Fourier Analysis Notes

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[1]Thanks to my family, my friend and freedom.

Abstract

This is a note about my understanding of Fourier Analysis This note won't include the last 2 part of Stein's Fourier Analysis[2]. Because study of number theory is not essential for me. I would probably study the last 2 chapters later.

Contents

1 Introduction to Fourier Series

Fourier Series are the very basis of Fourier Analysis. Following the Stein's instruction, I will start my note at Fourier Series. Fisrt, this note will introduce some basic concepts.

1.1 Basic concepts

1.1.1 Riemann integrable functions

A real-valued functions defined on $[0, L]$ is **Riemann integrable** if it is bounded and if $\forall \epsilon > 0, \exists 0 = x_0 < \cdots < x_N = L$, the subdivision satisfies following conditions. We denote $\mathcal{U}$ and $\mathcal{L}$ as the upper and lower sums of this subdivision.

$$\mathcal{U} = \sum_{j=1}^{N} [\sup_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

$$\mathcal{L} = \sum_{j=1}^{N} [\inf_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

then we have $\mathcal{U} - \mathcal{L} < \epsilon$.

1.1.2 Functions on the circle

Obviously, we can treat a 2π-periodic function $f \in \mathbb{R}$ as a function F on the circle.

$$f(\theta) = F(e^{i\theta})$$

and naturally, $f(0) = f(2\pi) \Rightarrow F(0) = F(e^{2\pi i})$

1.1.3 Fourier coefficient and series

For a function defined on $[a, b]$ and $b - a = L$, the n^{th} **Fourier coefficient** of f is defined by

$$\widehat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi inx/L} dx, \quad n \in \mathbb{Z}$$

The **Fourier series** of f is given by

$$\sum_{n=-\infty}^{n=+\infty} \widehat{f}(n) e^{2\pi inx/L}$$

Sometimes we denote a_n as $\widehat{f}(n)$, and we use this notation

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} a_n e^{2\pi inx/L}$$

When $L = 2\pi$, we can denote it by

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} a_n e^{2in\theta}$$

1.1.4 Trigonometric series and partial sum

Trigonometric series can be expressed as $\sum_{n=-\infty}^{n=+infty} c_n e^{2\pi inx/L}$, $c_n \in \mathbb{C}$. If for all large $|n|$, we get $c_n = 0$, it is called **Trigonometric polynomial**, its degree is the largest $|n|$ that $c_n \neq 0$.

For $N > 0$, the N^{th} **partial sum** are expressions of the form

$$S_N(f)(x) = \sum_{n=-N}^{n=N} \widehat{f}(n) e^{2\pi inx/L}$$

1.1.5 Euler's identity

Euler's identity is a useful formula in analysis, it connects e and trigonometric functions.

$$e^{i\theta} = \cos\theta + i \sin\theta$$

1.1.6 Jacobian matrix

Jacobian matrix of a vector-valued function $\{f_n\}$ is the matrix of all its first-order partial derivatives. It is defined by

$$\begin{aligned}\boldsymbol{J}(x_1,\cdots,x_n) &= \frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\end{aligned}$$

Jacobian is often used in multiple integration, we can change variables in multiple integration. For example, $x = f_x(a,b), y = f_y(a,b), F = (x,y)$, the integration on domain D is

$$\int\int_D F(x,y)dx\ dy = \int\int_T F(f_x(a,b), f_y(a,b))|\frac{\partial(x,y)}{\partial(a,b)}|da\ db$$

where T is the domain of f that $f(a,b) \in D$.

1.2 Convolutions and Kernels

There are types of functions. First, we introduce the convolution of 2π-periodic integrable functions f and g on $\mathbb{R}$, we define their **convolution** $f * g$ on $[-\pi,\pi]$ by

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$$

We can also change variables $t = x - y$ to get

$$\begin{aligned}(f*g)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)\frac{dy}{dt}dt \\ &= -\frac{1}{2\pi}\int_{\pi}^{-\pi} f(x-t)g(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)dt\end{aligned}$$

In the above equations, g is called the **kernel** of the convolution. Convolutions have several properties.

Proposition 1.1. *Suppose that f,g and h are 2π-periodic integrable functions. Then:*

1. $f*(g+h) = (f*g) + (f*h)$

2. $(cf) * g = c(f * g) = f * (cg), \quad \forall c \in \mathbb{C}$
3. $f * g = g * f$
4. $(f * g) * h = f * (g * h)$
5. $f * g$ *is continuous*
6. $\widehat{f * g}(n) = \widehat{f}(n) widehatg(n)$

Proof. Obviously, properties 1, 2, 3 are trivial. To prove 4, we write down

$$\begin{aligned} 4\pi^2 (f * g) * h(z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y) g(x-y) h(z-x) dy \ dx \\ 4\pi^2 f * (g * h)(z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y) g(x) h(z-x-y) dx \ dy \\ &= \int_{-\pi}^{\pi} \int_{y-\pi}^{y+\pi} f(y) g(x-y) h(z-x) dx \ dy \\ &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y) g(x-y) h(z-x) dx \ dy \end{aligned}$$

If g is continuous, the proof of 5 is trivial. We can also construct a series of continuous functions to approximate g, hence we have proved 5 for merely integrable function g.

Property 6 is the most critical property. For continuous g and f, we can just calculate and reformulate the equation. For merely integrable functions, we can apply the same technique used in proof of property 5. □

1.3 Uniqueness of Fourier series

It is critical to prove the uniqueness of fourier coefficients of a function. More precisely, if f and g have same fourier coefficients, then f and g are necessarily equal. To prove 2 items are equal, a useful technique is to prove their difference equals 0.

Theorem 1.2. *Suppose that f is an integrable function on the circle with $\widehat{f} = 0, \forall n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.*

Proof. First we suppose that f is real-valued and defined on $-[\pi, \pi]$. We will argue this by contradiction. Without loss of generality, we assume that $\theta = 0$, $f(0) > 0$ and f is continuous at 0.

The idea is to construct a family of trigonometric polynomials $\{p_k\}$ that 'peak' at 0, and $\int p_k(\theta) f(\theta) d\theta \to \infty$ as $k \to \infty$. This will be our contradiction since these integrals are equal to zero by assumption. (because if $\int p_k(\theta) f(\theta) \neq 0$, there exists $\widehat{f}(n) \neq 0$)

In this proof, we construct function $p(\theta) = \epsilon + \cos\theta$ and $p_k(\theta) = [p(\theta)]^k$, and calculate 3 intervals respectively. □

The next corollary is a deduced from the theorem 1.2.

Corollary 1.2.1. *Suppose that f is a continuous funtion on the circle and the Fourier series is absolutely convergent. Then the Fourier series converges uniformly to f.*

Corollary 1.2.2. *Suppose that f is a twice continuously differentiable function on the circle. Then*

$$\widehat{f}(n) = O(1/|n|^2), \quad \text{as } |n| \to \infty$$

Proof. To prove the corollary, we integrate py parts

$$\begin{aligned} 2\pi\widehat{f}(n) &= \int_0^{2\pi} f(\theta)e^{-in\theta}\, d\theta \\ &= \left[f(\theta)\cdot\frac{-e^{-in\theta}}{in}\right]_0^{2\pi} + \frac{1}{in}\int_0^{2\pi} f'(\theta)e^{-in\theta}\, d\theta \\ &= \frac{1}{in}\int_0^{2\pi} f'(\theta)e^{-in\theta}\, d\theta \\ &= \frac{1}{(in)^2}\int_0^{2\pi} f''(\theta)e^{-in\theta}\, d\theta \\ &= -\frac{1}{n^2}\int_0^{2\pi} f''(\theta)e^{-in\theta}\, d\theta \end{aligned}$$

The rest part of proof is trivial. □

Incidentally, we have got an important identity when $n \in \mathbb{Z}$

$$\widehat{f'}(n) = in\widehat{f}(n)$$

When $n \neq 0$, proof is given in the proof of corollary 1.2.2. When $n = 0$, $\int_0^{2\pi} f'(\theta)\, d\theta = 0$ since f is 2π-periodic function.

Since $f^{(m)}(n) \sim \sum a_n (in)^m e^{in\theta}$. Further smoothness conditions on f imply even better decay of the Fourier coefficients.

There are also stronger version of corollary 1.2.2.

Theorem 1.3. *The Fourier series converges absolutely if f satisfies a **Holder condition** of order α, with $\alpha > 1/2$*

$$\sup_\theta |f(\theta + t) - f(\theta)| \leq A|t|^\alpha$$

Proof. To be updated. □

1.4 Good kernels

We have introduced the notion of kernel in section 1.2. Now we introduce the notion of **good kernels** if a family of kernels satisfies the following properties:

Proposition 1.4. *Properties of good kernels:*

1. *For all* $n \geq 1$,
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\ dx = 1$$
2. *There exists* $M > 0$ *such that for all* $n \geq 1$,
$$\int_{-\pi}^{\pi} |K_n(n)|\ dx \leq M$$
3. $\forall \delta > 0$,
$$\int_{\delta<|x|<\pi} |K_n(x)|\ dx \to 0, \quad \textit{as } n \to \infty$$

Good kernels are important for their properties. The next theorem is deduced from those properties.

Theorem 1.5. *Let* $\{K)n\}_{n=1}^{\infty}$ *be a family of good kernels, and* f *an integrable function on the circle. Then*

$$\lim_{n\to\infty}(f * k_n)(x) = f(x)$$

whenever f *is continuous at* x. *If* f *is continuous everywhere, then the above limit is uniform.*

Proof. Using the property 2 and 3, the proof of the equation is trivial.

The second assertion depends on the periodic property. Since f is contiuous everywhere and f is periodic, then f is uniformly continuous. Hence δ can be chosen independently. So $f * K_n \to f$ uniformly. □

1.5 Some special kernels

In this section, we will introduce some special kernels.

1.5.1 Dirichlet kernel

The N^{th} **Dirichlet kernel** is defined by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

Another form is

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$$

The proof of the second form is not critical, we omit it. A critical fact is that Dirichlet kernel is not a good kernel. The proof is trivial, we omit it(calculate integrals for each item).

1.5.2 Cesaro means and summation

We define a series $\{c_n\}$, and partial sum $s_n = \sum_{k=0}^{n} c_k$. Since $\{c_n\}$ may not converge, we need a new series to measure the convergence. We define

$$\delta_N = \frac{\sum_{k=0}^{N-1} s_k}{N}$$

which is called the **Cesaro mean** of $\{s_n\}$, or the **Cesaro sum** of $\{c_n\}$. if δ_N converges to a limit, we say that $\sum c_n$ is **Cesaro summable**. It is easy to check that sum and Cesaro sum have the same limit, the proof is trivial.

1.5.3 Fejer's theorem

Fejer's kernel is defined by

$$F_N(x) = \frac{\sum_{n=0}^{N} D_n(x)}{N}$$

Lemma 1.6.

$$F_N(x) = \frac{\sin^2(Nx/2)}{N\sin^2(x/2)}$$

The Fejer kernel is a good kernel.

Proof. The proof of the first assertion is to be updated.

The proof of the second assertion is trivial since $F_N(x) > 0$. □

Applying the Theorem 1.5, we get the following theorem.

Theorem 1.7. *If f is integrals on circle, then Fourier series of f is Cesaro summable to f at every point of continuity of f.*

Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Cesaro summable to f.

Corollary 1.7.1. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

1.5.4 Abel means and Abel summation

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be **Abel summable** to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s$$

A critical fact is that a Cesaro summable series is always Abel summable, however, an Abel summable series is not necessarily Cesaro summable.

1.6 Poisson kernel

Poisson kernel is defined by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

and the convolution of Poisson kernel is defined by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} = (f * P_r)(\theta)$$

Lemma 1.8. *If $0 \leq r < 1$, then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

The Poisson kernel is a good kernel, as $r \to 1$

Theorem 1.9. *Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f. If f is integrals on circle, then Fourier series of f is Abel summable to f at every point of continuity of f.*

The proof is trivial, we omit it. But it is critical to notice that the Poiss kernel is indexed by r, not n.

Remark. *An important hint deduced from the above theorems is that when we calculate polynomials like $\sum \cos nx$, we can transform the polynomial to the form of $e^i n$ according to Euler's formula. Then we can apply the formula $\sum_{n=0}^{n=N} a^n = \frac{1-a^{n+1}}{1-a}$.*

In the following theorem, we apply Poisson kernel to steady heat equation.

Theorem 1.10. *Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by Poisson integral*

$$u(r, \theta) = (f * P_r)(\theta)$$

has the following properties:

1. *u has continuous derivatives in the unic unit disc and satisfies $\delta u = 0$*
2. *If θ is any point of continuity of f, then*

$$\lim_{r \to 1} u(r, \theta) = f(\theta)$$

3. *If f is continuous everywhere, then this limit is uniform.*

Proof. To prove 1, we need to review the definition of **Laplacian**

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and we get the following relationship between x, y, r, θ

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan \frac{y}{x} \\
\frac{\partial r}{\partial x} &= \frac{x}{r} \\
\frac{\partial r}{\partial y} &= \frac{y}{r} \\
\frac{\partial^2 r}{\partial x^2} &= \frac{y^2}{r^3} \\
\frac{\partial^2 r}{\partial y^2} &= \frac{x^2}{r^3} \\
\frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} \\
\frac{\partial \theta}{\partial y} &= \frac{x}{r^2} \\
\frac{\partial^2 \theta}{\partial x^2} &= -\frac{2xy}{r^4} \\
\frac{\partial^2 \theta}{\partial y^2} &= \frac{2xy}{r^4}
\end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} \left(\frac{\partial r}{\partial x} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2}$$

$$\begin{aligned}
\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 &= 1 \\
\frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 r}{\partial^2 y} &= \frac{1}{r} \\
\left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 &= \frac{1}{r^2} \\
\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} &= 0
\end{aligned}$$

Then, we get the Laplacian in form of r and θ.

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

and we calculate the formula term by term

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= r^{|m|-2}|m|(|m|-1)e^{im\theta} \\
\frac{1}{r}\frac{\partial u}{\partial r} &= r^{|m|-2}|m|e^{im\theta} \\
\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= r^{|m|-2}|m|^2 e^{im\theta+\pi} \\
&= -r^{|m|-2}|m|^2 e^{im\theta} \\
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} &= 0
\end{aligned}$$

The property 1 has been proved.

Because Poisson kernel is a good kernel, the proof of property 2 is trivial.

The proof of 3 is heuristic. We suppose function v solves the steady heat equation. v has a Fourier series

$$\sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta} \text{ where } a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v e^{-in\theta} d\theta$$

Taking into account that v solves

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

We get

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = 0$$

Applying principles of second ordinary differantial equation. We have $a_n(r) = A_n r^n + B_n r^{-n}$ for constant A_n and B_n. Because r^{-n} is not bounded when $r < 1$, we deduce that $B_n = 0$. Since v converges uniformly to f as $r \to 1$ we find that

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta} d\theta$$

Taking the above equation into $a_n(r)$, 3 has been proved. □

Remark. $(e^{ix})' = e^{i(x+\pi/2)} = ie^{ix}$

Remark. *There are 2 principles to judge whether the inner product of 2 series converge.*

- *Abel's test: If $\{a_n\}$ is monotonic and bounded, $\{\sum_{n=0}^{\infty} b_n\}$ converges. We have $\sum_{n=1}^{\infty} a_n b_n$ converges.*

- *Dirichlet's test: If* $\{a_n\}$ *decreases monotonically to* 0*, and* $\sum b_n$ *are bounded, then* $\sum_{n=1}^{\infty} a_n b_n$ *converges.*

Remark. *class* C_k *means the function* $f \in C_k$ *has 1st to k-th derivatives and their are all continuous.* f *is called k-th continuously differentiable.*

2 Convergence of Fourier Series

In this chapter, we continue to study convergence of Fourier series.

The first viewpoint is "global" and concerns the overall behavier of a function f over the $[0, 2\pi]$. If f is integrable on the circle, then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \to 0 \text{ as } N \to 0$$

The second viewpoint is "local" and concerns the behavier of f near a given point. Whether

$$S_N(f)(\theta) \to f(\theta)$$

2.1 Vector spaces and inner products

We have already been familiar with vector spaces. We only revview the definition of **inner product** here. An inner product on a vector space V over $\mathbb{R}$ associates to any pair X, Y of elements in V a real number which we denote by (X, Y). The inner product must be symmetric $(X, Y) = (Y, X)$ and linear in both variables

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z)$$

whenever $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in V$. Also, inner product must be positive-definite, $(X, X) \geq 0$. And we define the norm of X by

$$\|X\| = (X, X)^{1/2}$$

Additionally, $\|X\| = 0$ implies $X = 0$.

A classic $innerproductspace$ is $\mathbb{R}^d$, inner product is defined by

$$(X, Y) = x_1 y_1 + \cdots + x_d y_d$$

For vector spaces over complex numbers, the inner product of two elements is a complex number. Moreover, these inner products are called Hermitian since they satisfy $(X, Y) = \overline{(Y, X)}$. Hence the inner product is linear in the first variable, but conjugate-linear in the second:

$$\begin{aligned}
(\alpha X + \beta Y, Z) &= \alpha(X, Z) + \beta(Y, Z) \\
(Z, \alpha X + \beta Y) &= \overline{\alpha}(Z, X) + \overline{\beta}(Z, Y)
\end{aligned}$$

For example, the inner product of two elements Z, W in $\mathbb{C}^d$ is defined by

$$(Z, W) = z_1 \overline{w_1} + \cdots + z_d \overline{w_d}$$

If two elements X, Y are **orthogonal**, then $(X, Y) = 0$, and we write $X \perp Y$. Three critical results can be derived from the notion of orthogonality.

1. The Pythagorean theorem: if X, Y are orthogonal, then
$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$$
2. The Cauchy-Schwarz inequality: for any $X, Y \in V$ we have
$$|(X, Y)| \leq \|X\|\|Y\|$$
3. The traingle inequality: for any $X, Y \in V$ we have
$$\|X+Y\| \leq \|X\| + \|Y\|$$

Proof. The proof of 1 is trivial.

The proof of 2 is here: We assume $Y = kX + Z$, where $Z \in V, X \perp Z$. Hence we have

$$\begin{aligned} |(X,Y)| &= |(X, kX+Z)| \\ &= \|X\|^2|k| + |(X,Z)| \\ &= |k|\|X\|^2 \end{aligned}$$

$$\|X\| \cdot \|Y\| = \|X\| \cdot \|kX+Z\|$$

Obviously, $\|kX+Z\| \geq |k| \cdot \|X\|$, 2 is proved. The proof of 3 is also trivial, we omit it. $\square$

2.1.1 Infinite dimensional vector spaces

The vector space $\ell^2(\mathbb{Z})$ over $\mathbb{C}$ is the set of infinite sequences of complex numbers
$$(\cdots, a_{-n}, \cdots, a_0, \cdots, a_n, \cdots)$$
such that
$$\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$$
the sequence converges.

Remark. *To measure infinity, we often use limitation.*

In the three examples $\mathbb{R}^d, \mathbb{C}^d, \ell^2(\mathbb{Z})$, the vector spaces with their inner products and norms satisfy two important properties:

1. The inner product is strictly positive-definite, $\|X\| = 0$ implies $X = 0$.
2. the vector space is **complete**, which by definition means that every Cauchy sequence in the norm converges to a limit.

An inner product space with 1 and 2 is a **Hilbert space**. We see that $\mathbb{R}^d$ and $\mathbb{C}^d$ are examples of finite-dimensional Hilbert spaces, while $\ell^2(\mathbb{Z})$ is an example of an infinite-dimensional Hilbert space. If either of 1 and 2 fail, the space is called **pre-Hilbert space**.

We give an important example of a pre-Hilbert space where both 1 and 2 fail.

2.1.2 An example of pre-Hilbert space

Let $\mathcal{R}$ denote the set of complex-valued Riemann integrable functions on$[0, 2\pi]$. This is a vector space over C. Addition is defined pointwise by

$$(f + g)(\theta) = f(\theta) + g(\theta)$$

Naturally, multiplication by a scalar $\lambda \in \mathbb{C}$ is given by

$$(\lambda f)(\theta) = \lambda \cdot f(\theta)$$

An inner product is defined on this vector space by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)} d\theta$$

The norm of f is

$$\|f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta \right)^{1/2}$$

In $\mathcal{R}$, 1 fails, we have a simple example

$$f(\theta) = \begin{cases} 0, & \text{if } \theta \neq 0 \\ 1, & \text{if } \theta = 0 \end{cases}$$

where $\|f\| = 0$ but $f \neq 0$. But it does not matter since the points $f(\theta) \neq 0$ have measure 0.

Another fatal fact is that $\mathcal{R}$ is not complete. We hae function

$$f(\theta) = \begin{cases} 0, & \text{for } \theta = 0 \\ \log(1/\theta), & \text{for } 0 < \theta \leq 2\pi \end{cases}$$

Since f is not bounded, f is not integrable and $f \notin \mathcal{R}$. Moreover, the sequence of truncations f_n defined by

$$f_n(\theta) = \begin{cases} 0, & \text{for } 0 \leq \theta \leq 1/n \\ f(\theta), & \text{for } 1/n < \theta \leq 2\pi \end{cases}$$

can easily be seen to form a Cauchy sequence in $\mathcal{R}$. Further discussion of these problems remain in Real Analysis.

Remark. *In 2.1.2, f is the limit of f_n, but f is not Riemann integrable. Maybe f is Lebesgue integrable?*

2.2 Proof of mean-square convergence

Consider the space $\mathcal{R}$ mentioned above, we need to prove

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta$$

With this notation, we must prove $\|f - S_N(f)\| \to 0$ as $N \to \infty$.

For $n \in \mathbb{Z}$, let $e_n(\theta) = e^{in\theta}$, and observe that family $\{e_n\}_{n\in\mathbb{Z}}$ is **orthonormal**

$$(e_n, e_m) = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

Let f be an integrable function on the circle, we have

$$(f, e_n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta = a_n$$

Because $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal, we have

$$(f - S_N(f)) \perp \sum_{|n|\leq N} b_n e_n \tag{1}$$

for any complex number b_n. Applying Pythagorean theorem to 1, and we have

$$\|f\|^2 = \|f - \sum_{|n|\leq N} a_n e_n\|^2 + \|\sum_{|n|\leq N} a_n e_n\|^2$$

Since the orthonormal property of the family $\{e_n\}$ implies that

$$\|\sum_{|n|\leq N} a_n e_n\|^2 = \sum_{|n|\leq N} |a_n|^2$$

we deduce that

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n|\leq N} |a_n|^2 \tag{2}$$

Lemma 2.1 (**Best approximatation**). *If f is integrable on the circle with Fourier coefficients a_n, then*

$$\|f - S_N(f)\| \leq \|f - \sum_{|n|\leq N} c_n e_n\|$$

the equlity holds iff $c_n = a_n$ for all $|n| \leq N$.

Proof. Obviously, we have

$$f - \sum_{|n|\leq N} c_n e_n = f - S_N(f) + \sum_{|n|\leq N} b_n e_n$$

where $b_n = a_n - c_n$. Simply applying Pythagorean theorem, the lemma is proved. $\square$

Suppose that f is contiuous on the circle. Then, given $\epsilon > 0$, there exists a trigonometric polynomial P, say of degree M

$$|f(\theta) - P(\theta)| < \epsilon \text{ for all } \theta$$

Inparticular, taking squares and integrating the inequality yields $\|f - P\| < \epsilon$, and by the best approximation lemma we have

$$\|f - S_N(f)\| < \epsilon \text{ whenever } N \geq M$$

For merely integrable function f, we need to use the lemma

Lemma 2.2. *Suppose f is integrable on the circle and bounded by B. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions*

$$\sup_{x\in[-\pi,\pi]} |f_k(x)| \leq B$$

and

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)| dx \to 0 \text{ as } k \to \infty$$

Applying Lemma 2.2, we get

Theorem 2.3. *Let f be an integrable function on the circle with $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Then we have*

1. *Mean-square convergence of the Fourier series*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \to 0 \text{ as } N \to \infty$$

2. *Parseval's identity*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta$$

Remark. *If $\{e^n\}$ is nay orthonormal family of functions on the circle, and $a_n = (f, e_n)$, then we deduce*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2$$

*This is known as **Bessel's inequality**. Equality holds precisely when the family $\{e_n\}$ is also a "basis".*

Remark. *There exist sequences $\{a_n\}$ such that $\sum |a_n| < \infty$, yet no Riemann integrable function F has Fourier coefficients equals to a_n for all n. To be updated.*

Theorem 2.4 (**Riemann-Lebesgue lemma**). *If f is integrable on the circle, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.*

2.2.1 Review

To prove global convergence of integrable function f on the circle. We have several steps.

1. Applying lemma 2.2 to find function g to approximate f.
2. For contiuous function g, applying corollary 1.7.1 to get trigonometric polynomial P to approximate g.
3. Applying best approximation, because $\{e_n\}$ is a basis, we have proved theorem 2.3

Sometimes, to calculate Fourier coefficients of intricate funtions, we decompose the function to simpler functions.

Lemma 2.5. *Suppose F and G are integrable on the circle with*

$$F \sim \sum a_n e^{in\theta} \quad \text{and} \quad G \sim \sum b_n e^{in\theta}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} F(\theta)\overline{G(\theta)}d\theta = \sum a_n \overline{b_n}$$

2.3 Pointwise convergence

Theorem 2.6. *Let f be an integrable function on the circle which is differentiable at a point θ_0. Then $D_N(f)(\theta_0) \to f(\theta_0)$ as N tends to infinity.*

Proof. Construct a function F

$$F(t) = \begin{cases} \frac{f(\theta_0 - t) - f(\theta_0)}{t}, & \text{if } t \neq 0 \text{ and } |t| < \pi \\ -f'(\theta_0), & \text{if } t = 0. \end{cases}$$

Then use the F to replace f to prove $S_N(f)(\theta_0) - f(\theta_0)$. applying Lemma 2.4.

$$S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t) t D_N(t) dt$$

We recall that

$$tD_N(t) = \frac{t}{\sin(t/2)} \sin((N + 1/2)t)$$

Applying Lemma 2.4, we have proved that $S_N(f)(\theta_0) - f(\theta_0)$ as $N \to \infty$.

Why using F? Because we need $t/\sin(t/2)$ to be integrable.

□

Remark. *If f satisfies **Lipschitz condition** at θ_0*

$$|f(\theta) - f(\theta_0)| \leq M|\theta - \theta_0|$$

for a constant $M \geq 0$ and all θ. This is the same as saying that f satisfies Holder condition of order $\alpha = 1$.

2.3.1 A continuous function with diverging Fourier series

In order to understand convergence better, we need an example of a continuous periodic funtion whose Fourier series diverges at a point. Thus, theorem 2.6 fails if the differentiability is replaced by weaker assumption of continuity. Review the sawtooth function f

$$\sum_{n\neq 0} \frac{e^{in\theta}}{n}$$

We consider Fourier series

$$\sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n} \tag{3}$$

3 is no longer the Fourier series of a Riemann integrable function. Suppose it were the Fourier series of an integrable function, say $\widetilde{f}$, where in particular $\widetilde{f}$ is bounded. Using the Abel means, we have

$$|A_r(\widetilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n}$$

Because $\sum 1/n$ diverges, this gives contradiction since

$$|A_r(\widetilde{f})(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widetilde{f}(\theta)| P_r(\theta) d\theta \leq \sup_{\theta} |\widetilde{f}(\theta)|$$

Lemma 2.7. *Suppose that the Abel means $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum c_n$ are bounded as r tends to 1. If $C_n = (1/n)$, then the partial sums $S_N = \sum_{n=1}^{N} c_N$ are bounded.*

From lemma 2.7, we know $c_n = e^{in\theta}/n - e^{-in\theta}/n$ and hence $\sum_{c_n} isbounded$ So $S_N(f)(\theta)$ is uniformly bounded in N and θ.

Define $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$ and $\widetilde{P}_N(\theta) = e^{i(2N)\theta} \widetilde{f}_N(\theta)$ The rest is to be updated, because I am not fully understand yet.

Remark. ***Gibbs' phenomenon*** *is critical when processing signals with discontinuity.*

3 The Fourier Transform on $\mathbb{R}$

We have introduced the Fourier series of periodic functions. In this chapter, we will study Fourier transform on $\mathbb{R}$. Which means, we develop an analogous theory for non-periodic functions on $\mathbb{R}$.

Roughly speaking, the Fourier transform is a continuous version of Fourier coefficients. Hence we have the following equations

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx \tag{4}$$

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}d\xi \tag{5}$$

3.1 Elementary theory

3.1.1 Integration of functions on the real line

f defined on $\mathbb{R}$ is said to be of **moderate decrease** if f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \leq \frac{A}{1+x^2} \text{ for all } x \in \mathbb{R}$$

We denote $\mathcal{M}(\mathbb{R})$ as the set of functions of moderate decrease on $\mathbb{R}$.

Generally, we call a function f of moderate decrease whenever

$$|f(x)| \leq \frac{A}{1+|x|^{1+\epsilon}} \text{ for all } x \in \mathbb{R}$$

And f has the property

$$\int_{-\infty}^{\infty} f(x)dx < \infty$$

3.1.2 Schwartz space

The **Schwartz space** on $\mathbb{R}$ consists the set of all indefinitely differentiable functions f so that f and all its derivatives $f', \cdots, f^{(\ell)}, \cdots$ are **rapidly decreasing**, in the sense that

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty \quad \text{for every} \quad k, \ell \geq 0$$

We denote the space by $\mathcal{S} = \mathcal{S}(\mathbb{R})$. Moreover, if $f \in \mathcal{S}$, we have

$$f'(x) = \frac{df}{dx} \in \mathcal{S} \quad \text{and} \quad xf(x) \in \mathcal{S}$$

This expresses the important fact that the Schwartz space is closed under defferentiation and multiplication by polynomials.

A simple example of a function in $\mathcal{S}$ is the **Gaussian** defined by

$$f(x) = e^{-x^2}$$

Another example is $e^{-}|x|$, although it decreases rapidly, it is not differentiable at 0, so it does not belong to $\mathcal{S}$.

3.1.3 Fourier transform on $\mathcal{S}$

The **Fourier transform** of a function $f \in \mathcal{S}$ is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi} dx \tag{6}$$

We use notation

$$f(x) \longrightarrow \widehat{f}(\xi)$$

to mean that $\widehat{f}(\xi)$ denotes the Fourier transform of f.

Proposition 3.1. *If $f \in \mathcal{S}(\mathbb{R})$ then:*

1. $f(x+h) \longrightarrow \widehat{f}(\xi)e^{2\pi i h\xi}$ *whenever* $h \in \mathbb{R}$.
2. $f(x)e^{-2\pi i x h} \longrightarrow \widehat{f}(\xi + h)$ *whenever* $h \in \mathbb{R}$.
3. $f(\delta x) \longrightarrow \delta^{-1}\widehat{f}(\delta^{-1}\xi)$ *whenever* $h \in \mathbb{R}$.
4. $f'(x) \longrightarrow 2\pi i\xi\widehat{f}(\xi)$ *whenever* $h \in \mathbb{R}$.
5. $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi}\widehat{f}(\xi)$ *whenever* $h \in \mathbb{R}$.

Theorem 3.2. *If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$.*

The proof is trivial, we omit it.
An important formula is **Gaussian**, $e^{-\pi x^2}$

Theorem 3.3.

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\, dx\, dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\pi r^2} r\, dr\, d\theta \\
&= \int_{0}^{\infty} 2\pi r r^{-\pi r^2}\, dr \\
&= -e^{-\pi r^2}\Big|_0^\infty \\
&= 1
\end{aligned}$$

Theorem 3.4. *If $f(x) = e^{-\pi x^2}$, then $\widehat{f}(\xi) = f(\xi)$.*

Proof. Define
$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx$$
Because $f'(x) = -2\pi x f(x)$, we obtain
$$\widehat{f}'(\xi) = \int_{-\infty}^{\infty} f(x)(-2\pi i x) e^{-2\pi i x\xi}\, dx = i\int_{-\infty}^{\infty} f'(x) e^{-2\pi i x\xi}\, dx$$
By 4 in theorem 3.1, we find that
$$\widehat{f}'(\xi) = i(2\pi i\xi)\widehat{f}(\xi) = -2\pi\xi\widehat{f}(\xi)$$
Hence we have $\widehat{f}(\xi) = e^{-\pi\xi^2}$. □

Generally, for every $\delta > 0$, $K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$, then $\widehat{K_\delta}(\xi) = e^{-\pi\delta\xi^2}$. Similar to definition in Chapter 2, we define **good kernel** on $\mathbb{R}$

Proposition 3.5. *P_δ is a good kernel when*

1. $\int_{-\infty}^{\infty} P_\delta(x)\, dx = 1$.
2. $\int_{-\infty}^{\infty} |P_\delta(x)|\, dx \le M$.
3. *For every $\eta > 0$, we have $\int_{|x|>\eta} |P_\delta(x)|\, dx \to 0$ as $\delta \to 0$.*

Obviously, K_δ is a good kernel. Moreover, the **convolution** on $\mathbb{R}$ is defined by
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\, dt \tag{7}$$

Corollary 3.5.1. *If $f \in \mathcal{S}(\mathbb{R})$, then*
$$(f * K_\delta)(x) \longrightarrow f(x) \text{ uniformly in } x \text{ as } \delta \to 0$$

3.1.4 Fourier inversion

The next result is an identity sometimes called the multiplication formula.

Proposition 3.6. *If $f, g \in \mathcal{S}(\mathbb{R})$, then*
$$\int_{-\infty}^{\infty} f(x)\widehat{g}(x)\, dx = \int_{-\infty}^{\infty} \widehat{f}(x) g(x)\, dx$$

The proof is trivial, we omit it.

Theorem 3.7 (**Fourier inversion**)**.** *If $f \in \mathcal{S}(\mathbb{R})$, then*
$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

Proof. We have known that $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}, G_\delta(x) = \widehat{K_\delta}(x) = e^{-\pi\delta x^2}$.

Since K_δ is a good kernel, we obtain

$$\int_{-\infty}^{\infty} f(x)K_\delta(x)\,dx \longrightarrow f(0) \text{ as } \delta \to 0 \tag{8}$$

Since $G_\delta(x) \to 1$ as $\delta \to 1$, we obtain

$$\int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x)\,dx \longrightarrow \int_{-\infty}^{\infty} \widehat{f}(x)\,dx \text{ as } \delta \to 0 \tag{9}$$

Applying theorem 3.6, we have

$$\int_{-\infty}^{\infty} f(x)K_\delta(x)\,dx = \int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x)\,dx$$

After reviewed three equations above, we have

$$f(0) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\,d\xi$$

In general, let $F(y) = f(y + x)$ so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \widehat{F}(\xi)\,d\xi$$

Applying 2 in proposition 3.1, we have

$$\int_{-\infty}^{\infty} \widehat{F}(\xi)\,d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi$$

proof is completed. □

Remark. *If $f \notin \mathcal{S}(\mathbb{R})$, 8 and 9 would be wrong.*

We define two mappings $\mathcal{F}, \mathcal{F}^*$ by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \quad \text{and} \quad \mathcal{F}^*(f)(x) = \int_{-\infty}^{\infty} f(\xi)e^{2\pi i x\xi}\,d\xi$$

And we have $\mathcal{F}^* \circ \mathcal{F} = I$, $\mathcal{F} \circ \mathcal{F}^* = I$. Hence we deduce that

Corollary 3.7.1. *The Fourier transform is a bijective mapping on $\mathcal{S}(\mathbb{R})$.*

3.2 The Plancherel formula

The Fourier transform and Schwartz space has several properties

Proposition 3.8. *If $f, g \in \mathcal{S}(\mathbb{R})$, then*

1. $f * g \in \mathcal{S}(\mathcal{R})$.

2. $f * g = g * f$.

3. $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

The Schwartz space can be equipped with a Hermitian inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$$

Theorem 3.9 (Plancherel). *If* $f \in \mathcal{S}(\mathbb{R})$*, then* $\|\widehat{f}\| = \|f\|$.

Proof. The proof on the textbook is complicated. So I write it on my own. First, there is a formula about conjugate function $\overline{f}$.

$$\begin{aligned} f(x) &= \Re(f(x)) + i\Im(f(x)) \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f(\xi)}) + i\Im(\widehat{f(\xi)}))(\cos 2\pi x\xi - i \sin 2\pi x\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f(\xi)}) \cos 2\pi x\xi + \Im(\widehat{f(\xi)}) \sin 2\pi x\xi) \\ &\quad + i(\Im(\widehat{f(\xi)}) \cos 2\pi x\xi - \Re(\widehat{f(\xi)}) \sin 2\pi x\xi) \end{aligned} \tag{10}$$

$$\begin{aligned} \overline{f}(x) &= \Re(\overline{f}(x)) + \Im(\overline{f}(x)) \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f(\xi)}) - i\Im(\widehat{f(\xi)}))(\cos 2\pi x\xi + i \sin 2\pi x\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f(\xi)}) \cos 2\pi x\xi + \Im(\widehat{f(\xi)}) \sin 2\pi x\xi) \\ &\quad - i(\Im(\widehat{f(\xi)} \cos 2\pi x\xi - \Re(\widehat{f(\xi)}) \sin 2\pi x\xi) \end{aligned} \tag{11}$$

Applying 10 and 11, we get the formula

$$\overline{f}(x) = \int_{-\infty}^{\infty} \overline{\widehat{f(\xi)}} e^{2\pi i x\xi}\, d\xi \tag{12}$$

We write down the formula

$$
\begin{aligned}
\|f\|^2 &= \int_{-\infty}^{\infty} |f(x)|\, dx \\
&= \int_{-\infty}^{\infty} f(x)\overline{f(x)}\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f(\xi)} e^{-2\pi i x\xi}\, d\xi \int_{-\infty}^{\infty} \overline{\widehat{f(\xi')}} e^{2\pi i x\xi'}\, d\xi'\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f(\xi)} e^{-2\pi i x\xi} \overline{\widehat{f(\xi')}} e^{2\pi i x\xi'}\, d\xi\, d\xi'\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i x\xi} e^{2\pi i x\xi'}\, dx \widehat{f(\xi)}\overline{\widehat{f(\xi')}}\, d\xi\, d\xi' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\xi - \xi')\, \widehat{f(\xi)}\overline{\widehat{f(\xi')}}\, d\xi\, d\xi' \\
&= \int_{-\infty}^{\infty} \widehat{f(\xi)}\overline{\widehat{f(\xi)}}\, d\xi \\
&= \|\widehat{f}\|^2
\end{aligned} \tag{13}
$$

□

Remark. *In 13, we used the property that*

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-its}\, ds$$

and

$$\int_{t_1}^{t_2} \delta(t)\, dt = 1 \quad \text{if} \quad 0 \in [t_1, t_2]$$

More details are introduced in [1]

3.2.1 The Weierstrass approximation theorem

Theorem 3.10. *Let f be a continuous function on the closed and bounded interval $[a, b] \subset \mathbb{R}$. Then, $\forall \epsilon > 0$, there exists a polynomial P such that*

$$\sup_{x \in [a,b]} |f(x) - P(x)| < \epsilon$$

In other words, f can be uniformly approximated by polynomials.

Proof. The proof is classic, we first extend f to g in $\mathbb{R}$. Then, we use $g * K_\delta$ to approximate g. Finally, we use Taylor series and choose $R(x)$ as a polynomial to approximate K_δ. An important fact is, $(g * R)(x)$ is a polynomial in x. □

3.2.2 Functions of moderate decrease

In this situation, convolution $f * g$ of two functions of moderate decrease is also a function of moderate decrease. So, we have $\widehat{f * g} = \widehat{f}\widehat{g}$. Moreover, the multiplication formula $\int_{-\infty}^{\infty} f(x)\widehat{g}(x)\ dx = \int_{-\infty}^{\infty} \widehat{f}(y)g(y)\ dy$ continues to hold, and we deduce the Fourier inversion and Plancherel theorem when f and $\widehat{f}$ are both of moderate decrease.

3.3 Applications to some partial differential equations

3.3.1 The time-dependent heat equation on the real line

Consider an infinite rod, which we model by the real line, and suppose that we are given an initial temperature distribution $f(x)$ on the rod at time $t = 0$. We wish now to determine the temperature $u(x, t)$ at a point x at time $t > 0$. From some physical properties, we have the following partial differantial equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{14}$$

called the **heat equation**. The initial condition we impose is $u(x, 0) = f(x)$. Just as in the case of the circle, the solution is given in terms of a convolution. Define the **heat kernel** of the line by

$$\mathcal{H}_t(x) = K_\delta(x), \qquad \text{with } \delta = 4\pi t \tag{15}$$

so that

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \quad \text{and} \quad \widehat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 t\xi} \tag{16}$$

And we have

Theorem 3.11. *Given $f \in \mathcal{S}(\mathbb{R})$, let*

$$u(x, t) = (f * \mathcal{H}_t)(x) \quad \textit{for} \quad t > 0$$

where $\mathcal{H}_t$ is the heat kernel, then:

1. *The function u is C^2 when $x \in \mathbb{R}$ and $t > 0$, and u solves the heat equation.*
2. *$u(x, t) \to f(x)$ uniformly in x as $t \to 0$. Hence if we set $u(x, t) = f(x)$, then u is contiuous on the closure of the upper half-plane $\overline{\mathbb{R}^2_+} = \{(x, t) : x \in \mathbb{R}, t \geq 0\}$.*
3. *$\int_{-\infty}^{\infty} |u(x, t) - f(x)|^2\ dx \to 0$ as $t \to 0$.*

The theorem 3.11 guarantees the existence of a solution to the heat equation with initial data f. This solution is also unique. In this regard, we note that $u = f * \mathcal{H}_t, f \in \mathbb{R}$.

3.3.2 The steady-state heat equation in the upper hald-plane

The equation we are now concerned with is

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{17}$$

in the upper half-plane $\overline{\mathbb{R}^2_+} = \{(x,t) : x \in \mathbb{R}, t \geq 0\}$. The boundary condition we require is $u(x,0) = f(x)$. The kernel that solves the problem is called the **Poisson kernel** for the upper half-plane.

$$\mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \qquad \text{where} \quad x \in \mathbb{R} \text{ and } y > 0$$

Notably, for each fixed y, the kernel $\mathcal{P}_y$ is only of moderate decrease as a function of x. We take the Fourier transform of 17 in the x variable, thereby obtaining

$$-4\pi^2\xi^2\widehat{u}(\xi,y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi,y) = 0$$

by applying proposition 3.1. The general solution of this ordinary differantial equation in y (with ξ fixed) takes the form

$$\widehat{u}(\xi,y) = A(\xi)e^{-2\pi|\xi|y} + B(\xi)^{2\pi|\xi|y}$$

If we disregard the second term because its rapid exponential increase we find, after setting $y = 0$, that

$$\widehat{u}(\xi,y) = \widehat{f}(\xi)e^{-2\pi|\xi|y}$$

Remark. *In the above two examples, we apply Fourier transform in order to find some properties of ordinary differential equations. Normally, it would be a factor of exponent of* e*.*

Lemma 3.12. *The follwing identitis hold:*

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-2\pi|\xi|y}e^{2\pi i\xi x}\, d\xi &= \mathcal{P}_y(x) \\
\int_{-\infty}^{\infty} \mathcal{P}_y(x)e^{-2\pi i\xi x}\, dx &= e^{-2\pi|\xi|y}
\end{aligned}$$

Lemma 3.13. *The Poisson kernel is a good kernel as* $y \to 0$*.*

Lemma 3.14 (**Mean-value property**)**.** *Suppose* ω *is an open set in* $\mathbb{R}^2$ *and let* u *be a function of class* C^2 *with* $\triangle u = 0$ *in* ω*. If the closure of the disc centerned at* (x,y) *and of radius* R *is contained in* ω*, then*

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta$$

for all $0 \leq r \leq R$*.*

Proof. The proof is tricky. First, the equation $\triangle u = 0$ implies

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 u}{\partial \theta^2} &= 0 \\ \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= 0 \end{aligned}$$

If we define $F(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta$, we deduce that

$$r\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = \frac{1}{2\pi}\int_0^{2\pi} -\frac{\partial^2 u}{\partial \theta^2}(r,\theta)\, d\theta$$

Since $\partial u/\partial\theta$ is periodic, we have $r\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) = 0$. And consequently $r\partial F/\partial r$ must be a constant. Hence $\partial F/\partial r = 0$. Since $u(x,y) = F(0)$, we deduce that $F(r) = u(x,y)$. Proof completed. □

Hence we have the follwing theorem

Theorem 3.15. *Given $f \in \mathcal{S}(\mathbb{R})$, let $u(x,y) = (f * \mathcal{P}_y)(x)$. Then:*

1. *$u(x,y)$ is C^2 in $\mathbb{R}^2_+$ and $\triangle u = 0$.*
2. *$u(x,y) \to f(x)$ uniformly as $y \to 0$.*
3. *$\int_{-\infty}^{\infty} |u(x,y) - f(x)|^2\, dx \to 0$ as $y \to 0$.*
4. *If $u(x,0) = f(x)$, then u is contiuous on the closure $\overline{\mathbb{R}^2_+}$, and vanishes at infinity*

$$u(x,y) \longrightarrow 0 \text{ as } |x| + y \to \infty$$

3.4 The Poisson summation formula

This section reveals further connection between Fourier series and Fourier transform.

Given a function $f \in \mathcal{S}(\mathbb{R})$ on the real line, we can construct a new function on the circle by the recipe

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

Since f is rapidly decreasing, the series converges absolutely and uniformly on every compact subset of $\mathbb{R}$, so f is continuous.

We consider another form of F

$$F_2(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

The fundamental fact is that these two approached, which produce F_1 and F_2, actually lead to the same function.

Theorem 3.16 (Poisson summation formula). *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi inx}$$

In particular, setting $x = 0$ we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

Proof. The proof is trivial, but the theorem is very important. We can apply the property of Fourier series, and F_2 is the Fourier series of F_1. □

3.4.1 Theta and zeta functions

In this section, we inroduce some basic properties of theta and zeta functions. We define **theta function** $\vartheta(s)$ for $s > 0$ by

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$$

which satisfies the following equation

Theorem 3.17. $s^{-1/2}\vartheta(1/s) = \vartheta(s)$ *whenever* $s > 0$.

The proof is only a simple application of Poisson summation formula to the pair

$$f(x) = e^{-\pi s x^2} \qquad \text{and} \qquad \widehat{f}(\xi) = s^{1/2} e^{-\pi \xi^2/s}$$

The **Zeta function** $\zeta(s)$ defined for $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

ϑ and ζ have the following property

Theorem 3.18.

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^{\infty} t^{s/2-1}(\vartheta(t) - 1)\, dt$$

Proof. To be updated. □

Another important function $\Theta(z|\tau)$ is defined by

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau} e^{2\pi inz}$$

whenever $\Im(\tau) > 0$ and $z \in \mathbb{C}$. Taking $z = 0$ and $\tau = is$ we get $\Theta(z|\tau) = \vartheta(s)$.

3.4.2 Heat kernel

We define heat kernel on the circle by

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

Heat kernel on $\mathbb{R}$ is given by

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

Corollary 3.18.1. *The kernel $H_t(x)$ is a good kernel as $t \to 0$.*

3.4.3 Poisson kernels

We recall the definition of Poisson kernels

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} \quad \text{and} \quad \mathcal{P}_y = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

hence we have the theorem

Theorem 3.19. $P_r(2\pi x) = \sum_{n\in\mathbb{Z}} \mathcal{P}_y(x+n)$ *where* $r = e^{-2\pi y}$.

3.5 The Heisenberg uncertainty principle

Theorem 3.20. *Suppose ψ is a function is $\mathcal{S}(\mathbb{R})$ which satisfies the normalizing condition $\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1$. Then*

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \xi^2 |\widehat{\psi}(\xi)|^2\, d\xi\right) \geq \frac{1}{16\pi^2}$$

and equality holds iff $\psi(x) = Ae^{-Bx^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$. In fact, we have

$$\left(\int_{-\infty}^{\infty} (x-x_0)^2 |\psi(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} (\xi-\xi_0)^2 |\widehat{\psi}(\xi)|^2\, d\xi\right) \geq \frac{1}{16\pi^2}$$

for every $x_0, \xi_0 \in \mathbb{R}$.

4 The Fourier Transforms on $\mathbb{R}^d$

4.1 Preliminaries

We have $x \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \cdots, \alpha_d)$ of non-negative integers, the monomial x^α is defined by

$$x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}$$

Similarly, we define $(\partial/\partial x)^\alpha$ by

$$\left(\frac{\partial}{\partial x}\right) = \frac{\partial^|\alpha|}{\partial x_1^{\alpha_1} \cdots x_d^{\alpha_d}}$$

and $|\alpha| = \sum_{i=1}^d \alpha_i$.

4.1.1 Symmetries

There three important groups of symmetries in $\mathbb{R}^d$

1. Translations:$x \mapsto x + h$.
2. Dilations:$x \mapsto \delta x$.
3. Rotations:$R : \mathbb{R}^d \mapsto \mathbb{R}^d$ preserves the inner product. Using mathematical language,
$$R(ax + by) = aR(x) + bR(y)$$
for all $x, y \in \mathbb{R}^d$ and $a, b \in \mathbb{R}$.
$$R(x) \cdot R(y) = x \cdot y$$
for all $x, y \in \mathbb{R}^d$.

4.1.2 Integration on $\mathbb{R}^d$

First, we define **rapid decrease** and **moderate decrease** on $\mathbb{R}^d$. A continuous function f is of rapid decrease if

$$\sup_{x\in\mathbb{R}^d} |x|^k |f(x)| < \infty \quad \text{for every} \quad k \in \mathbb{N}^+$$

A continuous function f is of moderate decrease if

$$\sup_{x\in\mathbb{R}^d} |x|^{d+\epsilon} |f(x)| < \infty$$

The interaction of integration with the three important groups of symmetries is as follows: if f is of moderate decrease, then

1. $\int_{\mathbb{R}^d} f(x + h)\, dx = \int_{\mathbb{R}^d} f(x)\, dx$ for all $h \in \mathbb{R}^d$.
2. $\delta^d \int_{\mathbb{R}^d} f(\delta x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx$ for all $\delta > 0$.
3. $\int_{\mathbb{R}^d} f(R(x))\, dx = \int_{\mathbb{R}^d} f(x)\, dx$ for every rotation R.

4.1.3 Schwartz space and radial function

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists all infinitely differentiable functions f on $\mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta f(x)| < \infty$$

Radial function f depend only on $|x|$. In other words, f is a radial if there is a function $f_0(x)$, such that $f(x) = f_0(|x|)$.

4.2 Fourier transform on $\mathbb{R}^d$

The **Fourier transform** of a Schwartz function f si defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,dx$$

and we have proposition

Proposition 4.1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$.*

1. $f(x+h) \longrightarrow \widehat{f}(\xi)e^{2\pi i\xi\cdot h}$ *whenever* $h \in \mathbb{R}^d$.
2. $f(x)2^{-2\pi x\cdot h} \longrightarrow \widehat{f}(\xi+h)$ *whenever* $h \in \mathbb{R}^d$.
3. $f(\delta x) \longrightarrow \delta^{-d}\widehat{f}(\delta^{-1}\xi)$ *whenever* $\delta > 0$.
4. $\left(\frac{\partial}{\partial x}\right)^\alpha f(x) \longrightarrow (2\pi i\xi)^\alpha \widehat{f}(\xi)$.
5. $(-2\pi i x)^\alpha f(x) \longrightarrow \left(\frac{\partial}{\partial\xi}\right)^\alpha \widehat{f}(\xi)$.
6. $f(Rx) \longrightarrow \widehat{f}(R\xi)$.

Similar to Chapter 3, we have two corollaries

Corollary 4.1.1. *The Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ to itself.*

Corollary 4.1.2. *The Fourier transform of a radial function is radial.*

It is also simple to prove Fourier inversion and Plancherel formula on $\mathcal{S}(\mathbb{R}^d)$.

Theorem 4.2. *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)e^{2\pi x\cdot\xi}\,d\xi$$

Moreover

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2\,d\xi = \int_{\mathbb{R}^d} |f(x)|^2\,dx$$

Return to convolution, we also have theorem similar to proposition 3.8.

Theorem 4.3. *Suppose $f, g \in \mathcal{S}(\mathbb{R}^d)$, then*

1. $f * g \in \mathcal{S}(\mathbb{R}^d)$.
2. $f * g = g * f$.
3. $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.
4. $\int_{\mathbb{R}^d} f(x)\widehat{g}(x)\,dx = \int_{\mathbb{R}^d} \widehat{f}(x)g(x)\,dx$.

4.3 The wave equation on $\mathbb{R}^d \times \mathbb{R}$

Wave equation is an important part in physics, and we will introduce the wave equation on $\mathbb{R}^d \times \mathbb{R}$ and its Fourier transform.

4.3.1 Solutions in terms of Fourier transforms

The motion of a vibrating string satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

A natural generalization of this equation to d dimensional space is

$$\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

We now set $c = 1$ to simplify this problem, and it does not change the characteristics of the problem. Also, we define the **Laplacian** in d dimensional space by

$$\triangle = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \tag{18}$$

Hence we get

$$\triangle u = \frac{\partial^2 u}{\partial t^2} \tag{19}$$

The goal of this section is to find a solution to this equation, subject to the initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

where $f, g \in \mathcal{S}(\mathbb{R}^d)$. This is called the **Cauchy problem** for the wave equation. Applying proposition 4.1.4, we deduce that

$$-4\pi^2|\xi|^2\widehat{u} = \frac{\partial^2 u}{\partial t^2}\widehat{u} \tag{20}$$

Because we have known 19, the solution of the ordinary partial equation is

$$\widehat{u} = \widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$$

Theorem 4.4. *A solution of the Cauchy problem for the wave equation is*

$$u(x,t) = \int_{\mathbb{R}^d} \left[\widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right] e^{2\pi i x\cdot\xi}\, d\xi \tag{21}$$

Moreover, the solution is unique.

We define the **energy** of a solution by

$$E(t) = \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}\right|^2 + \sum_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right|^2 \, dx$$

In order to proceed the next deduction, we introduce a lemma.

Lemma 4.5. *Suppose a and b are complex numbers and α is real. Then*

$$|a\cos\alpha + b\sin\alpha|^2 + |-a\sin\alpha + b\cos\alpha|^2 = |a|^2 + |b|^2$$

Now by Plancherel's theorem,

$$\int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}\right|^2 \, dx = \int_{\mathbb{R}^d} \left|-2\pi|\xi|\widehat{f}(\xi)\sin(2\pi|\xi|t) + \widehat{g}(\xi)\cos(2\pi|\xi|t)\right|^2 \, d\xi$$

According to proposition 4.1, we have

$$\frac{\partial u}{\partial x_j} \longrightarrow (2\pi i\xi_j)\widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\frac{i\xi_j\sin(2\pi|\xi|t)}{|\xi|}$$

Applying proposition 4.1 again, we have

$$\int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}\right|^2 \, dx = \int_{\mathbb{R}^d} \left|-2\pi|\xi|\widehat{f}(\xi)\cos(2\pi|\xi|t) + \widehat{g}(\xi)\sin(2\pi|\xi|t)\right|^2 \, d\xi$$

We now apply lemma 4.5, the result is

$$E(t) = \int_{\mathbb{R}^d} (4\pi^2|\xi|^2|\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2) \, d\xi$$

So we have theorem

Theorem 4.6. *Suppose u is the solution of the wave equation given by theorem 4.4, then $E(t)$ is conserved, that is*

$$E(t) = E(0) \qquad \textit{for all } t \in \mathbb{R}$$

4.3.2 The wave equation in $\mathbb{R}^3 \times \mathbb{R}$

If S^2 denotes the **unit sphere** in $\mathbb{R}^3$, we define the **shperical mean** of function f over the shpere of radius t centered at x by

$$M_t(f)(x) = \frac{1}{4\pi}\int_{S^2} f(x - t\gamma) \, d\sigma(\gamma) \tag{22}$$

where $\sigma(\gamma)$ is the element of surface area for S^2. Notably, 4π is the area of the unit sphere.

There are two lemmas

Lemma 4.7. *If $f \in \mathcal{S}(\mathbb{R}^{\nvDash})$ and t is fixed, then $M_t(f) \in \mathcal{S}(\mathbb{R}^3)$. Moreover,$M_t(f)$ is infinitely differentiable in t, and each t-derivative also belongs to $\mathcal{S}(\mathbb{R}^3)$.*

Lemma 4.8.

$$\frac{1}{4\pi}\int_{S^2} e^{-2\pi i\xi\cdot\gamma}\, d\sigma(\gamma) = \frac{\sin(2\pi|\xi|)}{2\pi|\xi|}$$

Applying the above two lemmas, we have

$$\begin{aligned}
\widehat{M_t(f)}(\xi) &= \int_{\mathbb{R}^3} e^{-2\pi i x\cdot\xi}\left(\frac{1}{4\pi}\int_{S^2} f(x-\gamma t)\, d\sigma\gamma\right)\, dx \\
&= \frac{1}{4\pi}\int_{S^2}\int_{\mathbb{R}^3} e^{-2\pi i\gamma t\cdot\xi} f(x-\gamma t) e^{-2\pi i(x-\gamma t)\cdot\xi}\, d(x-\gamma t)\, d\sigma(\gamma) \\
&= \widehat{f}(\xi)\frac{\sin(2\pi|\xi|t)}{s\pi|\xi|t}
\end{aligned}$$

Theorem 4.9. *The solution when $d = 3$ of the Cauchy problem for the wave equation*

$$\triangle u = \frac{\partial^2 u}{\partial t^2} \quad \textit{subject to} \quad u(x,0) = f(x) \quad \textit{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

is given by

$$u(x,t) = \frac{\partial}{\partial t}(tM_t(f)(x)) + tM_t(g)(x)$$

The proof is trivial by applying theorem 4.4. And we should considet $f(x) = 0$ and $g(x) = 0$ seperately.

4.3.3 Huygens principle

The Huygens principle means the value of a point is only affect by its **backward light cone**, and a disturbance of a point only affect its **forward light cone**. The reason is the **finite speed of propagation**.

4.3.4 Radio symmetry and Bessel functions

We have already known that if $f(x) = f_0(|x|)$ for some f_0, then $\widehat{f}(\xi) = F_0(|\xi|)$ for some F_0. A natural problem is to determine a relation between f_0 and F_0. The relation is simple when $d = 1$

$$F_0(\rho) = 2\int_0^\infty \cos(2\pi\rho r) f_0(r)\, dr \tag{23}$$

In the case $d = 3$, the relation is also quite simple

$$F_0(\rho) = 2\rho^{-1}\int_0^\infty \sin(2\pi\rho r) f_0(r)\, dr \tag{24}$$

The **Bessel function** of order $n \in \mathbb{Z}$, denoted by $J_n(\rho)$, is defined as the n^{th} Fourier coefficient of the function $e^{i\rho \sin\theta}$. So

$$J_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\rho \sin\theta} e^{-in\theta}\, d\theta$$

When $d = 2$, we deduce that

$$F_0(\rho) = 2\pi \int_0^{\infty} J_0(2\pi r\rho) f_0(r)\, dr \tag{25}$$

4.4 The randon transform

In engineering, we often record the difference between input light and output light when crossing an object, and reconstruct the object. For example, when X-ray crosses an object, the relation between input light I_0 and output light I is

$$I = I_0 e^{-d\rho}$$

where d is the distance traveled by the beam in the object, and ρ denotes the **attenuation** coefficient (absorption coefficient). If the attenuation is different in the different of the object, we write the formula

$$I = I_0 e^{\int_L \rho}$$

We define **Randon transform** of ρ by

$$X(\rho)(L) = \int_L \rho$$

4.4.1 The Randon transform in $\mathbb{R}^3$

There exists a plane $\mathcal{P}$, and we define the Randon transform $\mathcal{R}(f)$ by

$$\mathcal{R}(f)(\mathcal{P}) = \int_{\mathcal{P}} f$$

To simplity the problem, we assume that $\mathcal{P} \in \mathcal{S}(\mathbb{R}^3)$. Given a unit vector $\gamma \in S^2$ and a number $t \in \mathbb{R}$, we define the plane $\mathcal{P}_{t,\gamma}$ by

$$\mathcal{P}_{t,\gamma} = \{x \in \mathbb{R}^3 : x \cdot \gamma = t\}$$

Obviously, we have the next proposition

Proposition 4.10. *If $f \in \mathcal{S}(\mathbb{R}^3)$, then for each γ, we have*

$$\int_{-\infty}^{\infty} \left(\int_{\mathcal{P}_{t,\gamma}} f \right) dt = \int_{\mathbb{R}^3} f(x)\, dx$$

Notably, **Randon transform** of a function $f \in \mathbb{R}^3$ is defined by

$$\mathcal{R}(f)(t,\gamma) = \int_{\mathcal{P}_{t,\gamma}} f$$

We need to prove Randon transform has two properties:

- **Uniqueness**: If $\mathcal{R}(f) = \mathcal{R}(g)$, then $f = g$.
- **Reconstruction**: Express f in terms of $\mathcal{R}(f)$.

Lemma 4.11. *If $f \in \mathcal{S}(\mathbb{R}^3)$, then $\mathcal{R}(f)(t,\gamma) \in \mathcal{S}(\mathbb{R})$ for each fixed γ. Moreover*

$$\widehat{\mathcal{R}}(f)(s,\gamma) = \widehat{f}(s\gamma)$$

Theorem 4.12. *If $f, g \in \mathcal{S}(\mathbb{R}^3)$ and $\mathcal{R}(f) = \mathcal{R}(g)$, then $f = g$.*

Proof. According to lemma 4.11, we have

$$\begin{aligned}\widehat{\mathcal{R}}(f)(s,\gamma) &= \widehat{f}(s\gamma) \\ \widehat{\mathcal{R}}(g)(s,\gamma) &= \widehat{g}(s\gamma)\end{aligned}$$

Because $\mathcal{R}(f) = \mathcal{R}(g)$, we have

$$\widehat{f} = \widehat{g}$$

Hence $f = g$. □

We next discover the inversion of Randon transform. Given a function F on $\mathbb{R} \times S^2$, we define its **dual Randon transform** by

$$\mathcal{R}^*(F)(x) = \int_{S^2} F(x \cdot \gamma, \gamma)\, d\sigma(\gamma) \tag{26}$$

The another form is

$$\mathcal{R}^*(F)(x) = \int_{\{\mathcal{P}_{t,\gamma} | x \in \mathcal{P}_{t\gamma}\}} F$$

We then state the reconstruction theorem

Theorem 4.13. *If $f \in \mathcal{S}(\mathbb{R}^3)$, then*

$$\triangle(\mathcal{R}^*\mathcal{R}(f)) = -8\pi^2 f$$

5 Finite Fourier Analysis

5.1 Fourier analysis on $\mathbb{Z}(N)$

5.1.1 Abelian group

Here is a aimple introduction to **abelian groups**. An abelian group is a set G with a binary operation $\cdot$, that satisfies

1. Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.
2. Identity: There exists an element $u \in G$ such that $a \cdot u = a$ for all $a \in G$.
3. Inverses: For every $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = u$.
4. Commutativity: For $a, b \in G$, $a \cdot b = b \cdot a$.

A **homomorphism** between two abelian groups in G and H is a map $f : G \to H$ which satisfies

$$f(a \cdot b) = f(a) \cdot f(b)$$

We say that two groups G and H are **isomorphic**, and write $G \approx H$, if there is a bijective homomorphism from G to H. Formally, if $f : G \to H$ and $\widetilde{f} : H \to G$ satisfies

$$(\widetilde{f} \circ f)(a) = a \qquad \text{and} \qquad (f \circ \widetilde{f})(b) = b$$

for every $a \in G, b \in H$,G and H are isomorphic.

5.1.2 The group $\mathbb{Z}(N)$

Let N be a positive integer. A complex number z is an N^{th} root of unity if $z^N = 1$. The set is denoted by $\mathbb{Z}(N)$. Moreover, the elements of set $\mathbb{Z}(N)$ is

$$\left\{1, e^{2\pi i/N}, \cdots, e^{2\pi i(N-1)/N}\right\}$$

Obviously, $\mathbb{Z}(N)$ is an abelian group. Then we introduce another abelian group, which is called the **group of integers modulo** N, denoted by $\mathbb{Z}/N\mathbb{Z}$. According to 5.1.1, we know $\mathbb{Z}(N)$ and $\mathbb{Z}/N\mathbb{Z}$ are isomorphic.

5.1.3 Fourier inversion and Plancherel identity on $\mathbb{Z}(N)$

We now define an important homomorphism from $\mathbb{Z}/N\mathbb{Z}$ to $\mathbb{Z}(N)$, which denoted by e_n. $e_n(x) = e^{2\pi inx/N}$. Similar to inner product defined on functions on $\mathbb{R}$. The Hermitian inner product

$$(F, G) = \sum_{k=0}^{N-1} F(k)\overline{G(k(}$$

on vector space V, where V is the set of homomorphism from $\mathbb{Z}/N\mathbb{Z}$ to $\mathbb{Z}(N)$. We can easily get the lemma

Theorem 5.1. *The family $\{e_0, \cdot, e_{N-1}\}$ is orthogonal.*

$$(e_m, e_l) = \begin{cases} N, & \text{if } m = l \\ 0, & \text{if } m \neq l \end{cases}$$

So, we define Fourier coefficient of F by

$$a_n = \frac{1}{N}\sum_{k=0}^{N-1} F(k)e^{-2\pi ikn/N}$$

and the Fourier inversion is

Theorem 5.2. *If F is a function on $\mathbb{Z}(N)$, then*

$$F(k) = \sum_{k=0}^{N-1} a_n e^{2\pi ink/N}$$

Moreover,

$$\sum_{n=0}^{N-1} |a_n|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |F(k)|^2$$

5.1.4 Fast Fourier transform

The fast Fourier transform is a method that was developed as a means of calculating coefficiently the Fourier coefficients of a function F on $\mathbb{Z}(N)$. We begin with a naive approach to the problem. Fix N, and suppose that we are given $F(0), \ldots, F(N-1)$ and $\omega_N = e^{-2\pi i/N}$. If we denote by $a_k^N(F)$ the k^{th} Fourier coefficient of F on $\mathbb{Z}(N)$, then by definition

$$a_k^N(F) = \frac{1}{N}\sum_{r=0}^{N-1} F(r)\omega_N^{kr} \tag{27}$$

To develop an algorithm to calculate Fourier coefficients coefficiently, we first need to prove a lemma.

Lemma 5.3. *If we are given $\omega_{2M} = e^{-2\pi i/(2M)}$, then*

$$\#(2M) \le 2\#(M) + 8M$$

where $\#(M)$ denotes the minimum operations needed to calculate all the Fourier coefficients of any functions on $\mathbb{Z}(M)$.

Proof. First, The calculation of $\omega_{2M}^0, \ldots, \omega_{2M}^{2M-1}$ requires no more than $2M$ operations. Then, we calculate $a_k^{2M}(F)$ by

$$\begin{aligned} a_k^{2M}(F) &= \frac{1}{2M}\sum_{r=0}^{2M-1} F(r)\omega_{2M}^{kr} \\ &= \frac{1}{2}\left(\frac{1}{M}\sum_{i=0}^{M-1} F(2i)\omega_{2M}^{k(2i)} + \frac{1}{M}\sum_{j=0}^{M-1} F(2j+1)\omega_{2M}^{k(2j+1)}\right) \\ &= \frac{1}{2}\left(\frac{1}{M}\sum_{i=0}^{M-1} F_0(i)\omega_M^{ki} + \frac{1}{M}\sum_{j=0}^{M-1} F_1(j)\omega_M^{kj}\omega_M^{k}\right) \end{aligned}$$

where we denote $F_0(x) = F(2x), F_1(x) = F(2x+1)$. Hence we have the formula

$$a_k^{2M}(F) = \frac{1}{2}\left(a_k^M(F_0) + a_k^M(F_1)\omega_{2M}^k\right)$$

Knowing $a_k^M(F_0), a_k^M(F_1), \omega_{2M}^k$, we see that $a_k^{2M}(F)$ can be computed using no more than three operations. So, the lemma has been proved. □

We also need to calculate the initial item $a_0^2(F), a_1^2(F)$

$$a_0^N(F) = \frac{1}{2}(F(0) + F(1)) \quad \text{and} \quad a_1^N(F) = \frac{1}{2}(F(0) - F(1))$$

In conclusion, we deduce the theorem

Theorem 5.4. *It is possible to calculate the Fourier coefficients of a function on $\mathbb{Z}(N)$ with at most $O(N \log N)$ operations.*

5.2 Extension of finite Fourier analysis

5.2.1 The group $\mathbb{Z}^*(q)$

Let q be a positive integer. An iteger $n \in \mathbb{Z}(q)$ is a **unit** if there exists an integer $m \in \mathbb{Z}(q)$ so that

$$nm \equiv 1 \qquad \text{mod } q$$

The set of all units in $\mathbb{Z}(q)$ is denoted by $\mathbb{Z}^*(q)$.

5.2.2 Characters

Let G be a finite abelian group and S^1 the unit circle in the complex plane. A **Characters** on G is a complex-valued function $e : G \to S^1$ which satisties the following condition:

$$e(a \cdot b) = e(a)e(b) \quad \text{for all} \quad a, b \in G \tag{28}$$

$e(a) = 1$ is called **trivial** or **unit character**.

If G is a finite abelian group, we denote by $\widehat{G}$ the set of all characters of G, and observe next this inherits the structure of an abelian group.

Lemma 5.5. *The set $\widehat{G}$ is an abelian group under multiplication defined by*

$$(e_1 \cdot e_2)(a) = e_1(a)e_2(b)$$

Lemma 5.6. *Let G be a finite abelian group, and $e :\to \mathbb{C} - \{0\}$ a multiplicative function, namely $e(a \cdot b) = e(a)e(b)$ for all $a, b \in G$. Then e si a character.*

5.2.3 The orthogonality relations

Let V denote the vector space of complex-valued functions defined on the finite abelian group G. Note that the dimension of V is $|G|$. (this proof is kind of complicated) We define a Hermitian inner product on V by

$$(f,g)=\frac{1}{|G|}\sum_{a\in G}f(a)\overline{g(a)} \quad \text{whenever} \quad f,g\in V$$

Hence we have two theorems

Theorem 5.7. *The characters of G form an orthonormal family with respect to the inner product defined above.*

Theorem 5.8. *If e is a non-trivial character of the group G, then $\sum_{a\in G}e(a)=0$.*

5.2.4 Fourier analysis on abelian groups

Given a function f on an abelian group G and character e of G, we define the Fourier coefficient of f with respect to e, by

$$\widehat{f}(e)=(f,e)=\sum 1|G|\sum_{a\in G}f(a)\overline{e(a)}$$

and the Fourier series of f as

$$f\sim\sum_{e\in\widehat{G}}\widehat{f}(e)e$$

Since the characters form a basis, we have

$$f=\sum_{e\in\widehat{G}}c_e e$$

And we have two theorem

Theorem 5.9. *Ler G be a finite abelian group. The vector space V of functions on G has inner product defined by*

$$(f,g)=\frac{1}{|G|}\sum_{a\in G}f(a)\overline{g(a)}$$

In particular, any functions f on G is equal to its Fourier series

$$f\sim\sum_{e\in\widehat{G}}\widehat{f}(e)e$$

Theorem 5.10 (**Plancherel identity**). *If f is a function on G, then $\|f\|^2=\sum_{e\in\widehat{G}}|\widehat{f}(e)|^2$.*

References

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