When is Phase Modulation (PM) synthesis equivalent to Frequency Modulation (FM) synthesis, and when do they differ?

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Contents

1 Oscillator Basics

The signal generated by an oscillator can be modeled as a function of time $O(t)$ where t denotes time, and $O\colon \mathbb{R} \to \mathbb{R}$.

An oscillator has a waveform associated with it which determines the shape of the signal. A waveform is a periodic function; for example, in the simplest case, it can be the good old trigonometric sine function. A bare $O(t) = \sin(t)$ signal can be thought of as a wave which oscillates once every 2π seconds.

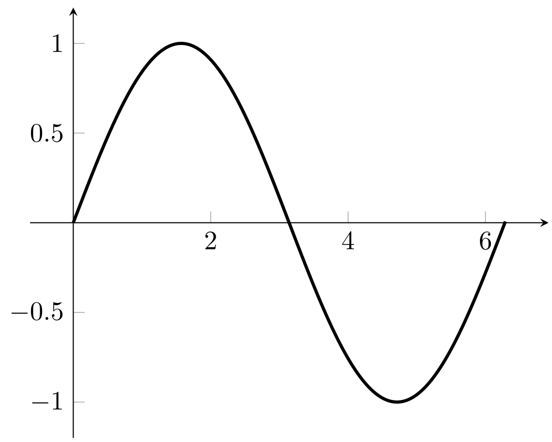

Figure 1: Plot of $O(t) = \sin(t)$.

To make it oscillate once every second instead (ie. at 1 Hz), its input needs to be scaled by 2π:

$$O(t) = \sin(2\pi \cdot t) \tag{1}$$

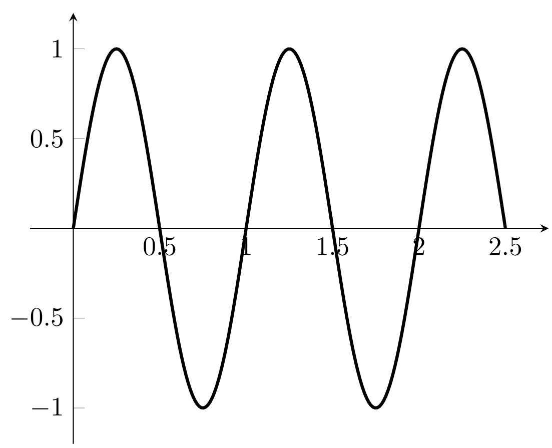

Figure 2: Plot of $O(t) = \sin(2\pi \cdot t)$.

To make it oscillate 3 times per second (ie. at 3 Hz), the input needs to be scaled even more:

$$O(t) = \sin(2\pi \cdot 3 \cdot t) \tag{2}$$

To make it oscillate at a constant frequency f, the input needs to be scaled by f:

$$O(t) = \sin(2\pi \cdot f \cdot t) \tag{3}$$

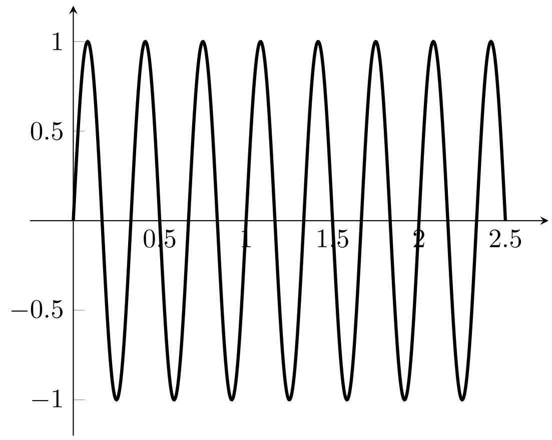

Figure 3: Plot of $O(t) = \sin(2\pi \cdot 3 \cdot t)$.

To start the signal at a different portion of the sine wave at $t = 0$ seconds, the input needs to be shifted by some number $\varphi \in \mathbb{R}$ which is called the "phase" or "phase offset":

$$O(t) = \sin(2\pi \cdot f \cdot t + \varphi) \tag{4}$$

The signal's amplitude can be changed as well by multiplying the whole thing by some number $A \in \mathbb{R}$:

$$O(t) = A \cdot \sin(2\pi \cdot f \cdot t + \varphi) \tag{5}$$

(See also: `https://en.wikipedia.org/wiki/Sine_wave`.)

But what does $2\pi \cdot f \cdot t + \varphi$ actually represent here?

The input to the sine function can be thought of as an angle measured in radians, hence the 2π term. Whatever this angle is measuring (e.g. it could be the rotation of a tonewheel), seems to be changing over time, since we have a time-dependent term in there. The trick is that what the frequency is measuring here is the "rate of change" (also known as the "time derivative") of this angle, telling how many full rotations are completed per second, similarly to how velocity in physics measures how many kilometers or miles are traveled per second.

Therefore $2\pi \cdot f \cdot t + \varphi$ here represents the total rotation of *something*, which has accumulated over a time span of t seconds. In other words, time and frequency aren't just multiplied together here: what's happening actually is that instantaneous changes of angle are being summed over a length of time which is divided into infinitesimally short intervals. So the equation should really look like this:

$$O(t) = A \cdot \sin\left(2\pi \cdot \int_0^t f \; d\tau + \varphi\right) \tag{6}$$

This isn't just unnecesessary, arbitrary pedantry, because the integration actually makes a huge difference when the constant frequency is replaced with one that is changing over time, which is necessary for modeling frequency modulation that is all about changing the frequency rapidly in each moment.

See also:

- `https://en.wikipedia.org/wiki/Radian`
- `https://en.wikipedia.org/wiki/Tonewheel`
- `https://en.wikipedia.org/wiki/Time_derivative`
- `https://en.wikipedia.org/wiki/Instantaneous_phase_and_frequency`

So let's replace the constant frequency with a function of time, $f\colon \mathbb{R} \to \mathbb{R}$:

$$O(t) = A \cdot \sin\left(2\pi \cdot \int_0^t f(\tau)\ d\tau + \varphi\right) \tag{7}$$

Finally, $\sin(t)$ can be replaced with some other periodic function $W\colon \mathbb{R} \to \mathbb{R}$ in order to get a different waveform (like sawtooth, triangle, etc.):

$$O(t) = A \cdot W\left(2\pi \cdot \int_0^t f(\tau)\ d\tau + \varphi\right) \tag{8}$$

2 Modulator and Carrier

The simplest case of modulation uses two oscillators: the Modulator and the Carrier. These are connected in a way which lets the Modulator affect one of the parameters of the Carrier. For example, in each moment in time, the momentary signal value of the Modulator is added to the selected parameter of the Carrier.

To see how Phase Modulation (PM) and Frequency Modulation (FM) are related to each other, we are going to mathematically model FM, and see if we can throw enough algebra at it to turn it into PM, then we will look at what happens to the modulator function in the process.

The signals generated by the two oscillators will be modeled as functions of time, similarly to equation 8: $M(t)$ and $C(t)$ for the Modulator and the Carrier respectively, where $M\colon \mathbb{R} \to \mathbb{R}$ and $C\colon \mathbb{R} \to \mathbb{R}$.

For simplicity's sake, let's define $M(t)$ with a constant frequency $f_M \in \mathbb{R}$, and a constant amplitude $A_M \in \mathbb{R}$, with zero phase offset ($\varphi_M = 0$), and with a waveform $W_M\colon \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned} M(t) =& A_M \cdot W_M(2\pi \cdot f_M \cdot t + \varphi_M) \\ =& A_M \cdot W_M(2\pi \cdot f_M \cdot t + 0) \\ =& A_M \cdot W_M(2\pi \cdot f_M \cdot t) \end{aligned} \tag{9}$$

Now let's define the Carrier's function with varying frequency; similarly to the above, $A_C \in \mathbb{R}$ is the amplitude, $\varphi_C \in \mathbb{R}$ is the phase offset, $f_{FM}\colon \mathbb{R} \to \mathbb{R}$ is the varying frequency, and $W_C\colon \mathbb{R} \to \mathbb{R}$ is the waveform:

$$C(t) = A_C \cdot W_C\left(2\pi \cdot \int_0^t f_{FM}(\tau)\ d\tau + \varphi_C\right) \tag{10}$$

Let's say that the Carrier's own frequency is a constant $f_C \in \mathbb{R}$, and this is what is being modulated with M, so $f_{FM}(\tau)$ can be expressed as:

$$f_{FM}(\tau) = M(\tau) + f_C \tag{11}$$

The fundamental theorem of calculus already seems to suggest that modulating the frequency by some function is equivalent to modulating the phase by an antiderivative of that function. Though there are special cases where those two are equivalent, that is not the general case, as it will be shown in the next section.

(See: `https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus`.)

3 Turning FM into PM

Substituting equation 11 into equation 10 and then expanding M, we get:

$$
\begin{aligned}
C(t) =&A_C \cdot W_C \left(2\pi \cdot \int_0^t \left(M(\tau) + f_C\right) \ d\tau + \varphi_C\right) \\
=&A_C \cdot W_C \left(2\pi \cdot \int_0^t M(\tau) \ d\tau + 2\pi \cdot \int_0^t f_C \ d\tau + \varphi_C\right) \\
=&A_C \cdot W_C \left(2\pi \cdot \int_0^t A_M \cdot W_M(2\pi \cdot f_M \cdot \tau) \ d\tau + 2\pi \cdot f_C \cdot t + \varphi_C\right) \\
=&A_C \cdot W_C \left(2\pi \cdot A_M \cdot \int_0^t W_M(2\pi \cdot f_M \cdot \tau) \ d\tau + 2\pi \cdot f_C \cdot t + \varphi_C\right)
\end{aligned} \tag{12}
$$

Thanks to Fourier, it is known that periodic functions can be expressed as sums of sinusoids, so for simplicity's sake, let's consider only those waveforms for the Modulator where, for some $N \in \mathbb{N}$, W_M can be written as:

$$
W_M(\tau) = \sum_{n=1}^{N} \left(B_n \cdot \sin(n\tau)\right) \tag{13}
$$

where $B_n \in \mathbb{R}$ are some constants for $n \in \mathbb{N},\ 1 \leq n \leq N$.
(See: `https://en.wikipedia.org/wiki/Fourier_series`)
Expressing W_M in the integral in equation 12 as a sum of sines yields:

$$
\begin{aligned}
\int_0^t W_M(2\pi \cdot f_M \cdot \tau) \ d\tau &= \int_0^t \sum_{n=1}^{N} B_n \cdot \sin(2\pi \cdot f_M \cdot n \cdot \tau) \ d\tau \\
&= \sum_{n=1}^{N} B_n \cdot \int_0^t \sin(2\pi \cdot f_M \cdot n \cdot \tau) \ d\tau
\end{aligned} \tag{14}
$$

Using the fact that for any constant $0 \neq \gamma \in \mathbb{R}$:

$$
\int \sin(\gamma x) \ dx = -\frac{1}{\gamma} \cos(\gamma x) + c \tag{15}
$$

where $c \in \mathbb{R}$ is the constant of integration, we can calculate the integral in the right side of equation 14:

$$
\begin{aligned}
\int_0^t \sin(2\pi \cdot f_M \cdot n \cdot \tau) \ d\tau &= \left[-\frac{1}{2\pi \cdot f_M \cdot n} \cdot \cos(2\pi \cdot f_M \cdot n \cdot \tau)\right]_0^t \\
&= -\frac{1}{2\pi \cdot f_M \cdot n} \cdot \left[\cos(2\pi \cdot f_M \cdot n \cdot \tau)\right]_0^t \\
&= -\frac{1}{2\pi \cdot f_M \cdot n} \cdot \left(\cos(2\pi \cdot f_M \cdot n \cdot t) - \cos(2\pi \cdot f_M \cdot n \cdot 0)\right) \\
&= -\frac{1}{2\pi \cdot f_M \cdot n} \cdot \left(\cos(2\pi \cdot f_M \cdot n \cdot t) - \cos(0)\right) \\
&= -\frac{1}{2\pi \cdot f_M \cdot n} \cdot \left(\cos(2\pi \cdot f_M \cdot n \cdot t) - 1\right) \\
&= -\frac{\cos(2\pi \cdot f_M \cdot n \cdot t) - 1}{2\pi \cdot f_M \cdot n} \\
&= \frac{1 - \cos(2\pi \cdot f_M \cdot n \cdot t)}{2\pi \cdot f_M \cdot n}
\end{aligned} \tag{16}
$$

Furthermore, since $\sin(x - \frac{\pi}{2}) = -\cos(x)$, the result in equation 16 can be written as:

$$\int_0^t \sin(2\pi \cdot f_M \cdot n \cdot \tau)\ d\tau = \frac{1 + \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right)}{2\pi \cdot f_M \cdot n} \tag{17}$$

Plugging equation 17 back into the right side of equation 14:

$$\begin{aligned}\sum_{n=1}^{N} B_n \cdot \int_0^t \sin(2\pi \cdot f_M \cdot n \cdot \tau)\ d\tau =& \sum_{n=1}^{N} B_n \cdot \frac{1 + \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right)}{2\pi \cdot f_M \cdot n} \\ =& \sum_{n=1}^{N} \frac{B_n}{2\pi \cdot f_M \cdot n} + \frac{1}{2\pi \cdot f_M} \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right)\end{aligned} \tag{18}$$

Let's define two more constants, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$:

$$\begin{aligned}\alpha =& \sum_{n=1}^{N} \frac{B_n}{2\pi \cdot f_M \cdot n} \\ \beta =& \frac{1}{2\pi \cdot f_M}\end{aligned} \tag{19}$$

Now we can rewrite equation 18 as:

$$\sum_{n=1}^{N} B_n \cdot \int_0^t \sin(2\pi \cdot f_M \cdot n \cdot \tau)\ d\tau = \alpha + \beta \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right) \tag{20}$$

Continuing equation 14 using equation 20:

$$\int_0^t W_M(2\pi \cdot f_M \cdot \tau)\ d\tau = \alpha + \beta \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right) \tag{21}$$

By plugging equation 21 back into equation 12, we obtain:

$$\begin{aligned}C(t) =& A_C \cdot W_C\left(2\pi \cdot A_M \cdot \int_0^t W_M\left(2\pi \cdot f_M \cdot \tau\right)\ d\tau + 2\pi \cdot f_C \cdot t + \varphi_C\right) \\ =& A_C \cdot W_C\left(2\pi \cdot A_M \cdot \left(\alpha + \beta \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right)\right) + 2\pi \cdot f_C \cdot t + \varphi_C\right) \\ =& A_C \cdot W_C\left(2\pi \cdot A_M \cdot \alpha + 2\pi \cdot A_M \cdot \beta \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right) + 2\pi \cdot f_C \cdot t + \varphi_C\right) \\ =& A_C \cdot W_C\left(2\pi \cdot f_C \cdot t + \varphi_C + 2\pi \cdot A_M \cdot \alpha + 2\pi \cdot A_M \cdot \beta \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right)\right)\end{aligned} \tag{22}$$

Finally, we can define two more constants $A_{FM} \in \mathbb{R}$ and $\varphi_{FM} \in \mathbb{R}$ as:

$$
\begin{aligned}
A_{FM} &= 2\pi \cdot A_M \cdot \beta \\
&= 2\pi \cdot A_M \cdot \frac{1}{2\pi \cdot f_M} \\
&= \frac{A_M}{f_M} \\
\varphi_{FM} &= \varphi_C + 2\pi \cdot A_M \cdot \alpha \\
&= \varphi_C + 2\pi \cdot A_M \cdot \sum_{n=1}^{N} \frac{B_n}{2\pi \cdot f_M \cdot n} \\
&= \varphi_C + \frac{A_M}{f_M} \cdot \sum_{n=1}^{N} \frac{B_n}{n}
\end{aligned}
\tag{23}
$$

With these, equation 22 takes the following form:

$$
C(t) = A_C \cdot W_C \left(2\pi \cdot f_C \cdot t + \varphi_{FM} + A_{FM} \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2} \right) \right) \tag{24}
$$

4 Conclusion

For the special case of $N = 1$ in equation 13, equation 24 becomes:

$$
\begin{aligned}
C(t) =& A_C \cdot W_C \left(2\pi \cdot f_C \cdot t + \varphi_{FM} + A_{FM} \cdot B_1 \cdot \sin\left(2\pi \cdot f_M \cdot t - \frac{\pi}{2} \right) \right) \\
=& A_C \cdot W_C \left(2\pi \cdot f_C \cdot t + \varphi_{FM} + A_{FM} \cdot W_M \left(2\pi \cdot f_M \cdot t - \frac{\pi}{2} \right) \right)
\end{aligned}
\tag{25}
$$

which is indeed the same as if we added a constant offset to φ_C and then modulated it by a slight variation of the original modulator signal which would have an amplitude of A_{FM} and a phase offset of $-\frac{\pi}{2}$ (see equation 9):

$$
\begin{aligned}
M(t) =& A_M \cdot W_M(2\pi \cdot f_M \cdot t + 0) \\
\hat{M}_1(t) =& A_{FM} \cdot W_M \left(2\pi \cdot f_M \cdot t - \frac{\pi}{2} \right) \\
C(t) =& A_C \cdot W_C \left(2\pi \cdot \int_0^t \left(M(\tau) + f_C \right) \, d\tau + \varphi_C \right) \\
=& A_C \cdot W_C \left(2\pi \cdot f_C \cdot t + \varphi_{FM} + \hat{M}_1(t) \right)
\end{aligned}
\tag{26}
$$

Therefore for $N = 1$, phase modulation and frequency modulation are indeed equivalent sonically with the right choice of amplitude and phase offset.

But for $N > 1$, if the frequency modulated $C(t)$ signal is expressed as a phase modulated signal, as can bee seen from equation 24, we get a modulator signal which has a significantly different harmonic content from the original $M(t)$, because the original B_n coefficients of W_M get replaced with $\frac{B_n}{n}$:

$$
\begin{aligned}
M(t) =& A_M \cdot \sum_{n=1}^{N} B_n \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t\right) \\
\hat{M_N}(t) =& A_{FM} \cdot \sum_{n=1}^{N} \frac{B_n}{n} \cdot \sin\left(2\pi \cdot f_M \cdot n \cdot t - \frac{\pi}{2}\right) \\
C(t) =& A_C \cdot W_C \left(2\pi \cdot \int_0^t \left(M(\tau) + f_C\right)\ d\tau + \varphi_C\right) \\
=& A_C \cdot W_C \left(2\pi \cdot f_C \cdot t + \varphi_{FM} + \hat{M_N}(t)\right)
\end{aligned}
\tag{27}
$$

(And indeed, $\hat{M_N}(t)$ is an antiderivative of $M(t)$.)

Thus, modulating the frequency with a harmonically complex signal is significantly different from modulating directly the phase with it. Therefore, in the general case, PM is not always equivalent to FM. ■