beta_{1i} D_{ij} - \beta_{2i} x_{ij} - \beta_{3i} D_{ij} x_{ij}) + \frac{\mu_0}{V_0^2} \right] \quad (1)$$

- $f(\beta_{1i}|\cdots) = \mathcal{N}(\beta_{1i}|\varphi_{1i}, \gamma_{1i})$ where

$$\gamma_{1i} = \left(\frac{1}{v_1^2} + \lambda \cdot \sum_{j=1}^{m} D_{ij}\right)^{-1}, \quad \varphi_{1i} = \gamma_{1i} \cdot \left[\lambda \sum_{j=1}^{m} D_{ij}\left(y_{ij} - \beta_{0i} - (\beta_{2i}+\beta_{3i})x_{ij}\right) + \frac{\mu_1}{v_1^2}\right] \quad (2)$$

- $f(\beta_{2i}|\cdots) = \mathcal{N}(\beta_{2i}|\varphi_{2i}, \gamma_{2i})$ where

$$\gamma_{2i} = \left(\frac{1}{v_2^2} + \lambda \sum_{j=1}^{m} x_{ij}^2\right)^{-1}, \quad \varphi_{2i} = \gamma_{2i} \cdot \left[\lambda \sum_j x_{ij}\left(y_{ij} - \beta_{0i} - \beta_{1i}D_{ij} - \beta_{3i}x_{ij}D_{ij}\right) + \frac{\mu_2}{v_2^2}\right]. \quad (3)$$

- $f(\beta_{3i}|\cdots) = \mathcal{N}(\beta_{3i}|\varphi_{3i}, \gamma_{3i})$ where

$$\gamma_{3i} = \left(\frac{1}{v_3^2} + \lambda \cdot \sum_{j=1}^{m} D_{ij} \cdot x_{ij}^2\right)^{-1},$$

$$\varphi_{3i} = \gamma_{3i} \cdot \left[\lambda \sum_j D_{ij} \cdot x_{ij}\left(y_{ij} - \beta_{0i} - \beta_{1i} - \beta_{2i}x_{ij}\right) + \frac{\mu_3}{v_3^2}\right]. \quad (4)$$

- $f(\lambda|\cdots) = Ga(\lambda|\lambda_a, \lambda_b)$ where

$$\lambda_a = \frac{mn}{2} + \frac{1}{2}, \quad \lambda_b = \frac{1}{2}\sum_{i,j}\left(y_{ij} - \beta_{0i} - \beta_{1i}D_{ij} - \beta_{2i}x_{ij} - \beta_{3i}x_{ij}D_{ij}\right)^2. \quad (5)$$

4. The Gibbs Sampler

1. Start;
2. Run ols of model on the data,
   initialize $\underline{\beta_0}$, $\underline{\beta_1}$, $\underline{\beta_2}$, $\underline{\beta_3}$, (μ_0, v_0), (μ_1, v_1), (μ_2, v_2) and (μ_3, v_3);
3. Initialize λ;
4. While (chain does not converge) do:
5.   Sample $\underline{\beta_0}$ from $f(\beta_{0i}|\cdots)$, $i=1,\cdots,n$, (Eq (1));
6.   Sample $\underline{\beta_1}$ from $f(\beta_{1i}|\cdots)$, $i=1,\cdots,n$, (Eq (2));
7.   Sample $\underline{\beta_2}$ from $f(\beta_{2i}|\cdots)$, $i=1,\cdots,n$, (Eq (3));
8.   Sample $\underline{\beta_3}$ from $f(\beta_{3i}|\cdots)$, $i=1,\cdots,n$, (Eq (4));
9.   Sample λ from $f(\lambda|\cdots)$, (Eq (5));
10. End While;
11. End.