

标量导数

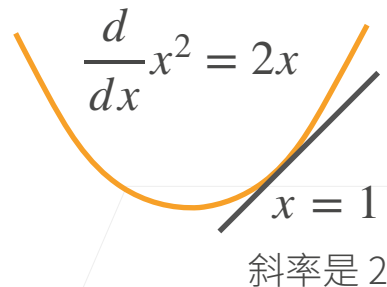


y	a	x^n	$\exp(x)$	$\log(x)$	$\sin(x)$
$\frac{dy}{dx}$	0	nx^{n-1}	$\exp(x)$	$\frac{1}{x}$	$\cos(x)$

a 不是 x 的函数

y	$u + v$	uv	$y = f(u), u = g(x)$
$\frac{dy}{dx}$	$\frac{du}{dx} + \frac{dv}{dx}$	$\frac{du}{dx}v + \frac{dv}{dx}u$	$\frac{dy}{du} \frac{du}{dx}$

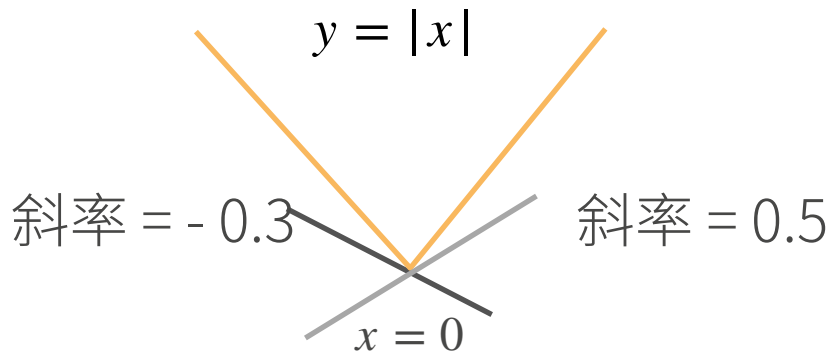
导数是切线的斜率



亚导数



- 将导数拓展到不可微的函数



$$\frac{\partial |x|}{\partial x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ a & \text{if } x = 0, \quad a \in [-1, 1] \end{cases}$$

另一个例子

$$\frac{\partial}{\partial x} \max(x, 0) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \\ a & \text{if } x = 0, \quad a \in [0, 1] \end{cases}$$

梯度

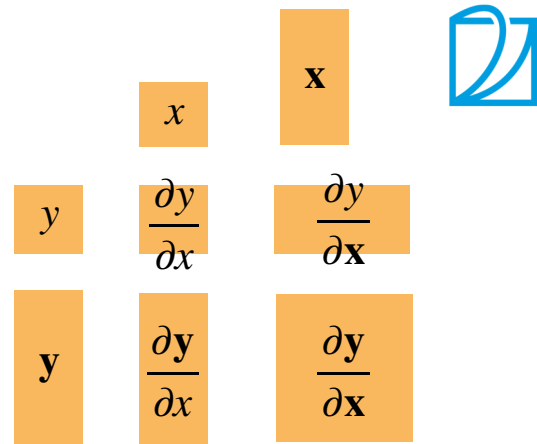


- 将导数拓展到向量

		标量		向量
		x		\mathbf{x}
标量	y	$\frac{\partial y}{\partial x}$		$\frac{\partial y}{\partial \mathbf{x}}$
向量	\mathbf{y}	$\frac{\partial \mathbf{y}}{\partial x}$		$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$

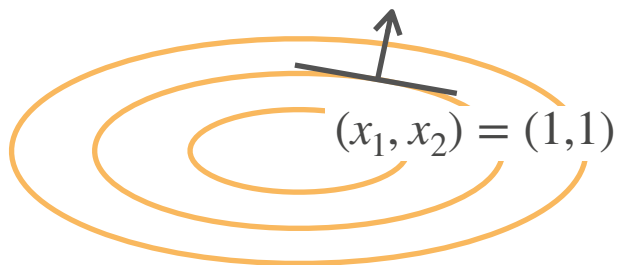
$$\partial y / \partial \mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \frac{\partial y}{\partial \mathbf{x}} = \left[\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n} \right]$$



$$\frac{\partial}{\partial \mathbf{x}} x_1^2 + 2x_2^2 = [2x_1, 4x_2]$$

方向 (2, 4) 跟等高线正交



样例



y	a	au	$\text{sum}(\mathbf{x})$	$\ \mathbf{x}\ ^2$
$\frac{\partial y}{\partial \mathbf{x}}$	$\mathbf{0}^T$	$a \frac{\partial u}{\partial \mathbf{x}}$	$\mathbf{1}^T$	$2\mathbf{x}^T$

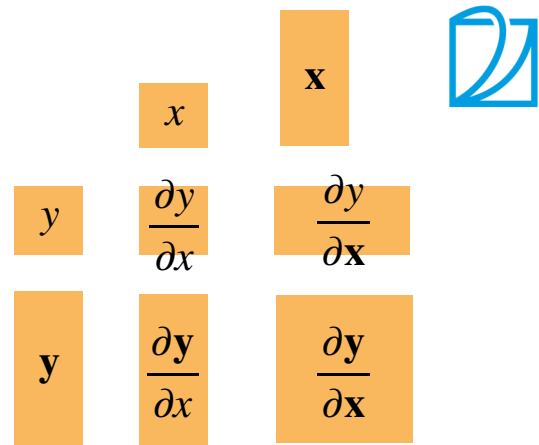
a is not a function of \mathbf{x}

$\mathbf{0}$ and $\mathbf{1}$ are vectors

y	$u + v$	uv	$\langle \mathbf{u}, \mathbf{v} \rangle$
$\frac{\partial y}{\partial \mathbf{x}}$	$\frac{\partial u}{\partial \mathbf{x}} + \frac{\partial v}{\partial \mathbf{x}}$	$\frac{\partial u}{\partial \mathbf{x}} v + \frac{\partial v}{\partial \mathbf{x}} u$	$\mathbf{u}^T \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v}^T \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$

$$\partial \mathbf{y} / \partial x$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad \frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$$



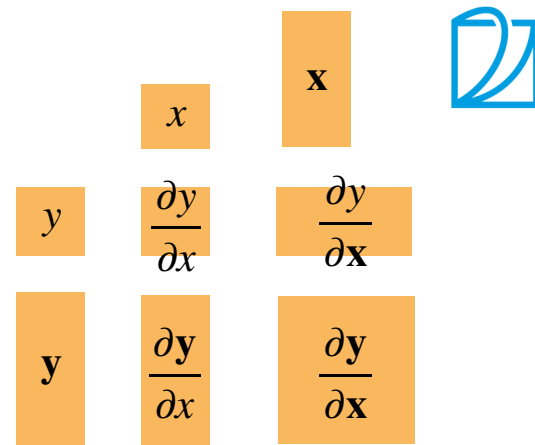
$\partial y / \partial \mathbf{x}$ 是行向量, $\partial \mathbf{y} / \partial x$ 是列向量

这个被称之为分子布局符号, 反过来的版本叫分母布局符号

$\partial \mathbf{y} / \partial \mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} \\ \frac{\partial y_2}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial y_m}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_1}{\partial x_2}, \dots, \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \dots, \frac{\partial y_2}{\partial x_n} \\ \vdots \\ \frac{\partial y_m}{\partial x_1}, \frac{\partial y_m}{\partial x_2}, \dots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$



样例



y	\mathbf{a}	\mathbf{x}	\mathbf{Ax}	$\mathbf{x}^T \mathbf{A}$
$\frac{\partial y}{\partial \mathbf{x}}$	$\mathbf{0}$	\mathbf{I}	\mathbf{A}	\mathbf{A}^T

$$\mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y} \in \mathbb{R}^m, \quad \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \in \mathbb{R}^{m \times n}$$

a , \mathbf{a} and \mathbf{A} are not functions of \mathbf{x}

$\mathbf{0}$ and \mathbf{I} are matrices

y	$a\mathbf{u}$	\mathbf{Au}	$\mathbf{u} + \mathbf{v}$
$\frac{\partial y}{\partial \mathbf{x}}$	$a \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\mathbf{A} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$



拓展到矩阵

	标量	向量	矩阵
	x (1,)	\mathbf{x} (n,1)	\mathbf{X} (n,k)
标量	y (1,)	$\frac{\partial y}{\partial x}$ (1,)	$\frac{\partial y}{\partial \mathbf{X}}$ (k,n)
向量	\mathbf{y} (m,1)	$\frac{\partial \mathbf{y}}{\partial x}$ (m,1)	$\frac{\partial \mathbf{y}}{\partial \mathbf{X}}$ (m,k,n)
矩阵	\mathbf{Y} (m,l)	$\frac{\partial \mathbf{Y}}{\partial x}$ (m,l)	$\frac{\partial \mathbf{Y}}{\partial \mathbf{X}}$ (m,l,k,n)



链式法则和自动求导

