ads 实验课 5

Zhejiang University, Advanced Data Structrue and Algorithm Analysis

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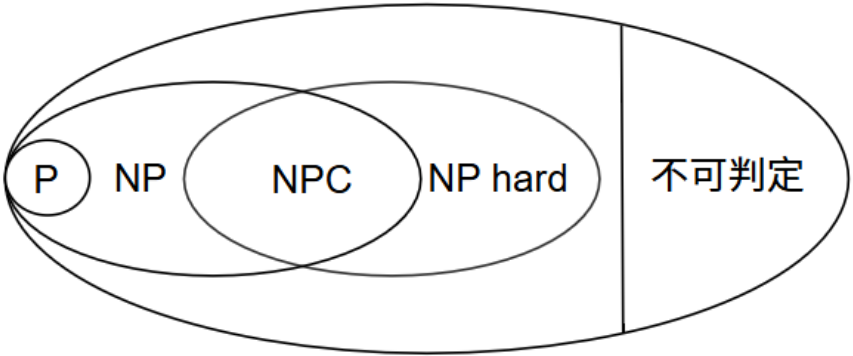
P and NP

What should we know?



decision problem.

P and NP

- Halting Problem.
- When Turing Machine get a input(string), it must go into one of three situations: loop, halt and accept, halt and reject.
- Every Turing Machine can de encoded as a string.
- We assume there is a Turing Machine T_1 can decide whether a Turing Machine M getting itself as input can halt
  - If M halts and accects itself or halts and rejects itself, T_1 halts and accepts M
  - If M loops, T_1 halts and rejects M
- Then we can construct a Turing Machine T_2, when it get a input M
  - Run T_1 on M, if T_1 halts and accepts M, then T_2 loops forever.
  - If T_1 halts and rejects M, then T_2 halts and accepts M
- What if T_2 get itself as input?

Approximation Algorithm

- What should we know?
- Bin packing: NF(2), FF(1.7), BF(1.7).
  - Offline Bin packing can not be α-approximation for any $\alpha < \frac{3}{2}$ unless $P \neq NP$
  - Online bin packing problem can not be α-approximation for any $\alpha < \frac{5}{3}$ unless $P \neq NP$
- Knapsack: 2-approximation, FPTAS
- K-center problem: 2-approx, two different versions(in metric graph and in Euclidean space).
  - If distances are unbounded, no approximation algorithm exists.
  - If we can only select given nodes in metric graph, 2-approximation algorithm can not be improved.
  - If we can select any nodes in 2-dimension Euclidean space, 2-approximation algorithm is known and no 1.82-approximation algorithm exists.
- TSP tour: 2-approximation, 1.5-approximation

Approximation Algorithm

- PTAS, EPTAS and FPTAS?
- PTAS(Polynomial Time Approximation Scheme) is polynomial of n if ϵ is given, may be in the form as $n^{\frac{1}{\epsilon}}$
- EPTAS(Efficient Polynomial Time Approximation Scheme) is polynomial of n multiplies function of $\frac{1}{\epsilon}$, such as $n^3(\frac{1}{\epsilon})^{\frac{1}{\epsilon}}$
- FPTAS(Fully Polynomial Time Approximation Scheme) is polynomial of n multiplies polynomial of $\frac{1}{\epsilon}$, such as $n^3(\frac{1}{\epsilon})^3$

Approximation Problem

多选题

Let $G = (V, E)$ be a undirected graph with non-negative edge weights. We assume that all edge weights are distinct. Let T_G be the maximum spanning tree of G. For each $v \in V$, define $\max(v)$ be the maximum-weight edge incident to v. Let $S_G = \{\max(v) : v \in V)\}$. ($S_G$ can be extened to a spanning tree of G in linear time. Can you see how to do this?) Which of the following statesments are true?

A. $S_G = T_G$.

B. $S_G \neq T_G$.

C. $S_G \subseteq T_G$**.**

D. $w(S_G) \geq w(T_G)/2$**.**

Approximation Problem

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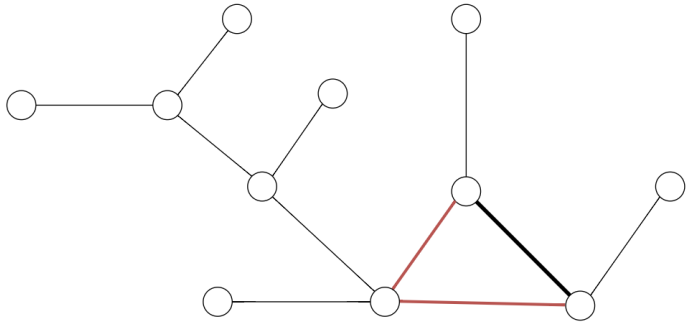
We can view the method as an approxization algorithm to solve this problem: find a maximum weighted edge set which has no loops.

Approximation Problem



We use Kruskal algorithm to solve the MST(Maximum Spanning Tree) problem.
If the black edge is in S_G, then the only possible reason why it is excluded from T_G is these two red edges have both been select before the black one, indicating they are both have bigger weight than the black one, there is a contradictory. Thus $S_G \subseteq T_G$.

Local Search

HW7 判断题 2

We are given a set of sites $S = \{s_1, s_2, \cdots, s_k\}$ in the plane, and we want to choose a set of k centers $C = \{c_1, c_2, \cdots, c_k\}$ so that the maximum distance from a site to the nearest center is minimized. Here c_i can be an arbitrary point in the plane. A local search algorithm arbitrarily choose k points in the plane to be the centers, then (1) divide S into k sets, where S_i is the set of all sites for which c_i is the nearest center; and (2) for each S_i, compute the central position as a new center for all the sites in S_i If steps (1) and (2) cause the covering radius to strictly decrease, we perform another iteration, otherwise the algorithm stops. When the above local search algorithm terminates, the covering radius of its solution is at most 2 times the optimal covering radius.

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We can give a counterexample.

Randomized Algorithm

HW7 判断题 4

Let $a = \{a_1, a_2, \cdots, a_i, \cdots, a_j, \cdots, a_n\}$ denote the list of elements we want to sort. In the quicksort algorithm, if the pivot is selected uniformly at random. Then any two elements get compared at most once and the probability of a_i and a_j being compared is $\frac{2}{j-i+2}$ for $j > i$, given that a_i or a_j is selected as the pivot.

Randomized Algorithm

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If array a is sorted, then it is a conclusion.

Randomized Algorithm

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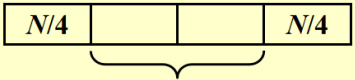
If array a is sorted, then it is a conclusion.

$$\sum_{i=1}^{n} \sum_{j>i} \frac{2}{j-i+2} = O(n \log n)$$

Local Search

Claim: The expected number of iterations needed until we find a central splitter is at most 2.



$\Pr[\textit{find a central splitter}] = 1/2$ ✓

Type j : the subproblem S is of *type j* if $N\left(\frac{3}{4}\right)^{j+1} \le |S| \le N\left(\frac{3}{4}\right)^{j}$

Claim: There are at most $\left(\frac{4}{3}\right)^{j+1}$ subproblems of *type j*.

$$E[T_{type\ j}] = O\left(N\left(\frac{3}{4}\right)^{j}\right) \times \left(\frac{4}{3}\right)^{j+1} = O(N)$$

Number of different types $= \log_{4/3} N = O(\log N)$

$\Big\}\ O(N \log N)$

Randomized Algorithm

HW7 判断题 5

Reviewing the randomized QuickSort in our course, we always select a central splitter as a pivot before recursions, make sure that each side contains at least n/4 elements. Hence, differing from the deterministic QuickSort, the worst case expected running time of the randomized QuickSort is $\Theta(n \log n)$.

Randomized Algorithm

HW7 判断题 5

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For any instance of Quick Sort, the expected running time is $\Theta(n \log n)$

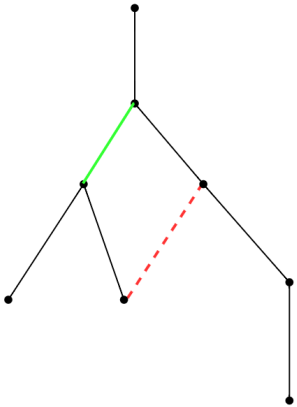
Local Search

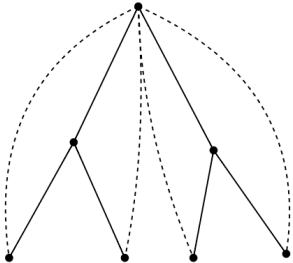
HW7 单选题 1

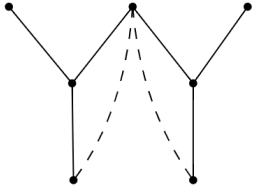
(1) Max Leaf Spanning Tree: find a spanning tree $T \in F$ with a maximum number of leaves.
(2) Minimum Spanning Tree: find a spanning tree $T \in F$ with a minimum total weight of all the edges in T.
(3) Minimum Degree Spanning Tree: find a spanning tree $T \in F$ such that its maximum degree of all the vertices is the smallest.

A. The local search always return an optimal solution for Max Leaf Spanning Tree.
B. The local search always return an optimal solution for Minimum Spanning Tree.
C. The local search always return an optimal solution for Minimum Degree Spanning Tree.
D. For neither of the problems that this local search always return an optimal solution.

Local Search







Local Search

Or you can solve this problem from the NP complexity.

- Minimum Spanning tree is P problem, because we can solve it using Prim or Kruskal algorithm.
- Max Leaf Spanning Tree is NP problem, because we can use reduction chain Vertex Cover $\rightarrow$ Connected Dominating Set Problem $\rightarrow$ Max Leaf Spanning Tree to prove its NP-complete.
- Minimum Degree Spanning Tree is also NP problem, because we can use reduction Hamilton Path $\rightarrow$ Minimum Degree Spanning Tree to prove its NP-complete.

If we can use local search to solve Max Leaf Spanning Tree or Minimum Degree Spanning Tree, we can solve all NP problem in polynomial time, which indicating P = NP.

Local Search

HW7 单选题 2

Max-cut problem: Given an undirected graph $G = (V, E)$ with positive integer edge weights w_e, find a node partition (A, B) such that $w(A, B)$, the total weight of edges crossing the cut, is maximized. Let us define S' be the neighbor of S such that S', can be obtained from S by moving one node from A to B, or one from B to A. We only choose a node which, when flipped, increases the cut value by at least $\frac{w(A,B)}{|V|}$. Then which of the following is true?

A. Upon the termination of the algorithm, the algorithm returns a cut (A,B) so that $2.5w(A, B) \geq w(A^*, B^*)$where (A^*, B^*) is an optimal partition.

B. The algorithm terminates after at most $O(\log |V| \log W)$ flips, where W is the total weight of edges.

C. Upon the termination of the algorithm, the algorithm returns a cut (A, B) so that $2s(A, B) \geq w(A^*, B^*)$.

D. The algorithm terminates after at most $O(|V|^2)$ flips.

Local Search

- **May NOT in polynomial time**

  ☝ **stop the algorithm when there are no "*big enough*" improvements.**

  Big-improvement-flip: Only choose a node which, when flipped, increases the cut value by at least

$$\frac{2\varepsilon}{|V|} w(A, B)$$

  Claim: Upon termination, the big-improvement-flip algorithm returns a cut (A, B) so that

$$(2+\varepsilon)\, w(A, B) \geq w(A^*, B^*)$$

  Claim: The big-improvement-flip algorithm terminates after at most $O(n/\varepsilon \log W)$ flips.

Local Search

- $\forall u \in A,\ \sum_{v\in A} w_{uv} < \sum_{v\in B} w_{uv} + \frac{2\epsilon}{|V|} w(A,B)$
- $2\sum_{u\in A}\sum_{v\in A} w_{uv} < \sum_{u\in A}\sum_{v\in B} w_{uv} + |A|\frac{2\epsilon}{|V|} w(A,B)$
- $2\sum_{u\in B}\sum_{v\in B} w_{uv} < \sum_{u\in B}\sum_{v\in A} w_{uv} + |B|\frac{2\epsilon}{|V|} w(A,B)$
- $\sum_{u\in A}\sum_{v\in A} w_{uv} + \sum_{u\in B}\sum_{v\in B} w_{uv} < \sum_{u\in B}\sum_{v\in A} w_{uv} + \epsilon w(A,B) = (1+\epsilon) w(A,B)$
- $w(A^*,B^*) \le \sum_{u\in A}\sum_{v\in A} w_{uv} + \sum_{u\in B}\sum_{v\in B} w_{uv} + w(A,B) < (2+\epsilon) w(A,B)$
- $\log_{1+\frac{2\epsilon}{|V|}} W = \frac{\log W}{\log(1+\frac{2\epsilon}{|V|})} \le \frac{|V|\log W}{2\epsilon} = O(\frac{n\log W}{\epsilon})$

Local Search

HW7 多选题 1

Recall the on-line Hiring Problem we introduced in class. Consider the following algorithm for this problem.

For n interviewees with distinct ratings, we always set an integer k with $0 \leq k < n$, and reject first k interviewees we encounter. In the remaining interviewees, we hire the first one we encounter with better rating than the best of the first k ones encountered. If none of the remaining interviewees is better than the best of the first k ones encountered, the last one encountered is hired.

Assume that the ratings of interviewees are uniformly random. Among the following statements about this algorithm, which is/are correct?

A. This algorithm hires the best of all the n interviewees if and only if this interviewee is not one of the first k ones encountered.

B. This algorithm hires the last interviewee encountered if and only if the best of all the n interviewees is in the first k ones encountered.

C .If $k = 1$, the probability of hiring the best of all the n interviewees is $1/n$.

D. If $k = n - 1$, the probability of hiring the best of all the n interviewees is $1/n$.

Local Search

- What should we know?
- LOCAL, SEARCH, always a feasible solution
- Hopfield Neural Networks
- Maximum Cut Problem(Upper bound 1.1382, lower bound 1.0625)
- State-flipping Algorithm
  - For Hopfield Neural Networks, no approximation-ratio is guaranteed, but a stable configuration must be found.
  - For Maximum Cut Problem, it is 2-approx.
- Big-improvement-flip Algorithm
  - It is only for Maximum Cut Problem, $(2+\epsilon)$-approx.
  - Its running time is polynomials.
- Simulated Annealing Algorithm, K-L heuristic Algorithm.

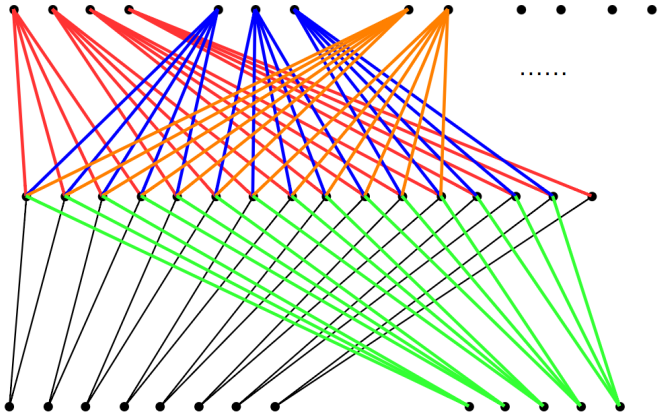
Randomized Problem

- What should we know?
- Monte Carlo algorithms: must terminate in polynomial time but may be wrong.
- Las Vegas algorithms: must get correct anwer but its running time may be not polynomials.
- Hiring Algorithm
- Quicksort: Type j. Las Vegas algorithm.

Problems

- Knapsack Problem 背包问题 FPTAS
- Bin Packing Problem 装箱问题 1.5-approximation
- Vertex Cover Problem 顶点覆盖问题 2-approximation
- Dominating Set Problem 支配集问题 $O(\log n)$-approximation
- Maximum Independent Set Problem 最大独立集问题 $O(n)$-approximation
- Maximum Clique Problem 最大团问题 $O(n)$-approximation

Vertex Cover



Degree greedy can be as bad as $O(\log n)$-approx.