1 An Introduction to Complex Field

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[1]Thanks to my family, my friend and freedom.

1 Complex Set

The set of all complex numbers is denoted by $\mathbb{C}$, which is similar to $\mathbb{R}^2$, but with some difference. We omit the definition and operation of complex numbers and only introduce the some important properties.

1.1 Cauchy sequence in $\mathbb{C}$

Completeness is essential for a metric space such that we can do operations. A metric space is complete iff Its Cauchy sequence converge. It is trivial to prove that every Cauchy sequence in $\mathbb{C}$ converge. We have thus the following result

Theorem 1.1. $\mathbb{C}$ *is complete.*

1.2 Sets in the comples plane

We first define the **open disc** $D_r(z_0)$ **of radius** r **centered at** z_0 by

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

Then we define **closed disc** $\overline{D}_r(z_0)$ **radius** r **centered at** z_0 by

$$\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

and the boundary of either the open or closed disc is the circle

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

Since the **unit disc** plays an important role in complex analysis, we often denote it by $\mathbb{D}$,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Gigen a set $\Omega \subset \mathbb{C}$, z_0 is an **interior point** of Ω if there exists $r > 0$ such that

$$D_r(z_0) \subset \Omega$$

We can also define the **interior** of a set by all its interior points.

Now we introduce the most critical concepts in this section: open set and closed set. A set Ω is open if $\forall z \in \Omega, \exists r > 0$ such that $D_r(z) \in \Omega$. A set Ω is closed if its complement $\Omega^c = \mathbb{C} - \Omega$ is open.

A point $z \in \mathbb{C}$ is a **limit point** of the set Ω if there exists a sequence of points $\{z_n\} \in \Omega$ such that $z_n \neq z$ and $\lim_{n\to\infty} z_n = z$. Then we ca define the **closure** of a set Ω by $\Omega \cup \Omega'$ where Ω' is the set of all limit points of Ω.

With definitions above, we define the **boundary** of a set Ω by its closure minus its interior.

A set Ω is bounded if there exists $M > 0$ such that $|z| < M$ whenever $z \in \Omega$. If Ω is bounded, we define its **diameter** by

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|$$

A set Ω is **compact** if it is closed and bounded. Arguing as in the case of real variables, one can prove the following

Theorem 1.2. *The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.*

An **open covering** of Ω is a family of open sets $\{U_\alpha\}$ such that

$$\Omega \subset \bigcup_{\alpha} U_\alpha$$

In analogy with the situation in $\mathbb{R}$, we have the following equivalent formulation of compactness.

Theorem 1.3. *A set Ω is compact if and only if every open covering of Ω has a finite subcovering.*

Proposition 1.3.1. *If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with property that*

$$diam(\Omega_n) \longrightarrow 0 \qquad as \quad n \to \infty$$

then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

The last notion is that of connectedness. An open set $\Omega \subset \mathbb{C}$ is said to be **connected** if it is not possible to find two disjoint non-empty open sets Ω_1 anf Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2$$

We also call the connected open set in $\mathbb{C}$ **region**. Similarly, we can define the connectedness of a closed set.