

Fourier Analysis Notes

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¹Thanks to my family, my friend and freedom.

Abstract

This is a note about my understanding of Fourier Analysis This note won't include the last 2 part of Stein's Fourier Analysis[2]. Because study of number theory is not essential for me. I would probably study the last 2 chapters later.

1 Introduction to Fourier Series

Fourier Series are the very basis of Fourier Analysis. Following the Stein's instruction, I will start my note at Fourier Series. Fisrt, this note will introduce some basic concepts.

1.1 Basic concepts

1.1.1 Riemann integrable functions

A real-valued functions defined on $[0, L]$ is **Riemann integrable** if it is bounded and if $\forall \epsilon > 0, \exists 0 = x_0 < \dots < x_N = L$, the subdivision satisfies following conditions. We denote \mathcal{U} and \mathcal{L} as the upper and lower sums of this subdivision.

$$\mathcal{U} = \sum_{j=1}^N [\sup_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

$$\mathcal{L} = \sum_{j=1}^N [\inf_{x_{j-1} \leq x \leq x_j} f(x)](x_j - x_{j-1})$$

then we have $\mathcal{U} - \mathcal{L} < \epsilon$.

1.1.2 Functions on the circle

Obviously, we can treat a 2π -periodic function $f \in \mathbb{R}$ as a function F on the circle.

$$f(\theta) = F(e^{i\theta})$$

and naturally, $f(0) = f(2\pi) \Rightarrow F(0) = F(e^{2\pi i})$

1.1.3 Fourier coefficient and series

For a function defined on $[a, b]$ and $b - a = L$, the n^{th} **Fourier coefficient** of f is defined by

$$\widehat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x / L} dx, \quad n \in \mathbb{Z}$$

The **Fourier series** of f is given by

$$\sum_{n=-\infty}^{n=+\infty} \widehat{f}(n) e^{2\pi i n x / L}$$

Sometimes we denote a_n as $\widehat{f}(n)$, and we use this notation

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} a_n e^{2\pi i n x / L}$$

When $L = 2\pi$, we can denote it by

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} a_n e^{2in\theta}$$

1.1.4 Trigonometric series and partial sum

Trigonometric series can be expressed as $\sum_{n=-\infty}^{n=+\infty} c_n e^{2\pi i n x / L}$, $c_n \in \mathbb{C}$. If for all large $|n|$, we get $c_n = 0$, it is called **Trigonometric polynomial**, its degree is the largest $|n|$ that $c_n \neq 0$.

For $N > 0$, the N^{th} **partial sum** are expressions of the form

$$S_N(f)(x) = \sum_{n=-N}^{n=N} \hat{f}(n) e^{2\pi i n x / L}$$

1.1.5 Euler's identity

Euler's identity is a useful formula in analysis, it connects e and trigonometric functions.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

1.1.6 Jacobian matrix

Jacobian matrix of a vector-valued function $\{f_n\}$ is the matrix of all its first-order partial derivatives. It is defined by

$$\begin{aligned} \mathbf{J}(x_1, \dots, x_n) &= \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \end{aligned}$$

Jacobian is often used in multiple integration, we can change variables in multiple integration. For example, $x = f_x(a, b)$, $y = f_y(a, b)$, $F = (x, y)$, the integration on domain D is

$$\int \int_D F(x, y) dx dy = \int \int_T F(f_x(a, b), f_y(a, b)) \left| \frac{\partial(x, y)}{\partial(a, b)} \right| da db$$

where T is the domain of f that $f(a, b) \in D$.

1.2 Convolutions and Kernels

There are types of functions. First, we introduce the convolution of 2π -periodic integrable functions f and g on \mathbb{R} , we define their **convolution** $f * g$ on $[-\pi, \pi]$ by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy$$

We can also change variables $t = x - y$ to get

$$\begin{aligned}
 (f * g)(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x-y)dy \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)\frac{dy}{dt}dt \\
 &= -\frac{1}{2\pi} \int_{\pi}^{-\pi} f(x-t)g(t)dt \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)g(t)dt
 \end{aligned}$$

In the above equations, g is called the **kernel** of the convolution. Convolutions have several properties.

Proposition 1.1. *Suppose that f, g and h are 2π -periodic integrable functions. Then:*

1. $f * (g + h) = (f * g) + (f * h)$
2. $(cf) * g = c(f * g) = f * (cg), \quad \forall c \in \mathbb{C}$
3. $f * g = g * f$
4. $(f * g) * h = f * (g * h)$
5. $f * g$ is continuous
6. $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$

Proof. Obviously, properties 1, 2, 3 are trivial. To prove 4, we write down

$$\begin{aligned}
 4\pi^2(f * g) * h(z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)g(x-y)h(z-x)dy \, dx \\
 4\pi^2 f * (g * h)(z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)g(x)h(z-x-y)dx \, dy \\
 &= \int_{-\pi}^{\pi} \int_{y-\pi}^{y+\pi} f(y)g(x-y)h(z-x)dx \, dy \\
 &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y)g(x-y)h(z-x)dx \, dy
 \end{aligned}$$

If g is continuous, the proof of 5 is trivial. We can also construct a series of continuous functions to approximate g , hence we have proved 5 for merely integrable function g .

Property 6 is the most critical property. For continuous g and f , we can just calculate and reformulate the equation. For merely integrable functions, we can apply the same technique used in proof of property 5. \square

1.3 Uniqueness of Fourier series

It is critical to prove the uniqueness of fourier coefficients of a function. More precisely, if f and g have same fourier coefficients, then f and g are necessarily equal. To prove 2 items are equal, a useful technique is to prove their difference equals 0.

Theorem 1.2. *Suppose that f is an integrable function on the circle with $\hat{f} = 0, \forall n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0 .*

Proof. First we suppose that f is real-valued and defined on $[-\pi, \pi]$. We will argue this by contradiction. Without loss of generality, we assume that $\theta = 0$, $f(0) > 0$ and f is continuous at 0.

The idea is to construct a family of trigonometric polynomials $\{p_k\}$ that 'peak' at 0, and $\int p_k(\theta)f(\theta)d\theta \rightarrow \infty$ as $k \rightarrow \infty$. This will be our contradiction since these integrals are equal to zero by assumption. (because if $\int p_k(\theta)f(\theta) \neq 0$, there exists $\hat{f}(n) \neq 0$)

In this proof, we construct function $p(\theta) = \epsilon + \cos \theta$ and $p_k(\theta) = [p(\theta)]^k$, and calculate 3 intervals respectively. \square

The next corollary is a deduced from the theorem 1.2.

Corollary 1.2.1. *Suppose that f is a continuous function on the circle and the Fourier series is absolutely convergent. Then the Fourier series converges uniformly to f .*

Corollary 1.2.2. *Suppose that f is a twice continuously differentiable function on the circle. Then*

$$\hat{f}(n) = O(1/|n|^2), \quad \text{as } |n| \rightarrow \infty$$

Proof. To prove the corollary, we integrate by parts

$$\begin{aligned} 2\pi \hat{f}(n) &= \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta \\ &= \left[f(\theta) \cdot \frac{-e^{-in\theta}}{in} \right]_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{(in)^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \\ &= -\frac{1}{n^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \end{aligned}$$

The rest part of proof is trivial. \square

Incidentally, we have got an important identity when $n \in \mathbb{Z}$

$$\widehat{f'}(n) = in\widehat{f}(n)$$

When $n \neq 0$, proof is given in the proof of corollary 1.2.2. When $n = 0$, $\int_0^{2\pi} f'(\theta) d\theta = 0$ since f is 2π -periodic function.

Since $f^{(m)}(n) \sim \sum a_n (in)^m e^{in\theta}$. Further smoothness conditions on f imply even better decay of the Fourier coefficients.

There are also stronger version of corollary 1.2.2.

Theorem 1.3. *The Fourier series converges absolutely if f satisfies a **Holder condition** of order α , with $\alpha > 1/2$*

$$\sup_{\theta} |f(\theta + t) - f(\theta)| \leq A|t|^\alpha$$

Proof. To be updated. □

1.4 Good kernels

We have introduced the notion of kernel in section 1.2. Now we introduce the notion of **good kernels** if a family of kernels satisfies the following properties:

Proposition 1.4. *Properties of good kernels:*

1. For all $n \geq 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$$

2. There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)| dx \leq M$$

3. $\forall \delta > 0$,

$$\int_{\delta < |x| < \pi} |K_n(x)| dx \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

Good kernels are important for their properties. The next theorem is deduced from those properties.

Theorem 1.5. *Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, and f an integrable function on the circle. Then*

$$\lim_{n \rightarrow \infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x . If f is continuous everywhere, then the above limit is uniform.

Proof. Using the property 2 and 3, the proof of the equation is trivial.

The second assertion depends on the periodic property. Since f is continuous everywhere and f is periodic, then f is uniformly continuous. Hence δ can be chosen independently. So $f * K_n \rightarrow f$ uniformly. □

1.5 Some special kernels

In this section, we will introduce some special kernels.

1.5.1 Dirichlet kernel

The N^{th} **Dirichlet kernel** is defined by

$$D_N(x) = \sum_{n=-N}^N e^{inx}$$

Another form is

$$D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})}$$

The proof of the second form is not critical, we omit it. A critical fact is that Dirichlet kernel is not a good kernel. The proof is trivial, we omit it (calculate integrals for each item).

1.5.2 Cesaro means and summation

We define a series $\{c_n\}$, and partial sum $s_n = \sum_{k=0}^n c_k$. Since $\{c_n\}$ may not converge, we need a new series to measure the convergence. We define

$$\delta_N = \frac{\sum_{k=0}^{N-1} s_k}{N}$$

which is called the **Cesaro mean** of $\{s_n\}$, or the **Cesaro sum** of $\{c_n\}$. if δ_N converges to a limit, we say that $\sum c_n$ is **Cesaro summable**. It is easy to check that sum and Cesaro sum have the same limit, the proof is trivial.

1.5.3 Fejer's theorem

Fejer's kernel is defined by

$$F_N(x) = \frac{\sum_{n=0}^N D_n(x)}{N}$$

Lemma 1.6.

$$F_N(x) = \frac{\sin^2(Nx/2)}{N \sin^2(x/2)}$$

The Fejer kernel is a good kernel.

Proof. The proof of the first assertion is to be updated.

The proof of the second assertion is trivial since $F_N(x) > 0$. □

Applying the Theorem 1.5, we get the following theorem.

Theorem 1.7. *If f is integrals on circle, then Fourier series of f is Cesaro summable to f at every point of continuity of f .*

Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Cesaro summable to f .

Corollary 1.7.1. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

1.5.4 Abel means and Abel summation

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be **Abel summable** to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \rightarrow 1} A(r) = s$$

A critical fact is that a Cesaro summable series is always Abel summable, however, an Abel summable series is not necessarily Cesaro summable.

1.6 Poisson kernel

Poisson kernel is defined by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

and the convolution of Poisson kernel is defined by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} = (f * P_r)(\theta)$$

Lemma 1.8. *If $0 \leq r < 1$, then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}$$

The Poisson kernel is a good kernel, as $r \rightarrow 1$

Theorem 1.9. *Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f . If f is integrals on circle, then Fourier series of f is Abel summable to f at every point of continuity of f .*

The proof is trivial, we omit it. But it is critical to notice that the Poiss kernel is indexed by r , not n .

Remark. An important hint deduced from the above theorems is that when we calculate polynomials like $\sum \cos nx$, we can transform the polynomial to the form of e^{in} according to Euler's formula. Then we can apply the formula $\sum_{n=0}^{n=N} a^n = \frac{1-a^{n+1}}{1-a}$.

In the following theorem, we apply Poisson kernel to steady heat equation.

Theorem 1.10. Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by Poisson integral

$$u(r, \theta) = (f * P_r)(\theta)$$

has the following properties:

1. u has continuous derivatives in the unit disc and satisfies $\delta u = 0$
2. If θ is any point of continuity of f , then

$$\lim_{r \rightarrow 1} u(r, \theta) = f(\theta)$$

3. If f is continuous everywhere, then this limit is uniform.

Proof. To prove 1, we need to review the definition of **Laplacian**

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and we get the following relationship between x, y, r, θ

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan \frac{y}{x} \\ \frac{\partial r}{\partial x} &= \frac{x}{r} \\ \frac{\partial r}{\partial y} &= \frac{y}{r} \\ \frac{\partial^2 r}{\partial x^2} &= -\frac{x}{r^3} \\ \frac{\partial^2 r}{\partial y^2} &= -\frac{y}{r^3} \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} \\ \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} \\ \frac{\partial^2 \theta}{\partial x^2} &= \frac{2xy}{r^4} \\ \frac{\partial^2 \theta}{\partial y^2} &= -\frac{2xy}{r^4} \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2} \left(\frac{\partial r}{\partial x} \right)^2 + \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2}$$

$$\begin{aligned} \left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 &= 1 \\ \frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 r}{\partial^2 y} &= \frac{1}{r} \\ \left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 &= \frac{1}{r^2} \\ \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} &= 0 \end{aligned}$$

Then, we get the Laplacian in form of r and θ .

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

and we calculate the formula term by term

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= r^{|m|-2} |m| (|m| - 1) e^{im\theta} \\ \frac{1}{r} \frac{\partial u}{\partial r} &= r^{|m|-2} |m| e^{im\theta} \\ \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= r^{|m|-2} |m|^2 e^{im\theta + \pi} \\ &= -r^{|m|-2} |m|^2 e^{im\theta} \\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} &= 0 \end{aligned}$$

The property 1 has been proved.

Because Poisson kernel is a good kernel, the proof of property 2 is trivial.

The proof of 3 is heuristic. We suppose function v solves the steady heat equation. v has a Fourier series

$$\sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \text{ where } a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v e^{-in\theta} d\theta$$

Taking into account that v solves

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

We get

$$a_n''(r) + \frac{1}{r} a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0$$

Applying principles of second ordinary differential equation. We have $a_n(r) = A_n r^n + B_n r^{-n}$ for constant A_n and B_n . Because r^{-n} is not bounded when $r < 1$, we deduce that $B_n = 0$. Since v converges uniformly to f as $r \rightarrow 1$ we find that

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

Taking the above equation into $a_n(r)$, 3 has been proved. \square

Remark. $(e^{ix})' = e^{i(x+\pi/2)} = ie^{ix}$

Remark. There are 2 principles to judge whether the inner product of 2 series converge.

- *Abel's test:* If $\{a_n\}$ is monotonic and bounded, $\{\sum_{n=0}^{\infty} b_n\}$ converges. We have $\sum_{n=1}^{\infty} a_n b_n$ converges.
- *Dirichlet's test:* If $\{a_n\}$ decreases monotonically to 0, and $\sum b_n$ are bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Remark. class C_k means the function $f \in C_k$ has 1st to k -th derivatives and their are all continuous. f is called k -th continuously differentiable.

2 Convergence of Fourier Series

In this chapter, we continue to study convergence of Fourier series.

The first viewpoint is "global" and concerns the overall behavior of a function f over the $[0, 2\pi]$. If f is integrable on the circle, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \rightarrow 0 \text{ as } N \rightarrow \infty$$

The second viewpoint is "local" and concerns the behavior of f near a given point. Whether

$$S_N(f)(\theta) \rightarrow f(\theta)$$

2.1 Vector spaces and inner products

We have already been familiar with vector spaces. We only review the definition of **inner product** here. An inner product on a vector space V over \mathbb{R} associates to any pair X, Y of elements in V a real number which we denote by (X, Y) . The inner product must be symmetric $(X, Y) = (Y, X)$ and linear in both variables

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z)$$

whenever $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in V$. Also, inner product must be positive-definite, $(X, X) \geq 0$. And we define the norm of X by

$$\|X\| = (X, X)^{1/2}$$

Additionally, $\|X\| = 0$ implies $X = 0$.

A classic *inner product space* is \mathbb{R}^d , inner product is defined by

$$(X, Y) = x_1 y_1 + \cdots + x_d y_d$$

For vector spaces over complex numbers, the inner product of two elements is a complex number. Moreover, these inner products are called Hermitian since they satisfy $(X, Y) = \overline{(Y, X)}$. Hence the inner product is linear in the first variable, but conjugate-linear in the second:

$$\begin{aligned} (\alpha X + \beta Y, Z) &= \alpha(X, Z) + \beta(Y, Z) \\ (Z, \alpha X + \beta Y) &= \overline{\alpha}(Z, X) + \overline{\beta}(Z, Y) \end{aligned}$$

For example, the inner product of two elements Z, W in \mathbb{C}^d is defined by

$$(Z, W) = z_1 \overline{w_1} + \cdots + z_d \overline{w_d}$$

If two elements X, Y are **orthogonal**, then $(X, Y) = 0$, and we write $X \perp Y$. Three critical results can be derived from the notion of orthogonality.

1. The Pythagorean theorem: if X, Y are orthogonal, then

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$$

2. The Cauchy-Schwarz inequality: for any $X, Y \in V$ we have

$$|(X, Y)| \leq \|X\| \|Y\|$$

3. The triangle inequality: for any $X, Y \in V$ we have

$$\|X + Y\| \leq \|X\| + \|Y\|$$

Proof. The proof of 1 is trivial.

The proof of 2 is here: We assume $Y = kX + Z$, where $Z \in V, X \perp Z$. Hence we have

$$\begin{aligned} |(X, Y)| &= |(X, kX + Z)| \\ &= \|X\|^2 |k| + |(X, Z)| \\ &= |k| \|X\|^2 \end{aligned}$$

$$\|X\| \cdot \|Y\| = \|X\| \cdot \|kX + Z\|$$

Obviously, $\|kX + Z\| \geq |k| \cdot \|X\|$, 2 is proved. The proof of 3 is also trivial, we omit it. \square

2.1.1 Infinite dimensional vector spaces

The vector space $\ell^2(\mathbb{Z})$ over \mathbb{C} is the set of infinite sequences of complex numbers

$$(\cdots, a_{-n}, \cdots, a_0, \cdots, a_n, \cdots)$$

such that

$$\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty$$

the sequence converges.

Remark. To measure infinity, we often use limitation.

In the three examples $\mathbb{R}^d, \mathbb{C}^d, \ell^2(\mathbb{Z})$, the vector spaces with their inner products and norms satisfy two important properties:

1. The inner product is strictly positive-definite, $\|X\| = 0$ implies $X = 0$.
2. the vector space is **complete**, which by definition means that every Cauchy sequence in the norm converges to a limit.

An inner product space with 1 and 2 is a **Hilbert space**. We see that \mathbb{R}^d and \mathbb{C}^d are examples of finite-dimensional Hilbert spaces, while $\ell^2(\mathbb{Z})$ is an example of an infinite-dimensional Hilbert space. If either of 1 and 2 fail, the space is called **pre-Hilbert space**.

We give an important example of a pre-Hilbert space where both 1 and 2 fail.

2.1.2 An example of pre-Hilbert space

Let \mathcal{R} denote the set of complex-valued Riemann integrable functions on $[0, 2\pi]$. This is a vector space over \mathbb{C} . Addition is defined pointwise by

$$(f + g)(\theta) = f(\theta) + g(\theta)$$

Naturally, multiplication by a scalar $\lambda \in \mathbb{C}$ is given by

$$(\lambda f)(\theta) = \lambda \cdot f(\theta)$$

An inner product is defined on this vector space by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \overline{g(\theta)} d\theta$$

The norm of f is

$$\|f\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta \right)^{1/2}$$

In \mathcal{R} , 1 fails, we have a simple example

$$f(\theta) = \begin{cases} 0, & \text{if } \theta \neq 0 \\ 1, & \text{if } \theta = 0 \end{cases}$$

where $\|f\| = 0$ but $f \neq 0$. But it does not matter since the points $f(\theta) \neq 0$ have measure 0.

Another fatal fact is that \mathcal{R} is not complete. We have function

$$f(\theta) = \begin{cases} 0, & \text{for } \theta = 0 \\ \log(1/\theta), & \text{for } 0 < \theta \leq 2\pi \end{cases}$$

Since f is not bounded, f is not integrable and $f \notin \mathcal{R}$. Moreover, the sequence of truncations f_n defined by

$$f_n(\theta) = \begin{cases} 0, & \text{for } 0 \leq \theta \leq 1/n \\ f(\theta), & \text{for } 1/n < \theta \leq 2\pi \end{cases}$$

can easily be seen to form a Cauchy sequence in \mathcal{R} . Further discussion of these problems remain in Real Analysis.

Remark. In 2.1.2, f is the limit of f_n , but f is not Riemann integrable. Maybe f is Lebesgue integrable?

2.2 Proof of mean-square convergence

Consider the space \mathcal{R} mentioned above, we need to prove

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta$$

With this notation, we must prove $\|f - S_N(f)\| \rightarrow 0$ as $N \rightarrow \infty$.

For $n \in \mathbb{Z}$, let $e_n(\theta) = e^{in\theta}$, and observe that family $\{e_n\}_{n \in \mathbb{Z}}$ is **orthonormal**

$$(e_n, e_m) = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

Let f be an integrable function on the circle, we have

$$(f, e_n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta = a_n$$

Because $\{e_n\}_{n \in \mathbb{Z}}$ is orthonormal, we have

$$(f - S_N(f)) \perp \sum_{|n| \leq N} b_n e_n \quad (1)$$

for any complex number b_n . Applying Pythagorean theorem to 1, and we have

$$\|f\|^2 = \|f - \sum_{|n| \leq N} a_n e_n\|^2 + \|\sum_{|n| \leq N} a_n e_n\|^2$$

Since the orthonormal property of the family $\{e_n\}$ implies that

$$\|\sum_{|n| \leq N} a_n e_n\|^2 = \sum_{|n| \leq N} |a_n|^2$$

we deduce that

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n| \leq N} |a_n|^2 \quad (2)$$

Lemma 2.1 (Best approximation). *If f is integrable on the circle with Fourier coefficients a_n , then*

$$\|f - S_N(f)\| \leq \|f - \sum_{|n| \leq N} c_n e_n\|$$

the equality holds iff $c_n = a_n$ for all $|n| \leq N$.

Proof. Obviously, we have

$$f - \sum_{|n| \leq N} c_n e_n = f - S_N(f) + \sum_{|n| \leq N} b_n e_n$$

where $b_n = a_n - c_n$. Simply applying Pythagorean theorem, the lemma is proved. \square

Suppose that f is continuous on the circle. Then, given $\epsilon > 0$, there exists a trigonometric polynomial P , say of degree M

$$|f(\theta) - P(\theta)| < \epsilon \text{ for all } \theta$$

Inparticular, taking squares and integrating the inequality yields $\|f - P\| < \epsilon$, and by the best approximation lemma we have

$$\|f - S_N(f)\| < \epsilon \text{ whenever } N \geq M$$

For merely integrable function f , we need to use the lemma

Lemma 2.2. *Suppose f is integrable on the circle and bounded by B . Then there exists a sequence $\{f_k\}_{k=1}^\infty$ of continuous functions*

$$\sup_{x \in [-\pi, \pi]} |f_k(x)| \leq B$$

and

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)| dx \rightarrow 0 \text{ as } k \rightarrow \infty$$

Applying Lemma 2.2, we get

Theorem 2.3. *Let f be an integrable function on the circle with $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Then we have*

1. Mean-square convergence of the Fourier series

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \rightarrow 0 \text{ as } N \rightarrow \infty$$

2. Parseval's identity

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta$$

Remark. *If $\{e^n\}$ is nay orthonormal family of functions on the circle, and $a_n = (f, e_n)$, then we deduce*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2$$

*This is known as **Bessel's inequality**. Equality holds precisely when the family $\{e_n\}$ is also a "basis".*

Remark. *There exist sequences $\{a_n\}$ such that $\sum |a_n| < \infty$, yet no Riemann integrable function F has Fourier coefficients equals to a_n for all n . To be updated.*

Theorem 2.4 (Riemann-Lebesgue lemma). *If f is integrable on the circle, then $\hat{f}(n) \rightarrow 0$ as $|n| \rightarrow \infty$.*

2.2.1 Review

To prove global convergence of integrable function f on the circle. We have several steps.

1. Applying lemma 2.2 to find function g to approximate f .
2. For continuous function g , applying corollary 1.7.1 to get trigonometric polynomial P to approximate g .
3. Applying best approximation, because $\{e_n\}$ is a basis, we have proved theorem 2.3

Sometimes, to calculate Fourier coefficients of intricate functions, we decompose the function to simpler functions.

Lemma 2.5. Suppose F and G are integrable on the circle with

$$F \sim \sum a_n e^{in\theta} \quad \text{and} \quad G \sim \sum b_n e^{in\theta}$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} F(\theta) \overline{G(\theta)} d\theta = \sum a_n \overline{b_n}$$

2.3 Pointwise convergence

Theorem 2.6. Let f be an integrable function on the circle which is differentiable at a point θ_0 . Then $D_N(f)(\theta_0) \rightarrow f(\theta_0)$ as N tends to infinity.

Proof. Construct a function F

$$F(t) = \begin{cases} \frac{f(\theta_0 - t) - f(\theta_0)}{t}, & \text{if } t \neq 0 \text{ and } |t| < \pi \\ -f'(\theta_0), & \text{if } t = 0. \end{cases}$$

Then use the F to replace f to prove $S_N(f)(\theta_0) - f(\theta_0)$. applying Lemma 2.4.

$$S_N(f)(\theta_0) - f(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) t D_N(t) dt$$

We recall that

$$t D_N(t) = \frac{t}{\sin(t/2)} \sin((N + 1/2)t)$$

Applying Lemma 2.4, we have proved that $S_N(f)(\theta_0) - f(\theta_0) \rightarrow 0$ as $N \rightarrow \infty$.

Why using F ? Because we need $t/\sin(t/2)$ to be integrable. □

Remark. If f satisfies **Lipschitz condition** at θ_0

$$|f(\theta) - f(\theta_0)| \leq M|\theta - \theta_0|$$

for a constant $M \geq 0$ and all θ . This is the same as saying that f satisfies Holder condition of order $\alpha = 1$.

2.3.1 A continuous function with diverging Fourier series

In order to understand convergence better, we need an example of a continuous periodic function whose Fourier series diverges at a point. Thus, theorem 2.6 fails if the differentiability is replaced by weaker assumption of continuity. Review the sawtooth function f

$$\sum_{n \neq 0} \frac{e^{in\theta}}{n}$$

We consider Fourier series

$$\sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n} \quad (3)$$

3 is no longer the Fourier series of a Riemann integrable function. Suppose it were the Fourier series of an integrable function, say \tilde{f} , where in particular \tilde{f} is bounded. Using the Abel means, we have

$$|A_r(\tilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n}$$

Because $\sum 1/n$ diverges, this gives contradiction since

$$|A_r(\tilde{f})(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta)| P_r(\theta) d\theta \leq \sup_{\theta} |\tilde{f}(\theta)|$$

Lemma 2.7. Suppose that the Abel means $A_r = \sum_{n=1}^{\infty} r^n c_n$ of the series $\sum c_n$ are bounded as r tends to 1. If $C_n = (1/n)$, then the partial sums $S_N = \sum_{n=1}^N c_n$ are bounded.

From lemma 2.7, we know $c_n = e^{in\theta}/n - e^{-in\theta}/n$ and hence $\sum c_n$ is bounded. So $S_N(f)(\theta)$ is uniformly bounded in N and θ .

Define $P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$ and $\tilde{P}_N(\theta) = e^{i(2N)\theta} \tilde{f}_N(\theta)$. The rest is to be updated, because I am not fully understand yet.

Remark. *Gibbs' phenomenon is critical when processing signals with discontinuity.*

3 The Fourier Transform on \mathbb{R}

We have introduced the Fourier series of periodic functions. In this chapter, we will study Fourier transform on \mathbb{R} . Which means, we develop an analogous theory for non-periodic functions on \mathbb{R} .

Roughly speaking, the Fourier transform is a continuous version of Fourier coefficients. Hence we have the following equations

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \quad (4)$$

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \quad (5)$$

3.1 Elementary theory

3.1.1 Integration of functions on the real line

f defined on \mathbb{R} is said to be of **moderate decrease** if f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \leq \frac{A}{1+x^2} \text{ for all } x \in \mathbb{R}$$

We denote $\mathcal{M}(\mathbb{R})$ as the set of functions of moderate decrease on \mathbb{R} . Generally, we call a function f of moderate decrease whenever

$$|f(x)| \leq \frac{A}{1+|x|^{1+\epsilon}} \text{ for all } x \in \mathbb{R}$$

And f has the property

$$\int_{-\infty}^{\infty} f(x) dx < \infty$$

3.1.2 Schwartz space

The **Schwartz space** on \mathbb{R} consists the set of all indefinitely differentiable functions f so that f and all its derivatives $f', \dots, f^{(\ell)}, \dots$ are **rapidly decreasing**, in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty \text{ for every } k, \ell \geq 0$$

We denote the space by $\mathcal{S} = \mathcal{S}(\mathbb{R})$. Moreover, if $f \in \mathcal{S}$, we have

$$f'(x) = \frac{df}{dx} \in \mathcal{S} \text{ and } xf(x) \in \mathcal{S}$$

This expresses the important fact that the Schwartz space is closed under differentiation and multiplication by polynomials.

A simple example of a function in \mathcal{S} is the **Gaussian** defined by

$$f(x) = e^{-x^2}$$

Another example is $e^{-|x|}$, although it decreases rapidly, it is not differentiable at 0, so it does not belong to \mathcal{S} .

3.1.3 Fourier transform on \mathcal{S}

The **Fourier transform** of a function $f \in \mathcal{S}$ is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \quad (6)$$

We use notation

$$f(x) \longrightarrow \widehat{f}(\xi)$$

to mean that $\widehat{f}(\xi)$ denotes the Fourier transform of f .

Proposition 3.1. *If $f \in \mathcal{S}(\mathbb{R})$ then:*

1. $f(x+h) \longrightarrow \widehat{f}(\xi) e^{2\pi i h \xi}$ whenever $h \in \mathbb{R}$.
2. $f(x) e^{-2\pi i x h} \longrightarrow \widehat{f}(\xi + h)$ whenever $h \in \mathbb{R}$.
3. $f(\delta x) \longrightarrow \delta^{-1} \widehat{f}(\delta^{-1} \xi)$ whenever $h \in \mathbb{R}$.
4. $f'(x) \longrightarrow 2\pi i \xi \widehat{f}(\xi)$ whenever $h \in \mathbb{R}$.
5. $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi} \widehat{f}(\xi)$ whenever $h \in \mathbb{R}$.

Theorem 3.2. *If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$.*

The proof is trivial, we omit it.

An important formula is **Gaussian**, $e^{-\pi x^2}$

Theorem 3.3.

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r dr d\theta \\ &= \int_0^{\infty} 2\pi r e^{-\pi r^2} dr \\ &= -e^{-\pi r^2} \Big|_0^{\infty} \\ &= 1 \end{aligned}$$

Theorem 3.4. *If $f(x) = e^{-\pi x^2}$, then $\widehat{f}(\xi) = f(\xi)$.*

Proof. Define

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx$$

Because $f'(x) = -2\pi x f(x)$, we obtain

$$\widehat{f}'(\xi) = \int_{-\infty}^{\infty} f(x)(-2\pi i x) e^{-2\pi i x \xi} dx = i \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi} dx$$

By 4 in theorem 3.1, we find that

$$\widehat{f}'(\xi) = i(2\pi i \xi) \widehat{f}(\xi) = -2\pi \xi \widehat{f}(\xi)$$

Hence we have $\widehat{f}(\xi) = e^{-\pi \xi^2}$. \square

Generally, for every $\delta > 0$, $K_\delta(x) = \delta^{-1/2} e^{-\pi x^2/\delta}$, then $\widehat{K}_\delta(\xi) = e^{-\pi \delta \xi^2}$. Similar to definition in Chapter 2, we define **good kernel** on \mathbb{R}

Proposition 3.5. P_δ is a good kernel when

1. $\int_{-\infty}^{\infty} P_\delta(x) dx = 1$.
2. $\int_{-\infty}^{\infty} |P_\delta(x)| dx \leq M$.
3. For every $\eta > 0$, we have $\int_{|x|>\eta} |P_\delta(x)| dx \rightarrow 0$ as $\delta \rightarrow 0$.

Obviously, K_δ is a good kernel. Moreover, the **convolution** on \mathbb{R} is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt \quad (7)$$

Corollary 3.5.1. If $f \in \mathcal{S}(\mathbb{R})$, then

$$(f * K_\delta)(x) \longrightarrow f(x) \text{ uniformly in } x \text{ as } \delta \rightarrow 0$$

3.1.4 Fourier inversion

The next result is an identity sometimes called the multiplication formula.

Proposition 3.6. If $f, g \in \mathcal{S}(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x) \widehat{g}(x) dx = \int_{-\infty}^{\infty} g(x) \widehat{f}(x) dx$$

The proof is trivial, we omit it.

Theorem 3.7 (Fourier inversion). If $f \in \mathcal{S}(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Proof. We have known that $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$, $G_\delta(x) = \widehat{K}_\delta(x) = e^{-\pi\delta x^2}$.

Since K_δ is a good kernel, we obtain

$$\int_{-\infty}^{\infty} f(x)K_\delta(x) dx \longrightarrow f(0) \text{ as } \delta \rightarrow 0 \quad (8)$$

Since $G_\delta(x) \rightarrow 1$ as $\delta \rightarrow 1$, we obtain

$$\int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x) dx \longrightarrow \int_{-\infty}^{\infty} \widehat{f}(x) dx \text{ as } \delta \rightarrow 0 \quad (9)$$

Applying theorem 3.6, we have

$$\int_{-\infty}^{\infty} f(x)K_\delta(x) dx \int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x) dx$$

After reviewed three equations above, we have

$$f(0) = \int_{-\infty}^{\infty} \widehat{f}(\xi) d\xi$$

In general, let $F(y) = f(y+x)$ so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \widehat{F}(\xi) d\xi$$

Applying 2 in proposition 3.1, we have

$$\int_{-\infty}^{\infty} \widehat{F}(\xi) d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x \xi} d\xi$$

proof is completed. □

Remark. If $f \notin \mathcal{S}(\mathbb{R})$, 8 and 9 would be wrong.

We define two mappings $\mathcal{F}, \mathcal{F}^*$ by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx \text{ and } \mathcal{F}^*(f)(x) = \int_{-\infty}^{\infty} f(\xi)e^{2\pi i x \xi} d\xi$$

And we have $\mathcal{F}^* \circ \mathcal{F} = I$, $\mathcal{F} \circ \mathcal{F}^* = I$. Hence we deduce that

Corollary 3.7.1. The Fourier transform is a bijective mapping on $\mathcal{S}(\mathbb{R})$.

3.2 The Plancherel formula

The Fourier transform and Schwartz space has several properties

Proposition 3.8. If $f, g \in \mathcal{S}(\mathbb{R})$, then

1. $f * g \in \mathcal{S}(\mathbb{R})$.

$$2. f * g = g * f.$$

$$3. \widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

The Schwartz space can be equipped with a Hermitian inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

Theorem 3.9 (Plancherel). *If $f \in \mathcal{S}(\mathbb{R})$, then $\|\widehat{f}\| = \|f\|$.*

Proof. The proof on the textbook is complicated. So I write it on my own. First, there is a formula about conjugate function \bar{f} .

$$\begin{aligned} f(x) &= \Re(f(x)) + i\Im(f(x)) \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) + i\Im(\widehat{f}(\xi)))(\cos 2\pi x\xi - i\sin 2\pi x\xi) d\xi \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) \cos 2\pi x\xi + \Im(\widehat{f}(\xi)) \sin 2\pi x\xi) \\ &\quad + i(\Im(\widehat{f}(\xi)) \cos 2\pi x\xi - \Re(\widehat{f}(\xi)) \sin 2\pi x\xi) \end{aligned} \tag{10}$$

$$\begin{aligned} \bar{f}(x) &= \Re(\bar{f}(x)) + i\Im(\bar{f}(x)) \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) - i\Im(\widehat{f}(\xi)))(\cos 2\pi x\xi + i\sin 2\pi x\xi) d\xi \\ &= \int_{-\infty}^{\infty} (\Re(\widehat{f}(\xi)) \cos 2\pi x\xi + \Im(\widehat{f}(\xi)) \sin 2\pi x\xi) \\ &\quad - i(\Im(\widehat{f}(\xi)) \cos 2\pi x\xi - \Re(\widehat{f}(\xi)) \sin 2\pi x\xi) \end{aligned} \tag{11}$$

Applying 10 and 11, we get the formula

$$\bar{f}(x) = \int_{-\infty}^{\infty} \overline{\widehat{f}(\xi)} e^{2\pi i x \xi} d\xi \tag{12}$$

We write down the formula

$$\begin{aligned}
\|f\|^2 &= \int_{-\infty}^{\infty} |f(x)|^2 dx \\
&= \int_{-\infty}^{\infty} f(x) \overline{f(x)} dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-2\pi i x \xi} d\xi \int_{-\infty}^{\infty} \overline{\widehat{f}(\xi')} e^{2\pi i x \xi'} d\xi' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-2\pi i x \xi} \overline{\widehat{f}(\xi')} e^{2\pi i x \xi'} d\xi d\xi' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i x \xi} e^{2\pi i x \xi'} dx \widehat{f}(\xi) \overline{\widehat{f}(\xi')} d\xi d\xi' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\xi - \xi') \widehat{f}(\xi) \overline{\widehat{f}(\xi')} d\xi d\xi' \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi) \overline{\widehat{f}(\xi)} d\xi \\
&= \|\widehat{f}\|^2
\end{aligned} \tag{13}$$

□

Remark. In 13, we used the property that

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-its} ds$$

and

$$\int_{t_1}^{t_2} \delta(t) dt = 1 \quad \text{if} \quad 0 \in [t_1, t_2]$$

More details are introduced in [1]

3.2.1 The Weierstrass approximation theorem

Theorem 3.10. Let f be a continuous function on the closed and bounded interval $[a, b] \subset \mathbb{R}$. Then, $\forall \epsilon > 0$, there exists a polynomial P such that

$$\sup_{x \in [a, b]} |f(x) - P(x)| < \epsilon$$

In other words, f can be uniformly approximated by polynomials.

Proof. The proof is classic, we first extend f to g in \mathbb{R} . Then, we use $g * K_\delta$ to approximate g . Finally, we use Taylor series and choose $R(x)$ as a polynomial to approximate K_δ . An important fact is, $(g * R)(x)$ is a polynomial in x . □

3.2.2 Functions of moderate decrease

In this situation, convolution $f * g$ of two functions of moderate decrease is also a function of moderate decrease. So, we have $\widehat{f * g} = \widehat{f}\widehat{g}$. Moreover, the multiplication formula $\int_{-\infty}^{\infty} f(x)\widehat{g}(x) dx = \int_{-\infty}^{\infty} \widehat{f}(y)g(y) dy$ continues to hold, and we deduce the Fourier inversion and Plancherel theorem when f and \widehat{f} are both of moderate decrease.

3.3 Applications to some partial differential equations

3.3.1 The time-dependent heat equation on the real line

Consider an infinite rod, which we model by the real line, and suppose that we are given an initial temperature distribution $f(x)$ on the rod at time $t = 0$. We wish now to determine the temperature $u(x, t)$ at a point x at time $t > 0$. From some physical properties, we have the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (14)$$

called the **heat equation**. The initial condition we impose is $u(x, 0) = f(x)$. Just as in the case of the circle, the solution is given in terms of a convolution. Define the **heat kernel** of the line by

$$\mathcal{H}_t(x) = K_\delta(x), \quad \text{with } \delta = 4\pi t \quad (15)$$

so that

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \quad \text{and} \quad \widehat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 t \xi^2} \quad (16)$$

And we have

Theorem 3.11. *Given $f \in \mathcal{S}(\mathbb{R})$, let*

$$u(x, t) = (f * \mathcal{H}_t)(x) \quad \text{for } t > 0$$

where \mathcal{H}_t is the heat kernel, then:

1. *The function u is C^2 when $x \in \mathbb{R}$ and $t > 0$, and u solves the heat equation.*
2. *$u(x, t) \rightarrow f(x)$ uniformly in x as $t \rightarrow 0$. Hence if we set $u(x, t) = f(x)$, then u is continuous on the closure of the upper half-plane $\mathbb{R}_+^2 = \{(x, t) : x \in \mathbb{R}, t \geq 0\}$.*
3. *$\int_{-\infty}^{\infty} |u(x, t) - f(x)|^2 dx \rightarrow 0$ as $t \rightarrow 0$.*

The theorem 3.11 guarantees the existence of a solution to the heat equation with initial data f . This solution is also unique. In this regard, we note that $u = f * \mathcal{H}_t, f \in \mathbb{R}$.

3.3.2 The steady-state heat equation in the upper half-plane

The equation we are now concerned with is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (17)$$

in the upper half-plane $\overline{\mathbb{R}_+^2} = \{(x, t) : x \in \mathbb{R}, t \geq 0\}$. The boundary condition we require is $u(x, 0) = f(x)$. The kernel that solves the problem is called the **Poisson kernel** for the upper half-plane.

$$\mathcal{P}_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \quad \text{where } x \in \mathbb{R} \text{ and } y > 0$$

Notably, for each fixed y , the kernel \mathcal{P}_y is only of moderate decrease as a function of x . We take the Fourier transform of 17 in the x variable, thereby obtaining

$$-4\pi^2 \xi^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}}{\partial y^2}(\xi, y) = 0$$

by applying proposition 3.1. The general solution of this ordinary differential equation in y (with ξ fixed) takes the form

$$\hat{u}(\xi, y) = A(\xi)e^{-2\pi|\xi|y} + B(\xi)e^{2\pi|\xi|y}$$

If we disregard the second term because its rapid exponential increase we find, after setting $y = 0$, that

$$\hat{u}(\xi, y) = \hat{f}(\xi)e^{-2\pi|\xi|y}$$

Remark. In the above two examples, we apply Fourier transform in order to find some properties of ordinary differential equations. Normally, it would be a factor of exponent of e .

Lemma 3.12. The following identities hold:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i \xi x} d\xi &= \mathcal{P}_y(x) \\ \int_{-\infty}^{\infty} \mathcal{P}_y(x) e^{-2\pi i \xi x} dx &= e^{-2\pi|\xi|y} \end{aligned}$$

Lemma 3.13. The Poisson kernel is a good kernel as $y \rightarrow 0$.

Lemma 3.14 (Mean-value property). Suppose ω is an open set in \mathbb{R}^2 and let u be a function of class C^2 with $\Delta u = 0$ in ω . If the closure of the disc centered at (x, y) and of radius R is contained in ω , then

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r \cos \theta, y + r \sin \theta) d\theta$$

for all $0 \leq r \leq R$.

Proof. The proof is tricky. First, the equation $\Delta u = 0$ implies

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 u}{\partial \theta^2} &= 0 \\ \frac{\partial^2 u}{\partial \theta^2} + r \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) &= 0\end{aligned}$$

If we define $F(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta$, we deduce that

$$r \frac{\partial}{\partial r} \left(r \frac{\partial F}{\partial r} \right) = \frac{1}{2\pi} \int_0^{2\pi} -\frac{\partial^2 u}{\partial \theta^2}(r, \theta) d\theta$$

Since $\partial u / \partial \theta$ is periodic, we have $r \frac{\partial}{\partial r} \left(r \frac{\partial F}{\partial r} \right) = 0$. And consequently $r \partial F / \partial r$ must be a constant. Hence $\partial F / \partial r = 0$. Since $u(x, y) = F(0)$, we deduce that $F(r) = u(x, y)$. Proof completed. \square

Hence we have the following theorem

Theorem 3.15. *Given $f \in \mathcal{S}(\mathbb{R})$, let $u(x, y) = (f * \mathcal{P}_y)(x)$. Then:*

1. $u(x, y)$ is C^2 in \mathbb{R}_+^2 and $\Delta u = 0$.
2. $u(x, y) \rightarrow f(x)$ uniformly as $y \rightarrow 0$.
3. $\int_{-\infty}^{\infty} |u(x, y) - f(x)|^2 dx \rightarrow 0$ as $y \rightarrow 0$.
4. If $u(x, 0) = f(x)$, then u is continuous on the closure $\overline{\mathbb{R}_+^2}$, and vanishes at infinity

$$u(x, y) \rightarrow 0 \text{ as } |x| + y \rightarrow \infty$$

3.4 The Poisson summation formula

This section reveals further connection between Fourier series and Fourier transform.

Given a function $f \in \mathcal{S}(\mathbb{R})$ on the real line, we can construct a new function on the circle by the recipe

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

Since f is rapidly decreasing, the series converges absolutely and uniformly on every compact subset of \mathbb{R} , so f is continuous.

We consider another form of F

$$F_2(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

The fundamental fact is that these two approaches, which produce F_1 and F_2 , actually lead to the same function.

Theorem 3.16 (Poisson summation formula). *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

In particular, setting $x = 0$ we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

Proof. The proof is trivial, but the theorem is very important. We can apply the property of Fourier series, and F_2 is the Fourier series of F_1 . \square

3.4.1 Theta and zeta functions

In this section, we introduce some basic properties of theta and zeta functions. We define **theta function** $\vartheta(s)$ for $s > 0$ by

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$$

which satisfies the following equation

Theorem 3.17. $s^{-1/2} \vartheta(1/s) = \vartheta(s)$ whenever $s > 0$.

The proof is only a simple application of Poisson summation formula to the pair

$$f(x) = e^{-\pi s x^2} \quad \text{and} \quad \widehat{f}(\xi) = s^{1/2} e^{-\pi \xi^2 / s}$$

The **Zeta function** $\zeta(s)$ defined for $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

ϑ and ζ have the following property

Theorem 3.18.

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{2} \int_0^{\infty} t^{s/2-1} (\vartheta(t) - 1) dt$$

Proof. To be updated. \square

Another important function $\Theta(z|\tau)$ is defined by

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau} e^{2\pi i n z}$$

whenever $\Im(\tau) > 0$ and $z \in \mathbb{C}$. Taking $z = 0$ and $\tau = is$ we get $\Theta(z|\tau) = \vartheta(s)$.

3.4.2 Heat kernel

We define heat kernel on the circle by

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

Heat kernel on \mathbb{R} is given by

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

Corollary 3.18.1. *The kernel $H_t(x)$ is a good kernel as $t \rightarrow 0$.*

3.4.3 Poisson kernels

We recall the definition of Poisson kernels

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} \quad \text{and} \quad \mathcal{P}_y = \frac{1}{\pi} \frac{y}{x^2+y^2}$$

hence we have the theorem

Theorem 3.19. $P_r(2\pi x) = \sum_{n \in \mathbb{Z}} \mathcal{P}_y(x+n)$ where $r = e^{-2\pi y}$.

3.5 The Heisenberg uncertainty principle

Theorem 3.20. *Suppose ψ is a function in $\mathcal{S}(\mathbb{R})$ which satisfies the normalizing condition $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. Then*

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} \xi^2 |\widehat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}$$

and equality holds iff $\psi(x) = Ae^{-Bx^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$. In fact, we have

$$\left(\int_{-\infty}^{\infty} (x-x_0)^2 |\psi(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} (\xi-\xi_0)^2 |\widehat{\psi}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}$$

for every $x_0, \xi_0 \in \mathbb{R}$.

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