1 Probability

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[1]Thanks to my family, my friend and freedom.

1 Four key items

The starting point for studying probability is the definition of four key terms: **experiment**, **sample outcome**, **sample space**, **event**.

1. Experiment is any procedure that can be repeated for infinite times, and has a well-defined set of possible outcomes.

2. Each of potential eventualities of an experiment is referred to as a sample outcome, and their totality is sample space.

3. Any designated collection of sample outcomes constitutes an event.

2 The probability function

Russian mathematician Kolmogorov claimed four axioms of probability in 20^{th} in order to define the probability function P:

1. Let A be any events defined over S. Then $P(A) \geq 0$.

2. $P(S) = 1$ where S is the entire sample space.

3. Let A abd B be any two mutually exclusice events defined over S. Then

$$P(A \cup B) = P(A) + P(B)$$

4. Let $A_1, A_2, \cdots$ be events defined over S, If $A_i \cap A_j = \emptyset$ for each $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

3 Conditional probability

Consider two events A and B, which are related, that is, the occurence of B will effect $P(A)$ in an experiment. Any probability that is revised to take into account the occurence of other events is said to be a **conditional probability**. The inner relation is that the occurence of the known fact has revised the sample space, which leads to the change of probability. We may parametrize the theorem.

Theorem 3.1.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Extend the theorem 3.1 to higher dimension, we have the corollary

Corollary 3.1.1. *We first define*

$$B_n = \bigcap_{i=1}^{n} A_n$$

Then we have

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \prod_{i=1}^{n-1} P(A_{i+1}|B_i)$$

Let $A_1, \cdots, A_n$ be partition of sample space S, that is, $\bigcup_i A_i = S$ and $A_i \cap A_j = \emptyset$. Then we can calculate $P(B)$ by sum the conditional probability $P(B|A_i)$.

Theorem 3.2. *Let $\{A_i\}_{i=1}^n$ be a set of events defined over S such that $S = \cup_i A_i, A_i \cap A_j = \emptyset (i \neq j)$. For any event B,*

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Concluding theorem 3.1 and theorem 3.2, we have deduced the **Bayes' theorem**.

Theorem 3.3 (**Bayes' theorem**)**.** *Let $\{A_i\}_{i=1}^n$ be a set of n events, each with positive probability, that partition S. For each event B where $P(B) > 0$,*

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

where $1 \leq j \leq n$.

4 Independence

We now deal with independent events.

Proposition 4.0.1. *Two events A and B are said to be **independent** if $P(A \cap B) = P(A) \cdot P(B)$.*

In analogy with proposition 4.0.1, we deduced the independence in generality.

Theorem 4.1. *Events $A_1, \cdots, A_n$ are said to be independent if for every indices $i_1, \cdots, i_k$ between 1 and n, inclusive*

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_k})$$

5 Combinatorics

In combinatorics, the most intuitive rule is the multiplication rule

Theorem 5.1 (multiplication rule). *If operation A can be performed in m different ways and operation B in n different ways, the sequence (A, B) can be performed in $m \cdot n$ different ways.*

One important application in combinatorics is counting permutations.

Theorem 5.2. *The number of permutations of length k that can be formed from a set of n distinct elements, repetitions not allowed, is denoted by the symbol P_k^n, where*

$$P_k^n = \frac{n!}{(n-k)!}$$

Corollary 5.2.1. *The number of ways to arrange n objects, n_1 being of one kind, n_2 of a second kind, ... , and n_r for r^{th} kind,*

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

where $\sum_{i=1}^{r} n_i = n$.

Another important application in combinatorics is counting combinations.

Theorem 5.3. *The number of combinations is denoted by $\binom{n}{k}$ or C_k^n, where*

$$\binom{n}{k} = \frac{n!}{k!(n-k!)}$$

Corollary 5.3.1. *Newton's binomial expansion is*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

6 Monte Carlo techniques

One can repeat a experiment for n times, and if event E occurs on m of those repetitions, then

$$P(E) = \lim_{n \to \infty} \frac{m}{n}$$