2 Functions on Complex Plane

ENSY SILVER[1]

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[1]Thanks to my family, my friend and freedom.

1 Continuous functions

The continuity of functions on the complex plane is analogous to the continuity of functions on the real line. Similarly, we have the theorem.

Theorem 1.1. *A continuous function on a compact set Ω is bounded and attains a maximum and minimum on Ω.*

2 Holomorphic functions

Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on Ω. The function f is **holomorphic at the point** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0+h)-f(z_0)}{h} \tag{1}$$

converges to a limit when $h \to 0$. The limit of the quotient is called the **derivative of f at** z_0 and denoted by $f'(z_0)$

$$f'(z_0) = \frac{f(z_0+h)-f(z_0)}{h}$$

The function f is said to be **holomorphic on** Ω if f is holomorphic at every point of Ω. If C is a closd subset of $\mathbb{C}$, we say that f is **holomorphic on** C if there exists a open set Ω that $C \subset \Omega$ and f is holomorphic on Ω.

Moreover, every holomorphic function is analytic and has infinitely derivatives.

Remark. *Holomorphic is defined by an open set because we need to promise that every holomorphic point is an interior point.*

It is clear from 1 that a function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0+h) - f(z_0) - ah = h\psi(h) \tag{2}$$

where ψ is defined for all small h and $\lim_{h\to 0}\psi(h) = 0$. Of course, we have $a = f'(z_0)$. In analogy with the situation in $\mathbb{R}$, we have the following proposition.

Proposition 2.0.1. *If f, g are holomorphic inΩ, then:*

1. *$f+g$ is holomorphic in Ω and $(f+g)' = f' + g'$.*
2. *fg is holomorphic in Ω and $(fg)' = f'g + fg'$.*
3. *If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

3 Derivatives of holomorphic functions

From 1, we know that $(f(z_0) + h - f(z_0))/h \to f'(z_0)$ whenever $h = x$ or $h = iy$, hence we deduce

$$f'(z_0) = \lim_{h\to 0} \frac{f(x_0 + x, y_0) - f(x_0, y_0)}{x} = \frac{\partial f}{\partial x} \tag{3}$$

$$f'(z_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + y) - f(x_0, y_0)}{iy} = \frac{1}{i}\frac{\partial f}{\partial y} \tag{4}$$

Writing $f = u + iv$, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5}$$

5 are called the **Cauchy-Riemann** equations, which link real and complex analysis. We can define two differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{1}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{1}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Proposition 3.0.1. *If f is holomorphic at z_0, then*

$$\frac{\partial f}{\partial \overline{z}} = 0 \quad \textit{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Also, if we write $F(x, y) = f(z)$, then F is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2$$

Proof.

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z}$$
$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial v}{\partial z} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{1}{i}\frac{\partial v}{\partial y}\right)$$

Applying 5, we have proved that $\frac{\partial u}{\partial z} = i\frac{\partial v}{\partial z}$, and $\frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$. Applying Cauchy-Riemann equations, we deduce that

$$\det J_F = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left|2\frac{\partial u}{\partial z}\right|^2 = |f'(z_0)|^2$$

□

Theorem 3.1. *Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f/\partial z$.*

Right now, we have known the necessity and sufficiency of a holomorphic function.

4 Power series

In general, a **power serie**s is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n \tag{6}$$

An important property of series is absolute convergence, in order to test it, we must investigate

$$\sum_{n=0}^{\infty} |a_n||z|^n$$

In analogy with the situation on the real line, we have two theorems.

Theorem 4.1. *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:*

1. *If $|z| < R$ the series converges absolutely.*
2. *if $|z| > R$ the series diverges*

Moreover, if we use the convention that $1/0 = \infty$ and $1/\infty = 0$, the R is given by Hadamard's formula[1]

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Theorem 4.2. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ define sa holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series for f*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Moreover, f' has the same radius of convergence as f.

Corollary 4.2.1. *A power series is infinitely complex differentiable in its disc of convergence, and the derivatives are als power series obtained by termwise differentiating.*

Remark. *A function f defined on an open set Ω is **analytic** at a point z_0 if there exists a power series centered at z_0, with positive radius of convergence. If f has power series expansion at every point in Ω, we say that f is **analytic on** Ω.*

5 Curves on complex plane

A **parametrized curve** is a function (t) which maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. We shall impose regularity conditions on the parametrization which are always verified in the situations that occur in this book. We say that the curve is **smooth** if $z'(t)$ exists and is continuous on $[a,b]$, ans $z'(t) \neq 0$ for $t \in [a,b]$. At the points $t = a, b$, the quantities $z'(a), z'(b)$ are interpreted as the one-sided limite.

Moreover, we claim that the parametrized curve is **piecewise-smooth** if z is continuous on $[a,b]$ and if there exist points

$$a = a_0 < a_1 < \cdots < a_n = b$$

where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$.

We now introduce the concept **equivalent**. Two parametrization $z : [a,b] \to \mathbb{C}$ and $\tilde{z} : [c,d] \to \mathbb{C}$ are equivalent if there exists a continuously differentiable bijection $s \mapsto t(s)$ from $[c,d]$ to $[a,b]$ so that $t'(s) > 0$ and $\tilde{z}(s) = z(t(s))$. The family of all parametrizations that are equivalent to $z(t)$ determines a **smooth** curve $\gamma \subset \mathbb{C}$.

A smooth or piecewise-smooth curve is **closed** if $z(a) = z(b)$ for any of its parametrization. Finally, a smooth or piecewise-smooth curve is **simple** if it is not self-intersecting.

5.1 Orientation and integration of curves

We can define a curve γ^- obtained from the curve γ by reversing the orientation. the parametrization for γ^- is defined by

$$z^-(t) = z(b + a - t)$$

where $z : [a,b] \to \mathbb{C}$ is the parametrization of γ.

A simple example is a circle on the complex plane. We define the circle $C_r(z_0)$ with radius r by

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

The **positive orientation** (counterclockwise) is defined by

$$z(t) = z_0 + re^{it} \qquad \text{where} \quad t \in [0, 2\pi]$$

while the **negative orientation** is defined by

$$z(t) = z_0 + re^{-it} \qquad \text{where} \quad t \in [0, 2\pi]$$

The properties of a holomorphic function f guarantees the equation

$$\int_\gamma f(z)\ dz = 0 \tag{7}$$

where γ is a closed curve on the complex plane. In order to calculate the integral of f along a smooth curve γ, we parametrize the γ by $z : [a,b] \to \mathbb{C}$, and deduce that

$$\int_\gamma f(z)\ d = \int_a^b f(z(t))'(t)\ dt$$

By definition, the **length** of the smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |z'(t)|\, dt$$

We have the following proposition

Theorem 5.1. *Integration of continuous functions over curves satisfies the following properties:*

1. *It is linear, if $\alpha, \beta \in \mathbb{C}$, then*

$$\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz$$

2.

$$\int_\gamma f(z)\, dz = -\int_{\gamma^-} f(z)\, dz$$

3.

$$\left| \int_\gamma f(z)\, dz \right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathit{length}(\gamma)$$

Further more, we define the **primitive** for f on Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$.

Theorem 5.2. *If a continuous function f has a primitive F in Ω, ans γ is a curve in Ω that begins at w_1 and ends at w_2, then*

$$\int_\gamma f(z); dz = F(w_2) - F(w_1)$$

This is a direct result from 7.

Corollary 5.2.1. *If γ is a closed curve in an open set Ω, and f is continuous and has a primitive in Ω, then*

$$\int_\gamma f(z)\, dz = 0$$

References

[1] encyclopedia. *Cauchy–Hadamard theorem.* `https://encyclopediaofmath.org/wiki/Cauchy-Hadamard_theorem`. Feb. 2006.