

# Fourier Analysis Notes

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<sup>1</sup>Thanks to my family, my friend and freedom.

### **Abstract**

This is a note about my understanding of Fourier Analysis This note won't include the last 2 part of Stein's Fourier Analysis[2]. Because study of number theory is not essential for me. I would probably study the last 2 chapters later.

# 1 Introduction to Fourier Series

Fourier Series are the very basis of Fourier Analysis. Following the Stein's instruction, I will start my note at Fourier Series. Fisrt, this note will introduce some basic concepts.

## 1.1 Basic concepts

### 1.1.1 Riemann integrable functions

A real-valued functions defined on  $[0, L]$  is **Riemann integrable** if it is bounded and if  $\forall \epsilon > 0, \exists 0 = x_0 < \dots < x_N = L$ , the subdivision satisfies following conditions. We denote  $\mathcal{U}$  and  $\mathcal{L}$  as the upper and lower sums of this subdivision.

$$\mathcal{U} = \sum_{j=1}^N [\sup_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

$$\mathcal{L} = \sum_{j=1}^N [\inf_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

then we have  $\mathcal{U} - \mathcal{L} < \epsilon$ .

### 1.1.2 Functions on the circle

Obviously, we can treat a  $2\pi$ -periodic function  $f \in \mathbb{R}$  as a function  $F$  on the circle.

$$f(\theta) = F(e^{i\theta})$$

and naturally,  $f(0) = f(2\pi) \Rightarrow F(0) = F(e^{2\pi i})$

### 1.1.3 Fourier coefficient and series

For a function defined on  $[a, b]$  and  $b - a = L$ , the  $n^{\text{th}}$  **Fourier coefficient** of  $f$  is defined by

$$\widehat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x / L} dx, \quad n \in \mathbb{Z}$$

The **Fourier series** of  $f$  is given by

$$\sum_{n=-\infty}^{n=+\infty} \widehat{f}(n) e^{2\pi i n x / L}$$

Sometimes we denote  $a_n$  as  $\widehat{f}(n)$ , and we use this notation

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} a_n e^{2\pi i n x / L}$$

When  $L = 2\pi$ , we can denote it by

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} a_n e^{2in\theta}$$

#### 1.1.4 Trigonometric series and partial sum

**Trigonometric series** can be expressed as  $\sum_{n=-\infty}^{n=+\infty} c_n e^{2\pi i n x / L}$ ,  $c_n \in \mathbb{C}$ . If for all large  $|n|$ , we get  $c_n = 0$ , it is called **Trigonometric polynomial**, its degree is the largest  $|n|$  that  $c_n \neq 0$ .

For  $N > 0$ , the  $N^{\text{th}}$  **partial sum** are expressions of the form

$$S_N(f)(x) = \sum_{n=-N}^{n=N} \hat{f}(n) e^{2\pi i n x / L}$$

#### 1.1.5 Euler's identity

**Euler's identity** is a useful formula in analysis, it connects  $e$  and trigonometric functions.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

#### 1.1.6 Jacobian matrix

**Jacobian matrix** of a vector-valued function  $\{f_n\}$  is the matrix of all its first-order partial derivatives. It is defined by

$$\begin{aligned} \mathbf{J}(x_1, \dots, x_n) &= \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \end{aligned}$$

Jacobian is often used in multiple integration, we can change variables in multiple integration. For example,  $x = f_x(a, b)$ ,  $y = f_y(a, b)$ ,  $F = (x, y)$ , the integration on domain  $D$  is

$$\int \int_D F(x, y) dx dy = \int \int_T F(f_x(a, b), f_y(a, b)) \left| \frac{\partial(x, y)}{\partial(a, b)} \right| da db$$

where  $T$  is the domain of  $f$  that  $f(a, b) \in D$ .

### 1.2 Convolutions and Kernels

There are types of functions. First, we introduce the convolution of  $2\pi$ -periodic integrable functions  $f$  and  $g$  on  $\mathbb{R}$ , we define their **convolution**  $f * g$  on  $[-\pi, \pi]$  by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy$$

We can also change variables  $t = x - y$  to get

$$\begin{aligned}
 (f * g)(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x-y)dy \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t) \frac{dy}{dt} dt \\
 &= -\frac{1}{2\pi} \int_{\pi}^{-\pi} f(x-t)g(t)dt \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)dt
 \end{aligned}$$

In the above equations,  $g$  is called the **kernel** of the convolution. Convolutions have several properties.

**Proposition 1.1.** *Suppose that  $f, g$  and  $h$  are  $2\pi$ -periodic integrable functions. Then:*

1.  $f * (g + h) = (f * g) + (f * h)$
2.  $(cf) * g = c(f * g) = f * (cg), \quad \forall c \in \mathbb{C}$
3.  $f * g = g * f$
4.  $(f * g) * h = f * (g * h)$
5.  $f * g$  is continuous
6.  $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$

*Proof.* Obviously, properties 1, 2, 3 are trivial. To prove 4, we write down

$$\begin{aligned}
 4\pi^2(f * g) * h(z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)g(x-y)h(z-x)dy \, dx \\
 4\pi^2 f * (g * h)(z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)g(x)h(z-x-y)dx \, dy \\
 &= \int_{-\pi}^{\pi} \int_{y-\pi}^{y+\pi} f(y)g(x-y)h(z-x)dx \, dy \\
 &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)g(x-y)h(z-x)dx \, dy
 \end{aligned}$$

If  $g$  is continuous, the proof of 5 is trivial. We can also construct a series of continuous functions to approximate  $g$ , hence we have proved 5 for merely integrable function  $g$ .

Property 6 is the most critical property. For continuous  $g$  and  $f$ , we can just calculate and reformulate the equation. For merely integrable functions, we can apply the same technique used in proof of property 5.  $\square$

### 1.3 Uniqueness of Fourier series

It is critical to prove the uniqueness of fourier coefficients of a function. More precisely, if  $f$  and  $g$  have same fourier coefficients, then  $f$  and  $g$  are necessarily equal. To prove 2 items are equal, a useful technique is to prove their difference equals 0.

**Theorem 1.2.** *Suppose that  $f$  is an integrable function on the circle with  $\hat{f} = 0, \forall n \in \mathbb{Z}$ . Then  $f(\theta_0) = 0$  whenever  $f$  is continuous at the point  $\theta_0$ .*

*Proof.* First we suppose that  $f$  is real-valued and defined on  $[-\pi, \pi]$ . We will argue this by contradiction. Without loss of generality, we assume that  $\theta = 0$ ,  $f(0) > 0$  and  $f$  is continuous at 0.

The idea is to construct a family of trigonometric polynomials  $\{p_k\}$  that 'peak' at 0, and  $\int p_k(\theta)f(\theta)d\theta \rightarrow \infty$  as  $k \rightarrow \infty$ . This will be our contradiction since these integrals are equal to zero by assumption. (because if  $\int p_k(\theta)f(\theta) \neq 0$ , there exists  $\hat{f}(n) \neq 0$ )

In this proof, we construct function  $p(\theta) = \epsilon + \cos \theta$  and  $p_k(\theta) = [p(\theta)]^k$ , and calculate 3 intervals respectively.  $\square$

The next corollary is a deduced from the theorem 1.2.

**Corollary 1.2.1.** *Suppose that  $f$  is a continuous function on the circle and the Fourier series is absolutely convergent. Then the Fourier series converges uniformly to  $f$ .*

**Corollary 1.2.2.** *Suppose that  $f$  is a twice continuously differentiable function on the circle. Then*

$$\hat{f}(n) = O(1/|n|^2), \quad \text{as } |n| \rightarrow \infty$$

*Proof.* To prove the corollary, we integrate by parts

$$\begin{aligned} 2\pi \hat{f}(n) &= \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta \\ &= \left[ f(\theta) \cdot \frac{-e^{-in\theta}}{in} \right]_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{(in)^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \\ &= -\frac{1}{n^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \end{aligned}$$

The rest part of proof is trivial.  $\square$

Incidentally, we have got an important identity when  $n \in \mathbb{Z}$

$$\widehat{f'}(n) = in\widehat{f}(n)$$

When  $n \neq 0$ , proof is given in the proof of corollary 1.2.2. When  $n = 0$ ,  $\int_0^{2\pi} f'(\theta) d\theta = 0$  since  $f$  is  $2\pi$ -periodic function.

Since  $f^{(m)}(n) \sim \sum a_n (in)^m e^{in\theta}$ . Further smoothness conditions on  $f$  imply even better decay of the Fourier coefficients.

There are also stronger version of corollary 1.2.2.

**Theorem 1.3.** *The Fourier series converges absolutely if  $f$  satisfies a **Holder condition** of order  $\alpha$ , with  $\alpha > 1/2$*

$$\sup_{\theta} |f(\theta + t) - f(\theta)| \leq A|t|^\alpha$$

*Proof.* To be updated. □

## 1.4 Good kernels

We have introduced the notion of kernel in section 1.2. Now we introduce the notion of **good kernels** if a family of kernels satisfies the following properties:

**Proposition 1.4.** *Properties of good kernels:*

1. For all  $n \geq 1$ ,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$$

2. There exists  $M > 0$  such that for all  $n \geq 1$ ,

$$\int_{-\pi}^{\pi} |K_n(x)| dx \leq M$$

3.  $\forall \delta > 0$ ,

$$\int_{\delta < |x| < \pi} |K_n(x)| dx \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

Good kernels are important for their properties. The next theorem is deduced from those properties.

**Theorem 1.5.** *Let  $\{K_n\}_{n=1}^{\infty}$  be a family of good kernels, and  $f$  an integrable function on the circle. Then*

$$\lim_{n \rightarrow \infty} (f * K_n)(x) = f(x)$$

*whenever  $f$  is continuous at  $x$ . If  $f$  is continuous everywhere, then the above limit is uniform.*

*Proof.* Using the property 2 and 3, the proof of the equation is trivial.

The second assertion depends on the periodic property. Since  $f$  is continuous everywhere and  $f$  is periodic, then  $f$  is uniformly continuous. Hence  $\delta$  can be chosen independently. So  $f * K_n \rightarrow f$  uniformly. □

## 1.5 Some special kernels

In this section, we will introduce some special kernels.

### 1.5.1 Dirichlet kernel

The  $N^{\text{th}}$  **Dirichlet kernel** is defined by

$$D_N(x) = \sum_{n=-N}^N e^{inx}$$

Another form is

$$D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})}$$

The proof of the second form is not critical, we omit it. A critical fact is that Dirichlet kernel is not a good kernel. The proof is trivial, we omit it (calculate integrals for each item).

### 1.5.2 Cesaro means and summation

We define a series  $\{c_n\}$ , and partial sum  $s_n = \sum_{k=0}^n c_k$ . Since  $\{c_n\}$  may not converge, we need a new series to measure the convergence. We define

$$\delta_N = \frac{\sum_{k=0}^{N-1} s_k}{N}$$

which is called the **Cesaro mean** of  $\{s_n\}$ , or the **Cesaro sum** of  $\{c_n\}$ . if  $\delta_N$  converges to a limit, we say that  $\sum c_n$  is **Cesaro summable**. It is easy to check that sum and Cesaro sum have the same limit, the proof is trivial.

### 1.5.3 Fejer's theorem

**Fejer's kernel** is defined by

$$F_N(x) = \frac{\sum_{n=0}^N D_n(x)}{N}$$

**Lemma 1.6.**

$$F_N(x) = \frac{\sin^2(Nx/2)}{N \sin^2(x/2)}$$

*The Fejer kernel is a good kernel.*

*Proof.* The proof of the first assertion is to be updated.

The proof of the second assertion is trivial since  $F_N(x) > 0$ . □

Applying the Theorem 1.5, we get the following theorem.



**Theorem 1.7.** *If  $f$  is integrals on circle, then Fourier series of  $f$  is Cesaro summable to  $f$  at every point of continuity of  $f$ .*

*Moreover, if  $f$  is continuous on the circle, then the Fourier series of  $f$  is uniformly Cesaro summable to  $f$ .*

**Corollary 1.7.1.** *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

#### 1.5.4 Abel means and Abel summation

A series of complex numbers  $\sum_{k=0}^{\infty} c_k$  is said to be **Abel summable** to  $s$  if for every  $0 \leq r < 1$ , the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \rightarrow 1} A(r) = s$$

A critical fact is that a Cesaro summable series is always Abel summable, however, an Abel summable series is not necessarily Cesaro summable.

#### 1.6 Poisson kernel

**Poisson kernel** is defined by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

and the convolution of Poisson kernel is defined by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} = (f * P_r)(\theta)$$

**Lemma 1.8.** *If  $0 \leq r < 1$ , then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}$$

*The Poisson kernel is a good kernel, as  $r \rightarrow 1$*

**Theorem 1.9.** *Moreover, if  $f$  is continuous on the circle, then the Fourier series of  $f$  is uniformly Abel summable to  $f$ . If  $f$  is integrals on circle, then Fourier series of  $f$  is Abel summable to  $f$  at every point of continuity of  $f$ .*

The proof is trivial, we omit it. But it is critical to notice that the Poiss kernel is indexed by  $r$ , not  $n$ .

**Remark.** An important hint deduced from the above theorems is that when we calculate polynomials like  $\sum \cos nx$ , we can transform the polynomial to the form of  $e^{in}$  according to Euler's formula. Then we can apply the formula  $\sum_{n=0}^{n=N} a^n = \frac{1-a^{n+1}}{1-a}$ .

In the following theorem, we apply Poisson kernel to steady heat equation.

**Theorem 1.10.** Let  $f$  be an integrable function defined on the unit circle. Then the function  $u$  defined in the unit disc by Poisson integral

$$u(r, \theta) = (f * P_r)(\theta)$$

has the following properties:

1.  $u$  has continuous derivatives in the unit disc and satisfies  $\delta u = 0$
2. If  $\theta$  is any point of continuity of  $f$ , then

$$\lim_{r \rightarrow 1} u(r, \theta) = f(\theta)$$

3. If  $f$  is continuous everywhere, then this limit is uniform.

*Proof.* To prove 1, we need to review the definition of **Laplacian**

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and we get the following relationship between  $x, y, r, \theta$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan \frac{y}{x} \\ \frac{\partial r}{\partial x} &= \frac{x}{r} \\ \frac{\partial r}{\partial y} &= \frac{y}{r} \\ \frac{\partial^2 r}{\partial x^2} &= -\frac{x}{r^3} \\ \frac{\partial^2 r}{\partial y^2} &= -\frac{y}{r^3} \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} \\ \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} \\ \frac{\partial^2 \theta}{\partial x^2} &= \frac{2xy}{r^4} \\ \frac{\partial^2 \theta}{\partial y^2} &= -\frac{2xy}{r^4} \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 u}{\partial \theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2}$$

$$\begin{aligned} \left( \frac{\partial r}{\partial x} \right)^2 + \left( \frac{\partial r}{\partial y} \right)^2 &= 1 \\ \frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 r}{\partial^2 y} &= \frac{1}{r} \\ \left( \frac{\partial \theta}{\partial x} \right)^2 + \left( \frac{\partial \theta}{\partial y} \right)^2 &= \frac{1}{r^2} \\ \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} &= 0 \end{aligned}$$

Then, we get the Laplacian in form of  $r$  and  $\theta$ .

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

and we calculate the formula term by term

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= r^{|m|-2} |m| (|m| - 1) e^{im\theta} \\ \frac{1}{r} \frac{\partial u}{\partial r} &= r^{|m|-2} |m| e^{im\theta} \\ \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= r^{|m|-2} |m|^2 e^{im\theta + \pi} \\ &= -r^{|m|-2} |m|^2 e^{im\theta} \\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} &= 0 \end{aligned}$$

The property 1 has been proved.

Because Poisson kernel is a good kernel, the proof of property 2 is trivial.

The proof of 3 is heuristic. We suppose function  $v$  solves the steady heat equation.  $v$  has a Fourier series

$$\sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \text{ where } a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v e^{-in\theta} d\theta$$

Taking into account that  $v$  solves

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

We get

$$a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0$$

Applying principles of second ordinary differential equation. We have  $a_n(r) = A_n r^n + B_n r^{-n}$  for constant  $A_n$  and  $B_n$ . Because  $r^{-n}$  is not bounded when  $r < 1$ , we deduce that  $B_n = 0$ . Since  $v$  converges uniformly to  $f$  as  $r \rightarrow 1$  we find that

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

Taking the above equation into  $a_n(r)$ , 3 has been proved.  $\square$

**Remark.**  $(e^{ix})' = e^{i(x+\pi/2)} = ie^{ix}$

**Remark.** There are 2 principles to judge whether the inner product of 2 series converge.

- *Abel's test:* If  $\{a_n\}$  is monotonic and bounded,  $\{\sum_{n=0}^{\infty} b_n\}$  converges. We have  $\sum_{n=1}^{\infty} a_n b_n$  converges.
- *Dirichlet's test:* If  $\{a_n\}$  decreases monotonically to 0, and  $\sum b_n$  are bounded, then  $\sum_{n=1}^{\infty} a_n b_n$  converges.

**Remark.** class  $C_k$  means the function  $f \in C_k$  has 1st to  $k$ -th derivatives and their are all continuous.  $f$  is called  $k$ -th continuously differentiable.

## 2 Convergence of Fourier Series

In this chapter, we continue to study convergence of Fourier series.

The first viewpoint is "global" and concerns the overall behavior of a function  $f$  over the  $[0, 2\pi]$ . If  $f$  is integrable on the circle, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \rightarrow 0 \text{ as } N \rightarrow \infty$$

The second viewpoint is "local" and concerns the behavior of  $f$  near a given point. Whether

$$S_N(f)(\theta) \rightarrow f(\theta)$$

### 2.1 Vector spaces and inner products

We have already been familiar with vector spaces. We only review the definition of **inner product** here. An inner product on a vector space  $V$  over  $\mathbb{R}$  associates to any pair  $X, Y$  of elements in  $V$  a real number which we denote by  $(X, Y)$ . The inner product must be symmetric  $(X, Y) = (Y, X)$  and linear in both variables

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z)$$

whenever  $\alpha, \beta \in \mathbb{R}$  and  $X, Y, Z \in V$ . Also, inner product must be positive-definite,  $(X, X) \geq 0$ . And we define the norm of  $X$  by

$$\|X\| = (X, X)^{1/2}$$

Additionally,  $\|X\| = 0$  implies  $X = 0$ .

A classic *inner product space* is  $\mathbb{R}^d$ , inner product is defined by

$$(X, Y) = x_1 y_1 + \cdots + x_d y_d$$

For vector spaces over complex numbers, the inner product of two elements is a complex number. Moreover, these inner products are called Hermitian since they satisfy  $(X, Y) = \overline{(Y, X)}$ . Hence the inner product is linear in the first variable, but conjugate-linear in the second:

$$\begin{aligned} (\alpha X + \beta Y, Z) &= \alpha(X, Z) + \beta(Y, Z) \\ (Z, \alpha X + \beta Y) &= \overline{\alpha}(Z, X) + \overline{\beta}(Z, Y) \end{aligned}$$

For example, the inner product of two elements  $Z, W$  in  $\mathbb{C}^d$  is defined by

$$(Z, W) = z_1 \overline{w_1} + \cdots + z_d \overline{w_d}$$

If two elements  $X, Y$  are **orthogonal**, then  $(X, Y) = 0$ , and we write  $X \perp Y$ . Three critical results can be derived from the notion of orthogonality.

1. The Pythagorean theorem: if  $X, Y$  are orthogonal, then

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$$

2. The Cauchy-Schwarz inequality: for any  $X, Y \in V$  we have

$$|(X, Y)| \leq \|X\| \|Y\|$$

3. The triangle inequality: for any  $X, Y \in V$  we have

$$\|X + Y\| \leq \|X\| + \|Y\|$$

*Proof.* The proof of 1 is trivial.

The proof of 2 is here: We assume  $Y = kX + Z$ , where  $Z \in V, X \perp Z$ . Hence we have

$$\begin{aligned} |(X, Y)| &= |(X, kX + Z)| \\ &= \|X\|^2 |k| + |(X, Z)| \\ &= |k| \|X\|^2 \end{aligned}$$

$$\|X\| \cdot \|Y\| = \|X\| \cdot \|kX + Z\|$$

Obviously,  $\|kX + Z\| \geq |k| \cdot \|X\|$ , 2 is proved. The proof of 3 is also trivial, we omit it.  $\square$

### 2.1.1 Infinite dimensional vector spaces

The vector space  $\ell^2(\mathbb{Z})$  over  $\mathbb{C}$  is the set of infinite sequences of complex numbers

$$(\cdots, a_{-n}, \cdots, a_0, \cdots, a_n, \cdots)$$

such that

$$\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty$$

the sequence converges.

**Remark.** To measure infinity, we often use limitation.

In the three examples  $\mathbb{R}^d, \mathbb{C}^d, \ell^2(\mathbb{Z})$ , the vector spaces with their inner products and norms satisfy two important properties:

1. The inner product is strictly positive-definite,  $\|X\| = 0$  implies  $X = 0$ .
2. the vector space is **complete**, which by definition means that every Cauchy sequence in the norm converges to a limit.

An inner product space with 1 and 2 is a **Hilbert space**. We see that  $\mathbb{R}^d$  and  $\mathbb{C}^d$  are examples of finite-dimensional Hilbert spaces, while  $\ell^2(\mathbb{Z})$  is an example of an infinite-dimensional Hilbert space. If either of 1 and 2 fail, the space is called **pre-Hilbert space**.

We give an important example of a pre-Hilbert space where both 1 and 2 fail.

### 2.1.2 An example of pre-Hilbert space

Let  $\mathcal{R}$  denote the set of complex-valued Riemann integrable functions on  $[0, 2\pi]$ . This is a vector space over  $\mathbb{C}$ . Addition is defined pointwise by

$$(f + g)(\theta) = f(\theta) + g(\theta)$$

Naturally, multiplication by a scalar  $\lambda \in \mathbb{C}$  is given by

$$(\lambda f)(\theta) = \lambda \cdot f(\theta)$$

An inner product is defined on this vector space by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \overline{g(\theta)} d\theta$$

The norm of  $f$  is

$$\|f\| = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta \right)^{1/2}$$

In  $\mathcal{R}$ , 1 fails, we have a simple example

$$f(\theta) = \begin{cases} 0, & \text{if } \theta \neq 0 \\ 1, & \text{if } \theta = 0 \end{cases}$$

where  $\|f\| = 0$  but  $f \neq 0$ . But it does not matter since the points  $f(\theta) \neq 0$  have measure 0.

Another fatal fact is that  $\mathcal{R}$  is not complete. We have function

$$f(\theta) = \begin{cases} 0, & \text{for } \theta = 0 \\ \log(1/\theta), & \text{for } 0 < \theta \leq 2\pi \end{cases}$$

Since  $f$  is not bounded,  $f$  is not integrable and  $f \notin \mathcal{R}$ . Moreover, the sequence of truncations  $f_n$  defined by

$$f_n(\theta) = \begin{cases} 0, & \text{for } 0 \leq \theta \leq 1/n \\ f(\theta), & \text{for } 1/n < \theta \leq 2\pi \end{cases}$$

can easily be seen to form a Cauchy sequence in  $\mathcal{R}$ . Further discussion of these problems remain in Real Analysis.

**Remark.** In 2.1.2,  $f$  is the limit of  $f_n$ , but  $f$  is not Riemann integrable. Maybe  $f$  is Lebesgue integrable?

## 2.2 Proof of mean-square convergence

Consider the space  $\mathcal{R}$  mentioned above, we need to prove

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta$$

With this notation, we must prove  $\|f - S_N(f)\| \rightarrow 0$  as  $N \rightarrow \infty$ .

For  $n \in \mathbb{Z}$ , let  $e_n(\theta) = e^{in\theta}$ , and observe that family  $\{e_n\}_{n \in \mathbb{Z}}$  is **orthonormal**

$$(e_n, e_m) = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

Let  $f$  be an integrable function on the circle, we have

$$(f, e_n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta = a_n$$

Because  $\{e_n\}_{n \in \mathbb{Z}}$  is orthonormal, we have

$$(f - S_N(f)) \perp \sum_{|n| \leq N} b_n e_n \quad (1)$$

for any complex number  $b_n$ . Applying Pythagorean theorem to 1, and we have

$$\|f\|^2 = \|f - \sum_{|n| \leq N} a_n e_n\|^2 + \|\sum_{|n| \leq N} a_n e_n\|^2$$

Since the orthonormal property of the family  $\{e_n\}$  implies that

$$\|\sum_{|n| \leq N} a_n e_n\|^2 = \sum_{|n| \leq N} |a_n|^2$$

we deduce that

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n| \leq N} |a_n|^2 \quad (2)$$

**Lemma 2.1 (Best approximation).** *If  $f$  is integrable on the circle with Fourier coefficients  $a_n$ , then*

$$\|f - S_N(f)\| \leq \|f - \sum_{|n| \leq N} c_n e_n\|$$

*the equality holds iff  $c_n = a_n$  for all  $|n| \leq N$ .*

*Proof.* Obviously, we have

$$f - \sum_{|n| \leq N} c_n e_n = f - S_N(f) + \sum_{|n| \leq N} b_n e_n$$

where  $b_n = a_n - c_n$ . Simply applying Pythagorean theorem, the lemma is proved.  $\square$

Suppose that  $f$  is continuous on the circle. Then, given  $\epsilon > 0$ , there exists a trigonometric polynomial  $P$ , say of degree  $M$

$$|f(\theta) - P(\theta)| < \epsilon \text{ for all } \theta$$



Inparticular, taking squares and integrating the inequality yields  $\|f - P\| < \epsilon$ , and by the best approximation lemma we have

$$\|f - S_N(f)\| < \epsilon \text{ whenever } N \geq M$$

For merely integrable function  $f$ , we need to use the lemma

**Lemma 2.2.** *Suppose  $f$  is integrable on the circle and bounded by  $B$ . Then there exists a sequence  $\{f_k\}_{k=1}^\infty$  of continuous functions*

$$\sup_{x \in [-\pi, \pi]} |f_k(x)| \leq B$$

and

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)| dx \rightarrow 0 \text{ as } k \rightarrow \infty$$

Applying Lemma 2.2, we get

**Theorem 2.3.** *Let  $f$  be an integrable function on the circle with  $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ . Then we have*

1. Mean-square convergence of the Fourier series

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \rightarrow 0 \text{ as } N \rightarrow \infty$$

2. Parseval's identity

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta$$

**Remark.** *If  $\{e^n\}$  is nay orthonormal family of functions on the circle, and  $a_n = (f, e_n)$ , then we deduce*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2$$

*This is known as **Bessel's inequality**. Equality holds precisely when the family  $\{e_n\}$  is also a "basis".*

**Remark.** *There exist sequences  $\{a_n\}$  such that  $\sum |a_n| < \infty$ , yet no Riemann integrable function  $F$  has Fourier coefficients equals to  $a_n$  for all  $n$ . To be updated.*

**Theorem 2.4 (Riemann-Lebesgue lemma).** *If  $f$  is integrable on the circle, then  $\hat{f}(n) \rightarrow 0$  as  $|n| \rightarrow \infty$ .*

### 2.2.1 Review

To prove global convergence of integrable function  $f$  on the circle. We have several steps.

1. Applying lemma 2.2 to find function  $g$  to approximate  $f$ .
2. For continuous function  $g$ , applying corollary 1.7.1 to get trigonometric polynomial  $P$  to approximate  $g$ .
3. Applying best approximation, because  $\{e_n\}$  is a basis, we have proved theorem 2.3

Sometimes, to calculate Fourier coefficients of intricate functions, we decompose the function to simpler functions.

**Lemma 2.5.** Suppose  $F$  and  $G$  are integrable on the circle with

$$F \sim \sum a_n e^{in\theta} \quad \text{and} \quad G \sim \sum b_n e^{in\theta}$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} F(\theta) \overline{G(\theta)} d\theta = \sum a_n \overline{b_n}$$

### 2.3 Pointwise convergence

**Theorem 2.6.** Let  $f$  be an integrable function on the circle which is differentiable at a point  $\theta_0$ . Then  $D_N(f)(\theta_0) \rightarrow f(\theta_0)$  as  $N$  tends to infinity.

*Proof.* Construct a function  $F$

$$F(t) = \begin{cases} \frac{f(\theta_0 - t) - f(\theta_0)}{t}, & \text{if } t \neq 0 \text{ and } |t| < \pi \\ -f'(\theta_0), & \text{if } t = 0. \end{cases}$$

Then use the  $F$  to replace  $f$  to prove  $S_N(f)(\theta_0) - f(\theta_0)$ . applying Lemma 2.4.

$$S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) t D_N(t) dt$$

We recall that

$$t D_N(t) = \frac{t}{\sin(t/2)} \sin((N + 1/2)t)$$

Applying Lemma 2.4, we have proved that  $S_N(f)(\theta_0) - f(\theta_0) \rightarrow 0$  as  $N \rightarrow \infty$ .

Why using  $F$ ? Because we need  $t/\sin(t/2)$  to be integrable. □

**Remark.** If  $f$  satisfies **Lipschitz condition** at  $\theta_0$

$$|f(\theta) - f(\theta_0)| \leq M|\theta - \theta_0|$$

for a constant  $M \geq 0$  and all  $\theta$ . This is the same as saying that  $f$  satisfies Holder condition of order  $\alpha = 1$ .

### 2.3.1 A continuous function with diverging Fourier series

In order to understand convergence better, we need an example of a continuous periodic function whose Fourier series diverges at a point. Thus, theorem 2.6 fails if the differentiability is replaced by weaker assumption of continuity. Review the sawtooth function  $f$

$$\sum_{n \neq 0} \frac{e^{in\theta}}{n}$$

We consider Fourier series

$$\sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n} \quad (3)$$

3 is no longer the Fourier series of a Riemann integrable function. Suppose it were the Fourier series of an integrable function, say  $\tilde{f}$ , where in particular  $\tilde{f}$  is bounded. Using the Abel means, we have

$$|A_r(\tilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n}$$

Because  $\sum 1/n$  diverges, this gives contradiction since

$$|A_r(\tilde{f})(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta)| P_r(\theta) d\theta \leq \sup_{\theta} |\tilde{f}(\theta)|$$

**Lemma 2.7.** Suppose that the Abel means  $A_r = \sum_{n=1}^{\infty} r^n c_n$  of the series  $\sum c_n$  are bounded as  $r$  tends to 1. If  $C_n = (1/n)$ , then the partial sums  $S_N = \sum_{n=1}^N c_n$  are bounded.

From lemma 2.7, we know  $c_n = e^{in\theta}/n - e^{-in\theta}/n$  and hence  $\sum c_n$  is bounded. So  $S_N(f)(\theta)$  is uniformly bounded in  $N$  and  $\theta$ .

Define  $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$  and  $\tilde{P}_N(\theta) = e^{i(2N)\theta} \tilde{f}_N(\theta)$ . The rest is to be updated, because I am not fully understand yet.

**Remark.** *Gibbs' phenomenon is critical when processing signals with discontinuity.*

### 3 The Fourier Transform on $\mathbb{R}$

We have introduced the Fourier series of periodic functions. In this chapter, we will study Fourier transform on  $\mathbb{R}$ . Which means, we develop an analogous theory for non-periodic functions on  $\mathbb{R}$ .

Roughly speaking, the Fourier transform is a continuous version of Fourier coefficients. Hence we have the following equations

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \quad (4)$$

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \quad (5)$$

#### 3.1 Elementary theory

##### 3.1.1 Integration of functions on the real line

$f$  defined on  $\mathbb{R}$  is said to be of **moderate decrease** if  $f$  is continuous and there exists a constant  $A > 0$  so that

$$|f(x)| \leq \frac{A}{1+x^2} \text{ for all } x \in \mathbb{R}$$

We denote  $\mathcal{M}(\mathbb{R})$  as the set of functions of moderate decrease on  $\mathbb{R}$ . Generally, we call a function  $f$  of moderate decrease whenever

$$|f(x)| \leq \frac{A}{1+|x|^{1+\epsilon}} \text{ for all } x \in \mathbb{R}$$

And  $f$  has the property

$$\int_{-\infty}^{\infty} f(x) dx < \infty$$

##### 3.1.2 Schwartz space

The **Schwartz space** on  $\mathbb{R}$  consists the set of all indefinitely differentiable functions  $f$  so that  $f$  and all its derivatives  $f', \dots, f^{(\ell)}, \dots$  are **rapidly decreasing**, in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty \text{ for every } k, \ell \geq 0$$

We denote the space by  $\mathcal{S} = \mathcal{S}(\mathbb{R})$ . Moreover, if  $f \in \mathcal{S}$ , we have

$$f'(x) = \frac{df}{dx} \in \mathcal{S} \text{ and } xf(x) \in \mathcal{S}$$

This expresses the important fact that the Schwartz space is closed under differentiation and multiplication by polynomials.

A simple example of a function in  $\mathcal{S}$  is the **Gaussian** defined by

$$f(x) = e^{-x^2}$$

Another example is  $e^{-|x|}$ , although it decreases rapidly, it is not differentiable at 0, so it does not belong to  $\mathcal{S}$ .

### 3.1.3 Fourier transform on $\mathcal{S}$

The **Fourier transform** of a function  $f \in \mathcal{S}$  is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \quad (6)$$

We use notation

$$f(x) \longrightarrow \widehat{f}(\xi)$$

to mean that  $\widehat{f}(\xi)$  denotes the Fourier transform of  $f$ .

**Proposition 3.1.** *If  $f \in \mathcal{S}(\mathbb{R})$  then:*

1.  $f(x+h) \longrightarrow \widehat{f}(\xi) e^{2\pi i h \xi}$  whenever  $h \in \mathbb{R}$ .
2.  $f(x) e^{-2\pi i x h} \longrightarrow \widehat{f}(\xi + h)$  whenever  $h \in \mathbb{R}$ .
3.  $f(\delta x) \longrightarrow \delta^{-1} \widehat{f}(\delta^{-1} \xi)$  whenever  $h \in \mathbb{R}$ .
4.  $f'(x) \longrightarrow 2\pi i \xi \widehat{f}(\xi)$  whenever  $h \in \mathbb{R}$ .
5.  $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi} \widehat{f}(\xi)$  whenever  $h \in \mathbb{R}$ .

**Theorem 3.2.** *If  $f \in \mathcal{S}(\mathbb{R})$ , then  $\widehat{f} \in \mathcal{S}(\mathbb{R})$ .*

The proof is trivial, we omit it.

An important formula is **Gaussian**,  $e^{-\pi x^2}$

**Theorem 3.3.**

$$\begin{aligned} \left( \int_{-\infty}^{\infty} e^{-\pi x^2} dx \right) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r dr d\theta \\ &= \int_0^{\infty} 2\pi r e^{-\pi r^2} dr \\ &= -e^{-\pi r^2} \Big|_0^{\infty} \\ &= 1 \end{aligned}$$

**Theorem 3.4.** *If  $f(x) = e^{-\pi x^2}$ , then  $\widehat{f}(\xi) = f(\xi)$ .*

*Proof.* Define

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx$$

Because  $f'(x) = -2\pi x f(x)$ , we obtain

$$\widehat{f}'(\xi) = \int_{-\infty}^{\infty} f(x)(-2\pi i x) e^{-2\pi i x \xi} dx = i \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi} dx$$

By 4 in theorem 3.1, we find that

$$\widehat{f}'(\xi) = i(2\pi i \xi) \widehat{f}(\xi) = -2\pi \xi \widehat{f}(\xi)$$

Hence we have  $\widehat{f}(\xi) = e^{-\pi \xi^2}$ .  $\square$

Generally, for every  $\delta > 0$ ,  $K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$ , then  $\widehat{K}_\delta(\xi) = e^{-\pi \delta \xi^2}$ . Similar to definition in Chapter 2, we define **good kernel** on  $\mathbb{R}$

**Proposition 3.5.**  $P_\delta$  is a good kernel when

1.  $\int_{-\infty}^{\infty} P_\delta(x) dx = 1$ .
2.  $\int_{-\infty}^{\infty} |P_\delta(x)| dx \leq M$ .
3. For every  $\eta > 0$ , we have  $\int_{|x|>\eta} |P_\delta(x)| dx \rightarrow 0$  as  $\delta \rightarrow 0$ .

Obviously,  $K_\delta$  is a good kernel. Moreover, the **convolution** on  $\mathbb{R}$  is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt \quad (7)$$

**Corollary 3.5.1.** If  $f \in \mathcal{S}(\mathbb{R})$ , then

$$(f * K_\delta)(x) \longrightarrow f(x) \text{ uniformly in } x \text{ as } \delta \rightarrow 0$$

### 3.1.4 Fourier inversion

The next result is an identity sometimes called the multiplication formula.

**Proposition 3.6.** If  $f, g \in \mathcal{S}(\mathbb{R})$ , then

$$\int_{-\infty}^{\infty} f(x) \widehat{g}(x) dx = \int_{-\infty}^{\infty} g(x) \widehat{f}(x) dx$$

The proof is trivial, we omit it.

**Theorem 3.7 (Fourier inversion).** If  $f \in \mathcal{S}(\mathbb{R})$ , then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$

*Proof.* We have known that  $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$ ,  $G_\delta(x) = \widehat{K}_\delta(x) = e^{-\pi\delta x^2}$ .

Since  $K_\delta$  is a good kernel, we obtain

$$\int_{-\infty}^{\infty} f(x)K_\delta(x) dx \longrightarrow f(0) \text{ as } \delta \rightarrow 0 \quad (8)$$

Since  $G_\delta(x) \rightarrow 1$  as  $\delta \rightarrow 1$ , we obtain

$$\int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x) dx \longrightarrow \int_{-\infty}^{\infty} \widehat{f}(x) dx \text{ as } \delta \rightarrow 0 \quad (9)$$

Applying theorem 3.6, we have

$$\int_{-\infty}^{\infty} f(x)K_\delta(x) dx = \int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x) dx$$

After reviewed three equations above, we have

$$f(0) = \int_{-\infty}^{\infty} \widehat{f}(\xi) d\xi$$

In general, let  $F(y) = f(y+x)$  so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \widehat{F}(\xi) d\xi$$

Applying 2 in proposition 3.1, we have

$$\int_{-\infty}^{\infty} \widehat{F}(\xi) d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x \xi} d\xi$$

proof is completed. □

**Remark.** If  $f \notin \mathcal{S}(\mathbb{R})$ , 8 and 9 would be wrong.

We define two mappings  $\mathcal{F}, \mathcal{F}^*$  by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx \text{ and } \mathcal{F}^*(f)(x) = \int_{-\infty}^{\infty} f(\xi)e^{2\pi i x \xi} d\xi$$

And we have  $\mathcal{F}^* \circ \mathcal{F} = I$ ,  $\mathcal{F} \circ \mathcal{F}^* = I$ . Hence we deduce that

**Corollary 3.7.1.** The Fourier transform is a bijective mapping on  $\mathcal{S}(\mathbb{R})$ .

### 3.2 The Plancherel formula

The Fourier transform and Schwartz space has several properties

**Proposition 3.8.** If  $f, g \in \mathcal{S}(\mathbb{R})$ , then

1.  $f * g \in \mathcal{S}(\mathbb{R})$ .

$$2. f * g = g * f.$$

$$3. \widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

The Schwartz space can be equipped with a Hermitian inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

**Theorem 3.9 (Plancherel).** *If  $f \in \mathcal{S}(\mathbb{R})$ , then  $\|\widehat{f}\| = \|f\|$ .*

*Proof.* The proof on the textbook is complicated. So I write it on my own. First, there is a formula about conjugate function  $\bar{f}$ .

$$\begin{aligned} f(x) &= \Re(f(x)) + i\Im(f(x)) \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) + i\Im(\widehat{f}(\xi)))(\cos 2\pi x\xi - i\sin 2\pi x\xi) d\xi \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) \cos 2\pi x\xi + \Im(\widehat{f}(\xi)) \sin 2\pi x\xi) \\ &\quad + i(\Im(\widehat{f}(\xi)) \cos 2\pi x\xi - \Re(\widehat{f}(\xi)) \sin 2\pi x\xi) \end{aligned} \tag{10}$$

$$\begin{aligned} \bar{f}(x) &= \Re(\bar{f}(x)) + i\Im(\bar{f}(x)) \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) - i\Im(\widehat{f}(\xi)))(\cos 2\pi x\xi + i\sin 2\pi x\xi) d\xi \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) \cos 2\pi x\xi + \Im(\widehat{f}(\xi)) \sin 2\pi x\xi) \\ &\quad - i(\Im(\widehat{f}(\xi)) \cos 2\pi x\xi - \Re(\widehat{f}(\xi)) \sin 2\pi x\xi) \end{aligned} \tag{11}$$

Applying 10 and 11, we get the formula

$$\bar{f}(x) = \int_{-\infty}^{\infty} \overline{\widehat{f}(\xi)} e^{2\pi i x \xi} d\xi \tag{12}$$



We write down the formula

$$\begin{aligned}
\|f\|^2 &= \int_{-\infty}^{\infty} |f(x)|^2 dx \\
&= \int_{-\infty}^{\infty} f(x) \overline{f(x)} dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-2\pi i x \xi} d\xi \int_{-\infty}^{\infty} \overline{\widehat{f}(\xi')} e^{2\pi i x \xi'} d\xi' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-2\pi i x \xi} \overline{\widehat{f}(\xi')} e^{2\pi i x \xi'} d\xi d\xi' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i x \xi} e^{2\pi i x \xi'} dx \widehat{f}(\xi) \overline{\widehat{f}(\xi')} d\xi d\xi' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\xi - \xi') \widehat{f}(\xi) \overline{\widehat{f}(\xi')} d\xi d\xi' \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi) \overline{\widehat{f}(\xi)} d\xi \\
&= \|\widehat{f}\|^2
\end{aligned} \tag{13}$$

□

**Remark.** In 13, we used the property that

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-its} ds$$

and

$$\int_{t_1}^{t_2} \delta(t) dt = 1 \quad \text{if} \quad 0 \in [t_1, t_2]$$

More details are introduced in [1]

### 3.2.1 The Weierstrass approximation theorem

**Theorem 3.10.** Let  $f$  be a continuous function on the closed and bounded interval  $[a, b] \subset \mathbb{R}$ . Then,  $\forall \epsilon > 0$ , there exists a polynomial  $P$  such that

$$\sup_{x \in [a, b]} |f(x) - P(x)| < \epsilon$$

In other words,  $f$  can be uniformly approximated by polynomials.

*Proof.* The proof is classic, we first extend  $f$  to  $g$  in  $\mathbb{R}$ . Then, we use  $g * K_\delta$  to approximate  $g$ . Finally, we use Taylor series and choose  $R(x)$  as a polynomial to approximate  $K_\delta$ . An important fact is,  $(g * R)(x)$  is a polynomial in  $x$ . □

### 3.2.2 Functions of moderate decrease

In this situation, convolution  $f * g$  of two functions of moderate decrease is also a function of moderate decrease. So, we have  $\widehat{f * g} = \widehat{f}\widehat{g}$ . Moreover, the multiplication formula  $\int_{-\infty}^{\infty} f(x)\widehat{g}(x) dx = \int_{-\infty}^{\infty} \widehat{f}(y)g(y) dy$  continues to hold, and we deduce the Fourier inversion and Plancherel theorem when  $f$  and  $\widehat{f}$  are both of moderate decrease.

## 3.3 Applications to some partial differential equations

### 3.3.1 The time-dependent heat equation on the real line

Consider an infinite rod, which we model by the real line, and suppose that we are given an initial temperature distribution  $f(x)$  on the rod at time  $t = 0$ . We wish now to determine the temperature  $u(x, t)$  at a point  $x$  at time  $t > 0$ . From some physical properties, we have the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (14)$$

called the **heat equation**. The initial condition we impose is  $u(x, 0) = f(x)$ . Just as in the case of the circle, the solution is given in terms of a convolution. Define the **heat kernel** of the line by

$$\mathcal{H}_t(x) = K_\delta(x), \quad \text{with } \delta = 4\pi t \quad (15)$$

so that

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \quad \text{and} \quad \widehat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 t \xi^2} \quad (16)$$

And we have

**Theorem 3.11.** *Given  $f \in \mathcal{S}(\mathbb{R})$ , let*

$$u(x, t) = (f * \mathcal{H}_t)(x) \quad \text{for } t > 0$$

*where  $\mathcal{H}_t$  is the heat kernel, then:*

1. *The function  $u$  is  $C^2$  when  $x \in \mathbb{R}$  and  $t > 0$ , and  $u$  solves the heat equation.*
2.  *$u(x, t) \rightarrow f(x)$  uniformly in  $x$  as  $t \rightarrow 0$ . Hence if we set  $u(x, t) = f(x)$ , then  $u$  is continuous on the closure of the upper half-plane  $\mathbb{R}_+^2 = \{(x, t) : x \in \mathbb{R}, t \geq 0\}$ .*
3.  *$\int_{-\infty}^{\infty} |u(x, t) - f(x)|^2 dx \rightarrow 0$  as  $t \rightarrow 0$ .*

The theorem 3.11 guarantees the existence of a solution to the heat equation with initial data  $f$ . This solution is also unique. In this regard, we note that  $u = f * \mathcal{H}_t, f \in \mathbb{R}$ .

### 3.3.2 The steady-state heat equation in the upper half-plane

The equation we are now concerned with is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (17)$$

in the upper half-plane  $\overline{\mathbb{R}_+^2} = \{(x, t) : x \in \mathbb{R}, t \geq 0\}$ . The boundary condition we require is  $u(x, 0) = f(x)$ . The kernel that solves the problem is called the **Poisson kernel** for the upper half-plane.

$$\mathcal{P}_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \quad \text{where } x \in \mathbb{R} \text{ and } y > 0$$

Notably, for each fixed  $y$ , the kernel  $\mathcal{P}_y$  is only of moderate decrease as a function of  $x$ . We take the Fourier transform of 17 in the  $x$  variable, thereby obtaining

$$-4\pi^2 \xi^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}}{\partial y^2}(\xi, y) = 0$$

by applying proposition 3.1. The general solution of this ordinary differential equation in  $y$  (with  $\xi$  fixed) takes the form

$$\hat{u}(\xi, y) = A(\xi)e^{-2\pi|\xi|y} + B(\xi)e^{2\pi|\xi|y}$$

If we disregard the second term because its rapid exponential increase we find, after setting  $y = 0$ , that

$$\hat{u}(\xi, y) = \hat{f}(\xi)e^{-2\pi|\xi|y}$$

**Remark.** In the above two examples, we apply Fourier transform in order to find some properties of ordinary differential equations. Normally, it would be a factor of exponent of  $e$ .

**Lemma 3.12.** The following identities hold:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i \xi x} d\xi &= \mathcal{P}_y(x) \\ \int_{-\infty}^{\infty} \mathcal{P}_y(x) e^{-2\pi i \xi x} dx &= e^{-2\pi|\xi|y} \end{aligned}$$

**Lemma 3.13.** The Poisson kernel is a good kernel as  $y \rightarrow 0$ .

**Lemma 3.14 (Mean-value property).** Suppose  $\omega$  is an open set in  $\mathbb{R}^2$  and let  $u$  be a function of class  $C^2$  with  $\Delta u = 0$  in  $\omega$ . If the closure of the disc centered at  $(x, y)$  and of radius  $R$  is contained in  $\omega$ , then

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r \cos \theta, y + r \sin \theta) d\theta$$

for all  $0 \leq r \leq R$ .

*Proof.* The proof is tricky. First, the equation  $\Delta u = 0$  implies

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 u}{\partial \theta^2} &= 0 \\ \frac{\partial^2 u}{\partial \theta^2} + r \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) &= 0\end{aligned}$$

If we define  $F(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta$ , we deduce that

$$r \frac{\partial}{\partial r} \left( r \frac{\partial F}{\partial r} \right) = \frac{1}{2\pi} \int_0^{2\pi} -\frac{\partial^2 u}{\partial \theta^2}(r, \theta) d\theta$$

Since  $\partial u / \partial \theta$  is periodic, we have  $r \frac{\partial}{\partial r} \left( r \frac{\partial F}{\partial r} \right) = 0$ . And consequently  $r \partial F / \partial r$  must be a constant. Hence  $\partial F / \partial r = 0$ . Since  $u(x, y) = F(0)$ , we deduce that  $F(r) = u(x, y)$ . Proof completed.  $\square$

Hence we have the following theorem

**Theorem 3.15.** *Given  $f \in \mathcal{S}(\mathbb{R})$ , let  $u(x, y) = (f * \mathcal{P}_y)(x)$ . Then:*

1.  $u(x, y)$  is  $C^2$  in  $\mathbb{R}_+^2$  and  $\Delta u = 0$ .
2.  $u(x, y) \rightarrow f(x)$  uniformly as  $y \rightarrow 0$ .
3.  $\int_{-\infty}^{\infty} |u(x, y) - f(x)|^2 dx \rightarrow 0$  as  $y \rightarrow 0$ .
4. If  $u(x, 0) = f(x)$ , then  $u$  is continuous on the closure  $\overline{\mathbb{R}_+^2}$ , and vanishes at infinity

$$u(x, y) \rightarrow 0 \text{ as } |x| + y \rightarrow \infty$$

### 3.4 The Poisson summation formula

This section reveals further connection between Fourier series and Fourier transform.

Given a function  $f \in \mathcal{S}(\mathbb{R})$  on the real line, we can construct a new function on the circle by the recipe

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

Since  $f$  is rapidly decreasing, the series converges absolutely and uniformly on every compact subset of  $\mathbb{R}$ , so  $f$  is continuous.

We consider another form of  $F$

$$F_2(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

The fundamental fact is that these two approaches, which produce  $F_1$  and  $F_2$ , actually lead to the same function.

**Theorem 3.16 (Poisson summation formula).** *If  $f \in \mathcal{S}(\mathbb{R})$ , then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

*In particular, setting  $x = 0$  we have*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

*Proof.* The proof is trivial, but the theorem is very important. We can apply the property of Fourier series, and  $F_2$  is the Fourier series of  $F_1$ .  $\square$

### 3.4.1 Theta and zeta functions

In this section, we introduce some basic properties of theta and zeta functions. We define **theta function**  $\vartheta(s)$  for  $s > 0$  by

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$$

which satisfies the following equation

**Theorem 3.17.**  $s^{-1/2} \vartheta(1/s) = \vartheta(s)$  whenever  $s > 0$ .

The proof is only a simple application of Poisson summation formula to the pair

$$f(x) = e^{-\pi s x^2} \quad \text{and} \quad \widehat{f}(\xi) = s^{1/2} e^{-\pi \xi^2 / s}$$

The **Zeta function**  $\zeta(s)$  defined for  $\Re(s) > 1$  by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$\vartheta$  and  $\zeta$  have the following property

**Theorem 3.18.**

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{2} \int_0^{\infty} t^{s/2-1} (\vartheta(t) - 1) dt$$

*Proof.* To be updated.  $\square$

Another important function  $\Theta(z|\tau)$  is defined by

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau} e^{2\pi i n z}$$

whenever  $\Im(\tau) > 0$  and  $z \in \mathbb{C}$ . Taking  $z = 0$  and  $\tau = is$  we get  $\Theta(z|\tau) = \vartheta(s)$ .

### 3.4.2 Heat kernel

We define heat kernel on the circle by

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

Heat kernel on  $\mathbb{R}$  is given by

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

**Corollary 3.18.1.** *The kernel  $H_t(x)$  is a good kernel as  $t \rightarrow 0$ .*

### 3.4.3 Poisson kernels

We recall the definition of Poisson kernels

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} \quad \text{and} \quad \mathcal{P}_y = \frac{1}{\pi} \frac{y}{x^2+y^2}$$

hence we have the theorem

**Theorem 3.19.**  $P_r(2\pi x) = \sum_{n \in \mathbb{Z}} \mathcal{P}_y(x+n)$  where  $r = e^{-2\pi y}$ .

## 3.5 The Heisenberg uncertainty principle

**Theorem 3.20.** *Suppose  $\psi$  is a function in  $\mathcal{S}(\mathbb{R})$  which satisfies the normalizing condition  $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$ . Then*

$$\left( \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right) \left( \int_{-\infty}^{\infty} \xi^2 |\widehat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}$$

and equality holds iff  $\psi(x) = Ae^{-Bx^2}$  where  $B > 0$  and  $|A|^2 = \sqrt{2B/\pi}$ . In fact, we have

$$\left( \int_{-\infty}^{\infty} (x-x_0)^2 |\psi(x)|^2 dx \right) \left( \int_{-\infty}^{\infty} (\xi-\xi_0)^2 |\widehat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}$$

for every  $x_0, \xi_0 \in \mathbb{R}$ .

## 4 The Fourier Transforms on $\mathbb{R}^d$

### 4.1 Preliminaries

We have  $x \in \mathbb{R}^d$  and  $\alpha = (\alpha_1, \dots, \alpha_d)$  of non-negative integers, the monomial  $x^\alpha$  is defined by

$$x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$$

Similarly, we define  $(\partial/\partial x)^\alpha$  by

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

and  $|\alpha| = \sum_{i=1}^d \alpha_i$ .

#### 4.1.1 Symmetries

There three important groups of symmetries in  $\mathbb{R}^d$

1. Translations:  $x \mapsto x + h$ .
2. Dilations:  $x \mapsto \delta x$ .
3. Rotations:  $R : \mathbb{R}^d \mapsto \mathbb{R}^d$  preserves the inner product. Using mathematical language,

$$R(ax + by) = aR(x) + bR(y)$$

for all  $x, y \in \mathbb{R}^d$  and  $a, b \in \mathbb{R}$ .

$$R(x) \cdot R(y) = x \cdot y$$

for all  $x, y \in \mathbb{R}^d$ .

#### 4.1.2 Integration on $\mathbb{R}^d$

First, we define **rapid decrease** and **moderate decrease** on  $\mathbb{R}^d$ . A continuous function  $f$  is of rapid decrease if

$$\sup_{x \in \mathbb{R}^d} |x|^k |f(x)| < \infty \quad \text{for every } k \in \mathbb{N}^+$$

A continuous function  $f$  is of moderate decrease if

$$\sup_{x \in \mathbb{R}^d} |x|^{d+\epsilon} |f(x)| < \infty$$

The interaction of integration with the three important groups of symmetries is as follows: if  $f$  is of moderate decrease, then

1.  $\int_{\mathbb{R}^d} f(x+h) dx = \int_{\mathbb{R}^d} f(x) dx$  for all  $h \in \mathbb{R}^d$ .
2.  $\delta^d \int_{\mathbb{R}^d} f(\delta x) dx = \int_{\mathbb{R}^d} f(x) dx$  for all  $\delta > 0$ .
3.  $\int_{\mathbb{R}^d} f(R(x)) dx = \int_{\mathbb{R}^d} f(x) dx$  for every rotation  $R$ .

#### 4.1.3 Schwartz space and radial function

The Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  consists all infinitely differentiable functions  $f$  on  $\mathbb{R}^d$  such that

$$\sup_{x \in \mathbb{R}^d} |x^\alpha| \left(\frac{\partial}{\partial x}\right)^\beta f(x) < \infty$$

**Radial function**  $f$  depend only on  $|x|$ . In other words,  $f$  is a radial if there is a function  $f_0(x)$ , such that  $f(x) = f_0(|x|)$ .

## 4.2 Fourier transform on $\mathbb{R}^d$

The **Fourier transform** of a Schwartz function  $f$  is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

and we have proposition

**Proposition 4.1.** *Let  $f \in \mathcal{S}(\mathbb{R}^d)$ .*

1.  $f(x+h) \longrightarrow \widehat{f}(\xi) e^{2\pi i \xi \cdot h}$  whenever  $h \in \mathbb{R}^d$ .
2.  $f(x) 2^{-2\pi x \cdot h} \longrightarrow \widehat{f}(\xi + h)$  whenever  $h \in \mathbb{R}^d$ .
3.  $f(\delta x) \longrightarrow \delta^{-d} \widehat{f}(\delta^{-1} \xi)$  whenever  $\delta > 0$ .
4.  $\left(\frac{\partial}{\partial x}\right) f(x) \longrightarrow (2\pi i \xi) \widehat{f}(\xi)$ .
5.  $(-2\pi i x)^\alpha f(x) \longrightarrow \left(\frac{\partial}{\partial \xi}\right)^\alpha \widehat{f}(\xi)$ .
6.  $f(Rx) \longrightarrow \widehat{f}(R\xi)$ .

Similar to Chapter 3, we have two corollaries

**Corollary 4.1.1.** *The Fourier transform maps  $\mathcal{S}(\mathbb{R}^d)$  to itself.*

**Corollary 4.1.2.** *The Fourier transform of a radial function is radial.*

It is also simple to prove Fourier inversion and Plancherel formula on  $\mathcal{S}(\mathbb{R}^d)$ .

**Theorem 4.2.** *Suppose  $f \in \mathcal{S}(\mathbb{R}^d)$ . Then*

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Moreover

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |f(x)|^2 dx$$

Return to convolution, we also have theorem similar to proposition 3.8.

**Theorem 4.3.** *Suppose  $f, g \in \mathcal{S}(\mathbb{R}^d)$ , then*

1.  $f * g \in \mathcal{S}(\mathbb{R}^d)$ .
2.  $f * g = g * f$ .
3.  $\widehat{(f * g)}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi)$ .

## References

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