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On the necessity of low-effective dimension

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Dedicated to Professor Niederreiter in honor of his 60th birthday

Abstract

We introduce a class of functions in high dimensions which have the maximum effective dimension, then prove that generalized Sobol' sequences provide the $O(N^{-1})$ convergence rate for the integration of this class of functions. An important consequence is that high-dimensional problems for which quasi-Monte Carlo outperforms Monte Carlo are not necessarily of low-effective dimension.

Keywords: ANOVA; Effective dimension; Generalized Sobol' sequences; High-dimensional integrals; Monte Carlo and quasi-Monte Carlo methods; Walsh functions

1. Introduction

In 1990, Sobol' [16,17] introduced the notion of global sensitivity for the analysis of non-linear mathematical problems, and pointed out that many practical problems can be well approximated using only low-order terms of the ANOVA (analysis of variance) decomposition of the problem. However, he did not use his results to explain the superiority of quasi-Monte Carlo (QMC) over Monte Carlo (MC) for some integration problems in practice. Around 1993, it was found by researchers at Columbia University that QMC outperforms MC for very high-dimensional problems in finance (see, e.g., [12,19]). After that, many researchers have paid considerable efforts in explaining this success of QMC [9,10,14,21,22]. Paskov [11] tried to answer this question using the notion of effective dimension. Caflisch et al. [1] are the first who formally defined effective dimension using ANOVA, and attempted

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to show empirically that low-effective dimension provides the key to understanding why QMC beats MC by a wide margin for certain very high-dimensional integrals. Recently, Owen [8] showed that scrambled $(0, m, d)$-nets have a much smaller variance than ordinary MC only for high-dimensional integration problems with low-effective dimension, and proposed that low-effective dimension is necessary for QMC to be much better than MC in high dimensions with practical sample sizes.

In this paper, we answer to Owen's proposal in the negative. To be precise, we prove that generalized Sobol' sequences provide the $O(N^{-1})$ convergence rate for a certain class of integration problems which have the maximum effective dimension. In the last section, we discuss the significance of this result and future research directions.

2. Main results

2.1. Definitions and notations

ANOVA (analysis of variance) is defined as follows [7]: let $u \subseteq \{1, 2, \ldots, d\}$ be a subset of the coordinates of $[0, 1)^d$ and let $\bar{u} = \{1, 2, \ldots, d\} - u$ be its complement. Also let $X = \{x_1, \ldots, x_d\}$ and $X^u = \{x_j ; j \in u\}$. Then, the ANOVA decomposition of $f(x_1, \ldots, x_d)$ is defined by

$$f(x_1, \ldots, x_d) = \sum_{u \subseteq \{1,2,\ldots,d\}} \alpha_u(x_1, \ldots, x_d),$$

where the sum is over all 2^d subsets of coordinates of $[0, 1)^d$. The terms $\alpha_u(x_1, \ldots, x_d)$ are defined recursively starting with

$$\alpha_\emptyset(x_1, \ldots, x_d) := I(f) \equiv \int_{[0,1)^d} f(z_1, \ldots, z_d)\, dz_1 \ldots dz_d,$$

and

$$\alpha_u(x_1, \ldots, x_d) := \int_{Z^u = X^u, Z^{\bar{u}} \in [0,1)^{\bar{u}}} \Big(f(z_1, \ldots, z_d) - \sum_{v \subset u} \alpha_v(z_1, \ldots, z_d)\Big) \prod_{j \in \bar{u}} dz_j,$$

where the sum is over proper subsets $v \neq u$. When $u = \{1, \ldots, d\}$,

$$\alpha_{\{1,\ldots,d\}}(x_1, \ldots, x_d) = f(x_1, \ldots, x_d) - \sum_{v \subset \{1,\ldots,d\}} \alpha_v(x_1, \ldots, x_d).$$

The meaning of $\alpha_u(x_1, \ldots, x_d)$ is the effect of the subset X^u on $f(x_1, \ldots, x_d)$ minus the effect of its proper subset X^v with $v \subset u$. The $\alpha_u(x_1, \ldots, x_d)$ have the following orthogonal property:

- Let $i \in u$. If we fix all the x_j, $j \neq i$, then

$$\int_0^1 \alpha_u(x_1, \ldots, x_d)\, dx_i = 0.$$

Thus, when $\emptyset \neq u \subset \{1, \ldots, d\}$,

$$\int_{[0,1)^d} \alpha_u(x_1, \ldots, x_d)\, dx_1 \ldots dx_d = 0.$$

- When $u \neq v$,

$$\int_{[0,1)^d} \alpha_u(x_1, \ldots, x_d)\alpha_v(x_1, \ldots, x_d)\, dx_1 \ldots dx_d = 0.$$

Hence, the variance of $f(x_1, \ldots, x_d)$ is given by

$$\sigma^2 = \int_{[0,1)^d} (f(x_1, \ldots, x_d) - \alpha_{\emptyset}(x_1, \ldots, x_d))^2\, dx_1 \ldots dx_d = \sum_{|u|>0} \sigma_u^2,$$

where

$$\sigma_u^2 := \sigma^2(\alpha_u) = \begin{cases} 0 & \text{if } u = \emptyset, \\ \int_{[0,1)^d} \alpha_u(x_1, \ldots, x_d)^2 dx_1 \ldots dx_d & \text{otherwise.} \end{cases}$$

The definition of effective dimension was introduced in two ways [1]:
Truncation sense:

$$D_{\text{trunc}} := \min\{i : 1 \leqslant i \leqslant d \text{ such that} \sum_{u \subseteq \{1,2,\ldots,i\}} \sigma_u^2 \geqslant 0.99\sigma^2\}.$$

Superposition sense:

$$D_{\text{super}} := \min \left\{ i : 1 \leqslant i \leqslant d \text{ such that} \sum_{|u| \leqslant i} \sigma_u^2 \geqslant 0.99\sigma^2 \right\}.$$

The Walsh functions [23] are defined by

$$\text{wal}(0, x) = 1 \quad \text{for } x \in [0, 1),$$

and for a nonnegative integer $m \geqslant 1$,

$$\text{wal}(m, x) = (-1)^{\sum_{j=1}^{\infty} m_{j-1} a_j} = (-1)^{(\mathbf{m}, X)},$$

where $m = m_0 + m_1 2 + \cdots$, and $x = a_1 2^{-1} + a_2 2^{-2} + \cdots$, and $\mathbf{m} = (m_0, m_1, \ldots)$ and $X = (a_1, a_2, \ldots)$ are the binary vector representations of m and x, respectively. The Rademacher functions are the subclass of the Walsh functions for which $m = 1, 2, \ldots, 2^k, \ldots$.

Let t_k be an integer with $2^{k-1} \leqslant t_k < 2^k$ for $k = 1, 2, \ldots$, and denote $t_k = t_{k,1} + t_{k,2} 2 + \cdots$. We define a nonsingular lower triangular matrix T as the (k, j)-element of T for $k \leqslant j$ being equal to $t_{k,j}$. Hereafter, we denote

$$r_0^{(T)}(x) = 1 \quad \text{for } x \in [0, 1),$$

and for $k = 1, 2, \ldots$,

$$r_k^{(T)}(x) = \text{wal}(t_k, x).$$

Note that the matrix T specifies uniquely a subclass of the Walsh functions, and that the identity matrix I corresponds to the Rademacher functions.

2.2. Functions with the maximum effective dimension

From now on, we fix d matrices $T_1, \ldots, T_d$ which specify d subclasses of the Walsh functions. First, we introduce the following functions in d dimensions:

Definition 1. We define

$$\phi_0(x_1, \ldots, x_d) = 1 \quad \text{for } (x_1, \ldots, x_d) \in [0, 1)^d,$$

and for $k = 1, 2, \ldots,$

$$\phi_k(x_1, \ldots, x_d) = \prod_{i=1}^{d} r_k^{(T_i)}(x_i).$$

The following lemma is straightforward from Definition 1.

Lemma 1. *For any $k = 1, 2, \ldots$, and any $1 \leqslant i \leqslant d$, if we fix the x_j, $(j \neq i)$,*

$$\int_0^1 \phi_k(x_1, \ldots, x_d)\, dx_i = 0.$$

We now define the class $\mathfrak{F}_d$ of functions in d dimensions.

Definition 2. We define a class $\mathfrak{F}_d$ which consists of functions

$$f(x_1, \ldots, x_d) = \sum_{k=0}^{\infty} c_k \phi_k(x_1, \ldots, x_d),$$

where c_k, $k = 0, 1, \ldots,$ are constants satisfying $|c_0| + \sum_{k=1}^{\infty} |c_k| 2^{k-1} \leqslant M < \infty$. Here M is a constant.

Let's consider the ANOVA decomposition of $f \in \mathfrak{F}_d$. First, we have

$$\alpha_\emptyset(x_1, \ldots, x_d) = \int_{[0,1)^d} f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d = c_0.$$

Since the Walsh functions are orthogonal [23],

$$\sigma(f)^2 = \int_{[0,1)^d} (f(x_1, \ldots, x_d) - c_0)^2\, dx_1 \ldots dx_d = \sum_{k=1}^{\infty} c_k^2.$$

From Lemma 1, for $\emptyset \neq u \subset \{1, \ldots, d\}$, we have

$$\begin{aligned}
\alpha_u(x_1, \ldots, x_d) &\equiv \int_{Z^u = X^u, Z^{\bar{u}} \in [0,1)^{\bar{u}}} (f(z_1, \ldots, z_d) - \sum_{v \subset u} \alpha_v(z_1, \ldots, z_d)) \prod_{j \in \bar{u}} dz_j \\
&= \int_{Z^u = X^u, Z^{\bar{u}} \in [0,1)^{\bar{u}}} (f(z_1, \ldots, z_d) - c_0) \prod_{j \in \bar{u}} dz_j \\
&= \int_{Z^u = X^u, Z^{\bar{u}} \in [0,1)^{\bar{u}}} \sum_{k=1}^{\infty} c_k \phi_k(z_1, \ldots, z_d) \prod_{j \in \bar{u}} dz_j \\
&= \sum_{k=1}^{\infty} c_k \int_{Z^u = X^u, Z^{\bar{u}} \in [0,1)^{\bar{u}}} \phi_k(z_1, \ldots, z_d) \prod_{j \in \bar{u}} dz_j = 0
\end{aligned}$$

and

$$\alpha_{\{1,\ldots,d\}}(x_1, \ldots, x_d) = f(x_1, \ldots, x_d) - c_0.$$

Thus, we have $\sigma(f) = \sigma_{\{1,\ldots,d\}}$. We now arrive at the following theorem:

Theorem 1. *For any function $f \in \mathfrak{F}_d$, its effective dimension, whether in the truncation or in the superposition sense, is equal to d.*

2.3. Convergence rate of generalized Sobol' sequences

QMC is the deterministic version of MC. The difference between the two is that MC uses random numbers, while QMC uses low-discrepancy sequences (for a detailed mathematical treatment of this topic, see [2,5,6,18]). Here, we consider a class of (t, d)-sequences in base two whose generator matrices are written as $(T_i)^{-1}U_i$, $i = 1, \ldots, d$, where T_i, $i = 1, \ldots, d$, are matrices specifying subclasses of the Walsh functions, and U_i, $i = 1, \ldots, d$, are arbitrary nonsingular upper-triangular matrices. Hereafter, we denote this class by $\mathfrak{S}_d$. By definition [18,20], the generalized Sobol' sequences are a subclass of $\mathfrak{S}_d$, where U_i, $i = 1, \ldots, d$, are constructed based on irreducible polynomials and the so-called direction numbers. Note that Sobol' sequences [13,15,18], which are defined using $T_i = I$, $i = 1, \ldots, d$, correspond to the Rademacher functions. We will prove the following theorem:

Theorem 2. *Suppose that the dimension d is odd. Then, for any sequence S in the class $\mathfrak{S}_d$, the integration error e_N of any function f in $\mathfrak{F}_d$ is given by*

$$e_N(S, f) < \frac{\sum_{k=1}^{\infty} |c_k| \min(2^{k-1}, N)}{N}.$$

Using the condition in Definition 2, we get $e_N(S, f) = O(N^{-1})$.

Corollary 1. *Suppose that the dimension d is odd. Then, for any generalized Sobol' sequence S in* $\mathfrak{S}_d$, *the integration error* e_N *of any function f in* $\mathfrak{F}_d$ *is given by*

$$e_N(S, f) = O(N^{-1}),$$

where the asymptotic constant is M.

Let $y_n, n = 0, 1, \ldots,$ be a sequence in $\mathfrak{S}_1$ whose generator matrix is nonsingular and upper triangular. We have the following lemma.

Lemma 2. *Suppose* $k \geqslant 1$. *Let* $y_{n,k}$ *denote the kth bit of* y_n. *For all* $n = h\ 2^{k-1}, h = 0, 1, \ldots,$

$$y_{n,k} = y_{n+1,k} = \cdots = y_{n+2^{k-1}-1,k},$$

and for all $n = h\ 2^k, h = 0, 1, \ldots,$

$$y_{n,k} = 1 - y_{n+2^{k-1},k}.$$

Proof. Let $n = n_0 + n_1 2 + n_2 2^2 + \cdots$ in the binary representation. Let the *k*th row of the generator matrix be $(g_1, g_2, \ldots)$. Note that $g_1 = g_2 = \cdots = g_{k-1} = 0$ and $g_k = 1$ since the matrix is nonsingular and upper triangular. Then we have

$$y_{n,k} = n_{k-1} + \sum_{j=k+1}^{\infty} g_j n_{j-1} \qquad (\text{mod } 2).$$

Observe that the value of n_{k-1} changes every 2^{k-1} times as the integer n increases, while the value of $\sum_{j=k+1}^{\infty} g_j n_{j-1}$ changes every 2^k (or more) times. This completes the proof. $\square$

For the subclass of the Walsh functions specified by a nonsingular lower triangular matrix T, we have

Lemma 3. *Suppose* $k \geqslant 1$. *Let* $s_n, n = 0, 1, \ldots,$ *be a sequence in* $\mathfrak{S}_1$ *whose generator matrix is* $T^{-1}U$, *where U denotes the generator matrix of* $y_n, n = 0, 1, \ldots$ *For all* $n = h\ 2^{k-1}, h = 0, 1, \ldots,$

$$r_k^{(T)}(s_n) = r_k^{(T)}(s_{n+1}) = \cdots = r_k^{(T)}(s_{n+2^{k-1}-1}),$$

and for all $n = h\ 2^k, h = 0, 1, \ldots,$

$$r_k^{(T)}(s_n) = -r_k^{(T)}(s_{n+2^{k-1}}).$$

Proof. The value of the Rademacher function $r_k^{(I)}(x)$ is determined by the *k*th bit of x. From Lemma 2, it follows that for all $n = h\ 2^{k-1}, h = 0, 1, \ldots,$

$$r_k^{(I)}(y_n) = r_k^{(I)}(y_{n+1}) = \cdots = r_k^{(I)}(y_{n+2^{k-1}-1}),$$

and for all $n = h\,2^k, h = 0, 1, \ldots,$

$$r_k^{(I)}(y_n) = -r_k^{(I)}(y_{n+2^{k-1}}).$$

We have

$$\begin{aligned} r_k^{(I)}(y_n) &= \mathrm{wal}(2^{k-1}, y_n) = (-1)^{(\mathbf{e}_k, Y_n)} = (-1)^{(\mathbf{t}_k T^{-1}, Y_n)} \\ &= (-1)^{(\mathbf{t}_k, Y_n (T^{-1})^{\mathrm{T}})} = (-1)^{(\mathbf{t}_k, S_n)} = \mathrm{wal}(t_k, s_n) = r_k^{(T)}(s_n), \end{aligned}$$

where $\mathbf{e}_k$ is the elementary row vector, i.e., only the kth element is one and all others are zero, and t_k is an integer corresponding to the row vector $\mathbf{t}_k = \mathbf{e}_k T$, and Y_n and S_n are the binary row vector representation of y_n and s_n, respectively. Here the superscript T denotes the transpose. Thus, the proof is complete. □

Remark that in case the vector S_n consists of finite 0's and infinite 1's, the generalized Sobol' point s_n is truncated with appropriate precision.

For d dimensions, we denote a sequence $S \in \mathfrak{S}_d$ by $\mathbf{s}_n = (s_n^{(1)}, \ldots, s_n^{(d)}), n = 0, 1, \ldots$ Then we have

Lemma 4. *Suppose $k \geqslant 1$. For all $n = h\,2^{k-1}, h = 0, 1, \ldots,$*

$$\phi_k(\mathbf{s}_n) = \phi_k(\mathbf{s}_{n+1}) = \cdots = \phi_k(\mathbf{s}_{n+2^{k-1}-1}),$$

and if the dimension d is odd, for all $n = h\,2^k, h = 0, 1, \ldots,$

$$\phi_k(\mathbf{s}_n) = -\phi_k(\mathbf{s}_{n+2^{k-1}}).$$

Proof. The first half follows from Lemma 3. The rest of the proof follows from the definition of ϕ_k as a product of an odd number of factors (d is odd) and Lemma 3. □

From $I(\phi_k) = 0$ for $k \geqslant 1$ and Lemma 4, it follows that for any $k = 1, 2, \ldots,$ the integration error e_N of ϕ_k for any sequence $S \in \mathfrak{S}_d$ is given by

$$e_N(S, \phi_k) = \left| \frac{1}{N} \sum_{n=0}^{N-1} \phi_k(\mathbf{s}_n) \right| = \left| \frac{1}{N} \sum_{n=N-N_k}^{N-1} \phi_k(\mathbf{s}_n) \right| \leqslant \frac{\min(2^{k-1}, N)}{N},$$

where N_k is the residue of N modulo 2^k.

We are now ready to prove Theorem 2.

Proof of Theorem 2.

$$\begin{aligned} e_N(S, f) &= \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{s}_n) - I(f) \right| = \left| \frac{1}{N} \sum_{n=0}^{N-1} (f(\mathbf{s}_n) - c_0) \right| \\ &= \left| \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=1}^{\infty} c_k \phi_k(\mathbf{s}_n) \right| \leqslant \sum_{k=1}^{\infty} \left| \frac{c_k}{N} \sum_{n=0}^{N-1} \phi_k(\mathbf{s}_n) \right| \leqslant \frac{\sum_{k=1}^{\infty} |c_k| \min(2^{k-1}, N)}{N}. \end{aligned}$$

Thus, the proof is complete. □

2.4. A general class of functions

Hereafter, we denote, for $k = 1, 2, \ldots,$

$$A_k^{(d)} = \{(k_1, \ldots, k_d) |\ k_i \in \mathbb{N}, 1 \leqslant i \leqslant d,\ k = \min_{1 \leqslant i \leqslant d} k_i, \quad \text{and} \quad \text{the number of } k_i \text{ with } k_i = k \text{ is odd}\}.$$

By using this notation, we generalize Definition 1 as follows:

Definition 3. We define

$$\Phi_0(x_1, \ldots, x_d) = c_0 \quad \text{for } (x_1, \ldots, x_d) \in [0, 1)^d,$$

and for $k = 1, 2, \ldots,$

$$\Phi_k(x_1, \ldots, x_d) = \sum_{(k_1, \ldots, k_d) \in A_k^{(d)}} c_{k_1, \ldots k_d} \prod_{1 \leqslant i \leqslant d} r_{k_i}^{(T_i)}(x_i),$$

where c_0 and $c_{k_1, \ldots, k_d}$ are constants satisfying $|c_0| + \sum_{(k_1, \ldots, k_d) \in A_k^{(d)}} |c_{k_1, \ldots k_d}| < \infty$.

If we denote

$$A_k^{(d)*} = \{(k_1, \ldots, k_d) |\ k_1 = \cdots = k_d = k\},$$

which was used in Definition 1, then $A_k^{(d)*} \subset A_k^{(d)}$ if d is odd.

Lemma 1 is generalized as follows.

Lemma 5. *For any $k = 1, 2, \ldots$, and any $1 \leqslant i \leqslant d$, if we fix the x_j, $(j \neq i)$,*

$$\int_0^1 \Phi_k(x_1, \ldots, x_d)\, dx_i = 0.$$

Lemma 4 is generalized as follows.

Lemma 6. *Let $\mathbf{s}_n, n = 0, 1, \ldots,$ be a sequence in $\mathfrak{S}_d$. Suppose $k \geqslant 1$. For all $n = h\ 2^{k-1}, h = 0, 1, \ldots$,*

$$\Phi_k(\mathbf{s}_n) = \Phi_k(\mathbf{s}_{n+1}) = \cdots = \Phi_k(\mathbf{s}_{n+2^{k-1}-1}),$$

and for all $n = h\ 2^k, h = 0, 1, \ldots$,

$$\Phi_k(\mathbf{s}_n) = -\Phi_k(\mathbf{s}_{n+2^{k-1}}).$$

Proof. The proof follows directly from Lemmas 3 and 4. $\square$

We now define the following class of functions:

Definition 4. We define the class $\mathfrak{G}_d$ which consists of the functions

$$g(x_1, \ldots, x_d) = \sum_{k=0}^{\infty} \Phi_k(x_1, \ldots, x_d),$$

where c_0 and $c_{k_1,\ldots,k_d}$ are constants satisfying

$$|c_0| + \sum_{k=1}^{\infty} \sum_{(k_1,\ldots,k_d)\in A_k^{(d)}} |c_{k_1,\ldots,k_d}| 2^{k-1} \leqslant M' < \infty.$$

Here M' is a constant.

Then, we have

Theorem 3. *For any function g in $\mathfrak{G}_d$, its effective dimension, whether in the truncation or in the superposition sense, is equal to d.*

Proof. Using Lemma 5, the proof follows in the same way as the proof of Theorem 1. □

Theorem 4. *For any sequence S in the class $\mathfrak{S}_d$, the integration error e_N of any function g in $\mathfrak{G}_d$ is given by*

$$e_N(S, g) < \frac{\sum_{k=1}^{\infty} \sum_{(k_1,\ldots,k_d)\in A_k^{(d)}} |c_{k_1,\ldots,k_d}| \min(2^{k-1}, N)}{N}.$$

Proof. Note that $|\Phi_k(x_1, \ldots, x_d)| \leqslant \sum_{(k_1,\ldots,k_d)\in A_k^{(d)}} |c_{k_1,\ldots,k_d}|$ for $k \geqslant 1$. From Lemma 6, we have

$$e_N(S, g) \leqslant \left| \frac{1}{N} \sum_{n=0}^{N-1} \Phi_k(\mathbf{s}_n) \right| = \left| \frac{1}{N} \sum_{n=N-N_k}^{N-1} \Phi_k(\mathbf{s}_n) \right|$$
$$\leqslant \frac{\sum_{(k_1,\ldots,k_d)\in A_k^{(d)}} |c_{k_1,\ldots,k_d}| \min(2^{k-1}, N)}{N},$$

where N_k is the residue of N modulo 2^k. Thus, the proof is complete. □

Corollary 1 is generalized as follows:

Corollary 2. *For any generalized Sobol' sequence S in $\mathfrak{S}_d$, the integration error e_N of any function g in $\mathfrak{G}_d$ is given by*

$$e_N(S, g) = O(N^{-1}),$$

where the asymptotic constant is M'.

We can compare the QMC error bounds in Theorems 2 and 4 with MC errors for $\mathfrak{F}_d$ and $\mathfrak{G}_d$, respectively. For example, let us take a function $f \in \mathfrak{F}_d$ with $c_0 = c_1 = \cdots = c_4 = 1$ and all others being zero. Suppose that $N = 10\,000$. Then, the QMC error bound is 0.0015, whereas the MC error is $\sigma(f)/\sqrt{N} = 0.02$. Thus, QMC is better than MC with a practical sample size for this function of maximum effective dimension. Note that the dimension size d does not appear in the error bound. If we use the Koksma–Hlawka bound to estimate the QMC error for the same example, it becomes

$$V(f)D_N^{(d)} \approxeq 2^{d+2}\left(\frac{2^t}{d!}\frac{(\log\, N)^d}{N}\right).$$

If we put $N = 10\,000$, $d = 21$, and $t \approx d$, then

$$2^{d+2}\frac{2^t}{d!}\frac{(\log\, N)^d}{N} = \frac{2^{44}}{21!}\frac{(13.3\cdots)^{21}}{10000} \gg 1,$$

which is a useless error bound.

3. Discussion

From the information-based complexity (IBC) viewpoint [22,24], the integration errors discussed in the previous section are written as follows: for odd dimension d,

$$e_N^{\text{worst}-\text{worst}}(\mathfrak{S}_d, \mathfrak{F}_d) \equiv \sup_{S\in\mathfrak{S}_d} \sup_{f\in\mathfrak{F}_d} e_N(S, f) = O(N^{-1}).$$

And for any dimension d,

$$e_N^{\text{worst}-\text{worst}}(\mathfrak{S}_d, \mathfrak{G}_d) \equiv \sup_{S\in\mathfrak{S}_d} \sup_{g\in\mathfrak{G}_d} e_N(S, g) = O(N^{-1}).$$

On the other hand, if the dimension d is even and $T_1, \ldots, T_d$ are identical, we have

$$e_N^{\text{worst}-\text{worst}}(\mathfrak{S}_d, \mathfrak{F}_d) = O(1),$$

because the worst-case is attained by the sequence in $\mathfrak{S}_d$ whose generator matrices are all identical. However, numerical experiments with randomly chosen generalized Sobol' sequences from $\mathfrak{S}_d$ for several functions in $\mathfrak{F}_d$ in even dimension d suggest the error $O(N^{-1})$. So, by using the notation from [4], we propose the following conjecture: For any dimension d,

$$e_N^{\text{worst}}(\mathfrak{S}_d, \mathfrak{F}_d) \equiv \text{rms}_{S\in\mathfrak{S}_d} \sup_{f\in\mathfrak{F}_d} e_N(S, f) = O(N^{-1}),$$

and/or

$$e_N^{\text{rand}}(\mathfrak{S}_d, \mathfrak{F}_d) \equiv \sup_{f\in\mathfrak{F}_d} \text{rms}_{S\in\mathfrak{S}_d} e_N(S, f) = O(N^{-1}).$$

It would be more challenging to consider the class $\bar{\mathfrak{G}}_d$ instead of $\mathfrak{F}_d$ in the conjecture, where $\bar{\mathfrak{G}}_d$ is defined by replacing $A_k^{(d)}$ with $\bar{A}_k^{(d)} = \{(k_1, \ldots, k_d)|\ k = \min_{1\leqslant i\leqslant d} k_i\}$ in Definition 4. Notice that the class $\bar{\mathfrak{G}}_d$ is also of maximum effective dimension.

For some other problems with the maximum effective dimension, empirically QMC has no advantage over MC [3,9]. However, for the class $\mathfrak{G}_d$ and the class $\mathfrak{F}_d$, we have proved that QMC beats MC. Owen [9] pointed out that low-effective dimension is not a sufficient condition for the superiority of QMC over MC. Now, we proved that it is not a necessary condition, either. So, it seems that "low-effective dimension" does not play a key role in explaining the success of QMC. Otherwise, the use of ANOVA for defining "effective dimension" is not appropriate for that purpose. These topics should be explored in more depth.

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